Note on solving Clairaut and d'Alembert (or Lagrange) first order ODE

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1 Introduction

This note is about how to solve two ODE's, the first is of the form

$$y(x) = x\frac{dy}{dx} + f\left(\frac{dy}{dx}\right) \tag{1}$$

And the second is of the form

$$y(x) = xg\left(\frac{dy}{dx}\right) + f\left(\frac{dy}{dx}\right) \tag{2}$$

The first ODE above is called the Clairaut ODE and the second is called d'Alembert (also called Lagrange ODE in some books). In the above f and g are functions of $p \equiv \frac{dy}{dx}$.

The Clairaut ODE is a special case of the d'Alembert ODE when the function $g(p) = p$.

Both ODE's are linear in $y(x)$ and in x. In Clairaut ODE $g(p)$ can only be p. Anything else, even $g(p) = 1$, means it is not Clairaut, but can be d'Alembert.

1.1 General method to solve Clairaut ODE

Starting with the Clairaut ODE as it is easier to solve than d'Alembert. Let $p \equiv \frac{dy}{dx}$. Eq. (1) now becomes

$$y(x) = xp + f(p) \tag{3}$$

Taking derivative w.r.t. x of the above gives

$$\begin{aligned} p &= p + x\frac{dp}{dx} + \frac{df\left(p\right)}{dp}\frac{dp}{dx} \\ 0 &= \left(x + \frac{df\left(p\right)}{dp}\right)\frac{dp}{dx} \end{aligned}$$

Therefore either $\frac{dp}{dx} = 0$ or $x + \frac{\partial f(p)}{\partial p} = 0$. The complete integral (i.e. the general solution) is obtained from $\frac{dp}{dx} = 0$ and the singular solution (if any) is obtained from solving $x + \frac{df(p)}{dp} = 0$. $\frac{dp}{dx} = 0$ implies that $p = C_1$ where C_1 is some constant to be found. Substituting $p = C_1$ in Eq. (3) gives

$$y(x) = C_1 x + f(C_1) \tag{4}$$

Now the singular solution is found by solving for p from the ODE $x + \frac{df(p)}{dp} = 0$ and substituting the solution p back into (3). This completes the solution for Clairaut ODE.

1.2 General method to solve the d'Alembert ODE

Let $p \equiv \frac{dy}{dx}$, then Eq(2) becomes

$$y(x) = xg\left(p\right) + f\left(p\right) \tag{5}$$

Taking derivative w.r.t. x gives

$$\begin{aligned} p &= g\left(p\right) + xg'\left(p\right)\frac{dp}{dx} + f'\left(p\right)\frac{dp}{dx} \\ p - g\left(p\right) &= \left(xg'\left(p\right) + f'\left(p\right)\right)\frac{dp}{dx} \end{aligned} \tag{6}$$

When $\frac{dp}{dx} = 0$, then p is constant, say p_0. These constants are found by solving for p from $p - g\left(p\right) = 0$ and substituting the result back into (5). This gives the singular solution. The general solution is found by rewriting (6) as

$$\begin{aligned} \frac{p - g\left(p\right)}{xg'\left(p\right) + f'\left(p\right)} &= \frac{dp}{dx} \\ \frac{dx}{dp} &= \frac{xg'\left(p\right) + f'\left(p\right)}{p - g\left(p\right)} \\ \frac{dx}{dp} - x\frac{g'\left(p\right)}{p - g\left(p\right)} &= \frac{f'\left(p\right)}{p - g\left(p\right)} \end{aligned}$$

$x(p)$ is now the dependent variable and p as the independent variable. The above is a linear first order in $x(p)$ since it is of the form

$$x' + xG(p) = Q(p)$$

This is easily solved using an integration factor

$$\begin{aligned}
\frac{d}{dp}\left(xe^{\int G(p)dp}\right) &= e^{\int G(p)dp}Q(p) \\
xe^{\int G(p)dp} &= \int e^{\int G(p)dp}Q(p) + C_1 \\
x(p) &= e^{-\int G(p)dp}\int e^{\int G(p)dp}Q(p) + C_1 e^{-\int G(p)dp}
\end{aligned}$$

Once $x(p)$ is found from the above as function of p, then p is found by inversion of the solution. Substituting this p back into (5) gives the general solution $y(x)$.

To show how these method work, the following ODE's are now solved.

