Animation of swinging pendulum on spring

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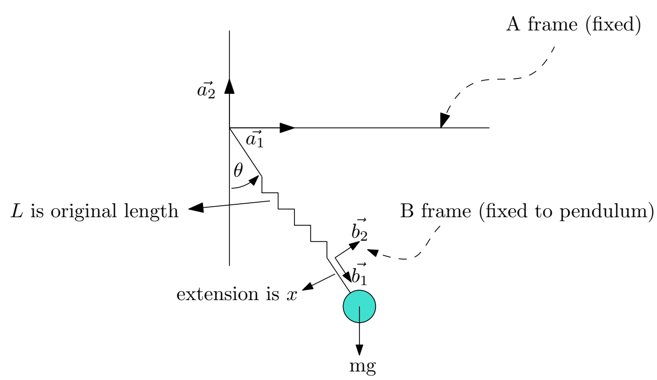

Figure 1: the Problem statement

Let original length of pendulum be L and extension at instance shown be x. The position vector of bob is

$$\bar{r} = (L + x)\,\bar{b}_1$$

Hence the velocity vector is

$$\begin{aligned}\left(\frac{d}{dt}\bar{r}\right)_A &= \left(\frac{d}{dt}\bar{r}\right)_B + \bar{\omega}\times\bar{r}\\ \bar{v}_A &= \left(\frac{d}{dt}(L+x)\,\bar{b}_1\right)_B + \bar{\omega}\times\big((L+x)\,\bar{b}_1\big)\\ &= \dot{x}\bar{b}_1 + \dot{\theta}\bar{b}_3\times\big((L+x)\,\bar{b}_1\big)\\ &= \dot{x}\bar{b}_1 + \dot{\theta}(L+x)\,\bar{b}_2\end{aligned}$$

And the acceleration is

$$\left(\frac{d}{dt}\bar{v}\right)_A = \left(\frac{d}{dt}\bar{v}\right)_B + \bar{\omega}\times\bar{v}$$

But

$$\begin{aligned}\left(\frac{d}{dt}\bar{v}\right)_B &= \frac{d}{dt}\big(\dot{x}\bar{b}_1 + \dot{\theta}(L+x)\,\bar{b}_2\big)_B\\ &= \ddot{x}\bar{b}_1 + \big(\ddot{\theta}(L+x) + \dot{\theta}\dot{x}\big)\,\bar{b}_2 \qquad (2)\end{aligned}$$

And

$$
\begin{aligned}
\bar{\omega} \times \bar{v} &= \bar{\omega} \times \left(\dot{x}\bar{b}_1 + \dot{\theta}(L+x)\,\bar{b}_2\right) \\
&= \dot{\theta}\bar{b}_3 \times \left(\dot{x}\bar{b}_1 + \dot{\theta}(L+x)\,\bar{b}_2\right) \\
&= \dot{\theta}\dot{x}\bar{b}_2 + \dot{\theta}\dot{x}\bar{b}_2 - \dot{\theta}^2(L+x)\,\bar{b}_1
\end{aligned} \tag{3}
$$

Substituting (2,3) into (1) gives the acceleration of the bob in frame A as

$$
\begin{aligned}
\bar{a}_A &= \ddot{x}\bar{b}_1 + \ddot{\theta}(L+x)\,\bar{b}_2 + \dot{\theta}\dot{x}\bar{b}_2 + \dot{\theta}\dot{x}\bar{b}_2 - \dot{\theta}^2(L+x)\,\bar{b}_1 \\
&= \ddot{x}\bar{b}_1 + \ddot{\theta}(L+x)\,\bar{b}_2 + 2\dot{\theta}\dot{x}\bar{b}_2 - \dot{\theta}^2(L+x)\,\bar{b}_1
\end{aligned}
$$

To obtain $\bar{F} = m\bar{a}$ we now just need to find $\bar{F}$. The force on bob is given by just the weight and the spring force acting on it. The spring force is proportional to spring coefficient k times the extension. $F_s = -kx$. Hence the force vector is

$$
\bar{F} = mg\cos\theta\bar{b}_1 - mg\sin\theta\bar{b}_2 - kx\bar{b}_1
$$

Therefore the equation of motion is

$$
\begin{aligned}
\bar{F} &= m\bar{a} \\
mg\cos\theta\bar{b}_1 - mg\sin\theta\bar{b}_2 - kx\bar{b}_1 &= m\left(\ddot{x}\bar{b}_1 + \ddot{\theta}(L+x)\,\bar{b}_2 + 2\dot{\theta}\dot{x}\bar{b}_2 - \dot{\theta}^2(L+x)\,\bar{b}_1\right) \\
g\cos\theta\bar{b}_1 - g\sin\theta\bar{b}_2 - \frac{k}{m}x\bar{b}_1 &= \ddot{x}\bar{b}_1 + \ddot{\theta}(L+x)\,\bar{b}_2 + 2\dot{\theta}\dot{x}\bar{b}_2 - \dot{\theta}^2(L+x)\,\bar{b}_1
\end{aligned}
$$

