Finding singular solutions of differential equations

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1 First order ode

Given first order ode $F(x, y, y') = 0$ the goal is find its singular solutions (if any). This method applies to first order ode's of degree not one.

Singular solution here, is meant to be the solution that can not be obtained from the general solution (hence called $y_c(x)$) for any value of c (including $\pm\infty$). This singular solution (called $y_s(x)$) will be the envelope of the family of solutions of the general solution $y_c(x)$. It will have no constant in it, unlike the general solution.

If the ode is an initial value problem, and if the uniqueness theorem says there is a unique in an interval around (x_0, y_0) then no singular solution exists as this will violate the uniqueness theorem. The main steps used to find singular solution are the following

1. Find y_s using p-discriminant method by eliminating y' from $F(x, y, y') = 0$ and $\frac{\partial F}{\partial y'} = 0$.
2. Verify that each y_s found satisfies the ode.
3. Find general solution to the ode $y_c(x)$. Written as $\Psi(x, y, c) = 0$
4. Verify that the two equations $y_c(x_0) = y_s(x_0)$ and $y_c'(x_0) = y_s'(x_0)$ are satisfied at an arbitrary point x_0 for each singular solution found in step 1. If so, then $y_s(x)$ is singular solution. (envelope of the family of curves of the general solution).
5. An alternative to (4) which seems to be more common, is to use the c-discriminant method method. In this we work directly with the implicit general solution $\Psi(x, y, c) = 0$. Then eliminate c from this and the equation $\frac{\partial \Psi(x,y,c)}{\partial c} = 0$. Then compare the resulting y_s with the one found from step (1) which is the p-discriminant method. If singular solution from p-discriminant and c-discriminant is the same, then this is indeed a singular solution. If they are different, then it is not a singular solution. Only the *common* singular solutions from the p-discriminant and the c-discriminant are valid. If p-discriminant does not yield solution, then we will use the solution from only c-discriminant. The Examples below show how to use these methods. In all the following examples, the plots will show the singular solution(s) as thick red dashed lines.

Given ode $F(x, y, p) = 0$ then necessary and sufficient conditions that singular solution exist are (see E.L.Ince page 88)

1. $F = 0$
2. $\frac{\partial F}{\partial p} = 0$

3. $\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial y} = 0$

The above should be satisfied simultaneously. However, I am not able to verify these now. However, Ince says that $\frac{\partial F}{\partial y} \neq 0$ is necessary for singular solution to exist. So will add this check below.

1.1 Example 1

$$9(y')^2 (2-y)^2 = 4(3-y)$$

Hence

$$\begin{aligned} F &= 9(y')^2 (2-y)^2 - 4(3-y) = 0 \\ \frac{\partial F}{\partial y'} &= 18y'(2-y)^2 = 0 \end{aligned}$$

We first check that $\frac{\partial F}{\partial y} = -18(y')^2 (2-y) + 4 \neq 0$. Now we apply p-discriminant. Eliminating y'. Second equation gives $y' = 0$. Substituting into the first equation gives these candidate singular solutions

$$y_s = 3 \tag{1}$$

We now have to check if this solution satisfies the ode. We see it does.

Now we have to find the general solution (also called the primitive). This comes out to be

$$\Psi(x,y,c) = (x+c)^2 - y^2(3-y)$$

Now we set up

$$\begin{aligned} \Psi(x,y,c) &= 0 = (x+c)^2 - y^2(3-y) \\ \frac{\partial \Psi}{\partial c} &= 0 = 2(x+c) \end{aligned}$$

Eliminating c. Second equation gives $c = -x$. Substituting into the first equation gives

$$0 = -y^2(3-y)$$

Hence the c-discriminant method gives

$$\begin{aligned} y_s &= 0 \\ y_s &= 3 \end{aligned} \tag{2}$$

Now we take the common y_s from the p-discriminant and the c-discriminant from (1,2). We see that $y_s = 3$ is common. Hence

$$y_s = 3$$

And $y_s = 0$ is removed. We also see that $y_s = 0$ does not even satisfy the ode. But even if it did, it is removed since it is not common with the p-discriminant .

If there is no common y_s found from applying the two method (p-discriminant and the c-discriminant) then it means there is no singular solution. The following plot shows the singular solution as the envelope of the family of general solution plotted using different values of c.

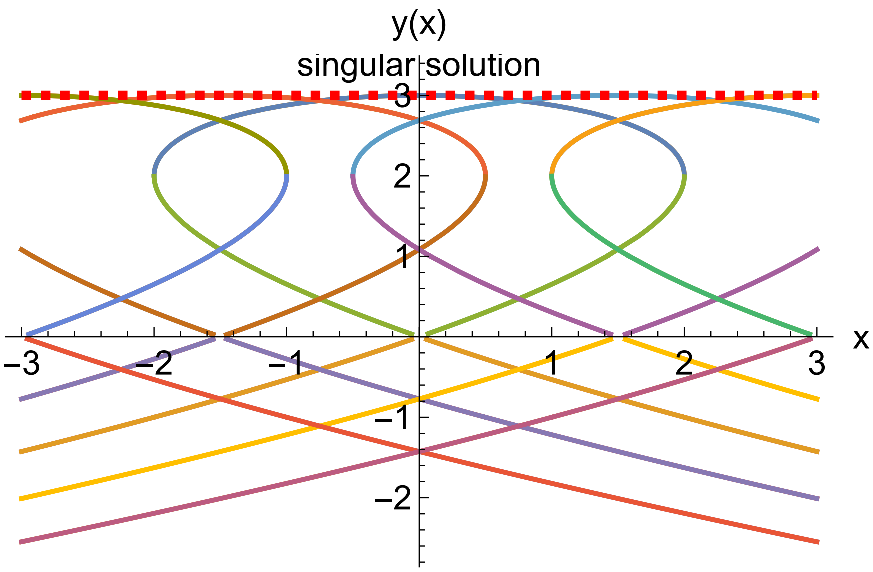

Figure 1: Singular solution envelope with general solution curves

1.2 Example 2

$$(y')^2 = xy$$

Applying p-discriminant method gives

$$F = (y')^2 - xy = 0$$
$$\frac{\partial F}{\partial y'} = 2y' = 0$$

We first check that $\frac{\partial F}{\partial y} = -x \neq 0$. Now we apply p-discriminant. Second equation gives $y' = 0$. Hence first equation now gives $xy = 0$ or $y_s = 0$. We see this satisfies the ode. Now we have to find the general solution. It will be

$$y = \frac{1}{36}\left(4x^3 - 12x^{\frac{3}{2}}c + 9c^2\right)$$
$$y = \frac{1}{36}\left(4x^3 + 12x^{\frac{3}{2}}c + 9c^2\right)$$

Hence we have two general solutions. These can be written as

$$\Psi_1(x,y,c) = y - \frac{1}{36}\left(4x^3 - 12x^{\frac{3}{2}}c + 9c^2\right) = 0$$
$$\Psi_2(x,y,c) = y - \frac{1}{36}\left(4x^3 + 12x^{\frac{3}{2}}c + 9c^2\right) = 0$$

Now we have to eliminate c from each and see if the resulting y solution agrees with the one found from the one found from the p-discriminant method. Starting with the first one

$$\Psi_1(x,y,c) = y - \frac{1}{36}\left(4x^3 - 12x^{\frac{3}{2}}c + 9c^2\right) = 0$$
$$\frac{\partial \Psi_1(x,y,c)}{\partial c} = -\frac{1}{36}\left(-12x^{\frac{3}{2}} + 18c\right) = 0$$

Second equation gives $c = \frac{12}{18}x^{\frac{3}{2}} = \frac{2}{3}x^{\frac{3}{2}}$. Substituting this in the first equation gives

$$y - \frac{1}{36}\left(4x^3 - 12x^{\frac{3}{2}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + 9\left(\frac{2}{3}x^{\frac{3}{2}}\right)^2\right) = 0$$
$$y = 0$$

Which agrees with $y_s = 0$ found from the p-discriminant method. For the second general solution

$$\Psi_2(x,y,c) = y - \frac{1}{36}\left(4x^3 + 12x^{\frac{3}{2}}c + 9c^2\right) = 0$$
$$\frac{\partial \Psi_2(x,y,c)}{\partial c} = -\frac{1}{36}\left(12x^{\frac{3}{2}} + 18c\right) = 0$$

Second equation gives $c = -\frac{12}{18}x^{\frac{3}{2}} = -\frac{2}{3}x^{\frac{3}{2}}$. Substituting this in the first equation gives

$$y - \frac{1}{36}\left(4x^3 + 12x^{\frac{3}{2}}\left(-\frac{2}{3}x^{\frac{3}{2}}\right) + 9\left(-\frac{2}{3}x^{\frac{3}{2}}\right)^2\right) = 0$$
$$y = 0$$

Which agrees with $y_s = 0$ found from the p-discriminant method. Hence $y_s = 0$ is singular solution. The following plot shows the singular solution as the envelope of the family of general solution plotted using different values of c.