1.3 General method to solve the d'Alembert ODE when f(p)=0

This is special case of the general d'Alembert $y = xg\left(\frac{dy}{dx}\right) + f(p)$. The ODE now reduces to

$$y = xg\left(\frac{dy}{dx}\right)$$

Where $f\left(\frac{dy}{dx}\right) = 0$ and $g\left(\frac{dy}{dx}\right)$ must be non-linear in $\frac{dy}{dx}$. For an example, $y(x) = x\left(\frac{dy}{dx}\right)^2$ is d'Alembert.

Let $p \equiv \frac{dy}{dx}$, then $y = xg\left(\frac{dy}{dx}\right)$ becomes

$$y(x) = xg(p) \tag{5}$$

Taking derivative w.r.t. x gives

$$\begin{aligned}
p &= g(p) + xg'(p)\frac{dp}{dx} \\
p - g(p) &= xg'(p)\frac{dp}{dx} \qquad (6)
\end{aligned}$$

When $\frac{dp}{dx} = 0$, then p is constant, say p_0. These constants are found by solving for p from $p - g(p) = 0$ and substituting the result back into (5). This gives the singular solution. The general solution is found by rewriting (6) as

$$\begin{aligned}
\frac{p-g\left(p\right)}{xg'\left(p\right)} &= \frac{dp}{dx} \\
\frac{dx}{dp} &= \frac{xg'\left(p\right)}{p-g\left(p\right)} \\
\frac{dx}{dp} - x\frac{g'\left(p\right)}{p-g\left(p\right)} &= 0
\end{aligned}$$

$x\left(p\right)$ is now the dependent variable and p as the independent variable. The above is a linear first order in $x\left(p\right)$ since it is of the form

$$x' + xG\left(p\right) = 0$$

This is easily solved using an integration factor

$$\begin{aligned}
\frac{d}{dp}\left(xe^{\int G\left(p\right)dp}\right) &= 0 \\
xe^{\int G\left(p\right)dp} &= C_1 \\
x\left(p\right) &= C_1 e^{-\int G\left(p\right)dp}
\end{aligned}$$

Once $x\left(p\right)$ is found from the above as function of p, then p is found by inversion of the solution. Substituting this p back into (5) gives the general solution $y(x)$.

To show how these method work, the following ODE's are now solved.

2 Solved examples

number	ode	transformed	$g(p)$	$f(p)$	type
1	$x\left(y'\right)^2 - yy' = -1$	$y = xp + \frac{1}{p}$	p	$\frac{1}{p}$	Clairaut
2	$y = xy' - \left(y'\right)^2$	$y = xp - p^2$	p	$-p^2$	Clairaut
3	$y = xy' - \frac{1}{4}\left(y'\right)^2$	$y = xp - \frac{1}{4}p^2$	p	$-\frac{1}{4}p^2$	Clairaut
4	$y = x\left(y'\right)^2$	$y = xp^2$	p^2	0	d'Alembert
5	$y = x + \left(y'\right)^2$	$y = x + p^2$	1	p^2	d'Alembert
6	$\left(y'\right)^2 - 1 - x - y = 0$	$y = -x + \left(p^2 - 1\right)$	-1	$p^2 - 1$	d'Alembert
7	$yy' - \left(y'\right)^2 = x$	$y = \frac{1}{p}x + p$	$\frac{1}{p}$	p	d'Alembert
8	$y = x\left(y'\right)^2 + \left(y'\right)^2$	$y = xp^2 + p^2$	p^2	p^2	d'Alembert
9	$y = \frac{x}{a}y' + \frac{b}{ay'}$	$y = \frac{x}{a}p + \frac{b}{a}p^{-1}$	$\frac{p}{a}$	$\frac{b}{ap}$	d'Alembert
10	$y = x\left(y' + a\sqrt{1 + \left(y'\right)^2}\right)$	$y = x\left(p + a\sqrt{1 + p^2}\right)$	$p + a\sqrt{1 + p^2}$	0	d'Alembert
11	$y = x\left(y'\right)^2$	$y = xp^2$	p^2	0	d'Alembert

Notice in the above table, that the ODE is Clairaut only when $g(p) = p$ and d'Alembert otherwise.

