Note on solving Abel first order ODE

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1 Solving Abel ode

This is a note about solving Abel ode. The purpose of this note is show to solve this ode using few examples.

1.1 Abel ode of first kind

This ODE has the form

$$y'(x) = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3 \tag{1}$$

Any one of the following forms is called an Abel ode of first kind

$$\begin{aligned}
y' &= f_0 + f_1 y + f_2 y^2 + f_3 y^3 \\
y' &= f_1 y + f_2 y^2 + f_3 y^3 \\
y' &= f_2 y^2 + f_3 y^3 \\
y' &= f_0 + f_2 y^2 + f_3 y^3 \\
y' &= f_0 + f_3 y^3 \\
y' &= f_0 + f_1 y + f_3 y^3 \\
y' &= f_2 y^2 + f_3 y^3
\end{aligned}$$

The case for both $f_0(x) = 0, f_2(x) = 0$ is not allowed, else it becomes Bernoulli ode. Either $f_0 = 0$ or $f_2 = 0$ is allowed but not both at same time. The term $f_3(x)$ must be there in all cases. When $f_2 = 0$ then Abel invariant is given by

$$\Delta = -\frac{\left(-f_0' f_3 + f_0 f_3' + 3 f_0 f_3 f_1\right)^3}{27 f_3^4 f_0^5}$$

In the case when $f_2 \neq 0$, then f_2 is removed from the original ode using the change of dependent variable $y = u(x) - \frac{f_2}{3 f_3}$. Now the new ode will not f_2 in it, and the above invariant applied to it.

There are two possibilities. Δ can be constant (does not depend on x) or not constant (function of x). The constant invariant is the easier case and can be solved. The non constant case is not fully solved and only few cases can be solved analytically.

1.1.1 Solution method

Find what is called the abel invariant and check if constant.

$$\Delta = -\frac{\left(-f_0' f_3 + f_0 f_3' + 3 f_0 f_3 f_1\right)^3}{27 f_3^4 f_0^5}$$

Then use the substitution $y = \frac{1}{u}$. Hence $y' = -\frac{1}{u^2} u'$. Substituting this in (1) gives

$$\begin{aligned}
-\frac{1}{u^2} u' &= f_0(x) + f_1(x)\frac{1}{u} + f_2(x)\frac{1}{u^2} + f_3(x)\frac{1}{u^3} \\
-uu' &= u^3 f_0(x) + u^2 f_1(x) + u f_2(x) + f_3(x) \\
uu' &= -u^3 f_0(x) - u^2 f_1(x) - u f_2(x) - f_3(x) \qquad (2)
\end{aligned}$$

Next, using substitution $u = \frac{1}{E}\left(y + \frac{f_2}{3f_3}\right)$ where $E = \exp\left(\int f_1 - \frac{f_2^2}{3f_3} dx\right)$ in the above gives

$$\frac{1}{E}\left(y + \frac{f_2}{3f_3}\right) u' = -u^3 f_0(x) - u^2 f_1(x) - u f_2(x) - f_3(x)$$

$$\begin{aligned}
u' &= \frac{1}{E^2}\frac{dE}{dx}\left(y + \frac{f_2}{3f_3}\right) + \frac{1}{E}\left(y' + \frac{1}{3}\frac{f_2' f_3 - f_2 f_3'}{f_3^2}\right) \\
&= \frac{1}{E^2}\frac{dE}{dx}\left(\frac{1}{u} + \frac{f_2}{3f_3}\right) + \frac{1}{E}\left(-\frac{1}{u^2}u' + \frac{1}{3}\frac{f_2' f_3 - f_2 f_3'}{f_3^2}\right) \\
u' + \frac{u'}{Eu^2} &= \frac{1}{E^2}\frac{dE}{dx}\left(\frac{1}{u} + \frac{f_2}{3f_3}\right) + \frac{1}{3E}\frac{f_2' f_3 - f_2 f_3'}{f_3^2} \\
u'\left(1 + \frac{1}{Eu^2}\right) &= \frac{1}{E^2}\frac{dE}{dx}\left(\frac{1}{u} + \frac{f_2}{3f_3}\right) + \frac{1}{3E}\frac{f_2' f_3 - f_2 f_3'}{f_3^2} \\
u' &= \frac{Eu^2}{1 + Eu^2}\left(\frac{1}{E^2}\frac{dE}{dx}\left(\frac{1}{u} + \frac{f_2}{3f_3}\right) + \frac{1}{3E}\frac{f_2' f_3 - f_2 f_3'}{f_3^2}\right) \\
u' &= \frac{u^2}{1 + Eu^2}\left(\frac{1}{E}\frac{dE}{dx}\left(\frac{1}{u} + \frac{f_2}{3f_3}\right) + \frac{1}{3}\frac{f_2' f_3 - f_2 f_3'}{f_3^2}\right)
\end{aligned}$$

Substituting the above into (2) gives

$$u\frac{u^2}{1 + Eu^2}\left(\frac{1}{E}\frac{dE}{dx}\left(\frac{1}{u} + \frac{f_2}{3f_3}\right) + \frac{1}{3}\frac{f_2' f_3 - f_2 f_3'}{f_3^2}\right) = -u^3 f_0 - u^2 f_1 - u f_2 - f_3$$

$$\begin{aligned}
E &= \exp\left(\int f_1(x) - \frac{f_2^2(x)}{3f_3(x)} dx\right) \\
\xi &= \int f_3(x) E^2 dx \\
u &= \frac{1}{E}\left(y + \frac{f_2(x)}{3f_3(x)}\right)
\end{aligned}$$

The above are used to convert the first kind Abel ode to canonical form. (To finish).