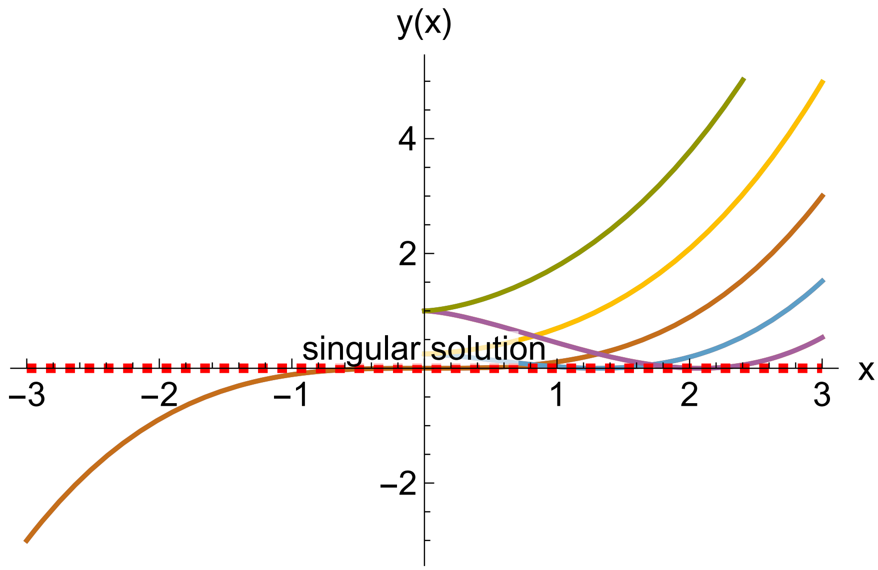

Figure 2: Singular solution envelope with general solution curves

1.3 Example 3

$$27y - 8(y')^3 = 0$$

Applying p-discriminant method gives

$$\begin{aligned} F &= 27y - 8(y')^3 = 0 \\ \frac{\partial F}{\partial y'} &= -24(y')^2 = 0 \end{aligned}$$

We first check that $\frac{\partial F}{\partial y} = 27 \neq 0$. Now we apply p-discriminant. Second equation gives $y' = 0$. First equation now gives $27y = 0$ or $y_s = 0$. We see this also satisfies the ode. The general solution can be found as

$$\Psi(x, y, c) = y^2 - (x + c)^3 = 0$$

Applying c-discriminant

$$\begin{aligned} \Psi(x, y, c) &= y^2 - (x + c)^3 = 0 \\ \frac{\partial \Psi(x, y, c)}{\partial c} &= -3(x + c)^2 = 0 \end{aligned}$$

Second equation gives $(x + c)^2 = 0$ or $c = -x$. From first equation this gives $y^2 = 0$ or $y = 0$. This is the same as y_s found from p-discriminant, hence

$$y_s = 0$$

The following plot shows the singular solution as the envelope of the family of general solution plotted using different values of c.

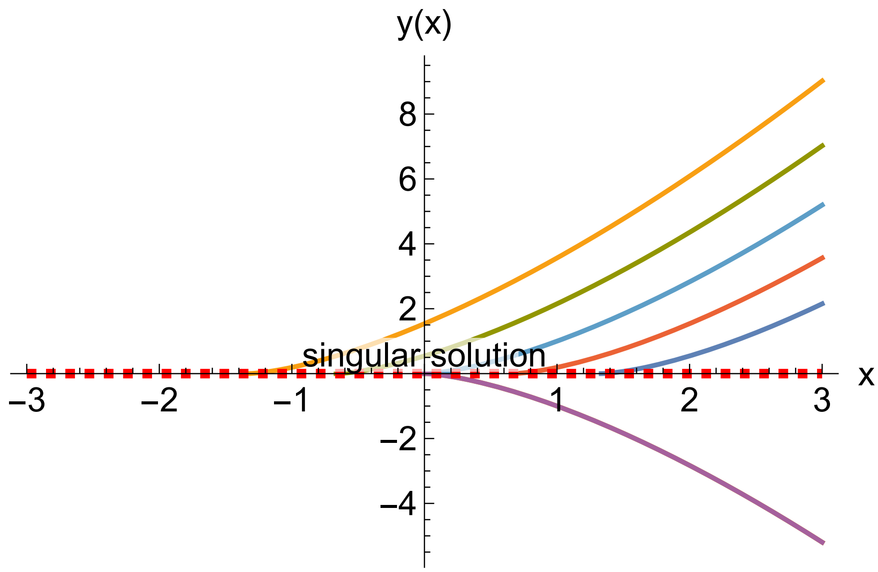

Figure 3: Singular solution envelope with general solution curves

1.4 Example 4

$$y - 2xy' - \ln y' = 0$$

Applying p-discriminant method gives

$$\begin{aligned} F = y - 2xy' - \ln y' &= 0 \\ \frac{\partial F}{\partial y'} = -2x - \frac{1}{y'} &= 0 \end{aligned}$$

Second equation gives $y' = -\frac{1}{2x}$. Substituting in the first equation gives

$$\begin{aligned} y + 1 - \ln \frac{-1}{2x} = 0 \\ y_s = \ln \left(\frac{-1}{2x}\right) - 1 \end{aligned}$$

This does not satisfy the ode. Hence no singular solution exist.

1.5 Example 5

$$y - x(1 + y') - (y')^2 = 0$$

Applying p-discriminant method gives

$$\begin{aligned} F &= y - x(1 + y') - (y')^2 = 0 \\ \frac{\partial F}{\partial y'} &= -x - 2y' = 0 \end{aligned}$$

We first check that $\frac{\partial F}{\partial y} = 1 \neq 0$. Now we apply p-discriminant. Second equation gives $y' = -\frac{x}{2}$. Substituting in the first equation gives $y - x\left(1 + \left(-\frac{x}{2}\right)\right) - \left(-\frac{x}{2}\right)^2 = 0$ or $\frac{1}{4}x^2 - x + y = 0$. Hence

$$y_s = x - \frac{1}{4}x^2$$

This does not satisfy the ode. Hence no singular solution exist.

1.6 Example 6

$$y - 2xy' - \sin(y') = 0$$

Applying p-discriminant method gives

$$\begin{aligned} F &= y - 2xy' - \sin(y') = 0 \\ \frac{\partial F}{\partial y'} &= -2x - \cos(y') = 0 \end{aligned}$$

We first check that $\frac{\partial F}{\partial y} = 1 \neq 0$. Now we apply p-discriminant. Second equation gives $-2x - \cos(y') = 0$ or $y' = \arccos(-2x)$. Substituting in the first equation gives $y - 2x \arccos(-2x) - \sin(\arccos(-2x)) = 0$. I need to look at this more. This should give $y_s = 0$ but now it does not.

1.7 Example 7

$$y - (y')^2 x + \frac{1}{y'} = 0$$

Applying p-discriminant method gives

$$\begin{aligned} F &= y - (y')^2 x + \frac{1}{y'} = 0 \\ \frac{\partial F}{\partial y'} &= -2xy' - \frac{1}{(y')^2} = 0 \end{aligned}$$

We first check that $\frac{\partial F}{\partial y} = 1 \neq 0$. Now we apply p-discriminant. Second equation gives 3 solutions for y'.

$$
\begin{aligned}
y' &= \frac{\left(-\frac{1}{2}\right)^{\frac{1}{3}}}{x^{\frac{1}{3}}} \\
y' &= \frac{1}{2^{\frac{1}{3}} x^{\frac{1}{3}}} \\
y' &= -\frac{(-1)^{\frac{2}{3}}}{2^{\frac{1}{3}} x^{\frac{1}{3}}}
\end{aligned}
$$

Using the first solution, then the first equation gives

$$
\begin{aligned}
y - \left(\frac{\left(-\frac{1}{2}\right)^{\frac{1}{3}}}{x^{\frac{1}{3}}}\right)^2 x + \frac{1}{\left(\frac{\left(-\frac{1}{2}\right)^{\frac{1}{3}}}{x^{\frac{1}{3}}}\right)} &= 0 \\
y_s &= \frac{3}{2}(-1)^{\frac{2}{3}} \sqrt[3]{2}\sqrt[3]{x}
\end{aligned}
$$

Now we check if this satisfies the ode $F = 0$. It does not. Trying the second solution $y' = \frac{1}{2^{\frac{1}{3}} x^{\frac{1}{3}}}$. Substituting into $F = 0$ gives

$$
\begin{aligned}
y - \left(\frac{1}{2^{\frac{1}{3}} x^{\frac{1}{3}}}\right)^2 x + \frac{1}{\left(\frac{1}{2^{\frac{1}{3}} x^{\frac{1}{3}}}\right)} &= 0 \\
y &= -\frac{1}{2}\sqrt[3]{2}\sqrt[3]{x}
\end{aligned}
$$

Now we check if this satisfies the ode $F = 0$. It does not. Trying the third solution $y' = -\frac{(-1)^{\frac{2}{3}}}{2^{\frac{1}{3}} x^{\frac{1}{3}}}$. Substituting into $F = 0$ gives

$$
\begin{aligned}
y - \left(-\frac{(-1)^{\frac{2}{3}}}{2^{\frac{1}{3}} x^{\frac{1}{3}}}\right)^2 x + \frac{1}{\left(-\frac{(-1)^{\frac{2}{3}}}{2^{\frac{1}{3}} x^{\frac{1}{3}}}\right)} &= 0 \\
y &= -\frac{3}{2}\sqrt[3]{-1}\sqrt[3]{2}\sqrt[3]{x}
\end{aligned}
$$

Now we check if this satisfies the ode $F = 0$. It does not. Hence no singular exist.