2.1 Example 1

To solve $x\left(y'\right)^2 - yy' = -1$, we first write it in normal form (by replacing y' with p), gives

$$y(x) = xp + \frac{1}{p} \tag{1}$$

Taking derivative w.r.t. x gives

$$\begin{aligned} p &= p + x\frac{dp}{dx} - \frac{1}{p^2}\frac{dp}{dx} \\ 0 &= \left(x - \frac{1}{p^2}\right)\frac{dp}{dx} \end{aligned}$$

The general solution is found from $\frac{dp}{dx} = 0$. This implies $p = C_1$. Substituting this into (1) gives

$$y(x) = xC_1 + \frac{1}{C_1} \tag{2}$$

The singular solution is found by solving $x - \frac{1}{p^2} = 0$. Hence $p^2 = \frac{1}{x}$ or $p = \pm\sqrt{\frac{1}{x}}$. Substituting these back in (1) gives

$$y_1(x) = x\sqrt{\frac{1}{x}} + \sqrt{x} = 2\sqrt{x} \tag{3}$$

$$y_2(x) = -x\sqrt{\frac{1}{x}} - \sqrt{x} = -2\sqrt{x} \tag{4}$$

Eq. (2) is the complete integral and (3,4) are the singular solutions.

2.2 Example 2

To solve $y = xy' - \left(y'\right)^2$, we first write it in normal form (by replacing y' with p), gives

$$y = xp - p^2 \tag{1}$$

Taking derivative w.r.t. x gives

$$\begin{aligned} p &= p + x\frac{dp}{dx} - 2p\frac{dp}{dx} \\ 0 &= \left(x - 2p\right)\frac{dp}{dx} \end{aligned}$$

The general solution is found from $\frac{dp}{dx} = 0$. This implies $p = C_1$. Substituting this into (1) gives

$$y(x) = xC_1 - C_1^2 \tag{2}$$

The singular solution is found by solving $x - 2p = 0$. Hence $p = \frac{x}{2}$. Substituting these back in (1) gives

$$\begin{aligned} y_1(x) &= \frac{x^2}{2} - \frac{x^2}{4} \\ &= \frac{x^2}{4} \end{aligned} \tag{3}$$

Eq. (2) is the complete integral and (3) is the singular solution.

2.3 Example 3

To solve $y = xy' - \frac{1}{4}\left(y'\right)^2$, we first write it in normal form (by replacing y' with p),gives

$$y = xp - \frac{1}{4}p^2 \tag{1}$$

Taking derivative w.r.t. x gives

$$\begin{aligned} p &= p + x\frac{dp}{dx} - \frac{1}{2}p\frac{dp}{dx} \\ 0 &= \left(x - \frac{1}{2}p\right)\frac{dp}{dx} \end{aligned}$$

The general solution is found from $\frac{dp}{dx} = 0$. This implies $p = C_1$. Substituting this into (1) gives

$$y(x) = xC_1 - \frac{1}{4}C_1^2 \tag{2}$$

The singular solution is found by solving $x - \frac{1}{2}p = 0$. Hence $p = 2x$. Substituting this back in (1) gives

$$y_1(x) = 2x^2 - x^2 = x^2 \tag{3}$$

Eq. (2) is the complete integral and (3) is the singular solutions.

2.4 Example 4

To solve $y = x\left(y'\right)^2$, we first write it in normal form (by replacing y' with p), gives

$$y = xp^2 \tag{1}$$

Taking derivative w.r.t. x gives

$$\begin{aligned} p &= p^2 + 2xp\frac{dp}{dx} \\ p - p^2 &= 2xp\frac{dp}{dx} \end{aligned} \tag{2}$$

To find the singular solution, we consider when $\frac{dp}{dx} = 0$, which implies that p is constant. Hence $p - p^2 = 0$ in this case. Solving for this gives $p = 0$ or $p = 1$. For each one of these values, we obtain a singular solution by substituting these values in (1), which gives

$$\begin{aligned} y_1(x) &= 0 \\ y_2(x) &= x \end{aligned}$$

Now we need to find the general solution which is when $\frac{dp}{dx} \neq 0$. From (2), writing it as

$$\begin{aligned} \frac{p - p^2}{2xp} &= \frac{dp}{dx} \\ \frac{dx}{dp} &= \frac{2xp}{p - p^2} \end{aligned}$$

