Solving first order nonlinear ode's of form $y' = f(x,y)^{\frac{1}{n}}$

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Contents

1 Introduction

This note is about solving a first order ode of the form $y' = (a + bx + cy)^{\frac{1}{n}}$ and $y' = (a + bx^r + cy)^m$ where $n, m \neq 1$ and are integers. This is of the form $y' = f(x,y)^{\frac{1}{n}}$ and $y' = f(x,y)^m$. Where $f(x,y)$ must be linear in both y and x. The reason it needs to be linear in x so that the transformed ode in z becomes separable.

One way to solve $y' = (a + bx + cy)^{\frac{1}{n}}$ is to raise both sides to n. For example for $n = 2$ the ode becomes $(y')^2 = (a + bx + cy)$ which can be solved as d'Alembert.

This is what Maple seems to do based on what the Maple advisor says about the type of this ode being d'Alembert.

But the problem with squaring both sides or raising both sides of ode to some power is that this will introduce extraneous to the squared ode. Hence it is will be better to avoid doing this. The following methods solves these odes without having to square or raise both sides to some power and eliminate the introduction of extraneous solutions.

2 Algorithm

2.1 Solving $y' = (a+bx+cy)^{\frac{1}{n}}$

For n integer $\neq 1$ which can be negative or positive, the ode is

$$\frac{dy}{dx} = (a+bx+cy)^{\frac{1}{n}} \tag{1}$$

Let $z = a+bx+cy$ then

$$\begin{aligned}\frac{dz}{dx} &= b + c\frac{dy}{dx}\\ \frac{dy}{dx} &= \left(\frac{dz}{dx} - b\right)\frac{1}{c}\end{aligned}$$

Hence (1) becomes

$$\begin{aligned}\left(\frac{dz}{dx} - b\right)\frac{1}{c} &= z^{\frac{1}{n}}\\ \frac{dz}{dx} &= cz^{\frac{1}{n}} + b\\ \int \frac{dz}{cz^{\frac{1}{n}} + b} &= \int dx \end{aligned} \tag{2}$$

If the left side is integrable, then the solution to (1) can be found. For n integer it is possible to find antiderivative. For example for $n = 2$ then (2) becomes

$$\frac{2}{c}\sqrt{z} - \frac{2b\ln\left(b + c\sqrt{z}\right)}{c^2} = x + C_1$$

Replacing back $z = a+bx+cy$ the above becomes

$$\frac{2}{c}\sqrt{a+bx+cy} - \frac{2b\ln\left(b + c\sqrt{a+bx+cy}\right)}{c^2} = x + C_1 \tag{3}$$

Which is the implicit solution to (1).

<u>for $n = 2$</u>. Using $a = 1, b = 1, c = 1$ Eq. (1) becomes

$$\frac{dy}{dx} = (1+x+y)^{\frac{1}{2}}$$

And (3) becomes

$$2\sqrt{1+x+y} - 2\ln\left(1 + \sqrt{1+x+y}\right) = x + C_1 \tag{4}$$

And <u>for $n = 3$</u> Eq. (2) becomes

$$\frac{3\left(-2b + cz^{\frac{1}{3}}\right)}{2c^2}z^{\frac{1}{3}} + \frac{3b^2\ln\left(b + cz^{\frac{1}{3}}\right)}{c^3} = x + C_1$$

Replacing back $z = a+bx+cy$ the above becomes

$$\frac{3\left(-2b + c(a+bx+cy)^{\frac{1}{3}}\right)}{2c^2}z^{\frac{1}{3}} + \frac{3b^2\ln\left(b + c(a+bx+cy)^{\frac{1}{3}}\right)}{c^3} = x + C_1 \tag{5}$$

Which is the implicit solution to (1) for $n=3$. Using $a=1,b=1,c=1$ then (1) becomes

$$\frac{dy}{dx}=(1+x+y)^{\frac{1}{3}}$$

And its solution (5) becomes

$$\frac{3}{2}\left(-2+(1+x+y)^{\frac{1}{3}}\right)(1+x+y)^{\frac{1}{3}}+3\ln\left(1+(1+x+y)^{\frac{1}{3}}\right)=x+C_1$$

And so on for higher values of n. This also works negative values of n. For example, for $n=-2$ then (1) becomes

$$\frac{dy}{dx}=(a+bx+cy)^{\frac{-1}{2}}$$

And the integral equation (2) now becomes

$$\int\frac{dz}{cz^{\frac{-1}{n}}+b}=\int dx$$

Which for $n=2$ gives

$$\frac{1}{b^3}\left(-2bc\sqrt{z}+b^2z+2c^2\ln\left(c+b\sqrt{z}\right)\right)=x+C_1$$

Replacing back $z=a+bx+cy$ the above becomes

$$\frac{1}{b^3}\left(-2bc\sqrt{a+bx+cy}+b^2(a+bx+cy)+2c^2\ln\left(c+b\sqrt{a+bx+cy}\right)\right)=x+C_1$$

For $a=1,b=1,c=1$ the above becomes

$$\left(-2\sqrt{1+x+y}+(1+x+y)+2\ln\left(1+\sqrt{1+x+y}\right)\right)=x+C_1$$

And so on.