1.8 Example 8

$$xy' + y' - (y')^2 - y = 0$$

Applying p-discriminant method gives

$$\begin{aligned}
F &= xy' + y' - (y')^2 - y = 0 \\
\frac{\partial F}{\partial y'} &= x + 1 - 2y' = 0
\end{aligned}$$

We first check that $\frac{\partial F}{\partial y} = -1 \neq 0$. Now we apply p-discriminant. Eliminating y'. Second equation gives $y' = \frac{x+1}{2}$. Substituting in the first equation gives

$$\begin{aligned}
x\left(\frac{x+1}{2}\right) + \frac{x+1}{2} - \left(\frac{x+1}{2}\right)^2 - y = 0 \\
y_s = \frac{1}{4}x^2 + \frac{1}{2}x + \frac{1}{4}
\end{aligned}$$

Now we verify this satisfies the ode. We see it does. Now we have to find the general solution. It will be

$$y_c = c + cx - c^2$$

Now we have to eliminate c using the c-discriminant method

$$\begin{aligned}
\Psi(x, y, c) &= -y + c + cx - c^2 = 0 \\
\frac{\partial \Psi(x, y, c)}{\partial c} &= 1 + x - 2c = 0
\end{aligned}$$

Second equation gives $1 + x - 2c = 0$ or $c = \frac{1+x}{2}$. Substituting into the first gives

$$\begin{aligned}
-y + \left(\frac{1+x}{2}\right) + \left(\frac{1+x}{2}\right)x - \left(\frac{1+x}{2}\right)^2 = 0 \\
\frac{1}{4}x^2 + \frac{1}{2}x - y + \frac{1}{4} = 0 \\
y_s = \frac{1}{4}x^2 + \frac{1}{2}x + \frac{1}{4}
\end{aligned}$$

We see this is the same as y_s from the p-discriminant method. Hence it is a singular solution. The following plot shows the singular solution as the envelope of the family of general solution plotted using different values of c.

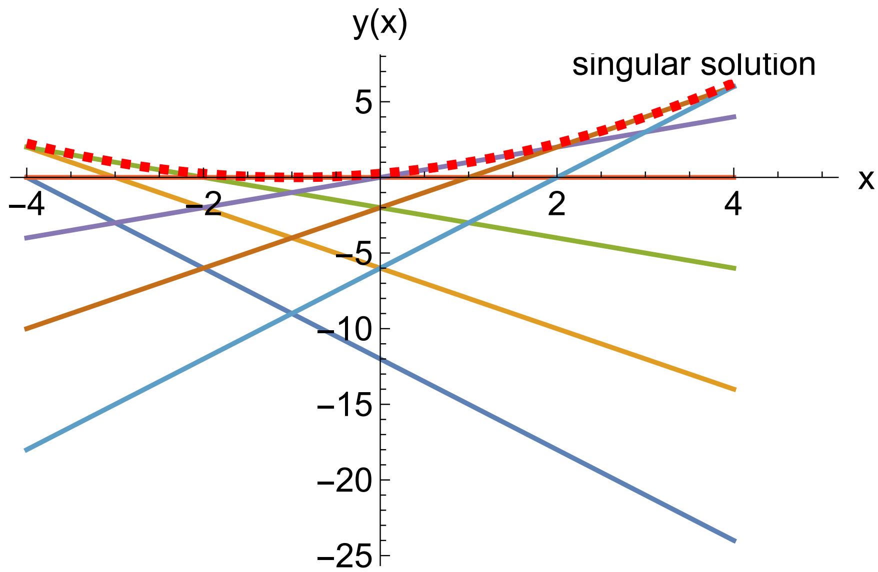

Figure 4: Singular solution envelope with general solution curves

1.9 Example 9

$$y = y'x + \sqrt{4 + y'^2}$$

Applying p-discriminant method gives

$$F = y - y'x - \sqrt{4 + y'^2} = 0$$
$$\frac{\partial F}{\partial y'} = -x - \frac{y'}{\sqrt{4 + y'^2}} = 0$$

We first check that $\frac{\partial F}{\partial y} = 1 \neq 0$. Now we apply p-discriminant. Second equation gives $x\sqrt{4 + y'^2} + y' = 0$ which gives

$$y' = \pm \frac{2x}{\sqrt{1 - x^2}}$$

Trying the negative root and substituting it in $F=0$ gives

$$
\begin{aligned}
y-\left(-\frac{2x}{\sqrt{1-x^2}}\right)x-\sqrt{4+\left(-\frac{2x}{\sqrt{1-x^2}}\right)^2}&=0\\
y+\frac{2x^2}{\sqrt{1-x^2}}-\sqrt{4+\frac{4x^2}{1-x^2}}&=0\\
y+\frac{2x^2}{\sqrt{1-x^2}}-2\frac{\sqrt{1-x^2+x^2}}{\sqrt{1-x^2}}&=0\\
y+\frac{2x^2}{\sqrt{1-x^2}}-\frac{2}{\sqrt{1-x^2}}&=0\\
y+\frac{2(x^2-1)}{\sqrt{1-x^2}}&=0\\
y+\frac{2(x^2-1)\sqrt{1-x^2}}{1-x^2}&=0\\
y+\frac{-2(1-x^2)\sqrt{1-x^2}}{1-x^2}&=0\\
y-2\sqrt{1-x^2}&=0\\
y_s&=2\sqrt{1-x^2}
\end{aligned}
$$

Which satisfies the ode. The general solution can be found to be

$$
\Psi(x,y,c)=y-xc-\sqrt{4+c^2}=0
$$

Now we have to eliminate c using the c-discriminant method

$$
\begin{aligned}
\Psi(x,y,c)&=y-xc-\sqrt{4+c^2}=0\\
\frac{\partial\Psi(x,y,c)}{\partial c}&=-x-\frac{2c}{2\sqrt{4+c^2}}=0
\end{aligned}
$$

Second equation gives

$$
c=\pm\frac{2x}{\sqrt{1-x^2}}
$$

Taking the negative root, and substituting into the first equation gives

$$
\begin{aligned}
y - x\left(\frac{-2x}{\sqrt{1-x^2}}\right) - \sqrt{4 + \left(\frac{2x}{\sqrt{1-x^2}}\right)^2} &= 0\\
y + \left(\frac{2x^2}{\sqrt{1-x^2}}\right) - 2\sqrt{1 + \frac{x^2}{1-x^2}} &= 0\\
y + \left(\frac{2x^2}{\sqrt{1-x^2}}\right) - \frac{2}{\sqrt{1-x^2}} &= 0\\
y - \frac{2(1-x^2)}{\sqrt{1-x^2}} &= 0\\
y - \frac{2(1-x^2)\sqrt{1-x^2}}{1-x^2} &= 0\\
y - 2\sqrt{1-x^2} &= 0\\
y_s &= 2\sqrt{1-x^2}
\end{aligned}
$$

Which is the same obtained using the p-discriminant. Hence

$$
y_s = 2\sqrt{1-x^2}
$$

Is singular solution. We have to try the other root also. But graphically, the above seems to be the only valid singular solution. The following plot shows the singular solution as the envelope of the family of general solution plotted using different values of c.