This is linear ODE in $x\left(p\right)$. Solving it gives $x = \frac{C_1}{\left(p-1\right)^2}$. This implies $\left(p-1\right)^2 = \frac{C_0}{x}$ or $p-1 = \pm\sqrt{\frac{C_0}{x}}$ or

$$
\begin{aligned}
p &= 1+\sqrt{\frac{C_0}{x}} \\
p &= 1-\sqrt{\frac{C_0}{x}}
\end{aligned}
$$

For each p, there is a solution. Substituting the above in (1) gives

$$
\begin{aligned}
y_3(x) &= x\left(1+\sqrt{\frac{C_0}{x}}\right)^2 \\
y_4(x) &= x\left(1-\sqrt{\frac{C_0}{x}}\right)^2
\end{aligned}
$$

Note however, that $y_2 = x$ can be obtained from $y_3(x)$ when $C_0 = 0$. Hence $y_2(x) = x$ is not singular solution. Therefore the final solution is

$$
\begin{aligned}
y(x) &= 0 \\
y(x) &= x\left(1+\sqrt{\frac{C_0}{x}}\right)^2 \\
y(x) &= x\left(1-\sqrt{\frac{C_0}{x}}\right)^2
\end{aligned}
$$

2.5 Example 5

To solve $y = x + \left(y'\right)^2$, we first write it in normal form (by replacing y' with p), gives

$$
y = x + p^2 \tag{1}
$$

Taking derivative w.r.t. x gives

$$
\begin{aligned}
p &= 1 + 2p\frac{dp}{dx} \\
p - 1 &= 2p\frac{dp}{dx} \qquad (2)
\end{aligned}
$$

To find the singular solution, we consider when $\frac{dp}{dx} = 0$, which implies that p is constant. Hence $p-1 = 0$ in this case. Solving for this gives $p = 1$. Substituting this values in (1), gives the solution

$$
y(x) = x + 1 \tag{3}
$$

Now we need to find the general solution which is when $\frac{dp}{dx} \neq 0$. From (2), writing it as

$$\begin{aligned} \frac{p-1}{2p} &= \frac{dp}{dx} \\ \frac{dx}{dp} &= \frac{2p}{p-1} \end{aligned}$$

Integrating gives

$$\begin{aligned} x &= 2p + 2\ln\left(p-1\right) + C \\ p &= \text{LambertW}\left(e^{\frac{x}{2}-1-\frac{C}{2}}\right) + 1 \\ &= \text{LambertW}\left(C_1 e^{\frac{x}{2}-1}\right) + 1 \end{aligned}$$

Substituting the above in (1) gives the general solution

$$y(x) = x + \left(\text{LambertW}\left(C_1 e^{\frac{x}{2}-1}\right) + 1\right)^2 \tag{4}$$

Note however that when $C_1 = 0$ then the general solution becomes $y(x) = x + 1$. Hence (3) is a particular solution and not a singular solution. Hence (4) is the only solution.

2.6 Example 6

To solve $\left(y'\right)^2 - 1 - x - y = 0$, we first write it in normal form (by replacing y' with p), gives

$$y = -x + \left(p^2 - 1\right) \tag{1}$$

Taking derivative w.r.t. x gives

$$\begin{aligned} p &= -1 + 2p\frac{dp}{dx} \\ p + 1 &= 2p\frac{dp}{dx} \end{aligned} \tag{2}$$

To find the singular solution, we consider when $\frac{dp}{dx} = 0$, which implies that p is constant. Hence $p + 1 = 0$ in this case. Solving for this gives $p = -1$. Substituting this values in (1), gives the solution

$$y(x) = -x \tag{3}$$

Now we need to find the general solution which is when $\frac{dp}{dx} \neq 0$. From (2), writing it as

$$\begin{aligned} \frac{p+1}{2p} &= \frac{dp}{dx} \\ \frac{dx}{dp} &= \frac{2p}{p+1} \end{aligned}$$

Integrating gives

$$
\begin{aligned}
x &= 2p - 2\ln\left(p+1\right) + C \\
p &= -\operatorname{LambertW}\left(-e^{-\frac{x}{2}-1+\frac{C}{2}}\right) - 1 \\
&= -\operatorname{LambertW}\left(-C_1 e^{-\frac{x}{2}-1}\right) + 1
\end{aligned}
$$

Substituting the above in (1) gives the general solution

$$
\begin{aligned}
y(x) &= -x + \left(p^2 - 1\right) \\
y(x) &= -x + \left(-\operatorname{LambertW}\left(-C_1 e^{-\frac{x}{2}-1}\right) + 1\right)^2 - 1 \qquad (4)
\end{aligned}
$$

Note however that when $C_1 = 0$ then the general solution becomes $y(x) = -x$. Hence (3) is a particular solution and not a singular solution. Solution (4) is the only solution.