Or, by equating each vector component

$$
\begin{aligned}
\ddot{x} - \dot{\theta}^2(L+x) + \frac{k}{m}x &= g\cos\theta \\
\ddot{\theta}(L+x) + 2\dot{\theta}\dot{x} &= -g\sin\theta
\end{aligned}
$$

Or

$$
\begin{aligned}
\ddot{x} &= \dot{\theta}^2(L+x) - \frac{k}{m}x + g\cos\theta \\
\ddot{\theta} &= -\frac{2\dot{\theta}\dot{x}}{L+x} - \frac{g}{L+x}\sin\theta
\end{aligned}
$$

Let initial conditions be $L = 1, x(0) = 0.1, \dot{x}(0) = 0, \theta(0) = 20^0, \dot{\theta}(0) = .1$ (rad/sec).

We can now solve for x, θ as function of time and make the animations. We can assume values for m, k and $g = 9.81$.

The solution will give $x(t), \theta(t)$. To plot the path, we need to express $x(t)$ in frame A coordinates of course which is the fixed frame. But this is easy, since the pendulum frame B is fixed at the base on the frame A origin.

The following is plot of the solution using $k = 0.99, m = 0.09$

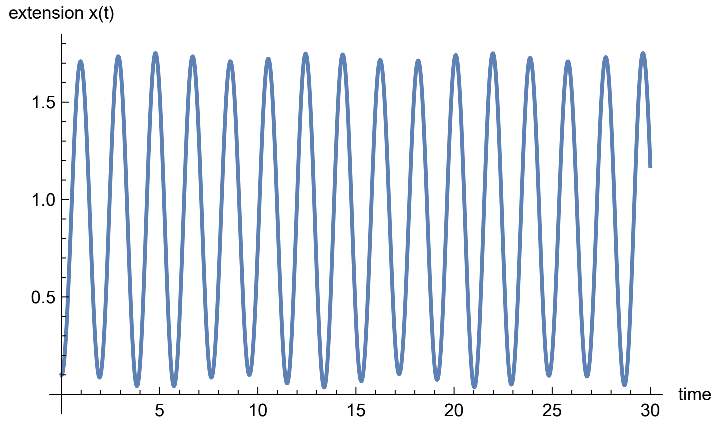

Figure 2: position of bob

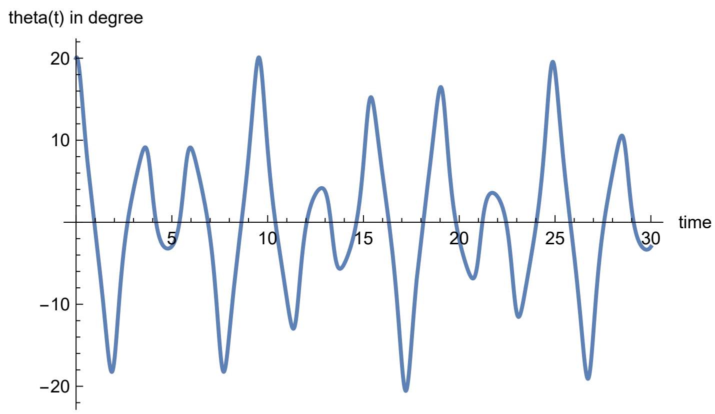

Figure 3: Angle θ of pendulum

The following is a quick animation of the above

The following is the code used

```
(*Nasser M. Abbasi, Jan 25, 2025*)

SetDirectory[NotebookDirectory[]]

L = 1;
k = .99;
m = .09;
g = 9.81;
ode1 = x''[t] == z'[t]^2*(L + x[t]) + g*Cos[z[t]] - k/m*x[t];
ode2 = z''[t] == -2 z'[t]*x'[t]/(L + x[t]) - g*Sin[z[t]]/(L + x[t]);
IC = {x[0] == .1, x'[0] == 0, z[0] == 20 Degree, z'[0] == .1};
sol = NDSolve[{ode1, ode2, IC}, {x, z}, {t, 0, 100}]

Manipulate[
 Module[{currentX, currentY},
  currentX = (L + Evaluate[x[t0] /. sol])*Sin[Evaluate[z[t0] /. sol]];
```

```
 currentY = (L + Evaluate[x[t0] /. sol])*Cos[Evaluate[z[t0] /. sol]];
 Grid[{
   {Graphics[{
      Line[{{0, 0}, {First@currentX, -First@currentY}}],
      {Blue, Disk[{First@currentX, -First@currentY}, .1]}
      },
     PlotRange -> {{-2, 2}, {-4, 1}}, GridLines -> Automatic,
     GridLinesStyle -> LightGray
     ]}}]
 ],
{{t0, 0, "time"}, 0, 10, .01},
TrackedSymbols :> {t0}
]
```