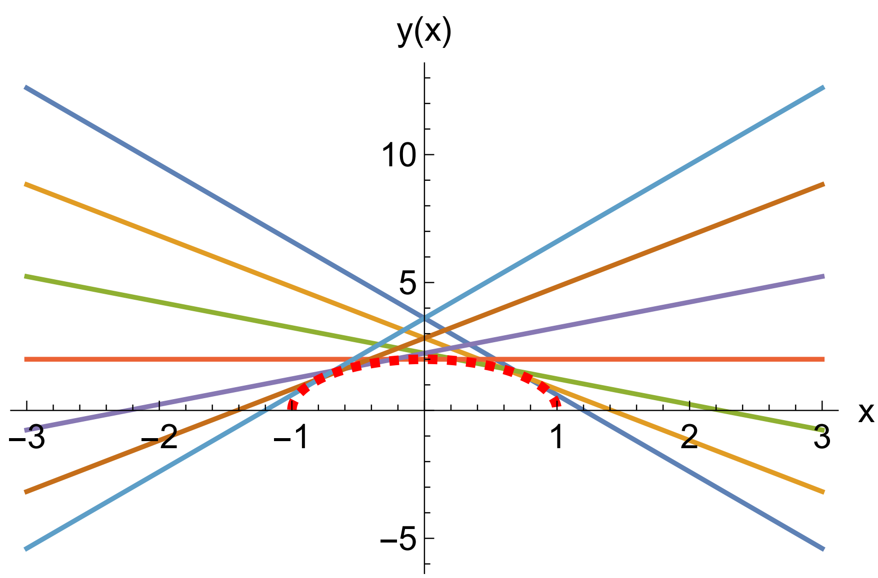

Figure 5: Singular solution envelope with general solution curves

1.10 Example 10

$$(y')^2 - xy' + y = 0$$

Applying p-discriminant method gives

$$\begin{aligned} F &= (y')^2 - xy' + y = 0 \\ \frac{\partial F}{\partial y'} &= 2y' - x = 0 \end{aligned}$$

We first check that $\frac{\partial F}{\partial y} = 1 \neq 0$. Now we apply p-discriminant. Eliminating y'. Second equation gives $y' = \frac{1}{2}x$. Substituting into the first equation gives

$$\begin{aligned} \left(\frac{1}{2}x\right)^2 - x\left(\frac{1}{2}x\right) + y = 0 \\ y_s = \frac{1}{4}x^2 \end{aligned}$$

This also satisfies the ode. Now we check using p-discriminant method. General solution can be found to be

$$\Psi(x, y, c) = xc - c^2 = 0$$

Now we have to eliminate c using the c-discriminant method

$$\begin{aligned} \Psi(x, y, c) &= y - xc + c^2 = 0 \\ \frac{\partial \Psi(x, y, c)}{\partial c} &= -x + 2c = 0 \end{aligned}$$

Second equation gives $c = \frac{1}{2}x$. First equation gives

$$\begin{aligned} y - x\left(\frac{1}{2}x\right) + \left(\frac{1}{2}x\right)^2 = 0 \\ y_s = \frac{1}{4}x^2 \end{aligned}$$

Since this is the same as y_s obtained using p-discriminant method then it is singular solution. The following plot shows the singular solution as the envelope of the family of general solution plotted using different values of c.

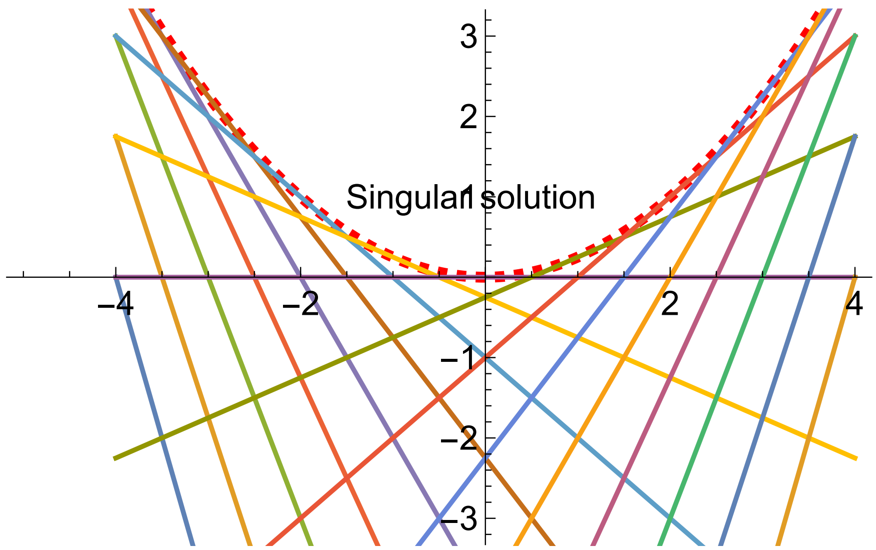

Figure 6: Singular solution envelope with general solution curves

1.11 Example 11

$$y' = y(1-y)$$

Since this is linear in y' we can not use the p-discriminant method. By inspection we see this is separable. Hence the candidate singular solutions are obtain when $y(1-y)=0$. This is because this is what we have to divide both sides by to integrate. Therefore

$$y_s = 0$$
$$y_s = 1$$

The general solution is found from

$$
\begin{aligned}
\int \frac{dy}{y\,(1-y)} &= \int dx \\
-\ln(y-1) + \ln y &= x + c \\
\ln \frac{y}{y-1} &= x + c \\
\frac{y}{y-1} &= c_1 e^{x} \\
y &= (y-1)\, c_1 e^{x} \\
y - y c_1 e^{x} &= -c_1 e^{x} \\
y(1 - c_1 e^{x}) &= -c_1 e^{x} \\
y &= \frac{c_1 e^{x}}{c_1 e^{x} - 1} \\
y &= \frac{c_1}{c_1 - e^{-x}} \\
y &= \frac{1}{1 - c_2 e^{-x}}
\end{aligned}
$$

Now we ask, can the singular solutions $y_s = 0, y_s = 1$ be obtained from the above general solution for any value of c_2? We see when $c_2 = 0$ then $y = 1$. Also when $c_2 = \infty$ then $y = 0$. So these are not really singular solutions. Some call these equilibrium solutions. But these should not be called singular solutions. Mathematica generates these when using the option *IncludeSingularSolutions*. But Maple does not give these when using the option *singsol=all*.

The following plot shows these equilibrium solutions with the general solution plotted using different values of c.

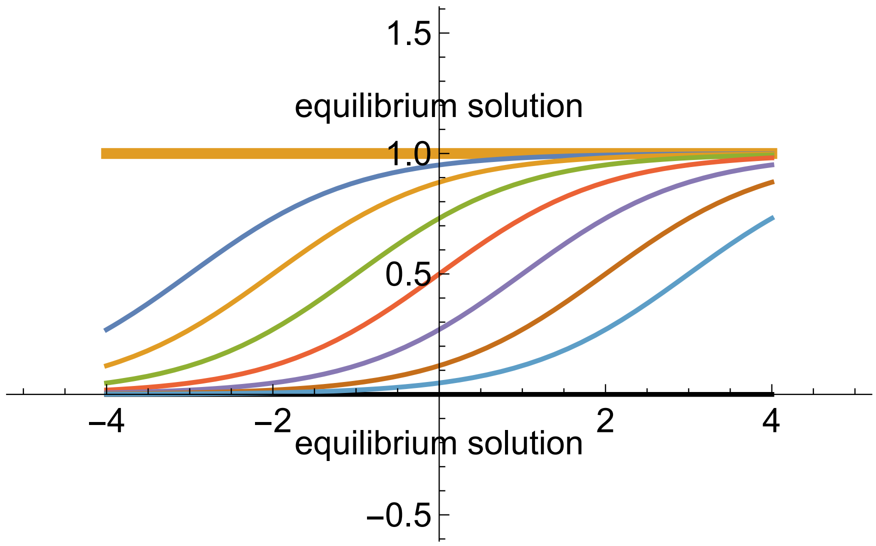

Figure 7: Equilibrium solution envelope with general solution curves

1.12 Example 12

$$(y')^2 x + y'y\ln y - y^2(\ln y)^4 = 0$$

Applying p-discriminant method gives

$$\begin{aligned} F &= (y')^2 x + y'y\ln y - y^2(\ln y)^4 = 0 \\ \frac{\partial F}{\partial y'} &= 2y'x + y\ln y = 0 \end{aligned}$$

We first check that $\frac{\partial F}{\partial y} \neq 0$. Now we apply p-discriminant. Eliminating y'. Second equation gives $y' = -\frac{y}{2x}\ln y$. Substituting into the first equation gives

$$\begin{aligned} \left(-\frac{y}{2x}\ln y\right)^2 x + \left(-\frac{y}{2x}\ln y\right) y\ln y - y^2(\ln y)^4 &= 0 \\ \frac{y^2}{4x}(\ln y)^2 - \frac{y^2}{2x}(\ln y)^2 - y^2(\ln y)^4 &= 0 \\ y^2\ln(y)^2\left(\frac{1}{4x} - \frac{1}{2x} - (\ln y)^2\right) &= 0 \end{aligned}$$

Hence we obtain the solutions

$$\begin{aligned} y &= 0 \\ y &= \infty \\ \frac{1}{4x} - \frac{1}{2x} - (\ln y)^2 &= 0 \end{aligned}$$

Or

$$y = 0 \tag{1}$$
$$y = \infty \tag{2}$$
$$1 + 4x(\ln y)^2 = 0 \tag{3}$$

The solution $y = 0$ does not satisfy the ode. Same for $y = \infty$. The solution $1 + 4x(\ln y)^2 = 0$ gives $y = \begin{cases} e^{\frac{-i}{2\sqrt{x}}} \\ e^{\frac{i}{2\sqrt{x}}} \end{cases}$ and only the first one satisfies the ode and for negative x only. The primitive can be found to be

$$\Psi(x, y, c) = y - e^{\frac{c}{c^2 - x}} = 0$$

Now we have to eliminate c using the c-discriminant method

$$\begin{aligned}
\Psi(x, y, c) &= y - e^{\frac{c}{c^2 - x}} = 0 \\
\frac{\partial \Psi(x, y, c)}{\partial c} &= \left(\frac{1}{c^2 - x} - \frac{2c^2}{(c^2 - x)^2} \right) e^{\frac{c}{c^2 - x}} = 0
\end{aligned}$$

Second equation gives $\frac{1}{c^2-x} - \frac{2c^2}{(c^2-x)^2} = 0$. Hence $c = \pm\sqrt{-x}$. Substituting $\sqrt{-x}$ in first equation gives

$$\begin{aligned}
y - e^{\frac{\sqrt{-x}}{-x-x}} &= 0 \\
y &= e^{\frac{\sqrt{-x}}{-2x}} \\
\ln y &= \frac{\sqrt{-x}}{-2x} \\
(\ln y)^2 &= \frac{-x}{4x^2} \\
4x(\ln y)^2 + 1 &= 0 \\
y_s &= \begin{cases} e^{\frac{-i}{2\sqrt{x}}} \\ e^{\frac{i}{2\sqrt{x}}} \end{cases}
\end{aligned} \tag{4}$$

Substituting $-\sqrt{-x}$ in first equation gives same solution. But only the first one satisfies the ode for $x < 0$. This is the same as obtained using p-discriminant, hence it is a singular solution.