2.2 Solving $y'=(a+bx+cy)^m$

For m integer $\neq 1$ which can be negative or positive, the ode is

$$\frac{dy}{dx}=(a+bx+cy)^m \tag{1}$$

Let $z=a+bx+cy$ then

$$\begin{aligned}\frac{dz}{dx}&=b+c\frac{dy}{dx}\\ \frac{dy}{dx}&=\left(\frac{dz}{dx}-b\right)\frac{1}{c}\end{aligned}$$

Hence (1) becomes

$$\begin{aligned}\left(\frac{dz}{dx}-b\right)\frac{1}{c}&=z^m\\ \frac{dz}{dx}&=cz^m+b\\ \int\frac{dz}{cz^m+b}&=\int dx \end{aligned}\tag{2}$$

If the left side is integrable, then the solution to (1) can be found. For m integer it is possible to find antiderivative. For example for $n=2$ then (2) becomes

$$\frac{1}{\sqrt{bc}}\arctan\left(\sqrt{\frac{c}{b}}z\right)=x+C_1$$

Replacing back $z=a+bx+cy$ the above becomes

$$\frac{1}{\sqrt{bc}}\arctan\left(\sqrt{\frac{c}{b}}\,(a+bx+cy)\right)=x+C_1 \tag{3}$$

Which is the implicit solution to (1).

<u>for $m=2$</u>. For an example, for $a=1,b=1,c=1$ Eq. (1) becomes

$$\frac{dy}{dx}=(1+x+y)^2$$

And (3) becomes

$$\begin{aligned}\arctan(1+x+y)&=x+C_1\\ 1+x+y&=\tan(x+C_1)\\ y&=\tan(x+C_1)-1-x \end{aligned}\tag{4}$$

And <u>for $m=3$</u> Eq. (2) becomes

$$\frac{-1}{6b^{\frac{2}{3}}c^{\frac{1}{3}}}\left(2\sqrt{3}\arctan\left(\frac{1-2\left(\frac{c}{b}\right)^{\frac{1}{3}}z}{\sqrt{3}}\right)-2\ln\left(b^{\frac{1}{3}}+c^{\frac{1}{3}}z\right)+\ln\left(b^{\frac{2}{3}}-b^{\frac{1}{3}}c^{\frac{1}{3}}+c^{\frac{2}{3}}z^2\right)\right)=x+C_1$$

Replacing back $z=a+bx+cy$ the above becomes

$$\frac{-1}{6b^{\frac{2}{3}}c^{\frac{1}{3}}}\left(2\sqrt{3}\arctan\left(\frac{1-2\left(\frac{c}{b}\right)^{\frac{1}{3}}(a+bx+cy)}{\sqrt{3}}\right)-2\ln\left(b^{\frac{1}{3}}+c^{\frac{1}{3}}(a+bx+cy)\right)+\ln\left(b^{\frac{2}{3}}-b^{\frac{1}{3}}c^{\frac{1}{3}}+c^{\frac{2}{3}}(a+bx+cy)^2\right)\right)=x+C_1 \tag{5}$$

Which is the implicit solution to (1) for $m=3$. Using $a=1,b=1,c=1$ then (1) becomes

$$\frac{dy}{dx}=(1+x+y)^3$$

And its solution (5) now simplifies to

$$\frac{-1}{6}\left(2\sqrt{3}\arctan\left(\frac{1-2(1+x+y)}{\sqrt{3}}\right)-2\ln(2+x+y)+\ln\left((1+x+y)^2\right)\right)=x+C_1$$

And so on for higher values of m, but solution get complicated very quickly. This method also works for <u>negative m</u>.

For example, for $m=-2$ then (1) becomes

$$\frac{dy}{dx}=(a+bx+cy)^{-2}$$

And the integral equation (2) now becomes

$$\int\frac{dz}{cz^{-2}+b}=\int dx$$

Which gives

$$\frac{z}{b}-\frac{\sqrt{c}\arctan\left(\sqrt{\frac{b}{c}}\,z\right)}{b^{\frac{3}{2}}}=x+C_1$$

Replacing back $z = a + bx + cy$ the above becomes

$$\frac{a+bx+cy}{b} - \frac{\sqrt{c}\,\arctan\left(\sqrt{\frac{b}{c}}\,(a+bx+cy)\right)}{b^{\frac{3}{2}}} = x + C_1$$

For $a = 1, b = 1, c = 1$ the above becomes

$$\begin{aligned}(1+x+y) - \arctan(1+x+y) &= x + C_1\\ \arctan(1+x+y) &= (1+x+y) - x - C_1\\ \arctan(1+x+y) &= 1 + y - C_1\\ \arctan(1+x+y) &= y + C_2\end{aligned}$$

And and so on for $= -3, -4, \cdots$ as all of these are integrable but become complicated very quickly and the computer is needed to find the antiderivatives in these cases.

