A Solution Manual For

Collection of Kovacic problems

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Contents

CHAPTER 1

Lookup tables for all problems in current book

1.1 section 1

Table 1.1: Lookup table for all problems in current section

ID	problem	ODE
9173	1	$(x^2-1)\,y''-2xy'+2y=0$
9174	2	$(x^2-1)\,y''-6xy'+12y=0$
9175	3	$(x^2+3)\,y''-7xy'+16y=0$
9176	4	$(x^2-1)\,y''+8xy'+12y=0$
9177	5	$3y''+xy'-4y=0$
9178	6	$5y''-2xy'+10y=0$
9179	7	$y''-x^2y'-3xy=0$
9180	8	$(x^2+1)\,y''+2xy'-2y=0$
9181	9	$y''+xy'-2y=0$
9182	10	$(x^2-6x+10)\,y''-4(x-3)\,y'+6y=0$
9183	11	$(x^2+6x)\,y''+(3x+9)\,y'-3y=0$
9184	12	$ty''+(t^2-1)\,y'+t^2y=0$
9185	13	$t^2y''-t(2+t)\,y'+(2+t)\,y=0$
9186	14	$ty''-(t+1)\,y'+y=0$
9187	15	$(-t+1)\,y''+y't-y=0$
9188	16	$x^2y''+xy'+\left(x^2-\frac{1}{4}\right)y=0$
9189	17	$ty''-(t+1)\,y'+y=0$
9190	18	$(-t+1)\,y''+y't-y=0$
9191	19	$y''+xy'+2y=0$
9192	20	$(x^2+1)\,y''-4xy'+6y=0$
9193	21	$(1-x)\,y''+xy'-y=0$
9194	22	$2y''+xy'+3y=0$
9195	23	$y''+xy'+2y=0$
9196	24	$(1-x)\,y''+xy'-y=0$
9197	25	$y''+xy'+2y=0$
9198	26	$(-x^2+4)\,y''+xy'+2y=0$
9199	27	$4x^2y''-4xy'+(-16x^2+3)\,y=0$
9200	28	$(x-1)\,y''-xy'+y=0$
9201	29	$x^2y''-2xy'+(x^2+2)\,y=0$
9202	31	$(x^2-2x)\,y''+(-x^2+2)\,y'+(2x-2)\,y=0$
9203	32	$4x^2y''+(-8x^2+4x)\,y'+(4x^2-4x-1)\,y=0$
9204	33	$y''+4xy'+(4x^2+2)\,y=0$
9205	34	$(2x+1)\,y''-2y'-(2x+3)\,y=0$
9206	35	$xy''-(2x+2)\,y'+(x+2)\,y=0$

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Table 1.1 Lookup table
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ID	problem	ODE
9207	36	$x^2y'' - 2xy' + (x^2 + 2)\, y = 0$
9208	38	$4x^2y'' - 4xy' + (-16x^2 + 3)\, y = 0$
9209	39	$4x^2y'' - 4xy' + (4x^2 + 3)\, y = 0$
9210	40	$x^2y'' - 2xy' - (x^2 - 2)\, y = 0$
9211	41	$x^2y'' - 2x(x + 1)\, y' + (x^2 + 2x + 2)\, y = 0$
9212	42	$x^2y'' - 2x(x + 2)\, y' + (x^2 + 4x + 6)\, y = 0$
9213	43	$x^2y'' - 4xy' + (x^2 + 6)\, y = 0$
9214	44	$(x - 1)\, y'' - xy' + y = 0$
9215	45	$4x^2y'' - 4x(x + 1)\, y' + (2x + 3)\, y = 0$
9216	46	$(3x - 1)\, y'' - (2 + 3x)\, y' - (6x - 8)\, y = 0$
9217	47	$(x + 2)\, y'' + xy' + 3y = 0$
9218	48	$x^2(1 - x)\, y'' + x(x + 4)\, y' + (2 - x)\, y = 0$
9219	49	$x^2(x + 1)\, y'' + x(2x + 1)\, y' - (4 + 6x)\, y = 0$
9220	50	$x^2(2x^2 + 1)\, y'' + x(2x^2 + 4)\, y' + 2(-x^2 + 1)\, y = 0$
9221	51	$x^2(x^2 + 2)\, y'' + 2x(x^2 + 5)\, y' + 2(-x^2 + 3)\, y = 0$
9222	52	$(x^2 + 1)\, y'' + 6xy' + 6y = 0$
9223	53	$(x^2 + 1)\, y'' + 2xy' - 2y = 0$
9224	54	$(x^2 + 1)\, y'' - 8xy' + 20y = 0$
9225	55	$(-x^2 + 1)\, y'' - 8xy' - 12y = 0$
9226	56	$(2x^2 + 1)\, y'' + 7xy' + 2y = 0$
9227	57	$(-x^2 + 1)\, y'' - 5xy' - 4y = 0$
9228	58	$(x^2 + 1)\, y'' - 10xy' + 28y = 0$
9229	59	$y'' + xy' + 2y = 0$
9230	60	$(2x^2 + 1)\, y'' - 9xy' - 6y = 0$
9231	61	$(2x^2 - 8x + 11)\, y'' - 16(x - 2)\, y' + 36y = 0$
9232	62	$y'' + (x - 3)\, y' + 3y = 0$
9233	63	$(x^2 - 8x + 14)\, y'' - 8(x - 4)\, y' + 20y = 0$
9234	64	$(2x^2 + 4x + 5)\, y'' - 20(x + 1)\, y' + 60y = 0$
9235	65	$(x^3 + 1)\, y'' + 7x^2y' + 9xy = 0$
9236	66	$(2x^5 + 1)\, y'' + 14x^4y' + 10x^3y = 0$
9237	67	$y'' + x^6y' + 7x^5y = 0$
9238	68	$(x^8 + 1)\, y'' - 16x^7y' + 72x^6y = 0$
9239	69	$y'' + x^5y' + 6x^4y = 0$
9240	70	$(3x + 1)\, y'' + xy' + 2y = 0$
9241	71	$(3x^2 + x + 1)\, y'' + (2 + 15x)\, y' + 12y = 0$

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ID	problem	ODE
9242	72	$(x+2)\,y''+(x+1)\,y'+3y=0$
9243	73	$(x+4)\,y''+(x+2)\,y'+2y=0$
9244	74	$(2x^2+3x)\,y''+10(x+1)\,y'+8y=0$
9245	75	$x^2y''-(6-7x)\,y'+8y=0$
9246	76	$(2x^2+x+1)\,y''+(1+7x)\,y'+2y=0$
9247	77	$(x+3)\,y''+(2x+1)\,y'-(2-x)\,y=0$
9248	78	$y''+3xy'+(2x^2+4)\,y=0$
9249	79	$(2+4x)\,y''-4y'-(6+4x)\,y=0$
9250	80	$y''-3xy'+(2x^2+5)\,y=0$
9251	81	$2y''+5xy'+(2x^2+4)\,y=0$
9252	82	$y''+4xy'+(4x^2+2)\,y=0$
9253	83	$y''+4xy'+(4x^2+2)\,y=0$
9254	84	$2x^2(x^2+x+1)\,y''+x(11x^2+11x+9)\,y'+(7x^2+10x+6)\,y=0$
9255	85	$3x^2y''+2x(-2x^2+x+1)\,y'+(-8x^2+2x)\,y=0$
9256	86	$12x^2(x+1)\,y''+x(3x^2+35x+11)\,y'-(-5x^2-10x+1)\,y=0$
9257	87	$x^2(10x^2+x+5)\,y''+x(48x^2+3x+4)\,y'+(36x^2+x)\,y=0$
9258	88	$18x^2(x+1)\,y''+3x(x^2+11x+5)\,y'-(-5x^2-2x+1)\,y=0$
9259	89	$2x^2y''+x(2x+3)\,y'-(1-x)\,y=0$
9260	90	$2x^2y''+x(x+5)\,y'-(2-3x)\,y=0$
9261	91	$3x^2y''+x(x+1)\,y'-y=0$
9262	92	$2x^2y''-xy'+(-2x+1)\,y=0$
9263	93	$3x^2y''+x(x+1)\,y'-(3x+1)\,y=0$
9264	94	$2x^2(x+3)\,y''+x(1+5x)\,y'+(x+1)\,y=0$
9265	95	$x^2(x+4)\,y''-x(1-3x)\,y'+y=0$
9266	96	$2x^2y''+5xy'+(x+1)\,y=0$
9267	97	$6x^2y''+x(10-x)\,y'-(x+2)\,y=0$
9268	98	$x^2(3+4x)\,y''+x(11+4x)\,y'-(3+4x)\,y=0$
9269	99	$2x^2(2+3x)\,y''+x(4+11x)\,y'-(1-x)\,y=0$
9270	100	$x^2(x+2)\,y''+5x(1-x)\,y'-(2-8x)\,y=0$
9271	101	$8x^2(-x^2+1)\,y''+2x(-13x^2+1)\,y'+(-9x^2+1)\,y=0$
9272	102	$x^2(x^2+1)\,y''-2x(-x^2+2)\,y'+4y=0$
9273	103	$x(x^2+3)\,y''+(-x^2+2)\,y'-8xy=0$
9274	104	$4x^2(-x^2+1)\,y''+x(-19x^2+7)\,y'-(14x^2+1)\,y=0$
9275	105	$3x^2(-x^2+2)\,y''+x(-11x^2+1)\,y'+(-5x^2+1)\,y=0$
9276	106	$2x^2(x^2+2)\,y''-x(-7x^2+12)\,y'+(3x^2+7)\,y=0$

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ID	problem	ODE
9277	107	$2x^2(x^2+2)\,y''+x(7x^2+4)\,y'-(-3x^2+1)\,y=0$
9278	108	$2x^2(2x^2+1)\,y''+5x(6x^2+1)\,y'-(-40x^2+2)\,y=0$
9279	109	$x(x^2+1)\,y''+(7x^2+4)\,y'+8xy=0$
9280	110	$2x^2(x^2+1)\,y''+x(8x^2+3)\,y'-(-4x^2+3)\,y=0$
9281	111	$9x^2y''+3x(x^2+3)\,y'-(-5x^2+1)\,y=0$
9282	112	$6x^2y''+x(6x^2+1)\,y'+(9x^2+1)\,y=0$
9283	113	$9x^2(x^2+1)\,y''+3x(13x^2+3)\,y'-(-25x^2+1)\,y=0$
9284	114	$4x^2(x^2+1)\,y''+4x(6x^2+1)\,y'-(-25x^2+1)\,y=0$
9285	115	$8x^2(2x^2+1)\,y''+2x(34x^2+5)\,y'-(-30x^2+1)\,y=0$
9286	116	$2x^2(x+1)\,y''-x(1-3x)\,y'+y=0$
9287	117	$6x^2(2x^2+1)\,y''+x(50x^2+1)\,y'+(30x^2+1)\,y=0$
9288	118	$28x^2(1-3x)\,y''-7x(5+9x)\,y'+7(2+9x)\,y=0$
9289	119	$8x^2(-x^2+2)\,y''+2x(-21x^2+10)\,y'-(35x^2+2)\,y=0$
9290	120	$4x^2(x^2+3x+1)\,y''-4x(-3x^2-3x+1)\,y'+3(x^2-x+1)\,y=0$
9291	121	$3x^2(x+1)^2\,y''-x(-11x^2-10x+1)\,y'+(5x^2+1)\,y=0$
9292	122	$4x^2(x^2+2x+3)\,y''-x(-15x^2-14x+3)\,y'+(7x^2+3)\,y=0$
9293	123	$x^2(x^2-2x+1)\,y''-x(x+3)\,y'+(x+4)\,y=0$
9294	124	$2x^2(x+2)\,y''+5x^2y'+(x+1)\,y=0$
9295	125	$x^2(-x^2+2)\,y''-2x(2x^2+1)\,y'+(-2x^2+2)\,y=0$
9296	126	$x^2y''-x(5-x)\,y'+(9-4x)\,y=0$
9297	127	$4x^2(x^2+x+1)\,y''+12x^2(x+1)\,y'+(3x^2+3x+1)\,y=0$
9298	128	$x^2(x^2+x+1)\,y''-x(-2x^2-4x+1)\,y'+y=0$
9299	129	$9x^2y''+3x(-2x^2+3x+5)\,y'+(-14x^2+12x+1)\,y=0$
9300	130	$x^2(2x+1)\,y''+x(3x^2+14x+5)\,y'+(12x^2+18x+4)\,y=0$
9301	131	$16x^2y''+4x(2x^2+x+6)\,y'+(18x^2+5x+1)\,y=0$
9302	132	$9x^2(x+1)\,y''+3x(-x^2+11x+5)\,y'+(-7x^2+16x+1)\,y=0$
9303	133	$36x^2(-2x+1)\,y''+24x(1-9x)\,y'+(1-70x)\,y=0$
9304	134	$x^2(x+1)\,y''-x(3-x)\,y'+4y=0$
9305	135	$x^2(-2x+1)\,y''-x(5-4x)\,y'+(9-4x)\,y=0$
9306	136	$2x^2(x+2)\,y''+x^2y'+(1-x)\,y=0$
9307	137	$2x^2(x+1)\,y''-x(6-x)\,y'+(8-x)\,y=0$
9308	138	$x^2(2x+1)\,y''+x(5+9x)\,y'+(3x+4)\,y=0$
9309	139	$x^2(-2x+1)\,y''-x(4x+5)\,y'+(9+4x)\,y=0$
9310	140	$x^2(1-x)\,y''+x(7+x)\,y'+(9-x)\,y=0$
9311	141	$x^2y''-x(-x^2+1)\,y'+(x^2+1)\,y=0$

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ID	problem	ODE
9312	142	$x^2(x^2+1)\,y''-3x(-x^2+1)\,y'+4y=0$
9313	143	$4x^2y''+2x^3y'+(3x^2+1)\,y=0$
9314	144	$x^2(x^2+1)\,y''-x(-2x^2+1)\,y'+y=0$
9315	145	$2x^2(x^2+2)\,y''+7x^3y'+(3x^2+1)\,y=0$
9316	146	$x^2(x^2+1)\,y''-x(-4x^2+1)\,y'+(2x^2+1)\,y=0$
9317	147	$4x^2(x^2+4)\,y''+3x(3x^2+8)\,y'+(-9x^2+1)\,y=0$
9318	148	$3x^2(x^2+3)\,y''+x(11x^2+3)\,y'+(5x^2+1)\,y=0$
9319	149	$9x^2y''-3x(-2x^2+7)\,y'+(2x^2+25)\,y=0$
9320	150	$x^2y''-x(-x^2+1)\,y'+(x^2+1)\,y=0$
9321	151	$x^2(-2x+1)\,y''+3xy'+(4x+1)\,y=0$
9322	152	$x(x+1)\,y''+(1-x)\,y'+y=0$
9323	153	$x^2(1-x)\,y''-x(3-5x)\,y'+(4-5x)\,y=0$
9324	154	$x^2(x^2+1)\,y''-x(9x^2+1)\,y'+(25x^2+1)\,y=0$
9325	155	$9x^2y''+3x(-x^2+1)\,y'+(7x^2+1)\,y=0$
9326	156	$x(x^2+1)\,y''+(-x^2+1)\,y'-8xy=0$
9327	157	$4x^2y''+2x(-x^2+4)\,y'+(7x^2+1)\,y=0$
9328	158	$4x^2(x+1)\,y''+8x^2y'+(x+1)\,y=0$
9329	159	$9x^2(x+3)\,y''+3x(3+7x)\,y'+(3+4x)\,y=0$
9330	160	$x^2(-x^2+2)\,y''-x(3x^2+2)\,y'+(-x^2+2)\,y=0$
9331	161	$16x^2(x^2+1)\,y''+8x(9x^2+1)\,y'+(49x^2+1)\,y=0$
9332	162	$x^2(3x+4)\,y''-x(4-3x)\,y'+4y=0$
9333	163	$4x^2(x^2+3x+1)\,y''+8x^2(2x+3)\,y'+(9x^2+3x+1)\,y=0$
9334	164	$x^2(1-x)^2\,y''-x(-3x^2+2x+1)\,y'+(x^2+1)\,y=0$
9335	165	$9x^2(x^2+x+1)\,y''+3x(13x^2+7x+1)\,y'+(25x^2+4x+1)\,y=0$
9336	166	$2x^2(x+2)\,y''-x(4-7x)\,y'-(5-3x)\,y=0$
9337	167	$x^2(-2x+1)\,y''+x(8-9x)\,y'+(6-3x)\,y=0$
9338	168	$x^2(x^2+1)\,y''+x(10x^2+3)\,y'-(-14x^2+15)\,y=0$
9339	169	$x^2(-2x^2+1)\,y''+x(-13x^2+7)\,y'-14x^2y=0$
9340	170	$4x^2(x+1)\,y''+4x(2x+1)\,y'-(3x+1)\,y=0$
9341	171	$2x^2(2+3x)\,y''+x(4+21x)\,y'-(1-9x)\,y=0$
9342	172	$x^2y''+x(x+2)\,y'-(2-3x)\,y=0$
9343	173	$4x^2(x+1)\,y''+4x(3+8x)\,y'-(5-49x)\,y=0$
9344	174	$x^2(x+1)\,y''-x(3+10x)\,y'+30xy=0$
9345	175	$x^2y''+x(x+1)\,y'-3(x+3)\,y=0$
9346	176	$x^2(2x+1)\,y''+x(9+13x)\,y'+(7+5x)\,y=0$

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Table 1.1 Lookup table
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ID	problem	ODE
9347	177	$4x^2(2x+1)\,y''-2x(4-x)\,y'-(7+5x)\,y=0$
9348	178	$3x^2(x+3)\,y''-x(15+x)\,y'-20y=0$
9349	179	$x^2(x+1)\,y''+x(1-10x)\,y'-(9-10x)\,y=0$
9350	180	$x^2(x+1)\,y''+3x^2y'-(6-x)\,y=0$
9351	181	$x^2(2x+1)\,y''-2x(3+14x)\,y'+(6+100x)\,y=0$
9352	182	$x^2(x+1)\,y''-x(6+11x)\,y'+(6+32x)\,y=0$
9353	183	$4x^2(x+1)\,y''+4x(4x+1)\,y'-(49+27x)\,y=0$
9354	184	$x^2(x^2+1)\,y''-x(-2x^2+7)\,y'+12y=0$
9355	185	$x^2y''-x(-x^2+7)\,y'+12y=0$
9356	186	$x^2y''+x(2x^2+1)\,y'-(-10x^2+1)\,y=0$
9357	187	$x^2y''+x(-2x^2+1)\,y'-4(2x^2+1)\,y=0$
9358	188	$x^2y''+x(-3x^2+1)\,y'-4(-3x^2+1)\,y=0$
9359	189	$x^2(x^2+1)\,y''+x(11x^2+5)\,y'+24x^2y=0$
9360	190	$4x^2(x^2+1)\,y''+8xy'-(-x^2+35)\,y=0$
9361	191	$x^2(x^2+1)\,y''-x(-x^2+5)\,y'-(25x^2+7)\,y=0$
9362	192	$x^2(x^2+1)\,y''+x(2x^2+5)\,y'-21y=0$
9363	193	$4x^2(x^2+1)\,y''+4x(x^2+2)\,y'-(x^2+15)\,y=0$
9364	194	$y''-\frac{2(t+1)y'}{t^2+2t-1}+\frac{2y}{t^2+2t-1}=0$
9365	195	$y''-4y't+(4t^2-2)\,y=0$
9366	196	$(-t^2+1)\,y''-2y't+2y=0$
9367	197	$(t^2+1)\,y''-2y't+2y=0$
9368	198	$(-t^2+1)\,y''-2y't+6y=0$
9369	199	$(2t+1)\,y''-4(t+1)\,y'+4y=0$
9370	200	$t^2y''+y't+\left(t^2-\frac{1}{4}\right)y=0$
9371	201	$y''-\frac{2ty'}{t^2+1}+\frac{2y}{t^2+1}=0$
9372	202	$y''+(t^2+2t+1)\,y'-(4+4t)\,y=0$
9373	204	$2ty''+(1-2t)\,y'-y=0$
9374	205	$2ty''+(t+1)\,y'-2y=0$
9375	206	$2t^2y''-y't+(t+1)\,y=0$
9376	207	$2t^2y''+(t^2-t)\,y'+y=0$
9377	208	$t^2y''+(-t^2+t)\,y'-y=0$
9378	209	$ty''-(t^2+2)\,y'+ty=0$
9379	210	$t^2y''+t(t+1)\,y'-y=0$
9380	211	$ty''-(4+t)\,y'+2y=0$
9381	212	$t^2y''+(t^2-3t)\,y'+3y=0$

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Table 1.1 Lookup table
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ID	problem	ODE
9382	213	$ty'' + y't + 2y = 0$
9383	214	$ty'' + (-t^2 + 1)\, y' + 4ty = 0$
9384	215	$t^2y'' - t(t + 1)\, y' + y = 0$
9385	216	$y'' + 4xy' + (4x^2 + 6)\, y = 0$
9386	217	$(-z^2 + 1)\, y'' - 3zy' + \lambda y = 0$
9387	218	$4zy'' + 2(1 - z)\, y' - y = 0$
9388	219	$f'' + 2(z - 1)\, f' + 4f = 0$
9389	220	$zy'' - 2y' + yz = 0$
9390	221	$zy'' + (2z - 3)\, y' + \frac{4y}{z} = 0$
9391	222	$y'' + 2xy' + 4y = 0$
9392	223	$y'' + xy' + 3y = 0$
9393	224	$y'' - x^2y' - 3xy = 0$
9394	225	$(-4x^2 + 1)\, y'' - 20xy' - 16y = 0$
9395	226	$(x^2 - 1)\, y'' - 6xy' + 12y = 0$
9396	227	$y'' + xy' + (x + 2)\, y = 0$
9397	228	$(2x^2 + 1)\, y'' + 7xy' + 2y = 0$
9398	229	$4y'' + xy' + 4y = 0$
9399	230	$y'' + xy' - 4y = 0$
9400	231	$4xy'' - xy' + 2y = 0$
9401	232	$6x^2y'' + x(1 + 18x)\, y' + (1 + 12x)\, y = 0$
9402	233	$3x^2y'' - x(8 + x)\, y' + 6y = 0$
9403	234	$2x^2y'' - x(2x + 1)\, y' + 2(4x - 1)\, y = 0$
9404	235	$4x^2y'' - 4x^2y' + (2x + 1)\, y = 0$
9405	236	$x^2y'' + x(3 - 2x)\, y' + (-2x + 1)\, y = 0$
9406	237	$x^2y'' - x(x + 3)\, y' + (4 - x)\, y = 0$
9407	238	$x^2y'' + x(3 - x)\, y' + y = 0$
9408	239	$x^2y'' - \left(2\sqrt{5} - 1\right) xy' + \left(\frac{19}{4} - 3x^2\right) y = 0$
9409	240	$x^2y'' + x(x - 3)\, y' + (4 - x)\, y = 0$
9410	241	$x^2y'' + x^2y' - (x + 2)\, y = 0$
9411	242	$x^2y'' + 2x^2y' + \left(x - \frac{3}{4}\right) y = 0$
9412	243	$x^2(x + 1)\, y'' + x^2y' - 2y = 0$
9413	244	$x^2y'' + x(x^2 + 6)\, y' + 6y = 0$
9414	245	$x^2y'' + x(1 - x)\, y' - y = 0$
9415	246	$x^2y'' - x(x + 3)\, y' + 4y = 0$
9416	247	$x^2y'' - x^2y' - 2y = 0$

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Table 1.1 Lookup table
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ID	problem	ODE
9417	248	$x^2y'' - x^2y' - (2+3x)\,y = 0$
9418	249	$x^2y'' + x(5-x)\,y' + 4y = 0$
9419	250	$4x^2y'' + 4x(1-x)\,y' + (2x-9)\,y = 0$
9420	251	$x^2y'' + 2x(x+2)\,y' + 2(x+1)\,y = 0$
9421	252	$x^2y'' - x(1-x)\,y' + (1-x)\,y = 0$
9422	253	$4x^2y'' + 4x(2x+1)\,y' + (4x-1)\,y = 0$
9423	254	$x^2y'' + x(x+4)\,y' + (x+2)\,y = 0$
9424	255	$x^2y'' + xy' + \left(x^2 - \frac{9}{4}\right)y = 0$
9425	256	$xy'' + 2y' + xy = 0$
9426	257	$2xy'' + 5(-2x+1)\,y' - 5y = 0$
9427	258	$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$
9428	259	$xy'' + (x+n)\,y' + (n+1)\,y = 0$
9429	260	$x^4y'' + xy' + y = 0$
9430	261	$x^2y'' + (2x^2+x)\,y' - 4y = 0$
9431	262	$(4x^3 - 14x^2 - 2x)\,y'' - (6x^2 - 7x + 1)\,y' + (6x-1)\,y = 0$
9432	263	$x^2y'' + x^2y' + (x-2)\,y = 0$
9433	264	$x^2y'' - x^2y' + (x-2)\,y = 0$
9434	265	$x^2(1-4x)\,y'' - \frac{xy'}{2} - \frac{3xy}{4} = 0$
9435	266	$x^2y'' + (x^2+x)\,y' + (x-9)\,y = 0$
9436	267	$x^2y'' + x(x+1)\,y' + (3x-1)\,y = 0$
9437	268	$x^2y'' - (x^2+4x)\,y' + 4y = 0$
9438	269	$2x^2y'' - (2+3x)\,y' + \frac{(2x-1)y}{x} = 0$
9439	270	$x(1-x)\,y'' + \left(\frac{3}{2} - 2x\right)y' - \frac{y}{4} = 0$
9440	271	$2x(1-x)\,y'' + xy' - y = 0$
9441	272	$2x(1-x)\,y'' + (1-11x)\,y' - 10y = 0$
9442	273	$x(1-x)\,y'' + \frac{(-2x+1)y'}{3} + \frac{20y}{9} = 0$
9443	274	$4y'' + \frac{3(-x^2+2)y}{(-x^2+1)^2} = 0$
9444	275	$u'' - \frac{2u'}{x} - a^2u = 0$
9445	276	$u'' + \frac{2u'}{x} - a^2u = 0$
9446	277	$u'' + \frac{2u'}{x} + a^2u = 0$
9447	278	$u'' + \frac{4u'}{x} - a^2u = 0$
9448	279	$u'' + \frac{4u'}{x} + a^2u = 0$
9449	280	$y'' - a^2y = \frac{6y}{x^2}$
9450	281	$y'' + n^2y = \frac{6y}{x^2}$
9451	282	$x^2y'' + xy' - \left(x^2 + \frac{1}{4}\right)y = 0$

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ID	problem	ODE
9452	283	$x^2y'' + xy' + \frac{(-9a^2+4x^2)y}{4a^2} = 0$
9453	284	$x^2y'' + xy' + \left(x^2 - \frac{25}{4}\right)y = 0$
9454	285	$y'' + qy' = \frac{2y}{x^2}$
9455	286	$xy'' + 3y' + 4x^3y = 0$
9456	287	$(x^2 + 2x)\,y'' - 2(x+1)\,y' + 2y = 0$
9457	288	$(x^2 + 2x)\,y'' - 2(x+1)\,y' + 2y = 0$
9458	289	$(x^2 + 1)\,y'' - 2xy' + 2y = 0$
9459	290	$(x^2 + 1)\,y'' - 2xy' + 2y = 0$
9460	291	$y'' - 4xy' + (4x^2 - 2)\,y = 0$
9461	292	$y'' - 4xy' + (4x^2 - 2)\,y = 0$
9462	293	$(2x - 3)\,y'' - xy' + y = 0$
9463	294	$y'' - xy' - 3y = 0$
9464	295	$(x^2 + 1)\,y'' - xy' + y = 0$
9465	296	$y'' - xy' + 2y = 0$
9466	297	$(-x^2 + 1)\,y'' - y' + y = 0$
9467	298	$x(x+1)^2\,y'' + (-x^2 + 1)\,y' + (x-1)\,y = 0$
9468	299	$2xy'' - y' + 2y = 0$
9469	300	$xy'' + xy' - 2y = 0$
9470	301	$x(x-1)^2\,y'' - 2y = 0$
9471	302	$y'' - 2xy' + x^2y = 0$
9472	303	$x(-x^2 + 2)\,y'' - (x^2 + 4x + 2)\,((1-x)\,y' + y) = 0$
9473	304	$x^2(x+1)\,y'' - (2x+1)\,(-y + xy') = 0$
9474	305	$2(2-x)\,x^2y'' - x(4-x)\,y' + (3-x)\,y = 0$
9475	306	$x^2(1-x)\,y'' + (5x-4)\,xy' + (6-9x)\,y = 0$
9476	307	$xy'' + (4x^2 + 1)\,y' + 4x(x^2 + 1)\,y = 0$
9477	309	$y'' - 2xy' + 8y = 0$
9478	310	$(-x^2 + 1)\,y'' - 2xy' + 12y = 0$
9479	311	$x(x+2)\,y'' + 2(x+1)\,y' - 2y = 0$
9480	313	$x(x+2)\,y'' + (x+1)\,y' - 4y = 0$
9481	314	$(x-1)\,y'' - xy' + y = 0$
9482	315	$(x^2 + 1)\,y'' - 2xy' + 2y = 0$
9483	316	$(x^2 - 2x + 10)\,y'' + xy' - 4y = 0$
9484	317	$(x^2 - 2x + 10)\,y'' + xy' - 4y = 0$
9485	318	$y'' - xy' + 2y = 0$
9486	319	$(x+2)\,y'' + xy' - y = 0$

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Table 1.1 Lookup table
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ID	problem	ODE
9487	320	$(x^2+1)\,y''-6y=0$
9488	321	$(x^2+2)\,y''+3xy'-y=0$
9489	322	$(x-1)\,y''-xy'+y=0$
9490	325	$x^2y''+\left(\frac{5}{3}x+x^2\right)y'-\frac{y}{3}=0$
9491	326	$2xy''-y'+2y=0$
9492	327	$2xy''-(2x+3)\,y'+y=0$
9493	328	$2x^2y''+3xy'+(2x-1)\,y=0$
9494	329	$xy''+2y'-xy=0$
9495	330	$x^2y''+xy'+\left(x^2-\frac{1}{4}\right)y=0$
9496	331	$xy''+(x-6)\,y'-3y=0$
9497	332	$x^4y''+\lambda y=0$
9498	333	$4x^2y''+4xy'+(4x^2-25)\,y=0$
9499	334	$x^2y''+xy'+\left(36x^2-\frac{1}{4}\right)y=0$
9500	335	$x^2y''+(x^2-2)\,y=0$
9501	336	$xy''+3y'+x^3y=0$
9502	337	$x^2y''+4xy'+(x^2+2)\,y=0$
9503	338	$16x^2y''+32xy'+(x^4-12)\,y=0$
9504	339	$y''-x^2y'+xy=0$
9505	340	$xy''-(x+2)\,y'+2y=0$
9506	341	$y''+xy'+2y=0$
9507	342	$(-x^2+1)\,y''-2xy'+2y=0$
9508	343	$y''-4xy'+(4x^2-2)\,y=0$
9509	344	$(-x^2+1)\,y''-2xy'+30y=0$
9510	345	$xy''+2y'+xy=0$
9511	346	$xy''+(2x+1)\,y'+(x+1)\,y=0$
9512	347	$2x(x-1)\,y''-(x+1)\,y'+y=0$
9513	348	$xy''+2y'+4xy=0$
9514	349	$xy''+(2-2x)\,y'+(x-2)\,y=0$
9515	350	$x^2y''+6xy'+(4x^2+6)\,y=0$
9516	351	$xy''+(-2x+1)\,y'+(x-1)\,y=0$
9517	352	$x(1-x)\,y''+\left(\frac{1}{2}+2x\right)y'-2y=0$
9518	353	$4(t^2-3t+2)\,y''-2y'+y=0$
9519	354	$2(t^2-5t+6)\,y''+(2t-3)\,y'-8y=0$
9520	355	$3t(t+1)\,y''+y't-y=0$
9521	356	$x^2y''+\frac{\left(x+\frac{3}{4}\right)y}{4}=0$

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Table 1.1 Lookup table
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ID	problem	ODE
9522	357	$x^2y'' + xy' + \frac{(x^2-1)y}{4} = 0$
9523	358	$xy'' + (-2x+1)\,y' + (x-1)\,y = 0$
9524	359	$xy'' - (x+1)\,y' + y = 0$
9525	360	$xy'' + 3y' + 4x^3y = 0$
9526	361	$x^2(-x^2+1)\,y'' + 2x(-x^2+1)\,y' - 2y = 0$
9527	362	$2xy'' + (x-2)\,y' - y = 0$
9528	363	$xy'' + 2y' + xy = 0$
9529	364	$y'' + 2x^2y' + (x^4+2x-1)\,y = 0$
9530	365	$u'' + \frac{u}{x^2} = 0$
9531	366	$u'' - (2x+1)\,u' + (x^2+x-1)\,u = 0$
9532	367	$y'' + 2y' + \left(1 + \frac{2}{(3x+1)^2}\right) y = 0$
9533	368	$x^2y'' - 2xy' + (x^2+2)\,y = 0$
9534	369	$y'' + \frac{2y'}{x} - \frac{2y}{(x+1)^2} = 0$
9535	370	$y'' + \frac{y}{2x^4} = 0$
9536	371	$y'' - xy' - xy = 0$
9537	372	$y'' - xy' - xy = 0$
9538	373	$y'' - xy' - xy = 0$
9539	374	$y'' - xy' - xy = 0$
9540	375	$y'' - xy' - xy = 0$
9541	376	$y'' - xy' - xy = 0$
9542	377	$y'' - xy' - xy = 0$
9543	378	$y'' - xy' - xy = 0$
9544	379	$y'' - xy' - xy = 0$
9545	380	$y'' - xy' - xy = 0$
9546	381	$y'' - xy' - xy = 0$
9547	382	$xy'' + 2y' + xy = 0$
9548	383	$2x^2y'' + 3xy' - xy = 0$
9549	384	$x^2y'' + (3x^2+2x)\,y' - 2y = 0$
9550	385	$2x^2(x^2+x+1)\,y'' + x(11x^2+11x+9)\,y' + (7x^2+10x+6)\,y = 0$
9551	388	$xy'' + (x+1)\,y' + 2y = 0$
9552	389	$x^2(x^2-2x+1)\,y'' - x(x+3)\,y' + (x+4)\,y = 0$
9553	390	$2x^2(x+2)\,y'' + 5x^2y' + (x+1)\,y = 0$
9554	391	$x^2y'' + 4xy' + (x^2+2)\,y = 0$
9555	392	$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$

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Table 1.1 Lookup table
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ID	problem	ODE
9556	394	$x^2y'' - xy' - \left(x^2 + \frac{5}{4}\right) y = 0$
9557	395	$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$
9558	396	$x^2y'' + 3xy' + 4x^4y = 0$
9559	398	$y'' = (x^2 + 3) y$
9560	399	$y'' + 2xy' + (x^2 + 1) y = 0$
9561	400	$x^3y'' + y' - \frac{y}{x} = 0$
9562	401	$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$
9563	402	$4x^2y'' + (-8x^2 + 4x) y' + (4x^2 - 4x - 1) y = 0$
9564	404	$y'' - y' + y = 0$
9565	405	$(x^2 - 1) y'' - 2xy' + 2y = 0$
9566	406	$x^2y'' - x(x + 2) y' + (x + 2) y = 0$
9567	407	$(x + 1) y'' - (x + 2) y' + y = 0$
9568	408	$(-x^2 + 1) y'' + 2xy' - 2y = 0$
9569	409	$(-x^2 + 1) y'' - 2xy' + 2y = 0$
9570	410	$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$
9571	411	$(x^2 - 1) y'' - 6xy' + 12y = 0$
9572	412	$(x^2 + 3) y'' - 7xy' + 16y = 0$
9573	413	$(x^2 - 1) y'' + 8xy' + 12y = 0$
9574	414	$3y'' + xy' - 4y = 0$
9575	415	$5y'' - 2xy' + 10y = 0$
9576	416	$y'' - x^2y' - 3xy = 0$
9577	417	$(x^2 + 1) y'' + 2xy' - 2y = 0$
9578	418	$y'' + xy' - 2y = 0$
9579	419	$(x^2 - 6x + 10) y'' - 4(x - 3) y' + 6y = 0$
9580	420	$(x^2 + 6x) y'' + (3x + 9) y' - 3y = 0$
9581	421	$ty'' + (t^2 - 1) y' + t^3y = 0$
9582	422	$t^2y'' - t(2 + t) y' + (2 + t) y = 0$
9583	423	$(x - 1) y'' - xy' + y = 0$
9584	424	$x^2y'' - \left(x - \frac{3}{16}\right) y = 0$
9585	425	$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$
9586	426	$t^2y'' - t(2 + t) y' + (2 + t) y = 0$
9587	427	$ty'' - (t + 1) y' + y = 0$
9588	428	$(-t + 1) y'' + y't - y = 0$
9589	429	$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$
9590	430	$ty'' - (t + 1) y' + y = 0$

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Table 1.1 Lookup table
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ID	problem	ODE
9591	431	$(-t+1)\,y''+y't-y=0$
9592	432	$y''+xy'+2y=0$
9593	433	$(x^2+1)\,y''-4xy'+6y=0$
9594	434	$(1-x)\,y''+xy'-y=0$
9595	435	$2y''+xy'+3y=0$
9596	436	$y''+xy'+2y=0$
9597	437	$(1-x)\,y''+xy'-y=0$
9598	438	$y''+xy'+2y=0$
9599	439	$(-x^2+4)\,y''+xy'+2y=0$
9600	440	$4x^2y''-4xy'+(-16x^2+3)\,y=0$
9601	441	$(x-1)\,y''-xy'+y=0$
9602	442	$x^2y''-2xy'+(x^2+2)\,y=0$
9603	444	$(x^2-2x)\,y''+(-x^2+2)\,y'+(2x-2)\,y=0$
9604	445	$(2x+1)\,y''-2y'-(2x+3)\,y=0$
9605	446	$4x^2y''+(-8x^2+4x)\,y'+(4x^2-4x-1)\,y=0$
9606	447	$y''+4xy'+(4x^2+2)\,y=0$
9607	448	$x^2y''+2x(x-1)\,y'+(x^2-2x+2)\,y=0$
9608	449	$x^2y''-x(2x-1)\,y'+(x^2-x-1)\,y=0$
9609	450	$(-2x+1)\,y''+2y'+(2x-3)\,y=0$
9610	451	$2xy''+(4x+1)\,y'+(2x+1)\,y=0$
9611	452	$xy''-(2x+1)\,y'+(x+1)\,y=0$
9612	453	$4x^2y''-4x(x+1)\,y'+(2x+3)\,y=0$
9613	454	$xy''+(2-2x)\,y'+(x-2)\,y=0$
9614	455	$x^2y''-2xy'+2y=0$
9615	456	$xy''-(2x+2)\,y'+(x+2)\,y=0$
9616	457	$x^2y''-2xy'+(x^2+2)\,y=0$
9617	458	$xy''-(4x+1)\,y'+(2+4x)\,y=0$
9618	460	$4x^2y''-4xy'+(-16x^2+3)\,y=0$
9619	461	$(2x+1)\,xy''-2(2x^2-1)\,y'-4(x+1)\,y=0$
9620	462	$(x^2-2x)\,y''+(-x^2+2)\,y'+(2x-2)\,y=0$
9621	463	$xy''-(4x+1)\,y'+(2+4x)\,y=0$
9622	464	$(3x-1)\,y''-(2+3x)\,y'-(6x-8)\,y=0$
9623	465	$(x+1)^2\,y''-2(x+1)\,y'-(x^2+2x-1)\,y=0$
9624	466	$4x^2y''+(-8x^2+4x)\,y'+(4x^2-4x-1)\,y=0$
9625	467	$y''+4xy'+(4x^2+2)\,y=0$

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Table 1.1 Lookup table
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ID	problem	ODE
9626	468	$(2x+1)\,y''-2y'-(2x+3)\,y=0$
9627	469	$xy''-(2x+2)\,y'+(x+2)\,y=0$
9628	470	$x^2y''-2xy'+(x^2+2)\,y=0$
9629	472	$4x^2y''-4xy'+(-16x^2+3)\,y=0$
9630	473	$4x^2y''-4xy'+(4x^2+3)\,y=0$
9631	474	$x^2y''-2xy'-(x^2-2)\,y=0$
9632	475	$x^2y''-2x(x+1)\,y'+(x^2+2x+2)\,y=0$
9633	476	$x^2y''-2x(x+2)\,y'+(x^2+4x+6)\,y=0$
9634	477	$x^2y''-4xy'+(x^2+6)\,y=0$
9635	478	$(x-1)\,y''-xy'+y=0$
9636	479	$4x^2y''-4x(x+1)\,y'+(2x+3)\,y=0$
9637	480	$(3x-1)\,y''-(2+3x)\,y'-(6x-8)\,y=0$
9638	481	$(x+2)\,y''+xy'+3y=0$
9639	482	$x^2(1-x)\,y''+x(x+4)\,y'+(2-x)\,y=0$
9640	483	$x^2(x+1)\,y''+x(2x+1)\,y'-(4+6x)\,y=0$
9641	484	$x^2(2x^2+1)\,y''+x(2x^2+4)\,y'+2(-x^2+1)\,y=0$
9642	485	$x^2(x^2+2)\,y''+2x(x^2+5)\,y'+2(-x^2+3)\,y=0$
9643	486	$(x^2+1)\,y''+6xy'+6y=0$
9644	487	$(x^2+1)\,y''+2xy'-2y=0$
9645	488	$(x^2+1)\,y''-8xy'+20y=0$
9646	489	$(-x^2+1)\,y''-8xy'-12y=0$
9647	490	$(2x^2+1)\,y''+7xy'+2y=0$
9648	491	$(-x^2+1)\,y''-5xy'-4y=0$
9649	492	$(x^2+1)\,y''-10xy'+28y=0$
9650	493	$y''+xy'+2y=0$
9651	495	$(2x^2-8x+11)\,y''-16(x-2)\,y'+36y=0$
9652	496	$y''+(x-3)\,y'+3y=0$
9653	497	$(x^2-8x+14)\,y''-8(x-4)\,y'+20y=0$
9654	498	$(2x^2+4x+5)\,y''-20(x+1)\,y'+60y=0$
9655	499	$(x^3+1)\,y''+7x^2y'+9xy=0$
9656	500	$(2x^5+1)\,y''+14x^4y'+10x^3y=0$
9657	501	$y''+x^6y'+7x^5y=0$
9658	502	$(x^8+1)\,y''-16x^7y'+72x^6y=0$
9659	503	$y''+x^5y'+6x^4y=0$
9660	504	$(3x+1)\,y''+xy'+2y=0$

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ID	problem	ODE
9661	505	$(3x^2+x+1)\,y''+(2+15x)\,y'+12y=0$
9662	506	$(x+2)\,y''+(x+1)\,y'+3y=0$
9663	507	$(x+4)\,y''+(x+2)\,y'+2y=0$
9664	508	$(2x^2+3x)\,y''+10(x+1)\,y'+8y=0$
9665	509	$x^2y''-(6-7x)\,y'+8y=0$
9666	510	$(2x^2+x+1)\,y''+(1+7x)\,y'+2y=0$
9667	511	$(x+3)\,y''+(2x+1)\,y'-(2-x)\,y=0$
9668	512	$y''+3xy'+(2x^2+4)\,y=0$
9669	513	$(2+4x)\,y''-4y'-(6+4x)\,y=0$
9670	514	$y''-3xy'+(2x^2+5)\,y=0$
9671	515	$2y''+5xy'+(2x^2+4)\,y=0$
9672	516	$y''+4xy'+(4x^2+2)\,y=0$
9673	517	$y''+4xy'+(4x^2+2)\,y=0$
9674	518	$2x^2(x^2+x+1)\,y''+x(11x^2+11x+9)\,y'+(7x^2+10x+6)\,y=0$
9675	519	$3x^2y''+2x(-2x^2+x+1)\,y'+(-8x^2+2x)\,y=0$
9676	520	$12x^2(x+1)\,y''+x(3x^2+35x+11)\,y'-(-5x^2-10x+1)\,y=0$
9677	521	$y''+3y'+4y=0$
9678	522	$18x^2(x+1)\,y''+3x(x^2+11x+5)\,y'-(-5x^2-2x+1)\,y=0$
9679	523	$2x^2y''+x(2x+3)\,y'-(1-x)\,y=0$
9680	524	$2x^2y''+x(x+5)\,y'-(2-3x)\,y=0$
9681	525	$3x^2y''+x(x+1)\,y'-y=0$
9682	526	$2x^2y''-xy'+(-2x+1)\,y=0$
9683	527	$3x^2y''+x(x+1)\,y'-(3x+1)\,y=0$
9684	528	$2x^2(x+3)\,y''+x(1+5x)\,y'+(x+1)\,y=0$
9685	529	$x^2(x+4)\,y''-x(1-3x)\,y'+y=0$
9686	530	$2x^2y''+5xy'+(x+1)\,y=0$
9687	531	$6x^2y''+x(10-x)\,y'-(x+2)\,y=0$
9688	532	$x^2(3+4x)\,y''+x(11+4x)\,y'-(3+4x)\,y=0$
9689	533	$2x^2(2+3x)\,y''+x(4+11x)\,y'-(1-x)\,y=0$
9690	534	$x^2(x+2)\,y''+5x(1-x)\,y'-(2-8x)\,y=0$
9691	535	$8x^2(-x^2+1)\,y''+2x(-13x^2+1)\,y'+(-9x^2+1)\,y=0$
9692	536	$x^2(x^2+1)\,y''-2x(-x^2+2)\,y'+4y=0$
9693	537	$x(x^2+3)\,y''+(-x^2+2)\,y'-8xy=0$
9694	538	$4x^2(-x^2+1)\,y''+x(-19x^2+7)\,y'-(14x^2+1)\,y=0$
9695	539	$3x^2(-x^2+2)\,y''+x(-11x^2+1)\,y'+(-5x^2+1)\,y=0$

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Table 1.1 Lookup table
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ID	problem	ODE
9696	540	$2x^2(x^2+2)\,y''-x(-7x^2+12)\,y'+(3x^2+7)\,y=0$
9697	541	$2x^2(x^2+2)\,y''+x(7x^2+4)\,y'-(-3x^2+1)\,y=0$
9698	542	$2x^2(2x^2+1)\,y''+5x(6x^2+1)\,y'-(-40x^2+2)\,y=0$
9699	543	$x(x^2+1)\,y''+(7x^2+4)\,y'+8xy=0$
9700	544	$2x^2(x^2+1)\,y''+x(8x^2+3)\,y'-(-4x^2+3)\,y=0$
9701	545	$9x^2y''+3x(x^2+3)\,y'-(-5x^2+1)\,y=0$
9702	546	$6x^2y''+x(6x^2+1)\,y'+(9x^2+1)\,y=0$
9703	547	$9x^2(x^2+1)\,y''+3x(13x^2+3)\,y'-(-25x^2+1)\,y=0$
9704	548	$4x^2(x^2+1)\,y''+4x(6x^2+1)\,y'-(-25x^2+1)\,y=0$
9705	549	$8x^2(2x^2+1)\,y''+2x(34x^2+5)\,y'-(-30x^2+1)\,y=0$
9706	550	$2x^2(x+1)\,y''-x(1-3x)\,y'+y=0$
9707	551	$6x^2(2x^2+1)\,y''+x(50x^2+1)\,y'+(30x^2+1)\,y=0$
9708	552	$28x^2(1-3x)\,y''-7x(5+9x)\,y'+7(2+9x)\,y=0$
9709	553	$8x^2(-x^2+2)\,y''+2x(-21x^2+10)\,y'-(35x^2+2)\,y=0$
9710	554	$4x^2(x^2+3x+1)\,y''-4x(-3x^2-3x+1)\,y'+3(x^2-x+1)\,y=0$
9711	555	$3x^2(x+1)^2\,y''-x(-11x^2-10x+1)\,y'+(5x^2+1)\,y=0$
9712	556	$4x^2(x^2+2x+3)\,y''-x(-15x^2-14x+3)\,y'+(7x^2+3)\,y=0$
9713	557	$x^2(x^2-2x+1)\,y''-x(x+3)\,y'+(x+4)\,y=0$
9714	558	$2x^2(x+2)\,y''+5x^2y'+(x+1)\,y=0$
9715	559	$x^2(-x^2+2)\,y''-2x(2x^2+1)\,y'+(-2x^2+2)\,y=0$
9716	560	$x^2y''-x(5-x)\,y'+(9-4x)\,y=0$
9717	561	$4x^2(x^2+x+1)\,y''+12x^2(x+1)\,y'+(3x^2+3x+1)\,y=0$
9718	562	$x^2(x^2+x+1)\,y''-x(-2x^2-4x+1)\,y'+y=0$
9719	563	$9x^2y''+3x(-2x^2+3x+5)\,y'+(-14x^2+12x+1)\,y=0$
9720	564	$x^2(2x+1)\,y''+x(3x^2+14x+5)\,y'+(12x^2+18x+4)\,y=0$
9721	565	$16x^2y''+4x(2x^2+x+6)\,y'+(18x^2+5x+1)\,y=0$
9722	566	$9x^2(x+1)\,y''+3x(-x^2+11x+5)\,y'+(-7x^2+16x+1)\,y=0$
9723	567	$36x^2(-2x+1)\,y''+24x(1-9x)\,y'+(1-70x)\,y=0$
9724	568	$x^2(x+1)\,y''-x(3-x)\,y'+4y=0$
9725	569	$x^2(-2x+1)\,y''-x(5-4x)\,y'+(9-4x)\,y=0$
9726	570	$2x^2(x+2)\,y''+x^2y'+(1-x)\,y=0$
9727	571	$2x^2(x+1)\,y''-x(6-x)\,y'+(8-x)\,y=0$
9728	572	$x^2(2x+1)\,y''+x(5+9x)\,y'+(3x+4)\,y=0$
9729	573	$x^2(-2x+1)\,y''-x(5+4x)\,y'+(9+4x)\,y=0$
9730	574	$x^2(1-x)\,y''+x(7+x)\,y'+(9-x)\,y=0$

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Table 1.1 Lookup table
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ID	problem	ODE
9731	575	$x^2y'' - x(-x^2+1)\,y' + (x^2+1)\,y = 0$
9732	576	$x^2(x^2+1)\,y'' - 3x(-x^2+1)\,y' + 4y = 0$
9733	577	$4x^2y'' + 2x^3y' + (3x^2+1)\,y = 0$
9734	578	$x^2(x^2+1)\,y'' - x(-2x^2+1)\,y' + y = 0$
9735	579	$2x^2(x^2+2)\,y'' + 7x^3y' + (3x^2+1)\,y = 0$
9736	580	$x^2(x^2+1)\,y'' - x(-4x^2+1)\,y' + (2x^2+1)\,y = 0$
9737	581	$4x^2(x^2+4)\,y'' + 3x(3x^2+8)\,y' + (-9x^2+1)\,y = 0$
9738	582	$3x^2(x^2+3)\,y'' + x(11x^2+3)\,y' + (5x^2+1)\,y = 0$
9739	583	$9x^2y'' - 3x(-2x^2+7)\,y' + (2x^2+25)\,y = 0$
9740	584	$x^2y'' - x(-x^2+1)\,y' + (x^2+1)\,y = 0$
9741	585	$x^2(-2x+1)\,y'' + 3xy' + (4x+1)\,y = 0$
9742	586	$x(x+1)\,y'' + (1-x)\,y' + y = 0$
9743	587	$x^2(1-x)\,y'' - x(3-5x)\,y' + (4-5x)\,y = 0$
9744	588	$x^2(x^2+1)\,y'' - x(9x^2+1)\,y' + (25x^2+1)\,y = 0$
9745	589	$9x^2y'' + 3x(-x^2+1)\,y' + (7x^2+1)\,y = 0$
9746	590	$x(x^2+1)\,y'' + (-x^2+1)\,y' - 8xy = 0$
9747	591	$4x^2y'' + 2x(-x^2+4)\,y' + (7x^2+1)\,y = 0$
9748	592	$4x^2(x+1)\,y'' + 8x^2y' + (x+1)\,y = 0$
9749	593	$9x^2(x+3)\,y'' + 3x(3+7x)\,y' + (3+4x)\,y = 0$
9750	594	$x^2(-x^2+2)\,y'' - x(3x^2+2)\,y' + (-x^2+2)\,y = 0$
9751	595	$16x^2(x^2+1)\,y'' + 8x(9x^2+1)\,y' + (49x^2+1)\,y = 0$
9752	596	$x^2(3x+4)\,y'' - x(4-3x)\,y' + 4y = 0$
9753	597	$4x^2(x^2+3x+1)\,y'' + 8x^2(2x+3)\,y' + (9x^2+3x+1)\,y = 0$
9754	598	$x^2(1-x)^2\,y'' - x(-3x^2+2x+1)\,y' + (x^2+1)\,y = 0$
9755	599	$9x^2(x^2+x+1)\,y'' + 3x(13x^2+7x+1)\,y' + (25x^2+4x+1)\,y = 0$
9756	600	$2x^2(x+2)\,y'' - x(4-7x)\,y' - (5-3x)\,y = 0$
9757	601	$x^2(-2x+1)\,y'' + x(8-9x)\,y' + (6-3x)\,y = 0$
9758	602	$x^2(x^2+1)\,y'' + x(10x^2+3)\,y' - (-14x^2+15)\,y = 0$
9759	603	$x^2(-2x^2+1)\,y'' + x(-13x^2+7)\,y' - 14x^2y = 0$
9760	604	$4x^2(x+1)\,y'' + 4x(2x+1)\,y' - (3x+1)\,y = 0$
9761	605	$2x^2(2+3x)\,y'' + x(4+21x)\,y' - (1-9x)\,y = 0$
9762	606	$x^2y'' + x(x+2)\,y' - (2-3x)\,y = 0$
9763	607	$4x^2(x+1)\,y'' + 4x(3+8x)\,y' - (5-49x)\,y = 0$
9764	608	$x^2(x+1)\,y'' - x(3+10x)\,y' + 30xy = 0$
9765	609	$x^2y'' + x(x+1)\,y' - 3(x+3)\,y = 0$

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Table 1.1 Lookup table
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ID	problem	ODE
9766	610	$x^2(2x+1)\,y''+x(9+13x)\,y'+(7+5x)\,y=0$
9767	611	$4x^2(2x+1)\,y''-2x(4-x)\,y'-(7+5x)\,y=0$
9768	612	$3x^2(x+3)\,y''-x(15+x)\,y'-20y=0$
9769	613	$x^2(x+1)\,y''+x(1-10x)\,y'-(9-10x)\,y=0$
9770	614	$x^2(x+1)\,y''+3x^2y'-(6-x)\,y=0$
9771	615	$x^2(2x+1)\,y''-2x(3+14x)\,y'+(6+100x)\,y=0$
9772	616	$x^2(x+1)\,y''-x(6+11x)\,y'+(6+32x)\,y=0$
9773	617	$4x^2(x+1)\,y''+4x(4x+1)\,y'-(49+27x)\,y=0$
9774	618	$x^2(x^2+1)\,y''-x(-2x^2+7)\,y'+12y=0$
9775	619	$x^2y''-x(-x^2+7)\,y'+12y=0$
9776	620	$x^2y''+x(2x^2+1)\,y'-(-10x^2+1)\,y=0$
9777	621	$x^2y''+x(-2x^2+1)\,y'-4(2x^2+1)\,y=0$
9778	622	$x^2y''+x(-3x^2+1)\,y'-4(-3x^2+1)\,y=0$
9779	623	$x^2(x^2+1)\,y''+x(11x^2+5)\,y'+24x^2y=0$
9780	624	$4x^2(x^2+1)\,y''+8xy'-(-x^2+35)\,y=0$
9781	625	$x^2(x^2+1)\,y''-x(-x^2+5)\,y'-(25x^2+7)\,y=0$
9782	626	$x^2(x^2+1)\,y''+x(2x^2+5)\,y'-21y=0$
9783	627	$4x^2(x^2+1)\,y''+4x(x^2+2)\,y'-(x^2+15)\,y=0$
9784	628	$y''-\frac{2(t+1)y'}{t^2+2t-1}+\frac{2y}{t^2+2t-1}=0$
9785	629	$y''-4y't+(4t^2-2)\,y=0$
9786	630	$(-t^2+1)\,y''-2y't+2y=0$
9787	631	$(t^2+1)\,y''-2y't+2y=0$
9788	632	$(-t^2+1)\,y''-2y't+6y=0$
9789	633	$(2t+1)\,y''-4(t+1)\,y'+4y=0$
9790	634	$t^2y''+y't+\left(t^2-\frac{1}{4}\right)y=0$
9791	635	$y''-\frac{2ty'}{t^2+1}+\frac{2y}{t^2+1}=0$
9792	636	$y''+(t^2+2t+1)\,y'-(4+4t)\,y=0$
9793	638	$2ty''+(1-2t)\,y'-y=0$
9794	639	$2ty''+(t+1)\,y'-2y=0$
9795	640	$2t^2y''-y't+(t+1)\,y=0$
9796	641	$2t^2y''+(t^2-t)\,y'+y=0$
9797	642	$t^2y''+(-t^2+t)\,y'-y=0$
9798	643	$ty''-(t^2+2)\,y'+ty=0$
9799	644	$t^2y''+t(t+1)\,y'-y=0$
9800	645	$ty''-(4+t)\,y'+2y=0$

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Table 1.1 Lookup table
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ID	problem	ODE
9801	646	$t^2y'' + (t^2 - 3t)\, y' + 3y = 0$
9802	647	$ty'' + y't + 2y = 0$
9803	648	$ty'' + (-t^2 + 1)\, y' + 4ty = 0$
9804	649	$t^2y'' - t(t + 1)\, y' + y = 0$
9805	650	$y'' + 4xy' + (4x^2 + 6)\, y = 0$
9806	651	$(-z^2 + 1)\, y'' - 3zy' + y = 0$
9807	652	$4zy'' + 2(1 - z)\, y' - y = 0$
9808	653	$f'' + 2(z - 1)\, f' + 4f = 0$
9809	654	$zy'' - 2y' + yz = 0$
9810	655	$zy'' + (2z - 3)\, y' + \frac{4y}{z} = 0$
9811	656	$xy'' + (-2x + 1)\, y' + (x - 1)\, y = 0$
9812	657	$x^2y'' - 2xy' + (x^2 + 2)\, y = 0$
9813	658	$(-x^2 + 1)\, y'' - 2xy' + 2y = 0$
9814	659	$4x^2y'' + 4xy' + (4x^2 - 1)\, y = 0$
9815	660	$xy'' - (2x + 1)\, y' + 2y = 0$
9816	661	$y'' + 2xy' + 4y = 0$
9817	662	$y'' + xy' + 3y = 0$
9818	663	$y'' - x^2y' - 3xy = 0$
9819	664	$(-4x^2 + 1)\, y'' - 20xy' - 16y = 0$
9820	665	$(x^2 - 1)\, y'' - 6xy' + 12y = 0$
9821	666	$y'' + xy' + (x + 2)\, y = 0$
9822	667	$(2x^2 + 1)\, y'' + 7xy' + 2y = 0$
9823	668	$4y'' + xy' + 4y = 0$
9824	669	$y'' + xy' - 4y = 0$
9825	670	$4xy'' - xy' + 2y = 0$
9826	671	$6x^2y'' + x(1 + 18x)\, y' + (1 + 12x)\, y = 0$
9827	672	$3x^2y'' - x(8 + x)\, y' + 6y = 0$
9828	673	$2x^2y'' - x(2x + 1)\, y' + 2(4x - 1)\, y = 0$
9829	674	$4x^2y'' - 4x^2y' + (2x + 1)\, y = 0$
9830	675	$x^2y'' + x(3 - 2x)\, y' + (-2x + 1)\, y = 0$
9831	676	$x^2y'' - x(x + 3)\, y' + (4 - x)\, y = 0$
9832	677	$x^2y'' + x(3 - x)\, y' + y = 0$
9833	678	$x^2y'' - \left(2\sqrt{5} - 1\right) xy' + \left(\frac{19}{4} - 3x^2\right) y = 0$
9834	679	$x^2y'' + x(x - 3)\, y' + (4 - x)\, y = 0$
9835	680	$x^2y'' + x^2y' - (x + 2)\, y = 0$

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Table 1.1 Lookup table
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ID	problem	ODE
9836	681	$x^2y'' + 2x^2y' + \left(x - \frac{3}{4}\right) y = 0$
9837	682	$x^2(x+1) y'' + x^2y' - 2y = 0$
9838	683	$x^2y'' + x(x^2+6) y' + 6y = 0$
9839	684	$x^2y'' + x(1-x) y' - y = 0$
9840	685	$x^2y'' - x(x+3) y' + 4y = 0$
9841	686	$x^2y'' - x^2y' - 2y = 0$
9842	687	$x^2y'' - x^2y' - (2+3x) y = 0$
9843	688	$x^2y'' + x(5-x) y' + 4y = 0$
9844	689	$4x^2y'' + 4x(1-x) y' + (2x-9) y = 0$
9845	690	$x^2y'' + 2x(x+2) y' + 2(x+1) y = 0$
9846	691	$x^2y'' - x(1-x) y' + (1-x) y = 0$
9847	692	$4x^2y'' + 4x(2x+1) y' + (4x-1) y = 0$
9848	693	$x^2y'' + x(x+4) y' + (x+2) y = 0$
9849	694	$x^2y'' + xy' + \left(x^2 - \frac{9}{4}\right) y = 0$
9850	695	$xy'' + 2y' + xy = 0$
9851	696	$2xy'' + 5(-2x+1) y' - 5y = 0$
9852	697	$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$
9853	698	$xy'' + (x+n) y' + (n+1) y = 0$
9854	699	$x^4y'' + xy' + y = 0$
9855	700	$x^2y'' + (2x^2+x) y' - 4y = 0$
9856	701	$(4x^3 - 14x^2 - 2x) y'' - (6x^2 - 7x + 1) y' + (6x-1) y = 0$
9857	702	$x^2y'' + x^2y' + (x-2) y = 0$
9858	703	$x^2y'' - x^2y' + (x-2) y = 0$
9859	704	$x^2(1-4x) y'' + \left(-\frac{1}{4}x - x^2\right) y' - \frac{5xy}{16} = 0$
9860	705	$x^2y'' + (x^2+x) y' + (x-9) y = 0$
9861	706	$x^2y'' + x(x+1) y' + (3x-1) y = 0$
9862	707	$x^2y'' - (x^2+4x) y' + 4y = 0$
9863	708	$2x^2y'' - (2+3x) y' + \frac{(2x-1)y}{x} = 0$
9864	709	$x(1-x) y'' + \left(\frac{3}{2} - 2x\right) y' - \frac{y}{4} = 0$
9865	710	$2x(1-x) y'' + xy' - y = 0$
9866	711	$2x(1-x) y'' + (1-11x) y' - 10y = 0$
9867	712	$x(1-x) y'' + \frac{(-2x+1)y'}{3} + \frac{20y}{9} = 0$
9868	713	$4y'' + \frac{3(-x^2+2)y}{(-x^2+1)^2} = 0$
9869	714	$u'' - \frac{2u'}{x} - a^2u = 0$
9870	715	$u'' + \frac{2u'}{x} - a^2u = 0$