There is another singular solution $y_s = 1$. But now I do not know how to find this. I need to look more into this. See paper by C.N. SRINIVASINGAR, example 4. The following plot shows the singular solution above as the envelope of the family of general solution plotted using different values of c. Added also $y_s = 1$.

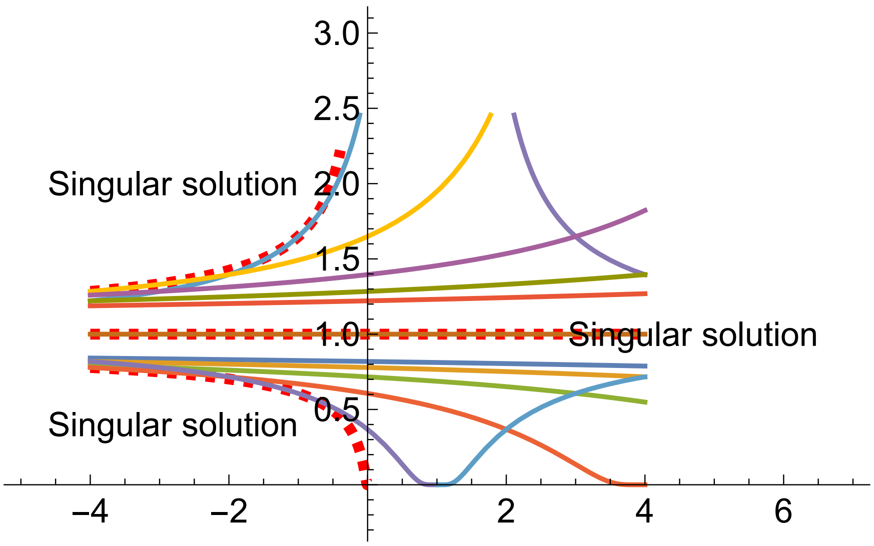

Figure 8: Singular solution envelope with general solution curves

1.13 Example 13

$$(y')^2 - 4y = 0$$

Applying p-discriminant method gives

$$F = (y')^2 - 4y = 0$$
$$\frac{\partial F}{\partial y'} = 2y' = 0$$

We first check that $\frac{\partial F}{\partial y} = -4 \neq 0$. Now we apply p-discriminant. Eliminating y'. Second equation gives $y' = 0$. Hence first equation now gives $y_s = 0$. We see this also satisfies the ode. The primitive can be found to be

$$\Psi(x, y, c) = y - (x + c)^2 = 0$$

Now we have to eliminate c using the c-discriminant method

$$\Psi(x, y, c) = y - (x + c)^2 = 0$$
$$\frac{\partial \Psi(x, y, c)}{\partial c} = -2(x + c) = 0$$

Second equation gives $c = -x$. Substituting this into the first equation gives

$$y - (x - x)^2 = 0$$
$$y_s = 0$$

Since this is the same as found by p-discriminant method then this is the singular solution. The following plot shows the singular solution as the envelope of the family of general solution plotted using different values of c.

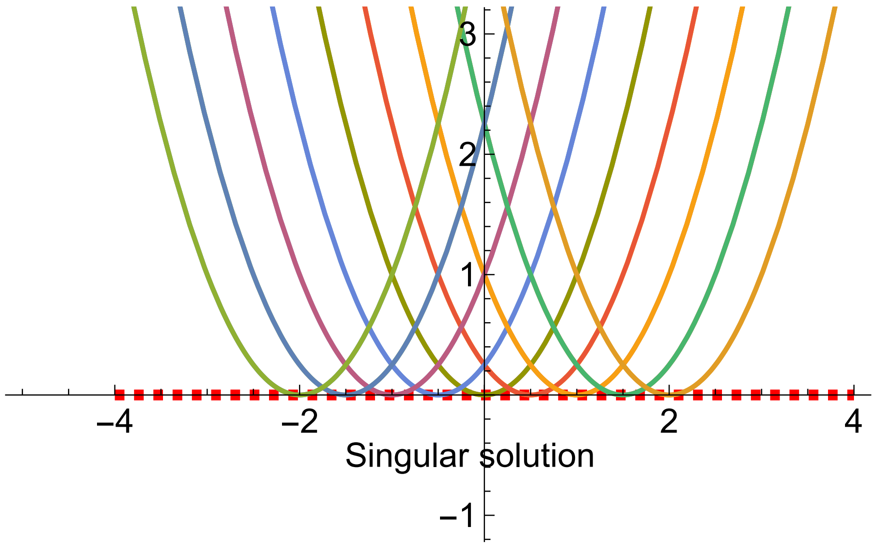

Figure 9: Singular solution envelope with general solution curves

1.14 Example 14

$$1 + (y')^2 - \frac{1}{y^2} = 0$$

Applying p-discriminant method gives

$$F = 1 + (y')^2 - \frac{1}{y^2} = 0$$
$$\frac{\partial F}{\partial y'} = 2y' = 0$$

We first check that $\frac{\partial F}{\partial y} = 2\frac{1}{y^3} \neq 0$. Now we apply p-discriminant. Eliminating y'. Second equation gives $y' = 0$. Hence first equation now gives $1 - \frac{1}{y^2} = 0$ or $y_s = \pm 1$. We see both solutions also satisfy the ode. The primitive can be found to be

$$\Psi(x, y, c) = y^2 + (x + c)^2 - 1 = 0$$

Now we have to eliminate c using the c-discriminant method

$$\Psi(x, y, c) = y^2 + (x + c)^2 - 1 = 0$$
$$\frac{\partial \Psi(x, y, c)}{\partial c} = 2(x + c) = 0$$

Second equation gives $c = -x$. Substituting this into the first equation gives

$$
\begin{aligned}
y^2 + (x - x)^2 - 1 &= 0 \\
y_s &= \pm 1
\end{aligned}
$$

Which agrees with the p-discriminant. Hence these are the singular solutions. The following plot shows the singular solution as the envelope of the family of general solution plotted using different values of c.

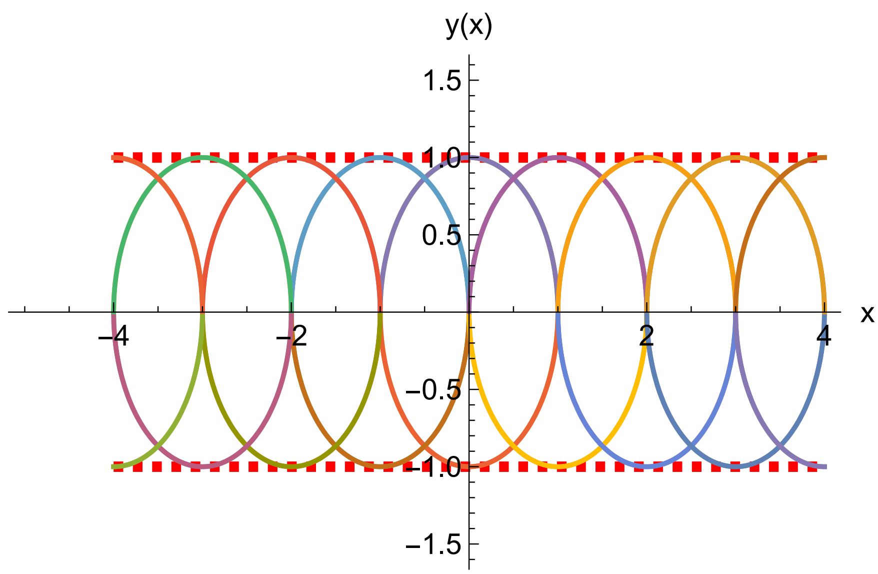

Figure 10: Singular solution envelope with general solution curves

1.15 Example 15

$$
y - (y')^2 + 3xy' - 3x^2 = 0
$$

Applying p-discriminant method gives

$$
\begin{aligned}
F = y - (y')^2 + 3xy' - 3x^2 &= 0 \\
\frac{\partial F}{\partial y'} = -2y' + 3x &= 0
\end{aligned}
$$