2.7 Example 7

Solving $yy' - \left(y'\right)^2 = x$. Writing it in normal form (by replacing y' with p), gives

$$
y = \frac{1}{p}x + p \qquad (1)
$$

Taking derivative w.r.t. x gives

$$
\begin{aligned}
p &= \frac{1}{p} - \frac{x}{p^2}\frac{dp}{dx} + \frac{dp}{dx} \\
p - \frac{1}{p} &= \left(1 - \frac{x}{p^2}\right)\frac{dp}{dx} \qquad (2)
\end{aligned}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$, which implies that p is a constant. In this case $p - \frac{1}{p} = 0$. Solving for this gives $p = \pm 1$. Substituting these values in (1) gives the solutions

$$
\begin{aligned}
y_1(x) &= x + 1 \qquad (3) \\
y_2(x) &= -(x + 1) \qquad (4)
\end{aligned}
$$

The general solution is found when $\frac{dp}{dx} \neq 0$. Writing (2) as

$$
\begin{aligned}
\frac{\frac{p^2-1}{p}}{1-\frac{x}{p^2}} &= \frac{dp}{dx} \\
\frac{dx}{dp} &= \frac{\frac{p^2-x}{p^2}}{\frac{p^2-1}{p}} \\
&= \frac{p^2-x}{p\left(p^2-1\right)} \\
\frac{dx}{dp} + x\frac{1}{p\left(p^2-1\right)} &= \frac{p}{\left(p^2-1\right)}
\end{aligned}
$$

This is linear ODE in $x\left(p\right)$. The solution is

$$
\begin{aligned}
x\left(p\right) &= \frac{p\sqrt{\left(p-1\right)\left(1+p\right)}\ln\left(p+\sqrt{p^2-1}\right)}{\left(1+p\right)\left(p-1\right)} + c_1\frac{p}{\sqrt{\left(1+p\right)\left(p-1\right)}} \\
&= \frac{p\sqrt{p^2-1}\ln\left(p+\sqrt{p^2-1}\right)}{p^2-1} + c_1\frac{p}{\sqrt{p^2-1}} \qquad (5)
\end{aligned}
$$

From (1) since $y = \frac{1}{p}x + p$ then solving for p gives

$$
\begin{aligned}
p_1 &= \frac{y}{2} + \frac{1}{2}\sqrt{y^2-4x} \\
p_2 &= \frac{y}{2} - \frac{1}{2}\sqrt{y^2-4x}
\end{aligned}
$$

For each p_i above, substituting into (5) gives solution (implicit) for $y\left(x\right)$. First solution is

$$
x = \frac{\left(\frac{y}{2}+\frac{1}{2}\sqrt{y^2-4x}\right)\sqrt{\left(\frac{y}{2}+\frac{1}{2}\sqrt{y^2-4x}\right)^2-1}\,\ln\left(\frac{y}{2}+\frac{1}{2}\sqrt{y^2-4x}+\sqrt{\left(\frac{y}{2}+\frac{1}{2}\sqrt{y^2-4x}\right)^2-1}\right)}{\left(\frac{y}{2}+\frac{1}{2}\sqrt{y^2-4x}\right)^2-1} + c_1\frac{\frac{y}{2}+\frac{1}{2}\sqrt{y^2-4x}}{\sqrt{\left(\frac{y}{2}+\frac{1}{2}\sqrt{y^2-4x}\right)^2}}
$$

And second solution is

$$
x = \frac{\left(\frac{y}{2}-\frac{1}{2}\sqrt{y^2-4x}\right)\sqrt{\left(\frac{y}{2}-\frac{1}{2}\sqrt{y^2-4x}\right)^2-1}\,\ln\left(\frac{y}{2}-\frac{1}{2}\sqrt{y^2-4x}+\sqrt{\left(\frac{y}{2}-\frac{1}{2}\sqrt{y^2-4x}\right)^2-1}\right)}{\left(\frac{y}{2}-\frac{1}{2}\sqrt{y^2-4x}\right)^2-1} + c_1\frac{\frac{y}{2}-\frac{1}{2}\sqrt{y^2-4x}}{\sqrt{\left(\frac{y}{2}-\frac{1}{2}\sqrt{y^2-4x}\right)^2}}
$$