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ID	problem	ODE
9871	716	$u'' + \frac{2u'}{x} + a^2u = 0$
9872	717	$u'' + \frac{4u'}{x} - a^2u = 0$
9873	718	$u'' + \frac{4u'}{x} + a^2u = 0$
9874	719	$y'' - a^2y = \frac{6y}{x^2}$
9875	720	$y'' + n^2y = \frac{6y}{x^2}$
9876	721	$x^2y'' + xy' - \left(x^2 + \frac{1}{4}\right)y = 0$
9877	722	$x^2y'' + xy' + \frac{(-9a^2+4x^2)y}{4a^2} = 0$
9878	723	$x^2y'' + xy' + \left(x^2 - \frac{25}{4}\right)y = 0$
9879	724	$y'' + qy' = \frac{2y}{x^2}$
9880	725	$xy'' + 3y' + 4x^3y = 0$
9881	726	$x^2(2-x)\,y'' + 2xy' - 2y = 0$
9882	727	$(x^2+1)\,y'' - 2xy' + 2y = 0$
9883	728	$xy'' - 2(x+1)\,y' + (x+2)\,y = 0$
9884	729	$3xy'' - 2(3x-1)\,y' + (3x-2)\,y = 0$
9885	730	$x(x+1)\,y'' - (x-1)\,y' + y = 0$
9886	731	$(x^2+2x)\,y'' - 2(x+1)\,y' + 2y = 0$
9887	732	$(x^2+2x)\,y'' - 2(x+1)\,y' + 2y = 0$
9888	733	$(x^2+1)\,y'' - 2xy' + 2y = 0$
9889	734	$(x^2+1)\,y'' - 2xy' + 2y = 0$
9890	735	$y'' - 4xy' + (4x^2-2)\,y = 0$
9891	736	$y'' - 4xy' + (4x^2-2)\,y = 0$
9892	737	$(2x-3)\,y'' - xy' + y = 0$
9893	738	$y'' - xy' - 3y = 0$
9894	739	$(x^2+1)\,y'' - xy' + y = 0$
9895	740	$y'' - xy' + 2y = 0$
9896	741	$(-x^2+1)\,y'' - y' + y = 0$
9897	742	$x(x+1)^2\,y'' + (-x^2+1)\,y' + (x-1)\,y = 0$
9898	743	$2xy'' - y' + 2y = 0$
9899	744	$xy'' + xy' - 2y = 0$
9900	745	$x(x-1)^2\,y'' - 2y = 0$
9901	746	$y'' - 2xy' + x^2y = 0$
9902	747	$x(-x^2+2)\,y'' - (x^2+4x+2)\left((1-x)\,y' + y\right) = 0$
9903	748	$x^2(x+1)\,y'' - (2x+1)\left(-y + xy'\right) = 0$
9904	749	$2x^2(2-x)\,y'' - x(4-x)\,y' + (3-x)\,y = 0$
9905	750	$x^2(1-x)\,y'' + (5x-4)\,xy' + (6-9x)\,y = 0$

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Table 1.1 Lookup table
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ID	problem	ODE
9906	751	$xy'' + (4x^2 + 1) y' + 4x(x^2 + 1) y = 0$
9907	754	$(-x^2 + 1) y'' - 2xy' + 12y = 0$
9908	755	$x(x + 2) y'' + 2(x + 1) y' - 2y = 0$
9909	757	$x(x + 2) y'' + (x + 1) y' - 4y = 0$
9910	758	$(x - 1) y'' - xy' + y = 0$
9911	759	$(x^2 + 1) y'' - 2xy' + 2y = 0$
9912	760	$(x^2 - 2x + 10) y'' + xy' - 4y = 0$
9913	761	$(x^2 - 2x + 10) y'' + xy' - 4y = 0$
9914	762	$y'' - xy' + 2y = 0$
9915	763	$(x + 2) y'' + xy' - y = 0$
9916	764	$(x^2 + 1) y'' - 6y = 0$
9917	765	$(x^2 + 2) y'' + 3xy' - y = 0$
9918	766	$(x - 1) y'' - xy' + y = 0$
9919	769	$x^2y'' + \left(\frac{5}{3}x + x^2\right) y' - \frac{y}{3} = 0$
9920	770	$2xy'' - y' + 2y = 0$
9921	771	$2xy'' - (2x + 3) y' + y = 0$
9922	772	$2x^2y'' + 3xy' + (2x - 1) y = 0$
9923	773	$xy'' + 2y' - xy = 0$
9924	774	$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$
9925	775	$xy'' + (x - 6) y' - 3y = 0$
9926	776	$x^4y'' + \lambda y = 0$
9927	777	$4x^2y'' + 4xy' + (4x^2 - 25) y = 0$
9928	778	$x^2y'' + xy' + \left(36x^2 - \frac{1}{4}\right) y = 0$
9929	779	$x^2y'' + (x^2 - 2) y = 0$
9930	780	$xy'' + 3y' + x^3y = 0$
9931	781	$x^2y'' + 4xy' + (x^2 + 2) y = 0$
9932	782	$16x^2y'' + 32xy' + (x^4 - 12) y = 0$
9933	783	$y'' - x^2y' + xy = 0$
9934	784	$xy'' - (x + 2) y' + 2y = 0$
9935	785	$y'' + xy' + 2y = 0$
9936	786	$(-x^2 + 1) y'' - 2xy' + 2y = 0$
9937	787	$y'' - 4xy' + (4x^2 - 2) y = 0$
9938	788	$(-x^2 + 1) y'' - 2xy' + 30y = 0$
9939	789	$xy'' + 2y' + xy = 0$
9940	790	$xy'' + (2x + 1) y' + (x + 1) y = 0$

Continued on next page

Table 1.1 Lookup table
Continued from previous page

ID	problem	ODE
9941	791	$2x(x-1)\,y''-(x+1)\,y'+y=0$
9942	792	$xy''+2y'+4xy=0$
9943	793	$xy''+(2-2x)\,y'+(x-2)\,y=0$
9944	794	$x^2y''+6xy'+(4x^2+6)\,y=0$
9945	795	$xy''+(-2x+1)\,y'+(x-1)\,y=0$
9946	796	$x(1-x)\,y''+\left(\frac{1}{2}+2x\right)y'-2y=0$
9947	797	$4(t^2-3t+2)\,y''-2y'+y=0$
9948	798	$2(t^2-5t+6)\,y''+(2t-3)\,y'-8y=0$
9949	799	$3t(t+1)\,y''+y't-y=0$
9950	800	$x^2y''+\frac{(x+\frac{3}{4})y}{4}=0$
9951	801	$x^2y''+xy'+\frac{(x^2-1)y}{4}=0$
9952	802	$xy''+(-2x+1)\,y'+(x-1)\,y=0$
9953	803	$xy''-(x+1)\,y'+y=0$
9954	804	$xy''+3y'+4x^3y=0$
9955	805	$x^2(-x^2+1)\,y''+2x(-x^2+1)\,y'-2y=0$
9956	806	$2xy''+(x-2)\,y'-y=0$
9957	807	$xy''+2y'+xy=0$
9958	808	$y''+2x^2y'+(x^4+2x-1)\,y=0$
9959	809	$u''+2u'+u=0$
9960	810	$u''-(2x+1)\,u'+(x^2+x-1)\,u=0$
9961	811	$y''+2y'+\left(1+\frac{2}{(3x+1)^2}\right)y=0$
9962	812	$x^2y''-2xy'+(x^2+2)\,y=0$
9963	813	$y''+\frac{2y'}{x}-\frac{2y}{(x+1)^2}=0$
9964	815	$y''-xy'-xy=0$
9965	816	$y''-xy'-xy=0$
9966	817	$y''-xy'-xy=0$
9967	818	$y''-xy'-xy=0$
9968	819	$y''-xy'-xy=0$
9969	820	$y''-xy'-xy=0$
9970	821	$y''-xy'-xy=0$
9971	822	$y''-xy'-xy=0$
9972	823	$y''-xy'-xy=0$
9973	824	$y''-xy'-xy=0$
9974	825	$y''-xy'-xy=0$

Continued on next page

Table 1.1 Lookup table
Continued from previous page

ID	problem	ODE
9975	826	$xy'' + 2y' + xy = 0$
9976	827	$2x^2y'' + 3xy' - xy = 0$
9977	828	$x^2y'' + (3x^2 + 2x)\, y' - 2y = 0$
9978	829	$2x^2(x^2 + x + 1)\, y'' + x(11x^2 + 11x + 9)\, y' + (7x^2 + 10x + 6)\, y = 0$
9979	830	$xy'' + (x + 1)\, y' + 2y = 0$
9980	831	$x^2(x^2 - 2x + 1)\, y'' - x(x + 3)\, y' + (x + 4)\, y = 0$
9981	832	$2x^2(x + 2)\, y'' + 5x^2y' + (x + 1)\, y = 0$
9982	833	$x^2y'' + 4xy' + (x^2 + 2)\, y = 0$
9983	834	$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$
9984	835	$x^2y'' - xy' - \left(x^2 + \frac{5}{4}\right) y = 0$
9985	836	$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$
9986	837	$x^2y'' + 3xy' + 4x^4y = 0$
9987	838	$y'' = (x^2 + 3)\, y$
9988	839	$y'' + 2xy' + (x^2 + 1)\, y = 0$
9989	840	$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$
9990	841	$4x^2y'' + (-8x^2 + 4x)\, y' + (4x^2 - 4x - 1)\, y = 0$
9991	843	$y'' = 0$
9992	844	$y'' = \frac{2y}{x^2}$
9993	845	$y'' = \frac{6y}{x^2}$

1.2 section 2. Solution found using all possible Kovacic cases

Table 1.2: Lookup table for all problems in current section

ID	problem	ODE
9994	1	$y'' = \left(-\frac{3}{16x^2} - \frac{2}{9(x-1)^2} + \frac{3}{16x(x-1)}\right) y$
9995	2	$y'' = \frac{20y}{x^2}$
9996	3	$y'' = \frac{12y}{x^2}$
9997	4	$y'' - \frac{y}{4x^2} = 0$
9998	5	$xy'' - (2x + 2)\, y' + (x + 2)\, y = 0$
9999	6	$y'' + \frac{y}{x^2} = 0$
10000	7	$(-x^2 + 1)\, y'' + y' + y = 0$
10001	8	$(x^2 - x)\, y'' - xy' + y = 0$
10002	9	$x^2(-x^2 + 2)\, y'' - x(4x^2 + 3)\, y' + (-2x^2 + 2)\, y = 0$

1.3 section 3. Problems from Kovacic related papers

Table 1.3: Lookup table for all problems in current section

ID	problem	ODE
10003	Kovacic 1985 paper. page 13. section 3.2, example 1	$y'' = \frac{(4x^6-8x^5+12x^4+4x^3+7x^2-20x+4)y}{4x^4}$
10004	Kovacic 1985 paper. page 14. section 3.2, example 2	$y'' = \left(\frac{6}{x^2}-1\right)y$
10005	Kovacic 1985 paper. page 15. Weber equation	$y'' = \left(\frac{x^2}{4}-\frac{11}{2}\right)y$
10006	Kovacic 1985 paper. page 19. section 4.2. Example 1	$y'' = \left(\frac{1}{x}-\frac{3}{16x^2}\right)y$
10007	Kovacic 1985 paper. page 23. section 5.2. Example 1	$y'' = \left(-\frac{3}{16x^2}-\frac{2}{9(x-1)^2}+\frac{3}{16x(x-1)}\right)y$

Continued on next page

Table 1.3 Lookup table
Continued from previous page

ID	problem	ODE
10008	Kovacic 1985 paper. page 25. section 5.2. Example 2	$y'' = -\frac{(5x^2+27)y}{36(x^2-1)^2}$
10009	Kovacic 2005 paper. Example 2	$y'' = -\frac{y}{4x^2}$
10010	David Saunders 1981 paper. Example 1	$y'' = (x^2+3)\,y$
10011	David Saunders 1981 paper. Example 3	$x^2y'' = 2y$
10012	Carolyn J. Smith 1984 paper. Appendix B examples and tests. Example 1	$y'' + 4xy' + (4x^2+2)\,y = 0$

Continued on next page

Table 1.3 Lookup table
Continued from previous page

ID	problem	ODE
10013	Carolyn J. Smith 1984 paper. Appendix B examples and tests. Example 2	$x^2y'' - 2xy' + (x^2+2)\,y = 0$
10014	Carolyn J. Smith 1984 paper. Appendix B examples and tests. Example 3	$(x-2)^2\,y'' - (x-2)\,y' - 3y = 0$

CHAPTER 2

Book Solved Problems

2.1 section 1

2.1.1 Problem 1

Internal problem ID [9173]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 1
Date solved : `Monday, January 27, 2025 at 05:51:43 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(x^2-1\right) y''-2xy'+2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.194 (sec)

Writing the ode as

$$\left(x^2-1\right) y''-2xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2-1 \\ B &= -2x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{3}{\left(x^2-1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= \left(x^2-1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{3}{\left(x^2-1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x+1)} - \frac{3}{4(x-1)} + \frac{3}{4(x+1)^2} + \frac{3}{4(x-1)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{(x^2-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\,(x-1)} + \frac{3}{2\,(x+1)} + (-)\,(0) \\
&= -\frac{1}{2\,(x-1)} + \frac{3}{2\,(x+1)} \\
&= \frac{x-2}{x^2-1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2(x-1)}+\frac{3}{2(x+1)}\right)(0)+\left(\left(\frac{1}{2(x-1)^2}-\frac{3}{2(x+1)^2}\right)+\left(-\frac{1}{2(x-1)}+\frac{3}{2(x+1)}\right)^2-\left(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\, dx}\\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x-1)}+\frac{3}{2(x+1)}\right)dx}\\
&= \frac{(x+1)^{3/2}}{\sqrt{x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}\\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2-1}\, dx}\\
&= z_1 e^{\frac{\ln(x-1)}{2}+\frac{\ln(x+1)}{2}}\\
&= z_1\left(\sqrt{x-1}\,\sqrt{x+1}\right)
\end{aligned}$$

Which simplifies to

$$y_1=(x+1)^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1\int \frac{e^{\int -\frac{-2x}{x^2-1}\, dx}}{(y_1)^2}\, dx\\
&= y_1\int \frac{e^{\ln(x-1)+\ln(x+1)}}{(y_1)^2}\, dx\\
&= y_1\left(-\frac{x\,\mathrm{e}^{\ln(x-1)+\ln(x+1)}}{(x+1)^3\,(x-1)}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1y_1+c_2y_2\\
&= c_1\left((x+1)^2\right)+c_2\left((x+1)^2\left(-\frac{x\,\mathrm{e}^{\ln(x-1)+\ln(x+1)}}{(x+1)^3\,(x-1)}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x^2-1)\left(\frac{d^2}{dx^2}y(x)\right)-2x\left(\frac{d}{dx}y(x)\right)+2y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{2y(x)}{x^2-1}+\frac{2\left(\frac{d}{dx}y(x)\right)x}{x^2-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x)-\frac{2\left(\frac{d}{dx}y(x)\right)x}{x^2-1}+\frac{2y(x)}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=-\frac{2x}{x^2-1}, P_3(x)=\frac{2}{x^2-1}\right]$

- ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

$\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=-1$

- ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

$\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

- ◦ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1$

- Multiply by denominators

$(x^2-1)\left(\frac{d^2}{dx^2}y(x)\right)-2x\left(\frac{d}{dx}y(x)\right)+2y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(-2u+2)\left(\frac{d}{du}y(u)\right)+2y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ◦ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

- ◦ Shift index using $k->k+2-m$

$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

Rewrite ODE with series expansions

$-2a_0r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)(k+r-1)+a_k(k+r-1)(k+r-2)\right)u^{k+r}\right)=$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-2r(-2+r)=0$

- Values of r that satisfy the indicial equation
$r \in \{0, 2\}$
- Each term in the series must be 0, giving the recursion relation
$(k+r-1)\left((-2k-2r-2)\,a_{k+1}+a_k(k+r-2)\right)=0$
- Recursion relation that defines series solution to ODE
$a_{k+1}=\frac{a_k(k+r-2)}{2(k+1+r)}$
- Recursion relation for $r=0$; series terminates at $k=2$
$a_{k+1}=\frac{a_k(k-2)}{2(k+1)}$
- Apply recursion relation for $k=0$
$a_1=-a_0$
- Apply recursion relation for $k=1$
$a_2=-\frac{a_1}{4}$
- Express in terms of a_0
$a_2=\frac{a_0}{4}$
- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li
$y(u)=a_0\cdot\left(1-u+\frac{1}{4}u^2\right)$
- Revert the change of variables $u=x+1$
$\left[y(x)=\frac{a_0(x-1)^2}{4}\right]$
- Recursion relation for $r=2$
$a_{k+1}=\frac{a_k k}{2(k+3)}$
- Solution for $r=2$
$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1}=\frac{a_k k}{2(k+3)}\right]$
- Revert the change of variables $u=x+1$
$\left[y(x)=\sum_{k=0}^{\infty} a_k(x+1)^{k+2}, a_{k+1}=\frac{a_k k}{2(k+3)}\right]$
- Combine solutions and rename parameters
$\left[y(x)=\frac{a_0(x-1)^2}{4}+\left(\sum_{k=0}^{\infty} b_k(x+1)^{k+2}\right), b_{k+1}=\frac{b_k k}{2(k+3)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 14

```
dsolve((x^2-1)*diff(diff(y(x),x),x)-2*diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y = c_2x^2 + c_1x + c_2$$

Mathematica DSolve solution

Solving time : 0.379 (sec)
Leaf size : 75

```
DSolve[{(x^2-1)*D[y[x],{x,2}]-2*x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True
```

$$y(x) \to \sqrt{x^2-1}\exp\left(\int_1^x \frac{K[1]+2}{K[1]^2-1}dK[1]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[2]}\frac{K[1]+2}{K[1]^2-1}dK[1]\right)dK[2] + c_1\right)$$

2.1.2 Problem 2

Internal problem ID [9174]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 2
Date solved : `Monday, January 27, 2025 at 05:51:43 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$(x^2-1)\,y''-6xy'+12y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.208 (sec)

Writing the ode as

$$(x^2-1)\,y''-6xy'+12y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2-1\\ B &= -6x \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{15}{(x^2-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15\\ t &= (x^2-1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{15}{(x^2-1)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.3: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4(x+1)^2} + \frac{15}{4(x+1)} + \frac{15}{4(x-1)^2} - \frac{15}{4(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15}{(x^2-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
-1	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{2(x-1)} + \frac{5}{2(x+1)} + (-)(0) \\
&= -\frac{3}{2(x-1)} + \frac{5}{2(x+1)} \\
&= \frac{x-4}{x^2-1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{3}{2(x-1)}+\frac{5}{2(x+1)}\right)(0)+\left(\left(\frac{3}{2(x-1)^2}-\frac{5}{2(x+1)^2}\right)+\left(-\frac{3}{2(x-1)}+\frac{5}{2(x+1)}\right)^2-\left(\frac{}{(x^2}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(-\frac{3}{2(x-1)}+\frac{5}{2(x+1)}\right)dx} \\ &= \frac{(x+1)^{5/2}}{(x-1)^{3/2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-6x}{x^2-1}\, dx} \\ &= z_1 e^{\frac{3\ln(x-1)}{2}+\frac{3\ln(x+1)}{2}} \\ &= z_1\left((x-1)^{3/2}(x+1)^{3/2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (x+1)^4$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-6x}{x^2-1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{\mathrm{e}^{3\ln(x-1)+3\ln(x+1)}}{(y_1)^2}\, dx \\ &= y_1\left(-\frac{x(x^2+1)\,\mathrm{e}^{3\ln(x-1)+3\ln(x+1)}}{(x+1)^7(x-1)^3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left((x+1)^4\right) + c_2\left((x+1)^4\left(-\frac{x(x^2+1)\,\mathrm{e}^{3\ln(x-1)+3\ln(x+1)}}{(x+1)^7(x-1)^3}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x^2-1)\left(\frac{d^2}{dx^2}y(x)\right)-6x\left(\frac{d}{dx}y(x)\right)+12y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{12y(x)}{x^2-1}+\frac{6\left(\frac{d}{dx}y(x)\right)x}{x^2-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)-\frac{6\left(\frac{d}{dx}y(x)\right)x}{x^2-1}+\frac{12y(x)}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=-\frac{6x}{x^2-1}, P_3(x)=\frac{12}{x^2-1}\right]$

- ○ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=-3$

- ○ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

- ○ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $(x^2-1)\left(\frac{d^2}{dx^2}y(x)\right)-6x\left(\frac{d}{dx}y(x)\right)+12y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(-6u+6)\left(\frac{d}{du}y(u)\right)+12y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ○ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

- ○ Shift index using $k->k+2-m$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r(-4+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)(k+r-3)+a_k(k+r-3)(k+r-4)\right)u^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-2r(-4+r)=0$

- Values of r that satisfy the indicial equation
  $r \in \{0,4\}$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-3)\left((-2k-2r-2)\,a_{k+1}+a_k(k+r-4)\right)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{a_k(k+r-4)}{2(k+1+r)}$
- Recursion relation for $r=0$; series terminates at $k=4$
  $a_{k+1}=\frac{a_k(k-4)}{2(k+1)}$
- Apply recursion relation for $k=0$
  $a_1=-2a_0$
- Apply recursion relation for $k=1$
  $a_2=-\frac{3a_1}{4}$
- Express in terms of a_0
  $a_2=\frac{3a_0}{2}$
- Apply recursion relation for $k=2$
  $a_3=-\frac{a_2}{3}$
- Express in terms of a_0
  $a_3=-\frac{a_0}{2}$
- Apply recursion relation for $k=3$
  $a_4=-\frac{a_3}{8}$
- Express in terms of a_0
  $a_4=\frac{a_0}{16}$
- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second linea
  $y(u)=a_0\cdot\left(1-2u+\frac{3}{2}u^2-\frac{1}{2}u^3+\frac{1}{16}u^4\right)$
- Revert the change of variables $u=x+1$
  $\left[y(x)=\frac{a_0(x-1)^4}{16}\right]$
- Recursion relation for $r=4$
  $a_{k+1}=\frac{a_k k}{2(k+5)}$
- Solution for $r=4$
  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+4}, a_{k+1}=\frac{a_k k}{2(k+5)}\right]$
- Revert the change of variables $u=x+1$
  $\left[y(x)=\sum_{k=0}^{\infty} a_k(x+1)^{k+4}, a_{k+1}=\frac{a_k k}{2(k+5)}\right]$
- Combine solutions and rename parameters
  $\left[y(x)=\frac{a_0(x-1)^4}{16}+\left(\sum_{k=0}^{\infty} b_k(x+1)^{4+k}\right), b_{k+1}=\frac{b_k k}{2(5+k)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 25

```
dsolve((x^2-1)*diff(diff(y(x),x),x)-6*diff(y(x),x)*x+12*y(x) = 0,y(x),singsol=all)
```

$$y = c_2x^4 + c_1x^3 + 6c_2x^2 + c_1x + c_2$$

Mathematica DSolve solution

Solving time : 0.399 (sec)
Leaf size : 75

```
DSolve[{(x^2-1)*D[y[x],{x,2}]-6*x*D[y[x],x]+12*y[x]==0,{}},y[x],x,IncludeSingularSolutions->
```

$$y(x) \to (x^2-1)^{3/2} \exp\left(\int_1^x \frac{K[1]+4}{K[1]^2-1}dK[1]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[2]} \frac{K[1]+4}{K[1]^2-1}dK[1]\right)dK[2] + c_1\right)$$

2.1.3 Problem 3

Internal problem ID [9175]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 3
Date solved : `Monday, January 27, 2025 at 05:51:44 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+3\right) y''-7xy'+16y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.362 (sec)

Writing the ode as

$$\left(x^2+3\right) y''-7xy'+16y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+3 \\ B &= -7x \\ C &= 16 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-x^2-234}{4\left(x^2+3\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2-234 \\ t &= 4\left(x^2+3\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-x^2-234}{4\left(x^2+3\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.5: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+3)^2$. There is a pole at $x = i\sqrt{3}$ of order 2. There is a pole at $x = -i\sqrt{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{77}{16\left(x - i\sqrt{3}\right)^2} + \frac{77}{16\left(x + i\sqrt{3}\right)^2} + \frac{79i\sqrt{3}}{48\left(x - i\sqrt{3}\right)} - \frac{79i\sqrt{3}}{48\left(x + i\sqrt{3}\right)}$$

For the pole at $x = i\sqrt{3}$ let b be the coefficient of $\frac{1}{\left(x - i\sqrt{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{77}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{7}{4} \end{aligned}$$

For the pole at $x = -i\sqrt{3}$ let b be the coefficient of $\frac{1}{\left(x + i\sqrt{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{77}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{7}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 234}{4\left(x^2+3\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 - 234}{4\left(x^2+3\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$i\sqrt{3}$	2	0	$\frac{11}{4}$	$-\frac{7}{4}$
$-i\sqrt{3}$	2	0	$\frac{11}{4}$	$-\frac{7}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{1}{2} - \left(-\frac{7}{2}\right) \\
&= 4
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{7}{4\left(x-i\sqrt{3}\right)} - \frac{7}{4\left(x+i\sqrt{3}\right)} + (-)(0) \\
&= -\frac{7}{4\left(x-i\sqrt{3}\right)} - \frac{7}{4\left(x+i\sqrt{3}\right)} \\
&= -\frac{7x}{2x^2+6}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(12x^2 + 6xa_3 + 2a_2\right) + 2\left(-\frac{7}{4\left(x - i\sqrt{3}\right)} - \frac{7}{4\left(x + i\sqrt{3}\right)}\right)\left(4x^3 + 3x^2 a_3 + 2xa_2 + a_1\right) + \left(\left(\frac{7}{4\left(x - i\sqrt{3}\right)^2}\right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{27}{8}, a_1 = 0, a_2 = -9, a_3 = 0\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 - 9x^2 + \frac{27}{8}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^4 - 9x^2 + \frac{27}{8}\right) e^{\int \left(-\frac{7}{4\left(x - i\sqrt{3}\right)} - \frac{7}{4\left(x + i\sqrt{3}\right)}\right) dx} \\
&= \left(x^4 - 9x^2 + \frac{27}{8}\right) \frac{1}{\left(\left(i\sqrt{3} - x\right)\left(x + i\sqrt{3}\right)\right)^{7/4}} \\
&= \frac{8x^4 - 72x^2 + 27}{8\left(-x^2 - 3\right)^{7/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-7x}{x^2+3} \, dx} \\
&= z_1 e^{\frac{7 \ln\left(x^2+3\right)}{4}} \\
&= z_1 \left(\left(x^2 + 3\right)^{7/4}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \left(\frac{1}{2} + \frac{i}{2}\right)\sqrt{2}\left(x^4 - 9x^2 + \frac{27}{8}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-7x}{x^2+3}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\frac{7\ln(x^2+3)}{2}}}{(y_1)^2}\,dx \\ &= y_1(\text{Expression too large to display}) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\left(\frac{1}{2}+\frac{i}{2}\right)\sqrt{2}\left(x^4-9x^2+\frac{27}{8}\right)\right) \\ &\quad + c_2\left(\left(\frac{1}{2}+\frac{i}{2}\right)\sqrt{2}\left(x^4-9x^2+\frac{27}{8}\right)(\text{Expression too large to display})\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 65

```
dsolve((x^2+3)*diff(diff(y(x),x),x)-7*diff(y(x),x)*x+16*y(x) = 0,y(x),singsol=all)
```

$$\begin{aligned} y &= 4c_2\left(x^4-9x^2+\frac{27}{8}\right)\ln\left(\sqrt{x^2+3}-x\right) \\ &\quad + \frac{5(10x^3-33x)\,c_2\sqrt{x^2+3}}{6} + \left(c_1+\frac{25c_2}{3}\right)\left(x^4-9x^2+\frac{27}{8}\right) \end{aligned}$$

Mathematica DSolve solution

Solving time : `0.613` (sec)
Leaf size : 69

```
DSolve[{(x^2+3)*D[y[x],{x,2}]-7*x*D[y[x],x]+16*y[x]==0,{}},y[x],x,IncludeSingularSolutions->
```

$$y(x) \to \frac{1}{24}c_2\left(3\left(8x^4-72x^2+27\right)\operatorname{arcsinh}\left(\frac{x}{\sqrt{3}}\right)+5x\sqrt{x^2+3}\left(33-10x^2\right)\right)+c_1\left(x^4-9x^2+\frac{27}{8}\right)$$

2.1.4 Problem 4

Internal problem ID [9176]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 4
Date solved : `Monday, January 27, 2025 at 05:51:45 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(x^{2}-1\right) y''+8 x y'+12 y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.183 (sec)

Writing the ode as

$$\left(x^{2}-1\right) y''+8 x y'+12 y=0 \tag{1}$$
$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^{2}-1 \\ B &= 8 x \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2 A}\, d x}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2 A B'-2 B A'+B^{2}-4 A C}{4 A^{2}} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{8}{\left(x^{2}-1\right)^{2}} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 8 \\ t &= \left(x^{2}-1\right)^{2} \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{8}{\left(x^{2}-1\right)^{2}}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.6: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{(x-1)^2} + \frac{2}{(x+1)^2} - \frac{2}{x-1} + \frac{2}{x+1}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= 0\\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{8}{(x^2-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	2	-1
-1	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right)\\ &= 1 - (1)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (-)[\sqrt{r}]_\infty\\ &= -\frac{1}{x-1} + \frac{2}{x+1} + (-)\,(0)\\ &= -\frac{1}{x-1} + \frac{2}{x+1}\\ &= \frac{x-3}{x^2-1}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(-\frac{1}{x-1} + \frac{2}{x+1}\right)(0) + \left(\left(\frac{1}{(x-1)^2} - \frac{2}{(x+1)^2}\right) + \left(-\frac{1}{x-1} + \frac{2}{x+1}\right)^2 - \left(\frac{8}{(x^2-1)^2}\right)\right) &= 0\\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= pe^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{x-1}+\frac{2}{x+1}\right)dx} \\
&= \frac{(x+1)^2}{x-1}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{8x}{x^2-1}\, dx} \\
&= z_1 e^{-2\ln(x-1)-2\ln(x+1)} \\
&= z_1 \left(\frac{1}{(x-1)^2 (x+1)^2}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{(x-1)^3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{8x}{x^2-1}\, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-4\ln(x-1)-4\ln(x+1)}}{(y_1)^2} \, dx \\
&= y_1 \left(-\frac{(x+1)(3x^2+1)(x-1)^4 \,\mathrm{e}^{-4\ln(x-1)-4\ln(x+1)}}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{(x-1)^3}\right) + c_2 \left(\frac{1}{(x-1)^3} \left(-\frac{(x+1)(3x^2+1)(x-1)^4 \,\mathrm{e}^{-4\ln(x-1)-4\ln(x+1)}}{3}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x^2-1)\left(\frac{d^2}{dx^2}y(x)\right)+8x\left(\frac{d}{dx}y(x)\right)+12y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{12y(x)}{x^2-1}-\frac{8\left(\frac{d}{dx}y(x)\right)x}{x^2-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x)+\frac{8\left(\frac{d}{dx}y(x)\right)x}{x^2-1}+\frac{12y(x)}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{8x}{x^2-1}, P_3(x)=\frac{12}{x^2-1}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=4$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1$

- Multiply by denominators

$(x^2-1)\left(\frac{d^2}{dx^2}y(x)\right)+8x\left(\frac{d}{dx}y(x)\right)+12y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(8u-8)\left(\frac{d}{du}y(u)\right)+12y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  ◦ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

  ◦ Shift index using $k->k+2-m$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

Rewrite ODE with series expansions

$-2a_0r(3+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)(k+r+4)+a_k(k+r+4)(k+r+3)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-2r(3+r)=0$

- Values of r that satisfy the indicial equation
  $r \in \{-3, 0\}$
- Each term in the series must be 0, giving the recursion relation
  $(k+r+4)\left((-2k-2r-2)\,a_{k+1}+a_k(k+r+3)\right)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{a_k(k+r+3)}{2(k+1+r)}$
- Recursion relation for $r=-3$
  $a_{k+1}=\frac{a_k k}{2(k-2)}$
- Series not valid for $r=-3$, division by 0 in the recursion relation at $k=2$
  $a_{k+1}=\frac{a_k k}{2(k-2)}$
- Recursion relation for $r=0$
  $a_{k+1}=\frac{a_k(k+3)}{2(k+1)}$
- Solution for $r=0$
  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+1}=\frac{a_k(k+3)}{2(k+1)}\right]$
- Revert the change of variables $u=x+1$
  $\left[y(x)=\sum_{k=0}^{\infty} a_k (x+1)^k, a_{k+1}=\frac{a_k(k+3)}{2(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 29

```
dsolve((x^2-1)*diff(diff(y(x),x),x)+8*diff(y(x),x)*x+12*y(x) = 0,y(x),singsol=all)
```

$$y=\frac{c_2 x^3+3c_1 x^2+3c_2 x+c_1}{(x^2-1)^3}$$

Mathematica DSolve solution

Solving time : 0.337 (sec)
Leaf size : 73

```
DSolve[{(x^2-1)*D[y[x],{x,2}]+8*x*D[y[x],x]+12*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tru
```

$$y(x) \to \frac{\exp\left(\int_1^x \frac{K[1]+3}{K[1]^2-1}dK[1]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[2]} \frac{K[1]+3}{K[1]^2-1}dK[1]\right)dK[2]+c_1\right)}{(x^2-1)^2}$$

2.1.5 Problem 5

Internal problem ID [9177]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 5
Date solved : `Monday, January 27, 2025 at 05:51:45 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$3y'' + xy' - 4y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.229 (sec)

Writing the ode as

$$3y'' + xy' - 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3 \\ B &= x \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 54}{36} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 54 \\ t &= 36 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{36} + \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.8: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{6} + \frac{9}{2x} - \frac{243}{4x^3} + \frac{6561}{4x^5} - \frac{885735}{16x^7} + \frac{33480783}{16x^9} - \frac{2711943423}{32x^{11}} + \frac{115063885233}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{6} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{36}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 54}{36} \\ &= Q + \frac{R}{36} \\ &= \left(\frac{x^2}{36} + \frac{3}{2}\right) + (0) \\ &= \frac{x^2}{36} + \frac{3}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{3}{2}\right) - (0) \\ &= \frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{6} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{3}{2}}{\frac{1}{6}} - 1\right) &= 4 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{3}{2}}{\frac{1}{6}} - 1\right) &= -5 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{36} + \frac{3}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{6}$	4	-5

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 4$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 4 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{x}{6}\right) \\ &= \frac{x}{6} \\ &= \frac{x}{6} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(12x^2 + 6xa_3 + 2a_2\right) + 2\left(\frac{x}{6}\right)\left(4x^3 + 3x^2a_3 + 2xa_2 + a_1\right) + \left(\left(\frac{1}{6}\right) + \left(\frac{x}{6}\right)^2 - \left(\frac{x^2}{36} + \frac{3}{2}\right)\right) = 0$$

$$-\frac{a_3x^3}{3} + \frac{2(18-a_2)\,x^2}{3} + (-a_1 + 6a_3)\,x - \frac{4a_0}{3} + 2a_2 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 27, a_1 = 0, a_2 = 18, a_3 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 + 18x^2 + 27$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\,dx} \\ &= \left(x^4 + 18x^2 + 27\right)e^{\int \frac{x}{6}dx} \\ &= \left(x^4 + 18x^2 + 27\right)e^{\frac{x^2}{12}} \\ &= \left(x^4 + 18x^2 + 27\right)e^{\frac{x^2}{12}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{3}\,dx} \\ &= z_1 e^{-\frac{x^2}{12}} \\ &= z_1\left(e^{-\frac{x^2}{12}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^4 + 18x^2 + 27$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{3}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{6}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{x^2}{6}}}{(x^4+18x^2+27)^2}dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 (x^4 + 18x^2 + 27) + c_2 \left(x^4 + 18x^2 + 27\left(\int \frac{\mathrm{e}^{-\frac{x^2}{6}}}{(x^4+18x^2+27)^2}dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special functio
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.040 (sec)
Leaf size : 47

```
dsolve(3*diff(diff(y(x),x),x)+diff(y(x),x)*x-4*y(x) = 0,y(x),singsol=all)
```

$$y = xc_1\left(x^2+15\right)\sqrt{6}\,\mathrm{e}^{-\frac{x^2}{6}}+\left(x^4+18x^2+27\right)\left(\sqrt{\pi}\,\mathrm{erf}\left(\frac{\sqrt{6}\,x}{6}\right)c_1+c_2\right)$$

Mathematica DSolve solution

Solving time : 0.023 (sec)
Leaf size : 43

```
DSolve[{3*D[y[x],{x,2}]+x*D[y[x],x]-4*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to c_1e^{-\frac{x^2}{6}}\,\mathrm{HermiteH}\left(-5,\frac{x}{\sqrt{6}}\right)+\frac{1}{27}c_2\left(x^4+18x^2+27\right)$$

2.1.6 Problem 6

Internal problem ID [9178]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 6
Date solved : `Monday, January 27, 2025 at 05:51:46 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$5y'' - 2xy' + 10y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.264 (sec)

Writing the ode as

$$5y'' - 2xy' + 10y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 5 \\ B &= -2x \\ C &= 10 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 55}{25} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 55 \\ t &= 25 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{25} - \frac{11}{5}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.9: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{5} - \frac{11}{2x} - \frac{605}{8x^3} - \frac{33275}{16x^5} - \frac{9150625}{128x^7} - \frac{704598125}{256x^9} - \frac{116258690625}{1024x^{11}} - \frac{10048072546875}{2048x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{5}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{5} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{25}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 55}{25} \\ &= Q + \frac{R}{25} \\ &= \left(\frac{x^2}{25} - \frac{11}{5}\right) + (0) \\ &= \frac{x^2}{25} - \frac{11}{5} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{11}{5}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{11}{5}\right) - (0) \\ &= -\frac{11}{5} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{5} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{11}{5}}{\frac{1}{5}} - 1\right) &&= -6 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{11}{5}}{\frac{1}{5}} - 1\right) &&= 5 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{25} - \frac{11}{5}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{5}$	-6	5

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 5$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 5 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{5}\right) \\ &= -\frac{x}{5} \\ &= -\frac{x}{5} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 5$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(20x^3 + 12x^2a_4 + 6xa_3 + 2a_2\right) + 2\left(-\frac{x}{5}\right)\left(5x^4 + 4x^3a_4 + 3x^2a_3 + 2xa_2 + a_1\right) + \left(\left(-\frac{1}{5}\right) + \left(-\frac{x}{5}\right)^2 - \left(\frac{x^2}{25} -\right.\right.$$

$$\frac{2a_4x^4}{5} + \frac{4(25+a_3)\,x^3}{5} + \frac{6(a_2+10a_4)\,x^2}{5} + \frac{2(4a_1+15a_3)\,x}{5} + 2a_0 +$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = 0, a_1 = \frac{375}{4}, a_2 = 0, a_3 = -25, a_4 = 0\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^5 - 25x^3 + \frac{375}{4}x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= \left(x^5 - 25x^3 + \frac{375}{4}x\right)e^{\int -\frac{x}{5}dx} \\ &= \left(x^5 - 25x^3 + \frac{375}{4}x\right)e^{-\frac{x^2}{10}} \\ &= \frac{\left(4x^5 - 100x^3 + 375x\right)e^{-\frac{x^2}{10}}}{4} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{5}\, dx} \\ &= z_1 e^{\frac{x^2}{10}} \\ &= z_1\left(e^{\frac{x^2}{10}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^5 - 25x^3 + \frac{375}{4}x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{5}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\frac{x^2}{5}}}{(y_1)^2}\,dx \\ &= y_1 \left(\int \frac{e^{\frac{x^2}{5}}}{\left(x^5 - 25x^3 + \frac{375}{4}x\right)^2}dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(x^5 - 25x^3 + \frac{375}{4}x\right) + c_2\left(x^5 - 25x^3 + \frac{375}{4}x\left(\int \frac{e^{\frac{x^2}{5}}}{\left(x^5 - 25x^3 + \frac{375}{4}x\right)^2}dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$5\frac{d^2}{dx^2}y(x) - 2x\left(\frac{d}{dx}y(x)\right) + 10y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{2x\left(\frac{d}{dx}y(x)\right)}{5} - 2y(x)$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) - \frac{2x\left(\frac{d}{dx}y(x)\right)}{5} + 2y(x) = 0$

- Multiply by denominators

  $5\frac{d^2}{dx^2}y(x) - 2x\left(\frac{d}{dx}y(x)\right) + 10y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite DE with series expansions

  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k->k+2$

$\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$

Rewrite DE with series expansions

$\sum_{k=0}^{\infty} (5a_{k+2}(k+2)(k+1) - 2a_k(k-5))x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $5(k^2+3k+2)a_{k+2} - 2a_k(k-5) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2a_k(k-5)}{5(k^2+3k+2)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solut
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returnin
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.022 (sec)
Leaf size : 31

```
dsolve(5*diff(diff(y(x),x),x)-2*diff(y(x),x)*x+10*y(x) = 0,y(x),singsol=all)
```

$$y = c_2 \operatorname{hypergeom}\left(\left[-\frac{5}{2}\right], \left[\frac{1}{2}\right], \frac{x^2}{5}\right) + \frac{4x\left(x^4 - 25x^2 + \frac{375}{4}\right)c_1}{375}$$

Mathematica DSolve solution

Solving time : 0.117 (sec)
Leaf size : 138

```
DSolve[{5*D[y[x],{x,2}]-2*x*D[y[x],x]+10*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{1}{200}\sqrt{\frac{\pi}{5}}c_2\sqrt{x^2}\left(4x^4-100x^2+375\right)\operatorname{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{5}}\right)+\frac{32c_1x^5}{25\sqrt{5}} -\frac{32c_1x^3}{\sqrt{5}}-\frac{9}{20}c_2e^{\frac{x^2}{5}}x^2+c_2e^{\frac{x^2}{5}}+\frac{1}{50}c_2e^{\frac{x^2}{5}}x^4+24\sqrt{5}c_1x$$

2.1.7 Problem 7

Internal problem ID [9179]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 7
Date solved : `Monday, January 27, 2025 at 05:51:47 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - x^2 y' - 3xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.308 (sec)

Writing the ode as

$$y'' - x^2 y' - 3xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x^2 \\ C &= -3x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x(x^3+8)}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x(x^3+8) \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x(x^3+8)}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.11: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 4 \\ &= -4 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{2} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x^2}{2} + \frac{2}{x} - \frac{4}{x^4} + \frac{16}{x^7} - \frac{80}{x^{10}} + \frac{448}{x^{13}} - \frac{2688}{x^{16}} + \frac{16896}{x^{19}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 2$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i x^i \\ &= \frac{x^2}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^1 = x$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^4}{4}$$

This shows that the coefficient of x in the above is 0. Now we need to find the coefficient of x in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x(x^3+8)}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^4 + 2x\right) + (0) \\ &= \frac{1}{4}x^4 + 2x \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is 2. Now b can be found.

$$\begin{aligned} b &= (2) - (0) \\ &= 2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x^2}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{2}{\frac{1}{2}} - 2\right) &&= 1 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{2}{\frac{1}{2}} - 2\right) &&= -3 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x(x^3+8)}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-4	$\frac{x^2}{2}$	1	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 1$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{x^2}{2}\right) \\ &= \frac{x^2}{2} \\ &= \frac{x^2}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{x^2}{2}\right)(1) + \left((x) + \left(\frac{x^2}{2}\right)^2 - \left(\frac{x(x^3+8)}{4}\right)\right) = 0$$
$$-xa_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= (x)\, e^{\int \frac{x^2}{2} dx} \\ &= (x)\, e^{\frac{x^3}{6}} \\ &= x\, e^{\frac{x^3}{6}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x^2}{1}\, dx} \\ &= z_1 e^{\frac{x^3}{6}} \\ &= z_1\left(e^{\frac{x^3}{6}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\frac{x^3}{3}} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{x^3}{3}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{x^3}{3}}}{x^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\frac{x^3}{3}} x\right) + c_2 \left(\mathrm{e}^{\frac{x^3}{3}} x \left(\int \frac{\mathrm{e}^{-\frac{x^3}{3}}}{x^2} dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2} y(x) - x^2 \left(\frac{d}{dx} y(x)\right) - 3xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2} y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

  - Convert $x \cdot y(x)$ to series expansion

    $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k- >k-1$

    $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert $x^2 \cdot \left(\frac{d}{dx} y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^{k+1}$

  - Shift index using $k- >k-1$

    $x^2 \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=1}^{\infty} a_{k-1} (k-1)\, x^k$

  - Convert $\frac{d^2}{dx^2} y(x)$ to series expansion

$\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- Shift index using $k->k+2$

  $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_{k-1}(k+2)\right) x^k\right) = 0$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+2)\,(k a_{k+2} - a_{k-1} + a_{k+2}) = 0$

- Shift index using $k->k+1$

  $(k+3)\,((k+1)\,a_{k+3} - a_k + a_{k+3}) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{a_k}{k+2}, 2a_2 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 58

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x^2-3*x*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{9\,\mathrm{e}^{\frac{x^3}{6}}\,\mathrm{WhittakerM}\left(\frac{1}{3}, \frac{5}{6}, \frac{x^3}{3}\right) c_2 x^3 + 9 c_1 x^2 \mathrm{e}^{\frac{x^3}{3}} + 5\, 3^{2/3} c_2 (x^3)^{1/3}\,(x^3+2)}{9x}$$

Mathematica DSolve solution

Solving time : 0.14 (sec)
Leaf size : 51

```
DSolve[{D[y[x],{x,2}]-x^2*D[y[x],x]-3*x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{9} e^{\frac{x^3}{3}} \left(9 c_1 x - 3^{2/3} c_2 \sqrt[3]{x^3}\,\Gamma\left(-\frac{1}{3}, \frac{x^3}{3}\right)\right)$$

2.1.8 Problem 8

Internal problem ID [9180]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 8
Date solved : `Monday, January 27, 2025 at 05:51:48 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+1\right) y''+2xy'-2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.280 (sec)

Writing the ode as

$$\left(x^2+1\right) y''+2xy'-2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= 2x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{2x^2+3}{\left(x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2x^2+3 \\ t &= \left(x^2+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{2x^2+3}{\left(x^2+1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.13: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(x-i)^2} - \frac{1}{4(x+i)^2} - \frac{5i}{4(x-i)} + \frac{5i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^2+3}{(x^2+1)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2x^2+3}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-i$	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x-2i} + \frac{1}{2x+2i} + (0) \\
&= \frac{1}{2x-2i} + \frac{1}{2x+2i} \\
&= \frac{x}{x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x-2i} + \frac{1}{2x+2i}\right)(1) + \left(\left(-\frac{1}{2(x-i)^2} - \frac{1}{2(x+i)^2}\right) + \left(\frac{1}{2x-2i} + \frac{1}{2x+2i}\right)^2 - \left(\frac{2x^2+3}{(x^2+1)^2}\right.\right.$$
$$-\frac{2(x^2+1)\,a_0}{(-x+i)^2\,(x+i)^2}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (x)\, e^{\int \left(\frac{1}{2x-2i} + \frac{1}{2x+2i}\right) dx} \\
&= (x)\sqrt{(-x+i)(x+i)} \\
&= x\sqrt{-x^2-1}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x}{x^2+1}\, dx} \\
&= z_1 e^{-\frac{\ln(x^2+1)}{2}} \\
&= z_1\left(\frac{1}{\sqrt{x^2+1}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = ix$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{x^2+1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\ln(x^2+1)}}{(y_1)^2}\, dx \\
&= y_1\left(\arctan(x) + \frac{1}{x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(ix) + c_2\left(ix\left(\arctan(x) + \frac{1}{x}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 14

```
dsolve((x^2+1)*diff(diff(y(x),x),x)+2*diff(y(x),x)*x-2*y(x) = 0,y(x),singsol=all)
```

$$y = c_1 x + \arctan(x) x c_2 + c_2$$

Mathematica DSolve solution

Solving time : 0.021 (sec)
Leaf size : 48

```
DSolve[{(1+x^2)*D[y[x],{x,2}]+2*x*D[y[x],x]-2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tru
```

$$y(x) \to \frac{1}{2} i(2c_1 x - c_2 x \log(1 - ix) + c_2 x \log(1 + ix) + 2ic_2)$$

2.1.9 Problem 9

Internal problem ID [9181]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 9
Date solved : `Monday, January 27, 2025 at 05:51:48 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + xy' - 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.227 (sec)

Writing the ode as

$$y'' + xy' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 10}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 10 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} + \frac{5}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.14: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + \frac{5}{2x} - \frac{25}{4x^3} + \frac{125}{4x^5} - \frac{3125}{16x^7} + \frac{21875}{16x^9} - \frac{328125}{32x^{11}} + \frac{2578125}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{2} \qquad (10)
\end{aligned}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}
$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 10}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} + \frac{5}{2}\right) + (0) \\
&= \frac{x^2}{4} + \frac{5}{2}
\end{aligned}
$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{5}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(\frac{5}{2}\right) - (0) \\
&= \frac{5}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{5}{2}}{\frac{1}{2}} - 1\right) &= 2 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{5}{2}}{\frac{1}{2}} - 1\right) &= -3
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2}{4} + \frac{5}{2}
$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}$	2	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{x}{2}\right) \\ &= \frac{x}{2} \\ &= \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (2) + 2\left(\frac{x}{2}\right)(2x + a_1) + \left(\left(\frac{1}{2}\right) + \left(\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} + \frac{5}{2}\right)\right) &= 0 \\ -a_1 x - 2a_0 + 2 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (x^2 + 1)\, e^{\int \frac{x}{2} dx} \\ &= (x^2 + 1)\, e^{\frac{x^2}{4}} \\ &= (x^2 + 1)\, e^{\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{1}\, dx} \\ &= z_1 e^{-\frac{x^2}{4}} \\ &= z_1\left(e^{-\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^2 + 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2}\,dx \\ &= y_1 \left(\int \frac{e^{-\frac{x^2}{2}}}{(x^2+1)^2} dx \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (x^2 + 1) + c_2 \left(x^2 + 1 \left(\int \frac{e^{-\frac{x^2}{2}}}{(x^2+1)^2} dx \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + x\left(\frac{d}{dx}y(x)\right) - 2y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite DE with series expansions
  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k+2$

    $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k-2)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} + a_k(k-2) = 0$

- Recursion relation; series terminates at $k = 2$

$a_{k+2} = -\frac{a_k(k-2)}{k^2+3k+2}$

- Apply recursion relation for $k = 0$

  $a_2 = a_0$

- Terminating series solution of the ODE. Use reduction of order to find the second linearly indepe

  $y(x) = A_2x^2 + A_1x + a_0$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.043 (sec)
Leaf size : 37

```
dsolve(diff(diff(y(x),x),x)+diff(y(x),x)*x-2*y(x) = 0,y(x),singsol=all)
```

$$y = \sqrt{2}\,\mathrm{e}^{-\frac{x^2}{2}} c_1 x + \left(x^2+1\right)\left(\sqrt{\pi}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right) c_1 + c_2\right)$$

Mathematica DSolve solution

Solving time : 0.021 (sec)
Leaf size : 35

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]-2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{-\frac{x^2}{2}} \,\mathrm{HermiteH}\left(-3, \frac{x}{\sqrt{2}}\right) + c_2\left(x^2+1\right)$$

2.1.10 Problem 10

Internal problem ID [9182]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 10
Date solved : `Monday, January 27, 2025 at 05:51:49 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^{2}-6x+10\right)y''-4(x-3)\,y'+6y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.305 (sec)

Writing the ode as

$$\left(x^{2}-6x+10\right)y''+(-4x+12)\,y'+6y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^{2}-6x+10\\ B &= -4x+12\\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t}\\ &= \frac{2AB'-2BA'+B^{2}-4AC}{4A^{2}} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-8}{\left(x^{2}-6x+10\right)^{2}} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -8\\ t &= \left(x^{2}-6x+10\right)^{2} \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{8}{\left(x^{2}-6x+10\right)^{2}}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.16: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 6x + 10)^2$. There is a pole at $x = 3 + i$ of order 2. There is a pole at $x = 3 - i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{(x-3-i)^2} + \frac{2}{(x-3+i)^2} + \frac{2i}{x-3-i} - \frac{2i}{x-3+i}$$

For the pole at $x = 3 + i$ let b be the coefficient of $\frac{1}{(x-3-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

For the pole at $x = 3 - i$ let b be the coefficient of $\frac{1}{(x-3+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= 0\\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{8}{(x^2 - 6x + 10)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$3+i$	2	0	2	-1
$3-i$	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right)\\ &= 1 - (1)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (-)[\sqrt{r}]_\infty\\ &= -\frac{1}{x-3-i} + \frac{2}{x-3+i} + (-)(0)\\ &= -\frac{1}{x-3-i} + \frac{2}{x-3+i}\\ &= \frac{x-3-3i}{x^2-6x+10}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x-3-i} + \frac{2}{x-3+i}\right)(0) + \left(\left(\frac{1}{(x-3-i)^2} - \frac{2}{(x-3+i)^2}\right) + \left(-\frac{1}{x-3-i} + \frac{2}{x-3+i}\right)^2\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{x-3-i}+\frac{2}{x-3+i}\right) dx} \\
&= \frac{\left(x^2-6x+10\right)^2}{\left(ix-3i+1\right)^3}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x+12}{x^2-6x+10}\, dx} \\
&= z_1 e^{\ln\left(x^2-6x+10\right)} \\
&= z_1 \left(x^2-6x+10\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(x^2-6x+10\right)^3}{\left(ix-3i+1\right)^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x+12}{x^2-6x+10}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{2\ln\left(x^2-6x+10\right)}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\frac{x^2-6x+\frac{26}{3}}{\left(x-3+i\right)^3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\left(x^2-6x+10\right)^3}{\left(ix-3i+1\right)^3}\right) + c_2 \left(\frac{\left(x^2-6x+10\right)^3}{\left(ix-3i+1\right)^3}\left(\frac{x^2-6x+\frac{26}{3}}{\left(x-3+i\right)^3}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 31

```
dsolve((x^2-6*x+10)*diff(diff(y(x),x),x)-4*(x-3)*diff(y(x),x)+6*y(x) = 0,y(x),singsol=
```

$$y = c_1x^3 + c_2x^2 + 6(-5c_1 - c_2)\,x + 60c_1 + \frac{26c_2}{3}$$

Mathematica DSolve solution

Solving time : 0.314 (sec)
Leaf size : 84

```
DSolve[{(x^2-6*x+10)*D[y[x],{x,2}]-4*(x-3)*D[y[x],x]+6*y[x]==0,{}},y[x],x,IncludeSingularSol
```

$$y(x) \to (x^2 - 6x + 10) \exp\left(\int_1^x \frac{K[1] - (3-3i)}{(K[1]-6)K[1]+10} dK[1]\right) \left(c_2 \int_1^x \exp\left(-2\int_1^{K[2]} \frac{K[1]-(3-3i)}{(K[1]-6)K[1]+10} dK[1]\right) dK[2] + c_1\right)$$

2.1.11 Problem 11

Internal problem ID [9183]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 11
Date solved : `Monday, January 27, 2025 at 05:51:50 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+6x\right)y''+(3x+9)\,y'-3y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.206 (sec)

Writing the ode as

$$\left(x^2+6x\right)y''+(3x+9)\,y'-3y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+6x \\ B &= 3x+9 \\ C &= -3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{15x^2+90x-27}{4\left(x^2+6x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15x^2+90x-27 \\ t &= 4\left(x^2+6x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{15x^2+90x-27}{4\left(x^2+6x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.17: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + 6x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -6$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2} + \frac{11}{16x} - \frac{11}{16\,(x+6)} - \frac{3}{16\,(x+6)^2}$$

For the pole at $x = -6$ let b be the coefficient of $\frac{1}{(x+6)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15x^2 + 90x - 27}{4\left(x^2 + 6x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15x^2 + 90x - 27}{4\left(x^2 + 6x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-6	2	0	$\frac{3}{4}$	$\frac{1}{4}$
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\ &= \frac{5}{2} - \left(\frac{3}{2}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{3}{4\left(x+6\right)} + \frac{3}{4x} + (0) \\ &= \frac{3}{4\left(x+6\right)} + \frac{3}{4x} \\ &= \frac{\frac{3x}{2} + \frac{9}{2}}{x\left(x+6\right)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{4(x+6)} + \frac{3}{4x}\right)(1) + \left(\left(-\frac{3}{4(x+6)^2} - \frac{3}{4x^2}\right) + \left(\frac{3}{4(x+6)} + \frac{3}{4x}\right)^2 - \left(\frac{15x^2 + 90x - 27}{4(x^2+6x)^2}\right)\right) = \frac{9 - 3a_0}{x(x+6)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 3\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 3$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+3) \, e^{\int \left(\frac{3}{4(x+6)} + \frac{3}{4x}\right) dx} \\
&= (x+3) \, (x(x+6))^{3/4} \\
&= (x+3) \, (x(x+6))^{3/4}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{3x+9}{x^2+6x} \, dx} \\
&= z_1 e^{-\frac{3\ln(x(x+6))}{4}} \\
&= z_1 \left(\frac{1}{(x(x+6))^{3/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x + 3$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x+9}{x^2+6x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{3\ln(x(x+6))}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(-\frac{(x+6) \, x(2x^2 + 12x + 9)}{81 (x+3) (x(x+6))^{3/2}}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x+3) + c_2\left(x + 3\left(-\frac{(x+6)\,x(2x^2+12x+9)}{81\,(x+3)\,(x\,(x+6))^{3/2}}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x^2+6x)\left(\frac{d^2}{dx^2}y(x)\right) + (3x+9)\left(\frac{d}{dx}y(x)\right) - 3y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{3y(x)}{x(6+x)} - \frac{3(x+3)\left(\frac{d}{dx}y(x)\right)}{x(6+x)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{3(x+3)\left(\frac{d}{dx}y(x)\right)}{x(6+x)} - \frac{3y(x)}{x(6+x)} = 0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{3(x+3)}{x(6+x)}, P_3(x) = -\frac{3}{x(6+x)}\right]$
- ○ $(6+x)\cdot P_2(x)$ is analytic at $x=-6$

  $\left((6+x)\cdot P_2(x)\right)\Big|_{x=-6} = \frac{3}{2}$
- ○ $(6+x)^2\cdot P_3(x)$ is analytic at $x=-6$

  $\left((6+x)^2\cdot P_3(x)\right)\Big|_{x=-6} = 0$
- ○ $x=-6$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -6$

- Multiply by denominators

  $x(6+x)\left(\frac{d^2}{dx^2}y(x)\right) + (3x+9)\left(\frac{d}{dx}y(x)\right) - 3y(x) = 0$
- Change variables using $x = u-6$ so that the regular singular point is at $u=0$

  $(u^2-6u)\left(\frac{d^2}{du^2}y(u)\right) + (3u-9)\left(\frac{d}{du}y(u)\right) - 3y(u) = 0$
- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
- ○ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$
- ○ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-3a_0r(1+2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} (-3a_{k+1}(k+1+r)(2k+3+2r) + a_k(k+r+3)(k+r-1))\, u^{k+r}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-3r(1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, -\frac{1}{2}\}$
- Each term in the series must be 0, giving the recursion relation

  $-6(k+1+r)\left(k+\frac{3}{2}+r\right)a_{k+1} + a_k(k+r+3)(k+r-1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r+3)(k+r-1)}{3(k+1+r)(2k+3+2r)}$
- Recursion relation for $r=0$; series terminates at $k=1$

  $a_{k+1} = \frac{a_k(k+3)(k-1)}{3(k+1)(2k+3)}$
- Apply recursion relation for $k=0$

  $a_1 = -\frac{a_0}{3}$
- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1-\frac{u}{3}\right)$
- Revert the change of variables $u = 6+x$

  $\left[y(x) = a_0\left(-\frac{x}{3}-1\right)\right]$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+1} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{3\left(k+\frac{1}{2}\right)(2k+2)}$
- Solution for $r = -\frac{1}{2}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{1}{2}}, a_{k+1} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{3\left(k+\frac{1}{2}\right)(2k+2)}\right]$
- Revert the change of variables $u = 6+x$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k(6+x)^{k-\frac{1}{2}}, a_{k+1} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{3\left(k+\frac{1}{2}\right)(2k+2)}\right]$
- Combine solutions and rename parameters

  $\left[y(x) = a_0\left(-\frac{x}{3}-1\right) + \left(\sum_{k=0}^{\infty} b_k(6+x)^{k-\frac{1}{2}}\right), b_{k+1} = \frac{b_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{3\left(k+\frac{1}{2}\right)(2k+2)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
```