We first check that $\frac{\partial F}{\partial y} = 1 \neq 0$. Now we apply p-discriminant. Eliminating y'. Second equation gives $y' = \frac{3}{2}x$. Substituting into the first equation gives

$$
\begin{aligned}
y - \left(\frac{3}{2}x\right)^2 + 3x\left(\frac{3}{2}x\right) - 3x^2 &= 0 \\
y_s &= \frac{3}{4}x^2
\end{aligned}
$$

The primitive can be found to be

$$\Psi(x, y, c) = y - cx - c^2 - x^2 = 0$$

Now we have to eliminate c using the c-discriminant method

$$\begin{aligned} \Psi(x, y, c) &= y - cx - c^2 - x^2 = 0 \\ \frac{\partial \Psi(x, y, c)}{\partial c} &= -x - 2c = 0 \end{aligned}$$

Second equation gives $c = -\frac{x}{2}$. Substituting this into the first equation gives

$$\begin{aligned} y - \left(-\frac{x}{2}\right) x - \left(-\frac{x}{2}\right)^2 - x^2 = 0 \\ y_s = \frac{3}{4} x^2 \end{aligned}$$

Which agrees with the p-discriminant curve. Hence this is a singular solution. The following plot shows the singular solution as the envelope of the family of general solution plotted using different values of c.

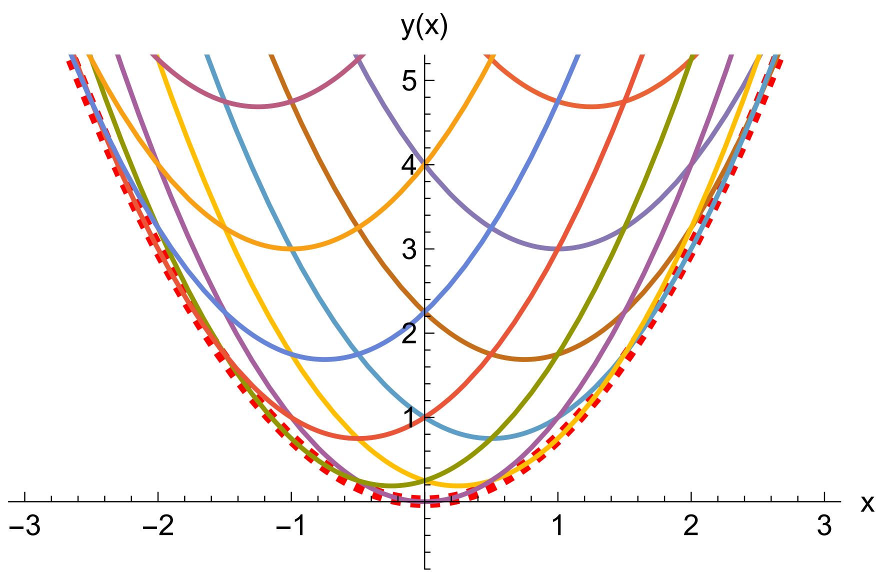

Figure 11: Singular solution envelope with general solution curves

1.16 Example 16

$$(y')^2 (1-y)^2 - 2 + y = 0$$

Applying p-discriminant method gives

$$\begin{aligned} F &= (y')^2 (1-y)^2 - 2 + y = 0 \\ \frac{\partial F}{\partial y'} &= 2(y') (1-y)^2 = 0 \end{aligned}$$

We first check that $\frac{\partial F}{\partial y} = -2(y')^2 (1-y) + 1 \neq 0$. Now we apply p-discriminant. Eliminating y'. Second equation gives $y' = 0$ or $(1-y)^2 = 0$. When $y' = 0$ then first equation gives

$$\begin{aligned} -2 + y &= 0 \\ y_s &= 2 \end{aligned}$$

And when $(1-y)^2 = 0$ then $y_s = 1$. We check now that $y_s = 2$ verifies the ode. But $y_s = 1$ does not. Hence the only candidate $y_s = 2$. The primitive can be found to be

$$\Psi(x,y,c) = 4(2-y)(y+1)^2 - 9(x+c)^2 = 0$$

Now we have to eliminate c using the c-discriminant method.

$$\begin{aligned} \Psi(x,y,c) &= 4(2-y)(y+1)^2 - 9(x+c)^2 = 0 \\ \frac{\partial \Psi(x,y,c)}{\partial c} &= -18(x+c) = 0 \end{aligned}$$

Second equation gives $c = -x$. Substituting this into the first equation gives

$$\begin{aligned} 4(2-y)(y+1)^2 - 9(x-x)^2 &= 0 \\ 4(2+y)(y+1)^2 &= 0 \end{aligned}$$

Hence $y_s = 2, y_s = -1$. But $y_s = -1$ does not verify the ode. Hence $y_s = 2$. Since this is the same y_s obtained from p-discriminant then it is singular solution.

$$y_s = 2$$

The following plot shows the singular solution as the envelope of the family of general solution plotted using different values of c.

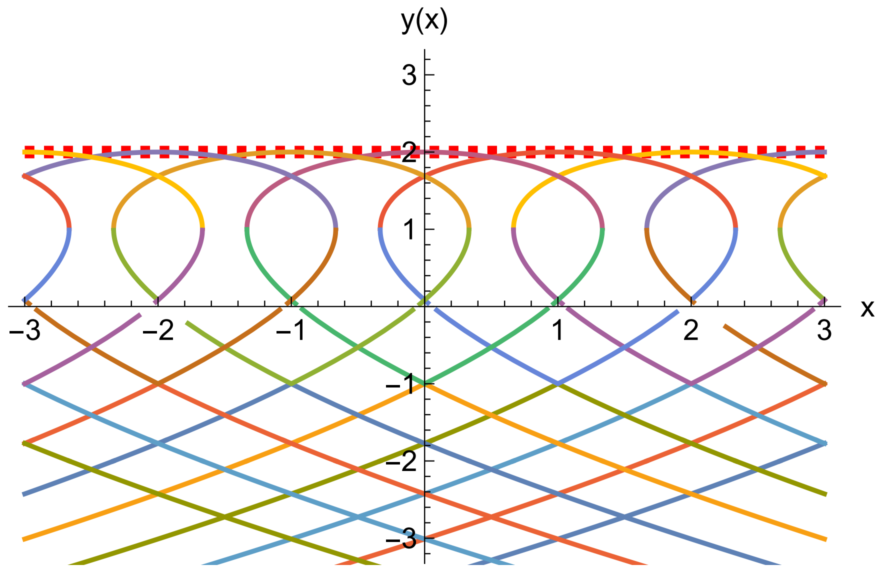

Figure 12: Singular solution envelope with general solution curves

1.17 Example 17

$$(y - xy')^2 - (y')^2 = 1$$

Applying p-discriminant method gives

$$\begin{aligned} F &= (y - xy')^2 - (y')^2 - 1 = 0 \\ \frac{\partial F}{\partial y'} &= 2(y - xy')(-x) - 2y' = 0 \\ &= -2yx + 2x^2y' - 2y' = 0 \end{aligned}$$

We first check that $\frac{\partial F}{\partial y} = 2(y - xy') \neq 0$. Now we apply p-discriminant. Eliminating y'. Second equation gives

$$\begin{aligned} -2yx + \left(2x^2 - 2\right) y' = 0 \\ y' = \frac{2yx}{(2x^2 - 2)} \end{aligned}$$

Substituting into $F=0$ gives

$$
\begin{aligned}
\left(y-x\left(\frac{2yx}{(2x^2-2)}\right)\right)^2-\left(\frac{2yx}{(2x^2-2)}\right)^2-1&=0\\
-\frac{1}{x^2-1}\left(x^2+y^2-1\right)&=0\\
x^2+y^2-1&=0\\
y^2&=1-x^2\\
y_s&=\pm\sqrt{1-x^2}
\end{aligned}
$$

Both of these solution verify the ode. The primitive can be found to be

$$
\Psi(x,y,c)=y-xc\pm\sqrt{1+c^2}=0
$$

Eliminating c. First solution gives

$$
\begin{aligned}
\Psi_1(x,y,c)&=y-xc+\sqrt{1+c^2}=0\\
\frac{\partial\Psi_1(x,y,c)}{\partial c}&=-x+\frac{1}{2}\frac{2c}{\sqrt{1+c^2}}=0
\end{aligned}
$$

Second equation gives $-x+\frac{1}{2}\frac{2c}{\sqrt{1+c^2}}=0$ or $c=x\sqrt{\frac{1}{1-x^2}}$. Substituting into the first equation above gives

$$
\begin{aligned}
y-x\left(x\sqrt{\frac{1}{1-x^2}}\right)+\sqrt{1+\left(x\sqrt{\frac{1}{1-x^2}}\right)^2}&=0\\
y-x^2\sqrt{\frac{1}{1-x^2}}+\sqrt{1+\frac{x^2}{1-x^2}}&=0\\
y-x^2\sqrt{\frac{1}{1-x^2}}+\sqrt{\frac{1-x^2+x^2}{1-x^2}}&=0\\
y-x^2\sqrt{\frac{1}{1-x^2}}+\sqrt{\frac{1}{1-x^2}}&=0\\
y+\sqrt{\frac{1}{1-x^2}}\left(1-x^2\right)&=0\\
y&=\sqrt{\frac{1}{1-x^2}}\left(x^2-1\right)\\
&=\frac{\left(x^2-1\right)}{\sqrt{1-x^2}}\\
&=\frac{\left(x^2-1\right)\sqrt{1-x^2}}{1-x^2}\\
&=-\sqrt{1-x^2}
\end{aligned}
$$

Which is given by p-discriminant above. Hence it is singular solution. If we try $\Psi_2(x,y,c) = y - xc - \sqrt{1+c^2} = 0$ we also can verify the second singular solution. Hence

$$y_s = \pm\sqrt{1-x^2}$$

The following plot shows the singular solution as the envelope of the family of general solution plotted using different values of c.