2.8 Example 8

Solving $y = x\left(y'\right)^2 + \left(y'\right)^2$. Writing it in normal form (by replacing y' with p), gives

$$y = xp^2 + p^2 \tag{1}$$

Taking derivative w.r.t. x gives

$$\begin{aligned} p &= p^2 + 2xp\frac{dp}{dx} + 2p\frac{dp}{dx} \\ p - p^2 &= \left(2xp + 2p\right)\frac{dp}{dx} \end{aligned} \tag{2}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$, which implies that p is a constant. In this case $p\left(1-p\right) = 0$. Solving for this gives $p = 0, p = 1$. Substituting these values in (1) gives the solutions

$$y_1(x) = 0 \tag{3}$$

$$y_2(x) = x + 1 \tag{4}$$

The general solution is found when $\frac{dp}{dx} \neq 0$. Writing (2) as

$$\begin{aligned} \frac{p\left(1-p\right)}{2p\left(x+1\right)} &= \frac{dp}{dx} \\ \frac{dx}{dp} &= \frac{2\left(x+1\right)}{\left(1-p\right)} \\ \frac{dx}{dp} - x\frac{2}{\left(1-p\right)} &= \frac{2}{\left(1-p\right)} \end{aligned}$$

This is linear ODE in $x\left(p\right)$. The solution is

$$x = \frac{C^2}{\left(p-1\right)^2} - 1$$

Hence

$$\begin{aligned} \frac{C^2}{\left(p-1\right)^2} &= x + 1 \\ \left(p-1\right)^2 &= \frac{C^2}{x+1} \\ \left(p-1\right) &= \pm\frac{C}{\sqrt{x+1}} \\ p &= 1 \pm \frac{C}{\sqrt{x+1}} \end{aligned}$$

Substituting the above in (1) gives the general solutions

$$y = (x+1)\,p^2$$

Therefore

$$\begin{aligned} y(x) &= (x+1)\left(1 + \frac{C}{\sqrt{x+1}}\right)^2 \\ y(x) &= (x+1)\left(1 - \frac{C}{\sqrt{x+1}}\right)^2 \end{aligned}$$

The solution $y_1(x) = 0$ found earlier is seen to be singular since it can not be obtained from the above general solution. But $y_2(x) = x+1$ can be obtained from the general solution when $C = 0$. Hence there are three solutions, they are

$$\begin{aligned} y_1(x) &= 0 \\ y_2(x) &= (x+1)\left(1 + \frac{C}{\sqrt{x+1}}\right)^2 \\ y_3(x) &= (x+1)\left(1 - \frac{C}{\sqrt{x+1}}\right)^2 \end{aligned}$$

2.9 Example 9

Solving $y = \frac{x}{a}y' + \frac{b}{ay'}$. Writing it in normal form (by replacing y' with p), gives

$$y = \frac{x}{a}p + \frac{b}{a}p^{-1} \tag{1}$$

Taking derivative w.r.t. x gives

$$\begin{aligned} p &= \frac{p}{a} + \frac{x}{a}\frac{dp}{dx} - \frac{b}{a}p^{-2}\frac{dp}{dx} \\ p - \frac{p}{a} &= \left(\frac{x}{a} - \frac{b}{a}p^{-2}\right)\frac{dp}{dx} \end{aligned} \tag{2}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$, which implies that p is a constant. In this case $p\left(1 - \frac{1}{a}\right) = 0$. Solving for this gives $p = 0$. Substituting these values in (1) does not generate any solutions due to division by zero. Hence no singular solution exist. The general solution is found when $\frac{dp}{dx} \neq 0$. Writing (2) as

$$\frac{p\left(1-\frac{1}{a}\right)}{\frac{x}{a}-\frac{b}{a}p^{-2}} = \frac{dp}{dx}$$

$$\frac{dx}{dp} = \frac{\frac{x}{a}-\frac{b}{a}p^{-2}}{p\left(1-\frac{1}{a}\right)}$$

$$\frac{dx}{dp} - x\frac{1}{p\,(a-1)} = -\frac{b}{a}\frac{1}{p^{3}\left(1-\frac{1}{a}\right)}$$