```
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.026 (sec)
Leaf size : 30

```
dsolve((x^2+6*x)*diff(diff(y(x),x),x)+(3*x+9)*diff(y(x),x)-3*y(x) = 0,y(x),singsol=all)
```

$$y = c_1(x+3) + \frac{c_2(2x^2+12x+9)}{\sqrt{x}\,\sqrt{6+x}}$$

Mathematica DSolve solution

Solving time : 0.675 (sec)
Leaf size : 103

```
DSolve[{(x^2+6*x)*D[y[x],{x,2}]+(3*x+9)*D[y[x],x]-3*y[x]==0,{}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{\left(9\sqrt{\pi}c_2\sqrt[4]{-x(x+6)}Q_{\frac{3}{2}}^{\frac{1}{2}}\left(\frac{x}{3}+1\right)+\sqrt{6}c_1(2x^2+12x+9)\right)\exp\left(\int_1^x -\frac{K[1]+3}{2K[1](K[1]+6)}dK[1]\right)}{9\sqrt{\pi}\sqrt[4]{-x(x+6)}}$$

2.1.12 Problem 12

Internal problem ID [9184]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 12
Date solved : `Monday, January 27, 2025 at 05:51:50 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$ty'' + (t^2 - 1)\,y' + t^2 y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.304 (sec)

Writing the ode as

$$ty'' + (t^2 - 1)\,y' + t^2 y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= t^2 - 1 \\ C &= t^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^4 - 4t^3 + 3}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^4 - 4t^3 + 3 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^4 - 4t^3 + 3}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.19: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{t^2}{4} - t + \frac{3}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{1} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{t}{2} - 1 - \frac{1}{t} - \frac{2}{t^2} - \frac{17}{4t^3} - \frac{25}{2t^4} - \frac{75}{2t^5} - \frac{117}{t^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i t^i \\ &= -1 + \frac{t}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1 - t + \frac{1}{4}t^2$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^4 - 4t^3 + 3}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}t^2 - t\right) + \left(\frac{3}{4t^2}\right) \\ &= \frac{t^2}{4} - t + \frac{3}{4t^2} \end{aligned}$$

We see that the coefficient of the term t in the quotient is 0. Now b can be found.

$$\begin{aligned} b &= (0) - (1) \\ &= -1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= -1 + \frac{t}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 1\right) &&= -\frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 1\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^4 - 4t^3 + 3}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$-1+\frac{t}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2t} + (-)\left(-1+\frac{t}{2}\right) \\ &= -\frac{1}{2t} + 1 - \frac{t}{2} \\ &= -\frac{(t-1)^2}{2t} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 1$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2t} + 1 - \frac{t}{2}\right)(1) + \left(\left(\frac{1}{2t^2} - \frac{1}{2}\right) + \left(-\frac{1}{2t} + 1 - \frac{t}{2}\right)^2 - \left(\frac{t^4 - 4t^3 + 3}{4t^2}\right) \right) = 0$$

$$\frac{(a_0+1)(t-1)}{t} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = t - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= pe^{\int \omega\, dt} \\
&= (t-1)\, e^{\int \left(-\frac{1}{2t}+1-\frac{t}{2}\right)dt} \\
&= (t-1)\, e^{-\frac{t^2}{4}+t-\frac{\ln(t)}{2}} \\
&= \frac{(t-1)\, e^{-\frac{t(t-4)}{4}}}{\sqrt{t}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{t^2-1}{t}\, dt} \\
&= z_1 e^{-\frac{t^2}{4}+\frac{\ln(t)}{2}} \\
&= z_1 \left(\sqrt{t}\, e^{-\frac{t^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (t-1)\, e^{-\frac{t(t-2)}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{t^2-1}{t}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{-\frac{t^2}{2}+\ln(t)}}{(y_1)^2}\, dt \\
&= y_1 \left(\int \frac{e^{-\frac{t^2}{2}+\ln(t)} e^{t(t-2)}}{(t-1)^2} dt \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left((t-1)\, e^{-\frac{t(t-2)}{2}} \right) + c_2 \left((t-1)\, e^{-\frac{t(t-2)}{2}} \left(\int \frac{e^{-\frac{t^2}{2}+\ln(t)} e^{t(t-2)}}{(t-1)^2} dt \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$t\left(\frac{d^2}{dt^2}y(t)\right)+(t^2-1)\left(\frac{d}{dt}y(t)\right)+t^2y(t)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dt^2}y(t)$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t)=-ty(t)-\frac{(t^2-1)\left(\frac{d}{dt}y(t)\right)}{t}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d^2}{dt^2}y(t)+\frac{(t^2-1)\left(\frac{d}{dt}y(t)\right)}{t}+ty(t)=0$

☐ Check to see if $t_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(t)=\frac{t^2-1}{t}, P_3(t)=t\right]$

  - $t\cdot P_2(t)$ is analytic at $t=0$

    $\left.(t\cdot P_2(t))\right|_{t=0}=-1$

  - $t^2\cdot P_3(t)$ is analytic at $t=0$

    $\left.(t^2\cdot P_3(t))\right|_{t=0}=0$

  - $t=0$is a regular singular point

  Check to see if $t_0=0$ is a regular singular point

  $t_0=0$

- Multiply by denominators

  $t\left(\frac{d^2}{dt^2}y(t)\right)+(t^2-1)\left(\frac{d}{dt}y(t)\right)+t^2y(t)=0$

- Assume series solution for $y(t)$

  $y(t)=\sum_{k=0}^{\infty}a_k t^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $t^2\cdot y(t)$ to series expansion

    $t^2\cdot y(t)=\sum_{k=0}^{\infty}a_k t^{k+r+2}$

  - Shift index using $k->k-2$

    $t^2\cdot y(t)=\sum_{k=2}^{\infty}a_{k-2}t^{k+r}$

  - Convert $t^m\cdot\left(\frac{d}{dt}y(t)\right)$ to series expansion for $m=0..2$

    $t^m\cdot\left(\frac{d}{dt}y(t)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,t^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $t^m\cdot\left(\frac{d}{dt}y(t)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,t^{k+r}$

  - Convert $t\cdot\left(\frac{d^2}{dt^2}y(t)\right)$ to series expansion

    $t\cdot\left(\frac{d^2}{dt^2}y(t)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,t^{k+r-1}$

  - Shift index using $k->k+1$

    $t\cdot\left(\frac{d^2}{dt^2}y(t)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,t^{k+r}$

  Rewrite ODE with series expansions

$$a_0 r(-2+r)\, t^{-1+r} + a_1(1+r)\,(-1+r)\, t^r + (a_2(2+r)\, r + a_0 r)\, t^{1+r} + \left(\sum_{k=2}^{\infty} (a_{k+1}(k+1+r)\,(k+$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 2\}$
- The coefficients of each power of t must be 0
  $[a_1(1+r)\,(-1+r) = 0, a_2(2+r)\, r + a_0 r = 0]$
- Solve for the dependent coefficient(s)
  $\left\{a_1 = 0, a_2 = -\frac{a_0}{2+r}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)\,(k+r-1) + a_{k-1}(k+r-1) + a_{k-2} = 0$
- Shift index using $k- >k+2$
  $a_{k+3}(k+3+r)\,(k+1+r) + a_{k+1}(k+1+r) + a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+3} = -\frac{k a_{k+1} + r a_{k+1} + a_k + a_{k+1}}{(k+3+r)(k+1+r)}$
- Recursion relation for $r = 0$
  $a_{k+3} = -\frac{k a_{k+1} + a_k + a_{k+1}}{(k+3)(k+1)}$
- Solution for $r = 0$
  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^k, a_{k+3} = -\frac{k a_{k+1} + a_k + a_{k+1}}{(k+3)(k+1)}, a_1 = 0, a_2 = -\frac{a_0}{2}\right]$
- Recursion relation for $r = 2$
  $a_{k+3} = -\frac{k a_{k+1} + a_k + 3a_{k+1}}{(k+5)(k+3)}$
- Solution for $r = 2$
  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k+2}, a_{k+3} = -\frac{k a_{k+1} + a_k + 3a_{k+1}}{(k+5)(k+3)}, a_1 = 0, a_2 = -\frac{a_0}{4}\right]$
- Combine solutions and rename parameters
  $\left[y(t) = \left(\sum_{k=0}^{\infty} a_k t^k\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+2}\right), a_{k+3} = -\frac{k a_{k+1} + a_k + a_{k+1}}{(k+3)(k+1)}, a_1 = 0, a_2 = -\frac{a_0}{2}, b_{k+3} = -\frac{k b_{k+1} + b_k +}{(5+k)(k}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
```

```
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
               <- hyper3 successful: indirect Equivalence to 0F1 under \`\`^ @ Moebius\`
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special fu
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.060` (sec)
Leaf size : 82

```
dsolve(t*diff(diff(y(t),t),t)+(t^2-1)*diff(y(t),t)+t^2*y(t) = 0,y(t),singsol=all)
```

$$y = \frac{e^{-\frac{(t-2)t}{2}} \sqrt{2} \left(c_2 \sqrt{\pi} (t-2)(t-1) \operatorname{erf}\left(\frac{\sqrt{2}\sqrt{-(t-2)^2}}{2}\right) - \sqrt{-(t-2)^2} \sqrt{2} \left(c_2 e^{\frac{(t-2)^2}{2}} - c_1 t + c_1\right)\right)}{2\sqrt{-(t-2)^2}}$$

Mathematica DSolve solution

Solving time : `0.557` (sec)
Leaf size : 54

```
DSolve[{t*D[y[t],{t,2}]+(t^2-1)*D[y[t],t]+t^2*y[t]==0,{}},y[t],t,IncludeSingularSolutions->True
```

$$y(t) \to e^{t-\frac{t^2}{2}}(t-1)\left(c_2 \int_1^t \frac{e^{\frac{1}{2}(K[1]-4)K[1]}K[1]}{(K[1]-1)^2} dK[1] + c_1\right)$$

2.1.13 Problem 13

Internal problem ID [9185]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 13
Date solved : `Monday, January 27, 2025 at 05:51:51 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$t^2y'' - t(t+2)\,y' + (t+2)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.071 (sec)

Writing the ode as

$$t^2y'' + \left(-t^2-2t\right)y' + (t+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= -t^2-2t \\ C &= t+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.21: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\frac{t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-t^2-2t}{t^2}\, dt} \\ &= z_1 e^{\frac{t}{2}+\ln(t)} \\ &= z_1 \left(t\, \mathrm{e}^{\frac{t}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-t^2-2t}{t^2} dt}}{(y_1)^2} dt \\ &= y_1 \int \frac{e^{t+2\ln(t)}}{(y_1)^2} dt \\ &= y_1 \left(\frac{e^{t+2\ln(t)}}{t^2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(t) + c_2 \left(t \left(\frac{e^{t+2\ln(t)}}{t^2} \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$t^2 \left(\frac{d^2}{dt^2} y(t) \right) - t(t+2) \left(\frac{d}{dt} y(t) \right) + (t+2)\, y(t) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dt^2} y(t)$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2} y(t) = -\frac{(t+2)y(t)}{t^2} + \frac{(t+2)\left(\frac{d}{dt} y(t)\right)}{t}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dt^2} y(t) - \frac{(t+2)\left(\frac{d}{dt} y(t)\right)}{t} + \frac{(t+2)y(t)}{t^2} = 0$

☐ Check to see if $t_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(t) = -\frac{t+2}{t}, P_3(t) = \frac{t+2}{t^2} \right]$

- ○ $t \cdot P_2(t)$ is analytic at $t = 0$

  $\left(t \cdot P_2(t)\right)\Big|_{t=0} = -2$

- ○ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $\left(t^2 \cdot P_3(t)\right)\Big|_{t=0} = 2$

- ○ $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $t^2 \left(\frac{d^2}{dt^2} y(t) \right) - t(t+2) \left(\frac{d}{dt} y(t) \right) + (t+2)\, y(t) = 0$

- Assume series solution for $y(t)$

  $y(t) = \sum_{k=0}^{\infty} a_k t^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $t^m \cdot y(t)$ to series expansion for $m = 0..1$

  $t^m \cdot y(t) = \sum_{k=0}^{\infty} a_k t^{k+r+m}$

- Shift index using $k->k-m$

  $t^m \cdot y(t) = \sum_{k=m}^{\infty} a_{k-m} t^{k+r}$

- Convert $t^m \cdot \left(\frac{d}{dt}y(t)\right)$ to series expansion for $m = 1..2$

  $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$

- Convert $t^2 \cdot \left(\frac{d^2}{dt^2}y(t)\right)$ to series expansion

  $t^2 \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)(-2+r)\, t^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)(k+r-2) - a_{k-1}(k+r-2)\right) t^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)\left(a_k(k+r-1) - a_{k-1}\right) = 0$

- Shift index using $k->k+1$

  $(k+r-1)\left(a_{k+1}(k+r) - a_k\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+r}$

- Recursion relation for $r = 1$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 1$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = \frac{a_k}{k+1}\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+2}$

- Solution for $r = 2$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k+2}, a_{k+1} = \frac{a_k}{k+2}\right]$

- Combine solutions and rename parameters

  $\left[y(t) = \left(\sum_{k=0}^{\infty} a_k t^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
```

```
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 12

```
dsolve(t^2*diff(diff(y(t),t),t)-t*(t+2)*diff(y(t),t)+(t+2)*y(t) = 0,y(t),singsol=all)
```

$$y = t(c_1 + c_2 e^t)$$

Mathematica DSolve solution

Solving time : 0.041 (sec)
Leaf size : 17

```
DSolve[{t^2*D[y[t],{t,2}]-t*(t+2)*D[y[t],t]+(t+2)*y[t] == 0,{}},y[t],t,IncludeSingularSoluti
```

$$y(t) \to et(c_2 e^t + c_1)$$

2.1.14 Problem 14

Internal problem ID [9186]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 14
Date solved : `Monday, January 27, 2025 at 05:51:51 PM`
CAS classification : `[_Laguerre]`

Solve

$$ty'' - (1+t)\,y' + y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.220 (sec)

Writing the ode as

$$ty'' + (-1-t)\,y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= -1-t \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 - 2t + 3}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^2 - 2t + 3 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 - 2t + 3}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.23: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4t^2} - \frac{1}{2t}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2t} + \frac{1}{2t^2} + \frac{1}{2t^3} + \frac{1}{4t^4} - \frac{1}{4t^5} - \frac{3}{4t^6} - \frac{3}{4t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 - 2t + 3}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2t + 3}{4t^2}\right) \\ &= \frac{1}{4} + \frac{-2t + 3}{4t^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 - 2t + 3}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{2t} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2} - \frac{1}{2t} \\ &= \frac{t-1}{2t} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2} - \frac{1}{2t}\right)(0) + \left(\left(\frac{1}{2t^2}\right) + \left(\frac{1}{2} - \frac{1}{2t}\right)^2 - \left(\frac{t^2-2t+3}{4t^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= e^{\int \left(\frac{1}{2} - \frac{1}{2t}\right) dt} \\ &= \frac{e^{\frac{t}{2}}}{\sqrt{t}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1-t}{t}\,dt} \\
&= z_1 e^{\frac{t}{2}+\frac{\ln(t)}{2}} \\
&= z_1\left(\sqrt{t}\,\mathrm{e}^{\frac{t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^t
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1-t}{t}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{t+\ln(t)}}{(y_1)^2}\,dt \\
&= y_1\left(-\frac{(1+t)\,\mathrm{e}^{t+\ln(t)}\mathrm{e}^{-2t}}{t}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^t\right) + c_2\left(\mathrm{e}^t\left(-\frac{(1+t)\,\mathrm{e}^{t+\ln(t)}\mathrm{e}^{-2t}}{t}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$t\left(\frac{d^2}{dt^2}y(t)\right) - (t+1)\left(\frac{d}{dt}y(t)\right) + y(t) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dt^2}y(t)$
- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = -\frac{y(t)}{t} + \frac{(t+1)\left(\frac{d}{dt}y(t)\right)}{t}$
- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d^2}{dt^2}y(t) - \frac{(t+1)\left(\frac{d}{dt}y(t)\right)}{t} + \frac{y(t)}{t} = 0$

☐ Check to see if $t_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(t) = -\frac{t+1}{t}, P_3(t) = \frac{1}{t}\right]$
- $t \cdot P_2(t)$ is analytic at $t = 0$

$(t \cdot P_2(t))\Big|_{t=0} = -1$

- $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $(t^2 \cdot P_3(t))\Big|_{t=0} = 0$

- $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $t\left(\frac{d^2}{dt^2}y(t)\right) + (-t-1)\left(\frac{d}{dt}y(t)\right) + y(t) = 0$

- Assume series solution for $y(t)$

  $y(t) = \sum_{k=0}^{\infty} a_k t^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $t^m \cdot \left(\frac{d}{dt}y(t)\right)$ to series expansion for $m = 0..1$

  $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$

- Convert $t \cdot \left(\frac{d^2}{dt^2}y(t)\right)$ to series expansion

  $t \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, t^{k+r-1}$

- Shift index using $k- >k+1$

  $t \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-2+r)\, t^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right) t^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1+r}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 0$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^k, a_{k+1} = \frac{a_k}{k+1}\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+3}$

- Solution for $r = 2$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$

- Combine solutions and rename parameters

$$\left[y(t) = \left(\sum_{k=0}^{\infty} a_k t^k\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 13

```
dsolve(t*diff(diff(y(t),t),t)-(t+1)*diff(y(t),t)+y(t) = 0,y(t),singsol=all)
```

$$y = c_2 e^t + c_1 t + c_1$$

Mathematica DSolve solution

Solving time : 0.422 (sec)
Leaf size : 78

```
DSolve[{t*D[y[t],{t,2}]-(1+t)*D[y[t],t]+y[t] == 0,{}},y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \sqrt{t} \exp\left(\frac{1}{2}\left(2\int_1^t \frac{K[1]-1}{2K[1]} dK[1] + t + 1\right)\right)\left(c_2 \int_1^t \exp\left(-2\int_1^{K[2]} \frac{K[1]-1}{2K[1]} dK[1]\right) dK[2] + c_1\right)$$

2.1.15 Problem 15

Internal problem ID [9187]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 15
Date solved : `Monday, January 27, 2025 at 05:51:52 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(1-t)\,y''+ty'-y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.232 (sec)

Writing the ode as

$$(1-t)\,y''+ty'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=1-t\\B&=t\\C&=-1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{t^2-4t+6}{4\,(-1+t)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=t^2-4t+6\\t&=4(-1+t)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(t)=\left(\frac{t^2-4t+6}{4\,(-1+t)^2}\right)z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.25: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(-1+t)^2$. There is a pole at $t = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4\,(-1+t)^2} - \frac{1}{2\,(-1+t)}$$

For the pole at $t = 1$ let b be the coefficient of $\frac{1}{(-1+t)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2t} + \frac{1}{t^3} + \frac{11}{4t^4} + \frac{21}{4t^5} + \frac{15}{2t^6} + \frac{6}{t^7} - \frac{117}{16t^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 - 4t + 6}{4t^2 - 8t + 4} \\ &= Q + \frac{R}{4t^2 - 8t + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2t + 5}{4t^2 - 8t + 4}\right) \\ &= \frac{1}{4} + \frac{-2t + 5}{4t^2 - 8t + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{t^2 - 4t + 6}{4\left(-1+t\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(-1+t\right)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2\left(-1+t\right)} + \frac{1}{2} \\
&= \frac{t-2}{2t-2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2(-1+t)}+\frac{1}{2}\right)(0)+\left(\left(\frac{1}{2(-1+t)^{2}}\right)+\left(-\frac{1}{2(-1+t)}+\frac{1}{2}\right)^{2}-\left(\frac{t^{2}-4t+6}{4(-1+t)^{2}}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega\,dt} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(-1+t)}+\frac{1}{2}\right)dt} \\
&= \frac{\mathrm{e}^{\frac{t}{2}}}{\sqrt{-1+t}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{t}{1-t}\,dt} \\
&= z_1 e^{\frac{t}{2}+\frac{\ln(-1+t)}{2}} \\
&= z_1\left(\sqrt{-1+t}\,\mathrm{e}^{\frac{t}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\mathrm{e}^{t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1\int \frac{e^{\int -\frac{t}{1-t}\,dt}}{(y_1)^2}\,dt \\
&= y_1\int \frac{e^{t+\ln(-1+t)}}{(y_1)^2}\,dt \\
&= y_1\left(-\frac{t\,\mathrm{e}^{t+\ln(-1+t)}\mathrm{e}^{-2t}}{-1+t}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1y_1+c_2y_2 \\
&= c_1\left(\mathrm{e}^{t}\right)+c_2\left(\mathrm{e}^{t}\left(-\frac{t\,\mathrm{e}^{t+\ln(-1+t)}\mathrm{e}^{-2t}}{-1+t}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(1-t)\left(\frac{d^2}{dt^2}y(t)\right)+t\left(\frac{d}{dt}y(t)\right)-y(t)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dt^2}y(t)$
- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t)=-\frac{y(t)}{t-1}+\frac{t\left(\frac{d}{dt}y(t)\right)}{t-1}$
- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d^2}{dt^2}y(t)-\frac{t\left(\frac{d}{dt}y(t)\right)}{t-1}+\frac{y(t)}{t-1}=0$

□ Check to see if $t_0=1$ is a regular singular point
  - Define functions

    $\left[P_2(t)=-\frac{t}{t-1}, P_3(t)=\frac{1}{t-1}\right]$
  - $(t-1)\cdot P_2(t)$ is analytic at $t=1$

    $\left((t-1)\cdot P_2(t)\right)\bigg|_{t=1}=-1$
  - $(t-1)^2\cdot P_3(t)$ is analytic at $t=1$

    $\left((t-1)^2\cdot P_3(t)\right)\bigg|_{t=1}=0$
  - $t=1$is a regular singular point

  Check to see if $t_0=1$ is a regular singular point

  $t_0=1$

- Multiply by denominators

  $(t-1)\left(\frac{d^2}{dt^2}y(t)\right)-t\left(\frac{d}{dt}y(t)\right)+y(t)=0$
- Change variables using $t=u+1$ so that the regular singular point is at $u=0$

  $u\left(\frac{d^2}{du^2}y(u)\right)+(-u-1)\left(\frac{d}{du}y(u)\right)+y(u)=0$
- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

    $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-1}$
  - Shift index using $k->k+1$

    $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)(k+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)(k+r-1)-a_k(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r) = 0$

- Values of r that satisfy the indicial equation
  $r \in \{0, 2\}$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_k}{k+1+r}$
- Recursion relation for $r = 0$
  $a_{k+1} = \frac{a_k}{k+1}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$
- Revert the change of variables $u = t-1$
  $\left[y(t) = \sum_{k=0}^{\infty} a_k (t-1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$
- Recursion relation for $r = 2$
  $a_{k+1} = \frac{a_k}{k+3}$
- Solution for $r = 2$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$
- Revert the change of variables $u = t-1$
  $\left[y(t) = \sum_{k=0}^{\infty} a_k (t-1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$
- Combine solutions and rename parameters
  $\left[y(t) = \left(\sum_{k=0}^{\infty} a_k (t-1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (t-1)^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 12

```
dsolve((1-t)*diff(diff(y(t),t),t)+t*diff(y(t),t)-y(t) = 0,y(t),singsol=all)
```

$$y = c_1 t + c_2 e^t$$

Mathematica DSolve solution

Solving time : 0.17 (sec)
Leaf size : 90

```
DSolve[{(1-t)*D[y[t],{t,2}]+t*D[y[t],t]-y[t] == 0,{}},y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \exp\left(\int_1^t \frac{K[1]-2}{2(K[1]-1)}dK[1] - \frac{1}{2}\int_1^t -\frac{K[2]}{K[2]-1}dK[2]\right)\left(c_2\int_1^t \exp\left(-2\int_1^{K[3]} \frac{K[1]-2}{2(K[1]-1)}dK[1]\right)dK[3] + c_1\right)$$

2.1.16 Problem 16

Internal problem ID [9188]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 16
Date solved : `Monday, January 27, 2025 at 05:51:53 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.148 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.27: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2\left(\frac{\cos(x)}{\sqrt{x}}(\tan(x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x\left(\frac{d}{dx}y(x)\right) + \left(x^2 - \frac{1}{4}\right)y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{(4x^2-1)y(x)}{4x^2} - \frac{\frac{d}{dx}y(x)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x) + \frac{\frac{d}{dx}y(x)}{x} + \frac{(4x^2-1)y(x)}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$

- Multiply by denominators
  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x\left(\frac{d}{dx}y(x)\right) + (4x^2 - 1)y(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-2})\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$
- Shift index using $k- >k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$
- Solution for $r = -\frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$
- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.036 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x+(x^2-1/4)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{\sin(x) c_1 + \cos(x) c_2}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.035 (sec)
Leaf size : 39

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-25/100)*y[x] == 0,{}},y[x],x,IncludeSingularSolut
```

$$y(x) \to \frac{e^{-ix}(2c_1 - ic_2e^{2ix})}{2\sqrt{x}}$$

2.1.17 Problem 17

Internal problem ID [9189]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 17
Date solved : `Monday, January 27, 2025 at 05:51:53 PM`
CAS classification : `[_Laguerre]`

Solve

$$
ty'' - (1+t)\,y' + y = 0
$$

Solved as second order ode using Kovacic algorithm

Time used: 0.229 (sec)

Writing the ode as

$$
\begin{align*}
ty'' + (-1-t)\,y' + y &= 0 \tag{1}\\
Ay'' + By' + Cy &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= t \\
B &= -1-t \tag{3}\\
C &= 1
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(t) = ye^{\int \frac{B}{2A}\,dt}
$$

Then (2) becomes

$$
z''(t) = rz(t) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5}\\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{t^2-2t+3}{4t^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align*}
s &= t^2-2t+3\\
t &= 4t^2
\end{align*}
$$

Therefore eq. (4) becomes

$$
z''(t) = \left(\frac{t^2-2t+3}{4t^2}\right) z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.29: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4t^2} - \frac{1}{2t}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2t} + \frac{1}{2t^2} + \frac{1}{2t^3} + \frac{1}{4t^4} - \frac{1}{4t^5} - \frac{3}{4t^6} - \frac{3}{4t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 - 2t + 3}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2t + 3}{4t^2}\right) \\ &= \frac{1}{4} + \frac{-2t + 3}{4t^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 - 2t + 3}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2t} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} - \frac{1}{2t} \\
&= \frac{t-1}{2t}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2} - \frac{1}{2t}\right)(0) + \left(\left(\frac{1}{2t^2}\right) + \left(\frac{1}{2} - \frac{1}{2t}\right)^2 - \left(\frac{t^2 - 2t + 3}{4t^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= e^{\int \left(\frac{1}{2} - \frac{1}{2t}\right) dt} \\
&= \frac{e^{\frac{t}{2}}}{\sqrt{t}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-1-t}{t}\,dt} \\ &= z_1 e^{\frac{t}{2}+\frac{\ln(t)}{2}} \\ &= z_1\left(\sqrt{t}\,\mathrm{e}^{\frac{t}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-1-t}{t}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{t+\ln(t)}}{(y_1)^2}\,dt \\ &= y_1\left(-\frac{(1+t)\,\mathrm{e}^{t+\ln(t)}\mathrm{e}^{-2t}}{t}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{t}\right) + c_2\left(\mathrm{e}^{t}\left(-\frac{(1+t)\,\mathrm{e}^{t+\ln(t)}\mathrm{e}^{-2t}}{t}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$t\left(\frac{d^2}{dt^2}y(t)\right) - (t+1)\left(\frac{d}{dt}y(t)\right) + y(t) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dt^2}y(t)$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = -\frac{y(t)}{t} + \frac{(t+1)\left(\frac{d}{dt}y(t)\right)}{t}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d^2}{dt^2}y(t) - \frac{(t+1)\left(\frac{d}{dt}y(t)\right)}{t} + \frac{y(t)}{t} = 0$

- ☐ Check to see if $t_0 = 0$ is a regular singular point
  - ◦ Define functions

    $\left[P_2(t) = -\frac{t+1}{t}, P_3(t) = \frac{1}{t}\right]$
  - ◦ $t \cdot P_2(t)$ is analytic at $t = 0$

$\left(t \cdot P_2(t)\right)\bigg|_{t=0} = -1$

- ◦ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $\left(t^2 \cdot P_3(t)\right)\bigg|_{t=0} = 0$

- ◦ $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $t\left(\frac{d^2}{dt^2}y(t)\right) + (-t-1)\left(\frac{d}{dt}y(t)\right) + y(t) = 0$

- Assume series solution for $y(t)$

  $y(t) = \sum_{k=0}^{\infty} a_k t^{k+r}$

- □ Rewrite ODE with series expansions

  - ◦ Convert $t^m \cdot \left(\frac{d}{dt}y(t)\right)$ to series expansion for $m = 0..1$

    $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$

  - ◦ Shift index using $k- >k+1-m$

    $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$

  - ◦ Convert $t \cdot \left(\frac{d^2}{dt^2}y(t)\right)$ to series expansion

    $t \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, t^{k+r-1}$

  - ◦ Shift index using $k- >k+1$

    $t \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-2+r)\, t^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right) t^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1+r}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 0$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^k, a_{k+1} = \frac{a_k}{k+1}\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+3}$

- Solution for $r = 2$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$

- Combine solutions and rename parameters

$$\left[y(t)=\left(\sum_{k=0}^{\infty} a_k t^k\right)+\left(\sum_{k=0}^{\infty} b_k t^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+3}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 13

```
dsolve(t*diff(diff(y(t),t),t)-(t+1)*diff(y(t),t)+y(t) = 0,y(t),singsol=all)
```

$$y=c_2 e^t+c_1 t+c_1$$

Mathematica DSolve solution

Solving time : 0.386 (sec)
Leaf size : 78

```
DSolve[{t*D[y[t],{t,2}]-(1+t)*D[y[t],t]+y[t] ==0,{}},y[t],t,IncludeSingularSolutions->True]
```

$$y(t)\to\sqrt{t}\exp\left(\frac{1}{2}\left(2\int_1^t\frac{K[1]-1}{2K[1]}dK[1]+t+1\right)\right)\left(c_2\int_1^t\exp\left(-2\int_1^{K[2]}\frac{K[1]-1}{2K[1]}dK[1]\right)dK[2]+c_1\right)$$

2.1.18 Problem 18

Internal problem ID [9190]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 18
Date solved : `Monday, January 27, 2025 at 05:51:54 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(1-t)\,y''+ty'-y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.223 (sec)

Writing the ode as

$$(1-t)\,y''+ty'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=1-t\\B&=t\\C&=-1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{t^2-4t+6}{4\left(-1+t\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=t^2-4t+6\\t&=4(-1+t)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(t)=\left(\frac{t^2-4t+6}{4\left(-1+t\right)^2}\right)z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.31: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(-1+t)^2$. There is a pole at $t = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4\,(-1+t)^2} - \frac{1}{2\,(-1+t)}$$

For the pole at $t = 1$ let b be the coefficient of $\frac{1}{(-1+t)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2t} + \frac{1}{t^3} + \frac{11}{4t^4} + \frac{21}{4t^5} + \frac{15}{2t^6} + \frac{6}{t^7} - \frac{117}{16t^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 - 4t + 6}{4t^2 - 8t + 4} \\ &= Q + \frac{R}{4t^2 - 8t + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2t + 5}{4t^2 - 8t + 4}\right) \\ &= \frac{1}{4} + \frac{-2t + 5}{4t^2 - 8t + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{t^2 - 4t + 6}{4(-1+t)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(-1+t)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2(-1+t)} + \frac{1}{2} \\
&= \frac{t-2}{2t-2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(-1+t)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2(-1+t)^2}\right) + \left(-\frac{1}{2(-1+t)} + \frac{1}{2}\right)^2 - \left(\frac{t^2-4t+6}{4(-1+t)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= pe^{\int \omega\, dt} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2(-1+t)} + \frac{1}{2}\right) dt} \\ &= \frac{\mathrm{e}^{\frac{t}{2}}}{\sqrt{-1+t}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{t}{1-t}\, dt} \\ &= z_1 e^{\frac{t}{2} + \frac{\ln(-1+t)}{2}} \\ &= z_1 \left(\sqrt{-1+t}\, \mathrm{e}^{\frac{t}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{t}{1-t}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{t+\ln(-1+t)}}{(y_1)^2}\, dt \\ &= y_1 \left(-\frac{t\,\mathrm{e}^{t+\ln(-1+t)}\mathrm{e}^{-2t}}{-1+t}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^t\right) + c_2\left(\mathrm{e}^t\left(-\frac{t\,\mathrm{e}^{t+\ln(-1+t)}\mathrm{e}^{-2t}}{-1+t}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(1-t)\left(\frac{d^2}{dt^2}y(t)\right)+t\left(\frac{d}{dt}y(t)\right)-y(t)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dt^2}y(t)$

- Isolate 2nd derivative

$\frac{d^2}{dt^2}y(t)=-\frac{y(t)}{t-1}+\frac{t\left(\frac{d}{dt}y(t)\right)}{t-1}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d^2}{dt^2}y(t)-\frac{t\left(\frac{d}{dt}y(t)\right)}{t-1}+\frac{y(t)}{t-1}=0$

□ Check to see if $t_0=1$ is a regular singular point

  ◦ Define functions

  $\left[P_2(t)=-\frac{t}{t-1}, P_3(t)=\frac{1}{t-1}\right]$

  ◦ $(t-1)\cdot P_2(t)$ is analytic at $t=1$

  $\left((t-1)\cdot P_2(t)\right)\Big|_{t=1}=-1$

  ◦ $(t-1)^2\cdot P_3(t)$ is analytic at $t=1$

  $\left((t-1)^2\cdot P_3(t)\right)\Big|_{t=1}=0$

  ◦ $t=1$is a regular singular point

  Check to see if $t_0=1$ is a regular singular point

  $t_0=1$

- Multiply by denominators

$(t-1)\left(\frac{d^2}{dt^2}y(t)\right)-t\left(\frac{d}{dt}y(t)\right)+y(t)=0$

- Change variables using $t=u+1$ so that the regular singular point is at $u=0$

$u\left(\frac{d^2}{du^2}y(u)\right)+(-u-1)\left(\frac{d}{du}y(u)\right)+y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  ◦ Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-1}$

  ◦ Shift index using $k->k+1$

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)(k+r-1)-a_k(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1+r}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$

- Revert the change of variables $u = t-1$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k (t-1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+3}$

- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$

- Revert the change of variables $u = t-1$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k (t-1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$

- Combine solutions and rename parameters

  $\left[y(t) = \left(\sum_{k=0}^{\infty} a_k (t-1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (t-1)^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 12

```
dsolve((1-t)*diff(diff(y(t),t),t)+t*diff(y(t),t)-y(t) = 0,y(t),singsol=all)
```

$$y = c_1 t + c_2 e^t$$

Mathematica DSolve solution

Solving time : 0.152 (sec)
Leaf size : 90

```
DSolve[{(1-t)*D[y[t],{t,2}]+t*D[y[t],t]-y[t] ==0,{}},y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \exp\left(\int_1^t \frac{K[1]-2}{2(K[1]-1)}dK[1] - \frac{1}{2}\int_1^t -\frac{K[2]}{K[2]-1}dK[2]\right)\left(c_2\int_1^t \exp\left(-2\int_1^{K[3]} \frac{K[1]-2}{2(K[1]-1)}dK[1]\right)dK[3] + c_1\right)$$

2.1.19 Problem 19

Internal problem ID [9191]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 19
Date solved : `Monday, January 27, 2025 at 05:51:55 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + xy' + 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.217 (sec)

Writing the ode as

$$y'' + xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 6 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.33: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{2x} - \frac{9}{4x^3} - \frac{27}{4x^5} - \frac{405}{16x^7} - \frac{1701}{16x^9} - \frac{15309}{32x^{11}} - \frac{72171}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v=1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1}=x^0=1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v=0$ or not. Since $v=1$ which is not zero, then starting $r=\frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2-6}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} - \frac{3}{2}\right) + (0) \\ &= \frac{x^2}{4} - \frac{3}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{2}\right) - (0) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{3}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{3}{2}\right)\right) &= 0 \\ a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (x)\, e^{\int -\frac{x}{2} dx} \\ &= (x)\, e^{-\frac{x^2}{4}} \\ &= x\, e^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{1}\, dx} \\ &= z_1 e^{-\frac{x^2}{4}} \\ &= z_1\left(e^{-\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^2}{2}} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2}\,dx \\ &= y_1 \left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-\frac{x^2}{2}} x\right) + c_2 \left(\mathrm{e}^{-\frac{x^2}{2}} x \left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} \right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + x\left(\frac{d}{dx}y(x)\right) + 2y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

  $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

  $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+2)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

$$(k+2)(ka_{k+2} + a_k + a_{k+2}) = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{k+1}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 37

```
dsolve(diff(diff(y(x),x),x)+diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y = -x\left(c_2 \operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)\pi - c_1\right)\mathrm{e}^{-\frac{x^2}{2}} + i\sqrt{\pi}\,\sqrt{2}\,c_2$$

Mathematica DSolve solution

Solving time : 0.052 (sec)
Leaf size : 69

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\sqrt{\frac{\pi}{2}}c_2 e^{-\frac{x^2}{2}}\sqrt{x^2}\operatorname{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{2}}\right) + \sqrt{2}c_1 e^{-\frac{x^2}{2}}x + c_2$$

2.1.20 Problem 20

Internal problem ID [9192]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 20
Date solved : `Monday, January 27, 2025 at 05:51:55 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+1\right) y''-4xy'+6y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.249 (sec)

Writing the ode as

$$\left(x^2+1\right) y''-4xy'+6y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= -4x \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{-8}{\left(x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -8 \\ t &= \left(x^2+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{8}{\left(x^2+1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.35: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{(x-i)^2} + \frac{2}{(x+i)^2} + \frac{2i}{x-i} - \frac{2i}{x+i}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{8}{\left(x^2+1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
i	2	0	2	-1
$-i$	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x-i} + \frac{2}{x+i} + (-)\,(0) \\
&= -\frac{1}{x-i} + \frac{2}{x+i} \\
&= \frac{x-3i}{x^2+1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x-i} + \frac{2}{x+i}\right)(0) + \left(\left(\frac{1}{(x-i)^2} - \frac{2}{(x+i)^2} \right) + \left(-\frac{1}{x-i} + \frac{2}{x+i} \right)^2 - \left(-\frac{8}{(x^2+1)^2} \right) \right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= pe^{\int \omega\,dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{x-i}+\frac{2}{x+i}\right)dx} \\
&= \frac{\left(x^2+1\right)^2}{\left(ix+1\right)^3}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x}{x^2+1}\,dx} \\
&= z_1 e^{\ln(x^2+1)} \\
&= z_1\left(x^2+1\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\left(x^2+1\right)^3}{\left(ix+1\right)^3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{x^2+1}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{2\ln(x^2+1)}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\frac{x^2-\frac{1}{3}}{\left(x+i\right)^3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\left(x^2+1\right)^3}{\left(ix+1\right)^3}\right) + c_2\left(\frac{\left(x^2+1\right)^3}{\left(ix+1\right)^3}\left(\frac{x^2-\frac{1}{3}}{\left(x+i\right)^3}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 21

```
dsolve((x^2+1)*diff(diff(y(x),x),x)-4*diff(y(x),x)*x+6*y(x) = 0,y(x),singsol=all)
```

$$y = c_2 x^3 - 3c_1 x^2 - 3c_2 x + c_1$$

Mathematica DSolve solution

Solving time : 0.267 (sec)
Leaf size : 75

```
DSolve[{(1+x^2)*D[y[x],{x,2}]-4*x*D[y[x],x]+6*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to (x^2 + 1) \exp\left(\int_1^x \frac{K[1]+3i}{K[1]^2+1} dK[1]\right) \left(c_2 \int_1^x \exp\left(-2\int_1^{K[2]} \frac{K[1]+3i}{K[1]^2+1} dK[1]\right) dK[2] + c_1\right)$$

2.1.21 Problem 21

Internal problem ID [9193]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 21
Date solved : `Monday, January 27, 2025 at 05:51:56 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(1-x)\,y''+xy'-y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.231 (sec)

Writing the ode as

$$(1-x)\,y''+xy'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1-x \\ B &= x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{x^2-4x+6}{4\left(-1+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2-4x+6 \\ t &= 4(-1+x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x^2-4x+6}{4\left(-1+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.36: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(-1 + x)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2\,(-1 + x)} + \frac{3}{4\,(-1 + x)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\
&= Q + \frac{R}{4x^2 - 8x + 4} \\
&= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\
&= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4}
\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (0) \\
&= -\frac{1}{2}
\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 4x + 6}{4\left(-1 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(-1 + x\right)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2\left(-1 + x\right)} + \frac{1}{2} \\
&= \frac{x - 2}{2x - 2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(-1+x)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2(-1+x)^2}\right) + \left(-\frac{1}{2(-1+x)} + \frac{1}{2}\right)^2 - \left(\frac{x^2-4x+6}{4(-1+x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2(-1+x)} + \frac{1}{2}\right) dx} \\ &= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{-1+x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{1-x}\, dx} \\ &= z_1 e^{\frac{x}{2} + \frac{\ln(-1+x)}{2}} \\ &= z_1\left(\sqrt{-1+x}\, \mathrm{e}^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1-x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x+\ln(-1+x)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{x\, \mathrm{e}^{x+\ln(-1+x)} \mathrm{e}^{-2x}}{-1+x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(-\frac{x\, \mathrm{e}^{x+\ln(-1+x)} \mathrm{e}^{-2x}}{-1+x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(1-x)\left(\frac{d^2}{dx^2}y(x)\right)+x\left(\frac{d}{dx}y(x)\right)-y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{y(x)}{x-1}+\frac{\left(\frac{d}{dx}y(x)\right)x}{x-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)-\frac{\left(\frac{d}{dx}y(x)\right)x}{x-1}+\frac{y(x)}{x-1}=0$

☐ Check to see if $x_0=1$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=-\frac{x}{x-1},P_3(x)=\frac{1}{x-1}\right]$

- ◦ $(x-1)\cdot P_2(x)$ is analytic at $x=1$

  $\left((x-1)\cdot P_2(x)\right)\bigg|_{x=1}=-1$

- ◦ $(x-1)^2\cdot P_3(x)$ is analytic at $x=1$

  $\left((x-1)^2\cdot P_3(x)\right)\bigg|_{x=1}=0$

- ◦ $x=1$is a regular singular point

  Check to see if $x_0=1$ is a regular singular point

  $x_0=1$

- Multiply by denominators

$(x-1)\left(\frac{d^2}{dx^2}y(x)\right)-x\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Change variables using $x=u+1$ so that the regular singular point is at $u=0$

$u\left(\frac{d^2}{du^2}y(u)\right)+(-u-1)\left(\frac{d}{du}y(u)\right)+y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ◦ Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-1}$

- ◦ Shift index using $k->k+1$

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(k+r-1)-a_k(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r)=0$

- Values of r that satisfy the indicial equation
  $r \in \{0,2\}$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-1)\left(a_{k+1}(k+1+r)-a_k\right)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{a_k}{k+1+r}$
- Recursion relation for $r=0$
  $a_{k+1}=\frac{a_k}{k+1}$
- Solution for $r=0$
  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+1}=\frac{a_k}{k+1}\right]$
- Revert the change of variables $u=x-1$
  $\left[y(x)=\sum_{k=0}^{\infty} a_k (x-1)^k, a_{k+1}=\frac{a_k}{k+1}\right]$
- Recursion relation for $r=2$
  $a_{k+1}=\frac{a_k}{k+3}$
- Solution for $r=2$
  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$
- Revert the change of variables $u=x-1$
  $\left[y(x)=\sum_{k=0}^{\infty} a_k (x-1)^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$
- Combine solutions and rename parameters
  $\left[y(x)=\left(\sum_{k=0}^{\infty} a_k (x-1)^k\right)+\left(\sum_{k=0}^{\infty} b_k (x-1)^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+3}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 12

```
dsolve((1-x)*diff(diff(y(x),x),x)+diff(y(x),x)*x-y(x) = 0,y(x),singsol=all)
```

$$y=c_1 x+e^{x} c_2$$

Mathematica DSolve solution

Solving time : 0.158 (sec)
Leaf size : 90

```
DSolve[{(1-x)*D[y[x],{x,2}]+x*D[y[x],x]-y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \exp\left(\int_1^x \frac{K[1]-2}{2(K[1]-1)}dK[1] - \frac{1}{2}\int_1^x -\frac{K[2]}{K[2]-1}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{K[1]-2}{2(K[1]-1)}dK[1]\right)dK[3] + c_1\right)$$

2.1.22 Problem 22

Internal problem ID [9194]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 22
Date solved : `Monday, January 27, 2025 at 05:51:56 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2y'' + xy' + 3y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.244 (sec)

Writing the ode as

$$2y'' + xy' + 3y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= x \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 20}{16} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 20$$
$$t = 16$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{16} - \frac{5}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.38: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \qquad (8) \end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{4} - \frac{5}{2x} - \frac{25}{2x^3} - \frac{125}{x^5} - \frac{3125}{2x^7} - \frac{21875}{x^9} - \frac{328125}{x^{11}} - \frac{5156250}{x^{13}} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{4} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{16}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 20}{16} \\ &= Q + \frac{R}{16} \\ &= \left(\frac{x^2}{16} - \frac{5}{4}\right) + (0) \\ &= \frac{x^2}{16} - \frac{5}{4} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{5}{4}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{5}{4}\right) - (0) \\ &= -\frac{5}{4} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{4} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{5}{4}}{\frac{1}{4}} - 1\right) &= -3 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{5}{4}}{\frac{1}{4}} - 1\right) &= 2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{16} - \frac{5}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{4}$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{4}\right) \\ &= -\frac{x}{4} \\ &= -\frac{x}{4} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered} (2) + 2\left(-\frac{x}{4}\right)(2x + a_1) + \left(\left(-\frac{1}{4}\right) + \left(-\frac{x}{4}\right)^2 - \left(\frac{x^2}{16} - \frac{5}{4}\right)\right) = 0 \\ 2 + \frac{a_1 x}{2} + a_0 = 0 \end{gathered}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -2, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x^2 - 2\right) e^{\int -\frac{x}{4} dx} \\ &= \left(x^2 - 2\right) e^{-\frac{x^2}{8}} \\ &= \left(x^2 - 2\right) e^{-\frac{x^2}{8}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x}{2} \, dx} \\ &= z_1 e^{-\frac{x^2}{8}} \\ &= z_1 \left(e^{-\frac{x^2}{8}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{-\frac{x^2}{4}}\left(x^2-2\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{4}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{e^{\frac{x^2}{4}}}{(x^2-2)^2}dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(e^{-\frac{x^2}{4}}\left(x^2-2\right)\right) + c_2\left(e^{-\frac{x^2}{4}}\left(x^2-2\right)\left(\int \frac{e^{\frac{x^2}{4}}}{(x^2-2)^2}dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$2\frac{d^2}{dx^2}y(x) + x\left(\frac{d}{dx}y(x)\right) + 3y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{x\left(\frac{d}{dx}y(x)\right)}{2} - \frac{3y(x)}{2}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) + \frac{x\left(\frac{d}{dx}y(x)\right)}{2} + \frac{3y(x)}{2} = 0$
- Multiply by denominators
  $2\frac{d^2}{dx^2}y(x) + x\left(\frac{d}{dx}y(x)\right) + 3y(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  - Convert $x\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion
    $x\cdot\left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$
  - Convert $\frac{d^2}{dx^2}y(x)$ to series expansion
    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$
  - Shift index using $k- >k+2$

$\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

Rewrite DE with series expansions

$\sum_{k=0}^{\infty} \left(2a_{k+2}(k+2)(k+1) + a_k(k+3)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(2k^2 + 6k + 4)\, a_{k+2} + a_k(k+3) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(k+3)}{2(k^2+3k+2)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.025 (sec)
Leaf size : 32

```
dsolve(2*diff(diff(y(x),x),x)+diff(y(x),x)*x+3*y(x) = 0,y(x),singsol=all)
```

$$y = (x^2-2)\left(c_1 \operatorname{erfi}\left(\frac{x}{2}\right)\sqrt{\pi} + c_2\right) \mathrm{e}^{-\frac{x^2}{4}} - 2c_1 x$$

Mathematica DSolve solution

Solving time : 0.408 (sec)
Leaf size : 52

```
DSolve[{2*D[y[x],{x,2}]+x*D[y[x],x]+3*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-\frac{x^2}{4}}(x^2-2)\left(c_2 \int_1^x \frac{e^{\frac{K[1]^2}{4}}}{(K[1]^2-2)^2} dK[1] + c_1\right)$$

2.1.23 Problem 23

Internal problem ID [9195]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 23
Date solved : `Monday, January 27, 2025 at 05:51:57 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + xy' + 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.210 (sec)

Writing the ode as

$$y'' + xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 6 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.40: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{2x} - \frac{9}{4x^3} - \frac{27}{4x^5} - \frac{405}{16x^7} - \frac{1701}{16x^9} - \frac{15309}{32x^{11}} - \frac{72171}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{2} \qquad (10)
\end{aligned}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}
$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 6}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} - \frac{3}{2}\right) + (0) \\
&= \frac{x^2}{4} - \frac{3}{2}
\end{aligned}
$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{3}{2}\right) - (0) \\
&= -\frac{3}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2}{4} - \frac{3}{2}
$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{3}{2}\right)\right) &= 0 \\ a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= (x)\,e^{\int -\frac{x}{2}dx} \\ &= (x)\,e^{-\frac{x^2}{4}} \\ &= x\,e^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{1}\, dx} \\ &= z_1 e^{-\frac{x^2}{4}} \\ &= z_1\left(e^{-\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^2}{2}} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-\frac{x^2}{2}} x\right) + c_2 \left(\mathrm{e}^{-\frac{x^2}{2}} x \left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} \right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + x\left(\frac{d}{dx}y(x)\right) + 2y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

- ☐ Rewrite DE with series expansions

  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k+2$

    $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+2)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

$$(k+2)\left(k a_{k+2}+a_k+a_{k+2}\right)=0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y(x)=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=-\frac{a_k}{k+1}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 37

```
dsolve(diff(diff(y(x),x),x)+diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y=-x\left(c_2 \operatorname{erf}\left(\frac{i\sqrt{2}\, x}{2}\right)\pi-c_1\right)\mathrm{e}^{-\frac{x^2}{2}}+i\sqrt{\pi}\,\sqrt{2}\, c_2$$

Mathematica DSolve solution

Solving time : 0.052 (sec)
Leaf size : 69

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to -\sqrt{\frac{\pi}{2}} c_2 e^{-\frac{x^2}{2}}\sqrt{x^2}\operatorname{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{2}}\right)+\sqrt{2} c_1 e^{-\frac{x^2}{2}} x+c_2$$

2.1.24 Problem 24

Internal problem ID [9196]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 24
Date solved : `Monday, January 27, 2025 at 05:51:58 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(1-x)\,y''+xy'-y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.242 (sec)

Writing the ode as

$$(1-x)\,y''+xy'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=1-x\\ B&=x\\ C&=-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{x^2-4x+6}{4\left(-1+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=x^2-4x+6\\ t&=4(-1+x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x^2-4x+6}{4\left(-1+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.42: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(-1 + x)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4\,(-1+x)^2} - \frac{1}{2\,(-1+x)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 4x + 6}{4(-1+x)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(-1+x)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2(-1+x)} + \frac{1}{2} \\
&= \frac{x-2}{2x-2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2(-1+x)}+\frac{1}{2}\right)(0)+\left(\left(\frac{1}{2(-1+x)^2}\right)+\left(-\frac{1}{2(-1+x)}+\frac{1}{2}\right)^2-\left(\frac{x^2-4x+6}{4(-1+x)^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(-1+x)}+\frac{1}{2}\right)dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{-1+x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{1-x}\, dx} \\
&= z_1 e^{\frac{x}{2}+\frac{\ln(-1+x)}{2}} \\
&= z_1\left(\sqrt{-1+x}\,\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1-x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x+\ln(-1+x)}}{(y_1)^2}\, dx \\
&= y_1\left(-\frac{x\,\mathrm{e}^{x+\ln(-1+x)}\mathrm{e}^{-2x}}{-1+x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^{x}) + c_2\left(\mathrm{e}^{x}\left(-\frac{x\,\mathrm{e}^{x+\ln(-1+x)}\mathrm{e}^{-2x}}{-1+x}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(1-x)\left(\frac{d^2}{dx^2}y(x)\right)+x\left(\frac{d}{dx}y(x)\right)-y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{y(x)}{x-1}+\frac{\left(\frac{d}{dx}y(x)\right)x}{x-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)-\frac{\left(\frac{d}{dx}y(x)\right)x}{x-1}+\frac{y(x)}{x-1}=0$

□ Check to see if $x_0=1$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=-\frac{x}{x-1}, P_3(x)=\frac{1}{x-1}\right]$

  ◦ $(x-1)\cdot P_2(x)$ is analytic at $x=1$

    $\left.((x-1)\cdot P_2(x))\right|_{x=1}=-1$

  ◦ $(x-1)^2\cdot P_3(x)$ is analytic at $x=1$

    $\left.((x-1)^2\cdot P_3(x))\right|_{x=1}=0$

  ◦ $x=1$is a regular singular point

  Check to see if $x_0=1$ is a regular singular point

  $x_0=1$

- Multiply by denominators

  $(x-1)\left(\frac{d^2}{dx^2}y(x)\right)-x\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Change variables using $x=u+1$ so that the regular singular point is at $u=0$

  $u\left(\frac{d^2}{du^2}y(u)\right)+(-u-1)\left(\frac{d}{du}y(u)\right)+y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  ◦ Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

    $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-1}$

  ◦ Shift index using $k->k+1$

    $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(k+r-1)-a_k(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1+r}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$

- Revert the change of variables $u = x - 1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x-1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+3}$

- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$

- Revert the change of variables $u = x - 1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x-1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k (x-1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x-1)^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 12

```
dsolve((1-x)*diff(diff(y(x),x),x)+diff(y(x),x)*x-y(x) = 0,y(x),singsol=all)
```

$$y = c_1 x + e^x c_2$$

Mathematica DSolve solution

Solving time : `0.154` (sec)
Leaf size : 90

```
DSolve[{(1-x)*D[y[x],{x,2}]+x*D[y[x],x]-y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \exp\left(\int_1^x \frac{K[1]-2}{2(K[1]-1)}dK[1]-\frac{1}{2}\int_1^x -\frac{K[2]}{K[2]-1}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{K[1]-2}{2(K[1]-1)}dK[1]\right)dK[3]+c_1\right)$$

2.1.25 Problem 25

Internal problem ID [9197]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 25
Date solved : `Monday, January 27, 2025 at 05:51:58 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + xy' + 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.218 (sec)

Writing the ode as

$$y'' + xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 6 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.44: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{2x} - \frac{9}{4x^3} - \frac{27}{4x^5} - \frac{405}{16x^7} - \frac{1701}{16x^9} - \frac{15309}{32x^{11}} - \frac{72171}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 6}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} - \frac{3}{2}\right) + (0) \\ &= \frac{x^2}{4} - \frac{3}{2}\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{3}{2}\right) - (0) \\ &= -\frac{3}{2}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{3}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{3}{2}\right)\right) &= 0 \\ a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x)\, \mathrm{e}^{\int -\frac{x}{2} dx} \\ &= (x)\, \mathrm{e}^{-\frac{x^2}{4}} \\ &= x\, \mathrm{e}^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{1} \, dx} \\ &= z_1 e^{-\frac{x^2}{4}} \\ &= z_1\left(\mathrm{e}^{-\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{-\frac{x^2}{2}} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{e^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(e^{-\frac{x^2}{2}} x \right) + c_2 \left(e^{-\frac{x^2}{2}} x \left(-\frac{e^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + x\left(\frac{d}{dx}y(x)\right) + 2y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

  $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

  $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+2)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

$$(k+2)\left(k a_{k+2}+a_k+a_{k+2}\right)=0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y(x)=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=-\frac{a_k}{k+1}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.007` (sec)
Leaf size : 37

```
dsolve(diff(diff(y(x),x),x)+diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y=-x\left(c_2 \operatorname{erf}\left(\frac{i\sqrt{2}\, x}{2}\right)\pi-c_1\right)\mathrm{e}^{-\frac{x^2}{2}}+i\sqrt{\pi}\,\sqrt{2}\, c_2$$

Mathematica DSolve solution

Solving time : `0.046` (sec)
Leaf size : 69

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to -\sqrt{\frac{\pi}{2}}c_2 e^{-\frac{x^2}{2}}\sqrt{x^2}\operatorname{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{2}}\right)+\sqrt{2}c_1 e^{-\frac{x^2}{2}}x+c_2$$

2.1.26 Problem 26

Internal problem ID [9198]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 26
Date solved : `Monday, January 27, 2025 at 05:51:59 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(-x^2+4\right) y''+x y'+2 y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 1.058 (sec)

Writing the ode as

$$\left(-x^2+4\right) y''+x y'+2 y=0 \tag{1}$$

$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+4 \\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{11x^2-24}{4\left(x^2-4\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 11x^2-24 \\ t &= 4\left(x^2-4\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{11x^2-24}{4\left(x^2-4\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.46: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2-4)^2$. There is a pole at $x = 2$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{17}{32\,(x-2)} - \frac{17}{32\,(x+2)} + \frac{5}{16\,(x+2)^2} + \frac{5}{16\,(x-2)^2}$$

For the pole at $x = 2$ let b be the coefficient of $\frac{1}{(x-2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{11x^2 - 24}{4\left(x^2-4\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{11}{4}$. Hence

$$\begin{aligned} E_\infty &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{2\} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
2	2	$\{-1, 2, 5\}$
-2	2	$\{-1, 2, 5\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = -1,\ e_2 = -1,\ e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(2 - (-1 + (-1))) \\ &= 2 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{-1}{(x-(2))} + \frac{-1}{(x-(-2))}\right) \\ &= -\frac{1}{2(x-2)} - \frac{1}{2(x+2)} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 2$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 2$, then letting

$$p = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$\frac{11x^2 a_1 + 16(6 + a_0)\,x + 36a_1}{\left(x^2-4\right)^2} = 0$$

And solving for p gives

$$p = x^2 - 6$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p}\\ &= \frac{2x}{x^2-6} - \frac{1}{2(x-2)} - \frac{1}{2(x+2)}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{2x}{x^2-6} - \frac{1}{2(x-2)} - \frac{1}{2(x+2)}\right)w + \frac{-11x^4+74x^2-128}{4x^6-56x^4+256x^2-384} = 0$$

Solving for ω gives

$$\omega = \frac{2\sqrt{3}\,x^2\sqrt{x^2-4} + x^3 - 8\sqrt{3}\,\sqrt{x^2-4} - 2x}{2(x^2-6)(x-2)(x+2)}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega\,dx}\\ &= e^{\int \frac{2\sqrt{3}\,x^2\sqrt{x^2-4}+x^3-8\sqrt{3}\,\sqrt{x^2-4}-2x}{2\left(x^2-6\right)(x-2)(x+2)}dx}\\ &= \frac{\sqrt{x^2-6}\left(x+\sqrt{x^2-4}\right)^{\sqrt{3}} \mathrm{e}^{-\frac{\operatorname{arctanh}\left(\frac{\left(\sqrt{2}\,\sqrt{3}\,x-4\right)\sqrt{2}}{2\sqrt{x^2-4}}\right)}{2} - \frac{\operatorname{arctanh}\left(\frac{\left(4+\sqrt{2}\,\sqrt{3}\,x\right)\sqrt{2}}{2\sqrt{x^2-4}}\right)}{2}}}{(x-2)^{1/4}(x+2)^{1/4}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{-x^2+4}\,dx}\\ &= z_1 e^{\frac{\ln\left(x^2-4\right)}{4}}\\ &= z_1\left(\left(x^2-4\right)^{1/4}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x^2-6}\left(x+\sqrt{x^2-4}\right)^{\sqrt{3}} \mathrm{e}^{-\frac{\operatorname{arctanh}\left(\frac{x\sqrt{6}-4}{\sqrt{2x^2-8}}\right)}{2} - \frac{\operatorname{arctanh}\left(\frac{4+x\sqrt{6}}{\sqrt{2x^2-8}}\right)}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1\int \frac{e^{\int -\frac{x}{-x^2+4}\,dx}}{(y_1)^2}\,dx\\ &= y_1\int \frac{e^{\frac{\ln\left(x^2-4\right)}{2}}}{(y_1)^2}\,dx\\ &= y_1\left(\int \frac{\sqrt{x^2-4}\left(x+\sqrt{x^2-4}\right)^{-2\sqrt{3}} \mathrm{e}^{\operatorname{arctanh}\left(\frac{x\sqrt{6}-4}{\sqrt{2x^2-8}}\right)+\operatorname{arctanh}\left(\frac{4+x\sqrt{6}}{\sqrt{2x^2-8}}\right)}}{x^2-6}dx\right)\end{aligned}$$

Therefore the solution is

$$y = c_1y_1 + c_2y_2$$

$$= c_1\left(\sqrt{x^2-6}\left(x+\sqrt{x^2-4}\right)^{\sqrt{3}} e^{-\frac{\operatorname{arctanh}\left(\frac{x\sqrt{6}-4}{\sqrt{2x^2-8}}\right)}{2}-\frac{\operatorname{arctanh}\left(\frac{4+x\sqrt{6}}{\sqrt{2x^2-8}}\right)}{2}}\right) + c_2\left(\sqrt{x^2-6}\left(x\right.\right.$$

$$\left.+\sqrt{x^2-4}\right)^{\sqrt{3}} e^{-\frac{\operatorname{arctanh}\left(\frac{x\sqrt{6}-4}{\sqrt{2x^2-8}}\right)}{2}-\frac{\operatorname{arctanh}\left(\frac{4+x\sqrt{6}}{\sqrt{2x^2-8}}\right)}{2}}\left(\int \frac{\sqrt{x^2-4}\left(x+\sqrt{x^2-4}\right)^{-2\sqrt{3}} e^{\operatorname{arctanh}\left(\frac{x\sqrt{6}-4}{\sqrt{2x^2-8}}\right)+\operatorname{arctanh}\left(\frac{4}{\sqrt{}}\right.}}{x^2-6}\right.$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(-x^2+4)\left(\frac{d^2}{dx^2}y(x)\right)+x\left(\frac{d}{dx}y(x)\right)+2y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{2y(x)}{x^2-4}+\frac{\left(\frac{d}{dx}y(x)\right)x}{x^2-4}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)-\frac{\left(\frac{d}{dx}y(x)\right)x}{x^2-4}-\frac{2y(x)}{x^2-4}=0$
- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x)=-\frac{x}{x^2-4}, P_3(x)=-\frac{2}{x^2-4}\right]$
  - $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