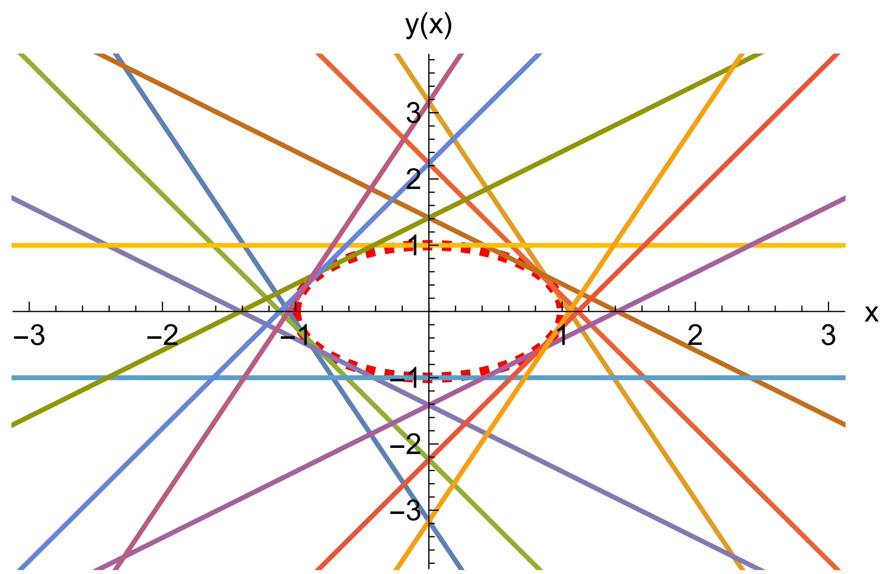

Figure 13: Singular solution envelope with general solution curves

1.18 Example 18

$$y = xy' + ay'(1-y')$$

Applying p-discriminant method gives

$$\begin{aligned} F &= -y + xy' + ay'(1-y') = 0 \\ \frac{\partial F}{\partial y'} &= x + a(1-y') - ay' = 0 \\ &= a + x - 2ay' = 0 \end{aligned}$$

We first check that $\frac{\partial F}{\partial y} = -1 \neq 0$. Now we apply p-discriminant. Eliminating y'.

Second equation gives $y' = \frac{a+x}{2a}$. Substituting in the first equation gives

$$
\begin{aligned}
-y + x\left(\frac{a+x}{2a}\right) + a\left(\frac{a+x}{2a}\right)\left(1-\left(\frac{a+x}{2a}\right)\right) &= 0 \\
\frac{1}{4a}\left(a^2 + 2ax - 4ya + x^2\right) &= 0 \\
y_s &= \frac{(x+a)^2}{4a} \qquad a \neq 0
\end{aligned}
$$

Now we check if this satisfies the ode itself. We see it does. The general solution can be found to be

$$
\Psi(x,y,c) = y - cx - ac(1-c) = 0
$$

Hence

$$
\begin{aligned}
\Psi(x,y,c) &= y - cx - ac(1-c) = 0 \\
\frac{\partial \Psi(x,y,c)}{\partial c} &= -x - a(1-c) + ac = 0
\end{aligned}
$$

Eliminating c. Second equation gives $c = \frac{a+x}{2a}$. Substituting into the general solution gives

$$
\begin{aligned}
y - \left(\frac{a+x}{2a}\right)x - a\left(\frac{a+x}{2a}\right)\left(1-\left(\frac{a+x}{2a}\right)\right) &= 0 \\
y_s &= \frac{(x+a)^2}{4a} \qquad a \neq 0
\end{aligned}
$$

Which is the same obtained using p-discriminant. Hence this is the singular solution. The following plot shows the singular solution as the envelope of the family of general solution plotted using different values of c. For this, $a = 1$ was used.

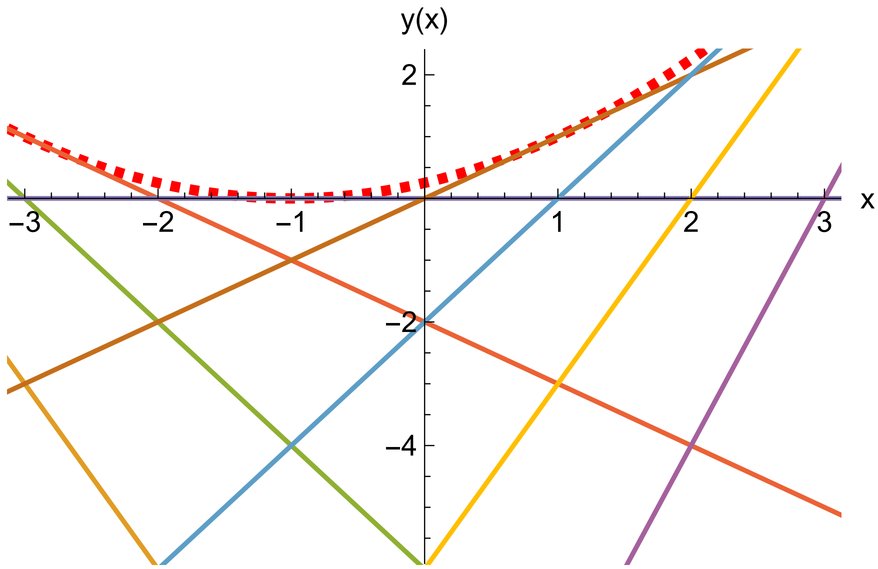

Figure 14: Singular solution envelope with general solution curves

1.19 Example 19

$$(y')^3 - 4xyy' + 8y^2 = 0$$

Applying p-discriminant method gives

$$F = (y')^3 - 4xyy' + 8y^2 = 0$$
$$\frac{\partial F}{\partial y'} = 3(y')^2 - 4xy = 0$$

We first check that $\frac{\partial F}{\partial y} = -4xy' + 16y \neq 0$. Now we apply p-discriminant. Eliminating y'. Second equation gives $y' = \pm\left(\frac{4xy}{3}\right)^{\frac{1}{2}}$. Substituting first solution in the first equation gives

$$\left(\frac{4xy}{3}\right)^{\frac{3}{2}} - 4xy\left(\frac{4xy}{3}\right)^{\frac{1}{2}} + 8y^2 = 0$$
$$y_s = \frac{4}{27}x^3$$

Which satisfies the ode. The general solution can be found to be

$$\Psi(x, y, c) = y - \frac{1}{4}\frac{x^2}{c} + \frac{1}{8}\frac{x}{c^2} - \frac{1}{64c^3} = 0$$

Hence

$$
\begin{aligned}
\Psi(x,y,c) &= y - \frac{1}{4}\frac{x^2}{c} + \frac{1}{8}\frac{x}{c^2} - \frac{1}{64c^3} = 0\\
\frac{\partial \Psi(x,y,c)}{\partial c} &= \frac{1}{4}\frac{x^2}{c^2} - \frac{1}{4}\frac{x}{c^3} + \frac{3}{64c^4} = 0
\end{aligned}
$$

Eliminating c. Second equation gives $c = \frac{1}{4x}$ or $c = \frac{3}{4x}$. Substituting $c = \frac{1}{4x}$ in the first equation above gives

$$
\begin{aligned}
y - \frac{1}{4}\frac{x^2}{\left(\frac{1}{4x}\right)} + \frac{1}{8}\frac{x}{\left(\frac{1}{4x}\right)^2} - \frac{1}{64\left(\frac{1}{4x}\right)^3} &= 0\\
y_s &= 0
\end{aligned}
$$

Which satisfies the ode. But $y = 0$ can be obtained from the general solution above when $c = \infty$ so it is not singular solution. Substituting $c = \frac{3}{4x}$ in the first equation above gives

$$
\begin{aligned}
y - \frac{1}{4}\frac{x^2}{\left(\frac{3}{4x}\right)} + \frac{1}{8}\frac{x}{\left(\frac{3}{4x}\right)^2} - \frac{1}{64\left(\frac{3}{4x}\right)^3} &= 0\\
y &= \frac{4}{27}x^3
\end{aligned}
$$

Which is the same obtained by p-discriminant. Hence this is the singular solution. The following plot shows the singular solution as the envelope of the family of general solution plotted using different values of c.