This is linear ODE in $x\left(p\right)$. The solution is

$$x\left(p\right) = \frac{b}{(3a-2)p^{2}} + C_{1}p^{\frac{1}{a-1}} \tag{3}$$

Solving for p from above gives (using the computer) the following root object

$$p = \operatorname{RootOf}\left(2\_Z^{\frac{1}{a-1}}C_{1}\_Z^{2}a - \_Z^{\frac{1}{a-1}}C_{1}\_Z^{2} - 2\_Z^{2}ax + \_Z^{2}x + b\right)$$

Substituting the above back into (1) gives the solution as

$$y = \frac{x}{a}\operatorname{RootOf}\left(2\_Z^{\frac{1}{a-1}}C_{1}\_Z^{2}a - \_Z^{\frac{1}{a-1}}C_{1}\_Z^{2} - 2\_Z^{2}ax + \_Z^{2}x + b\right) + \frac{b}{a\operatorname{RootOf}\left(2\_Z^{\frac{1}{a-1}}C_{1}\_Z^{2}a - \_Z^{\frac{1}{a-1}}C_{1}\_Z^{2} - 2\_Z\right.}$$

The hardest step in this method is the inversion of the solution of (3) to obtain p. The above solution was verified using the computer and it satisfies the ode $y = \frac{x}{a}y' + \frac{b}{ay'}$

2.10 Example 10

Solving $y = xy' + ax\sqrt{1+\left(y'\right)^{2}}$. Writing it in normal form (by replacing y' with p), gives

$$y = x\left(p + a\sqrt{1+p^{2}}\right) \tag{1}$$

Hence $g\left(p\right) = p + a\sqrt{1+p^{2}}, f\left(p\right) = 0$. This is the special case of d'Alembert when $f\left(p\right) = 0$. Taking derivative w.r.t. x gives

$$p = p + a\sqrt{1+p^{2}} + x\left(\frac{dp}{dx} + \frac{1}{2}\frac{2ap}{\sqrt{1+p^{2}}}\frac{dp}{dx}\right)$$

$$0 = a\sqrt{1+p^{2}} + x\frac{dp}{dx} + \frac{1}{2}x\frac{2ap}{\sqrt{1+p^{2}}}\frac{dp}{dx}$$

$$0 = a\sqrt{1+p^{2}} + \left(x + \frac{1}{2}x\frac{2ap}{\sqrt{1+p^{2}}}\right)\frac{dp}{dx}$$

$$-a\sqrt{1+p^{2}} = \left(x + \frac{1}{2}x\frac{2ap}{\sqrt{1+p^{2}}}\right)\frac{dp}{dx} \tag{2}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$, which implies that $-a\sqrt{1+p^2} = 0$ or $\sqrt{1+p^2} = 0$. This gives no real solution for p. Hence no singular solution.

The general solution is when $\frac{dp}{dx} \neq 0$. Writing (2) as

$$\frac{-a\sqrt{1+p^2}}{x + \frac{1}{2}x\frac{2ap}{\sqrt{1+p^2}}} = \frac{dp}{dx}$$

$$\frac{dx}{dp} = \frac{x\left(1 + \frac{1}{2}\frac{2ap}{\sqrt{1+p^2}}\right)}{-a\sqrt{1+p^2}}$$

$$\frac{dx}{x} = \frac{1 + \frac{1}{2}\frac{2ap}{\sqrt{1+p^2}}}{-a\sqrt{1+p^2}}dp$$

$$\frac{dx}{x} = \frac{\sqrt{1+p^2} + \frac{1}{2}2ap}{-a\left(1+p^2\right)}dp$$

$$\frac{dx}{x} = \left(-\frac{1}{a\sqrt{1+p^2}} - \frac{p}{\left(1+p^2\right)}\right)dp$$

Integrating gives

$$\ln x\left(p\right) = -\frac{1}{2}\ln\left(p^2+1\right) - \frac{1}{a}\operatorname{arcsinh}\left(p\right)$$

Hence

$$x = c_1\frac{-e^{-\frac{1}{a}\left(\operatorname{arcsinh}\left(p\right)\right)}}{\sqrt{p^2+1}} \tag{3}$$