    $\left((x+2)\cdot P_2(x)\right)\Big|_{x=-2}=-\frac{1}{2}$
  - $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

    $\left((x+2)^2\cdot P_3(x)\right)\Big|_{x=-2}=0$
  - $x=-2$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=-2$
- Multiply by denominators

  $(x^2-4)\left(\frac{d^2}{dx^2}y(x)\right)-x\left(\frac{d}{dx}y(x)\right)-2y(x)=0$
- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

  $(u^2-4u)\left(\frac{d^2}{du^2}y(u)\right)+(-u+2)\left(\frac{d}{du}y(u)\right)-2y(u)=0$
- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$
- □ Rewrite ODE with series expansions
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- ◦ Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- ◦ Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0r(-3+2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)\,(2k-1+2r) + a_k(k^2+2kr+r^2-2k-2r-\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
$-2r(-3+2r) = 0$
- Values of r that satisfy the indicial equation
$r \in \left\{0, \frac{3}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation
$-4\left(k+r-\frac{1}{2}\right)(k+1+r)\, a_{k+1} + \left(k^2+(2r-2)\, k+r^2-2r-2\right) a_k = 0$
- Recursion relation that defines series solution to ODE
$a_{k+1} = \frac{(k^2+2kr+r^2-2k-2r-2)a_k}{2(2k-1+2r)(k+1+r)}$
- Recursion relation for $r = 0$
$a_{k+1} = \frac{(k^2-2k-2)a_k}{2(2k-1)(k+1)}$
- Solution for $r = 0$
$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{(k^2-2k-2)a_k}{2(2k-1)(k+1)}\right]$
- Revert the change of variables $u = x+2$
$\left[y(x) = \sum_{k=0}^{\infty} a_k(x+2)^k, a_{k+1} = \frac{(k^2-2k-2)a_k}{2(2k-1)(k+1)}\right]$
- Recursion relation for $r = \frac{3}{2}$
$a_{k+1} = \frac{(k^2+k-\frac{11}{4})a_k}{2(2k+2)(k+\frac{5}{2})}$
- Solution for $r = \frac{3}{2}$
$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{3}{2}}, a_{k+1} = \frac{(k^2+k-\frac{11}{4})a_k}{2(2k+2)(k+\frac{5}{2})}\right]$
- Revert the change of variables $u = x+2$
$\left[y(x) = \sum_{k=0}^{\infty} a_k(x+2)^{k+\frac{3}{2}}, a_{k+1} = \frac{(k^2+k-\frac{11}{4})a_k}{2(2k+2)(k+\frac{5}{2})}\right]$
- Combine solutions and rename parameters
$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k(x+2)^k\right) + \left(\sum_{k=0}^{\infty} b_k(x+2)^{k+\frac{3}{2}}\right), a_{k+1} = \frac{(k^2-2k-2)a_k}{2(2k-1)(k+1)}, b_{k+1} = \frac{(k^2+k-\frac{11}{4})b_k}{2(2k+2)(k+\frac{5}{2})}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Legendre successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special fu
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.012 (sec)
Leaf size : 37

```
dsolve((-x^2+4)*diff(diff(y(x),x),x)+diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y = (x^2-4)^{3/4}\left(\text{LegendreP}\left(\sqrt{3}-\frac{1}{2},\frac{3}{2},\frac{x}{2}\right)c_1+\text{LegendreQ}\left(\sqrt{3}-\frac{1}{2},\frac{3}{2},\frac{x}{2}\right)c_2\right)$$

Mathematica DSolve solution

Solving time : 0.05 (sec)
Leaf size : 58

```
DSolve[{(4-x^2)*D[y[x],{x,2}]+x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to (x^2-4)^{3/4}\left(c_1P^{\frac{3}{2}}_{-\frac{1}{2}+\sqrt{3}}\left(\frac{x}{2}\right)+c_2Q^{\frac{3}{2}}_{-\frac{1}{2}+\sqrt{3}}\left(\frac{x}{2}\right)\right)$$

2.1.27 Problem 27

Internal problem ID [9199]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 27
Date solved : `Monday, January 27, 2025 at 05:52:00 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' - 4xy' + \left(-16x^2+3\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.087 (sec)

Writing the ode as

$$4x^2y'' - 4xy' + \left(-16x^2+3\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x \\ C &= -16x^2+3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.48: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{4x^2}\, dx} \\ &= z_1 e^{\frac{\ln(x)}{2}} \\ &= z_1\left(\sqrt{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}\, \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{e^{4x}}{4}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\sqrt{x}\,e^{-2x}\right) + c_2 \left(\sqrt{x}\,e^{-2x}\left(\frac{e^{4x}}{4}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2\left(\frac{d^2}{dx^2}y(x)\right) - 4x\left(\frac{d}{dx}y(x)\right) + (-16x^2+3)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{(16x^2-3)y(x)}{4x^2} + \frac{\frac{d}{dx}y(x)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) - \frac{\frac{d}{dx}y(x)}{x} - \frac{(16x^2-3)y(x)}{4x^2} = 0$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{1}{x}, P_3(x) = -\frac{16x^2-3}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{3}{4}$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) - 4x\left(\frac{d}{dx}y(x)\right) + (-16x^2+3)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ∘ Shift index using $k- >k-m$

  $x^{m}\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ∘ Convert $x\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_{k}(k+r)\,x^{k+r}$

- ∘ Convert $x^{2}\cdot\left(\frac{d^{2}}{dx^{2}}y(x)\right)$ to series expansion

  $x^{2}\cdot\left(\frac{d^{2}}{dx^{2}}y(x)\right)=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(-1+2r)\,(-3+2r)\,x^{r}+a_{1}(1+2r)\,(-1+2r)\,x^{1+r}+\left(\sum_{k=2}^{\infty}(a_{k}(2k+2r-1)\,(2k+2r-3)-16a$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)\,(-3+2r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{\frac{1}{2},\frac{3}{2}\right\}$

- Each term must be 0

  $a_{1}(1+2r)\,(-1+2r)=0$

- Solve for the dependent coefficient(s)

  $a_{1}=0$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r-\frac{1}{2}\right)\left(k+r-\frac{3}{2}\right)a_{k}-16a_{k-2}=0$

- Shift index using $k- >k+2$

  $4\left(k+\frac{3}{2}+r\right)\left(k+\frac{1}{2}+r\right)a_{k+2}-16a_{k}=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{16a_{k}}{(2k+3+2r)(2k+1+2r)}$

- Recursion relation for $r=\frac{1}{2}$

  $a_{k+2}=\frac{16a_{k}}{(2k+4)(2k+2)}$

- Solution for $r=\frac{1}{2}$

  $\left[y(x)=\sum_{k=0}^{\infty}a_{k}x^{k+\frac{1}{2}},a_{k+2}=\frac{16a_{k}}{(2k+4)(2k+2)},a_{1}=0\right]$

- Recursion relation for $r=\frac{3}{2}$

  $a_{k+2}=\frac{16a_{k}}{(2k+6)(2k+4)}$

- Solution for $r=\frac{3}{2}$

  $\left[y(x)=\sum_{k=0}^{\infty}a_{k}x^{k+\frac{3}{2}},a_{k+2}=\frac{16a_{k}}{(2k+6)(2k+4)},a_{1}=0\right]$

- Combine solutions and rename parameters

  $\left[y(x)=\left(\sum_{k=0}^{\infty}a_{k}x^{k+\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_{k}x^{k+\frac{3}{2}}\right),a_{k+2}=\frac{16a_{k}}{(2k+4)(2k+2)},a_{1}=0,b_{k+2}=\frac{16b_{k}}{(2k+6)(2k+4)},b_{1}=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 21

```
dsolve(4*x^2*diff(diff(y(x),x),x)-4*diff(y(x),x)*x+(-16*x^2+3)*y(x) = 0,y(x),singsol=a
```

$$y = \sqrt{x}\,(c_1 \sinh(2x) + c_2 \cosh(2x))$$

Mathematica DSolve solution

Solving time : 0.036 (sec)
Leaf size : 32

```
DSolve[{4*x^2*D[y[x],{x,2}]-4*x*D[y[x],x]+(3-16*x^2)*y[x]==0,{}},y[x],x,IncludeSingularSolut
```

$$y(x) \to \frac{1}{4}e^{-2x}\sqrt{x}\left(c_2 e^{4x} + 4c_1\right)$$

2.1.28 Problem 28

Internal problem ID [9200]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 28
Date solved : `Monday, January 27, 2025 at 05:52:01 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(x-1)\, y'' - x y' + y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.228 (sec)

Writing the ode as

$$(x-1)\, y'' - x y' + y = 0 \tag{1}$$
$$A y'' + B y' + C y = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x-1 \\ B &= -x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 6}{4\,(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 4x + 6 \\ t &= 4(x-1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 6}{4\,(x-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.50: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4(x-1)^2} - \frac{1}{2(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 4x + 6}{4(x-1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(x-1)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2(x-1)} + \frac{1}{2} \\
&= \frac{x-2}{2x-2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2(x-1)^2}\right) + \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)^2 - \left(\frac{x^2-4x+6}{4(x-1)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right) dx} \\
&= \frac{e^{\frac{x}{2}}}{\sqrt{x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{x-1} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x-1)}{2}} \\
&= z_1 \left(\sqrt{x-1}\, e^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x-1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{x+\ln(x-1)}}{(y_1)^2} \, dx \\
&= y_1 \left(-\frac{x\, e^{x+\ln(x-1)} e^{-2x}}{x-1}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 (e^x) + c_2 \left(e^x \left(-\frac{x\, e^{x+\ln(x-1)} e^{-2x}}{x-1}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x-1)\left(\frac{d^2}{dx^2}y(x)\right)-x\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{y(x)}{x-1}+\frac{\left(\frac{d}{dx}y(x)\right)x}{x-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)-\frac{\left(\frac{d}{dx}y(x)\right)x}{x-1}+\frac{y(x)}{x-1}=0$

□ Check to see if $x_0=1$ is a regular singular point

- Define functions

$\left[P_2(x)=-\frac{x}{x-1},P_3(x)=\frac{1}{x-1}\right]$

- $(x-1)\cdot P_2(x)$ is analytic at $x=1$

$\left((x-1)\cdot P_2(x)\right)\Big|_{x=1}=-1$

- $(x-1)^2\cdot P_3(x)$ is analytic at $x=1$

$\left((x-1)^2\cdot P_3(x)\right)\Big|_{x=1}=0$

- $x=1$is a regular singular point

Check to see if $x_0=1$ is a regular singular point

$x_0=1$

- Multiply by denominators

$(x-1)\left(\frac{d^2}{dx^2}y(x)\right)-x\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Change variables using $x=u+1$ so that the regular singular point is at $u=0$

$u\left(\frac{d^2}{du^2}y(u)\right)+(-u-1)\left(\frac{d}{du}y(u)\right)+y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-1}$

- Shift index using $k->k+1$

$u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$

Rewrite ODE with series expansions

$a_0r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(k+r-1)-a_k(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r) = 0$

- Values of r that satisfy the indicial equation
  $r \in \{0, 2\}$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_k}{k+1+r}$
- Recursion relation for $r = 0$
  $a_{k+1} = \frac{a_k}{k+1}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$
- Revert the change of variables $u = x - 1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x-1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$
- Recursion relation for $r = 2$
  $a_{k+1} = \frac{a_k}{k+3}$
- Solution for $r = 2$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$
- Revert the change of variables $u = x - 1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x-1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k (x-1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x-1)^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 12

```
dsolve((x-1)*diff(diff(y(x),x),x)-diff(y(x),x)*x+y(x) = 0,y(x),singsol=all)
```

$$y = c_1 x + e^x c_2$$

Mathematica DSolve solution

Solving time : 0.152 (sec)
Leaf size : 90

```
DSolve[{(x-1)*D[y[x],{x,2}]-x*D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \exp\left(\int_1^x \frac{K[1]-2}{2(K[1]-1)}dK[1] - \frac{1}{2}\int_1^x -\frac{K[2]}{K[2]-1}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{K[1]-2}{2(K[1]-1)}dK[1]\right)dK[3] + c_1\right)$$

2.1.29 Problem 29

Internal problem ID [9201]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 29
Date solved : `Monday, January 27, 2025 at 05:52:02 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - 2xy' + \left(x^2+2\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.125 (sec)

Writing the ode as

$$x^2y'' - 2xy' + \left(x^2+2\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.52: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\, dx} \\ &= z_1 e^{\ln(x)} \\ &= z_1(x) \end{aligned}$$

Which simplifies to

$$y_1 = x\cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2} dx}}{(y_1)^2} dx \\
&= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2} dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x\cos(x)) + c_2(x\cos(x)(\tan(x)))
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right) - 2x\left(\frac{d}{dx}y(x)\right) + (x^2+2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{(x^2+2)y(x)}{x^2} + \frac{2\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d^2}{dx^2}y(x) - \frac{2\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(x^2+2)y(x)}{x^2} = 0$
- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{2}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = -2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $x^2\left(\frac{d^2}{dx^2}y(x)\right) - 2x\left(\frac{d}{dx}y(x)\right) + (x^2+2)\,y(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$
    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- > k - m$

$$x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

$$x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

$$x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+r)\,(-2+r)\, x^r + a_1 r(-1+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)\,(k+r-2) + a_{k-2}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+r)\,(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{1, 2\}$$

- Each term must be 0

$$a_1 r(-1+r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+r-1)\,(k+r-2) + a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$a_{k+2}(k+1+r)\,(k+r) + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k}{(k+1+r)(k+r)}$$

- Recursion relation for $r = 1$

$$a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$$

- Solution for $r = 1$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$$

- Recursion relation for $r = 2$

$$a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$$

- Solution for $r = 2$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+3)(k+2)}, b_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
```

```
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 15

```
dsolve(x^2*diff(diff(y(x),x),x)-2*diff(y(x),x)*x+(x^2+2)*y(x) = 0,y(x),singsol=all)
```

$$y = x(\sin(x)\, c_1 + \cos(x)\, c_2)$$

Mathematica DSolve solution

Solving time : 0.028 (sec)
Leaf size : 33

```
DSolve[{x^2*D[y[x],{x,2}]-2*x*D[y[x],x]+(x^2+2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tr
```

$$y(x) \to c_1 e^{-ix} x - \frac{1}{2} i c_2 e^{ix} x$$

2.1.30 Problem 31

Internal problem ID [9202]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 31
Date solved : `Monday, January 27, 2025 at 05:52:02 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2-2x\right)y''+\left(-x^2+2\right)y'+(2x-2)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.314 (sec)

Writing the ode as

$$\left(x^2-2x\right)y''+\left(-x^2+2\right)y'+(2x-2)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2-2x\\B&=-x^2+2\\C&=2x-2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{x^4-8x^3+24x^2-24x+12}{4\left(x^2-2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=x^4-8x^3+24x^2-24x+12\\t&=4\left(x^2-2x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x^4-8x^3+24x^2-24x+12}{4\left(x^2-2x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.54: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{3}{4x} - \frac{1}{4(x-2)} + \frac{3}{4x^2} + \frac{3}{4(x-2)^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = 2$ let b be the coefficient of $\frac{1}{(x-2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{2}{x^3} + \frac{11}{x^4} + \frac{42}{x^5} + \frac{132}{x^6} + \frac{348}{x^7} + \frac{711}{x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4x^4 - 16x^3 + 16x^2} \\
&= Q + \frac{R}{4x^4 - 16x^3 + 16x^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{-4x^3 + 20x^2 - 24x + 12}{4x^4 - 16x^3 + 16x^2}\right) \\
&= \frac{1}{4} + \frac{-4x^3 + 20x^2 - 24x + 12}{4x^4 - 16x^3 + 16x^2}
\end{aligned}$$

Since the degree of t is 4, then we see that the coefficient of the term x^3 in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned}
b &= (-1) - (0) \\
&= -1
\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 0\right) &&= -1 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 0\right) &&= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4\left(x^2 - 2x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
2	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-}\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} - \frac{1}{2\left(x - 2\right)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2x} - \frac{1}{2\left(x - 2\right)} + \frac{1}{2} \\
&= -\frac{1}{2x} - \frac{1}{2x - 4} + \frac{1}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x} - \frac{1}{2(x-2)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2x^2} + \frac{1}{2(x-2)^2}\right) + \left(-\frac{1}{2x} - \frac{1}{2(x-2)} + \frac{1}{2}\right)^2 - \left(\frac{x^4 - 8x^3 +}{4(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2x} - \frac{1}{2(x-2)} + \frac{1}{2}\right) dx} \\ &= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x-2}\,\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x^2+2}{x^2-2x} \, dx} \\ &= z_1 e^{\frac{x}{2} + \frac{\ln(x-2)}{2} + \frac{\ln(x)}{2}} \\ &= z_1 \left(\sqrt{x-2}\,\sqrt{x}\,\mathrm{e}^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x-2}\,\sqrt{x}\,\mathrm{e}^{x}}{\sqrt{x(x-2)}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+2}{x^2-2x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{x+\ln(x-2)+\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{x\,\mathrm{e}^{x+\ln(x-2)+\ln(x)}\,\mathrm{e}^{-2x}}{x-2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\sqrt{x-2}\,\sqrt{x}\,\mathrm{e}^{x}}{\sqrt{x(x-2)}}\right) + c_2 \left(\frac{\sqrt{x-2}\,\sqrt{x}\,\mathrm{e}^{x}}{\sqrt{x(x-2)}} \left(-\frac{x\,\mathrm{e}^{x+\ln(x-2)+\ln(x)}\,\mathrm{e}^{-2x}}{x-2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x^2-2x)\left(\frac{d^2}{dx^2}y(x)\right)+(-x^2+2)\left(\frac{d}{dx}y(x)\right)+(2x-2)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{2(x-1)y(x)}{x(x-2)}+\frac{(x^2-2)\left(\frac{d}{dx}y(x)\right)}{x(x-2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)-\frac{(x^2-2)\left(\frac{d}{dx}y(x)\right)}{x(x-2)}+\frac{2(x-1)y(x)}{x(x-2)}=0$

☐ Check to see if x_0 is a regular singular point

- ∘ Define functions

  $\left[P_2(x)=-\frac{x^2-2}{x(x-2)},P_3(x)=\frac{2(x-1)}{x(x-2)}\right]$

- ∘ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\Big|_{x=0}=-1$

- ∘ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=0$

- ∘ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x(x-2)\left(\frac{d^2}{dx^2}y(x)\right)+(-x^2+2)\left(\frac{d}{dx}y(x)\right)+(2x-2)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

☐ Rewrite ODE with series expansions

- ∘ Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ∘ Shift index using $k->k-m$

  $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ∘ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..2$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ∘ Shift index using $k->k+1-m$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ∘ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=1..2$

  $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- ∘ Shift index using $k->k+2-m$

  $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$-2a_0r(-2+r)\,x^{-1+r}+(-2a_1(1+r)\,(-1+r)+a_0(1+r)\,(-2+r))\,x^r+\left(\sum_{k=1}^{\infty}(-2a_{k+1}(k+r+\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-2r(-2+r)=0$
- Values of r that satisfy the indicial equation
  $r\in\{0,2\}$
- Each term must be 0
  $-2a_1(1+r)\,(-1+r)+a_0(1+r)\,(-2+r)=0$
- Each term in the series must be 0, giving the recursion relation
  $a_k(k+r+1)\,(k+r-2)-2k^2a_{k+1}+(-4ra_{k+1}-a_{k-1})\,k-2r^2a_{k+1}-a_{k-1}r+3a_{k-1}+2a_{k+1}=$
- Shift index using $k->k+1$
  $a_{k+1}(k+2+r)\,(k+r-1)-2(k+1)^2\,a_{k+2}+(-4ra_{k+2}-a_k)\,(k+1)-2r^2a_{k+2}-ra_k+3a_k+$
- Recursion relation that defines series solution to ODE
  $a_{k+2}=\frac{k^2a_{k+1}+2kra_{k+1}+r^2a_{k+1}-ka_k+ka_{k+1}-ra_k+ra_{k+1}+2a_k-2a_{k+1}}{2(k^2+2kr+r^2+2k+2r)}$
- Recursion relation for $r=0$
  $a_{k+2}=\frac{k^2a_{k+1}-ka_k+ka_{k+1}+2a_k-2a_{k+1}}{2(k^2+2k)}$
- Series not valid for $r=0$, division by 0 in the recursion relation at $k=0$
  $a_{k+2}=\frac{k^2a_{k+1}-ka_k+ka_{k+1}+2a_k-2a_{k+1}}{2(k^2+2k)}$
- Recursion relation for $r=2$
  $a_{k+2}=\frac{k^2a_{k+1}-ka_k+5ka_{k+1}+4a_{k+1}}{2(k^2+6k+8)}$
- Solution for $r=2$
  $\left[y(x)=\sum_{k=0}^{\infty}a_kx^{k+2},a_{k+2}=\frac{k^2a_{k+1}-ka_k+5ka_{k+1}+4a_{k+1}}{2(k^2+6k+8)},-6a_1=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 14

```
dsolve((x^2-2*x)*diff(diff(y(x),x),x)+(-x^2+2)*diff(y(x),x)+(2*x-2)*y(x) = 0,y(x),sing
```

$$y=c_1x^2+e^xc_2$$

Mathematica DSolve solution

Solving time : 0.309 (sec)
Leaf size : 115

```
DSolve[{(x^2-2*x)*D[y[x],{x,2}]+(2-x^2)*D[y[x],x]+(2*x-2)*y[x]==0,{}},y[x],x,IncludeSingularSol
```

$$
y(x) \to \exp\left(\int_1^x \frac{(K[1]-4)K[1]+2}{2(K[1]-2)K[1]}dK[1] - \frac{1}{2}\int_1^x \left(-\frac{1}{K[2]} - 1 + \frac{1}{2-K[2]}\right)dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{K[1]^2-4K[1]+2}{2(K[1]-2)K[1]}dK[1]\right)dK[3] + c_1\right)
$$

2.1.31 Problem 32

Internal problem ID [9203]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 32
Date solved : `Monday, January 27, 2025 at 05:52:03 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' + \left(-8x^2+4x\right)y' + \left(4x^2-4x-1\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.105 (sec)

Writing the ode as

$$4x^2y'' + \left(-8x^2+4x\right)y' + \left(4x^2-4x-1\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -8x^2+4x \\ C &= 4x^2-4x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.56: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-8x^2+4x}{4x^2}\, dx} \\ &= z_1 e^{x - \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{e^x}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^x}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-8x^2+4x}{4x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2x-\ln(x)}}{(y_1)^2}\, dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^x}{\sqrt{x}}\right) + c_2 \left(\frac{e^x}{\sqrt{x}}(x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2\left(\frac{d^2}{dx^2}y(x)\right) + (-8x^2+4x)\left(\frac{d}{dx}y(x)\right) + (4x^2-4x-1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(4x^2-4x-1)y(x)}{4x^2} + \frac{(2x-1)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) - \frac{(2x-1)\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(4x^2-4x-1)y(x)}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2x-1}{x}, P_3(x) = \frac{4x^2-4x-1}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{1}{4}$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) - 4x(2x-1)\left(\frac{d}{dx}y(x)\right) + (4x^2-4x-1)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx} y(x)\right)$ to series expansion for $m = 1..2$

  $x^m \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)(-1+2r)\, x^r + (a_1(3+2r)(1+2r) - 4a_0(1+2r))\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)(2k+2\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3+2r)(1+2r) - 4a_0(1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = \frac{4a_0}{3+2r}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + (-8k - 8r + 4)\, a_{k-1} + 4a_{k-2} = 0$

- Shift index using $k->k+2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + (-8k - 12 - 8r)\, a_{k+1} + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{4(2ka_{k+1} + 2ra_{k+1} - a_k + 3a_{k+1})}{4k^2 + 8kr + 4r^2 + 16k + 16r + 15}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = \frac{4(2ka_{k+1} - a_k + 2a_{k+1})}{4k^2 + 12k + 8}$

- Solution for $r = -\frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = \frac{4(2ka_{k+1} - a_k + 2a_{k+1})}{4k^2 + 12k + 8}, a_1 = 2a_0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = \frac{4(2ka_{k+1} - a_k + 4a_{k+1})}{4k^2 + 20k + 24}$

- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{4(2ka_{k+1} - a_k + 4a_{k+1})}{4k^2 + 20k + 24}, a_1 = a_0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = \frac{4(2ka_{k+1} - a_k + 2a_{k+1})}{4k^2 + 12k + 8}, a_1 = 2a_0, b_{k+2} = \frac{4(2kb_{k+1} - b_k + 4b_{k+1})}{4k^2 + 20k + 24}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.022 (sec)
Leaf size : 15

```
dsolve(4*x^2*diff(diff(y(x),x),x)+(-8*x^2+4*x)*diff(y(x),x)+(4*x^2-4*x-1)*y(x) = 0,y(x
```

$$y = \frac{e^x (c_2 x + c_1)}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.031 (sec)
Leaf size : 21

```
DSolve[{4*x^2*D[y[x],{x,2}]+(4*x-8*x^2)*D[y[x],x]+(4*x^2-4*x-1)*y[x]==0,{}},y[x],x,IncludeSi
```

$$y(x) \to \frac{e^x (c_2 x + c_1)}{\sqrt{x}}$$

2.1.32 Problem 33

Internal problem ID [9204]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 33
Date solved : `Monday, January 27, 2025 at 05:52:03 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + 4xy' + \left(4x^2 + 2\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.055 (sec)

Writing the ode as

$$y'' + 4xy' + \left(4x^2 + 2\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4x \\ C &= 4x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.58: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x}{1}\, dx} \\ &= z_1 e^{-x^2} \\ &= z_1 \left(\mathrm{e}^{-x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-2x^2}}{(y_1)^2}\, dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-x^2}\right) + c_2 \left(\mathrm{e}^{-x^2}(x)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2}y(x) + 4x\left(\frac{d}{dx}y(x)\right) + (4x^2+2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$
  $x^m \cdot y(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$
- ◦ Shift index using $k- >k-m$
  $x^m \cdot y(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$
- ◦ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion
  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$
- ◦ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion
  $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$
- ◦ Shift index using $k- >k+2$
  $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions
$2a_2 + 2a_0 + (6a_3 + 6a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 2a_k(2k+1) + 4a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0
  $[2a_2 + 2a_0 = 0, 6a_3 + 6a_1 = 0]$
- Solve for the dependent coefficient(s)
  $\{a_2 = -a_0, a_3 = -a_1\}$
- Each term in the series must be 0, giving the recursion relation
  $(k^2 + 3k + 2)\, a_{k+2} + 4a_k k + 2a_k + 4a_{k-2} = 0$

- Shift index using $k- >k+2$

  $$\left((k+2)^2+3k+8\right)a_{k+4}+4a_{k+2}(k+2)+2a_{k+2}+4a_k=0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y(x)=\sum_{k=0}^{\infty}a_k x^k, a_{k+4}=-\frac{2(2ka_{k+2}+2a_k+5a_{k+2})}{k^2+7k+12}, a_2=-a_0, a_3=-a_1\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 16

```
dsolve(diff(diff(y(x),x),x)+4*diff(y(x),x)*x+(4*x^2+2)*y(x) = 0,y(x),singsol=all)
```

$$y=\mathrm{e}^{-x^2}(c_2x+c_1)$$

Mathematica DSolve solution

Solving time : 0.026 (sec)
Leaf size : 20

```
DSolve[{D[y[x],{x,2}]+4*x*D[y[x],x]+(4*x^2+2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x)\to e^{-x^2}(c_2x+c_1)$$

2.1.33 Problem 34

Internal problem ID [9205]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 34
Date solved : `Monday, January 27, 2025 at 05:52:04 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(2x+1)\,y''-2y'-(2x+3)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.224 (sec)

Writing the ode as

$$(2x+1)\,y''-2y'+(-2x-3)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x+1\\ B &= -2 \\ C &= -2x-3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{4x^2+8x+6}{(2x+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2+8x+6\\ t &= (2x+1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{4x^2+8x+6}{(2x+1)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.60: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x+1)^2$. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{3}{4\left(x + \frac{1}{2}\right)^2} + \frac{1}{x + \frac{1}{2}}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{1}{2x} - \frac{1}{4x^3} + \frac{11}{32x^4} - \frac{21}{64x^5} + \frac{15}{64x^6} - \frac{3}{32x^7} - \frac{117}{2048x^8} + \ldots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 + 8x + 6}{4x^2 + 4x + 1} \\ &= Q + \frac{R}{4x^2 + 4x + 1} \\ &= (1) + \left(\frac{4x + 5}{4x^2 + 4x + 1}\right) \\ &= 1 + \frac{4x + 5}{4x^2 + 4x + 1} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$\begin{aligned} b &= (1) - (0) \\ &= 1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{1}{1} - 0\right) &&= \frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{1}{1} - 0\right) &&= -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^2+8x+6}{(2x+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2\left(x+\frac{1}{2}\right)} + (-)(1) \\ &= -\frac{1}{2\left(x+\frac{1}{2}\right)} - 1 \\ &= -\frac{2(x+1)}{2x+1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right)(0) + \left(\left(\frac{1}{2\left(x+\frac{1}{2}\right)^2}\right) + \left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right)^2 - \left(\frac{4x^2+8x+6}{(2x+1)^2}\right) \right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2\left(x+\frac{1}{2}\right)}-1\right)dx} \\
&= \frac{\mathrm{e}^{-x}}{\sqrt{2x+1}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{2x+1}\, dx} \\
&= z_1 e^{\frac{\ln(2x+1)}{2}} \\
&= z_1 \left(\sqrt{2x+1}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{2x+1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\ln(2x+1)}}{(y_1)^2}\, dx \\
&= y_1 \left(x\, \mathrm{e}^{2x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x}\left(x\, \mathrm{e}^{2x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(2x+1)\left(\frac{d^2}{dx^2}y(x)\right) - 2\frac{d}{dx}y(x) - (2x+3)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{(2x+3)y(x)}{2x+1} + \frac{2\left(\frac{d}{dx}y(x)\right)}{2x+1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x) - \frac{2\left(\frac{d}{dx}y(x)\right)}{2x+1} - \frac{(2x+3)y(x)}{2x+1} = 0$

- ☐ Check to see if $x_0 = -\frac{1}{2}$ is a regular singular point
  - ◦ Define functions
    $\left[P_2(x) = -\frac{2}{2x+1}, P_3(x) = -\frac{2x+3}{2x+1}\right]$
  - ◦ $\left(x+\frac{1}{2}\right) \cdot P_2(x)$ is analytic at $x = -\frac{1}{2}$
    $\left(\left(x+\frac{1}{2}\right) \cdot P_2(x)\right)\Big|_{x=-\frac{1}{2}} = -1$
  - ◦ $\left(x+\frac{1}{2}\right)^2 \cdot P_3(x)$ is analytic at $x = -\frac{1}{2}$
    $\left(\left(x+\frac{1}{2}\right)^2 \cdot P_3(x)\right)\Big|_{x=-\frac{1}{2}} = 0$
  - ◦ $x = -\frac{1}{2}$is a regular singular point
    Check to see if $x_0 = -\frac{1}{2}$ is a regular singular point
    $x_0 = -\frac{1}{2}$
- • Multiply by denominators
  $(2x+1)\left(\frac{d^2}{dx^2}y(x)\right) - 2\frac{d}{dx}y(x) + (-2x-3)\,y(x) = 0$
- • Change variables using $x = u - \frac{1}{2}$ so that the regular singular point is at $u = 0$
  $2u\left(\frac{d^2}{du^2}y(u)\right) - 2\frac{d}{du}y(u) + (-2u-2)\,y(u) = 0$
- • Assume series solution for $y(u)$
  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$
- ☐ Rewrite ODE with series expansions
  - ◦ Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$
    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$
  - ◦ Shift index using $k->k-m$
    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
  - ◦ Convert $\frac{d}{du}y(u)$ to series expansion
    $\frac{d}{du}y(u) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1}$
  - ◦ Shift index using $k->k+1$
    $\frac{d}{du}y(u) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,u^{k+r}$
  - ◦ Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion
    $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-1}$
  - ◦ Shift index using $k->k+1$
    $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$
    Rewrite ODE with series expansions
    $2a_0r(-2+r)\,u^{-1+r} + (2a_1(1+r)\,(-1+r) - 2a_0)\,u^r + \left(\sum_{k=1}^{\infty} (2a_{k+1}(k+1+r)\,(k+r-1) - 2a_k\right.$
- • $a_0$cannot be 0 by assumption, giving the indicial equation
  $2r(-2+r) = 0$
- • Values of r that satisfy the indicial equation
  $r \in \{0, 2\}$

- Each term must be 0
  $2a_1(1+r)(-1+r)-2a_0=0$
- Each term in the series must be 0, giving the recursion relation
  $2a_{k+1}(k+1+r)(k+r-1)-2a_k-2a_{k-1}=0$
- Shift index using $k->k+1$
  $2a_{k+2}(k+2+r)(k+r)-2a_{k+1}-2a_k=0$
- Recursion relation that defines series solution to ODE
  $a_{k+2}=\frac{a_{k+1}+a_k}{(k+2+r)(k+r)}$
- Recursion relation for $r=0$
  $a_{k+2}=\frac{a_{k+1}+a_k}{(k+2)k}$
- Series not valid for $r=0$, division by 0 in the recursion relation at $k=0$
  $a_{k+2}=\frac{a_{k+1}+a_k}{(k+2)k}$
- Recursion relation for $r=2$
  $a_{k+2}=\frac{a_{k+1}+a_k}{(k+4)(k+2)}$
- Solution for $r=2$
  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2}=\frac{a_{k+1}+a_k}{(k+4)(k+2)}, 6a_1-2a_0=0\right]$
- Revert the change of variables $u=x+\frac{1}{2}$
  $\left[y(x)=\sum_{k=0}^{\infty} a_k\left(x+\frac{1}{2}\right)^{k+2}, a_{k+2}=\frac{a_{k+1}+a_k}{(k+4)(k+2)}, 6a_1-2a_0=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 16

```
dsolve((2*x+1)*diff(diff(y(x),x),x)-2*diff(y(x),x)-(2*x+3)*y(x) = 0,y(x),singsol=all)
```

$$y=c_1\mathrm{e}^{-x}+c_2\mathrm{e}^{x}x$$

Mathematica DSolve solution

Solving time : `0.358` (sec)
Leaf size : 69

```
DSolve[{(2*x+1)*D[y[x],{x,2}]-2*D[y[x],x]-(2*x+3)*y[x]==0,{}},y[x],x,IncludeSingularSolution
```

$$y(x) \to \sqrt{2x+1} \exp \left(\int_1^x \left(\frac{1}{-2K[1]-1} - 1 \right) dK[1] \right) \left(c_2 \int_1^x \exp \left(-2 \int_1^{K[2]} \left(\frac{1}{-2K[1]-1} - 1 \right) dK[1] \right) dK[2] + c_1 \right)$$

2.1.34 Problem 35

Internal problem ID [9206]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 35
Date solved : `Monday, January 27, 2025 at 05:52:04 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' - (2x + 2) y' + (x + 2) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.129 (sec)

Writing the ode as

$$xy'' + (-2x - 2) y' + (x + 2) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -2x - 2 \\ C &= x + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.62: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the gcd$(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned}d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right)\\ &= -1 - (-1)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty\\ &= -\frac{1}{x} + (-)(0)\\ &= -\frac{1}{x}\\ &= -\frac{1}{x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) &= 0\\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{1}{x} dx} \\
&= \frac{1}{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x-2}{x} \, dx} \\
&= z_1 e^{x+\ln(x)} \\
&= z_1 (x \, \mathrm{e}^{x})
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-2}{x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{2x+2\ln(x)}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{x \, \mathrm{e}^{2x+2\ln(x)} \mathrm{e}^{-2x}}{3} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^{x}) + c_2 \left(\mathrm{e}^{x} \left(\frac{x \, \mathrm{e}^{2x+2\ln(x)} \mathrm{e}^{-2x}}{3} \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dx^2} y(x)\right) x - (2x+2) \left(\frac{d}{dx} y(x)\right) + (x+2) \, y(x) = 0$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dx^2} y(x)$
- Isolate 2nd derivative
$\frac{d^2}{dx^2} y(x) = -\frac{(x+2)y(x)}{x} + \frac{2(x+1)\left(\frac{d}{dx} y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x) - \frac{2(x+1)\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(x+2)y(x)}{x} = 0$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - ◦ Define functions

    $\left[P_2(x) = -\frac{2(x+1)}{x}, P_3(x) = \frac{x+2}{x}\right]$
  - ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -2$
  - ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$
  - ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- • Multiply by denominators

  $\left(\frac{d^2}{dx^2}y(x)\right)x + (-2-2x)\left(\frac{d}{dx}y(x)\right) + (x+2)\,y(x) = 0$
- • Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- □ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - ◦ Shift index using $k- >k-m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..1$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - ◦ Shift index using $k- >k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - ◦ Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$
  - ◦ Shift index using $k- >k+1$

    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

    Rewrite ODE with series expansions

    $a_0 r(-3+r)\,x^{-1+r} + (a_1(1+r)\,(-2+r) - 2a_0(-1+r))\,x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(k-2+r) - 2\right.$
- • $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+r) = 0$
- • Values of r that satisfy the indicial equation

  $r \in \{0, 3\}$
- • Each term must be 0

  $a_1(1+r)\,(-2+r) - 2a_0(-1+r) = 0$
- • Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k-2+r) - 2a_k k - 2a_k r + 2a_k + a_{k-1} = 0$

- Shift index using $k- >k+1$

  $a_{k+2}(k+2+r)(k+r-1)-2a_{k+1}(k+1)-2ra_{k+1}+2a_{k+1}+a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{2ka_{k+1}+2ra_{k+1}-a_k}{(k+2+r)(k+r-1)}$
- Recursion relation for $r=0$

  $a_{k+2}=\frac{2ka_{k+1}-a_k}{(k+2)(k-1)}$
- Series not valid for $r=0$, division by 0 in the recursion relation at $k=1$

  $a_{k+2}=\frac{2ka_{k+1}-a_k}{(k+2)(k-1)}$
- Recursion relation for $r=3$

  $a_{k+2}=\frac{2ka_{k+1}-a_k+6a_{k+1}}{(k+5)(k+2)}$
- Solution for $r=3$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2}=\frac{2ka_{k+1}-a_k+6a_{k+1}}{(k+5)(k+2)}, 4a_1-4a_0=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 14

```
dsolve(x*diff(diff(y(x),x),x)-(2+2*x)*diff(y(x),x)+(x+2)*y(x) = 0,y(x),singsol=all)
```

$$y = e^x (c_2 x^3 + c_1)$$

Mathematica DSolve solution

Solving time : 0.047 (sec)
Leaf size : 25

```
DSolve[{x*D[y[x],{x,2}]-(2*x+2)*D[y[x],x]+(x+2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to \frac{1}{3} e^{x+1} (c_2 x^3 + 3c_1)$$

2.1.35 Problem 36

Internal problem ID [9207]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 36
Date solved : `Monday, January 27, 2025 at 05:52:05 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - 2xy' + (x^2 + 2)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.126 (sec)

Writing the ode as

$$x^2y'' - 2xy' + (x^2 + 2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.64: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\, dx} \\ &= z_1 e^{\ln(x)} \\ &= z_1(x) \end{aligned}$$

Which simplifies to

$$y_1 = x\cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x\cos(x)) + c_2(x\cos(x)(\tan(x)))
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) - 2x\left(\frac{d}{dx}y(x)\right) + (x^2+2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(x^2+2)y(x)}{x^2} + \frac{2\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) - \frac{2\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(x^2+2)y(x)}{x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{2}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) - 2x\left(\frac{d}{dx}y(x)\right) + (x^2+2)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k - m$

$$x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

$$x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

$$x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+r)\,(-2+r)\, x^r + a_1 r(-1+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-1)\,(k+r-2) + a_{k-2})\, x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
$(-1+r)\,(-2+r) = 0$
- Values of r that satisfy the indicial equation
$r \in \{1, 2\}$
- Each term must be 0
$a_1 r(-1+r) = 0$
- Solve for the dependent coefficient(s)
$a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
$a_k(k+r-1)\,(k+r-2) + a_{k-2} = 0$
- Shift index using $k- >k+2$
$a_{k+2}(k+1+r)\,(k+r) + a_k = 0$
- Recursion relation that defines series solution to ODE
$a_{k+2} = -\frac{a_k}{(k+1+r)(k+r)}$
- Recursion relation for $r=1$
$a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$
- Solution for $r=1$
$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$
- Recursion relation for $r=2$
$a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$
- Solution for $r=2$
$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$
- Combine solutions and rename parameters
$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+3)(k+2)}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
```

```
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 15

```
dsolve(x^2*diff(diff(y(x),x),x)-2*diff(y(x),x)*x+(x^2+2)*y(x) = 0,y(x),singsol=all)
```

$$y = x(\sin(x)\, c_1 + \cos(x)\, c_2)$$

Mathematica DSolve solution

Solving time : 0.029 (sec)
Leaf size : 33

```
DSolve[{x^2*D[y[x],{x,2}]-2*x*D[y[x],x]+(x^2+2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tr
```

$$y(x) \to c_1 e^{-ix} x - \frac{1}{2} i c_2 e^{ix} x$$

2.1.36 Problem 38

Internal problem ID [9208]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 38
Date solved : `Monday, January 27, 2025 at 05:52:06 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' - 4xy' + \left(-16x^2+3\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.089 (sec)

Writing the ode as

$$4x^2y'' - 4xy' + \left(-16x^2+3\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x \\ C &= -16x^2+3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.66: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{4x^2}\, dx} \\ &= z_1 e^{\frac{\ln(x)}{2}} \\ &= z_1\left(\sqrt{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}\,\mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{e^{4x}}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\sqrt{x}\, e^{-2x}\right) + c_2 \left(\sqrt{x}\, e^{-2x} \left(\frac{e^{4x}}{4}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$4x^2 \left(\frac{d^2}{dx^2}y(x)\right) - 4x\left(\frac{d}{dx}y(x)\right) + (-16x^2+3)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = \frac{(16x^2-3)y(x)}{4x^2} + \frac{\frac{d}{dx}y(x)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x) - \frac{\frac{d}{dx}y(x)}{x} - \frac{(16x^2-3)y(x)}{4x^2} = 0$
- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{1}{x}, P_3(x) = -\frac{16x^2-3}{4x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = -1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{3}{4}$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $4x^2 \left(\frac{d^2}{dx^2}y(x)\right) - 4x\left(\frac{d}{dx}y(x)\right) + (-16x^2+3)\,y(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$
    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx} y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)\,(-3+2r)\, x^r + a_1(1+2r)\,(-1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r-1)\,(2k+2r-3) - 16a\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)\,(-3+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$
- Each term must be 0

  $a_1(1+2r)\,(-1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r-\frac{1}{2}\right)\left(k+r-\frac{3}{2}\right) a_k - 16a_{k-2} = 0$
- Shift index using $k- >k+2$

  $4\left(k+\frac{3}{2}+r\right)\left(k+\frac{1}{2}+r\right) a_{k+2} - 16a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{16a_k}{(2k+3+2r)(2k+1+2r)}$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = \frac{16a_k}{(2k+4)(2k+2)}$
- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{16a_k}{(2k+4)(2k+2)}, a_1 = 0\right]$
- Recursion relation for $r = \frac{3}{2}$

  $a_{k+2} = \frac{16a_k}{(2k+6)(2k+4)}$
- Solution for $r = \frac{3}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2} = \frac{16a_k}{(2k+6)(2k+4)}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+2} = \frac{16a_k}{(2k+4)(2k+2)}, a_1 = 0, b_{k+2} = \frac{16b_k}{(2k+6)(2k+4)}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 21

```
dsolve(4*x^2*diff(diff(y(x),x),x)-4*diff(y(x),x)*x+(-16*x^2+3)*y(x) = 0,y(x),singsol=a
```

$$y = \sqrt{x}\, (c_1 \sinh (2x) + c_2 \cosh (2x))$$

Mathematica DSolve solution

Solving time : 0.035 (sec)
Leaf size : 32

```
DSolve[{4*x^2*D[y[x],{x,2}]-4*x*D[y[x],x]+(3-16*x^2)*y[x]==0,{}},y[x],x,IncludeSingularSolut
```

$$y(x) \to \frac{1}{4} e^{-2x} \sqrt{x} \left(c_2 e^{4x} + 4c_1\right)$$

2.1.37 Problem 39

Internal problem ID [9209]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 39
Date solved : `Monday, January 27, 2025 at 05:52:06 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' - 4xy' + \left(4x^2+3\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.142 (sec)

Writing the ode as

$$4x^2y'' - 4xy' + \left(4x^2+3\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x \\ C &= 4x^2+3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.68: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{4x^2}\, dx} \\ &= z_1 e^{\frac{\ln(x)}{2}} \\ &= z_1\left(\sqrt{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}\, \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{4x^2} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2} dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\sqrt{x}\,\cos(x)\right) + c_2\left(\sqrt{x}\,\cos(x)\,(\tan(x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$4x^2\left(\frac{d^2}{dx^2}y(x)\right) - 4x\left(\frac{d}{dx}y(x)\right) + (4x^2+3)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{(4x^2+3)y(x)}{4x^2} + \frac{\frac{d}{dx}y(x)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) - \frac{\frac{d}{dx}y(x)}{x} + \frac{(4x^2+3)y(x)}{4x^2} = 0$
- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{1}{x}, P_3(x) = \frac{4x^2+3}{4x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = -1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{3}{4}$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) - 4x\left(\frac{d}{dx}y(x)\right) + (4x^2+3)\,y(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$
    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k - m$

$x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx} y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)\,(-3+2r)\, x^r + a_1(1+2r)\,(-1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r-1)\,(2k+2r-3) + 4\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)\,(-3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$

- Each term must be 0

  $a_1(1+2r)\,(-1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r-\frac{1}{2}\right)\left(k+r-\frac{3}{2}\right) a_k + 4a_{k-2} = 0$

- Shift index using $k- >k+2$

  $4\left(k+\frac{3}{2}+r\right)\left(k+\frac{1}{2}+r\right) a_{k+2} + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{(2k+3+2r)(2k+1+2r)}$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{(2k+4)(2k+2)}$

- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{(2k+4)(2k+2)}, a_1 = 0\right]$

- Recursion relation for $r = \frac{3}{2}$

  $a_{k+2} = -\frac{4a_k}{(2k+6)(2k+4)}$

- Solution for $r = \frac{3}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2} = -\frac{4a_k}{(2k+6)(2k+4)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+2} = -\frac{4a_k}{(2k+4)(2k+2)}, a_1 = 0, b_{k+2} = -\frac{4b_k}{(2k+6)(2k+4)}, b_1 =\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 17

```
dsolve(4*x^2*diff(diff(y(x),x),x)-4*diff(y(x),x)*x+(4*x^2+3)*y(x) = 0,y(x),singsol=all)
```

$$y = \sqrt{x}\,(\sin(x)\,c_1 + \cos(x)\,c_2)$$

Mathematica DSolve solution

Solving time : 0.032 (sec)
Leaf size : 39

```
DSolve[{4*x^2*D[y[x],{x,2}]-4*x*D[y[x],x]+(4*x^2+3)*y[x]==0,{}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{1}{2}e^{-ix}\sqrt{x}\left(2c_1 - ic_2e^{2ix}\right)$$

2.1.38 Problem 40

Internal problem ID [9210]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 40
Date solved : `Monday, January 27, 2025 at 05:52:07 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - 2xy' - \left(x^2-2\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.072 (sec)

Writing the ode as

$$x^2y'' - 2xy' + \left(-x^2+2\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= -x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.70: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\, dx} \\ &= z_1 e^{\ln(x)} \\ &= z_1(x) \end{aligned}$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{e^{2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x\, e^{-x}\right) + c_2 \left(x\, e^{-x} \left(\frac{e^{2x}}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right) - 2x\left(\frac{d}{dx}y(x)\right) - (x^2-2)\, y(x) = 0$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
$\frac{d^2}{dx^2}y(x) = \frac{(x^2-2)y(x)}{x^2} + \frac{2\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
$\frac{d^2}{dx^2}y(x) - \frac{2\left(\frac{d}{dx}y(x)\right)}{x} - \frac{(x^2-2)y(x)}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
  $\left[P_2(x) = -\frac{2}{x}, P_3(x) = -\frac{x^2-2}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
  $\left.(x \cdot P_2(x))\right|_{x=0} = -2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$
  - $x = 0$is a regular singular point
  Check to see if $x_0 = 0$ is a regular singular point
  $x_0 = 0$
- Multiply by denominators
$x^2\left(\frac{d^2}{dx^2}y(x)\right) - 2x\left(\frac{d}{dx}y(x)\right) + (-x^2+2)\, y(x) = 0$
- Assume series solution for $y(x)$
$y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

$x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)\,(-2+r)\, x^r + a_1 r(-1+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)\,(k+r-2) - a_{k-2}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$

- Each term must be 0

  $a_1 r(-1+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)\,(k+r-2) - a_{k-2} = 0$

- Shift index using $k- >k+2$

  $a_{k+2}(k+1+r)\,(k+r) - a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k}{(k+1+r)(k+r)}$

- Recursion relation for $r = 1$

  $a_{k+2} = \frac{a_k}{(k+2)(k+1)}$

- Solution for $r = 1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = \frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$

- Recursion relation for $r = 2$

  $a_{k+2} = \frac{a_k}{(k+3)(k+2)}$

- Solution for $r = 2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = \frac{a_k}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = \frac{b_k}{(k+3)(k+2)}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 15

```
dsolve(x^2*diff(diff(y(x),x),x)-2*diff(y(x),x)*x-(x^2-2)*y(x) = 0,y(x),singsol=all)
```

$$y = x(c_1 \sinh(x) + c_2 \cosh(x))$$

Mathematica DSolve solution

Solving time : 0.028 (sec)
Leaf size : 25

```
DSolve[{x^2*D[y[x],{x,2}]-2*x*D[y[x],x]-(x^2-2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to c_1 e^{-x} x + \frac{1}{2} c_2 e^x x$$

2.1.39 Problem 41

Internal problem ID [9211]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 41
Date solved : `Monday, January 27, 2025 at 05:52:07 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - 2x(x+1)\,y' + \left(x^2+2x+2\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.077 (sec)

Writing the ode as

$$x^2y'' + \left(-2x^2-2x\right)y' + \left(x^2+2x+2\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x^2-2x \\ C &= x^2+2x+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.72: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x^2-2x}{x^2}\, dx} \\ &= z_1 e^{x+\ln(x)} \\ &= z_1 (x\, \mathrm{e}^{x}) \end{aligned}$$

Which simplifies to

$$y_1 = x\, \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^2-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x+2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x\,\mathrm{e}^{x}) + c_2(x\,\mathrm{e}^{x}(x))
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right) - 2x(x+1)\left(\frac{d}{dx}y(x)\right) + (x^2+2x+2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{(x^2+2x+2)y(x)}{x^2} + \frac{2(x+1)\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) - \frac{2(x+1)\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(x^2+2x+2)y(x)}{x^2} = 0$
- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{2(x+1)}{x}, P_3(x) = \frac{x^2+2x+2}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = -2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $x^2\left(\frac{d^2}{dx^2}y(x)\right) - 2x(x+1)\left(\frac{d}{dx}y(x)\right) + (x^2+2x+2)\,y(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$
    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k-m$

$x^{m}\cdot y(x)=\sum_{k=m}^{\infty} a_{k-m}x^{k+r}$

- Convert $x^{m}\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

  $x^{m}\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty} a_{k}(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m}\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^{2}\cdot\left(\frac{d^{2}}{dx^{2}}y(x)\right)$ to series expansion

  $x^{2}\cdot\left(\frac{d^{2}}{dx^{2}}y(x)\right)=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(-1+r)\,(-2+r)\,x^{r}+\left(a_{1}r(-1+r)-2a_{0}(-1+r)\right)x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_{k}(k+r-1)\,(k+r-2)-\right.\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation
  $(-1+r)\,(-2+r)=0$
- Values of r that satisfy the indicial equation
  $r\in\{1,2\}$
- Each term must be 0
  $a_{1}r(-1+r)-2a_{0}(-1+r)=0$
- Solve for the dependent coefficient(s)
  $a_{1}=\frac{2a_{0}}{r}$
- Each term in the series must be 0, giving the recursion relation
  $a_{k}(k+r-1)\,(k+r-2)-2a_{k-1}k-2a_{k-1}r+a_{k-2}+4a_{k-1}=0$
- Shift index using $k->k+2$
  $a_{k+2}(k+1+r)\,(k+r)-2a_{k+1}(k+2)-2a_{k+1}r+a_{k}+4a_{k+1}=0$
- Recursion relation that defines series solution to ODE
  $a_{k+2}=\frac{2ka_{k+1}+2a_{k+1}r-a_{k}}{(k+1+r)(k+r)}$
- Recursion relation for $r=1$
  $a_{k+2}=\frac{2ka_{k+1}-a_{k}+2a_{k+1}}{(k+2)(k+1)}$
- Solution for $r=1$
  $\left[y(x)=\sum_{k=0}^{\infty} a_{k}x^{k+1},a_{k+2}=\frac{2ka_{k+1}-a_{k}+2a_{k+1}}{(k+2)(k+1)},a_{1}=2a_{0}\right]$
- Recursion relation for $r=2$
  $a_{k+2}=\frac{2ka_{k+1}-a_{k}+4a_{k+1}}{(k+3)(k+2)}$
- Solution for $r=2$
  $\left[y(x)=\sum_{k=0}^{\infty} a_{k}x^{k+2},a_{k+2}=\frac{2ka_{k+1}-a_{k}+4a_{k+1}}{(k+3)(k+2)},a_{1}=a_{0}\right]$
- Combine solutions and rename parameters
  $\left[y(x)=\left(\sum_{k=0}^{\infty} a_{k}x^{k+1}\right)+\left(\sum_{k=0}^{\infty} b_{k}x^{k+2}\right),a_{k+2}=\frac{2ka_{k+1}-a_{k}+2a_{k+1}}{(k+2)(k+1)},a_{1}=2a_{0},b_{k+2}=\frac{2kb_{k+1}-b_{k}+4b_{k+1}}{(k+3)(k+2)},\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 13

```
dsolve(x^2*diff(diff(y(x),x),x)-2*x*(x+1)*diff(y(x),x)+(x^2+2*x+2)*y(x) = 0,y(x),singsol
```

$$y = e^x x(c_2 x + c_1)$$

Mathematica DSolve solution

Solving time : 0.045 (sec)
Leaf size : 19

```
DSolve[{x^2*D[y[x],{x,2}]-2*x*(x+1)*D[y[x],x]+(x^2+2*x+2)*y[x]==0,{}},y[x],x,IncludeSingularSo
```

$$y(x) \to e^{x+1} x(c_2 x + c_1)$$

2.1.40 Problem 42

Internal problem ID [9212]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 42
Date solved : `Monday, January 27, 2025 at 05:52:08 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - 2x(x+2)\,y' + \left(x^2+4x+6\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.097 (sec)

Writing the ode as

$$x^2y'' + \left(-2x^2-4x\right)y' + \left(x^2+4x+6\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x^2-4x \\ C &= x^2+4x+6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.74: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2x^2-4x}{x^2}\, dx} \\ &= z_1 e^{x+2\ln(x)} \\ &= z_1 \left(x^2 \mathrm{e}^x\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^2 \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^2-4x}{x^2} dx}}{(y_1)^2} dx \\
&= y_1 \int \frac{e^{2x+4\ln(x)}}{(y_1)^2} dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 (x^2 e^x) + c_2 (x^2 e^x (x))
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2 \left(\frac{d^2}{dx^2} y(x)\right) - 2x(x+2) \left(\frac{d}{dx} y(x)\right) + (x^2+4x+6)\, y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2} y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2} y(x) = -\frac{(x^2+4x+6) y(x)}{x^2} + \frac{2(x+2)\left(\frac{d}{dx} y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2} y(x) - \frac{2(x+2)\left(\frac{d}{dx} y(x)\right)}{x} + \frac{(x^2+4x+6) y(x)}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{2(x+2)}{x}, P_3(x) = \frac{x^2+4x+6}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -4$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 6$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 \left(\frac{d^2}{dx^2} y(x)\right) - 2x(x+2) \left(\frac{d}{dx} y(x)\right) + (x^2+4x+6)\, y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

$x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx} y(x)\right)$ to series expansion for $m = 1..2$

  $x^m \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k- > k+1-m$

  $x^m \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)(-3+r)\, x^r + (a_1(-1+r)(-2+r) - 2a_0(-2+r))\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-2)(k+r-\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{2, 3\}$

- Each term must be 0

  $a_1(-1+r)(-2+r) - 2a_0(-2+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = \frac{2a_0}{-1+r}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-2)(k+r-3) - 2a_{k-1}k - 2a_{k-1}r + a_{k-2} + 6a_{k-1} = 0$

- Shift index using $k- > k+2$

  $a_{k+2}(k+r)(k+r-1) - 2a_{k+1}(k+2) - 2a_{k+1}r + a_k + 6a_{k+1} = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2k a_{k+1} + 2a_{k+1} r - a_k - 2a_{k+1}}{(k+r)(k+r-1)}$

- Recursion relation for $r = 2$

  $a_{k+2} = \frac{2k a_{k+1} - a_k + 2a_{k+1}}{(k+2)(k+1)}$

- Solution for $r = 2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{2k a_{k+1} - a_k + 2a_{k+1}}{(k+2)(k+1)}, a_1 = 2a_0\right]$

- Recursion relation for $r = 3$

  $a_{k+2} = \frac{2k a_{k+1} - a_k + 4a_{k+1}}{(k+3)(k+2)}$

- Solution for $r = 3$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = \frac{2k a_{k+1} - a_k + 4a_{k+1}}{(k+3)(k+2)}, a_1 = a_0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), a_{k+2} = \frac{2k a_{k+1} - a_k + 2a_{k+1}}{(k+2)(k+1)}, a_1 = 2a_0, b_{k+2} = \frac{2k b_{k+1} - b_k + 4b_{k+1}}{(k+3)(k+2)}, b_1\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 15

```
dsolve(x^2*diff(diff(y(x),x),x)-2*x*(x+2)*diff(y(x),x)+(x^2+4*x+6)*y(x) = 0,y(x),sings
```

$$y = e^x x^2 (c_2 x + c_1)$$

Mathematica DSolve solution

Solving time : 0.049 (sec)
Leaf size : 21

```
DSolve[{x^2*D[y[x],{x,2}]-2*x*(x+2)*D[y[x],x]+(x^2+4*x+6)*y[x]==0,{}},y[x],x,IncludeSingular
```

$$y(x) \to e^{x+2} x^2 (c_2 x + c_1)$$

2.1.41 Problem 43

Internal problem ID [9213]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 43
Date solved : `Monday, January 27, 2025 at 05:52:08 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - 4xy' + \left(x^2+6\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.137 (sec)

Writing the ode as

$$x^2y'' - 4xy' + \left(x^2+6\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -4x \\ C &= x^2+6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.76: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{x^2}\, dx} \\ &= z_1 e^{2\ln(x)} \\ &= z_1\left(x^2\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^2 \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{4\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(x^2\cos(x)\right) + c_2\left(x^2\cos(x)(\tan(x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right) - 4x\left(\frac{d}{dx}y(x)\right) + (x^2+6)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{(x^2+6)y(x)}{x^2} + \frac{4\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) - \frac{4\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(x^2+6)y(x)}{x^2} = 0$
- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{4}{x}, P_3(x) = \frac{x^2+6}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = -4$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 6$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $x^2\left(\frac{d^2}{dx^2}y(x)\right) - 4x\left(\frac{d}{dx}y(x)\right) + (x^2+6)\,y(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$
    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k - m$

$x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx} y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)\,(-3+r)\,x^r + a_1(-1+r)\,(-2+r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-2)\,(k+r-3) + a_{k-2})\, x^{k}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)\,(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{2, 3\}$

- Each term must be 0

  $a_1(-1+r)\,(-2+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-2)\,(k+r-3) + a_{k-2} = 0$

- Shift index using $k- >k+2$

  $a_{k+2}(k+r)\,(k+r-1) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+r)(k+r-1)}$

- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$

- Solution for $r = 2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$

- Recursion relation for $r = 3$

  $a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$

- Solution for $r = 3$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+3)(k+2)}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
```

```
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)-4*diff(y(x),x)*x+(x^2+6)*y(x) = 0,y(x),singsol=all)
```

$$y = x^2(\sin(x) c_1 + \cos(x) c_2)$$

Mathematica DSolve solution

Solving time : 0.03 (sec)
Leaf size : 37

```
DSolve[{x^2*D[y[x],{x,2}]-4*x*D[y[x],x]+(x^2+6)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tr
```

$$y(x) \to \frac{1}{2} e^{-ix} x^2 \left(2c_1 - ic_2 e^{2ix}\right)$$

2.1.42 Problem 44

Internal problem ID [9214]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 44
Date solved : `Monday, January 27, 2025 at 05:52:09 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(x-1)\,y''-xy'+y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.243 (sec)

Writing the ode as

$$(x-1)\,y''-xy'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x-1\\ B&=-x\\ C&=1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{x^2-4x+6}{4\,(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=x^2-4x+6\\ t&=4(x-1)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x^2-4x+6}{4\,(x-1)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.78: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4(x-1)^2} - \frac{1}{2(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(x-1)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2(x-1)} + \frac{1}{2} \\
&= \frac{x-2}{2x-2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2\left(x-1\right)}+\frac{1}{2}\right)(0)+\left(\left(\frac{1}{2\left(x-1\right)^{2}}\right)+\left(-\frac{1}{2\left(x-1\right)}+\frac{1}{2}\right)^{2}-\left(\frac{x^{2}-4x+6}{4\left(x-1\right)^{2}}\right)\right)=0$$