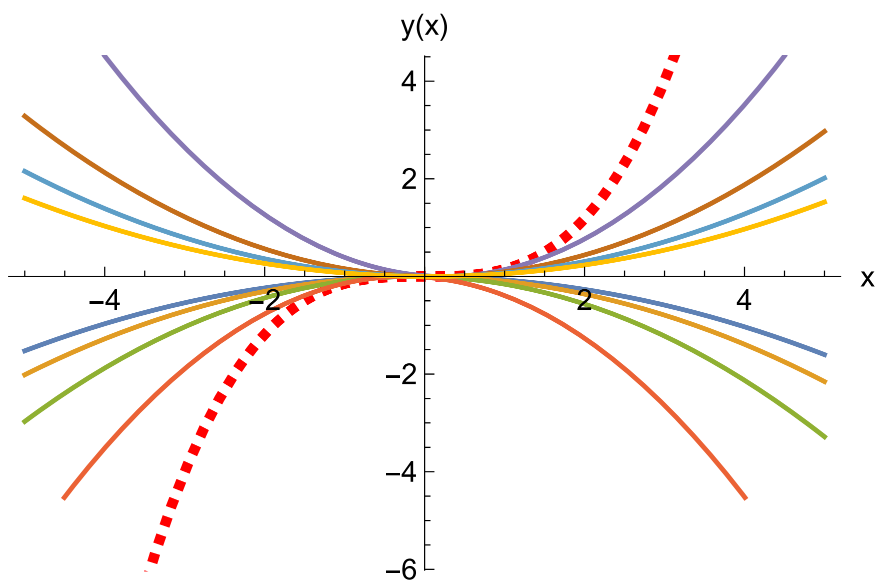

Figure 15: Singular solution envelope with general solution curves

1.20 Example 20

$$4x(y')^2 = (2x-1)^2$$

Applying p-discriminant method gives

$$\begin{aligned} F &= 4x(y')^2 - (2x-1)^2 = 0 \\ \frac{\partial F}{\partial y'} &= 8xy' = 0 \end{aligned}$$

We first check that $\frac{\partial F}{\partial y}$ and see it is zero. Hence no singular solution exists.

1.21 Example 21

$$y' = 2x\left(1-y^2\right)^{\frac{1}{2}}$$

Applying p-discriminant method gives

$$\begin{aligned} F &= y' - 2x\left(1-y^2\right)^{\frac{1}{2}} = 0 \\ \frac{\partial F}{\partial y'} &= 1 = 0 \end{aligned}$$

The p-discriminant does not yield result since it gives $1=0$. Lets try C-discriminant. The general solution can be found to be

$$\Psi(x,y,c) = y - \sin\left(x^2+2c\right) = 0$$

Hence

$$\begin{aligned} \Psi(x,y,c) &= y - \sin\left(x^2+2c\right) = 0 \\ \frac{\partial \Psi(x,y,c)}{\partial c} &= -2\cos\left(x^2+2c\right) = 0 \end{aligned}$$

Hence $x^2+2c = \frac{\pi}{2}$ (there are infinite solutions). Hence $c = \frac{\pi}{4} - \frac{x^2}{2}$. Substituting in the first equation gives

$$y - \sin\left(x^2 + 2\left(\frac{\pi}{4} - \frac{x^2}{2}\right)\right) = 0$$

$$\begin{aligned} y_s &= \sin\left(x^2 + 2\left(\frac{\pi}{4} - \frac{x^2}{2}\right)\right) \\ &= \sin\left(\frac{\pi}{2}\right) \\ &= 1 \end{aligned}$$

And if we took $x^2+2c=-\frac{\pi}{2}$ then we now obtain $y_s=-1$. Now we check that $y=\pm 1$ satisfy the ode itself. We see that they do. This is an example where

The following plot shows the singular solution as the envelope of the family of general solution plotted using different values of c. p-discriminant does not yield result but C-discriminant does.

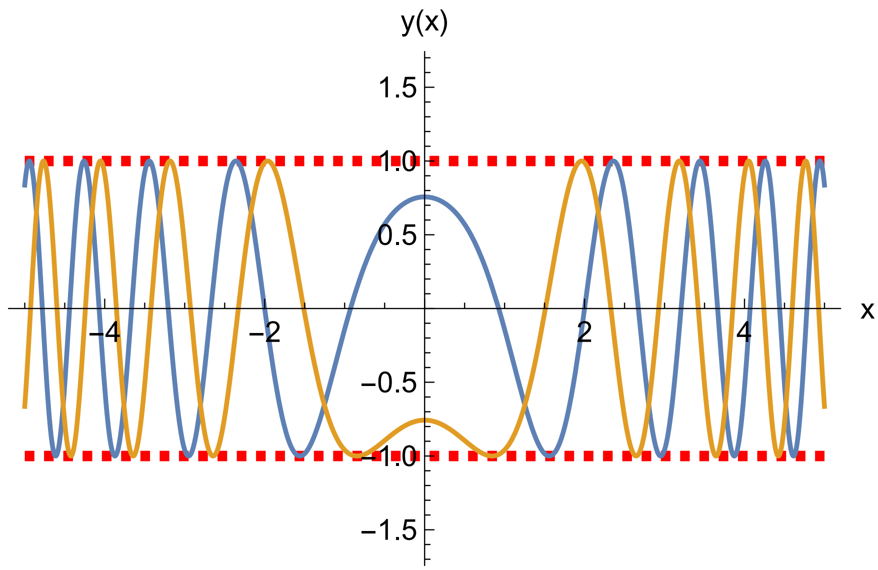

Figure 16: Singular solution envelope with general solution curves

1.22 Example 22

$$(y')^2+2xy'-y=0$$

Applying p-discriminant method gives

$$\begin{aligned}F&=(y')^2+2xy'-y=0\\\frac{\partial F}{\partial y'}&=2y'+2x=0\end{aligned}$$

Eliminating y'. Second equation gives $y'=-x$. Substituting first solution in the first equation gives

$$\begin{aligned}x^2-2x^2-y&=0\\y_s&=-x^2\end{aligned}$$

Now we check that this satisfies the ode itself. We see it does not. Now we try the c-discriminant method. The general solution is too complicated to write here. But Mathematica and Maple claim there is no singular solution. So will leave it there for now. The paper I took this example from is wrong. It claimed $y = x^2$ is the envelope. It is not.

1.23 Example 23

$$(y')^2 (2-3y)^2 - 4(1-y) = 0$$

Applying p-discriminant method gives

$$
\begin{aligned}
F &= (y')^2 (2-3y)^2 - 4(1-y) = 0 \\
\frac{\partial F}{\partial y'} &= 2y'(2-3y)^2 = 0
\end{aligned}
$$

Eliminating y'. Second equation gives $y' = 0$ and $y = \frac{2}{3}$. The solution $y = \frac{2}{3}$ does not satisfy the ode. Substituting $y' = 0$ in the first equation gives

$$
\begin{aligned}
(1-y) &= 0 \\
y_s &= 1
\end{aligned}
$$

This solution does satisfy the ode. The following plot shows the singular solution as the envelope of the family of general solution plotted using different values of c.

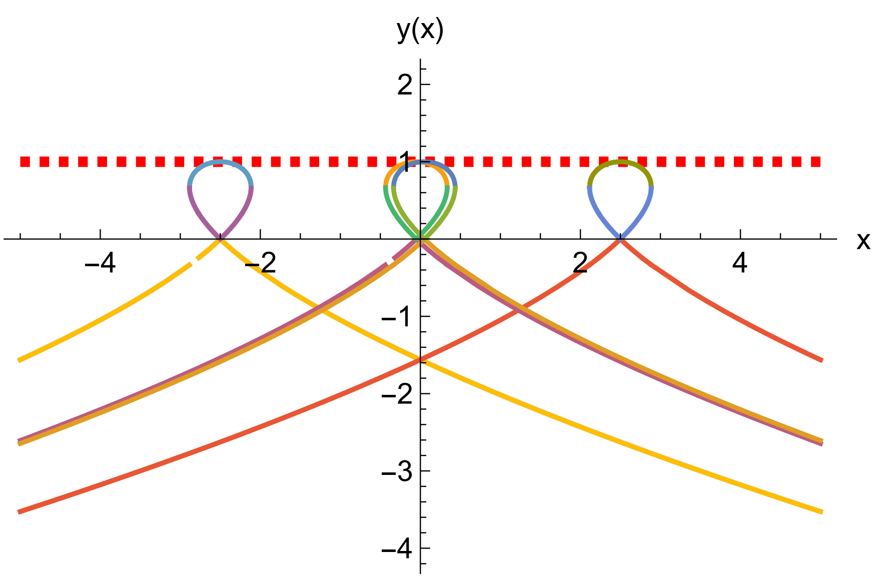

Figure 17: Singular solution envelope with general solution curves