3 Examples

3.1 Example 1 $y' = (1+5x+y)^{\frac{1}{2}}$

Let $z = 1 + 5x + y$, then $\frac{dz}{dx} = 5 + y'$. This simplifies to

$$\begin{aligned}y' &= z' - 5\\ (1+x^2+y)^{\frac{1}{2}} &= z' - 5\\ z^{\frac{1}{2}} &= z' - 5\\ \frac{dz}{dx} &= z^{\frac{1}{2}} + 5\end{aligned}$$

Which is separable. Hence

$$\begin{aligned}\frac{dz}{z^{\frac{1}{2}}+5} &= dx\\ 2\sqrt{z} - 5\ln\left(5+\sqrt{z}\right) + 5\ln\left(\sqrt{z}-5\right) - 5\ln(z-25) &= x + C_1\end{aligned}$$

Hence the implicit solution is

$$\begin{aligned}2\sqrt{1+5x+y} - 5\ln\left(5+\sqrt{1+5x+y}\right) + 5\ln\left(\sqrt{1+5x+y}-5\right) - 5\ln(1+5x+y-25) &= x + C_1\\ 2\sqrt{1+5x+y} - 5\ln\left(5+\sqrt{1+5x+y}\right) + 5\ln\left(\sqrt{1+5x+y}-5\right) - 5\ln(5x+y-24) &= x + C_1\end{aligned} \tag{1}$$

The above method is now compared to using d'Alembert for solving the ode, which results after squaring both sides of the given ode. Squaring the ode gives

$$\begin{aligned}(y')^2 &= (1+5x+y)\\ y &= (y')^2 - 1 - 5x\\ &= x(-5) + \left(p^2-1\right)\\ &= xf(p) + g(p)\end{aligned} \tag{2}$$

Where $p=\frac{dy}{dx}$. This is d'Alembert of the form $y=xf(p)+g(p)$ where $f(p)=5$ and $g(p)=p^2-1$. Taking derivative of (2) w.r.t. x gives

$$\begin{aligned}p&=f(p)+x\frac{df}{dp}\frac{dp}{dx}+\frac{dg}{dp}\frac{dp}{dx}\\ p-f(p)&=\left(x\frac{df}{dp}+\frac{dg}{dp}\right)\frac{dp}{dx}\end{aligned}\tag{3}$$

Using $f(p)=5$ and $g(p)=p^2-1$ the above becomes

$$\begin{aligned}p-5&=2p\frac{dp}{dx}\\ \frac{dp}{dx}&=\frac{p-5}{2p}\end{aligned}$$

Which is separable. Solving for p gives

$$p=5\,\mathrm{LambertW}\left(\frac{C}{5}e^{\frac{x}{10}-1}\right)+5$$

Substituting this back into (2) gives

$$y=-5x+\left(\left(5\,\mathrm{LambertW}\left(\frac{C}{5}e^{\frac{x}{10}-1}\right)+5\right)^2-1\right)\tag{4}$$

This is an explicit general solution for the ode $y'=(1+5x+y)^{\frac{1}{2}}$. The singular solution is found when $\frac{dp}{dx}=0$ in (3) which gives

$$\begin{aligned}p-5&=0\\ p&=5\end{aligned}$$

Eq (2) now becomes

$$\begin{aligned}y&=-5x+\left(5^2-1\right)\\ &=24-5x\end{aligned}\tag{5}$$

However, and this is the problem with squaring the ode, it can be shown that both solution (4) and (5) do not verify the given $y'=(1+5x+y)^{\frac{1}{2}}$. What went wrong? They do verify the ode $y'=-(1+5x+y)^{\frac{1}{2}}$ (with minus sign). This example shows why one must be careful when squaring both sides of an ode and solving the squared version. Because the squared version of the ode, when also squaring it, results with same solutions. Therefore It is better to avoid the squaring operation and to try to find a method to solve the original ode in its original form.

4 References

1. will-squaring-both-sides-of-the-ode-change-its-type Thanks to this answer which gave the main hint on how to solve such ode. I expanded this idea for a more general cases and different exponents.
2. Wikipedia entry on D'Alembert's equation This show alternative method to solve the ode for $\frac{1}{2}$.
3. Note on solving Clairaut and d'Alembert rst order dierential equations
4. Wikipedia entry on Riccati equation
5. Wikipedia entry on Abel ode