$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x-1)}+\frac{1}{2}\right)dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{x-1}\, dx} \\
&= z_1 e^{\frac{x}{2}+\frac{\ln(x-1)}{2}} \\
&= z_1 \left(\sqrt{x-1}\, \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x-1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x+\ln(x-1)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{x\, \mathrm{e}^{x+\ln(x-1)}\mathrm{e}^{-2x}}{x-1}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^{x}) + c_2\left(\mathrm{e}^{x}\left(-\frac{x\, \mathrm{e}^{x+\ln(x-1)}\mathrm{e}^{-2x}}{x-1}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x-1)\left(\frac{d^2}{dx^2}y(x)\right)-x\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{y(x)}{x-1}+\frac{\left(\frac{d}{dx}y(x)\right)x}{x-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)-\frac{\left(\frac{d}{dx}y(x)\right)x}{x-1}+\frac{y(x)}{x-1}=0$

□ Check to see if $x_0=1$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{x}{x-1}, P_3(x)=\frac{1}{x-1}\right]$

  ◦ $(x-1)\cdot P_2(x)$ is analytic at $x=1$

  $\left((x-1)\cdot P_2(x)\right)\bigg|_{x=1}=-1$

  ◦ $(x-1)^2\cdot P_3(x)$ is analytic at $x=1$

  $\left((x-1)^2\cdot P_3(x)\right)\bigg|_{x=1}=0$

  ◦ $x=1$is a regular singular point

  Check to see if $x_0=1$ is a regular singular point

  $x_0=1$

- Multiply by denominators

  $(x-1)\left(\frac{d^2}{dx^2}y(x)\right)-x\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Change variables using $x=u+1$ so that the regular singular point is at $u=0$

  $u\left(\frac{d^2}{du^2}y(u)\right)+(-u-1)\left(\frac{d}{du}y(u)\right)+y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  ◦ Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-1}$

  ◦ Shift index using $k->k+1$

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)(k+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)(k+r-1)-a_k(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r) = 0$

- Values of r that satisfy the indicial equation
  $r \in \{0, 2\}$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-1)\,(a_{k+1}(k+1+r) - a_k) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_k}{k+1+r}$
- Recursion relation for $r = 0$
  $a_{k+1} = \frac{a_k}{k+1}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$
- Revert the change of variables $u = x - 1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x-1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$
- Recursion relation for $r = 2$
  $a_{k+1} = \frac{a_k}{k+3}$
- Solution for $r = 2$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$
- Revert the change of variables $u = x - 1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x-1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k (x-1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x-1)^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 12

```
dsolve((x-1)*diff(diff(y(x),x),x)-diff(y(x),x)*x+y(x) = 0,y(x),singsol=all)
```

$$y = c_1 x + e^x c_2$$

Mathematica DSolve solution

Solving time : 0.153 (sec)
Leaf size : 90

```
DSolve[{(x-1)*D[y[x],{x,2}]-x*D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to \exp\left(\int_1^x \frac{K[1]-2}{2(K[1]-1)}dK[1]-\frac{1}{2}\int_1^x -\frac{K[2]}{K[2]-1}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{K[1]-2}{2(K[1]-1)}dK[1]\right)dK[3]+c_1\right)$$

2.1.43 Problem 45

Internal problem ID [9215]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 45
Date solved : `Monday, January 27, 2025 at 05:52:09 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' - 4x(x+1)\,y' + (2x+3)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.100 (sec)

Writing the ode as

$$4x^2y'' + \left(-4x^2 - 4x\right)y' + (2x+3)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x^2 - 4x \\ C &= 2x+3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.80: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-4x^2 - 4x}{4x^2}\, dx} \\ &= z_1 e^{\frac{x}{2} + \frac{\ln(x)}{2}} \\ &= z_1 \left(\sqrt{x}\, \mathrm{e}^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^2-4x}{4x^2} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{x+\ln(x)}}{(y_1)^2} dx \\ &= y_1 \left(\frac{e^{x+\ln(x)}}{x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\sqrt{x}\right) + c_2 \left(\sqrt{x} \left(\frac{e^{x+\ln(x)}}{x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2 \left(\frac{d^2}{dx^2} y(x)\right) - 4x(x+1) \left(\frac{d}{dx} y(x)\right) + (2x+3)\, y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2} y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2} y(x) = -\frac{(2x+3)y(x)}{4x^2} + \frac{(x+1)\left(\frac{d}{dx} y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2} y(x) - \frac{(x+1)\left(\frac{d}{dx} y(x)\right)}{x} + \frac{(2x+3)y(x)}{4x^2} = 0$
- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{x+1}{x}, P_3(x) = \frac{2x+3}{4x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{3}{4}$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $4x^2 \left(\frac{d^2}{dx^2} y(x)\right) - 4x(x+1) \left(\frac{d}{dx} y(x)\right) + (2x+3)\, y(x) = 0$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ◦ Shift index using $k- > k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- ◦ Shift index using $k- > k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- ◦ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)\,(-3+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)\,(2k+2r-3) - 2a_{k-1}(2k+2r-3)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)\,(-3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $4\left(\left(k+r-\frac{1}{2}\right) a_k - a_{k-1}\right)\left(k+r-\frac{3}{2}\right) = 0$

- Shift index using $k- > k+1$

  $4\left(\left(k+\frac{1}{2}+r\right) a_{k+1} - a_k\right)\left(k+r-\frac{1}{2}\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k}{2k+1+2r}$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = \frac{2a_k}{2k+2}$

- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{2a_k}{2k+2}\right]$

- Recursion relation for $r = \frac{3}{2}$

  $a_{k+1} = \frac{2a_k}{2k+4}$

- Solution for $r = \frac{3}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1} = \frac{2a_k}{2k+4}\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+1} = \frac{2a_k}{2k+2}, b_{k+1} = \frac{2b_k}{2k+4}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 14

```
dsolve(4*x^2*diff(diff(y(x),x),x)-4*x*(x+1)*diff(y(x),x)+(2*x+3)*y(x) = 0,y(x),singsol
```

$$y = (c_1 + e^x c_2)\sqrt{x}$$

Mathematica DSolve solution

Solving time : 0.046 (sec)
Leaf size : 25

```
DSolve[{4*x^2*D[y[x],{x,2}]-4*x*(x+1)*D[y[x],x]+(2*x+3)*y[x]==0,{}},y[x],x,IncludeSingularSo
```

$$y(x) \to \sqrt{e}\sqrt{x}(c_2 e^x + c_1)$$

2.1.44 Problem 46

Internal problem ID [9216]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 46
Date solved : `Monday, January 27, 2025 at 05:52:10 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(3x-1)\,y''-(3x+2)\,y'-(6x-8)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.254 (sec)

Writing the ode as

$$(3x-1)\,y''+(-3x-2)\,y'+(-6x+8)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x-1 \\ B &= -3x-2 \\ C &= -6x+8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{81x^2-108x+54}{4\,(3x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 81x^2-108x+54 \\ t &= 4(3x-1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{81x^2-108x+54}{4\,(3x-1)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.82: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(3x-1)^2$. There is a pole at $x = \frac{1}{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{4} + \frac{3}{4\left(x-\frac{1}{3}\right)^2} - \frac{3}{2\left(x-\frac{1}{3}\right)}$$

For the pole at $x = \frac{1}{3}$ let b be the coefficient of $\frac{1}{\left(x-\frac{1}{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3}{2} - \frac{1}{2x} + \frac{1}{9x^3} + \frac{11}{108x^4} + \frac{7}{108x^5} + \frac{5}{162x^6} + \frac{2}{243x^7} - \frac{13}{3888x^8} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{3}{2} \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{81x^2 - 108x + 54}{36x^2 - 24x + 4} \\
&= Q + \frac{R}{36x^2 - 24x + 4} \\
&= \left(\frac{9}{4}\right) + \left(\frac{-54x + 45}{36x^2 - 24x + 4}\right) \\
&= \frac{9}{4} + \frac{-54x + 45}{36x^2 - 24x + 4}
\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -54. Dividing this by leading coefficient in t which is 36 gives $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{3}{2}\right) - (0) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{3}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{3}{2}} - 0\right) &= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{3}{2}} - 0\right) &= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{81x^2 - 108x + 54}{4(3x-1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{1}{3}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(x-\frac{1}{3}\right)} + \left(\frac{3}{2}\right) \\
&= -\frac{1}{2\left(x-\frac{1}{3}\right)} + \frac{3}{2} \\
&= \frac{9x-6}{6x-2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2\left(x-\frac{1}{3}\right)}+\frac{3}{2}\right)(0)+\left(\left(\frac{1}{2\left(x-\frac{1}{3}\right)^2}\right)+\left(-\frac{1}{2\left(x-\frac{1}{3}\right)}+\frac{3}{2}\right)^2-\left(\frac{81x^2-108x+54}{4\left(3x-1\right)^2}\right)\right)=0$$

$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx}\\
&= \mathrm{e}^{\int \left(-\frac{1}{2\left(x-\frac{1}{3}\right)}+\frac{3}{2}\right)dx}\\
&= \frac{\mathrm{e}^{\frac{3x}{2}}}{\sqrt{3x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}\\
&= z_1 e^{-\int \frac{1}{2}\frac{-3x-2}{3x-1}\, dx}\\
&= z_1 e^{\frac{x}{2}+\frac{\ln(3x-1)}{2}}\\
&= z_1\left(\sqrt{3x-1}\,\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\mathrm{e}^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1\int \frac{e^{\int -\frac{-3x-2}{3x-1}\, dx}}{\left(y_1\right)^2}\, dx\\
&= y_1\int \frac{e^{x+\ln(3x-1)}}{\left(y_1\right)^2}\, dx\\
&= y_1\left(-\frac{x\,\mathrm{e}^{x+\ln(3x-1)}\mathrm{e}^{-4x}}{3x-1}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1y_1+c_2y_2\\
&= c_1\left(\mathrm{e}^{2x}\right)+c_2\left(\mathrm{e}^{2x}\left(-\frac{x\,\mathrm{e}^{x+\ln(3x-1)}\mathrm{e}^{-4x}}{3x-1}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(3x-1)\left(\frac{d^2}{dx^2}y(x)\right)-(3x+2)\left(\frac{d}{dx}y(x)\right)-(6x-8)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{2(3x-4)y(x)}{3x-1}+\frac{(3x+2)\left(\frac{d}{dx}y(x)\right)}{3x-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)-\frac{(3x+2)\left(\frac{d}{dx}y(x)\right)}{3x-1}-\frac{2(3x-4)y(x)}{3x-1}=0$

□ Check to see if $x_0=\frac{1}{3}$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=-\frac{3x+2}{3x-1}, P_3(x)=-\frac{2(3x-4)}{3x-1}\right]$

  ◦ $\left(x-\frac{1}{3}\right)\cdot P_2(x)$ is analytic at $x=\frac{1}{3}$

    $\left(\left(x-\frac{1}{3}\right)\cdot P_2(x)\right)\bigg|_{x=\frac{1}{3}}=-1$

  ◦ $\left(x-\frac{1}{3}\right)^2\cdot P_3(x)$ is analytic at $x=\frac{1}{3}$

    $\left(\left(x-\frac{1}{3}\right)^2\cdot P_3(x)\right)\bigg|_{x=\frac{1}{3}}=0$

  ◦ $x=\frac{1}{3}$is a regular singular point

  Check to see if $x_0=\frac{1}{3}$ is a regular singular point

  $x_0=\frac{1}{3}$

- Multiply by denominators

  $(3x-1)\left(\frac{d^2}{dx^2}y(x)\right)+(-2-3x)\left(\frac{d}{dx}y(x)\right)+(-6x+8)\,y(x)=0$

- Change variables using $x=u+\frac{1}{3}$ so that the regular singular point is at $u=0$

  $3u\left(\frac{d^2}{du^2}y(u)\right)+(-3-3u)\left(\frac{d}{du}y(u)\right)+(-6u+6)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

    $u^m\cdot y(u)=\sum_{k=0}^{\infty} a_k u^{k+r+m}$

  ◦ Shift index using $k->k-m$

    $u^m\cdot y(u)=\sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  ◦ Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

    $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-1}$

- Shift index using $k- >k+1$

  $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $3a_0 r(-2+r)\,u^{-1+r} + (3a_1(1+r)(-1+r) - 3a_0(-2+r))\,u^r + \left(\sum_{k=1}^{\infty} (3a_{k+1}(k+1+r)(k+r-1)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $3r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term must be 0

  $3a_1(1+r)(-1+r) - 3a_0(-2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $3a_{k+1}(k+1+r)(k+r-1) + a_k(-3k-3r+6) - 6a_{k-1} = 0$

- Shift index using $k- >k+1$

  $3a_{k+2}(k+2+r)(k+r) + a_{k+1}(-3k+3-3r) - 6a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{ka_{k+1}+ra_{k+1}+2a_k-a_{k+1}}{(k+2+r)(k+r)}$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{ka_{k+1}+2a_k-a_{k+1}}{(k+2)k}$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$

  $a_{k+2} = \frac{ka_{k+1}+2a_k-a_{k+1}}{(k+2)k}$

- Recursion relation for $r = 2$

  $a_{k+2} = \frac{ka_{k+1}+2a_k+a_{k+1}}{(k+4)(k+2)}$

- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2} = \frac{ka_{k+1}+2a_k+a_{k+1}}{(k+4)(k+2)}, 9a_1 = 0\right]$

- Revert the change of variables $u = x - \frac{1}{3}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k \left(x-\frac{1}{3}\right)^{k+2}, a_{k+2} = \frac{ka_{k+1}+2a_k+a_{k+1}}{(k+4)(k+2)}, 9a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 18

```
dsolve((3*x-1)*diff(diff(y(x),x),x)-(2+3*x)*diff(y(x),x)-(6*x-8)*y(x) = 0,y(x),singsol
```

$$y = e^{2x}c_1 + c_2 x\, e^{-x}$$

Mathematica DSolve solution

Solving time : 0.243 (sec)
Leaf size : 94

```
DSolve[{(3*x-1)*D[y[x],{x,2}]-(3*x+2)*D[y[x],x]-(6*x-8)*y[x]==0,{}},y[x],x,IncludeSingularSo
```

$$y(x) \to \exp\left(\int_1^x \frac{6-9K[1]}{2-6K[1]}dK[1] - \frac{1}{2}\int_1^x \left(\frac{3}{1-3K[2]}-1\right)dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{6-9K[1]}{2-6K[1]}dK[1]\right)dK[3] + c_1\right)$$

2.1.45 Problem 47

Internal problem ID [9217]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 47
Date solved : `Monday, January 27, 2025 at 05:52:11 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(2+x)\,y''+xy'+3y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.319 (sec)

Writing the ode as

$$(2+x)\,y''+xy'+3y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=2+x\\ B&=x\\ C&=3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{x^2-12x-20}{4\,(2+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=x^2-12x-20\\ t&=4(2+x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x^2-12x-20}{4\,(2+x)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.84: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2+x)^2$. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{(2+x)^2} - \frac{4}{2+x}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(2+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{4}{x} - \frac{6}{x^2} - \frac{72}{x^3} - \frac{556}{x^4} - \frac{5440}{x^5} - \frac{55088}{x^6} - \frac{586688}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 12x - 20}{4x^2 + 16x + 16} \\ &= Q + \frac{R}{4x^2 + 16x + 16} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-16x - 24}{4x^2 + 16x + 16}\right) \\ &= \frac{1}{4} + \frac{-16x - 24}{4x^2 + 16x + 16} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -16. Dividing this by leading coefficient in t which is 4 gives -4. Now b can be found.

$$\begin{aligned} b &= (-4) - (0) \\ &= -4 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-4}{\frac{1}{2}} - 0\right) &&= -4 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-4}{\frac{1}{2}} - 0\right) &&= 4 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 12x - 20}{4(2+x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	-4	4

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 4$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^+) \\
&= 4 - (2) \\
&= 2
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{2}{2+x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{2}{2+x} - \frac{1}{2} \\
&= -\frac{x-2}{2(2+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{2}{2+x} - \frac{1}{2}\right)(2x + a_1) + \left(\left(-\frac{2}{(2+x)^2}\right) + \left(\frac{2}{2+x} - \frac{1}{2}\right)^2 - \left(\frac{x^2 - 12x - 20}{4(2+x)^2}\right)\right) = 0$$

$$\frac{(a_1 + 6)\, x + 2a_0 + 2a_1 + 4}{2+x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 4, a_1 = -6\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 6x + 4$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= \left(x^2 - 6x + 4\right) e^{\int \left(\frac{2}{2+x} - \frac{1}{2}\right) dx} \\ &= \left(x^2 - 6x + 4\right) e^{-\frac{x}{2} + 2\ln(2+x)} \\ &= \left(x^2 - 6x + 4\right) (2+x)^2 e^{-\frac{x}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{2+x}\, dx} \\ &= z_1 e^{-\frac{x}{2} + \ln(2+x)} \\ &= z_1\left((2+x)\, e^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2+x)^3 e^{-x} \left(x^2 - 6x + 4\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{2+x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-x + 2\ln(2+x)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{e^x (x^4 - x^3 - 18x^2 - 22x + 8)}{240\, (x^2 - 6x + 4)\, (2+x)^3} - \frac{e^{-2}\, \mathrm{Ei}_1(-2-x)}{240}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left((2+x)^3 e^{-x} \left(x^2 - 6x + 4\right)\right) \\ &\quad + c_2\left((2+x)^3 e^{-x} \left(x^2 - 6x + 4\right) \left(-\frac{e^x (x^4 - x^3 - 18x^2 - 22x + 8)}{240\, (x^2 - 6x + 4)\, (2+x)^3} - \frac{e^{-2}\, \mathrm{Ei}_1(-2-x)}{240}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x+2)\left(\frac{d^2}{dx^2}y(x)\right)+x\left(\frac{d}{dx}y(x)\right)+3y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{3y(x)}{x+2}-\frac{\left(\frac{d}{dx}y(x)\right)x}{x+2}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{\left(\frac{d}{dx}y(x)\right)x}{x+2}+\frac{3y(x)}{x+2}=0$

☐ Check to see if $x_0=-2$ is a regular singular point

- ○ Define functions

  $\left[P_2(x)=\frac{x}{x+2},P_3(x)=\frac{3}{x+2}\right]$

- ○ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

  $\left((x+2)\cdot P_2(x)\right)\Big|_{x=-2}=-2$

- ○ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

  $\left((x+2)^2\cdot P_3(x)\right)\Big|_{x=-2}=0$

- ○ $x=-2$is a regular singular point

  Check to see if $x_0=-2$ is a regular singular point

  $x_0=-2$

- Multiply by denominators

  $(x+2)\left(\frac{d^2}{dx^2}y(x)\right)+x\left(\frac{d}{dx}y(x)\right)+3y(x)=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

  $u\left(\frac{d^2}{du^2}y(u)\right)+(u-2)\left(\frac{d}{du}y(u)\right)+3y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ○ Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-1}$

- ○ Shift index using $k->k+1$

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-3+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(k-2+r)+a_k(k+r+3)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-3+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{0,3\}$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k-2+r)+a_k(k+r+3)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{a_k(k+r+3)}{(k+1+r)(k-2+r)}$
- Recursion relation for $r=0$

  $a_{k+1}=-\frac{a_k(k+3)}{(k+1)(k-2)}$
- Series not valid for $r=0$, division by 0 in the recursion relation at $k=2$

  $a_{k+1}=-\frac{a_k(k+3)}{(k+1)(k-2)}$
- Recursion relation for $r=3$

  $a_{k+1}=-\frac{a_k(k+6)}{(k+4)(k+1)}$
- Solution for $r=3$

  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+3}, a_{k+1}=-\frac{a_k(k+6)}{(k+4)(k+1)}\right]$
- Revert the change of variables $u=x+2$

  $\left[y(x)=\sum_{k=0}^{\infty}a_k(x+2)^{k+3}, a_{k+1}=-\frac{a_k(k+6)}{(k+4)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 72

```
dsolve((x+2)*diff(diff(y(x),x),x)+diff(y(x),x)*x+3*y(x) = 0,y(x),singsol=all)
```

$$y=e^{-x-2}c_2(x^2-6x+4)(x+2)^3\operatorname{Ei}_1(-x-2)+c_1e^{-x}(x^2-6x+4)(x+2)^3+c_2(x^4-x^3-18x^2-22x+8)$$

Mathematica DSolve solution

Solving time : 0.688 (sec)
Leaf size : 106

```
DSolve[{(2+x)*D[y[x],{x,2}]+x*D[y[x],x]+3*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to (x^2 - 6x + 4) \exp\left(\int_1^x \left(\frac{2}{K[1]+2} - \frac{1}{2}\right) dK[1] - \frac{1}{2}\int_1^x \frac{K[2]}{K[2]+2} dK[2]\right)\left(c_2 \int_1^x \frac{\exp\left(-2\int_1^{K[3]} \left(\frac{2}{K[1]+2} - \frac{1}{2}\right) dK[1]\right)}{(K[3]^2 - 6K[3]+4)^2} dK[3] + c_1\right)$$

2.1.46 Problem 48

Internal problem ID [9218]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 48
Date solved : `Monday, January 27, 2025 at 05:52:11 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1-x)\,y''+x(4+x)\,y'+(2-x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.236 (sec)

Writing the ode as

$$\left(-x^3+x^2\right)y''+\left(x^2+4x\right)y'+(2-x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-x^3+x^2\\B&=x^2+4x\\C&=2-x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-x+36}{4x\left(-1+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=-x+36\\t&=4x(-1+x)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-x+36}{4x\left(-1+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.86: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x(-1+x)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{9}{-1+x} + \frac{9}{x} + \frac{35}{4\,(-1+x)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x+36}{4x\left(-1+x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x+36}{4x\left(-1+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1
1	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{1}{2} - \left(-\frac{3}{2}\right) \\
&= 2
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{x} - \frac{5}{2\left(-1+x\right)} + (-)\left(0\right) \\
&= \frac{1}{x} - \frac{5}{2\left(-1+x\right)} \\
&= \frac{1}{x} - \frac{5}{-2+2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{1}{x} - \frac{5}{2(-1+x)}\right)(2x + a_1) + \left(\left(-\frac{1}{x^2} + \frac{5}{2(-1+x)^2}\right) + \left(\frac{1}{x} - \frac{5}{2(-1+x)}\right)^2 - \left(\frac{-x+36}{4x(-1+x)^2}\right.\right.$$

$$\frac{(a_1-6)x + 4a_0 - 2a_1}{x(-1+x)}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 3, a_1 = 6\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + 6x + 3$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (x^2 + 6x + 3)\, e^{\int \left(\frac{1}{x} - \frac{5}{2(-1+x)}\right) dx} \\
&= (x^2 + 6x + 3)\, e^{\ln(x) - \frac{5\ln(-1+x)}{2}} \\
&= \frac{(x^2 + 6x + 3)\, x}{(-1+x)^{5/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2+4x}{-x^3+x^2}\, dx} \\
&= z_1 e^{-2\ln(x) + \frac{5\ln(-1+x)}{2}} \\
&= z_1 \left(\frac{(-1+x)^{5/2}}{x^2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2 + 6x + 3}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+4x}{-x^3+x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-4\ln(x) + 5\ln(-1+x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\ln(x) + \frac{1}{9x} - \frac{4\left(-38x - \frac{69}{2}\right)}{9(x^2 + 6x + 3)}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^2+6x+3}{x}\right) + c_2 \left(\frac{x^2+6x+3}{x}\left(\ln(x) + \frac{1}{9x} - \frac{4\left(-38x - \frac{69}{2}\right)}{9\,(x^2+6x+3)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 49

```
dsolve(x^2*(1-x)*diff(diff(y(x),x),x)+x*(x+4)*diff(y(x),x)+(-x+2)*y(x) = 0,y(x),singsol=
```

$$
y = \frac{3xc_2(x^2+6x+3)\ln(x) + c_1 x^3 + (6c_1 + 51c_2)\,x^2 + (3c_1 + 48c_2)\,x + c_2}{x^2}
$$

Mathematica DSolve solution

Solving time : 0.853 (sec)
Leaf size : 119

```
DSolve[{x^2*(1-x)*D[y[x],{x,2}]+x*(4+x)*D[y[x],x]+(2-x)*y[x]==0,{}},y[x],x,IncludeSingularSolu
```

$$
y(x) \to (x^2+6x+3)\exp\left(\int_1^x \left(\frac{1}{K[1]} + \frac{5}{2-2K[1]}\right)dK[1] - \frac{1}{2}\int_1^x \frac{K[2]+4}{K[2]-K[2]^2}dK[2]\right)\left(c_2\int_1^x \frac{\exp\left(-2\int_1^{K[3]}\left(\frac{1}{K[1]} + \frac{5}{2-2K[1]}\right)dK[1]\right)}{(K[3]^2+6K[3]+3)^2}dK[3] + c_1\right)
$$

2.1.47 Problem 49

Internal problem ID [9219]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 49
Date solved : `Monday, January 27, 2025 at 05:52:12 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+x)\,y''+x(1+2x)\,y'-(4+6x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.230 (sec)

Writing the ode as

$$x^2(1+x)\,y''+\left(2x^2+x\right)y'+(-6x-4)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(1+x) \\ B &= 2x^2+x \\ C &= -6x-4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{24x^2+40x+15}{4\left(x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 24x^2+40x+15 \\ t &= 4\left(x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{24x^2+40x+15}{4\left(x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.87: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{2\,(1+x)} - \frac{1}{4\,(1+x)^2} + \frac{15}{4x^2} + \frac{5}{2x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{24x^2 + 40x + 15}{4(x^2 + x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{24x^2 + 40x + 15}{4(x^2 + x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	3	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 3$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\
&= 3 - (3) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2+2x} + \frac{5}{2x} + (0) \\
&= \frac{1}{2+2x} + \frac{5}{2x} \\
&= \frac{6x+5}{2x(1+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2+2x} + \frac{5}{2x}\right)(0) + \left(\left(-\frac{1}{2(1+x)^2} - \frac{5}{2x^2}\right) + \left(\frac{1}{2+2x} + \frac{5}{2x}\right)^2 - \left(\frac{24x^2+40x+15}{4(x^2+x)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{2+2x} + \frac{5}{2x}\right) dx} \\
&= x^{5/2}\sqrt{1+x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^2+x}{x^2(1+x)} \, dx} \\
&= z_1 e^{-\frac{\ln(x(1+x))}{2}} \\
&= z_1\left(\frac{1}{\sqrt{x(1+x)}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{5/2}\sqrt{1+x}}{\sqrt{x(1+x)}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2+x}{x^2(1+x)} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\ln(x(1+x))}}{(y_1)^2} \, dx \\
&= y_1\left(-\frac{1}{4x^4} - \frac{1}{2x^2} + \ln(x) + \frac{1}{3x^3} + \frac{1}{x} - \ln(1+x)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^{5/2}\sqrt{1+x}}{\sqrt{x(1+x)}}\right) + c_2\left(\frac{x^{5/2}\sqrt{1+x}}{\sqrt{x(1+x)}}\left(-\frac{1}{4x^4} - \frac{1}{2x^2} + \ln(x) + \frac{1}{3x^3} + \frac{1}{x} - \ln(1+x)\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(2x+1)\left(\frac{d}{dx}y(x)\right)-(4+6x)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{2(3x+2)y(x)}{(x+1)x^2}-\frac{(2x+1)\left(\frac{d}{dx}y(x)\right)}{x(x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{(2x+1)\left(\frac{d}{dx}y(x)\right)}{x(x+1)}-\frac{2(3x+2)y(x)}{(x+1)x^2}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{2x+1}{x(x+1)},P_3(x)=-\frac{2(3x+2)}{(x+1)x^2}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=1$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(2x+1)\left(\frac{d}{dx}y(x)\right)+(-6x-4)\,y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^3-2u^2+u)\left(\frac{d^2}{du^2}y(u)\right)+(2u^2-3u+1)\left(\frac{d}{du}y(u)\right)+(-6u+2)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

  ◦ Shift index using $k->k-m$

  $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  ◦ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  ◦ Shift index using $k->k+2-m$

$$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$a_0r^2u^{-1+r}+\left(a_1(1+r)^2-a_0(2r^2+r-2)\right)u^r+\left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+1+r)^2-a_k(2k^2+4kr+2r^2+k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r^2=0$
- Values of r that satisfy the indicial equation
  $r=0$
- Each term must be 0
  $a_1(1+r)^2-a_0(2r^2+r-2)=0$
- Each term in the series must be 0, giving the recursion relation
  $(k^2-k-6)\,a_{k-1}+(-2k^2-k+2)\,a_k+a_{k+1}(k+1)^2=0$
- Shift index using $k->k+1$
  $((k+1)^2-k-7)\,a_k+\left(-2(k+1)^2-k+1\right)a_{k+1}+a_{k+2}(k+2)^2=0$
- Recursion relation that defines series solution to ODE
  $a_{k+2}=-\frac{k^2a_k-2k^2a_{k+1}+ka_k-5ka_{k+1}-6a_k-a_{k+1}}{(k+2)^2}$
- Recursion relation for $r=0$
  $a_{k+2}=-\frac{k^2a_k-2k^2a_{k+1}+ka_k-5ka_{k+1}-6a_k-a_{k+1}}{(k+2)^2}$
- Solution for $r=0$
  $\left[y(u)=\sum_{k=0}^{\infty}a_ku^k,a_{k+2}=-\frac{k^2a_k-2k^2a_{k+1}+ka_k-5ka_{k+1}-6a_k-a_{k+1}}{(k+2)^2},a_1+2a_0=0\right]$
- Revert the change of variables $u=x+1$
  $\left[y(x)=\sum_{k=0}^{\infty}a_k(x+1)^k,a_{k+2}=-\frac{k^2a_k-2k^2a_{k+1}+ka_k-5ka_{k+1}-6a_k-a_{k+1}}{(k+2)^2},a_1+2a_0=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 46

```
dsolve(x^2*(x+1)*diff(diff(y(x),x),x)+x*(2*x+1)*diff(y(x),x)-(4+6*x)*y(x) = 0,y(x),sings
```

$$y=c_1x^2+\frac{c_2(12\ln(x)\,x^4-12\ln(x+1)\,x^4+12x^3-6x^2+4x-3)}{x^2}$$

Mathematica DSolve solution

Solving time : 0.253 (sec)
Leaf size : 103

```
DSolve[{x^2*(1+x)*D[y[x],{x,2}]+x*(1+2*x)*D[y[x],x]-(4+6*x)*y[x]==0,{}},y[x],x,IncludeSingul
```

$$y(x) \to \exp\left(\int_1^x \frac{6K[1]+5}{2K[1]^2+2K[1]}dK[1] - \frac{1}{2}\int_1^x \left(\frac{1}{K[2]+1} + \frac{1}{K[2]}\right) dK[2]\right) \left(c_2 \int_1^x \exp\left(-2\int_1^{K[3]} \frac{6K[1]+5}{2K[1]^2+2K[1]}dK[1]\right) dK[3] + c_1\right)$$

2.1.48 Problem 50

Internal problem ID [9220]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 50
Date solved : Monday, January 27, 2025 at 05:52:13 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$x^2\left(2x^2+1\right)y''+x\left(2x^2+4\right)y'+2\left(-x^2+1\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.431 (sec)

Writing the ode as

$$\left(2x^4+x^2\right)y''+\left(2x^3+4x\right)y'+\left(-2x^2+2\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^4+x^2 \\ B &= 2x^3+4x \\ C &= -2x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{3x^2-9}{\left(2x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3x^2-9 \\ t &= \left(2x^2+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{3x^2-9}{\left(2x^2+1\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.89: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x^2 + 1)^2$. There is a pole at $x = \frac{i\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{i\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{21}{16\left(x - \frac{i\sqrt{2}}{2}\right)^2} + \frac{21}{16\left(x + \frac{i\sqrt{2}}{2}\right)^2} + \frac{15i\sqrt{2}}{16\left(x - \frac{i\sqrt{2}}{2}\right)} - \frac{15i\sqrt{2}}{16\left(x + \frac{i\sqrt{2}}{2}\right)}$$

For the pole at $x = \frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{4} \end{aligned}$$

For the pole at $x = -\frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{3x^2 - 9}{(2x^2+1)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3x^2 - 9}{(2x^2+1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{i\sqrt{2}}{2}$	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
$-\frac{i\sqrt{2}}{2}$	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{3}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{4\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{3}{4\left(x+\frac{i\sqrt{2}}{2}\right)} + (-)(0) \\
&= -\frac{3}{4\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{3}{4\left(x+\frac{i\sqrt{2}}{2}\right)} \\
&= -\frac{3x}{2x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{4\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{3}{4\left(x+\frac{i\sqrt{2}}{2}\right)}\right)(1) + \left(\left(\frac{3}{4\left(x-\frac{i\sqrt{2}}{2}\right)^2} + \frac{3}{4\left(x+\frac{i\sqrt{2}}{2}\right)^2}\right) + \left(-\frac{3}{4\left(x-\frac{i\sqrt{2}}{2}\right)} -\right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (x)\, \mathrm{e}^{\int \left(-\frac{3}{4\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{3}{4\left(x+\frac{i\sqrt{2}}{2}\right)}\right) dx} \\
&= (x)\, \frac{1}{\left((i\sqrt{2}-2x)\,(2x+i\sqrt{2})\right)^{3/4}} \\
&= \frac{x}{(-4x^2-2)^{3/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^3+4x}{2x^4+x^2}\, dx} \\
&= z_1 e^{\frac{3\ln(2x^2+1)}{4} - 2\ln(x)} \\
&= z_1 \left(\frac{(2x^2+1)^{3/4}}{x^2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{2^{1/4}(4x^2+2)^{3/4}}{2x\,(-4x^2-2)^{3/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3+4x}{2x^4+x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{3\ln(2x^2+1)}{2}-4\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{2i(2x^4-x^2-1)\sqrt{2x^2+1}\,\sqrt{2}\,\sqrt{\frac{(-4x^2-2)(4x^2+2)}{(2x^2+1)^2}}}{x\sqrt{-4x^2-2}\,\sqrt{4x^2+2}} \right. \\
&\quad \left. - \frac{6i\,\mathrm{arcsinh}\left(\sqrt{2}\,x\right)\sqrt{\frac{(-4x^2-2)(4x^2+2)}{(2x^2+1)^2}}\,(2x^2+1)}{\sqrt{-4x^2-2}\,\sqrt{4x^2+2}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{2^{1/4}(4x^2+2)^{3/4}}{2x\,(-4x^2-2)^{3/4}}\right) + c_2\left(\frac{2^{1/4}(4x^2+2)^{3/4}}{2x\,(-4x^2-2)^{3/4}}\right)\left(-\frac{2i(2x^4-x^2-1)\sqrt{2x^2+1}\,\sqrt{2}\,\sqrt{\frac{(-4x^2-2)(4x^2+2)}{(2x^2+1)^2}}}{x\sqrt{-4x^2-2}\,\sqrt{4x^2+2}} - \frac{6i\,\mathrm{ar}}{}\right.
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(2x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + x(2x^2+4)\left(\frac{d}{dx}y(x)\right) + 2(-x^2+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{2(x^2-1)y(x)}{x^2(2x^2+1)} - \frac{2(x^2+2)\left(\frac{d}{dx}y(x)\right)}{x(2x^2+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{2(x^2+2)\left(\frac{d}{dx}y(x)\right)}{x(2x^2+1)} - \frac{2(x^2-1)y(x)}{x^2(2x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ○ Define functions

    $\left[P_2(x) = \frac{2(x^2+2)}{x(2x^2+1)}, P_3(x) = -\frac{2(x^2-1)}{x^2(2x^2+1)}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 4$

  ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$

  ○ $x = 0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

$x^2(2x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + 2x(x^2+2)\left(\frac{d}{dx}y(x)\right) + (-2x^2+2)\,y(x) = 0$

- Assume series solution for $y(x)$

$y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

$x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

◦ Shift index using $k- >k-m$

$x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

◦ Shift index using $k- >k+1-m$

$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

◦ Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

◦ Shift index using $k- >k+2-m$

$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

Rewrite ODE with series expansions

$a_0(2+r)\,(1+r)\,x^r + a_1(3+r)\,(2+r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+2)\,(k+r+1) + 2a_{k-2}(k+r-1\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(2+r)\,(1+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{-2, -1\}$

- Each term must be 0

$a_1(3+r)\,(2+r) = 0$

- Solve for the dependent coefficient(s)

$a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

$a_k(k+r+2)\,(k+r+1) + 2a_{k-2}(k+r-1)\,(k-3+r) = 0$

- Shift index using $k- >k+2$

$a_{k+2}(k+4+r)\,(k+3+r) + 2a_k(k+r+1)\,(k+r-1) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+2} = -\frac{2a_k(k+r+1)(k+r-1)}{(k+4+r)(k+3+r)}$

- Recursion relation for $r = -2$

$a_{k+2} = -\frac{2a_k(k-1)(k-3)}{(k+2)(k+1)}$

- Solution for $r = -2$

$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{2a_k(k-1)(k-3)}{(k+2)(k+1)}, a_1 = 0\right]$

- Recursion relation for $r = -1$; series terminates at $k = 2$

$a_{k+2} = -\frac{2a_k k(k-2)}{(k+3)(k+2)}$

- Solution for $r = -1$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{2a_k k(k-2)}{(k+3)(k+2)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), a_{k+2} = -\frac{2a_k(k-1)(k-3)}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{2b_k k(k-2)}{(k+3)(k+2)}, b_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.009 (sec)
Leaf size : 42

```
dsolve(x^2*(2*x^2+1)*diff(diff(y(x),x),x)+x*(2*x^2+4)*diff(y(x),x)+2*(-x^2+1)*y(x) = 0,y
```

$$y = \frac{c_2\sqrt{2}\,(x-1)\,(x+1)\,\sqrt{2x^2+1} + x\left(3\,\text{arcsinh}\left(\sqrt{2}\,x\right)c_2 + c_1\right)}{x^2}$$

Mathematica DSolve solution

Solving time : 0.788 (sec)
Leaf size : 73

```
DSolve[{x^2*(1+2*x^2)*D[y[x],{x,2}]+x*(4+2*x^2)*D[y[x],x]+2*(1-x^2)*y[x]==0,{}},y[x],x,IncludeS
```

$$y(x) \to \frac{\left(c_1 x - c_2\,\text{Hypergeometric2F1}\left(-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, -2x^2\right)\right)\exp\left(-\frac{1}{2}\int_1^x \frac{2(K[1]^2+2)}{2K[1]^3+K[1]}dK[1]\right)}{(2x^2+1)^{3/4}}$$

2.1.49 Problem 51

Internal problem ID [9221]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 51
Date solved : `Monday, January 27, 2025 at 05:52:13 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(x^2+2\right)y''+2x\left(x^2+5\right)y'+2\left(-x^2+3\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.472 (sec)

Writing the ode as

$$\left(x^4+2x^2\right)y''+\left(2x^3+10x\right)y'+\left(-2x^2+6\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+2x^2 \\ B &= 2x^3+10x \\ C &= -2x^2+6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{2x^4-5x^2+3}{\left(x^3+2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2x^4-5x^2+3 \\ t &= \left(x^3+2x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{2x^4-5x^2+3}{\left(x^3+2x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.91: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^3 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i\sqrt{2}$ of order 2. There is a pole at $x = -i\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2} + \frac{21}{16\left(x - i\sqrt{2}\right)^2} + \frac{21}{16\left(x + i\sqrt{2}\right)^2} + \frac{11i\sqrt{2}}{32\left(x - i\sqrt{2}\right)} - \frac{11i\sqrt{2}}{32\left(x + i\sqrt{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x-i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}$$

For the pole at $x = -i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x+i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^4 - 5x^2 + 3}{\left(x^3 + 2x\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2x^4 - 5x^2 + 3}{\left(x^3 + 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$i\sqrt{2}$	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
$-i\sqrt{2}$	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= 2 - (0) \\
&= 2
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{3}{2x} - \frac{3}{4\left(x-i\sqrt{2}\right)} - \frac{3}{4\left(x+i\sqrt{2}\right)} + (0) \\
&= \frac{3}{2x} - \frac{3}{4\left(x-i\sqrt{2}\right)} - \frac{3}{4\left(x+i\sqrt{2}\right)} \\
&= \frac{3}{x^3+2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{3}{2x} - \frac{3}{4\left(x-i\sqrt{2}\right)} - \frac{3}{4\left(x+i\sqrt{2}\right)} \right)(2x+a_1) + \left(\left(-\frac{3}{2x^2} + \frac{3}{4\left(x-i\sqrt{2}\right)^2} + \frac{3}{4\left(x+i\sqrt{2}\right)^2} \right) + \left(\frac{3}{2 \right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 8, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + 8$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \left(x^2+8\right) e^{\int \left(\frac{3}{2x} - \frac{3}{4\left(x-i\sqrt{2}\right)} - \frac{3}{4\left(x+i\sqrt{2}\right)} \right) dx} \\
&= \left(x^2+8\right) e^{-\frac{3\ln\left(x^2+2\right)}{4} + \frac{3\ln(x)}{2}} \\
&= \frac{\left(x^2+8\right) x^{3/2}}{\left(x^2+2\right)^{3/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^3+10x}{x^4+2x^2}\, dx} \\
&= z_1 e^{\frac{3\ln\left(x^2+2\right)}{4} - \frac{5\ln(x)}{2}} \\
&= z_1 \left(\frac{\left(x^2+2\right)^{3/4}}{x^{5/2}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2+8}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3+10x}{x^4+2x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{3\ln(x^2+2)}{2}-5\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{(x^2+2)^{5/2}}{256x^2} + \frac{(x^2+2)^{3/2}}{384} + \frac{\sqrt{x^2+2}}{96} - \frac{\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2}}{\sqrt{x^2+2}}\right)}{64} + \frac{3\sqrt{x^2+2}}{64\,(x^2+8)} \right. \\
&\qquad \left. + \frac{x^2\sqrt{x^2+2}}{768}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^2+8}{x}\right) + c_2\left(\frac{x^2+8}{x}\left(-\frac{(x^2+2)^{5/2}}{256x^2} + \frac{(x^2+2)^{3/2}}{384} + \frac{\sqrt{x^2+2}}{96}\right.\right. \\
&\qquad \left.\left. - \frac{\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2}}{\sqrt{x^2+2}}\right)}{64} + \frac{3\sqrt{x^2+2}}{64\,(x^2+8)} + \frac{x^2\sqrt{x^2+2}}{768}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x^2+2)\left(\frac{d^2}{dx^2}y(x)\right) + 2x(x^2+5)\left(\frac{d}{dx}y(x)\right) + 2(-x^2+3)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{2(x^2-3)y(x)}{(x^2+2)x^2} - \frac{2(x^2+5)\left(\frac{d}{dx}y(x)\right)}{x(x^2+2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{2(x^2+5)\left(\frac{d}{dx}y(x)\right)}{x(x^2+2)} - \frac{2(x^2-3)y(x)}{(x^2+2)x^2} = 0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{2(x^2+5)}{x(x^2+2)}, P_3(x) = -\frac{2(x^2-3)}{(x^2+2)x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$(x \cdot P_2(x)) \bigg|_{x=0} = 5$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x)) \bigg|_{x=0} = 3$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2+2)\left(\frac{d^2}{dx^2}y(x)\right) + 2x(x^2+5)\left(\frac{d}{dx}y(x)\right) + (-2x^2+6)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $2a_0(3+r)\,(1+r)\,x^r + 2a_1(4+r)\,(2+r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (2a_k(k+r+3)\,(k+r+1) + a_{k-2}(k+r)\,(k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $2(3+r)\,(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-3, -1\}$

- Each term must be 0

  $2a_1(4+r)\,(2+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $2a_k(k+r+3)\,(k+r+1) + a_{k-2}(k+r)\,(k-3+r) = 0$

- Shift index using $k- >k+2$

  $2a_{k+2}(k+5+r)\,(k+r+3) + a_k(k+r+2)\,(k+r-1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r+2)(k+r-1)}{2(k+5+r)(k+r+3)}$

- Recursion relation for $r=-3$; series terminates at $k=4$

  $a_{k+2}=-\frac{a_k(k-1)(k-4)}{2(k+2)k}$

- Solution for $r=-3$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2}=-\frac{a_k(k-1)(k-4)}{2(k+2)k}, a_1=0\right]$

- Recursion relation for $r=-1$; series terminates at $k=2$

  $a_{k+2}=-\frac{a_k(k+1)(k-2)}{2(k+4)(k+2)}$

- Solution for $r=-1$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2}=-\frac{a_k(k+1)(k-2)}{2(k+4)(k+2)}, a_1=0\right]$

- Combine solutions and rename parameters

  $\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^{k-3}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), a_{k+2}=-\frac{a_k(k-1)(-4+k)}{2(k+2)k}, a_1=0, b_{k+2}=-\frac{b_k(k+1)(k-2)}{2(4+k)(k+2)}, b_1=\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 53

```
dsolve(x^2*(x^2+2)*diff(diff(y(x),x),x)+2*x*(x^2+5)*diff(y(x),x)+2*(-x^2+3)*y(x) = 0,y
```

$$y=\frac{-c_2(x-2)\sqrt{x^2+2}\,(x+2)\sqrt{2}+(x^2+8)\,x^2\left(\operatorname{arctanh}\left(\frac{\sqrt{2}}{\sqrt{x^2+2}}\right)c_2+c_1\right)}{x^3}$$

Mathematica DSolve solution

Solving time : `0.628` (sec)
Leaf size : 112

```
DSolve[{x^2*(2+x^2)*D[y[x],{x,2}]+2*x*(x^2+5)*D[y[x],x]+2*(3-x^2)*y[x]==0,{}},y[x],x,IncludeSi
```

$$y(x) \to (x^2+8)\exp\left(\int_1^x \frac{3}{K[1]^3+2K[1]}dK[1] - \frac{1}{2}\int_1^x \frac{2K[2]^2+10}{K[2]^3+2K[2]}dK[2]\right)\left(c_2\int_1^x \frac{\exp\left(-2\int_1^{K[3]}\frac{3}{K[1]^3+2K[1]}dK[1]\right)}{(K[3]^2+8)^2}dK[3] + c_1\right)$$

2.1.50 Problem 52

Internal problem ID [9222]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 52
Date solved : `Monday, January 27, 2025 at 05:52:14 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+1\right) y''+6xy'+6y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.296 (sec)

Writing the ode as

$$\left(x^2+1\right) y''+6xy'+6y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= 6x \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{-3}{\left(x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= \left(x^2+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{3}{\left(x^2+1\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.93: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 + 1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4 (x - i)^2} + \frac{3}{4 (x + i)^2} + \frac{3i}{4 (x - i)} - \frac{3i}{4 (x + i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{\left(x^2+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)} + (-)\left(0\right) \\
&= -\frac{1}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)} \\
&= \frac{x-2i}{x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2\left(x-i\right)}+\frac{3}{2\left(x+i\right)}\right)(0)+\left(\left(\frac{1}{2\left(x-i\right)^{2}}-\frac{3}{2\left(x+i\right)^{2}}\right)+\left(-\frac{1}{2\left(x-i\right)}+\frac{3}{2\left(x+i\right)}\right)^{2}-\left(-\frac{\quad}{(x^2}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2(x-i)}+\frac{3}{2(x+i)}\right)dx} \\ &= \frac{\left(x^{2}+1\right)^{3/2}}{\left(ix+1\right)^{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{6x}{x^{2}+1}\, dx} \\ &= z_1 e^{-\frac{3\ln\left(x^{2}+1\right)}{2}} \\ &= z_1\left(\frac{1}{\left(x^{2}+1\right)^{3/2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{\left(ix+1\right)^{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^{2}}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{6x}{x^{2}+1}\, dx}}{\left(y_1\right)^{2}}\, dx \\ &= y_1 \int \frac{e^{-3\ln\left(x^{2}+1\right)}}{\left(y_1\right)^{2}}\, dx \\ &= y_1\left(-\frac{x}{\left(x+i\right)^{2}}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{1}{\left(ix+1\right)^{2}}\right)+c_2\left(\frac{1}{\left(ix+1\right)^{2}}\left(-\frac{x}{\left(x+i\right)^{2}}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 24

```
dsolve((x^2+1)*diff(diff(y(x),x),x)+6*diff(y(x),x)*x+6*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_2 x^2 + c_1 x - c_2}{\left(x^2+1\right)^2}$$

Mathematica DSolve solution

Solving time : 0.346 (sec)
Leaf size : 79

```
DSolve[{(1+x^2)*D[y[x],{x,2}]+6*x*D[y[x],x]+6*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to \frac{\exp\left(\int_1^x \frac{K[1]+2i}{K[1]^2+1}dK[1]\right)\left(c_2 \int_1^x \exp\left(-2\int_1^{K[2]} \frac{K[1]+2i}{K[1]^2+1}dK[1]\right)dK[2] + c_1\right)}{\left(x^2+1\right)^{3/2}}$$

2.1.51 Problem 53

Internal problem ID [9223]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 53
Date solved : `Monday, January 27, 2025 at 05:52:15 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+1\right)y''+2xy'-2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.268 (sec)

Writing the ode as

$$\left(x^2+1\right)y''+2xy'-2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=x^2+1\\ B&=2x\\ C&=-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{2x^2+3}{\left(x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=2x^2+3\\ t&=\left(x^2+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{2x^2+3}{\left(x^2+1\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.94: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(x-i)^2} - \frac{1}{4(x+i)^2} - \frac{5i}{4(x-i)} + \frac{5i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^2+3}{(x^2+1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2x^2+3}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-i$	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x-2i} + \frac{1}{2x+2i} + (0) \\
&= \frac{1}{2x-2i} + \frac{1}{2x+2i} \\
&= \frac{x}{x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x-2i} + \frac{1}{2x+2i}\right)(1) + \left(\left(-\frac{1}{2(x-i)^2} - \frac{1}{2(x+i)^2}\right) + \left(\frac{1}{2x-2i} + \frac{1}{2x+2i}\right)^2 - \left(\frac{2x^2+3}{(x^2+1)^2}\right.\right.$$
$$-\frac{2(x^2+1)\,a_0}{(-x+i)^2\,(x+i)^2}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (x)\, e^{\int \left(\frac{1}{2x-2i} + \frac{1}{2x+2i}\right) dx} \\ &= (x)\, \sqrt{(-x+i)\,(x+i)} \\ &= x\sqrt{-x^2-1} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x}{x^2+1}\, dx} \\ &= z_1 e^{-\frac{\ln(x^2+1)}{2}} \\ &= z_1\left(\frac{1}{\sqrt{x^2+1}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = ix$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{x^2+1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\ln(x^2+1)}}{(y_1)^2}\, dx \\ &= y_1\left(\arctan(x) + \frac{1}{x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(ix) + c_2\left(ix\left(\arctan(x) + \frac{1}{x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 14

```
dsolve((x^2+1)*diff(diff(y(x),x),x)+2*diff(y(x),x)*x-2*y(x) = 0,y(x),singsol=all)
```

$$y = c_1 x + \arctan(x) x c_2 + c_2$$

Mathematica DSolve solution

Solving time : 0.021 (sec)
Leaf size : 48

```
DSolve[{(1+x^2)*D[y[x],{x,2}]+2*x*D[y[x],x]-2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tru
```

$$y(x) \to \frac{1}{2} i(2c_1 x - c_2 x \log(1 - ix) + c_2 x \log(1 + ix) + 2ic_2)$$

2.1.52 Problem 54

Internal problem ID [9224]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 54
Date solved : `Monday, January 27, 2025 at 05:52:16 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+1\right) y''-8 x y'+20 y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.265 (sec)

Writing the ode as

$$\left(x^2+1\right) y''-8 x y'+20 y=0 \tag{1}$$

$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= -8x \\ C &= 20 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2 A} d x}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2 A B'-2 B A'+B^2-4 A C}{4 A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{-24}{\left(x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -24 \\ t &= \left(x^2+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{24}{\left(x^2+1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.95: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{(x-i)^2} + \frac{6}{(x+i)^2} + \frac{6i}{x-i} - \frac{6i}{x+i}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{24}{(x^2+1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	3	-2
$-i$	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{2}{x-i} + \frac{3}{x+i} + (-)(0) \\
&= -\frac{2}{x-i} + \frac{3}{x+i} \\
&= \frac{x-5i}{x^2+1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{2}{x-i} + \frac{3}{x+i}\right)(0) + \left(\left(\frac{2}{(x-i)^2} - \frac{3}{(x+i)^2} \right) + \left(-\frac{2}{x-i} + \frac{3}{x+i} \right)^2 - \left(-\frac{24}{(x^2+1)^2} \right) \right) = 0
$$

$$
0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{2}{x-i}+\frac{3}{x+i}\right) dx} \\
&= \frac{(x^2+1)^3}{(ix+1)^5}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-8x}{x^2+1} \, dx} \\
&= z_1 e^{2\ln(x^2+1)} \\
&= z_1 \left((x^2+1)^2\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(x^2+1)^5}{(ix+1)^5}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-8x}{x^2+1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{4\ln(x^2+1)}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{x^4 - 2x^2 + \frac{1}{5}}{(x+i)^5}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x^2+1)^5}{(ix+1)^5}\right) + c_2 \left(\frac{(x^2+1)^5}{(ix+1)^5} \left(\frac{x^4 - 2x^2 + \frac{1}{5}}{(x+i)^5}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 33

```
dsolve((x^2+1)*diff(diff(y(x),x),x)-8*diff(y(x),x)*x+20*y(x) = 0,y(x),singsol=all)
```

$$y = c_2x^5 + 5c_1x^4 - 10c_2x^3 - 10c_1x^2 + 5c_2x + c_1$$

Mathematica DSolve solution

Solving time : 0.281 (sec)
Leaf size : 77

```
DSolve[{(1+x^2)*D[y[x],{x,2}]-8*x*D[y[x],x]+20*y[x]==0,{}},y[x],x,IncludeSingularSolutions->
```

$$y(x) \to (x^2+1)^2 \exp\left(\int_1^x \frac{K[1]+5i}{K[1]^2+1}dK[1]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[2]}\frac{K[1]+5i}{K[1]^2+1}dK[1]\right)dK[2]+c_1\right)$$

2.1.53 Problem 55

Internal problem ID [9225]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 55
Date solved : `Monday, January 27, 2025 at 05:52:16 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-x^2+1\right) y''-8 x y'-12 y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.178 (sec)

Writing the ode as

$$\left(-x^2+1\right) y''-8 x y'-12 y=0 \tag{1}$$
$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+1 \\ B &= -8x \\ C &= -12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{8}{\left(x^2-1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 8 \\ t &= \left(x^2-1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{8}{\left(x^2-1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.96: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x+1} + \frac{2}{(x+1)^2} - \frac{2}{x-1} + \frac{2}{(x-1)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{8}{(x^2-1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	2	-1
-1	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x-1} + \frac{2}{x+1} + (-)(0) \\
&= -\frac{1}{x-1} + \frac{2}{x+1} \\
&= \frac{x-3}{x^2-1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x-1} + \frac{2}{x+1}\right)(0) + \left(\left(\frac{1}{(x-1)^2} - \frac{2}{(x+1)^2}\right) + \left(-\frac{1}{x-1} + \frac{2}{x+1}\right)^2 - \left(\frac{8}{(x^2-1)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{x-1}+\frac{2}{x+1}\right)dx} \\ &= \frac{(x+1)^2}{x-1} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-8x}{-x^2+1}\, dx} \\ &= z_1 e^{-2\ln(x-1)-2\ln(x+1)} \\ &= z_1 \left(\frac{1}{(x-1)^2 (x+1)^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{(x-1)^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-8x}{-x^2+1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-4\ln(x-1)-4\ln(x+1)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{(x+1)(3x^2+1)(x-1)^4 \,\mathrm{e}^{-4\ln(x-1)-4\ln(x+1)}}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{1}{(x-1)^3}\right) + c_2\left(\frac{1}{(x-1)^3}\left(-\frac{(x+1)(3x^2+1)(x-1)^4 \,\mathrm{e}^{-4\ln(x-1)-4\ln(x+1)}}{3}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(-x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)-8x\left(\frac{d}{dx}y(x)\right)-12y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{12y(x)}{x^2-1}-\frac{8\left(\frac{d}{dx}y(x)\right)x}{x^2-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{8\left(\frac{d}{dx}y(x)\right)x}{x^2-1}+\frac{12y(x)}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{8x}{x^2-1}, P_3(x)=\frac{12}{x^2-1}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=4$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $(x^2-1)\left(\frac{d^2}{dx^2}y(x)\right)+8x\left(\frac{d}{dx}y(x)\right)+12y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(8u-8)\left(\frac{d}{du}y(u)\right)+12y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  ◦ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

  ◦ Shift index using $k->k+2-m$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r(3+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)(k+r+4)+a_k(k+r+4)(k+r+3)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-2r(3+r)=0$

- Values of r that satisfy the indicial equation
  $r\in\{-3,0\}$
- Each term in the series must be 0, giving the recursion relation
  $(k+r+4)\left(\left(-2k-2r-2\right)a_{k+1}+a_k(k+r+3)\right)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{a_k(k+r+3)}{2(k+1+r)}$
- Recursion relation for $r=-3$
  $a_{k+1}=\frac{a_k k}{2(k-2)}$
- Series not valid for $r=-3$, division by 0 in the recursion relation at $k=2$
  $a_{k+1}=\frac{a_k k}{2(k-2)}$
- Recursion relation for $r=0$
  $a_{k+1}=\frac{a_k(k+3)}{2(k+1)}$
- Solution for $r=0$
  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^k, a_{k+1}=\frac{a_k(k+3)}{2(k+1)}\right]$
- Revert the change of variables $u=x+1$
  $\left[y(x)=\sum_{k=0}^{\infty}a_k(x+1)^k, a_{k+1}=\frac{a_k(k+3)}{2(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 29

```
dsolve((-x^2+1)*diff(diff(y(x),x),x)-8*diff(y(x),x)*x-12*y(x) = 0,y(x),singsol=all)
```

$$y=\frac{c_2x^3+3c_1x^2+3c_2x+c_1}{\left(x^2-1\right)^3}$$

Mathematica DSolve solution

Solving time : `0.318` (sec)
Leaf size : 73

```
DSolve[{(1-x^2)*D[y[x],{x,2}]-8*x*D[y[x],x]-12*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tru
```

$$y(x) \to \frac{\exp\left(\int_1^x \frac{K[1]+3}{K[1]^2-1} dK[1]\right)\left(c_2 \int_1^x \exp\left(-2\int_1^{K[2]} \frac{K[1]+3}{K[1]^2-1} dK[1]\right) dK[2] + c_1\right)}{(x^2-1)^2}$$

2.1.54 Problem 56

Internal problem ID [9226]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 56
Date solved : `Monday, January 27, 2025 at 05:52:17 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(2x^2+1)\,y''+7xy'+2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.337 (sec)

Writing the ode as

$$(2x^2+1)\,y''+7xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x^2+1\\B&=7x\\C&=2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{5x^2+6}{4\,(2x^2+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=5x^2+6\\t&=4(2x^2+1)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{5x^2+6}{4\,(2x^2+1)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.98: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2+1)^2$. There is a pole at $x = \frac{i\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{i\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{64\left(x - \frac{i\sqrt{2}}{2}\right)^2} - \frac{7}{64\left(x + \frac{i\sqrt{2}}{2}\right)^2} - \frac{17i\sqrt{2}}{64\left(x - \frac{i\sqrt{2}}{2}\right)} + \frac{17i\sqrt{2}}{64\left(x + \frac{i\sqrt{2}}{2}\right)}$$

For the pole at $x = \frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8} \end{aligned}$$

For the pole at $x = -\frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{5x^2+6}{4\left(2x^2+1\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{5x^2+6}{4\left(2x^2+1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$\frac{i\sqrt{2}}{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$
$-\frac{i\sqrt{2}}{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{5}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-}\right) \\
&= \frac{5}{4} - \left(\frac{1}{4}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{8x-4i\sqrt{2}} + \frac{1}{8x+4i\sqrt{2}} + (0) \\
&= \frac{1}{8x-4i\sqrt{2}} + \frac{1}{8x+4i\sqrt{2}} \\
&= \frac{x}{4x^2+2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{8x-4i\sqrt{2}} + \frac{1}{8x+4i\sqrt{2}}\right)(1) + \left(\left(-\frac{1}{8\left(x-\frac{i\sqrt{2}}{2}\right)^2} - \frac{1}{8\left(x+\frac{i\sqrt{2}}{2}\right)^2}\right) + \left(\frac{1}{8x-4i\sqrt{2}} + \frac{1}{8x+4i\sqrt{}}\right.\right.
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, e^{\int \left(\frac{1}{8x-4i\sqrt{2}} + \frac{1}{8x+4i\sqrt{2}}\right) dx} \\
&= (x)\left(\left(i\sqrt{2}-2x\right)\left(2x+i\sqrt{2}\right)\right)^{1/8} \\
&= x\left(-4x^2-2\right)^{1/8}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7x}{2x^2+1}\, dx} \\
&= z_1 e^{-\frac{7\ln\left(2x^2+1\right)}{8}} \\
&= z_1 \left(\frac{1}{\left(2x^2+1\right)^{7/8}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{2^{7/8} x \left(-4x^2-2\right)^{1/8}}{\left(4x^2+2\right)^{7/8}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{7x}{2x^2+1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{7\ln(2x^2+1)}{4}}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{2^{1/4}(4x^2+2)^{7/4}}{4\,(2x^2+1)^{7/4}\, x^2\, (-4x^2-2)^{1/4}} dx \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{2^{7/8} x (-4x^2-2)^{1/8}}{(4x^2+2)^{7/8}} \right) + c_2 \left(\frac{2^{7/8} x (-4x^2-2)^{1/8}}{(4x^2+2)^{7/8}} \left(\int \frac{2^{1/4}(4x^2+2)^{7/4}}{4\,(2x^2+1)^{7/4}\, x^2\, (-4x^2-2)^{1/4}} dx \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Legendre successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special functio
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.039 (sec)
Leaf size : 37

```
dsolve((2*x^2+1)*diff(diff(y(x),x),x)+7*diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_1 \operatorname{LegendreP}\left(\frac{1}{4}, \frac{3}{4}, i\sqrt{2}\, x\right) + c_2 \operatorname{LegendreQ}\left(\frac{1}{4}, \frac{3}{4}, i\sqrt{2}\, x\right)}{\left(2x^2+1\right)^{3/8}}$$

Mathematica DSolve solution

Solving time : 0.073 (sec)
Leaf size : 66