But from (1), we see that, since $y = x\left(p + a\sqrt{1+p^2}\right)$ then solving for p from this gives solutions

$$p_1 = -\frac{1}{x}\frac{ay + \sqrt{-a^2x^2 + x^2 + y^2}a - y}{a^2-1}$$
$$p_2 = \frac{1}{x}\frac{-ay + \sqrt{-a^2x^2 + x^2 + y^2}a - y}{a^2-1}$$

We substitute back each one of the above solutions p_i into Eq (3) to obtain two implicit solutions for $y(x)$. The first solution is when $p = p_1$

$$x = c_1\frac{-e^{-\frac{1}{a}\left(\operatorname{arcsinh}\left(-\frac{1}{x}\frac{ay+\sqrt{-a^2x^2+x^2+y^2}a-y}{a^2-1}\right)\right)}}{\sqrt{\left(-\frac{1}{x}\frac{ay+\sqrt{-a^2x^2+x^2+y^2}a-y}{a^2-1}\right)^2+1}}$$

And the second solution when $p = p_2$ is

$$x = c_1 \frac{-e^{-\frac{1}{a}\left(\operatorname{arcsinh}\left(\frac{1}{x}\frac{-ay+\sqrt{-a^2x^2+x^2+y^2}a-y}{a^2-1}\right)\right)}}{\sqrt{\left(\frac{1}{x}\frac{-ay+\sqrt{-a^2x^2+x^2+y^2}a-y}{a^2-1}\right)^2+1}}$$

2.11 Example 11

Solving $y = x\left(y'\right)^2$. Writing it in normal form (by replacing y' with p), gives

$$y = xp^2 \tag{1}$$

Hence $g\left(p\right) = p^2, f\left(p\right) = 0$. This is the special case of d'Alembert when $f\left(p\right) = 0$. Taking derivative w.r.t. x gives

$$\begin{aligned} p &= p^2 + 2xp\frac{dp}{dx} \\ p - p^2 &= 2xp\frac{dp}{dx} \end{aligned} \tag{2}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$, which implies that $p - p^2 = 0$ or $p\left(1 - p^2\right) = 0$. This gives $p = 0$ or $p^2 = 1$. The first gives $y = 0$ and the second gives $p = 1$ or $p = -1$. Therefore $y = x$ or $y = -x$. But this second solution does not satisfy the ODE. Hence only $y = 0$ and $y = x$ are singular solutions.

The general solution is when $\frac{dp}{dx} \neq 0$. Writing (2) as

$$\begin{aligned} \frac{p-p^2}{2xp} &= \frac{dp}{dx} \\ \frac{dx}{dp} &= \frac{2xp}{-p-p^2} \\ \frac{dx}{x} &= \frac{2p}{-p-p^2}dp \\ &= \frac{-2}{1+p}dp \end{aligned}$$

Integrating gives

$$\ln x\left(p\right) = -2\ln\left(1+p\right) + c$$

Hence

$$
\begin{aligned}
x &= c_1 \frac{1}{\left(1+p\right)^2} \\
\left(1+p\right)^2 &= c_1 \frac{1}{x} \\
1+p &= \pm c_1 \sqrt{\frac{1}{x}} \\
p &= \left(\pm c_1 \sqrt{\frac{1}{x}}\right) - 1
\end{aligned} \tag{3}
$$

From (1) $y = xp^2$ therefore the general solution is

$$
\begin{aligned}
y_1 &= x\left(c_1\sqrt{\frac{1}{x}} - 1\right)^2 = x\left(\frac{1}{x}c_1^2 - 2c_1\sqrt{\frac{1}{x}} + 1\right) = c_1^2 - 2xc_1\sqrt{\frac{1}{x}} + x \\
y_2 &= x\left(-c_1\sqrt{\frac{1}{x}} - 1\right)^2 = x\left(\frac{1}{x}c_1^2 + 2c_1\sqrt{\frac{1}{x}} + 1\right) = c_1^2 + 2xc_1\sqrt{\frac{1}{x}} + x
\end{aligned}
$$

Therefore the solutions are

$$
\begin{aligned}
y_1(x) &= c_1^2 - 2xc_1\sqrt{\frac{1}{x}} + x \\
y_2(x) &= c_1^2 + 2xc_1\sqrt{\frac{1}{x}} + x \\
y_3(x) &= x \\
y_4(x) &= 0
\end{aligned}
$$

The last two above are singular solutions.

2.12 references

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