Finite difference approximation formulas

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1 Approximation to first derivative

These formulas below approximate u' at $x = x_j$ where j is the grid point number.

	formula	truncation	Truncation	common name and
		error	error order	common notation
1	$u_j' \approx \frac{1}{h}(u_{j+1} - u_j)$	$-u_j''\frac{h}{2} - u_j^{(3)}\frac{h^2}{3!} - \cdots$	$O(h)$	one point forward D_+
2	$u_j' \approx \frac{1}{h}(u_j - u_{j-1})$	$u_j''\frac{h}{2} - u_j^{(3)}\frac{h^2}{3!} + \cdots$	$O(h)$	one point backward $D\_$
3	$u_j' \approx \frac{1}{2h}(u_{j+1} - u_{j-1})$	$-u_j^{(3)}\frac{h^2}{6} - u_j^{(6)}\frac{h^5}{6!} - \cdots$	$O(h^2)$	centered difference, $D_0 =$
4	$u_j' \approx \frac{1}{h}\left(\frac{3}{2}u_j - 2u_{j+1} + \frac{1}{2}u_{j+2}\right)$	to do	$O(h^2)$	3 points forward differenc
5	$u_j' \approx \frac{1}{6}(2u_{j+1} + 3u_j - 6u_{j-1} + u_{j-2})$	to do	$O(h^3)$	

For example, to obtain the third formula above, we start from Taylor series and obtain

$$u_{j+1} = u_j + hu_j' + \frac{h^2}{2!}u_j'' + \frac{h^3}{3!}u_j''' + \cdots$$

then we write it again for the previous point

$$u_{j-1} = u_j - hu_j' + \frac{h^2}{2!}u_j'' - \frac{h^3}{3!}u_j''' \cdots$$

Notice the sign change in the expressions. We now *subtract* the second formula above from the above resulting in

$$u_{j+1} - u_{j-1} = 2hu_j' + 2\frac{h^3}{3!}u_j''' + \cdots$$

Or

$$u_{j+1} - u_{j-1} = 2hu_j' + 2\frac{h^3}{3!}u_j''' + \cdots$$

$$\frac{u_{j+1} - u_{j-1}}{2h} = u_j' + \overbrace{h^2\frac{u_j'''}{3!}}^{O(h^2)\text{ error}} + \cdots$$

2 Approximation to second derivative

These formulas below approximate u'' at $x = x_j$ where j is the grid point number. For approximation to u'' the accuracy of the approximation formula must be no less than 2.

	formula	truncation	Truncation	common name
		error	error order	
1	$u''_j \approx \frac{1}{h^2}(U_{j-1} - 2U_j + U_{j+1})$	$-u^{(4)}\frac{h^2}{12} - u^{(6)}\frac{h^4}{360} - \cdots$	$O(h^2)$	3 points centered difference

To obtain the third formula above, we start from Taylor series. This results in

$$u_{j+1} = u_j + hu'_j + \frac{h^2}{2!}u''_j + \frac{h^3}{3!}u'''_j + \frac{h^4}{4!}u''''_j \cdots$$

Then we write it again for the previous point

$$u_{j-1} = u_j - hu'_j + \frac{h^2}{2!}u''_j - \frac{h^3}{3!}u'''_j + \frac{h^4}{4!}u''''_j \cdots$$

Notice the sign change in the expressions. We now *add* the second formula above from the above resulting in

$$u_{j+1} + u_{j-1} = 2u_j + 2h^2 u''_j + 2\frac{h^4}{4!}u''''_j + \cdots$$

$$\frac{u_{j-1} - 2u_j + u_{j-1}}{2h^2} = u''_j + \overbrace{h^2\frac{u'''_j}{4!}}^{O(h^2)\text{ error}} + \cdots$$