```
DSolve[{(1+2*x^2)*D[y[x],{x,2}]+7*x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to \frac{c_2 Q_{\frac{1}{4}}^{\frac{3}{4}}\left(i\sqrt{2}x\right)}{\left(2x^2+1\right)^{3/8}} + \frac{2i\sqrt[4]{2}c_1 x}{\left(2x^2+1\right)^{3/4} \operatorname{Gamma}\left(\frac{1}{4}\right)}$$

2.1.55 Problem 57

Internal problem ID [9227]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 57
Date solved : `Monday, January 27, 2025 at 05:52:18 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-x^2+1\right) y''-5 x y'-4 y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.241 (sec)

Writing the ode as

$$\left(-x^2+1\right) y''-5 x y'-4 y=0 \tag{1}$$
$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+1 \\ B &= -5x \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{-x^2+6}{4\left(x^2-1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2+6 \\ t &= 4\left(x^2-1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-x^2+6}{4\left(x^2-1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.99: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{16\,(x-1)} + \frac{5}{16\,(x-1)^2} + \frac{5}{16\,(x+1)^2} + \frac{7}{16\,(x+1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 + 6}{4(x^2 - 1)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 + 6}{4(x^2 - 1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
-1	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^- + \alpha_{c_2}^-) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{4(x - 1)} - \frac{1}{4(x + 1)} + (-)(0) \\
&= -\frac{1}{4(x - 1)} - \frac{1}{4(x + 1)} \\
&= -\frac{x}{2x^2 - 2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{4(x-1)} - \frac{1}{4(x+1)}\right)(1) + \left(\left(\frac{1}{4(x-1)^2} + \frac{1}{4(x+1)^2}\right) + \left(-\frac{1}{4(x-1)} - \frac{1}{4(x+1)}\right)^2 - \left(\frac{-3}{4(x\right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (x)\, e^{\int \left(-\frac{1}{4(x-1)} - \frac{1}{4(x+1)}\right) dx} \\ &= (x) \frac{1}{((x-1)(x+1))^{1/4}} \\ &= \frac{x}{(x^2-1)^{1/4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-5x}{-x^2+1}\, dx} \\ &= z_1 e^{-\frac{5\ln(x-1)}{4} - \frac{5\ln(x+1)}{4}} \\ &= z_1 \left(\frac{1}{(x-1)^{5/4}(x+1)^{5/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x}{(x-1)^{5/4}(x+1)^{5/4}(x^2-1)^{1/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-5x}{-x^2+1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{5\ln(x-1)}{2} - \frac{5\ln(x+1)}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{(x^2-1)^{3/2}}{x} - x\sqrt{x^2-1} + \ln\left(x + \sqrt{x^2-1}\right)\right) \end{aligned}$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{x}{(x-1)^{5/4}(x+1)^{5/4}(x^2-1)^{1/4}} \right) + c_2 \left(\frac{x}{(x-1)^{5/4}(x+1)^{5/4}(x^2-1)^{1/4}} \left(\frac{(x^2-1)^{3/2}}{x} - x\sqrt{x^2-1} + \right. \right.$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(-x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) - 5x\left(\frac{d}{dx}y(x)\right) - 4y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{4y(x)}{x^2-1} - \frac{5\left(\frac{d}{dx}y(x)\right)x}{x^2-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{5\left(\frac{d}{dx}y(x)\right)x}{x^2-1} + \frac{4y(x)}{x^2-1} = 0$

- □ Check to see if x_0 is a regular singular point
  - ◦ Define functions

    $\left[P_2(x) = \frac{5x}{x^2-1}, P_3(x) = \frac{4}{x^2-1}\right]$
  - ◦ $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((x+1) \cdot P_2(x)\right)\Big|_{x=-1} = \frac{5}{2}$
  - ◦ $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $\left((x+1)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$
  - ◦ $x = -1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -1$

- Multiply by denominators

  $(x^2-1)\left(\frac{d^2}{dx^2}y(x)\right) + 5x\left(\frac{d}{dx}y(x)\right) + 4y(x) = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u = 0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (5u-5)\left(\frac{d}{du}y(u)\right) + 4y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

- □ Rewrite ODE with series expansions
  - ◦ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$
  - ◦ Shift index using $k->k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$
  - ◦ Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$-a_0 r(3+2r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty}\left(-a_{k+1}(k+1+r)(2k+5+2r) + a_k(k+r+2)^2\right)u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
$$-r(3+2r) = 0$$
- Values of r that satisfy the indicial equation
$$r \in \left\{0, -\frac{3}{2}\right\}$$
- Each term in the series must be 0, giving the recursion relation
$$a_k(k+r+2)^2 - 2(k+1+r)\,a_{k+1}\left(k+\frac{5}{2}+r\right) = 0$$
- Recursion relation that defines series solution to ODE
$$a_{k+1} = \frac{a_k(k+r+2)^2}{(k+1+r)(2k+5+2r)}$$
- Recursion relation for $r=0$
$$a_{k+1} = \frac{a_k(k+2)^2}{(k+1)(2k+5)}$$
- Solution for $r=0$
$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(k+2)^2}{(k+1)(2k+5)}\right]$$
- Revert the change of variables $u = x+1$
$$\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+1} = \frac{a_k(k+2)^2}{(k+1)(2k+5)}\right]$$
- Recursion relation for $r = -\frac{3}{2}$
$$a_{k+1} = \frac{a_k\left(k+\frac{1}{2}\right)^2}{\left(k-\frac{1}{2}\right)(2k+2)}$$
- Solution for $r = -\frac{3}{2}$
$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{3}{2}}, a_{k+1} = \frac{a_k\left(k+\frac{1}{2}\right)^2}{\left(k-\frac{1}{2}\right)(2k+2)}\right]$$
- Revert the change of variables $u = x+1$
$$\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^{k-\frac{3}{2}}, a_{k+1} = \frac{a_k\left(k+\frac{1}{2}\right)^2}{\left(k-\frac{1}{2}\right)(2k+2)}\right]$$
- Combine solutions and rename parameters
$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k(x+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k(x+1)^{k-\frac{3}{2}}\right), a_{k+1} = \frac{a_k(k+2)^2}{(k+1)(2k+5)}, b_{k+1} = \frac{b_k\left(k+\frac{1}{2}\right)^2}{\left(k-\frac{1}{2}\right)(2k+2)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
```

```
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.033 (sec)
Leaf size : 39

```
dsolve((-x^2+1)*diff(diff(y(x),x),x)-5*diff(y(x),x)*x-4*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{\ln\left(x+\sqrt{x^2-1}\right) c_2 x + c_1 x - \sqrt{x^2-1}\, c_2}{\left(x^2-1\right)^{3/2}}$$

Mathematica DSolve solution

Solving time : 0.155 (sec)
Leaf size : 47

```
DSolve[{(1-x^2)*D[y[x],{x,2}]-5*x*D[y[x],x]-4*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to \frac{c_2 x \arcsin(x)}{(1-x^2)^{3/2}} + \frac{c_1 x}{(x^2-1)^{3/2}} - \frac{c_2}{x^2-1}$$

2.1.56 Problem 58

Internal problem ID [9228]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 58
Date solved : Monday, January 27, 2025 at 05:52:18 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$\left(x^2+1\right)y''-10xy'+28y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.329 (sec)

Writing the ode as

$$\left(x^2+1\right)y''-10xy'+28y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= -10x \\ C &= 28 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{2x^2-33}{\left(x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2x^2-33 \\ t &= \left(x^2+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{2x^2-33}{\left(x^2+1\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.101: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{35}{4(x-i)^2} + \frac{35}{4(x+i)^2} + \frac{31i}{4(x-i)} - \frac{31i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^2 - 33}{(x^2+1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2x^2 - 33}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{7}{2}$	$-\frac{5}{2}$
$-i$	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$\begin{aligned}d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right)\\ &= 2 - (1)\\ &= 1\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (+)[\sqrt{r}]_\infty\\ &= -\frac{5}{2\,(x-i)} + \frac{7}{2\,(x+i)} + (0)\\ &= -\frac{5}{2\,(x-i)} + \frac{7}{2\,(x+i)}\\ &= \frac{x-6i}{x^2+1}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{5}{2(x-i)} + \frac{7}{2(x+i)}\right)(1) + \left(\left(\frac{5}{2(x-i)^2} - \frac{7}{2(x+i)^2}\right) + \left(-\frac{5}{2(x-i)} + \frac{7}{2(x+i)}\right)^2 - \left(\frac{2x^{2}}{(x^{2}}\right.\right.$$
$$-\frac{2(6i+a_0)(x^{2}}{(-x+i)^2(x}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -6i\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - 6i$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\,dx} \\
&= (x-6i)\,e^{\int \left(-\frac{5}{2(x-i)} + \frac{7}{2(x+i)}\right)dx} \\
&= (x-6i)\,e^{\frac{\ln(x^2+1)}{2} - 6i\arctan(x)} \\
&= \frac{(-x+6i)(x^2+1)^{7/2}}{(-x+i)^6}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-10x}{x^2+1}\,dx} \\
&= z_1 e^{\frac{5\ln(x^2+1)}{2}} \\
&= z_1\left((x^2+1)^{5/2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2+1)^6(-x+6i)}{(-x+i)^6}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-10x}{x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{5\ln(x^2+1)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{724i}{2401(x+i)^4} - \frac{16i}{147(x+i)^6} - \frac{3125i}{117649(x+i)^2} + \frac{496}{1715(x+i)^5} - \frac{7432}{50421(x+i)^3}\right. \\
&\qquad \left. - \frac{3125}{823543(x+i)} + \frac{3125}{823543(x-6i)}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{(x^2+1)^6(-x+6i)}{(-x+i)^6}\right) \\
&+ c_2\left(\frac{(x^2+1)^6(-x+6i)}{(-x+i)^6}\left(\frac{724i}{2401(x+i)^4} - \frac{16i}{147(x+i)^6} - \frac{3125i}{117649(x+i)^2}\right.\right. \\
&\left.\left.+ \frac{496}{1715(x+i)^5} - \frac{7432}{50421(x+i)^3} - \frac{3125}{823543(x+i)} + \frac{3125}{823543(x-6i)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 39

```
dsolve((x^2+1)*diff(diff(y(x),x),x)-10*diff(y(x),x)*x+28*y(x) = 0,y(x),singsol=all)
```

$$
y = c_1 + \frac{35}{3}c_1x^4 - 14c_1x^2 + c_2x^7 + 21c_2x^5 - 105c_2x^3 + 35c_2x
$$

Mathematica DSolve solution

Solving time : 0.352 (sec)
Leaf size : 93

```
DSolve[{(1+x^2)*D[y[x],{x,2}]-10*x*D[y[x],x]+28*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$
y(x) \to (x+6i)(x^2+1)^{5/2}\exp\left(\int_1^x \frac{K[1]+6i}{K[1]^2+1}dK[1]\right)\left(c_2\int_1^x \frac{\exp\left(-2\int_1^{K[2]}\frac{K[1]+6i}{K[1]^2+1}dK[1]\right)}{(K[2]+6i)^2}dK[2] + c_1\right)
$$

2.1.57 Problem 59

Internal problem ID [9229]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 59
Date solved : `Monday, January 27, 2025 at 05:52:19 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + xy' + 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.207 (sec)

Writing the ode as

$$y'' + xy' + 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 6 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.102: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{2x} - \frac{9}{4x^3} - \frac{27}{4x^5} - \frac{405}{16x^7} - \frac{1701}{16x^9} - \frac{15309}{32x^{11}} - \frac{72171}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 6}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} - \frac{3}{2}\right) + (0) \\ &= \frac{x^2}{4} - \frac{3}{2}\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{3}{2}\right) - (0) \\ &= -\frac{3}{2}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{3}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{3}{2}\right)\right) &= 0 \\ a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= (x)\, e^{\int -\frac{x}{2} dx} \\ &= (x)\, e^{-\frac{x^2}{4}} \\ &= x\, e^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{1}\, dx} \\ &= z_1 e^{-\frac{x^2}{4}} \\ &= z_1\left(e^{-\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^2}{2}} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x^2}{2}} x \right) + c_2 \left(\mathrm{e}^{-\frac{x^2}{2}} x \left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} \right) \right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + x\left(\frac{d}{dx}y(x)\right) + 2y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite DE with series expansions

  ◦ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

  $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

  $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+2)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

$(k+2)(ka_{k+2}+a_k+a_{k+2})=0$

- Recursion relation that defines the series solution to the ODE

$$\left[y(x)=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=-\frac{a_k}{k+1}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 37

```
dsolve(diff(diff(y(x),x),x)+diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y=-x\left(c_2\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)\pi-c_1\right)\mathrm{e}^{-\frac{x^2}{2}}+i\sqrt{\pi}\,\sqrt{2}\,c_2$$

Mathematica DSolve solution

Solving time : `0.048` (sec)
Leaf size : 69

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to -\sqrt{\frac{\pi}{2}}c_2e^{-\frac{x^2}{2}}\sqrt{x^2}\mathrm{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{2}}\right)+\sqrt{2}c_1e^{-\frac{x^2}{2}}x+c_2$$

2.1.58 Problem 60

Internal problem ID [9230]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 60
Date solved : `Monday, January 27, 2025 at 05:52:20 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(2x^2+1\right)y''-9xy'-6y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.470 (sec)

Writing the ode as

$$\left(2x^2+1\right)y''-9xy'-6y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=2x^2+1\\ B&=-9x\\ C&=-6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{165x^2+6}{4\left(2x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=165x^2+6\\ t&=4\left(2x^2+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{165x^2+6}{4\left(2x^2+1\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.104: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 + 1)^2$. There is a pole at $x = \frac{i\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{i\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{153}{64 \left(x - \frac{i\sqrt{2}}{2}\right)^2} + \frac{153}{64 \left(x + \frac{i\sqrt{2}}{2}\right)^2} - \frac{177i\sqrt{2}}{64 \left(x - \frac{i\sqrt{2}}{2}\right)} + \frac{177i\sqrt{2}}{64 \left(x + \frac{i\sqrt{2}}{2}\right)}$$

For the pole at $x = \frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{153}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{17}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{9}{8} \end{aligned}$$

For the pole at $x = -\frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{153}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{17}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{9}{8}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{165x^2+6}{4\left(2x^2+1\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{165}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{15}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{11}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{165x^2+6}{4\left(2x^2+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{i\sqrt{2}}{2}$	2	0	$\frac{17}{8}$	$-\frac{9}{8}$
$-\frac{i\sqrt{2}}{2}$	2	0	$\frac{17}{8}$	$-\frac{9}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{15}{4}$	$-\frac{11}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{15}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{15}{4} - \left(-\frac{9}{4}\right) \\
&= 6
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{9}{8\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{9}{8\left(x+\frac{i\sqrt{2}}{2}\right)} + (0) \\ &= -\frac{9}{8\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{9}{8\left(x+\frac{i\sqrt{2}}{2}\right)} \\ &= -\frac{9x}{4x^2+2}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 6$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(30x^4 + 20x^3a_5 + 12x^2a_4 + 6xa_3 + 2a_2) + 2\left(-\frac{9}{8\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{9}{8\left(x+\frac{i\sqrt{2}}{2}\right)}\right)(6x^5 + 5x^4a_5 + 4x^3a_4 + 3x^2a_3$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{1}{3}, a_1 = 0, a_2 = 1, a_3 = 0, a_4 = \frac{5}{3}, a_5 = 0\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^6 + \frac{5}{3}x^4 + x^2 + \frac{1}{3}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= pe^{\int \omega\, dx} \\ &= \left(x^6 + \frac{5}{3}x^4 + x^2 + \frac{1}{3}\right) \mathrm{e}^{\int \left(-\frac{9}{8\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{9}{8\left(x+\frac{i\sqrt{2}}{2}\right)}\right) dx} \\ &= \left(x^6 + \frac{5}{3}x^4 + x^2 + \frac{1}{3}\right) \frac{1}{\left((i\sqrt{2}-2x)\,(2x+i\sqrt{2})\right)^{9/8}} \\ &= \frac{-3x^6 - 5x^4 - 3x^2 - 1}{(-4x^2-2)^{1/8}\,(12x^2+6)}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-9x}{2x^2+1}\, dx} \\ &= z_1 e^{\frac{9\ln(2x^2+1)}{8}} \\ &= z_1\left((2x^2+1)^{9/8}\right)\end{aligned}$$

Which simplifies to

$$y_1 = -\frac{2^{7/8}(4x^2+2)^{1/8}(3x^6+5x^4+3x^2+1)}{12(-4x^2-2)^{1/8}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-9x}{2x^2+1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\frac{9\ln(2x^2+1)}{4}}}{(y_1)^2}\,dx \\ &= y_1 \left(\int \frac{36(2x^2+1)^{9/4}\, 2^{1/4}(-4x^2-2)^{1/4}}{(4x^2+2)^{1/4}(3x^6+5x^4+3x^2+1)^2}dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1y_1 + c_2y_2 \\ &= c_1\left(-\frac{2^{7/8}(4x^2+2)^{1/8}(3x^6+5x^4+3x^2+1)}{12(-4x^2-2)^{1/8}}\right) + c_2\left(-\frac{2^{7/8}(4x^2+2)^{1/8}(3x^6+5x^4+3x^2+1)}{12(-4x^2-2)^{1/8}}\left(\int \frac{36\cdots}{(4x^2\cdots}\right.\right. \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Legendre successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special functic
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.013` (sec)
Leaf size : 37

```
dsolve((2*x^2+1)*diff(diff(y(x),x),x)-9*diff(y(x),x)*x-6*y(x) = 0,y(x),singsol=all)
```

$$y = \left(\text{LegendreP}\left(\frac{11}{4}, \frac{13}{4}, i\sqrt{2}\,x\right) c_1 + \text{LegendreQ}\left(\frac{11}{4}, \frac{13}{4}, i\sqrt{2}\,x\right) c_2\right) \left(2x^2+1\right)^{13/8}$$

Mathematica DSolve solution

Solving time : `0.236` (sec)
Leaf size : 71

```
DSolve[{(1+2*x^2)*D[y[x],{x,2}]-9*x*D[y[x],x]-6*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tr
```

$$y(x) \to c_2 \left(2x^2+1\right)^{13/8} Q_{\frac{11}{4}}^{\frac{13}{4}}\left(i\sqrt{2}x\right) + \frac{64\sqrt[4]{2}c_1(3x^6+5x^4+3x^2+1)}{3\,\text{Gamma}\left(-\frac{9}{4}\right)}$$

2.1.59 Problem 61

Internal problem ID [9231]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 61
Date solved : `Monday, January 27, 2025 at 05:52:21 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(2x^2-8x+11\right)y''-16(x-2)\,y'+36y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.599 (sec)

Writing the ode as

$$\left(2x^2-8x+11\right)y''+(-16x+32)\,y'+36y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=2x^2-8x+11\\ B&=-16x+32\\ C&=36 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{8x^2-32x-100}{\left(2x^2-8x+11\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=8x^2-32x-100\\ t&=\left(2x^2-8x+11\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{8x^2-32x-100}{\left(2x^2-8x+11\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.105: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x^2 - 8x + 11)^2$. There is a pole at $x = 2 + \frac{i\sqrt{6}}{2}$ of order 2. There is a pole at $x = 2 - \frac{i\sqrt{6}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{\left(x - 2 - \frac{i\sqrt{6}}{2}\right)^2} + \frac{6}{\left(x - 2 + \frac{i\sqrt{6}}{2}\right)^2} + \frac{5i\sqrt{6}}{3\left(x - 2 - \frac{i\sqrt{6}}{2}\right)} - \frac{5i\sqrt{6}}{3\left(x - 2 + \frac{i\sqrt{6}}{2}\right)}$$

For the pole at $x = 2 + \frac{i\sqrt{6}}{2}$ let b be the coefficient of $\frac{1}{\left(x-2-\frac{i\sqrt{6}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2 \end{aligned}$$

For the pole at $x = 2 - \frac{i\sqrt{6}}{2}$ let b be the coefficient of $\frac{1}{\left(x-2+\frac{i\sqrt{6}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{8x^2 - 32x - 100}{(2x^2 - 8x + 11)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{8x^2 - 32x - 100}{(2x^2 - 8x + 11)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$2 + \frac{i\sqrt{6}}{2}$	2	0	3	-2
$2 - \frac{i\sqrt{6}}{2}$	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{2}{x-2-\frac{i\sqrt{6}}{2}} + \frac{3}{x-2+\frac{i\sqrt{6}}{2}} + (0) \\
&= -\frac{2}{x-2-\frac{i\sqrt{6}}{2}} + \frac{3}{x-2+\frac{i\sqrt{6}}{2}} \\
&= \frac{-5i\sqrt{6}+2x-4}{2x^2-8x+11}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{2}{x-2-\frac{i\sqrt{6}}{2}} + \frac{3}{x-2+\frac{i\sqrt{6}}{2}}\right)(1) + \left(\left(\frac{2}{\left(x-2-\frac{i\sqrt{6}}{2}\right)^2} - \frac{3}{\left(x-2+\frac{i\sqrt{6}}{2}\right)^2}\right) + \left(-\frac{2}{x-2-\frac{i\sqrt{6}}{2}} +\right.\right.
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = -\frac{5i\sqrt{6}}{2} - 2\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x - 2 - \frac{5i\sqrt{6}}{2}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - 2 - \frac{5i\sqrt{6}}{2}\right) e^{\int \left(-\frac{2}{x-2-\frac{i\sqrt{6}}{2}} + \frac{3}{x-2+\frac{i\sqrt{6}}{2}}\right) dx} \\
&= \left(x - 2 - \frac{5i\sqrt{6}}{2}\right) e^{\frac{\ln\left(4x^2-16x+22\right)}{2} - 5i \arctan\left(\frac{(2x-4)\sqrt{6}}{6}\right)} \\
&= \frac{9\left(5\sqrt{6}+2ix-4i\right)\left(2x^2-8x+11\right)^3 \sqrt{6}}{2\left(-x\sqrt{6}+2\sqrt{6}+3i\right)^5}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-16x+32}{2x^2-8x+11} \, dx} \\
&= z_1 e^{2\ln\left(2x^2-8x+11\right)} \\
&= z_1 \left(\left(2x^2-8x+11\right)^2\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = -\frac{9(2x^2-8x+11)^5 \left(5\sqrt{6}+2ix-4i\right)\sqrt{6}}{2\left(x\sqrt{6}-2\sqrt{6}-3i\right)^5}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-16x+32}{2x^2-8x+11}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{4\ln(2x^2-8x+11)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{10i\sqrt{6}}{27\left(2x-4+i\sqrt{6}\right)^4} + \frac{8i\sqrt{6}}{729\left(2x-4+i\sqrt{6}\right)^2} - \frac{16}{15\left(2x-4+i\sqrt{6}\right)^5} \right. \\
&\quad \left. + \frac{22}{81\left(2x-4+i\sqrt{6}\right)^3} + \frac{4}{2187\left(2x-4+i\sqrt{6}\right)} - \frac{4}{2187\left(-5i\sqrt{6}+2x-4\right)}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(-\frac{9(2x^2-8x+11)^5 \left(5\sqrt{6}+2ix-4i\right)\sqrt{6}}{2\left(x\sqrt{6}-2\sqrt{6}-3i\right)^5}\right) \\
&\quad + c_2 \left(-\frac{9(2x^2-8x+11)^5 \left(5\sqrt{6}+2ix-4i\right)\sqrt{6}}{2\left(x\sqrt{6}-2\sqrt{6}-3i\right)^5} \left(-\frac{10i\sqrt{6}}{27\left(2x-4+i\sqrt{6}\right)^4} \right.\right. \\
&\quad + \frac{8i\sqrt{6}}{729\left(2x-4+i\sqrt{6}\right)^2} - \frac{16}{15\left(2x-4+i\sqrt{6}\right)^5} + \frac{22}{81\left(2x-4+i\sqrt{6}\right)^3} \\
&\quad \left.\left. + \frac{4}{2187\left(2x-4+i\sqrt{6}\right)} - \frac{4}{2187\left(-5i\sqrt{6}+2x-4\right)}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(2x^2-8x+11)\left(\frac{d^2}{dx^2}y(x)\right) - 16(x-2)\left(\frac{d}{dx}y(x)\right) + 36y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{36y(x)}{2x^2-8x+11} + \frac{16(x-2)\left(\frac{d}{dx}y(x)\right)}{2x^2-8x+11}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lir

  $\frac{d^2}{dx^2}y(x) - \frac{16(x-2)\left(\frac{d}{dx}y(x)\right)}{2x^2-8x+11} + \frac{36y(x)}{2x^2-8x+11} = 0$

□ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = -\frac{16(x-2)}{2x^2-8x+11}, P_3(x) = \frac{36}{2x^2-8x+11}\right]$

- $\left(x - 2 + \frac{I\sqrt{6}}{2}\right) \cdot P_2(x)$ is analytic at $x = 2 - \frac{I\sqrt{6}}{2}$

  $\left(\left(x - 2 + \frac{I\sqrt{6}}{2}\right) \cdot P_2(x)\right)\bigg|_{x=2-\frac{I\sqrt{6}}{2}} = 0$

- $\left(x - 2 + \frac{I\sqrt{6}}{2}\right)^2 \cdot P_3(x)$ is analytic at $x = 2 - \frac{I\sqrt{6}}{2}$

  $\left(\left(x - 2 + \frac{I\sqrt{6}}{2}\right)^2 \cdot P_3(x)\right)\bigg|_{x=2-\frac{I\sqrt{6}}{2}} = 0$

- $x = 2 - \frac{I\sqrt{6}}{2}$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 2 - \frac{I\sqrt{6}}{2}$

- Multiply by denominators

  $(2x^2 - 8x + 11)\left(\frac{d^2}{dx^2}y(x)\right) + (-16x + 32)\left(\frac{d}{dx}y(x)\right) + 36y(x) = 0$

- Change variables using $x = u + 2 - \frac{I\sqrt{6}}{2}$ so that the regular singular point is at $u = 0$

  $(2u^2 - 2\,Iu\sqrt{6})\left(\frac{d^2}{du^2}y(u)\right) + (-16u + 8\,I\sqrt{6})\left(\frac{d}{du}y(u)\right) + 36y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2\,I\sqrt{6}\,r(r-5)\,a_0u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2\,I\sqrt{6}\,(k+1+r)(k-4+r)\,a_{k+1} + 2a_k(k+r-3)(k+r-6)\right)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2\,I\sqrt{6}\,r(r-5) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 5\}$

- Each term in the series must be 0, giving the recursion relation

  $-2\,I\sqrt{6}\,(k+1+r)(k-4+r)\,a_{k+1} + 2a_k(k+r-3)(k+r-6) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{-\frac{I}{6}a_k(k^2+2kr+r^2-9k-9r+18)\sqrt{6}}{k^2+2kr+r^2-3k-3r-4}$

- Recursion relation for $r = 0$; series terminates at $k = 3$

  $a_{k+1} = \frac{-\frac{I}{6}a_k(k^2-9k+18)\sqrt{6}}{k^2-3k-4}$

- Apply recursion relation for $k = 0$
  $a_1 = \frac{3 I}{4} a_0 \sqrt{6}$
- Apply recursion relation for $k = 1$
  $a_2 = \frac{5 I}{18} a_1 \sqrt{6}$
- Express in terms of a_0
  $a_2 = -\frac{5 a_0}{4}$
- Apply recursion relation for $k = 2$
  $a_3 = \frac{I}{9} a_2 \sqrt{6}$
- Express in terms of a_0
  $a_3 = -\frac{5 I}{36} a_0 \sqrt{6}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li
  $y(u) = a_0 \cdot \left(1 + \frac{3 I \sqrt{6}\, u}{4} - \frac{5 u^2}{4} - \frac{5 I \sqrt{6}\, u^3}{36}\right)$
- Revert the change of variables $u = x - 2 + \frac{I \sqrt{6}}{2}$
  $\left[y(x) = -\frac{I}{72} a_0 \sqrt{6}\, (10 x^3 - 60 x^2 + 111 x - 62)\right]$
- Recursion relation for $r = 5$; series terminates at $k = 1$
  $a_{k+1} = \frac{-\frac{I}{6} a_k (k^2+k-2) \sqrt{6}}{k^2+7k+6}$
- Apply recursion relation for $k = 0$
  $a_1 = \frac{I}{18} a_0 \sqrt{6}$
- Terminating series solution of the ODE for $r = 5$. Use reduction of order to find the second li
  $y(u) = a_0 \cdot \left(1 + \frac{I \sqrt{6}\, u}{18}\right)$
- Revert the change of variables $u = x - 2 + \frac{I \sqrt{6}}{2}$
  $\left[y(x) = a_0 \left(\frac{5}{6} + \frac{I(x-2)\sqrt{6}}{18}\right)\right]$
- Combine solutions and rename parameters
  $\left[y(x) = -\frac{I a_0 \sqrt{6}\, (10 x^3 - 60 x^2 + 111 x - 62)}{72} + b_0 \left(\frac{5}{6} + \frac{I(x-2)\sqrt{6}}{18}\right)\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 55

```
dsolve((2*x^2-8*x+11)*diff(diff(y(x),x),x)-16*(x-2)*diff(y(x),x)+36*y(x) = 0,y(x),singso
```

$$y = c_2x^6 - 12c_2x^5 + \frac{165c_2x^4}{2} + c_1x^3 + \frac{3(-8c_1 - 1815c_2)\,x^2}{4} + \frac{3(37c_1 + 10890c_2)\,x}{10} - \frac{31c_1}{5} - \frac{16577c_2}{8}$$

Mathematica DSolve solution

Solving time : 0.625 (sec)
Leaf size : 146

```
DSolve[{(11-8*x+2*x^2)*D[y[x],{x,2}]-16*(x-2)*D[y[x],x]+36*y[x]==0,{}},y[x],x,IncludeSingularSo
```

$$y(x) \to \frac{1}{2}\left(2x + 5i\sqrt{6} - 4\right)\left(2x^2 - 8x + 11\right)^2 \exp\left(\int_1^x \frac{2K[1] + 5i\sqrt{6} - 4}{2(K[1]-4)K[1] + 11}dK[1]\right)\left(c_2\int_1^x - \frac{4\exp\left(-2\int_1^{K[2]} \frac{2K[1]+5i\sqrt{6}-4}{2(K[1]-4)K[1]+11}dK[1]\right)}{\left(-2iK[2] + 5\sqrt{6} + 4i\right)^2}dK[2] + c_1\right)$$

2.1.60 Problem 62

Internal problem ID [9232]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 62
Date solved : `Monday, January 27, 2025 at 05:52:22 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + (x-3)\,y' + 3y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.282 (sec)

Writing the ode as

$$y'' + (x-3)\,y' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x-3 \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6x - 1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 6x - 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4} + \frac{1}{4}x^2 - \frac{3}{2}x\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.107: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{2} - \frac{5}{2x} - \frac{15}{2x^2} - \frac{115}{4x^3} - \frac{495}{4x^4} - \frac{2285}{4x^5} - \frac{11055}{4x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} - \frac{3}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 - \frac{3}{2}x + \frac{9}{4}$$

This shows that the coefficient of 1 in the above is $\frac{9}{4}$. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 6x - 1}{4} \\ &= Q + \frac{R}{4} \\ &= \left(-\frac{1}{4} + \frac{1}{4}x^2 - \frac{3}{2}x\right) + (0) \\ &= -\frac{1}{4} + \frac{1}{4}x^2 - \frac{3}{2}x \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{4}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{4}\right) - \left(\frac{9}{4}\right) \\ &= -\frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} - \frac{3}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &= -3 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &= 2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4} + \frac{1}{4}x^2 - \frac{3}{2}x$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2} - \frac{3}{2}$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2} - \frac{3}{2}\right) \\ &= \frac{3}{2} - \frac{x}{2} \\ &= \frac{3}{2} - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{3}{2} - \frac{x}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2}\right) + \left(\frac{3}{2} - \frac{x}{2}\right)^2 - \left(-\frac{1}{4} + \frac{1}{4}x^2 - \frac{3}{2}x\right)\right) = 0$$

$$(x+3)\,a_1 + 6x + 2a_0 + 2 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 8, a_1 = -6\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 6x + 8$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\,dx} \\ &= \left(x^2 - 6x + 8\right) e^{\int \left(\frac{3}{2} - \frac{x}{2}\right)dx} \\ &= \left(x^2 - 6x + 8\right) e^{\frac{3}{2}x - \frac{1}{4}x^2} \\ &= \left(x^2 - 6x + 8\right) e^{-\frac{x(-6+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x-3}{1}\,dx} \\ &= z_1 e^{\frac{3}{2}x - \frac{1}{4}x^2} \\ &= z_1\left(e^{-\frac{x(-6+x)}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x(-6+x)}{2}} \left(x^2 - 6x + 8\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x-3}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{1}{2}x^2+3x}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{-\frac{1}{2}x^2+3x} \mathrm{e}^{x(-6+x)}}{(x^2-6x+8)^2} dx \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-\frac{x(-6+x)}{2}} (x^2 - 6x + 8) \right) + c_2 \left(\mathrm{e}^{-\frac{x(-6+x)}{2}} (x^2 - 6x + 8) \left(\int \frac{\mathrm{e}^{-\frac{1}{2}x^2+3x} \mathrm{e}^{x(-6+x)}}{(x^2-6x+8)^2} dx \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + (x-3)\left(\frac{d}{dx}y(x)\right) + 3y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite DE with series expansions

  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..1$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$

  - Shift index using $k- >k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, x^k$

  - Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k+2$

    $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite DE with series expansions

$$\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - 3a_{k+1}(k+1) + a_k(k+3)\right) x^k = 0$$

- Each term in the series must be 0, giving the recursion relation
$$k^2 a_{k+2} + \left(a_k - 3a_{k+1} + 3a_{k+2}\right) k + 3a_k - 3a_{k+1} + 2a_{k+2} = 0$$
- Recursion relation that defines the series solution to the ODE
$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k k - 3a_{k+1}k + 3a_k - 3a_{k+1}}{k^2+3k+2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special fu
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.017 (sec)
Leaf size : 73

```
dsolve(diff(diff(y(x),x),x)+(x-3)*diff(y(x),x)+3*y(x) = 0,y(x),singsol=all)
```

$$y = (x-2)\,c_2 e^{-\frac{(x-3)^2}{2}}(x-4)\left(\operatorname{erf}\left(\frac{\sqrt{2}\sqrt{-(x-3)^2}}{2}\right) - 1\right)\sqrt{\pi} - \sqrt{2}\sqrt{-(x-3)^2}\,c_2 - c_1 e^{-\frac{(x-3)^2}{2}}(x-2)\,(x-4)$$

Mathematica DSolve solution

Solving time : 0.69 (sec)
Leaf size : 62

```
DSolve[{D[y[x],{x,2}]+(x-3)*D[y[x],x]+3*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-\frac{1}{2}(x-6)x}\left(x^2 - 6x + 8\right)\left(c_2 \int_1^x \frac{e^{\frac{1}{2}(K[1]-6)K[1]}}{(K[1]-4)^2(K[1]-2)^2} dK[1] + c_1\right)$$

2.1.61 Problem 63

Internal problem ID [9233]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 63
Date solved : `Monday, January 27, 2025 at 05:52:22 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2-8x+14\right) y''-8(x-4)\, y'+20y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.285 (sec)

Writing the ode as

$$\left(x^2-8x+14\right) y''+(-8x+32)\, y'+20y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=x^2-8x+14\\ B&=-8x+32\\ C&=20 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{48}{\left(x^2-8x+14\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=48\\ t&=\left(x^2-8x+14\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{48}{\left(x^2-8x+14\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.109: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 8x + 14)^2$. There is a pole at $x = 4 + \sqrt{2}$ of order 2. There is a pole at $x = 4 - \sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{\left(x - 4 + \sqrt{2}\right)^2} + \frac{6}{\left(x - 4 - \sqrt{2}\right)^2} + \frac{3\sqrt{2}}{x - 4 + \sqrt{2}} - \frac{3\sqrt{2}}{x - 4 - \sqrt{2}}$$

For the pole at $x = 4 + \sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x-4-\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

For the pole at $x = 4 - \sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x-4+\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{48}{(x^2 - 8x + 14)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$4+\sqrt{2}$	2	0	3	-2
$4-\sqrt{2}$	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{2}{x-4-\sqrt{2}} + \frac{3}{x-4+\sqrt{2}} + (-)(0) \\
&= -\frac{2}{x-4-\sqrt{2}} + \frac{3}{x-4+\sqrt{2}} \\
&= \frac{x-4-5\sqrt{2}}{x^2-8x+14}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{2}{x-4-\sqrt{2}}+\frac{3}{x-4+\sqrt{2}}\right)(0)+\left(\left(\frac{2}{\left(x-4-\sqrt{2}\right)^{2}}-\frac{3}{\left(x-4+\sqrt{2}\right)^{2}}\right)+\left(-\frac{2}{x-4-\sqrt{2}}+\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}z_1(x)&=pe^{\int \omega\,dx}\\&=\mathrm{e}^{\int\left(-\frac{2}{x-4-\sqrt{2}}+\frac{3}{x-4+\sqrt{2}}\right)dx}\\&=\frac{\left(x-4+\sqrt{2}\right)^{3}}{\left(-x+4+\sqrt{2}\right)^{2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1&=z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\&=z_1e^{-\int \frac{1}{2}\frac{-8x+32}{x^2-8x+14}\,dx}\\&=z_1e^{2\ln(x^2-8x+14)}\\&=z_1\left(\left(x^2-8x+14\right)^2\right)\end{aligned}$$

Which simplifies to

$$y_1=\frac{\left(x^2-8x+14\right)^2\left(x-4+\sqrt{2}\right)^{3}}{\left(-x+4+\sqrt{2}\right)^{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int\frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2&=y_1\int\frac{e^{\int -\frac{-8x+32}{x^2-8x+14}\,dx}}{(y_1)^2}\,dx\\&=y_1\int\frac{e^{4\ln(x^2-8x+14)}}{(y_1)^2}\,dx\\&=y_1\left(-\frac{1}{x-4+\sqrt{2}}-\frac{64}{5\left(x-4+\sqrt{2}\right)^{5}}+\frac{16\sqrt{2}}{\left(x-4+\sqrt{2}\right)^{4}}-\frac{16}{\left(x-4+\sqrt{2}\right)^{3}}\right.\\&\qquad\left.+\frac{4\sqrt{2}}{\left(x-4+\sqrt{2}\right)^{2}}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y&=c_1y_1+c_2y_2\\&=c_1\left(\frac{\left(x^2-8x+14\right)^2\left(x-4+\sqrt{2}\right)^{3}}{\left(-x+4+\sqrt{2}\right)^{2}}\right)\\&\quad+c_2\left(\frac{\left(x^2-8x+14\right)^2\left(x-4+\sqrt{2}\right)^{3}}{\left(-x+4+\sqrt{2}\right)^{2}}\left(-\frac{1}{x-4+\sqrt{2}}-\frac{64}{5\left(x-4+\sqrt{2}\right)^{5}}\right.\right.\\&\qquad\left.\left.+\frac{16\sqrt{2}}{\left(x-4+\sqrt{2}\right)^{4}}-\frac{16}{\left(x-4+\sqrt{2}\right)^{3}}+\frac{4\sqrt{2}}{\left(x-4+\sqrt{2}\right)^{2}}\right)\right)\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x^2-8x+14)\left(\frac{d^2}{dx^2}y(x)\right)-8(-4+x)\left(\frac{d}{dx}y(x)\right)+20y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{20y(x)}{x^2-8x+14}+\frac{8(-4+x)\left(\frac{d}{dx}y(x)\right)}{x^2-8x+14}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)-\frac{8(-4+x)\left(\frac{d}{dx}y(x)\right)}{x^2-8x+14}+\frac{20y(x)}{x^2-8x+14}=0$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x)=-\frac{8(-4+x)}{x^2-8x+14}, P_3(x)=\frac{20}{x^2-8x+14}\right]$

  - $(x-4+\sqrt{2})\cdot P_2(x)$ is analytic at $x=4-\sqrt{2}$

    $\left((x-4+\sqrt{2})\cdot P_2(x)\right)\Big|_{x=4-\sqrt{2}}=0$

  - $(x-4+\sqrt{2})^2\cdot P_3(x)$ is analytic at $x=4-\sqrt{2}$

    $\left((x-4+\sqrt{2})^2\cdot P_3(x)\right)\Big|_{x=4-\sqrt{2}}=0$

  - $x=4-\sqrt{2}$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=4-\sqrt{2}$

- Multiply by denominators

  $(x^2-8x+14)\left(\frac{d^2}{dx^2}y(x)\right)+(-8x+32)\left(\frac{d}{dx}y(x)\right)+20y(x)=0$

- Change variables using $x=u+4-\sqrt{2}$ so that the regular singular point is at $u=0$

  $(u^2-2u\sqrt{2})\left(\frac{d^2}{du^2}y(u)\right)+(-8u+8\sqrt{2})\left(\frac{d}{du}y(u)\right)+20y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2\sqrt{2}(r-5)ra_0u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2\sqrt{2}(k+r-4)(k+1+r)a_{k+1}+a_k(k+r-4)(k+r-5)\right)u^{k+}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-2\sqrt{2}\,(r-5)\,r=0$
- Values of r that satisfy the indicial equation
  $r\in\{0,5\}$
- Each term in the series must be 0, giving the recursion relation
  $\left(-2a_{k+1}(k+1+r)\sqrt{2}+a_k(k+r-5)\right)(k+r-4)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{a_k(k+r-5)\sqrt{2}}{4(k+1+r)}$
- Recursion relation for $r=0$; series terminates at $k=5$
  $a_{k+1}=\frac{a_k(k-5)\sqrt{2}}{4(k+1)}$
- Apply recursion relation for $k=0$
  $a_1=-\frac{5a_0\sqrt{2}}{4}$
- Apply recursion relation for $k=1$
  $a_2=-\frac{a_1\sqrt{2}}{2}$
- Express in terms of a_0
  $a_2=\frac{5a_0}{4}$
- Apply recursion relation for $k=2$
  $a_3=-\frac{a_2\sqrt{2}}{4}$
- Express in terms of a_0
  $a_3=-\frac{5a_0\sqrt{2}}{16}$
- Apply recursion relation for $k=3$
  $a_4=-\frac{a_3\sqrt{2}}{8}$
- Express in terms of a_0
  $a_4=\frac{5a_0}{64}$
- Apply recursion relation for $k=4$
  $a_5=-\frac{a_4\sqrt{2}}{20}$
- Express in terms of a_0
  $a_5=-\frac{a_0\sqrt{2}}{256}$
- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li
  $y(u)=a_0\cdot\left(1-\frac{5u\sqrt{2}}{4}+\frac{5u^2}{4}-\frac{5\sqrt{2}\,u^3}{16}+\frac{5u^4}{64}-\frac{\sqrt{2}\,u^5}{256}\right)$
- Revert the change of variables $u=x-4+\sqrt{2}$
  $\left[y(x)=a_0\left(\frac{(-x^5+20x^4-180x^3+880x^2-2260x+2384)\sqrt{2}}{256}+\frac{5x^4}{128}-\frac{5x^3}{8}+\frac{125x^2}{32}-\frac{45x}{4}+\frac{401}{32}\right)\right]$
- Recursion relation for $r=5$
  $a_{k+1}=\frac{a_k k\sqrt{2}}{4(k+6)}$
- Solution for $r=5$
  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+5},a_{k+1}=\frac{a_k k\sqrt{2}}{4(k+6)}\right]$
- Revert the change of variables $u=x-4+\sqrt{2}$
  $\left[y(x)=\sum_{k=0}^{\infty}a_k\left(x-4+\sqrt{2}\right)^{k+5},a_{k+1}=\frac{a_k k\sqrt{2}}{4(k+6)}\right]$
- Combine solutions and rename parameters
  $\Big[y(x)=a_0\left(\frac{(-x^5+20x^4-180x^3+880x^2-2260x+2384)\sqrt{2}}{256}+\frac{5x^4}{128}-\frac{5x^3}{8}+\frac{125x^2}{32}-\frac{45x}{4}+\frac{401}{32}\right)+\Big(\sum_{k=0}^{\infty}b_k(x-4$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 55

```
dsolve((x^2-8*x+14)*diff(diff(y(x),x),x)-8*(x-4)*diff(y(x),x)+20*y(x) = 0,y(x),singsol=a
```

$$y = c_1 x^5 + c_2 x^4 + 4(-35c_1 - 4c_2)\, x^3 + 20(56c_1 + 5c_2)\, x^2 + 4(-875c_1 - 72c_2)\, x + 4032c_1 + \frac{1604c_2}{5}$$

Mathematica DSolve solution

Solving time : 0.094 (sec)
Leaf size : 77

```
DSolve[{(x^2-8*x+14)*D[y[x],{x,2}]+8*(x-4)*D[y[x],x]+20*y[x]==0,{}},y[x],x,IncludeSingularSolu
```

$$y(x) \to \frac{c_1 P^3_{\frac{1}{2}i\left(i+\sqrt{31}\right)}\left(\frac{x-4}{\sqrt{2}}\right) + c_2 Q^3_{\frac{1}{2}i\left(i+\sqrt{31}\right)}\left(\frac{x-4}{\sqrt{2}}\right)}{\left(x^2-8x+14\right)^{3/2}}$$

2.1.62 Problem 64

Internal problem ID [9234]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 64
Date solved : `Monday, January 27, 2025 at 05:52:23 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(2x^2+4x+5\right)y''-20(x+1)\,y'+60y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.598 (sec)

Writing the ode as

$$\left(2x^2+4x+5\right)y''+(-20x-20)\,y'+60y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2+4x+5 \\ B &= -20x-20 \\ C &= 60 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-210}{\left(2x^2+4x+5\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -210 \\ t &= \left(2x^2+4x+5\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{210}{\left(2x^2+4x+5\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.111: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x^2 + 4x + 5)^2$. There is a pole at $x = -1 + \frac{i\sqrt{6}}{2}$ of order 2. There is a pole at $x = -1 - \frac{i\sqrt{6}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{35}{4\left(x + 1 - \frac{i\sqrt{6}}{2}\right)^2} + \frac{35}{4\left(x + 1 + \frac{i\sqrt{6}}{2}\right)^2} + \frac{35i\sqrt{6}}{12\left(x + 1 - \frac{i\sqrt{6}}{2}\right)} - \frac{35i\sqrt{6}}{12\left(x + 1 + \frac{i\sqrt{6}}{2}\right)}$$

For the pole at $x = -1 + \frac{i\sqrt{6}}{2}$ let b be the coefficient of $\frac{1}{\left(x+1-\frac{i\sqrt{6}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{5}{2} \end{aligned}$$

For the pole at $x = -1 - \frac{i\sqrt{6}}{2}$ let b be the coefficient of $\frac{1}{\left(x+1+\frac{i\sqrt{6}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{210}{(2x^2+4x+5)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$-1+\frac{i\sqrt{6}}{2}$	2	0	$\frac{7}{2}$	$-\frac{5}{2}$
$-1-\frac{i\sqrt{6}}{2}$	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{5}{2\left(x+1-\frac{i\sqrt{6}}{2}\right)} + \frac{7}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)} + (-)(0) \\
&= -\frac{5}{2\left(x+1-\frac{i\sqrt{6}}{2}\right)} + \frac{7}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)} \\
&= \frac{-6i\sqrt{6}+2x+2}{2x^2+4x+5}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{5}{2\left(x+1-\frac{i\sqrt{6}}{2}\right)} + \frac{7}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)}\right)(0) + \left(\left(\frac{5}{2\left(x+1-\frac{i\sqrt{6}}{2}\right)^2} - \frac{7}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)^2}\right) + \left(-\frac{}{2}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{5}{2\left(x+1-\frac{i\sqrt{6}}{2}\right)} + \frac{7}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)}\right) dx} \\
&= \frac{27\sqrt{2}\left(2x^2+4x+5\right)^{7/2}}{\left(3+i\left(x+1\right)\sqrt{6}\right)^6}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-20x-20}{2x^2+4x+5} \, dx} \\
&= z_1 e^{\frac{5\ln\left(2x^2+4x+5\right)}{2}} \\
&= z_1 \left(\left(2x^2+4x+5\right)^{5/2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = -\frac{\left(2x^2+4x+5\right)^6 \sqrt{2}}{27\left(i - \frac{(x+1)\sqrt{2}\sqrt{3}}{3}\right)^6}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-20x-20}{2x^2+4x+5} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{5\ln\left(2x^2+4x+5\right)}}{\left(y_1\right)^2} \, dx \\
&= y_1 \left(-\frac{\frac{1}{2}x^5 + \frac{5}{2}x^4 + \frac{5}{2}x^3 - \frac{5}{2}x^2 - \frac{31}{8}x - \frac{7}{8}}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)^6}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(-\frac{(2x^2+4x+5)^6 \sqrt{2}}{27 \left(i - \frac{(x+1)\sqrt{2}\sqrt{3}}{3}\right)^6} \right) \\
&\quad + c_2 \left(-\frac{(2x^2+4x+5)^6 \sqrt{2}}{27 \left(i - \frac{(x+1)\sqrt{2}\sqrt{3}}{3}\right)^6} \left(-\frac{\frac{1}{2}x^5 + \frac{5}{2}x^4 + \frac{5}{2}x^3 - \frac{5}{2}x^2 - \frac{31}{8}x - \frac{7}{8}}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)^6} \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(2x^2+4x+5)\left(\frac{d^2}{dx^2}y(x)\right) - 20(x+1)\left(\frac{d}{dx}y(x)\right) + 60y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{60y(x)}{2x^2+4x+5} + \frac{20(x+1)\left(\frac{d}{dx}y(x)\right)}{2x^2+4x+5}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) - \frac{20(x+1)\left(\frac{d}{dx}y(x)\right)}{2x^2+4x+5} + \frac{60y(x)}{2x^2+4x+5} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{20(x+1)}{2x^2+4x+5}, P_3(x) = \frac{60}{2x^2+4x+5}\right]$

  ◦ $\left(x+1+\frac{\mathrm{I}\sqrt{6}}{2}\right) \cdot P_2(x)$ is analytic at $x = -1 - \frac{\mathrm{I}\sqrt{6}}{2}$

    $\left(\left(x+1+\frac{\mathrm{I}\sqrt{6}}{2}\right) \cdot P_2(x)\right)\Bigg|_{x=-1-\frac{\mathrm{I}\sqrt{6}}{2}} = 0$

  ◦ $\left(x+1+\frac{\mathrm{I}\sqrt{6}}{2}\right)^2 \cdot P_3(x)$ is analytic at $x = -1 - \frac{\mathrm{I}\sqrt{6}}{2}$

    $\left(\left(x+1+\frac{\mathrm{I}\sqrt{6}}{2}\right)^2 \cdot P_3(x)\right)\Bigg|_{x=-1-\frac{\mathrm{I}\sqrt{6}}{2}} = 0$

  ◦ $x = -1 - \frac{\mathrm{I}\sqrt{6}}{2}$ is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1 - \frac{\mathrm{I}\sqrt{6}}{2}$

- Multiply by denominators

  $(2x^2+4x+5)\left(\frac{d^2}{dx^2}y(x)\right) + (-20x-20)\left(\frac{d}{dx}y(x)\right) + 60y(x) = 0$

- Change variables using $x = u - 1 - \frac{\mathrm{I}\sqrt{6}}{2}$ so that the regular singular point is at $u = 0$

  $\left(2u^2 - 2\mathrm{I}u\sqrt{6}\right)\left(\frac{d^2}{du^2}y(u)\right) + \left(-20u + 10\mathrm{I}\sqrt{6}\right)\left(\frac{d}{du}y(u)\right) + 60y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-2\,\mathrm{I}\sqrt{6}\,(r-6)\, r a_0 u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2\,\mathrm{I}\sqrt{6}\,(k+r-5)(k+1+r)\, a_{k+1} + 2a_k(k+r-5)(k+r-6)\right)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2\,\mathrm{I}\sqrt{6}\,(r-6)\, r = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 6\}$$

- Each term in the series must be 0, giving the recursion relation

$$-2\left(\mathrm{I}a_{k+1}(k+1+r)\sqrt{6} - a_k(k+r-6)\right)(k+r-5) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{-\frac{\mathrm{I}}{6}a_k(k+r-6)\sqrt{6}}{k+1+r}$$

- Recursion relation for $r = 0$; series terminates at $k = 6$

$$a_{k+1} = \frac{-\frac{\mathrm{I}}{6}a_k(k-6)\sqrt{6}}{k+1}$$

- Recursion relation that defines the terminating series solution of the ODE for $r = 0$

$$\left[y(u) = \sum_{k=0}^{5} a_k u^k, a_{k+1} = \frac{-\frac{\mathrm{I}}{6}a_k(k-6)\sqrt{6}}{k+1}\right]$$

- Revert the change of variables $u = x + 1 + \frac{\mathrm{I}\sqrt{6}}{2}$

$$\left[y(x) = \sum_{k=0}^{5} a_k \left(x + 1 + \frac{\mathrm{I}\sqrt{6}}{2}\right)^k, a_{k+1} = \frac{-\frac{\mathrm{I}}{6}a_k(k-6)\sqrt{6}}{k+1}\right]$$

- Recursion relation for $r = 6$

$$a_{k+1} = \frac{-\frac{\mathrm{I}}{6}a_k k\sqrt{6}}{k+7}$$

- Solution for $r = 6$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+6}, a_{k+1} = \frac{-\frac{\mathrm{I}}{6}a_k k\sqrt{6}}{k+7}\right]$$

- Revert the change of variables $u = x + 1 + \frac{\mathrm{I}\sqrt{6}}{2}$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k \left(x + 1 + \frac{\mathrm{I}\sqrt{6}}{2}\right)^{k+6}, a_{k+1} = \frac{-\frac{\mathrm{I}}{6}a_k k\sqrt{6}}{k+7}\right]$$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{5} a_k \left(x + 1 + \frac{\mathrm{I}\sqrt{6}}{2}\right)^k\right) + \left(\sum_{k=0}^{\infty} b_k \left(x + 1 + \frac{\mathrm{I}\sqrt{6}}{2}\right)^{k+6}\right), a_{k+1} = \frac{-\frac{\mathrm{I}}{6}a_k(k-6)\sqrt{6}}{k+1}, b_{k+1} = \frac{-\frac{\mathrm{I}}{6}b_k k\sqrt{6}}{k+7}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 65

```
dsolve((2*x^2+4*x+5)*diff(diff(y(x),x),x)-20*(x+1)*diff(y(x),x)+60*y(x) = 0,y(x),sings
```

$$y = c_2 x^6 + c_1 x^5 + \frac{5(2c_1 - 15c_2)\, x^4}{2} + 5(c_1 - 20c_2)\, x^3 + \frac{5(-4c_1 - 45c_2)\, x^2}{4} + \frac{(-31c_1 + 120c_2)\, x}{4} - \frac{7c_1}{4} + \frac{155c_2}{8}$$

Mathematica DSolve solution

Solving time : 0.582 (sec)
Leaf size : 108

```
DSolve[{(2*x^2+4*x+5)*D[y[x],{x,2}]-20*(x+1)*D[y[x],x]+60*y[x]==0,{}},y[x],x,IncludeSingular
```

$$y(x) \to \left(2x^2 + 4x + 5\right)^{5/2} \exp\left(\int_1^x \frac{2\left(K[1] + 3i\sqrt{6} + 1\right)}{2K[1](K[1]+2)+5} dK[1]\right) \left(c_2 \int_1^x \exp\left(-2\int_1^{K[2]} \frac{2\left(K[1] + 3i\sqrt{6} + 1\right)}{2K[1](K[1]+2)+5} dK[1]\right) dK[+ c_1\right)$$

2.1.63 Problem 65

Internal problem ID [9235]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 65
Date solved : `Monday, January 27, 2025 at 05:52:24 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^3+1\right)y''+7x^2y'+9xy=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.415 (sec)

Writing the ode as

$$\left(x^3+1\right)y''+7x^2y'+9xy=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^3+1 \\ B &= 7x^2 \\ C &= 9x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-x(x^3+8)}{4\left(x^3+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x\left(x^3+8\right) \\ t &= 4\left(x^3+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{x(x^3+8)}{4\left(x^3+1\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.113: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3+1)^2$. There is a pole at $x = -1$ of order 2. There is a pole at $x = \frac{1}{2} - \frac{i\sqrt{3}}{2}$ of order 2. There is a pole at $x = \frac{1}{2} + \frac{i\sqrt{3}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$\begin{aligned} r = {} & \frac{7}{36\left(x - \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} + \frac{7}{36\left(x - \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{5}{36} + \frac{5i\sqrt{3}}{36}}{x - \frac{1}{2} - \frac{i\sqrt{3}}{2}} \\ & + \frac{-\frac{5}{36} - \frac{5i\sqrt{3}}{36}}{x - \frac{1}{2} + \frac{i\sqrt{3}}{2}} + \frac{5}{18(x+1)} + \frac{7}{36(x+1)^2} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6} \end{aligned}$$

For the pole at $x = \frac{1}{2} - \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6}
\end{aligned}
$$

For the pole at $x = \frac{1}{2} + \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{x(x^3+8)}{4\,(x^3+1)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{x(x^3+8)}{4\,(x^3+1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{7}{6}$	$-\frac{1}{6}$
$\frac{1}{2} - \frac{i\sqrt{3}}{2}$	2	0	$\frac{7}{6}$	$-\frac{1}{6}$
$\frac{1}{2} + \frac{i\sqrt{3}}{2}$	2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{6\,(x+1)} - \frac{1}{6\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)} - \frac{1}{6\left(x-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)} + (-)\,(0) \\
&= -\frac{1}{6\,(x+1)} - \frac{1}{6\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)} - \frac{1}{6\left(x-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)} \\
&= -\frac{x^2}{2x^3+2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{6\,(x+1)} - \frac{1}{6\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)} - \frac{1}{6\left(x-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)} \right)(1) + \left(\left(\frac{1}{6\,(x+1)^2} + \frac{1}{6\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2} + \right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\,dx} \\
&= (x)\,\mathrm{e}^{\int \left(-\frac{1}{6(x+1)} - \frac{1}{6\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)} - \frac{1}{6\left(x-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)} \right) dx} \\
&= (x)\,\frac{1}{\left((x+1)\left(2x-1+i\sqrt{3}\right)\left(i\sqrt{3}-2x+1\right)\right)^{1/6}} \\
&= \frac{x}{\left((x+1)\left(2x-1+i\sqrt{3}\right)\left(i\sqrt{3}-2x+1\right)\right)^{1/6}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7x^2}{x^3+1}\,dx} \\
&= z_1 e^{-\frac{7\ln\left(x^3+1\right)}{6}} \\
&= z_1 \left(\frac{1}{\left(x^3+1\right)^{7/6}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x}{(x^3+1)^{7/6}(-4x^3-4)^{1/6}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{7x^2}{x^3+1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\frac{7\ln(x^3+1)}{3}}}{(y_1)^2}\,dx \\ &= y_1 \left(\int \frac{(-4x^3-4)^{1/3}}{x^2}dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{x}{(x^3+1)^{7/6}(-4x^3-4)^{1/6}}\right) + c_2\left(\frac{x}{(x^3+1)^{7/6}(-4x^3-4)^{1/6}}\left(\int \frac{(-4x^3-4)^{1/3}}{x^2}dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x^3+1)\left(\frac{d^2}{dx^2}y(x)\right) + 7x^2\left(\frac{d}{dx}y(x)\right) + 9xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{9xy(x)}{x^3+1} - \frac{7x^2\left(\frac{d}{dx}y(x)\right)}{x^3+1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{7x^2\left(\frac{d}{dx}y(x)\right)}{x^3+1} + \frac{9xy(x)}{x^3+1} = 0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{7x^2}{x^3+1}, P_3(x) = \frac{9x}{x^3+1}\right]$

  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1} = \frac{7}{3}$

  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$

  - $x=-1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -1$

- Multiply by denominators

  $(x^3+1)\left(\frac{d^2}{dx^2}y(x)\right)+7x^2\left(\frac{d}{dx}y(x)\right)+9xy(x)=0$
- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^3-3u^2+3u)\left(\frac{d^2}{du^2}y(u)\right)+(7u^2-14u+7)\left(\frac{d}{du}y(u)\right)+(9u-9)\,y(u)=0$
- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$
- ○ Shift index using $k->k-m$

  $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$
- ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$
- ○ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$
- ○ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$
- ○ Shift index using $k->k+2-m$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

Rewrite ODE with series expansions

$a_0r(4+3r)\,u^{-1+r}+\left(a_1(1+r)\,(7+3r)-a_0(3r^2+11r+9)\right)u^r+\left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+1+r)\,(3k+7\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(4+3r)=0$
- Values of r that satisfy the indicial equation

  $r\in\left\{0,-\frac{4}{3}\right\}$
- Each term must be 0

  $a_1(1+r)\,(7+3r)-a_0(3r^2+11r+9)=0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(3k+7+3r)-a_k(3k^2+6kr+3r^2+11k+11r+9)+a_{k-1}(k+2+r)^2=0$
- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)\,(3k+10+3r)-a_{k+1}\left(3(k+1)^2+6(k+1)\,r+3r^2+11k+20+11r\right)+a_k(k+$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{k^2a_k-3k^2a_{k+1}+2kra_k-6kra_{k+1}+r^2a_k-3r^2a_{k+1}+6ka_k-17ka_{k+1}+6ra_k-17ra_{k+1}+9a_k-23a_{k+1}}{(k+2+r)(3k+10+3r)}$
- Recursion relation for $r=0$

  $a_{k+2}=-\frac{k^2a_k-3k^2a_{k+1}+6ka_k-17ka_{k+1}+9a_k-23a_{k+1}}{(k+2)(3k+10)}$
- Solution for $r=0$

  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^k,a_{k+2}=-\frac{k^2a_k-3k^2a_{k+1}+6ka_k-17ka_{k+1}+9a_k-23a_{k+1}}{(k+2)(3k+10)},7a_1-9a_0=0\right]$
- Revert the change of variables $u=x+1$

$$\left[y(x)=\sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+2}=-\frac{k^2 a_k-3k^2 a_{k+1}+6k a_k-17k a_{k+1}+9a_k-23a_{k+1}}{(k+2)(3k+10)}, 7a_1-9a_0=0\right]$$

- Recursion relation for $r=-\frac{4}{3}$

$$a_{k+2}=-\frac{k^2 a_k-3k^2 a_{k+1}+\frac{10}{3}k a_k-9k a_{k+1}+\frac{25}{9}a_k-\frac{17}{3}a_{k+1}}{\left(k+\frac{2}{3}\right)(3k+6)}$$

- Solution for $r=-\frac{4}{3}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k-\frac{4}{3}}, a_{k+2}=-\frac{k^2 a_k-3k^2 a_{k+1}+\frac{10}{3}k a_k-9k a_{k+1}+\frac{25}{9}a_k-\frac{17}{3}a_{k+1}}{\left(k+\frac{2}{3}\right)(3k+6)}, -a_1+\frac{a_0}{3}=0\right]$$

- Revert the change of variables $u=x+1$

$$\left[y(x)=\sum_{k=0}^{\infty} a_k(x+1)^{k-\frac{4}{3}}, a_{k+2}=-\frac{k^2 a_k-3k^2 a_{k+1}+\frac{10}{3}k a_k-9k a_{k+1}+\frac{25}{9}a_k-\frac{17}{3}a_{k+1}}{\left(k+\frac{2}{3}\right)(3k+6)}, -a_1+\frac{a_0}{3}=0\right]$$

- Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty} a_k(x+1)^k\right)+\left(\sum_{k=0}^{\infty} b_k(x+1)^{k-\frac{4}{3}}\right), a_{k+2}=-\frac{k^2 a_k-3k^2 a_{k+1}+6k a_k-17k a_{k+1}+9a_k-23a_{k+1}}{(k+2)(3k+10)}, 7a_1\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returnin
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.168 (sec)
Leaf size : 28

```
dsolve((x^3+1)*diff(diff(y(x),x),x)+7*diff(y(x),x)*x^2+9*x*y(x) = 0,y(x),singsol=all)
```

$$y=c_1 \operatorname{hypergeom}\left([1,1],\left[\frac{2}{3}\right],-x^3\right)+\frac{c_2 x}{(x^3+1)^{4/3}}$$

Mathematica DSolve solution

Solving time : 0.638 (sec)
Leaf size : 39

```
DSolve[{(1+x^3)*D[y[x],{x,2}]+7*x^2*D[y[x],x]+9*x*y[x]==0,{}},y[x],x,IncludeSingularSolution
```

$$y(x) \to \frac{c_1 x - c_2 \,\text{Hypergeometric2F1}\left(-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}, -x^3\right)}{(x^3+1)^{4/3}}$$

2.1.64 Problem 66

Internal problem ID [9236]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 66
Date solved : `Monday, January 27, 2025 at 05:52:25 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(2x^{5}+1\right) y''+14x^{4}y'+10x^{3}y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 1.110 (sec)

Writing the ode as

$$\left(2x^{5}+1\right) y''+14x^{4}y'+10x^{3}y = 0 \tag{1}$$
$$Ay''+By'+Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^{5}+1 \\ B &= 14x^{4} \\ C &= 10x^{3} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^{2} - 4AC}{4A^{2}} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^{3}(5x^{5}+6)}{\left(2x^{5}+1\right)^{2}} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3x^{3}\left(5x^{5}+6\right) \\ t &= \left(2x^{5}+1\right)^{2} \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^{3}(5x^{5}+6)}{\left(2x^{5}+1\right)^{2}}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.115: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 10 - 8 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x^5+1)^2$. There is a pole at $x = \frac{2^{4/5}\sqrt{5}}{8} + \frac{2^{4/5}}{8} + \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}$ of order 2. There is a pole at $x = \frac{2^{4/5}}{8} - \frac{2^{4/5}\sqrt{5}}{8} + \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8} + \frac{i2^{3/10}\sqrt{5-\sqrt{5}}\,\sqrt{5}}{8}$ of order 2. There is a pole at $x = -\frac{2^{4/5}}{2}$ of order 2. There is a pole at $x = \frac{2^{4/5}}{8} - \frac{2^{4/5}\sqrt{5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}\,\sqrt{5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8}$ of order 2. There is a pole at $x = \frac{2^{4/5}\sqrt{5}}{8} + \frac{2^{4/5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \text{Expression too large to display}$$

For the pole at $x = \frac{2^{4/5}\sqrt{5}}{8} + \frac{2^{4/5}}{8} + \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}$ let b be the coefficient of $\frac{1}{\left(x - \frac{2^{4/5}\sqrt{5}}{8} - \frac{2^{4/5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{21}{100}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{10} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{10} \end{aligned}$$

For the pole at $x = \frac{2^{4/5}}{8} - \frac{2^{4/5}\sqrt{5}}{8} + \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8} + \frac{i2^{3/10}\sqrt{5-\sqrt{5}}\,\sqrt{5}}{8}$ let b be the coefficient of $\frac{1}{\left(x-\frac{2^{4/5}}{8}+\frac{2^{4/5}\sqrt{5}}{8}-\frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8}-\frac{i2^{3/10}\sqrt{5-\sqrt{5}}\,\sqrt{5}}{8}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{21}{100}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{10}\\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{10}
\end{aligned}
$$

For the pole at $x = -\frac{2^{4/5}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{2^{4/5}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{21}{100}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{10}\\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{10}
\end{aligned}
$$

For the pole at $x = \frac{2^{4/5}}{8} - \frac{2^{4/5}\sqrt{5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}\,\sqrt{5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8}$ let b be the coefficient of $\frac{1}{\left(x-\frac{2^{4/5}}{8}+\frac{2^{4/5}\sqrt{5}}{8}+\frac{i2^{3/10}\sqrt{5-\sqrt{5}}\,\sqrt{5}}{8}+\frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{21}{100}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{10}\\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{10}
\end{aligned}
$$

For the pole at $x = \frac{2^{4/5}\sqrt{5}}{8} + \frac{2^{4/5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}$ let b be the coefficient of $\frac{1}{\left(x-\frac{2^{4/5}\sqrt{5}}{8}-\frac{2^{4/5}}{8}+\frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{21}{100}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{10}\\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{10}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{3x^3(5x^5+6)}{(2x^5+1)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2}\\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3x^3(5x^5+6)}{(2x^5+1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{2^{4/5}\sqrt{5}}{8}+\frac{2^{4/5}}{8}+\frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}$	2	0	$\frac{7}{10}$	$\frac{3}{10}$
$\frac{2^{4/5}}{8}-\frac{2^{4/5}\sqrt{5}}{8}+\frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8}+\frac{i2^{3/10}\sqrt{5-\sqrt{5}}\,\sqrt{5}}{8}$	2	0	$\frac{7}{10}$	$\frac{3}{10}$
$-\frac{2^{4/5}}{2}$	2	0	$\frac{7}{10}$	$\frac{3}{10}$
$\frac{2^{4/5}}{8}-\frac{2^{4/5}\sqrt{5}}{8}-\frac{i2^{3/10}\sqrt{5-\sqrt{5}}\,\sqrt{5}}{8}-\frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8}$	2	0	$\frac{7}{10}$	$\frac{3}{10}$
$\frac{2^{4/5}\sqrt{5}}{8}+\frac{2^{4/5}}{8}-\frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}$	2	0	$\frac{7}{10}$	$\frac{3}{10}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=\frac{5}{2}$ then

$$\begin{aligned}d&=\alpha_\infty^+-\left(\alpha_{c_1}^-+\alpha_{c_2}^-+\alpha_{c_3}^-+\alpha_{c_4}^-+\alpha_{c_5}^-\right)\\&=\frac{5}{2}-\left(\frac{3}{2}\right)\\&=1\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega&=\left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^-}{x-c_1}\right)+\left((-)[\sqrt{r}]_{c_2}+\frac{\alpha_{c_2}^-}{x-c_2}\right)+\left((-)[\sqrt{r}]_{c_3}+\frac{\alpha_{c_3}^-}{x-c_3}\right)+\left((-)[\sqrt{r}]_{c_4}+\frac{\alpha_{c_4}^-}{x-c_4}\right)\\&=\frac{3}{10\left(x-\frac{2^{4/5}\sqrt{5}}{8}-\frac{2^{4/5}}{8}-\frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}\right)}+\frac{3}{10\left(x-\frac{2^{4/5}}{8}+\frac{2^{4/5}\sqrt{5}}{8}-\frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8}-\frac{i2^{3/10}\sqrt{5-\sqrt{5}}\,\sqrt{5}}{8}\right)}+\frac{}{10(}\\&=\frac{3}{10\left(x-\frac{2^{4/5}\sqrt{5}}{8}-\frac{2^{4/5}}{8}-\frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}\right)}+\frac{3}{10\left(x-\frac{2^{4/5}}{8}+\frac{2^{4/5}\sqrt{5}}{8}-\frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8}-\frac{i2^{3/10}\sqrt{5-\sqrt{5}}\,\sqrt{5}}{8}\right)}+\frac{}{10(}\\&=\frac{3x^4}{2x^5+1}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=x+a_0 \tag{2A}$$

Substituting the above in eq. (1A) and Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0=0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int \left(\frac{3}{10\left(x - \frac{2^{4/5}\sqrt{5}}{8} - \frac{2^{4/5}}{8} - \frac{i 2^{3/10}\sqrt{5-\sqrt{5}}}{4}\right)} + \frac{3}{10\left(x - \frac{2^{4/5}}{8} + \frac{2^{4/5}\sqrt{5}}{8} - \frac{i 2^{3/10}\sqrt{5-\sqrt{5}}}{8} - \frac{i 2^{3/10}\sqrt{5-\sqrt{5}}\,\sqrt{5}}{8}\right)} + \frac{3}{10\left(x + \frac{2^{4/5}}{2}\right)} + \frac{}{10\left(x - \frac{2^{4/5}}{8} + 2^{4/}\right.} \right.} \\
&= (x) \left(\left(2^{4/5} + 2x\right) \left(2^{4/5}\sqrt{5} + 2i 2^{3/10}\sqrt{5 - \sqrt{5}} + 2^{4/5} - 8x\right) \left(8x - 2^{4/5}\sqrt{5} - 2^{4/5} + 2i 2^{3/10}\sqrt{5 - \sqrt{5}}\right.\right. \\
&= x 8^{3/10} \left(\left(x + \frac{2^{4/5}}{2}\right) \left(i 2^{3/10}\sqrt{5 - \sqrt{5}} + \frac{\left(-\sqrt{5} - 1\right) 2^{4/5}}{2} + 4x\right) \left(i\left(\sqrt{5} + 1\right) 2^{3/10}\sqrt{5 - \sqrt{5}} + \left(-\sqrt{\right.\right.\right.
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{14x^4}{2x^5+1}\, dx} \\
&= z_1 e^{-\frac{7\ln\left(2x^5+1\right)}{10}} \\
&= z_1 \left(\frac{1}{\left(2x^5+1\right)^{7/10}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{x 8^{3/10} \left(1024x^5 + 512\right)^{3/10}}{\left(2x^5 + 1\right)^{7/10}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{14x^4}{2x^5+1}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{7\ln\left(2x^5+1\right)}{5}}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\int \frac{8^{2/5}}{8x^2 \left(1024x^5 + 512\right)^{3/5}} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x 8^{3/10} \left(1024x^5 + 512\right)^{3/10}}{\left(2x^5 + 1\right)^{7/10}}\right) + c_2 \left(\frac{x 8^{3/10} \left(1024x^5 + 512\right)^{3/10}}{\left(2x^5 + 1\right)^{7/10}} \left(\int \frac{8^{2/5}}{8x^2 \left(1024x^5 + 512\right)^{3/5}} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returr
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.098 (sec)
Leaf size : 30

```
dsolve((2*x^5+1)*diff(diff(y(x),x),x)+14*x^4*diff(y(x),x)+10*x^3*y(x) = 0,y(x),singsol
```

$$y = \frac{c_1 x}{\left(2x^5+1\right)^{2/5}} + c_2 \operatorname{hypergeom}\left(\left[\frac{1}{5}, 1\right], \left[\frac{4}{5}\right], -2x^5\right)$$

Mathematica DSolve solution

Solving time : 0.0 (sec)
Leaf size : 0

```
DSolve[{(1+2*x^5)*D[y[x],{x,2}]+14*x^4*D[y[x],x]+10*x^3*y[x]==0,{}},y[x],x,IncludeSingularSo
```

Timed out

2.1.65 Problem 67

Internal problem ID [9237]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 67
Date solved : Monday, January 27, 2025 at 05:52:27 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$y'' + x^6 y' + 7x^5 y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.413 (sec)

Writing the ode as

$$y'' + x^6 y' + 7x^5 y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x^6 \\ C &= 7x^5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^5(x^7 - 16)}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^5\left(x^7 - 16\right) \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^5(x^7 - 16)}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.116: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 12 \\ &= -12 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -12 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -12$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{12}{2} = 6$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{6} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^6$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x^6}{2} - \frac{4}{x} - \frac{16}{x^8} - \frac{128}{x^{15}} - \frac{1280}{x^{22}} - \frac{14336}{x^{29}} - \frac{172032}{x^{36}} - \frac{2162688}{x^{43}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 6$ gives

$$[\sqrt{r}]_\infty = \sum_{i=0}^{6} a_i x^i = \frac{x^6}{2} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^5 = x^5$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^{12}}{4}$$

This shows that the coefficient of x^5 in the above is 0. Now we need to find the coefficient of x^5 in r. How this is done depends on if $v = 0$ or not. Since $v = 6$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x^5 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^5(x^7 - 16)}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^{12} - 4x^5\right) + (0) \\ &= \frac{1}{4}x^{12} - 4x^5 \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is -4. Now b can be found.

$$\begin{aligned} b &= (-4) - (0) \\ &= -4 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x^6}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-4}{\frac{1}{2}} - 6\right) &= -7 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-4}{\frac{1}{2}} - 6\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^5(x^7 - 16)}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-12	$\frac{x^6}{2}$	-7	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$
\begin{aligned}
d &= \alpha_\infty^- \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= (-)[\sqrt{r}]_\infty \\
&= 0 + (-)\left(\frac{x^6}{2}\right) \\
&= -\frac{x^6}{2} \\
&= -\frac{x^6}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{x^6}{2}\right)(1) + \left((-3x^5) + \left(-\frac{x^6}{2}\right)^2 - \left(\frac{x^5(x^7-16)}{4}\right)\right) = 0
$$

$$
x^5 a_0 = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= pe^{\int \omega\, dx} \\
&= (x)\,\mathrm{e}^{\int -\frac{x^6}{2} dx} \\
&= (x)\,\mathrm{e}^{-\frac{x^7}{14}} \\
&= x\,\mathrm{e}^{-\frac{x^7}{14}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^6}{1} dx} \\
&= z_1 e^{-\frac{x^7}{14}} \\
&= z_1\left(\mathrm{e}^{-\frac{x^7}{14}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^7}{7}} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^6}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{x^7}{7}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{7^{6/7}(-1)^{1/7}\left(-\frac{7x^6(-1)^{6/7}\Gamma\left(\frac{6}{7}\right)}{(-x^7)^{6/7}} + \frac{7\,7^{1/7}(-1)^{6/7}\mathrm{e}^{\frac{x^7}{7}}}{x} + \frac{7x^6(-1)^{6/7}\Gamma\left(\frac{6}{7}, -\frac{x^7}{7}\right)}{(-x^7)^{6/7}}\right)}{49} \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x^7}{7}} x\right) \\
&\quad + c_2 \left(\mathrm{e}^{-\frac{x^7}{7}} x \left(\frac{7^{6/7}(-1)^{1/7}\left(-\frac{7x^6(-1)^{6/7}\Gamma\left(\frac{6}{7}\right)}{(-x^7)^{6/7}} + \frac{7\,7^{1/7}(-1)^{6/7}\mathrm{e}^{\frac{x^7}{7}}}{x} + \frac{7x^6(-1)^{6/7}\Gamma\left(\frac{6}{7}, -\frac{x^7}{7}\right)}{(-x^7)^{6/7}}\right)}{49} \right) \right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2} y(x) + x^6 \left(\frac{d}{dx} y(x)\right) + 7x^5 y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2} y(x)$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

- ◦ Convert $x^5 \cdot y(x)$ to series expansion
  $x^5 \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+5}$
- ◦ Shift index using $k- >k-5$
  $x^5 \cdot y(x) = \sum_{k=5}^{\infty} a_{k-5} x^k$
- ◦ Convert $x^6 \cdot \left(\frac{d}{dx} y(x)\right)$ to series expansion
  $x^6 \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^{k+5}$

- Shift index using $k->k-5$

  $x^6 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=5}^{\infty} a_{k-5}(k-5)\, x^k$

- Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

  $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- Shift index using $k->k+2$

  $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $30a_6x^4 + 20a_5x^3 + 12a_4x^2 + 6a_3x + 2a_2 + \left(\sum_{k=5}^{\infty} (a_{k+2}(k+2)\,(k+1) + a_{k-5}(k+2))\, x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 = 0, 6a_3 = 0, 12a_4 = 0, 20a_5 = 0, 30a_6 = 0]$

- Solve for the dependent coefficient(s)

  $\{a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0, a_6 = 0\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+2)\,(ka_{k+2} + a_{k-5} + a_{k+2}) = 0$

- Shift index using $k->k+5$

  $(k+7)\,((k+5)\, a_{k+7} + a_k + a_{k+7}) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+7} = -\frac{a_k}{k+6}, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0, a_6 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 62

```
dsolve(diff(diff(y(x),x),x)+x^6*diff(y(x),x)+7*x^5*y(x) = 0,y(x),singsol=all)
```

$$y = -\frac{\left(-c_1 \mathrm{e}^{-\frac{x^7}{7}} x - c_2 7^{1/7}\right) \left(-x^7\right)^{6/7} + x^7 c_2 \mathrm{e}^{-\frac{x^7}{7}} \left(\Gamma\left(\frac{6}{7}\right) - \Gamma\left(\frac{6}{7}, -\frac{x^7}{7}\right)\right)}{\left(-x^7\right)^{6/7}}$$

Mathematica DSolve solution

Solving time : 0.26 (sec)
Leaf size : 53

```
DSolve[{D[y[x],{x,2}]+x^6*D[y[x],x]+7*x^5*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{49}e^{-\frac{x^7}{7}}\left(49c_1x - 7^{6/7}c_2\sqrt[7]{-x^7}\Gamma\left(-\frac{1}{7}, -\frac{x^7}{7}\right)\right)$$

2.1.66 Problem 68

Internal problem ID [9238]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 68
Date solved : `Tuesday, January 28, 2025 at 04:00:13 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^8+1\right) y''-16x^7y'+72x^6y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 459.996 (sec)

Writing the ode as

$$\left(x^8+1\right) y''-16x^7y'+72x^6y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^8+1 \\ B &= -16x^7 \\ C &= 72x^6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{-128x^6}{\left(x^8+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -128x^6 \\ t &= \left(x^8+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{128x^6}{\left(x^8+1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.118: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 16 - 6 \\ &= 10 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^8+1)^2$. There is a pole at $x = \frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2} + \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = -\frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = -\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = -\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = -\frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = \frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2} + \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 10 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 10 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
\begin{aligned}
r = {} & \frac{2}{\left(x-\frac{\sqrt{2+\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^{2}}+\frac{2}{\left(x-\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2+\sqrt{2}}}{2}\right)^{2}}+\frac{2}{\left(x+\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2+\sqrt{2}}}{2}\right)^{2}} \\
& +\frac{2}{\left(x+\frac{\sqrt{2+\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^{2}}+\frac{2}{\left(x+\frac{\sqrt{2+\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^{2}} \\
& +\frac{2}{\left(x+\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2+\sqrt{2}}}{2}\right)^{2}}+\frac{2}{\left(x-\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2+\sqrt{2}}}{2}\right)^{2}} \\
& +\frac{2}{\left(x-\frac{\sqrt{2+\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^{2}}+\frac{2\left(\frac{\sqrt{2+\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^{7}}{x-\frac{\sqrt{2+\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}}+\frac{2\left(\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2+\sqrt{2}}}{2}\right)^{7}}{x-\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2+\sqrt{2}}}{2}} \\
& +\frac{2\left(-\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2+\sqrt{2}}}{2}\right)^{7}}{x+\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2+\sqrt{2}}}{2}}+\frac{2\left(-\frac{\sqrt{2+\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^{7}}{x+\frac{\sqrt{2+\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}}+\frac{2\left(-\frac{\sqrt{2+\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^{7}}{x+\frac{\sqrt{2+\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}} \\
& +\frac{2\left(-\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2+\sqrt{2}}}{2}\right)^{7}}{x+\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2+\sqrt{2}}}{2}}+\frac{2\left(\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2+\sqrt{2}}}{2}\right)^{7}}{x-\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2+\sqrt{2}}}{2}}+\frac{2\left(\frac{\sqrt{2+\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^{7}}{x-\frac{\sqrt{2+\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}}
\end{aligned}
$$

For the pole at $x=\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2}+\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2}-\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b=2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b}=2 \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b}=-1
\end{aligned}
$$

For the pole at $x=\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b=2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b}=2 \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b}=-1
\end{aligned}
$$

For the pole at $x=-\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b=2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b}=2 \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b}=-1
\end{aligned}
$$

For the pole at $x=-\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2}-\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2}+\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b=2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b}=2 \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b}=-1
\end{aligned}
$$

For the pole at $x = -\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2}+\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

For the pole at $x = -\frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

For the pole at $x = \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

For the pole at $x = \frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2} + \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2}-\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

Since the order of r at ∞ is $10 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{128x^6}{(x^8+1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2} + \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$\frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$-\frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$-\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$-\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$-\frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$\frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2} + \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
10	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^- + \alpha_{c_4}^- + \alpha_{c_5}^- + \alpha_{c_6}^- + \alpha_{c_7}^- + \alpha_{c_8}^+\right) \\ &= 1 - (-5) \\ &= 6 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + \left((-)[\sqrt{r}]_{c_4} + \frac{\alpha_{c_4}^-}{x-c_4}\right) \\ &= -\frac{1}{x - \frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}} - \frac{1}{x - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}} - \frac{1}{x + \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2} -} \\ &= -\frac{1}{x - \frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}} - \frac{1}{x - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}} - \frac{1}{x + \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2} -} \\ &= \frac{\left((3x^6 - 3ix^4 - 3ix^2 - 3)\sqrt{2} - 3(x^2+1)\left((-1+i)x^4 + 1 + i\right)\right)\sqrt{2-\sqrt{2}} - 3\left(\left((-1+i)x^4 + 1 + i\right)\right.}{2\left(x\sqrt{2-\sqrt{2}} + x^2 + 1\right)\left(x\left(1+\sqrt{2}\right)\sqrt{2-\sqrt{2}} + x^2 + 1\right)\left(x^2 - x\sqrt{2-\sqrt{2}} + 1\right)\left(-x\left(1+\sqrt{2}\right)\right.} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 6$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^6 + x^5 a_5 + x^4 a_4 + x^3 a_3 + x^2 a_2 + x a_1 + a_0 \tag{2A}$$

Substituting the above in eq. (1A) and Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{i\sqrt{2} - 1 + i}{i\sqrt{2} + 1 + i}, a_1 = \frac{\left(\frac{12}{7} - \frac{12i}{7}\right)\sqrt{2}}{\left(i\sqrt{2} + 1 + i\right)\sqrt{2-\sqrt{2}}}, a_2 = -\frac{15\left(-\sqrt{2} - 1 + i\right)}{7\left(i\sqrt{2} + 1 + i\right)}, a_3 = \frac{32}{7\sqrt{2-\sqrt{2}}\left(i\sqrt{2} + 1\right.}\right.$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^6 + \frac{\left(\frac{12}{7} + \frac{12i}{7}\right)x^5\sqrt{2}}{\left(i\sqrt{2} + 1 + i\right)\sqrt{2-\sqrt{2}}} + \frac{15x^4\left(\sqrt{2} + 1 + i\right)}{7\left(i\sqrt{2} + 1 + i\right)} + \frac{32x^3}{7\sqrt{2-\sqrt{2}}\left(i\sqrt{2} + 1 + i\right)} - \frac{15x^2\left(-\sqrt{2} - 1\right.}{7\left(i\sqrt{2} + 1 +\right.}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \left(x^6 + \frac{\left(\frac{12}{7} + \frac{12i}{7}\right)x^5\sqrt{2}}{\left(i\sqrt{2} + 1 + i\right)\sqrt{2-\sqrt{2}}} + \frac{15x^4\left(\sqrt{2} + 1 + i\right)}{7\left(i\sqrt{2} + 1 + i\right)} + \frac{32x^3}{7\sqrt{2-\sqrt{2}}\left(i\sqrt{2} + 1 + i\right)} - \frac{15x^2\left(-\sqrt{2} -\right.}{7\left(i\sqrt{2} + 1\right.}\right. \\ &= \left(x^6 + \frac{\left(\frac{12}{7} + \frac{12i}{7}\right)x^5\sqrt{2}}{\left(i\sqrt{2} + 1 + i\right)\sqrt{2-\sqrt{2}}} + \frac{15x^4\left(\sqrt{2} + 1 + i\right)}{7\left(i\sqrt{2} + 1 + i\right)} + \frac{32x^3}{7\sqrt{2-\sqrt{2}}\left(i\sqrt{2} + 1 + i\right)} - \frac{15x^2\left(-\sqrt{2} -\right.}{7\left(i\sqrt{2} + 1\right.}\right. \\ &= \text{Expression too large to display} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-16x^7}{x^8+1}\,dx} \\ &= z_1 e^{\ln(x^8+1)} \\ &= z_1\left(x^8+1\right) \end{aligned}$$

Which simplifies to

$$y_1 = \text{Expression too large to display}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-16x^7}{x^8+1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2\ln(x^8+1)}}{(y_1)^2}\,dx \\ &= y_1(\text{Expression too large to display}) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\text{Expression too large to display}) \\ &\quad + c_2(\text{Expression too large to display}(\text{Expression too large to display})) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 22

```
dsolve((x^8+1)*diff(diff(y(x),x),x)-16*x^7*diff(y(x),x)+72*y(x)*x^6 = 0,y(x),singsol=all
```

$$y = -\frac{7}{9}c_1 + c_1 x^8 + c_2 x^9 - \frac{9}{7}c_2 x$$

Mathematica DSolve solution

Solving time : 0.0 (sec)
Leaf size : 0

```
DSolve[{(1+x^8)*D[y[x],{x,2}]-16*x^7*D[y[x],x]+72*x^6*y[x]==0,{}},y[x],x,IncludeSingularSolu
```

Timed out

2.1.67 Problem 69

Internal problem ID [9239]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 69
Date solved : `Monday, January 27, 2025 at 05:59:53 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + x^5 y' + 6x^4 y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.429 (sec)

Writing the ode as

$$y'' + x^5 y' + 6x^4 y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x^5 \\ C &= 6x^4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4(x^6 - 14)}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4(x^6 - 14) \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4(x^6 - 14)}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.119: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 10 \\ &= -10 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -10 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -10$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{10}{2} = 5$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{5} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^5$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x^5}{2} - \frac{7}{2x} - \frac{49}{4x^7} - \frac{343}{4x^{13}} - \frac{12005}{16x^{19}} - \frac{117649}{16x^{25}} - \frac{2470629}{32x^{31}} - \frac{27176919}{32x^{37}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 5$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{5} a_i x^i \\
&= \frac{x^5}{2}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^4 = x^4$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{x^{10}}{4}
$$

This shows that the coefficient of x^4 in the above is 0. Now we need to find the coefficient of x^4 in r. How this is done depends on if $v = 0$ or not. Since $v = 5$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of x^4 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^4(x^6 - 14)}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^{10} - \frac{7}{2}x^4\right) + (0) \\
&= \frac{1}{4}x^{10} - \frac{7}{2}x^4
\end{aligned}
$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{7}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{7}{2}\right) - (0) \\
&= -\frac{7}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x^5}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{7}{2}}{\frac{1}{2}} - 5\right) &= -6 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{7}{2}}{\frac{1}{2}} - 5\right) &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^4(x^6 - 14)}{4}
$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-10	$\frac{x^5}{2}$	-6	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x^5}{2}\right) \\ &= -\frac{x^5}{2} \\ &= -\frac{x^5}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{x^5}{2}\right)(1) + \left(\left(-\frac{5x^4}{2}\right) + \left(-\frac{x^5}{2}\right)^2 - \left(\frac{x^4(x^6-14)}{4}\right)\right) &= 0 \\ x^4 a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x)\, \mathrm{e}^{\int -\frac{x^5}{2} dx} \\ &= (x)\, \mathrm{e}^{-\frac{x^6}{12}} \\ &= x\, \mathrm{e}^{-\frac{x^6}{12}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^5}{1} dx} \\ &= z_1 e^{-\frac{x^6}{12}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x^6}{12}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^6}{6}} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^5}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{x^6}{6}}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{6^{5/6}(-1)^{1/6}\left(-\frac{6x^5(-1)^{5/6}\Gamma\left(\frac{5}{6}\right)}{(-x^6)^{5/6}} + \frac{6\, 6^{1/6}(-1)^{5/6}\mathrm{e}^{\frac{x^6}{6}}}{x} + \frac{6x^5(-1)^{5/6}\Gamma\left(\frac{5}{6}, -\frac{x^6}{6}\right)}{(-x^6)^{5/6}}\right)}{36} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-\frac{x^6}{6}} x\right) \\ &\quad + c_2 \left(\mathrm{e}^{-\frac{x^6}{6}} x \left(\frac{6^{5/6}(-1)^{1/6}\left(-\frac{6x^5(-1)^{5/6}\Gamma\left(\frac{5}{6}\right)}{(-x^6)^{5/6}} + \frac{6\, 6^{1/6}(-1)^{5/6}\mathrm{e}^{\frac{x^6}{6}}}{x} + \frac{6x^5(-1)^{5/6}\Gamma\left(\frac{5}{6}, -\frac{x^6}{6}\right)}{(-x^6)^{5/6}}\right)}{36} \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + x^5\left(\frac{d}{dx}y(x)\right) + 6x^4 y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

- ☐ Rewrite ODE with series expansions
  - Convert $x^4 \cdot y(x)$ to series expansion

    $x^4 \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+4}$
  - Shift index using $k- > k-4$

    $x^4 \cdot y(x) = \sum_{k=4}^{\infty} a_{k-4} x^k$
  - Convert $x^5 \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x^5 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^{k+4}$

- Shift index using $k->k-4$

  $x^5 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=4}^{\infty} a_{k-4}(k-4)\, x^k$

- Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

  $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- Shift index using $k->k+2$

  $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $20a_5x^3 + 12a_4x^2 + 6a_3x + 2a_2 + \left(\sum_{k=4}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_{k-4}(k+2)\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 = 0, 6a_3 = 0, 12a_4 = 0, 20a_5 = 0]$

- Solve for the dependent coefficient(s)

  $\{a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+2)\,(ka_{k+2} + a_{k-4} + a_{k+2}) = 0$

- Shift index using $k->k+4$

  $(k+6)\,((k+4)\, a_{k+6} + a_k + a_{k+6}) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+6} = -\frac{a_k}{k+5}, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.007` (sec)
Leaf size : 63

```
dsolve(diff(diff(y(x),x),x)+x^5*diff(y(x),x)+6*x^4*y(x) = 0,y(x),singsol=all)
```

$$y = c_2 e^{-\frac{x^6}{6}} \left(-x^6\right)^{1/6} 6^{5/6} \Gamma\left(\frac{5}{6}\right) - c_2 e^{-\frac{x^6}{6}} \left(-x^6\right)^{1/6} 6^{5/6} \Gamma\left(\frac{5}{6}, -\frac{x^6}{6}\right) + c_1 e^{-\frac{x^6}{6}} x + 6c_2$$

Mathematica DSolve solution

Solving time : 0.233 (sec)
Leaf size : 53

```
DSolve[{D[y[x],{x,2}]+x^5*D[y[x],x]+6*x^4*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{36} e^{-\frac{x^6}{6}} \left(36 c_1 x - 6^{5/6} c_2 \sqrt[6]{-x^6} \Gamma\left(-\frac{1}{6}, -\frac{x^6}{6}\right)\right)$$

2.1.68 Problem 70

Internal problem ID [9240]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 70
Date solved : `Monday, January 27, 2025 at 05:59:53 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(1+3x)\,y''+xy'+2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 30.965 (sec)

Writing the ode as

$$(1+3x)\,y''+xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=1+3x\\ B&=x\\ C&=2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{x^2-24x-6}{4\left(1+3x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=x^2-24x-6\\ t&=4(1+3x)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x^2-24x-6}{4\left(1+3x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.121: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(1 + 3x)^2$. There is a pole at $x = -\frac{1}{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{36} + \frac{19}{324\left(x + \frac{1}{3}\right)^2} - \frac{37}{54\left(x + \frac{1}{3}\right)}$$

For the pole at $x = -\frac{1}{3}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{19}{324}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{19}{18} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{18} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r} \approx \frac{1}{6} - \frac{37}{18x} - \frac{319}{27x^2} - \frac{11831}{81x^3} - \frac{2157901}{972x^4} - \frac{110035199}{2916x^5} - \frac{1501983319}{2187x^6} - \frac{85889060456}{6561x^7} + \dots \tag{9}
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a = \frac{1}{6}
$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{6}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{36}
$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 24x - 6}{36x^2 + 24x + 4} \\
&= Q + \frac{R}{36x^2 + 24x + 4} \\
&= \left(\frac{1}{36}\right) + \left(\frac{-\frac{74x}{3} - \frac{55}{9}}{36x^2 + 24x + 4}\right) \\
&= \frac{1}{36} + \frac{-\frac{74x}{3} - \frac{55}{9}}{36x^2 + 24x + 4}
\end{aligned}
$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is $-\frac{74}{3}$. Dividing this by leading coefficient in t which is 36 gives $-\frac{37}{54}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{37}{54}\right) - (0) \\
&= -\frac{37}{54}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{6} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{37}{54}}{\frac{1}{6}} - 0\right) &&= -\frac{37}{18} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{-\frac{37}{54}}{\frac{1}{6}} - 0\right) = \frac{37}{18}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 24x - 6}{4\left(1+3x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{3}$	2	0	$\frac{19}{18}$	$-\frac{1}{18}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{6}$	$-\frac{37}{18}$	$\frac{37}{18}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{37}{18}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\
&= \frac{37}{18} - \left(\frac{19}{18}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{19}{18\left(x+\frac{1}{3}\right)} + (-)\left(\frac{1}{6}\right) \\
&= \frac{19}{18\left(x+\frac{1}{3}\right)} - \frac{1}{6} \\
&= -\frac{-6+x}{2\left(1+3x\right)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{19}{18\left(x+\frac{1}{3}\right)}-\frac{1}{6}\right)(1)+\left(\left(-\frac{19}{18\left(x+\frac{1}{3}\right)^{2}}\right)+\left(\frac{19}{18\left(x+\frac{1}{3}\right)}-\frac{1}{6}\right)^{2}-\left(\frac{x^{2}-24x-6}{4\left(1+3x\right)^{2}}\right)\right)=0$$

$$\frac{a_0+6}{1+3x}=0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0=-6\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x)=-6+x$$

Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\, dx}\\
&= (-6+x)\,\mathrm{e}^{\int\left(\frac{19}{18\left(x+\frac{1}{3}\right)}-\frac{1}{6}\right)dx}\\
&= (-6+x)\,\mathrm{e}^{-\frac{x}{6}+\frac{19\ln(1+3x)}{18}}\\
&= (-6+x)\,(1+3x)^{19/18}\,\mathrm{e}^{-\frac{x}{6}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}\\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{1+3x}\, dx}\\
&= z_1 e^{-\frac{x}{6}+\frac{\ln(1+3x)}{18}}\\
&= z_1\left((1+3x)^{1/18}\,\mathrm{e}^{-\frac{x}{6}}\right)
\end{aligned}$$

Which simplifies to

$$y_1=(1+3x)^{10/9}\,\mathrm{e}^{-\frac{x}{3}}(-6+x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int\frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1\int\frac{e^{\int -\frac{x}{1+3x}\, dx}}{(y_1)^2}\, dx\\
&= y_1\int\frac{e^{-\frac{x}{3}+\frac{\ln(1+3x)}{9}}}{(y_1)^2}\, dx\\
&= y_1\left(\int\frac{\mathrm{e}^{-\frac{x}{3}+\frac{\ln(1+3x)}{9}}\,\mathrm{e}^{\frac{2x}{3}}}{(1+3x)^{20/9}\,(-6+x)^2}dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1y_1+c_2y_2\\
&= c_1\left((1+3x)^{10/9}\,\mathrm{e}^{-\frac{x}{3}}(-6+x)\right)\\
&\quad + c_2\left((1+3x)^{10/9}\,\mathrm{e}^{-\frac{x}{3}}(-6+x)\left(\int\frac{\mathrm{e}^{-\frac{x}{3}+\frac{\ln(1+3x)}{9}}\,\mathrm{e}^{\frac{2x}{3}}}{(1+3x)^{20/9}\,(-6+x)^2}dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(3x+1)\left(\frac{d^2}{dx^2}y(x)\right)+x\left(\frac{d}{dx}y(x)\right)+2y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{2y(x)}{3x+1}-\frac{x\left(\frac{d}{dx}y(x)\right)}{3x+1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{x\left(\frac{d}{dx}y(x)\right)}{3x+1}+\frac{2y(x)}{3x+1}=0$

- ☐ Check to see if $x_0=-\frac{1}{3}$ is a regular singular point
  - Define functions

    $\left[P_2(x)=\frac{x}{3x+1},P_3(x)=\frac{2}{3x+1}\right]$
  - $\left(x+\frac{1}{3}\right)\cdot P_2(x)$ is analytic at $x=-\frac{1}{3}$

    $\left(\left(x+\frac{1}{3}\right)\cdot P_2(x)\right)\bigg|_{x=-\frac{1}{3}}=-\frac{1}{9}$
  - $\left(x+\frac{1}{3}\right)^2\cdot P_3(x)$ is analytic at $x=-\frac{1}{3}$

    $\left(\left(x+\frac{1}{3}\right)^2\cdot P_3(x)\right)\bigg|_{x=-\frac{1}{3}}=0$
  - $x=-\frac{1}{3}$is a regular singular point

  Check to see if $x_0=-\frac{1}{3}$ is a regular singular point

  $x_0=-\frac{1}{3}$

- Multiply by denominators

  $(3x+1)\left(\frac{d^2}{dx^2}y(x)\right)+x\left(\frac{d}{dx}y(x)\right)+2y(x)=0$

- Change variables using $x=u-\frac{1}{3}$ so that the regular singular point is at $u=0$

  $3u\left(\frac{d^2}{du^2}y(u)\right)+\left(u-\frac{1}{3}\right)\left(\frac{d}{du}y(u)\right)+2y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

    $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-1}$
  - Shift index using $k->k+1$

    $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)(k+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $\frac{a_0r(-10+9r)u^{-1+r}}{3}+\left(\sum_{k=0}^{\infty}\left(\frac{a_{k+1}(k+1+r)(9k-1+9r)}{3}+a_k(k+r+2)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $\frac{r(-10+9r)}{3} = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{0, \frac{10}{9}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $3(k+1+r)\left(k-\frac{1}{9}+r\right)a_{k+1} + a_k(k+r+2) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = -\frac{3a_k(k+r+2)}{(k+1+r)(9k-1+9r)}$
- Recursion relation for $r = 0$
  $a_{k+1} = -\frac{3a_k(k+2)}{(k+1)(9k-1)}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = -\frac{3a_k(k+2)}{(k+1)(9k-1)}\right]$
- Revert the change of variables $u = x + \frac{1}{3}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k \left(x+\frac{1}{3}\right)^k, a_{k+1} = -\frac{3a_k(k+2)}{(k+1)(9k-1)}\right]$
- Recursion relation for $r = \frac{10}{9}$
  $a_{k+1} = -\frac{3a_k\left(k+\frac{28}{9}\right)}{\left(k+\frac{19}{9}\right)(9k+9)}$
- Solution for $r = \frac{10}{9}$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{10}{9}}, a_{k+1} = -\frac{3a_k\left(k+\frac{28}{9}\right)}{\left(k+\frac{19}{9}\right)(9k+9)}\right]$
- Revert the change of variables $u = x + \frac{1}{3}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k \left(x+\frac{1}{3}\right)^{k+\frac{10}{9}}, a_{k+1} = -\frac{3a_k\left(k+\frac{28}{9}\right)}{\left(k+\frac{19}{9}\right)(9k+9)}\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k \left(x+\frac{1}{3}\right)^k\right) + \left(\sum_{k=0}^{\infty} b_k \left(x+\frac{1}{3}\right)^{k+\frac{10}{9}}\right), a_{k+1} = -\frac{3a_k(k+2)}{(k+1)(9k-1)}, b_{k+1} = -\frac{3b_k\left(k+\frac{28}{9}\right)}{\left(k+\frac{19}{9}\right)(9k+9)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
```

```
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returnin
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.032 (sec)
Leaf size : 62

```
dsolve((3*x+1)*diff(diff(y(x),x),x)+diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{\left(x+\frac{1}{3}\right)(x-6)\, c_1 \mathrm{e}^{-\frac{x}{3}} \left(\Gamma\left(-\frac{1}{9}\right)+\frac{10\Gamma\left(-\frac{10}{9},-\frac{x}{3}-\frac{1}{9}\right)}{9}\right)\left(-\frac{x}{3}-\frac{1}{9}\right)^{1/9}}{9} + 3c_2\left(x+\frac{1}{3}\right)(x-6)\,\mathrm{e}^{-\frac{x}{3}}\left(\frac{x}{3}+\frac{1}{9}\right)^{1/9} - \frac{10c_1\mathrm{e}^{\frac{1}{9}}}{9}$$

Mathematica DSolve solution

Solving time : 0.421 (sec)
Leaf size : 103

```
DSolve[{(1+3*x)*D[y[x],{x,2}]+x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to (x-6)\exp\left(\int_1^x \frac{6-K[1]}{6K[1]+2}dK[1] - \frac{1}{2}\int_1^x \frac{K[2]}{3K[2]+1}dK[2]\right)\left(c_2\int_1^x \frac{\exp\left(-2\int_1^{K[3]}\frac{6-K[1]}{6K[1]+2}dK[1]\right)}{(K[3]-6)^2}dK[3]+c_1\right)$$

2.1.69 Problem 71

Internal problem ID [9241]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 71
Date solved : `Monday, January 27, 2025 at 06:00:25 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(3x^2+x+1\right)y''+(2+15x)\,y'+12y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.763 (sec)

Writing the ode as

$$\left(3x^2+x+1\right)y''+(2+15x)\,y'+12y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=3x^2+x+1\\ B&=2+15x\\ C&=12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-9x^2-12x-18}{4\left(3x^2+x+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=-9x^2-12x-18\\ t&=4\left(3x^2+x+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-9x^2-12x-18}{4\left(3x^2+x+1\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.123: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(3x^2 + x + 1)^2$. There is a pole at $x = -\frac{1}{6} + \frac{i\sqrt{11}}{6}$ of order 2. There is a pole at $x = -\frac{1}{6} - \frac{i\sqrt{11}}{6}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{\frac{27}{88} + \frac{3i\sqrt{11}}{88}}{\left(x + \frac{1}{6} - \frac{i\sqrt{11}}{6}\right)^2} + \frac{\frac{27}{88} - \frac{3i\sqrt{11}}{88}}{\left(x + \frac{1}{6} + \frac{i\sqrt{11}}{6}\right)^2} + \frac{57i\sqrt{11}}{242\left(x + \frac{1}{6} - \frac{i\sqrt{11}}{6}\right)} - \frac{57i\sqrt{11}}{242\left(x + \frac{1}{6} + \frac{i\sqrt{11}}{6}\right)}$$

For the pole at $x = -\frac{1}{6} + \frac{i\sqrt{11}}{6}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{6}-\frac{i\sqrt{11}}{6}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{27}{88} + \frac{3i\sqrt{11}}{88}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} + \frac{\sqrt{1078 + 66i\sqrt{11}}}{44} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} - \frac{\sqrt{1078 + 66i\sqrt{11}}}{44} \end{aligned}$$

For the pole at $x = -\frac{1}{6} - \frac{i\sqrt{11}}{6}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{6}+\frac{i\sqrt{11}}{6}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{27}{88} - \frac{3i\sqrt{11}}{88}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{1078 - 66i\sqrt{11}}}{44} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{1078 - 66i\sqrt{11}}}{44}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{-9x^2 - 12x - 18}{4\left(3x^2 + x + 1\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-9x^2 - 12x - 18}{4\left(3x^2 + x + 1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{6} + \frac{i\sqrt{11}}{6}$	2	0	$\frac{1}{2} + \frac{\sqrt{1078+66i\sqrt{11}}}{44}$	$\frac{1}{2} - \frac{\sqrt{1078+66i\sqrt{11}}}{44}$
$-\frac{1}{6} - \frac{i\sqrt{11}}{6}$	2	0	$\frac{1}{2} + \frac{\sqrt{1078-66i\sqrt{11}}}{44}$	$\frac{1}{2} - \frac{\sqrt{1078-66i\sqrt{11}}}{44}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{\frac{1}{2}-\frac{\sqrt{1078+66i\sqrt{11}}}{44}}{x+\frac{1}{6}-\frac{i\sqrt{11}}{6}} + \frac{\frac{1}{2}-\frac{\sqrt{1078-66i\sqrt{11}}}{44}}{x+\frac{1}{6}+\frac{i\sqrt{11}}{6}} + (-)(0) \\
&= \frac{\frac{1}{2}-\frac{\sqrt{1078+66i\sqrt{11}}}{44}}{x+\frac{1}{6}-\frac{i\sqrt{11}}{6}} + \frac{\frac{1}{2}-\frac{\sqrt{1078-66i\sqrt{11}}}{44}}{x+\frac{1}{6}+\frac{i\sqrt{11}}{6}} \\
&= -\frac{3x}{6x^2+2x+2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{\frac{1}{2}-\frac{\sqrt{1078+66i\sqrt{11}}}{44}}{x+\frac{1}{6}-\frac{i\sqrt{11}}{6}} + \frac{\frac{1}{2}-\frac{\sqrt{1078-66i\sqrt{11}}}{44}}{x+\frac{1}{6}+\frac{i\sqrt{11}}{6}}\right)(1) + \left(\left(-\frac{\frac{1}{2}-\frac{\sqrt{1078+66i\sqrt{11}}}{44}}{\left(x+\frac{1}{6}-\frac{i\sqrt{11}}{6}\right)^2} - \frac{\frac{1}{2}-\frac{\sqrt{1078-66i\sqrt{11}}}{44}}{\left(x+\frac{1}{6}+\frac{i\sqrt{11}}{6}\right)^2}\right) + \left(\frac{\frac{1}{2}-}{x}\right.\right.
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int \left(\frac{\frac{1}{2}-\frac{\sqrt{1078+66i\sqrt{11}}}{44}}{x+\frac{1}{6}-\frac{i\sqrt{11}}{6}} + \frac{\frac{1}{2}-\frac{\sqrt{1078-66i\sqrt{11}}}{44}}{x+\frac{1}{6}+\frac{i\sqrt{11}}{6}}\right) dx} \\
&= (x)\, \mathrm{e}^{-\frac{\ln\left(36x^2+12x+12\right)}{4} + \frac{\sqrt{11}\arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{22}} \\
&= \frac{x\sqrt{2}\, 3^{3/4} \mathrm{e}^{\frac{\sqrt{11}\arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{22}}}{6\left(3x^2+x+1\right)^{1/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2+15x}{3x^2+x+1}\, dx} \\
&= z_1 e^{-\frac{5\ln\left(3x^2+x+1\right)}{4} + \frac{\sqrt{11}\arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{22}} \\
&= z_1\left(\frac{\mathrm{e}^{\frac{\sqrt{11}\arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{22}}}{\left(3x^2+x+1\right)^{5/4}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{\frac{\sqrt{11}\, \arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{11}} x\sqrt{2}\, 3^{3/4}}{6\left(3x^2+x+1\right)^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2+15x}{3x^2+x+1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{5\ln\left(3x^2+x+1\right)}{2}+\frac{\sqrt{11}\, \arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{11}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{2\,\mathrm{e}^{-\frac{5\ln\left(3x^2+x+1\right)}{2}+\frac{\sqrt{11}\, \arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{11}} \left(3x^2+x+1\right)^3 \mathrm{e}^{-\frac{2\sqrt{11}\, \arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{11}} \sqrt{3}}{x^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{\frac{\sqrt{11}\, \arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{11}} x\sqrt{2}\, 3^{3/4}}{6\left(3x^2+x+1\right)^{3/2}}\right) \\
&+ c_2 \left(\frac{\mathrm{e}^{\frac{\sqrt{11}\, \arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{11}} x\sqrt{2}\, 3^{3/4}}{6\left(3x^2+x+1\right)^{3/2}} \left(\int \frac{2\,\mathrm{e}^{-\frac{5\ln\left(3x^2+x+1\right)}{2}+\frac{\sqrt{11}\, \arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{11}} \left(3x^2+x+1\right)^3 \mathrm{e}^{-\frac{2\sqrt{11}\, \arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{11}}}{x^2}\right.\right.
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(3x^2+x+1\right)\left(\frac{d^2}{dx^2}y(x)\right) + (2+15x)\left(\frac{d}{dx}y(x)\right) + 12y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{12y(x)}{3x^2+x+1} - \frac{(2+15x)\left(\frac{d}{dx}y(x)\right)}{3x^2+x+1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(2+15x)\left(\frac{d}{dx}y(x)\right)}{3x^2+x+1} + \frac{12y(x)}{3x^2+x+1} = 0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{2+15x}{3x^2+x+1}, P_3(x) = \frac{12}{3x^2+x+1}\right]$

- ○ $\left(\frac{\mathrm{I}\sqrt{11}}{6}+x+\frac{1}{6}\right)\cdot P_2(x)$ is analytic at $x = -\frac{1}{6}-\frac{\mathrm{I}\sqrt{11}}{6}$

$\left(\left(\frac{\mathrm{I}\sqrt{11}}{6}+x+\frac{1}{6}\right)\cdot P_2(x)\right)\Big|_{x=-\frac{1}{6}-\frac{\mathrm{I}\sqrt{11}}{6}}=0$

- $\left(\frac{\mathrm{I}\sqrt{11}}{6}+x+\frac{1}{6}\right)^2\cdot P_3(x)$ is analytic at $x=-\frac{1}{6}-\frac{\mathrm{I}\sqrt{11}}{6}$

  $\left(\left(\frac{\mathrm{I}\sqrt{11}}{6}+x+\frac{1}{6}\right)^2\cdot P_3(x)\right)\Big|_{x=-\frac{1}{6}-\frac{\mathrm{I}\sqrt{11}}{6}}=0$

- $x=-\frac{1}{6}-\frac{\mathrm{I}\sqrt{11}}{6}$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-\frac{1}{6}-\frac{\mathrm{I}\sqrt{11}}{6}$

- Multiply by denominators

  $(3x^2+x+1)\left(\frac{d^2}{dx^2}y(x)\right)+(2+15x)\left(\frac{d}{dx}y(x)\right)+12y(x)=0$

- Change variables using $x=u-\frac{1}{6}-\frac{\mathrm{I}\sqrt{11}}{6}$ so that the regular singular point is at $u=0$

  $(3u^2-\mathrm{I}u\sqrt{11})\left(\frac{d^2}{du^2}y(u)\right)+\left(-\frac{1}{2}+15u-\frac{5\,\mathrm{I}\sqrt{11}}{2}\right)\left(\frac{d}{du}y(u)\right)+12y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$
  - Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $\frac{\mathrm{I}\sqrt{11}\,r\left(\mathrm{I}\sqrt{11}-33-22r\right)a_0u^{-1+r}}{22}+\left(\sum_{k=0}^{\infty}\left(\frac{\mathrm{I}\sqrt{11}\,(k+1+r)\left(\mathrm{I}\sqrt{11}-22k-55-22r\right)a_{k+1}}{22}+3a_k(k+r+2)^2\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $\frac{\mathrm{I}}{22}\sqrt{11}\,r\left(\mathrm{I}\sqrt{11}-33-22r\right)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{0,-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $3a_k(k+r+2)^2-(k+1+r)\,a_{k+1}\left(\frac{1}{2}+\mathrm{I}\left(k+r+\frac{5}{2}\right)\sqrt{11}\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{6a_k\left(k^2+2kr+r^2+4k+4r+4\right)}{2\,\mathrm{I}\sqrt{11}\,k^2+4\,\mathrm{I}kr\sqrt{11}+2\,\mathrm{I}\sqrt{11}\,r^2+7\,\mathrm{I}k\sqrt{11}+7\,\mathrm{I}r\sqrt{11}+5\,\mathrm{I}\sqrt{11}+k+r+1}$

- Recursion relation for $r=0$

  $a_{k+1}=\frac{6a_k\left(k^2+4k+4\right)}{2\,\mathrm{I}\sqrt{11}\,k^2+1+7\,\mathrm{I}k\sqrt{11}+5\,\mathrm{I}\sqrt{11}+k}$

- Solution for $r=0$

  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^k, a_{k+1}=\frac{6a_k\left(k^2+4k+4\right)}{2\,\mathrm{I}\sqrt{11}\,k^2+1+7\,\mathrm{I}k\sqrt{11}+5\,\mathrm{I}\sqrt{11}+k}\right]$

- Revert the change of variables $u=\frac{I\sqrt{11}}{6}+x+\frac{1}{6}$

$$\left[y(x)=\sum_{k=0}^{\infty}a_k\left(\frac{I\sqrt{11}}{6}+x+\frac{1}{6}\right)^k, a_{k+1}=\frac{6a_k(k^2+4k+4)}{2I\sqrt{11}\,k^2+1+7Ik\sqrt{11}+5I\sqrt{11}+k}\right]$$

- Recursion relation for $r=-\frac{3}{2}+\frac{I\sqrt{11}}{22}$

$$a_{k+1}=\frac{6a_k\left(k^2+2k\left(-\frac{3}{2}+\frac{I\sqrt{11}}{22}\right)+\left(-\frac{3}{2}+\frac{I\sqrt{11}}{22}\right)^2+4k-2+\frac{2I\sqrt{11}}{11}\right)}{2I\sqrt{11}\,k^2+4Ik\left(-\frac{3}{2}+\frac{I\sqrt{11}}{22}\right)\sqrt{11}+2I\sqrt{11}\left(-\frac{3}{2}+\frac{I\sqrt{11}}{22}\right)^2+7Ik\sqrt{11}+7I\left(-\frac{3}{2}+\frac{I\sqrt{11}}{22}\right)\sqrt{11}+\frac{111I\sqrt{11}}{22}+k-\frac{1}{2}}$$

- Solution for $r=-\frac{3}{2}+\frac{I\sqrt{11}}{22}$

$$\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k-\frac{3}{2}+\frac{I\sqrt{11}}{22}}, a_{k+1}=\frac{6a_k\left(k^2+2k\left(-\frac{3}{2}+\frac{I\sqrt{11}}{22}\right)+\left(-\frac{3}{2}+\frac{I\sqrt{11}}{22}\right)^2+4k-2+\frac{2I\sqrt{11}}{11}\right)}{2I\sqrt{11}\,k^2+4Ik\left(-\frac{3}{2}+\frac{I\sqrt{11}}{22}\right)\sqrt{11}+2I\sqrt{11}\left(-\frac{3}{2}+\frac{I\sqrt{11}}{22}\right)^2+7Ik\sqrt{11}+7I\left(-\frac{3}{2}+\frac{I\sqrt{11}}{22}\right)\sqrt{11}\right.$$

- Revert the change of variables $u=\frac{I\sqrt{11}}{6}+x+\frac{1}{6}$

$$\left[y(x)=\sum_{k=0}^{\infty}a_k\left(\frac{I\sqrt{11}}{6}+x+\frac{1}{6}\right)^{k-\frac{3}{2}+\frac{I\sqrt{11}}{22}}, a_{k+1}=\frac{6a_k\left(k^2+2k\left(-\frac{3}{2}+\frac{I\sqrt{11}}{22}\right)+\left(-\frac{3}{2}+\frac{I\sqrt{11}}{22}\right)^2+4\right.}{2I\sqrt{11}\,k^2+4Ik\left(-\frac{3}{2}+\frac{I\sqrt{11}}{22}\right)\sqrt{11}+2I\sqrt{11}\left(-\frac{3}{2}+\frac{I\sqrt{11}}{22}\right)^2+7Ik\sqrt{11}+}\right.$$

- Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty}a_k\left(\frac{I\sqrt{11}}{6}+x+\frac{1}{6}\right)^k\right)+\left(\sum_{k=0}^{\infty}b_k\left(\frac{I\sqrt{11}}{6}+x+\frac{1}{6}\right)^{k-\frac{3}{2}+\frac{I\sqrt{11}}{22}}\right), a_{k+1}=\frac{6a_k(k^2+4k+}{2I\sqrt{11}\,k^2+1+7Ik\sqrt{11}}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special functio
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.220 (sec)
Leaf size : 163

```
dsolve((3*x^2+x+1)*diff(diff(y(x),x),x)+(2+15*x)*diff(y(x),x)+12*y(x) = 0,y(x),singsol=a
```

$$y = \frac{e^{\frac{\sqrt{11}\,\arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{22}}\left(\left(-36x^2-12x-12\right)^{-\frac{1}{4}+\frac{i\sqrt{11}}{44}}\left(i\sqrt{11}-6x-1\right)^{3/2}c_1\,\text{hypergeom}\left(\left[\frac{1}{2}+\frac{i\sqrt{11}}{22},\frac{1}{2}+\frac{i\sqrt{11}}{22}\right],\right.\right.}{}$$

Mathematica DSolve solution

Solving time : 1.147 (sec)
Leaf size : 113

```
DSolve[{(1+x+3*x^2)*D[y[x],{x,2}]+(2+15*x)*D[y[x],x]+12*y[x]==0,{}},y[x],x,IncludeSingularSolu
```

$$y(x) \to x\exp\left(\int_1^x -\frac{3K[1]}{6K[1]^2+2K[1]+2}dK[1] - \frac{1}{2}\int_1^x \frac{15K[2]+2}{3K[2]^2+K[2]+1}dK[2]\right)\left(c_2\int_1^x \frac{\exp\left(-2\int_1^{K[3]}-\frac{3K[1]}{6K[1]^2+2K[1]+2}dK[1]\right)}{K[3]^2}dK[3] + c_1\right)$$

2.1.70 Problem 72

Internal problem ID [9242]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 72
Date solved : `Monday, January 27, 2025 at 06:00:26 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(2+x)\,y''+(1+x)\,y'+3y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.286 (sec)

Writing the ode as

$$(2+x)\,y''+(1+x)\,y'+3y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=2+x\\ B&=1+x\\ C&=3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{x^2-10x-21}{4\,(2+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=x^2-10x-21\\ t&=4(2+x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x^2-10x-21}{4\,(2+x)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.125: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2+x)^2$. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{7}{2\,(2+x)} + \frac{3}{4\,(2+x)^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(2+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r} \approx \frac{1}{2} - \frac{7}{2x} - \frac{9}{2x^2} - \frac{97}{2x^3} - \frac{1291}{4x^4} - \frac{11103}{4x^5} - \frac{98061}{4x^6} - \frac{913053}{4x^7} + \dots \tag{9}
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a = \frac{1}{2}
$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}
$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 10x - 21}{4x^2 + 16x + 16} \\
&= Q + \frac{R}{4x^2 + 16x + 16} \\
&= \left(\frac{1}{4}\right) + \left(\frac{-14x - 25}{4x^2 + 16x + 16}\right) \\
&= \frac{1}{4} + \frac{-14x - 25}{4x^2 + 16x + 16}
\end{aligned}
$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -14. Dividing this by leading coefficient in t which is 4 gives $-\frac{7}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{7}{2}\right) - (0) \\
&= -\frac{7}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{7}{2}}{\frac{1}{2}} - 0\right) &= -\frac{7}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{7}{2}}{\frac{1}{2}} - 0\right) &= \frac{7}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 10x - 21}{4\left(2+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-2	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{7}{2}$	$\frac{7}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{7}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{7}{2} - \left(\frac{3}{2}\right) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{3}{2\left(2+x\right)} + (-)\left(\frac{1}{2}\right) \\
&= \frac{3}{2\left(2+x\right)} - \frac{1}{2} \\
&= -\frac{-1+x}{2\left(2+x\right)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(2)+2\left(\frac{3}{2(2+x)}-\frac{1}{2}\right)(2x+a_1)+\left(\left(-\frac{3}{2(2+x)^2}\right)+\left(\frac{3}{2(2+x)}-\frac{1}{2}\right)^2-\left(\frac{x^2-10x-21}{4(2+x)^2}\right)\right)=0$$

$$\frac{(a_1+4)x+2a_0+a_1+4}{2+x}=0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0=0, a_1=-4\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x)=x^2-4x$$

Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\,dx} \\
&= (x^2-4x)\,\mathrm{e}^{\int\left(\frac{3}{2(2+x)}-\frac{1}{2}\right)dx} \\
&= (x^2-4x)\,\mathrm{e}^{-\frac{x}{2}+\frac{3\ln(2+x)}{2}} \\
&= x(x-4)(2+x)^{3/2}\,\mathrm{e}^{-\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1+x}{2+x}\,dx} \\
&= z_1 e^{-\frac{x}{2}+\frac{\ln(2+x)}{2}} \\
&= z_1\left(\sqrt{2+x}\,\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1=(2+x)^2\,\mathrm{e}^{-x}x(x-4)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int\frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1\int\frac{e^{\int -\frac{1+x}{2+x}\,dx}}{(y_1)^2}\,dx \\
&= y_1\int\frac{e^{-x+\ln(2+x)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{\mathrm{e}^{-2}\,\mathrm{Ei}_1(-2-x)}{48}-\frac{\mathrm{e}^{x}}{288(2+x)^2}-\frac{11\,\mathrm{e}^{x}}{864(2+x)}-\frac{\mathrm{e}^{x}}{3456(x-4)}-\frac{\mathrm{e}^{x}}{128x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1y_1+c_2y_2 \\
&= c_1\left((2+x)^2\,\mathrm{e}^{-x}x(x-4)\right)+c_2\left((2+x)^2\,\mathrm{e}^{-x}x(x-4)\left(-\frac{\mathrm{e}^{-2}\,\mathrm{Ei}_1(-2-x)}{48}-\frac{\mathrm{e}^{x}}{288(2+x)^2}\right.\right. \\
&\quad \left.\left.-\frac{11\,\mathrm{e}^{x}}{864(2+x)}-\frac{\mathrm{e}^{x}}{3456(x-4)}-\frac{\mathrm{e}^{x}}{128x}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x+2)\left(\frac{d^2}{dx^2}y(x)\right)+(x+1)\left(\frac{d}{dx}y(x)\right)+3y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{3y(x)}{x+2}-\frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{x+2}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{x+2}+\frac{3y(x)}{x+2}=0$
- □ Check to see if $x_0=-2$ is a regular singular point
  - Define functions

    $\left[P_2(x)=\frac{x+1}{x+2}, P_3(x)=\frac{3}{x+2}\right]$
  - $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

    $\left((x+2)\cdot P_2(x)\right)\bigg|_{x=-2}=-1$
  - $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

    $\left((x+2)^2\cdot P_3(x)\right)\bigg|_{x=-2}=0$
  - $x=-2$is a regular singular point

  Check to see if $x_0=-2$ is a regular singular point

  $x_0=-2$
- Multiply by denominators

  $(x+2)\left(\frac{d^2}{dx^2}y(x)\right)+(x+1)\left(\frac{d}{dx}y(x)\right)+3y(x)=0$
- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

  $u\left(\frac{d^2}{du^2}y(u)\right)+(u-1)\left(\frac{d}{du}y(u)\right)+3y(u)=0$
- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$
- □ Rewrite ODE with series expansions
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

    $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-1}$
  - Shift index using $k->k+1$

    $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)(k+r-1)+a_k(k+r+3)\right)u^{k+r}\right)=0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r)=0$

- Values of r that satisfy the indicial equation
  $r \in \{0,2\}$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)(k+r-1)+a_k(k+r+3)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=-\frac{a_k(k+r+3)}{(k+1+r)(k+r-1)}$
- Recursion relation for $r=0$
  $a_{k+1}=-\frac{a_k(k+3)}{(k+1)(k-1)}$
- Series not valid for $r=0$, division by 0 in the recursion relation at $k=1$
  $a_{k+1}=-\frac{a_k(k+3)}{(k+1)(k-1)}$
- Recursion relation for $r=2$
  $a_{k+1}=-\frac{a_k(k+5)}{(k+3)(k+1)}$
- Solution for $r=2$
  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1}=-\frac{a_k(k+5)}{(k+3)(k+1)}\right]$
- Revert the change of variables $u=x+2$
  $\left[y(x)=\sum_{k=0}^{\infty} a_k(x+2)^{k+2}, a_{k+1}=-\frac{a_k(k+5)}{(k+3)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 59

```
dsolve((x+2)*diff(diff(y(x),x),x)+(x+1)*diff(y(x),x)+3*y(x) = 0,y(x),singsol=all)
```

$$y=\mathrm{e}^{-x-2}c_2x(x-4)(x+2)^2\,\mathrm{Ei}_1(-x-2)+c_1\mathrm{e}^{-x}x(x-4)(x+2)^2+c_2(x^3-x^2-10x-6)$$

Mathematica DSolve solution

Solving time : 0.542 (sec)
Leaf size : 106

```
DSolve[{(2+x)*D[y[x],{x,2}]+(1+x)*D[y[x],x]+3*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True
```

$$y(x) \to (x-4)x \exp\left(\int_1^x \left(\frac{3}{2(K[1]+2)} - \frac{1}{2}\right) dK[1] - \frac{1}{2}\int_1^x \frac{K[2]+1}{K[2]+2} dK[2]\right) \left(c_2 \int_1^x \frac{\exp\left(-2\int_1^{K[3]} \left(\frac{3}{2(K[1]+2)} - \frac{1}{2}\right) dK[1]\right)}{(K[3]-4)^2 K[3]^2} dK[3] + c_1\right)$$

2.1.71 Problem 73

Internal problem ID [9243]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 73
Date solved : `Monday, January 27, 2025 at 06:00:27 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(4+x)\,y''+(2+x)\,y'+2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.276 (sec)

Writing the ode as

$$(4+x)\,y''+(2+x)\,y'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=4+x\\B&=2+x\\C&=2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{x^2-4x-24}{4\,(4+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=x^2-4x-24\\t&=4(4+x)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x^2-4x-24}{4\,(4+x)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.127: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(4+x)^2$. There is a pole at $x = -4$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

<u>Attempting to find a solution using case $n = 1$.</u>

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{(4+x)^2} - \frac{3}{4+x}$$

For the <u>pole at $x = -4$</u> let b be the coefficient of $\frac{1}{(4+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{3}{x} + \frac{5}{x^2} - \frac{34}{x^3} + \frac{59}{x^4} - \frac{586}{x^5} + \frac{370}{x^6} - \frac{12484}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x - 24}{4x^2 + 32x + 64} \\ &= Q + \frac{R}{4x^2 + 32x + 64} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-12x - 40}{4x^2 + 32x + 64}\right) \\ &= \frac{1}{4} + \frac{-12x - 40}{4x^2 + 32x + 64} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -12. Dividing this by leading coefficient in t which is 4 gives -3. Now b can be found.

$$\begin{aligned} b &= (-3) - (0) \\ &= -3 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-3}{\frac{1}{2}} - 0\right) &&= -3 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-3}{\frac{1}{2}} - 0\right) &&= 3 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x - 24}{4\left(4+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-4	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	-3	3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 3$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\
&= 3 - (2) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{2}{4+x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{2}{4+x} - \frac{1}{2} \\
&= -\frac{x}{2\left(4+x\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{2}{4+x} - \frac{1}{2}\right)(1) + \left(\left(-\frac{2}{\left(4+x\right)^2}\right) + \left(\frac{2}{4+x} - \frac{1}{2}\right)^2 - \left(\frac{x^2-4x-24}{4\left(4+x\right)^2}\right) \right) &= 0 \\
\frac{a_0}{4+x} &= 0
\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x) \, e^{\int \left(\frac{2}{4+x} - \frac{1}{2}\right) dx} \\ &= (x) \, e^{-\frac{x}{2} + 2\ln(4+x)} \\ &= x(4+x)^2 \, e^{-\frac{x}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2+x}{4+x} \, dx} \\ &= z_1 e^{-\frac{x}{2} + \ln(4+x)} \\ &= z_1 \left((4+x) \, e^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (4+x)^3 \, e^{-x} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2+x}{4+x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-x + 2\ln(4+x)}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{29 \, e^{x}}{768 \, (4+x)} - \frac{e^{-4} \, \mathrm{Ei}_1(-4-x)}{24} - \frac{5 \, e^{x}}{192 \, (4+x)^2} - \frac{e^{x}}{48 \, (4+x)^3} - \frac{e^{x}}{256x} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left((4+x)^3 \, e^{-x} x\right) + c_2 \left((4+x)^3 \, e^{-x} x \left(-\frac{29 \, e^{x}}{768 \, (4+x)} - \frac{e^{-4} \, \mathrm{Ei}_1(-4-x)}{24} - \frac{5 \, e^{x}}{192 \, (4+x)^2} \right.\right. \\ &\quad \left.\left. - \frac{e^{x}}{48 \, (4+x)^3} - \frac{e^{x}}{256x} \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x+4)\left(\frac{d^2}{dx^2}y(x)\right)+(x+2)\left(\frac{d}{dx}y(x)\right)+2y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{2y(x)}{x+4}-\frac{(x+2)\left(\frac{d}{dx}y(x)\right)}{x+4}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{(x+2)\left(\frac{d}{dx}y(x)\right)}{x+4}+\frac{2y(x)}{x+4}=0$

□ Check to see if $x_0=-4$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{x+2}{x+4}, P_3(x)=\frac{2}{x+4}\right]$

  ◦ $(x+4)\cdot P_2(x)$ is analytic at $x=-4$

  $\left.((x+4)\cdot P_2(x))\right|_{x=-4}=-2$

  ◦ $(x+4)^2\cdot P_3(x)$ is analytic at $x=-4$

  $\left.((x+4)^2\cdot P_3(x))\right|_{x=-4}=0$

  ◦ $x=-4$is a regular singular point

  Check to see if $x_0=-4$ is a regular singular point

  $x_0=-4$

- Multiply by denominators

  $(x+4)\left(\frac{d^2}{dx^2}y(x)\right)+(x+2)\left(\frac{d}{dx}y(x)\right)+2y(x)=0$

- Change variables using $x=u-4$ so that the regular singular point is at $u=0$

  $u\left(\frac{d^2}{du^2}y(u)\right)+(u-2)\left(\frac{d}{du}y(u)\right)+2y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  ◦ Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-1}$

  ◦ Shift index using $k->k+1$

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-3+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(k-2+r)+a_k(k+r+2)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-3+r) = 0$

- Values of r that satisfy the indicial equation
  $r \in \{0, 3\}$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)(k-2+r) + a_k(k+r+2) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = -\frac{a_k(k+r+2)}{(k+1+r)(k-2+r)}$
- Recursion relation for $r = 0$
  $a_{k+1} = -\frac{a_k(k+2)}{(k+1)(k-2)}$
- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 2$
  $a_{k+1} = -\frac{a_k(k+2)}{(k+1)(k-2)}$
- Recursion relation for $r = 3$
  $a_{k+1} = -\frac{a_k(k+5)}{(k+4)(k+1)}$
- Solution for $r = 3$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+3}, a_{k+1} = -\frac{a_k(k+5)}{(k+4)(k+1)}\right]$
- Revert the change of variables $u = x+4$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x+4)^{k+3}, a_{k+1} = -\frac{a_k(k+5)}{(k+4)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 53

```
dsolve((x+4)*diff(diff(y(x),x),x)+(x+2)*diff(y(x),x)+2*y(x) = 0,y(x),singsol=all)
```

$$y = e^{-x-4} c_2 x(x+4)^3 \operatorname{Ei}_1(-x-4) + c_1 e^{-x} x(x+4)^3 + c_2(x^3 + 9x^2 + 22x + 6)$$

Mathematica DSolve solution

Solving time : 0.375 (sec)
Leaf size : 93

```
DSolve[{(4+x)*D[y[x],{x,2}]+(2+x)*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True
```

$$y(x) \to x \exp \left(\int_1^x -\frac{K[1]}{2(K[1]+4)} dK[1] - \frac{1}{2} \int_1^x \frac{K[2]+2}{K[2]+4} dK[2] \right) \left(c_2 \int_1^x \frac{\exp \left(-2 \int_1^{K[3]} -\frac{K[1]}{2(K[1]+4)} dK[1] \right)}{K[3]^2} dK[3] + c_1 \right)$$

2.1.72 Problem 74

Internal problem ID [9244]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 74
Date solved : `Monday, January 27, 2025 at 06:00:27 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(2x^2+3x\right)y''+10(1+x)\,y'+8y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.290 (sec)

Writing the ode as

$$\left(2x^2+3x\right)y''+(10x+10)\,y'+8y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=2x^2+3x\\ B&=10x+10\\ C&=8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-x^2+6x+10}{\left(2x^2+3x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=-x^2+6x+10\\ t&=\left(2x^2+3x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-x^2+6x+10}{\left(2x^2+3x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.129: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x^2 + 3x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{3}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{36\left(x+\frac{3}{2}\right)^2} + \frac{22}{27\left(x+\frac{3}{2}\right)} - \frac{22}{27x} + \frac{10}{9x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{10}{9}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{3} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{2}{3} \end{aligned}$$

For the pole at $x = -\frac{3}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{3}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 + 6x + 10}{(2x^2 + 3x)^2}$$

Since the gcd$(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 + 6x + 10}{(2x^2 + 3x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{3}$	$-\frac{2}{3}$
$-\frac{3}{2}$	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{2}{3x} + \frac{1}{6x+9} + (-)(0) \\
&= -\frac{2}{3x} + \frac{1}{6x+9} \\
&= -\frac{x+2}{x(2x+3)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{2}{3x} + \frac{1}{6x+9}\right)(1) + \left(\left(\frac{2}{3x^2} - \frac{1}{6\left(x+\frac{3}{2}\right)^2}\right) + \left(-\frac{2}{3x} + \frac{1}{6x+9}\right)^2 - \left(\frac{-x^2+6x+10}{\left(2x^2+3x\right)^2}\right)\right) = 0$$

$$\frac{-4 + 2a_0}{x\left(2x+3\right)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+2)\,\mathrm{e}^{\int \left(-\frac{2}{3x} + \frac{1}{6x+9}\right) dx} \\
&= (x+2)\,\mathrm{e}^{\frac{\ln(2x+3)}{6} - \frac{2\ln(x)}{3}} \\
&= \frac{(x+2)\left(2x+3\right)^{1/6}}{x^{2/3}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{10x+10}{2x^2+3x}\, dx} \\
&= z_1 e^{-\frac{5\ln(2x+3)}{6} - \frac{5\ln(x)}{3}} \\
&= z_1 \left(\frac{1}{\left(2x+3\right)^{5/6} x^{5/3}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x+2}{\left(2x+3\right)^{2/3} x^{7/3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{10x+10}{2x^2+3x}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{\mathrm{e}^{-\frac{5\ln(2x+3)}{3} - \frac{10\ln(x)}{3}}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{5\ln(2x+3)}{3} - \frac{10\ln(x)}{3}} \left(2x+3\right)^{4/3} x^{14/3}}{\left(x+2\right)^2}\, dx\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x+2}{(2x+3)^{2/3}\, x^{7/3}}\right) + c_2\left(\frac{x+2}{(2x+3)^{2/3}\, x^{7/3}}\left(\int \frac{e^{-\frac{5\ln(2x+3)}{3}-\frac{10\ln(x)}{3}}(2x+3)^{4/3}\, x^{14/3}}{(x+2)^2}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(2x^2+3x)\left(\frac{d^2}{dx^2}y(x)\right)+10(x+1)\left(\frac{d}{dx}y(x)\right)+8y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{8y(x)}{x(2x+3)}-\frac{10(x+1)\left(\frac{d}{dx}y(x)\right)}{x(2x+3)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{10(x+1)\left(\frac{d}{dx}y(x)\right)}{x(2x+3)}+\frac{8y(x)}{x(2x+3)}=0$
- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x)=\frac{10(x+1)}{x(2x+3)}, P_3(x)=\frac{8}{x(2x+3)}\right]$
  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{10}{3}$
  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$
  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$
- Multiply by denominators

  $x(2x+3)\left(\frac{d^2}{dx^2}y(x)\right)+(10x+10)\left(\frac{d}{dx}y(x)\right)+8y(x)=0$
- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty} a_k x^{k+r}$
- □ Rewrite ODE with series expansions
  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..1$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=1..2$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$
  - Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(7+3r)\,x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(3k+10+3r) + 2a_k(k+r+2)^2\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(7+3r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{0, -\frac{7}{3}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)\,(3k+10+3r) + 2a_k(k+r+2)^2 = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = -\frac{2a_k(k+r+2)^2}{(k+1+r)(3k+10+3r)}$
- Recursion relation for $r = 0$
  $a_{k+1} = -\frac{2a_k(k+2)^2}{(k+1)(3k+10)}$
- Solution for $r = 0$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{2a_k(k+2)^2}{(k+1)(3k+10)}\right]$
- Recursion relation for $r = -\frac{7}{3}$
  $a_{k+1} = -\frac{2a_k\left(k-\frac{1}{3}\right)^2}{\left(k-\frac{4}{3}\right)(3k+3)}$
- Solution for $r = -\frac{7}{3}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{7}{3}}, a_{k+1} = -\frac{2a_k\left(k-\frac{1}{3}\right)^2}{\left(k-\frac{4}{3}\right)(3k+3)}\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{7}{3}}\right), a_{k+1} = -\frac{2a_k(k+2)^2}{(k+1)(3k+10)}, b_{k+1} = -\frac{2b_k\left(k-\frac{1}{3}\right)^2}{\left(k-\frac{4}{3}\right)(3k+3)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
```

```
      <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
   <- hypergeometric successful
<- special function solution successful
   -> Trying to convert hypergeometric functions to elementary form...
   <- elementary form for at least one hypergeometric solution is achieved - return
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.147 (sec)
Leaf size : 31

```
dsolve((2*x^2+3*x)*diff(diff(y(x),x),x)+10*(x+1)*diff(y(x),x)+8*y(x) = 0,y(x),singsol=
```

$$y = \frac{c_1(x+2)}{\left(1+\frac{2x}{3}\right)^{2/3} x^{7/3}} + c_2 \operatorname{hypergeom}\left([2,2], \left[\frac{10}{3}\right], -\frac{2x}{3}\right)$$

Mathematica DSolve solution

Solving time : 0.439 (sec)
Leaf size : 118

```
DSolve[{(3*x+2*x^2)*D[y[x],{x,2}]+10*(1+x)*D[y[x],x]+8*y[x]==0,{}},y[x],x,IncludeSingularSol
```

$$y(x) \to (x+2) \exp\left(\int_1^x -\frac{K[1]+2}{2K[1]^2+3K[1]} dK[1] - \frac{1}{2}\int_1^x \frac{10(K[2]+1)}{K[2](2K[2]+3)} dK[2]\right)\left(c_2 \int_1^x \frac{\exp\left(-2\int_1^{K[3]} -\frac{K[1]+2}{2K[1]^2+3K[1]} dK[1]\right)}{(K[3]+2)^2} dK[3] + c_1\right)$$

2.1.73 Problem 75

Internal problem ID [9245]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 75
Date solved : `Monday, January 27, 2025 at 06:00:28 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - (6-7x)\,y' + 8y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.263 (sec)

Writing the ode as

$$x^2y'' + (-6+7x)\,y' + 8y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -6+7x \\ C &= 8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^2 - 60x + 36}{4x^4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3x^2 - 60x + 36 \\ t &= 4x^4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^2 - 60x + 36}{4x^4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.131: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^4$. There is a pole at $x = 0$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned} \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) \end{aligned}$$

The partial fraction decomposition of r is

$$r = -\frac{15}{x^3} + \frac{3}{4x^2} + \frac{9}{x^4}$$

There is pole in r at $x = 0$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{3}{x^2} - \frac{5}{2x} - \frac{11}{12} - \frac{55x}{72} - \frac{671x^2}{864} - \frac{4565x^3}{5184} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{3}{x^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^2}$ is

$$a = 3$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be -15. Therefore

$$\begin{aligned} b &= (-15) - (0) \\ &= -15 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{3}{x^2} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &= \frac{1}{2}\left(\frac{-15}{3} + 2\right) &= -\frac{3}{2} \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) = \frac{1}{2}\left(-\frac{-15}{3} + 2\right) = \frac{7}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x^2 - 60x + 36}{4x^4}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3x^2 - 60x + 36}{4x^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	4	$\frac{3}{x^2}$	$-\frac{3}{2}$	$\frac{7}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\
&= -\frac{1}{2} - \left(-\frac{3}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{3}{x^2} - \frac{3}{2x} + (-)\,(0) \\
&= \frac{3}{x^2} - \frac{3}{2x} \\
&= -\frac{3(-2+x)}{2x^2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{3}{x^2} - \frac{3}{2x}\right)(1) + \left(\left(-\frac{6}{x^3} + \frac{3}{2x^2}\right) + \left(\frac{3}{x^2} - \frac{3}{2x}\right)^2 - \left(\frac{3x^2 - 60x + 36}{4x^4}\right)\right) = 0
$$

$$
\frac{6 + 3a_0}{x^2} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -2\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = -2 + x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (-2+x)\, e^{\int \left(\frac{3}{x^2} - \frac{3}{2x}\right) dx} \\
&= (-2+x)\, e^{-\frac{3\ln(x)}{2} - \frac{3}{x}} \\
&= \frac{(-2+x)\, e^{-\frac{3}{x}}}{x^{3/2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-6+7x}{x^2}\,dx} \\
&= z_1 e^{-\frac{7\ln(x)}{2}-\frac{3}{x}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{3}{x}}}{x^{7/2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{-\frac{6}{x}}(-2+x)}{x^5}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-6+7x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-7\ln(x)-\frac{6}{x}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{12\,\mathrm{e}^{\frac{6}{x}}}{\frac{6}{x}-3} + 54\,\mathrm{Ei}_1\left(-\frac{6}{x}\right) + \frac{x^2\mathrm{e}^{\frac{6}{x}}}{2} + 7x\,\mathrm{e}^{\frac{6}{x}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{-\frac{6}{x}}(-2+x)}{x^5}\right) + c_2 \left(\frac{\mathrm{e}^{-\frac{6}{x}}(-2+x)}{x^5}\left(\frac{12\,\mathrm{e}^{\frac{6}{x}}}{\frac{6}{x}-3} + 54\,\mathrm{Ei}_1\left(-\frac{6}{x}\right) + \frac{x^2\mathrm{e}^{\frac{6}{x}}}{2} + 7x\,\mathrm{e}^{\frac{6}{x}}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 50

```
dsolve(x^2*diff(diff(y(x),x),x)-(6-7*x)*diff(y(x),x)+8*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{108c_2 e^{-\frac{6}{x}}(x-2)\operatorname{Ei}_1\left(-\frac{6}{x}\right) + c_1 e^{-\frac{6}{x}}(x-2) + xc_2(x^2+12x-36)}{x^5}$$

Mathematica DSolve solution

Solving time : 0.361 (sec)
Leaf size : 55

```
DSolve[{x^2*D[y[x],{x,2}]-(6-7*x)*D[y[x],x]+8*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to \frac{e^{5-\frac{6}{x}}(x-2)\left(c_2 \int_1^x \frac{e^{\frac{6}{K[1]}-3}K[1]^3}{(K[1]-2)^2}dK[1] + c_1\right)}{x^5}$$

2.1.74 Problem 76

Internal problem ID [9246]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 76
Date solved : `Monday, January 27, 2025 at 06:00:29 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(2x^2+x+1\right)y''+(1+7x)\,y'+2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.984 (sec)

Writing the ode as

$$\left(2x^2+x+1\right)y''+(1+7x)\,y'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2+x+1 \\ B &= 1+7x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{5x^2-2x+5}{4\left(2x^2+x+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 5x^2-2x+5 \\ t &= 4\left(2x^2+x+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{5x^2-2x+5}{4\left(2x^2+x+1\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.132: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 + x + 1)^2$. There is a pole at $x = -\frac{1}{4} + \frac{i\sqrt{7}}{4}$ of order 2. There is a pole at $x = -\frac{1}{4} - \frac{i\sqrt{7}}{4}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{-\frac{29}{224} + \frac{9i\sqrt{7}}{224}}{\left(x + \frac{1}{4} - \frac{i\sqrt{7}}{4}\right)^2} + \frac{-\frac{29}{224} - \frac{9i\sqrt{7}}{224}}{\left(x + \frac{1}{4} + \frac{i\sqrt{7}}{4}\right)^2} - \frac{8i\sqrt{7}}{49\left(x + \frac{1}{4} - \frac{i\sqrt{7}}{4}\right)} + \frac{8i\sqrt{7}}{49\left(x + \frac{1}{4} + \frac{i\sqrt{7}}{4}\right)}$$

For the pole at $x = -\frac{1}{4} + \frac{i\sqrt{7}}{4}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{4}-\frac{i\sqrt{7}}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{29}{224} + \frac{9i\sqrt{7}}{224}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} + \frac{3\sqrt{42 + 14i\sqrt{7}}}{56} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} - \frac{3\sqrt{42 + 14i\sqrt{7}}}{56} \end{aligned}$$

For the pole at $x=-\frac{1}{4}-\frac{i\sqrt{7}}{4}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{4}+\frac{i\sqrt{7}}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{29}{224}-\frac{9i\sqrt{7}}{224}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2}+\frac{3\sqrt{42-14i\sqrt{7}}}{56}\\
\alpha_c^- &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}-\frac{3\sqrt{42-14i\sqrt{7}}}{56}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty=0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r=\frac{s}{t}=\frac{5x^2-2x+5}{4\left(2x^2+x+1\right)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=\frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{4}\\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{5x^2-2x+5}{4\left(2x^2+x+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{4}+\frac{i\sqrt{7}}{4}$	2	0	$\frac{1}{2}+\frac{3\sqrt{42+14i\sqrt{7}}}{56}$	$\frac{1}{2}-\frac{3\sqrt{42+14i\sqrt{7}}}{56}$
$-\frac{1}{4}-\frac{i\sqrt{7}}{4}$	2	0	$\frac{1}{2}+\frac{3\sqrt{42-14i\sqrt{7}}}{56}$	$\frac{1}{2}-\frac{3\sqrt{42-14i\sqrt{7}}}{56}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=\frac{5}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+-\left(\alpha_{c_1}^-+\alpha_{c_2}^-\right)\\
&= \frac{5}{4}-\left(\frac{1}{4}\right)\\
&= 1
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{\frac{1}{2}-\frac{3\sqrt{42+14i\sqrt{7}}}{56}}{x+\frac{1}{4}-\frac{i\sqrt{7}}{4}} + \frac{\frac{1}{2}-\frac{3\sqrt{42-14i\sqrt{7}}}{56}}{x+\frac{1}{4}+\frac{i\sqrt{7}}{4}} + (0) \\
&= \frac{\frac{1}{2}-\frac{3\sqrt{42+14i\sqrt{7}}}{56}}{x+\frac{1}{4}-\frac{i\sqrt{7}}{4}} + \frac{\frac{1}{2}-\frac{3\sqrt{42-14i\sqrt{7}}}{56}}{x+\frac{1}{4}+\frac{i\sqrt{7}}{4}} \\
&= \frac{x+1}{4x^2+2x+2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{1}{2}-\frac{3\sqrt{42+14i\sqrt{7}}}{56}}{x+\frac{1}{4}-\frac{i\sqrt{7}}{4}} + \frac{\frac{1}{2}-\frac{3\sqrt{42-14i\sqrt{7}}}{56}}{x+\frac{1}{4}+\frac{i\sqrt{7}}{4}}\right)(1) + \left(\left(-\frac{\frac{1}{2}-\frac{3\sqrt{42+14i\sqrt{7}}}{56}}{\left(x+\frac{1}{4}-\frac{i\sqrt{7}}{4}\right)^2} - \frac{\frac{1}{2}-\frac{3\sqrt{42-14i\sqrt{7}}}{56}}{\left(x+\frac{1}{4}+\frac{i\sqrt{7}}{4}\right)^2}\right) + \left(\frac{\frac{1}{2}-3\sqrt{}}{x+\frac{1}{4}}\right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (x+1)\, e^{\int \left(\frac{\frac{1}{2}-\frac{3\sqrt{42+14i\sqrt{7}}}{56}}{x+\frac{1}{4}-\frac{i\sqrt{7}}{4}} + \frac{\frac{1}{2}-\frac{3\sqrt{42-14i\sqrt{7}}}{56}}{x+\frac{1}{4}+\frac{i\sqrt{7}}{4}}\right) dx} \\
&= (x+1)\, e^{\frac{\ln\left(16x^2+8x+8\right)}{8} + \frac{3\sqrt{7}\, \arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{28}} \\
&= (x+1)\, 2^{3/8} \left(2x^2+x+1\right)^{1/8} e^{\frac{3\sqrt{7}\, \arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{28}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1+7x}{2x^2+x+1}\, dx} \\
&= z_1 e^{-\frac{7\ln\left(2x^2+x+1\right)}{8} + \frac{3\sqrt{7}\, \arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{28}} \\
&= z_1 \left(\frac{e^{\frac{3\sqrt{7}\, \arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{28}}}{\left(2x^2+x+1\right)^{7/8}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{\frac{3\sqrt{7}\, \arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{14}} (x+1)\, 2^{3/8}}{(2x^2+x+1)^{3/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1+7x}{2x^2+x+1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{7\ln(2x^2+x+1)}{4}+\frac{3\sqrt{7}\, \arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{14}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{7\ln(2x^2+x+1)}{4}+\frac{3\sqrt{7}\, \arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{14}} (2x^2+x+1)^{3/2}\, \mathrm{e}^{-\frac{3\sqrt{7}\, \arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{7}}\, 2^{1/4}}{2(x+1)^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{\frac{3\sqrt{7}\, \arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{14}} (x+1)\, 2^{3/8}}{(2x^2+x+1)^{3/4}}\right) \\
&+ c_2 \left(\frac{\mathrm{e}^{\frac{3\sqrt{7}\, \arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{14}} (x+1)\, 2^{3/8}}{(2x^2+x+1)^{3/4}} \left(\int \frac{\mathrm{e}^{-\frac{7\ln(2x^2+x+1)}{4}+\frac{3\sqrt{7}\, \arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{14}} (2x^2+x+1)^{3/2}\, \mathrm{e}^{-\frac{3\sqrt{7}\, \arctan\left(\frac{(4x+1)}{7}\right.}{7}}}{2(x+1)^2}\right.\right.
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(2x^2+x+1)\left(\frac{d^2}{dx^2}y(x)\right) + (1+7x)\left(\frac{d}{dx}y(x)\right) + 2y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{2y(x)}{2x^2+x+1} - \frac{(1+7x)\left(\frac{d}{dx}y(x)\right)}{2x^2+x+1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(1+7x)\left(\frac{d}{dx}y(x)\right)}{2x^2+x+1} + \frac{2y(x)}{2x^2+x+1} = 0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1+7x}{2x^2+x+1}, P_3(x) = \frac{2}{2x^2+x+1}\right]$

  - $\left(\frac{\mathrm{I}\sqrt{7}}{4} + x + \frac{1}{4}\right) \cdot P_2(x)$ is analytic at $x = -\frac{1}{4} - \frac{\mathrm{I}\sqrt{7}}{4}$

$$\left(\left(\frac{I\sqrt{7}}{4}+x+\frac{1}{4}\right)\cdot P_2(x)\right)\Big|_{x=-\frac{1}{4}-\frac{I\sqrt{7}}{4}}=0$$

- $\left(\frac{I\sqrt{7}}{4}+x+\frac{1}{4}\right)^2\cdot P_3(x)$ is analytic at $x=-\frac{1}{4}-\frac{I\sqrt{7}}{4}$

$$\left(\left(\frac{I\sqrt{7}}{4}+x+\frac{1}{4}\right)^2\cdot P_3(x)\right)\Big|_{x=-\frac{1}{4}-\frac{I\sqrt{7}}{4}}=0$$

- $x=-\frac{1}{4}-\frac{I\sqrt{7}}{4}$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-\frac{1}{4}-\frac{I\sqrt{7}}{4}$

- Multiply by denominators

$$(2x^2+x+1)\left(\frac{d^2}{dx^2}y(x)\right)+(1+7x)\left(\frac{d}{dx}y(x)\right)+2y(x)=0$$

- Change variables using $x=u-\frac{1}{4}-\frac{I\sqrt{7}}{4}$ so that the regular singular point is at $u=0$

$$(2u^2-Iu\sqrt{7})\left(\frac{d^2}{du^2}y(u)\right)+\left(-\frac{3}{4}+7u-\frac{7I\sqrt{7}}{4}\right)\left(\frac{d}{du}y(u)\right)+2y(u)=0$$

- Assume series solution for $y(u)$

$$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

$$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$\frac{I\sqrt{7}\left(3I\sqrt{7}-21-28r\right)ra_0u^{-1+r}}{28}+\left(\sum_{k=0}^{\infty}\left(\frac{I\sqrt{7}\left(3I\sqrt{7}-28k-49-28r\right)(k+1+r)a_{k+1}}{28}+a_k(k+r+2)(2k+2r+1)\right)\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$\frac{I}{28}\sqrt{7}\left(3I\sqrt{7}-21-28r\right)r=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,\frac{3I\sqrt{7}}{28}-\frac{3}{4}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-Ia_{k+1}(k+1+r)\left(k+r+\frac{7}{4}\right)\sqrt{7}+\frac{(-3k-3r-3)a_{k+1}}{4}+2(k+r+2)\,a_k\left(k+r+\frac{1}{2}\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{4a_k\left(2k^2+4kr+2r^2+5k+5r+2\right)}{3+4I\sqrt{7}\,k^2+8I\sqrt{7}\,kr+4I\sqrt{7}\,r^2+11I\sqrt{7}\,k+11I\sqrt{7}\,r+7I\sqrt{7}+3k+3r}$$

- Recursion relation for $r=0$

$$a_{k+1}=\frac{4a_k\left(2k^2+5k+2\right)}{3+4I\sqrt{7}\,k^2+11I\sqrt{7}\,k+7I\sqrt{7}+3k}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty}a_k u^k, a_{k+1}=\frac{4a_k\left(2k^2+5k+2\right)}{3+4I\sqrt{7}\,k^2+11I\sqrt{7}\,k+7I\sqrt{7}+3k}\right]$$

- Revert the change of variables $u = \frac{I\sqrt{7}}{4} + x + \frac{1}{4}$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k \left(\frac{I\sqrt{7}}{4} + x + \frac{1}{4}\right)^k, a_{k+1} = \frac{4a_k\,(2k^2+5k+2)}{3+4\,I\sqrt{7}\,k^2+11\,I\sqrt{7}\,k+7\,I\sqrt{7}+3k}\right]$$

- Recursion relation for $r = \frac{3\,I\sqrt{7}}{28} - \frac{3}{4}$

$$a_{k+1} = \frac{4a_k\left(2k^2+4k\left(\frac{3\,I\sqrt{7}}{28}-\frac{3}{4}\right)+2\left(\frac{3\,I\sqrt{7}}{28}-\frac{3}{4}\right)^2+5k+\frac{15\,I\sqrt{7}}{28}-\frac{7}{4}\right)}{\frac{3}{4}+4\,I\sqrt{7}\,k^2+8\,I\sqrt{7}\,k\left(\frac{3\,I\sqrt{7}}{28}-\frac{3}{4}\right)+4\,I\sqrt{7}\left(\frac{3\,I\sqrt{7}}{28}-\frac{3}{4}\right)^2+11\,I\sqrt{7}\,k+11\,I\sqrt{7}\left(\frac{3\,I\sqrt{7}}{28}-\frac{3}{4}\right)+\frac{205\,I\sqrt{7}}{28}+3k}$$

- Solution for $r = \frac{3\,I\sqrt{7}}{28} - \frac{3}{4}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{3\,I\sqrt{7}}{28}-\frac{3}{4}}, a_{k+1} = \frac{4a_k\left(2k^2+4k\left(\frac{3\,I\sqrt{7}}{28}-\frac{3}{4}\right)+2\left(\frac{3\,I\sqrt{7}}{28}-\frac{3}{4}\right)^2+5k+\frac{15\,I\sqrt{7}}{28}-\frac{7}{4}\right)}{\frac{3}{4}+4\,I\sqrt{7}\,k^2+8\,I\sqrt{7}\,k\left(\frac{3\,I\sqrt{7}}{28}-\frac{3}{4}\right)+4\,I\sqrt{7}\left(\frac{3\,I\sqrt{7}}{28}-\frac{3}{4}\right)^2+11\,I\sqrt{7}\,k+11\,I\sqrt{7}\left(\frac{3\,I\sqrt{7}}{28}-\frac{3}{4}\right)+\frac{205\,I}{28}}$$

- Revert the change of variables $u = \frac{I\sqrt{7}}{4} + x + \frac{1}{4}$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k \left(\frac{I\sqrt{7}}{4} + x + \frac{1}{4}\right)^{k+\frac{3\,I\sqrt{7}}{28}-\frac{3}{4}}, a_{k+1} = \frac{4a_k\left(2k^2+4k\left(\frac{3\,I\sqrt{7}}{28}-\frac{3}{4}\right)+2\left(\frac{3\,I\sqrt{7}}{28}-\frac{3}{4}\right)^2+5k+\frac{15\,I\sqrt{}}{28}\right.}{\frac{3}{4}+4\,I\sqrt{7}\,k^2+8\,I\sqrt{7}\,k\left(\frac{3\,I\sqrt{7}}{28}-\frac{3}{4}\right)+4\,I\sqrt{7}\left(\frac{3\,I\sqrt{7}}{28}-\frac{3}{4}\right)^2+11\,I\sqrt{7}\,k+11\,I\sqrt{7}}$$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k \left(\frac{I\sqrt{7}}{4} + x + \frac{1}{4}\right)^k\right) + \left(\sum_{k=0}^{\infty} b_k \left(\frac{I\sqrt{7}}{4} + x + \frac{1}{4}\right)^{k+\frac{3\,I\sqrt{7}}{28}-\frac{3}{4}}\right), a_{k+1} = \frac{4a_k\,(2k^2+5k+2)}{3+4\,I\sqrt{7}\,k^2+11\,I\sqrt{7}\,k+7\,I\sqrt{}}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returnin
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.064 (sec)
Leaf size : 77

```
dsolve((2*x^2+x+1)*diff(diff(y(x),x),x)+(1+7*x)*diff(y(x),x)+2*y(x) = 0,y(x),singsol=a
```

$$y = c_1 \,\mathrm{hypergeom}\left(\left[\frac{1}{2}, 2\right], \left[-\frac{\left(-7\sqrt{7}+3i\right)\sqrt{7}}{28}\right], \frac{1}{2}+\frac{i(-4x-1)\sqrt{7}}{14}\right) + c_2\left(i\sqrt{7}+4x+1\right)^{-\frac{3}{4}+\frac{3i\sqrt{7}}{28}}\left(i\sqrt{7}-4x-1\right)^{-\frac{3}{4}-\frac{3i\sqrt{7}}{28}}(x+1)$$

Mathematica DSolve solution

Solving time : 1.008 (sec)
Leaf size : 119

```
DSolve[{(1+x+2*x^2)*D[y[x],{x,2}]+(1+7*x)*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolu
```

$$y(x) \to (x+1)\exp\left(\int_1^x \frac{K[1]+1}{4K[1]^2+2K[1]+2}dK[1] - \frac{1}{2}\int_1^x \frac{7K[2]+1}{2K[2]^2+K[2]+1}dK[2]\right)\left(c_2\int_1^x \frac{\exp\left(-2\int_1^{K[3]}\frac{K[1]+1}{4K[1]^2+2K[1]+2}dK[1]\right)}{(K[3]+1)^2}dK[3] + c_1\right)$$

2.1.75 Problem 77

Internal problem ID [9247]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 77
Date solved : `Monday, January 27, 2025 at 06:00:30 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(3+x)\,y''+(1+2x)\,y'-(2-x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.161 (sec)

Writing the ode as

$$(3+x)\,y''+(1+2x)\,y'+(x-2)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=3+x\\B&=1+2x\\C&=x-2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{35}{4\,(3+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=35\\t&=4(3+x)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{35}{4\,(3+x)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.134: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(3+x)^2$. There is a pole at $x = -3$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{35}{4\,(3+x)^2}$$

For the pole at $x = -3$ let b be the coefficient of $\frac{1}{(3+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{35}{4\,(3+x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{35}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{7}{2} \\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=-\frac{5}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{35}{4\,(3+x)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-3	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=-\frac{5}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{5}{2}-\left(-\frac{5}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^-}{x-c_1}\right)+(-)[\sqrt{r}]_\infty \\
&= -\frac{5}{2\,(3+x)}+(-)\,(0) \\
&= -\frac{5}{2\,(3+x)} \\
&= -\frac{5}{2\,(3+x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}
$$

Let

$$
p(x)=1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{5}{2\left(3+x\right)}\right)(0)+\left(\left(\frac{5}{2\left(3+x\right)^{2}}\right)+\left(-\frac{5}{2\left(3+x\right)}\right)^{2}-\left(\frac{35}{4\left(3+x\right)^{2}}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{5}{2(3+x)}dx} \\ &= \frac{1}{\left(3+x\right)^{5/2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1+2x}{3+x}\,dx} \\ &= z_1 e^{-x+\frac{5\ln(3+x)}{2}} \\ &= z_1\left(\left(3+x\right)^{5/2}\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$y_1=\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1+2x}{3+x}\,dx}}{\left(y_1\right)^2}\,dx \\ &= y_1 \int \frac{e^{-2x+5\ln(3+x)}}{\left(y_1\right)^2}\,dx \\ &= y_1\left(\frac{x\left(x^5+18x^4+135x^3+540x^2+1215x+1458\right)\mathrm{e}^{-2x+5\ln(3+x)}\mathrm{e}^{2x}}{6\left(3+x\right)^5}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-x}\right)+c_2\left(\mathrm{e}^{-x}\left(\frac{x\left(x^5+18x^4+135x^3+540x^2+1215x+1458\right)\mathrm{e}^{-2x+5\ln(3+x)}\mathrm{e}^{2x}}{6\left(3+x\right)^5}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x+3)\left(\frac{d^2}{dx^2}y(x)\right)+(2x+1)\left(\frac{d}{dx}y(x)\right)-(-x+2)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{(x-2)y(x)}{x+3}-\frac{(2x+1)\left(\frac{d}{dx}y(x)\right)}{x+3}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x)+\frac{(2x+1)\left(\frac{d}{dx}y(x)\right)}{x+3}+\frac{(x-2)y(x)}{x+3}=0$

□ Check to see if $x_0=-3$ is a regular singular point

  - Define functions

  $\left[P_2(x)=\frac{2x+1}{x+3}, P_3(x)=\frac{x-2}{x+3}\right]$

  - $(x+3)\cdot P_2(x)$ is analytic at $x=-3$

  $\left((x+3)\cdot P_2(x)\right)\Big|_{x=-3}=-5$

  - $(x+3)^2\cdot P_3(x)$ is analytic at $x=-3$

  $\left((x+3)^2\cdot P_3(x)\right)\Big|_{x=-3}=0$

  - $x=-3$is a regular singular point

Check to see if $x_0=-3$ is a regular singular point

$x_0=-3$

- Multiply by denominators

$(x+3)\left(\frac{d^2}{dx^2}y(x)\right)+(2x+1)\left(\frac{d}{dx}y(x)\right)+(x-2)\,y(x)=0$

- Change variables using $x=u-3$ so that the regular singular point is at $u=0$

$u\left(\frac{d^2}{du^2}y(u)\right)+(2u-5)\left(\frac{d}{du}y(u)\right)+(u-5)\,y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

  - Shift index using $k->k-m$

  $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-1}$

  - Shift index using $k->k+1$

$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)u^{k+r}$

Rewrite ODE with series expansions

$a_0r(-6+r)u^{-1+r} + (a_1(1+r)(-5+r) + a_0(-5+2r))u^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)(k-5+r)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-6+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 6\}$
- Each term must be 0
  $a_1(1+r)(-5+r) + a_0(-5+2r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)(k-5+r) + 2a_kk + 2a_kr - 5a_k + a_{k-1} = 0$
- Shift index using $k- >k+1$
  $a_{k+2}(k+2+r)(k-4+r) + 2a_{k+1}(k+1) + 2ra_{k+1} - 5a_{k+1} + a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{2ka_{k+1}+2ra_{k+1}+a_k-3a_{k+1}}{(k+2+r)(k-4+r)}$
- Recursion relation for $r = 0$
  $a_{k+2} = -\frac{2ka_{k+1}+a_k-3a_{k+1}}{(k+2)(k-4)}$
- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 4$
  $a_{k+2} = -\frac{2ka_{k+1}+a_k-3a_{k+1}}{(k+2)(k-4)}$
- Recursion relation for $r = 6$
  $a_{k+2} = -\frac{2ka_{k+1}+a_k+9a_{k+1}}{(k+8)(k+2)}$
- Solution for $r = 6$
  $\left[y(u) = \sum_{k=0}^{\infty} a_ku^{k+6}, a_{k+2} = -\frac{2ka_{k+1}+a_k+9a_{k+1}}{(k+8)(k+2)}, 7a_1 + 7a_0 = 0\right]$
- Revert the change of variables $u = x+3$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+3)^{k+6}, a_{k+2} = -\frac{2ka_{k+1}+a_k+9a_{k+1}}{(k+8)(k+2)}, 7a_1 + 7a_0 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 33

```
dsolve((x+3)*diff(diff(y(x),x),x)+(2*x+1)*diff(y(x),x)-(-x+2)*y(x) = 0,y(x),singsol=all)
```

$$y = \left(\left(x^2+3x+9\right)\left(x^2+9x+27\right)(6+x)\, c_2 x + c_1\right) \mathrm{e}^{-x}$$

Mathematica DSolve solution

Solving time : 0.329 (sec)
Leaf size : 52

```
DSolve[{(3+x)*D[y[x],{x,2}]+(1+2*x)*D[y[x],x]-(2-x)*y[x]==0,{}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{(c_2(x+3)^6 + 6c_1)\exp\left(-\frac{1}{2}\int_1^x \left(2 - \frac{5}{K[1]+3}\right) dK[1]\right)}{6(x+3)^{5/2}}$$

2.1.76 Problem 78

Internal problem ID [9248]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 78
Date solved : `Monday, January 27, 2025 at 06:00:31 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + 3xy' + \left(2x^2 + 4\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.242 (sec)

Writing the ode as

$$y'' + 3xy' + \left(2x^2 + 4\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3x \\ C &= 2x^2 + 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 10}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 10 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{5}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.136: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{5}{2x} - \frac{25}{4x^3} - \frac{125}{4x^5} - \frac{3125}{16x^7} - \frac{21875}{16x^9} - \frac{328125}{32x^{11}} - \frac{2578125}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 10}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} - \frac{5}{2}\right) + (0) \\ &= \frac{x^2}{4} - \frac{5}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{5}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{5}{2}\right) - (0) \\ &= -\frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= -3 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= 2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{5}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either + or − and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (2) + 2\left(-\frac{x}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{5}{2}\right)\right) &= 0 \\ a_1 x + 2a_0 + 2 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \left(x^2 - 1\right) e^{\int -\frac{x}{2} dx} \\ &= \left(x^2 - 1\right) e^{-\frac{x^2}{4}} \\ &= \left(x^2 - 1\right) e^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3x}{1}\, dx} \\ &= z_1 e^{-\frac{3x^2}{4}} \\ &= z_1\left(e^{-\frac{3x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (x^2 - 1)\,\mathrm{e}^{-x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{3x^2}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{3x^2}{2}}\,\mathrm{e}^{2x^2}}{(x^2-1)^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((x^2-1)\,\mathrm{e}^{-x^2}\right) + c_2\left((x^2-1)\,\mathrm{e}^{-x^2}\left(\int \frac{\mathrm{e}^{-\frac{3x^2}{2}}\,\mathrm{e}^{2x^2}}{(x^2-1)^2} dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + 3x\left(\frac{d}{dx}y(x)\right) + (2x^2+4)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- > k - m$

    $x^m \cdot y(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- > k + 2$

$\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

Rewrite ODE with series expansions

$2a_2 + 4a_0 + (6a_3 + 7a_1)\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)(k+1) + a_k(3k+4) + 2a_{k-2})\, x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 + 4a_0 = 0, 6a_3 + 7a_1 = 0]$

- Solve for the dependent coefficient(s)

  $\left\{a_2 = -2a_0, a_3 = -\frac{7a_1}{6}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} + 3a_k k + 4a_k + 2a_{k-2} = 0$

- Shift index using $k- > k+2$

  $\left((k+2)^2 + 3k + 8\right) a_{k+4} + 3a_{k+2}(k+2) + 4a_{k+2} + 2a_k = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{3ka_{k+2}+2a_k+10a_{k+2}}{k^2+7k+12}, a_2 = -2a_0, a_3 = -\frac{7a_1}{6}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special fu
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.026 (sec)
Leaf size : 48

```
dsolve(diff(diff(y(x),x),x)+3*diff(y(x),x)*x+(2*x^2+4)*y(x) = 0,y(x),singsol=all)
```

$$y = 2\,\mathrm{e}^{-\frac{x^2}{2}} c_1 x - \mathrm{e}^{-x^2}(x-1)(x+1)\left(c_1\sqrt{\pi}\,\mathrm{erfi}\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2} - c_2\right)$$

Mathematica DSolve solution

Solving time : 0.391 (sec)
Leaf size : 50

```
DSolve[{D[y[x],{x,2}]+3*x*D[y[x],x]+(4+2*x^2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to e^{-x^2}\left(x^2-1\right)\left(c_2\int_1^x \frac{e^{\frac{K[1]^2}{2}}}{\left(K[1]^2-1\right)^2}dK[1]+c_1\right)$$

2.1.77 Problem 79

Internal problem ID [9249]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 79
Date solved : `Monday, January 27, 2025 at 06:00:31 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(2+4x)\, y''-4y'-(6+4x)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.223 (sec)

Writing the ode as

$$(2+4x)\, y''-4y'+(-4x-6)\, y = 0 \tag{1}$$
$$Ay''+By'+Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2+4x \\ B &= -4 \\ C &= -4x-6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2+8x+6}{(1+2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2+8x+6 \\ t &= (1+2x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2+8x+6}{(1+2x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.138: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (1 + 2x)^2$. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{3}{4\left(x + \frac{1}{2}\right)^2} + \frac{1}{x + \frac{1}{2}}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{1}{2x} - \frac{1}{4x^3} + \frac{11}{32x^4} - \frac{21}{64x^5} + \frac{15}{64x^6} - \frac{3}{32x^7} - \frac{117}{2048x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 + 8x + 6}{4x^2 + 4x + 1} \\ &= Q + \frac{R}{4x^2 + 4x + 1} \\ &= (1) + \left(\frac{4x + 5}{4x^2 + 4x + 1}\right) \\ &= 1 + \frac{4x + 5}{4x^2 + 4x + 1} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$\begin{aligned} b &= (1) - (0) \\ &= 1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{1}{1} - 0\right) &&= \frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{1}{1} - 0\right) &&= -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^2+8x+6}{(1+2x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2\left(x+\frac{1}{2}\right)} + (-)(1) \\ &= -\frac{1}{2\left(x+\frac{1}{2}\right)} - 1 \\ &= -\frac{2(x+1)}{1+2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right)(0) + \left(\left(\frac{1}{2\left(x+\frac{1}{2}\right)^2}\right) + \left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right)^2 - \left(\frac{4x^2+8x+6}{(1+2x)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2\left(x+\frac{1}{2}\right)}-1\right) dx} \\ &= \frac{\mathrm{e}^{-x}}{\sqrt{1+2x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4}{2+4x} \, dx} \\ &= z_1 e^{\frac{\ln(1+2x)}{2}} \\ &= z_1 \left(\sqrt{1+2x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{2+4x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\ln(1+2x)}}{(y_1)^2} \, dx \\ &= y_1 \left(x \, \mathrm{e}^{2x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(x \, \mathrm{e}^{2x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(2+4x)\left(\frac{d^2}{dx^2}y(x)\right) - 4\frac{d}{dx}y(x) - (6+4x)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{(2x+3)y(x)}{2x+1} + \frac{2\left(\frac{d}{dx}y(x)\right)}{2x+1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x) - \frac{2\left(\frac{d}{dx}y(x)\right)}{2x+1} - \frac{(2x+3)y(x)}{2x+1} = 0$

- ☐ Check to see if $x_0 = -\frac{1}{2}$ is a regular singular point
  - ◦ Define functions
    $\left[P_2(x) = -\frac{2}{2x+1}, P_3(x) = -\frac{2x+3}{2x+1}\right]$
  - ◦ $\left(x+\frac{1}{2}\right)\cdot P_2(x)$ is analytic at $x = -\frac{1}{2}$
    $\left(\left(x+\frac{1}{2}\right)\cdot P_2(x)\right)\Big|_{x=-\frac{1}{2}} = -1$
  - ◦ $\left(x+\frac{1}{2}\right)^2\cdot P_3(x)$ is analytic at $x = -\frac{1}{2}$
    $\left(\left(x+\frac{1}{2}\right)^2\cdot P_3(x)\right)\Big|_{x=-\frac{1}{2}} = 0$
  - ◦ $x = -\frac{1}{2}$is a regular singular point
    Check to see if $x_0 = -\frac{1}{2}$ is a regular singular point
    $x_0 = -\frac{1}{2}$
- • Multiply by denominators
  $(2x+1)\left(\frac{d^2}{dx^2}y(x)\right) - 2\frac{d}{dx}y(x) + (-2x-3)\,y(x) = 0$
- • Change variables using $x = u - \frac{1}{2}$ so that the regular singular point is at $u = 0$
  $2u\left(\frac{d^2}{du^2}y(u)\right) - 2\frac{d}{du}y(u) + (-2u-2)\,y(u) = 0$
- • Assume series solution for $y(u)$
  $y(u) = \sum\limits_{k=0}^{\infty} a_k u^{k+r}$
- ☐ Rewrite ODE with series expansions
  - ◦ Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$
    $u^m \cdot y(u) = \sum\limits_{k=0}^{\infty} a_k u^{k+r+m}$
  - ◦ Shift index using $k- >k-m$
    $u^m \cdot y(u) = \sum\limits_{k=m}^{\infty} a_{k-m} u^{k+r}$
  - ◦ Convert $\frac{d}{du}y(u)$ to series expansion
    $\frac{d}{du}y(u) = \sum\limits_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1}$
  - ◦ Shift index using $k- >k+1$
    $\frac{d}{du}y(u) = \sum\limits_{k=-1}^{\infty} a_{k+1}(k+1+r)\,u^{k+r}$
  - ◦ Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion
    $u\cdot\left(\frac{d^2}{du^2}y(u)\right) = \sum\limits_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-1}$
  - ◦ Shift index using $k- >k+1$
    $u\cdot\left(\frac{d^2}{du^2}y(u)\right) = \sum\limits_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$
    Rewrite ODE with series expansions
    $2a_0r(-2+r)\,u^{-1+r} + (2a_1(1+r)\,(-1+r) - 2a_0)\,u^r + \left(\sum\limits_{k=1}^{\infty} (2a_{k+1}(k+1+r)\,(k+r-1) - 2a_k\right.$
- • $a_0$cannot be 0 by assumption, giving the indicial equation
  $2r(-2+r) = 0$
- • Values of r that satisfy the indicial equation
  $r \in \{0, 2\}$

- Each term must be 0
  $2a_1(1+r)(-1+r) - 2a_0 = 0$
- Each term in the series must be 0, giving the recursion relation
  $2a_{k+1}(k+1+r)(k+r-1) - 2a_k - 2a_{k-1} = 0$
- Shift index using $k- > k+1$
  $2a_{k+2}(k+2+r)(k+r) - 2a_{k+1} - 2a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{a_{k+1}+a_k}{(k+2+r)(k+r)}$
- Recursion relation for $r = 0$
  $a_{k+2} = \frac{a_{k+1}+a_k}{(k+2)k}$
- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$
  $a_{k+2} = \frac{a_{k+1}+a_k}{(k+2)k}$
- Recursion relation for $r = 2$
  $a_{k+2} = \frac{a_{k+1}+a_k}{(k+4)(k+2)}$
- Solution for $r = 2$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2} = \frac{a_{k+1}+a_k}{(k+4)(k+2)}, 6a_1 - 2a_0 = 0\right]$
- Revert the change of variables $u = x + \frac{1}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k \left(x+\frac{1}{2}\right)^{k+2}, a_{k+2} = \frac{a_{k+1}+a_k}{(k+4)(k+2)}, 6a_1 - 2a_0 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 16

```
dsolve((4*x+2)*diff(diff(y(x),x),x)-4*diff(y(x),x)-(6+4*x)*y(x) = 0,y(x),singsol=all)
```

$$y = c_1 e^{-x} + c_2 e^{x} x$$

Mathematica DSolve solution

Solving time : 0.345 (sec)
Leaf size : 69

```
DSolve[{(2+4*x)*D[y[x],{x,2}]-4*D[y[x],x]-(6+4*x)*y[x]==0,{}},y[x],x,IncludeSingularSolution
```

$$y(x) \to \sqrt{2x+1}\exp\left(\int_1^x \left(\frac{1}{-2K[1]-1} - 1\right) dK[1]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[2]}\left(\frac{1}{-2K[1]-1} - 1\right) dK[1]\right) dK[2] + c_1\right)$$

2.1.78 Problem 80

Internal problem ID [9250]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 80
Date solved : `Monday, January 27, 2025 at 06:00:32 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - 3xy' + (2x^2 + 5)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.231 (sec)

Writing the ode as

$$y'' - 3xy' + (2x^2 + 5)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3x \\ C &= 2x^2 + 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 26}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 26 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{13}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.140: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{13}{2x} - \frac{169}{4x^3} - \frac{2197}{4x^5} - \frac{142805}{16x^7} - \frac{2599051}{16x^9} - \frac{101362989}{32x^{11}} - \frac{2070701061}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 26}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} - \frac{13}{2}\right) + (0) \\ &= \frac{x^2}{4} - \frac{13}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{13}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{13}{2}\right) - (0) \\ &= -\frac{13}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{13}{2}}{\frac{1}{2}} - 1\right) &&= -7 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{13}{2}}{\frac{1}{2}} - 1\right) &&= 6 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{13}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-7	6

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 6$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 6 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 6$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(30x^4 + 20x^3a_5 + 12x^2a_4 + 6xa_3 + 2a_2\right) + 2\left(-\frac{x}{2}\right)\left(6x^5 + 5x^4a_5 + 4x^3a_4 + 3x^2a_3 + 2xa_2 + a_1\right) + \left(\left(-\frac{1}{2}\right.\right.$$

$$a_5x^5 + 2(15 + a_4)x^4 + (3a_3 + 20a_5)x^3 + 4(a_2 + 3a_4)x^2 + (5$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -15, a_1 = 0, a_2 = 45, a_3 = 0, a_4 = -15, a_5 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^6 - 15x^4 + 45x^2 - 15$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= \left(x^6 - 15x^4 + 45x^2 - 15\right)e^{\int -\frac{x}{2} dx} \\ &= \left(x^6 - 15x^4 + 45x^2 - 15\right)e^{-\frac{x^2}{4}} \\ &= \left(x^6 - 15x^4 + 45x^2 - 15\right)e^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-3x}{1}\, dx} \\ &= z_1 e^{\frac{3x^2}{4}} \\ &= z_1\left(e^{\frac{3x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{\frac{x^2}{2}}\left(x^6 - 15x^4 + 45x^2 - 15\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3x}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\frac{3x^2}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{e^{\frac{3x^2}{2}} e^{-x^2}}{\left(x^6 - 15x^4 + 45x^2 - 15\right)^2} dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(e^{\frac{x^2}{2}}\left(x^6 - 15x^4 + 45x^2 - 15\right)\right) \\ &\quad + c_2\left(e^{\frac{x^2}{2}}\left(x^6 - 15x^4 + 45x^2 - 15\right)\left(\int \frac{e^{\frac{3x^2}{2}} e^{-x^2}}{\left(x^6 - 15x^4 + 45x^2 - 15\right)^2} dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) - 3x\left(\frac{d}{dx}y(x)\right) + \left(2x^2 + 5\right) y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- > k - m$

  $x^m \cdot y(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

  $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- Shift index using $k->k+2$

  $\frac{d^2}{dx^2}y(x)=\sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$

  Rewrite ODE with series expansions

  $2a_2+5a_0+(6a_3+2a_1)x+\left(\sum_{k=2}^{\infty}(a_{k+2}(k+2)(k+1)-a_k(3k-5)+2a_{k-2})x^k\right)=0$

- The coefficients of each power of x must be 0

  $[2a_2+5a_0=0, 6a_3+2a_1=0]$

- Solve for the dependent coefficient(s)

  $\left\{a_2=-\frac{5a_0}{2}, a_3=-\frac{a_1}{3}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)a_{k+2}-3a_kk+5a_k+2a_{k-2}=0$

- Shift index using $k->k+2$

  $\left((k+2)^2+3k+8\right)a_{k+4}-3a_{k+2}(k+2)+5a_{k+2}+2a_k=0$

- Recursion relation that defines the series solution to the ODE

  $\left[y(x)=\sum_{k=0}^{\infty}a_kx^k, a_{k+4}=\frac{3ka_{k+2}-2a_k+a_{k+2}}{k^2+7k+12}, a_2=-\frac{5a_0}{2}, a_3=-\frac{a_1}{3}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.029 (sec)
Leaf size : 62

```
dsolve(diff(diff(y(x),x),x)-3*diff(y(x),x)*x+(2*x^2+5)*y(x) = 0,y(x),singsol=all)
```

$$y=(x^6-15x^4+45x^2-15)\left(c_1\sqrt{\pi}\,\text{erfi}\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}+c_2\right)e^{\frac{x^2}{2}}-2\,e^{x^2}c_1x(x^2-11)(x^2-3)$$

Mathematica DSolve solution

Solving time : 0.871 (sec)
Leaf size : 74

```
DSolve[{D[y[x],{x,2}]-3*x*D[y[x],x]+(5+2*x^2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tru
```

$$y(x) \to e^{\frac{x^2}{2}}\left(x^6-15x^4+45x^2-15\right)\left(c_2\int_1^x \frac{e^{\frac{K[1]^2}{2}}}{\left(K[1]^6-15K[1]^4+45K[1]^2-15\right)^2}dK[1]+c_1\right)$$

2.1.79 Problem 81

Internal problem ID [9251]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 81
Date solved : `Monday, January 27, 2025 at 06:00:33 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2y'' + 5xy' + \left(2x^2 + 4\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.187 (sec)

Writing the ode as

$$2y'' + 5xy' + \left(2x^2 + 4\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= 5x \\ C &= 2x^2 + 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9x^2 - 12}{16} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9x^2 - 12 \\ t &= 16 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{9x^2}{16} - \frac{3}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.142: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3x}{4} - \frac{1}{2x} - \frac{1}{6x^3} - \frac{1}{9x^5} - \frac{5}{54x^7} - \frac{7}{81x^9} - \frac{7}{81x^{11}} - \frac{22}{243x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{4}$$

From Eq. (9) the sum up to $v = 1$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{3x}{4}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9x^2}{16}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{9x^2 - 12}{16} \\
&= Q + \frac{R}{16} \\
&= \left(\frac{9x^2}{16} - \frac{3}{4}\right) + (0) \\
&= \frac{9x^2}{16} - \frac{3}{4}
\end{aligned}
$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{4}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{3}{4}\right) - (0) \\
&= -\frac{3}{4}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{3x}{4} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{4}}{\frac{3}{4}} - 1\right) &&= -1 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{4}}{\frac{3}{4}} - 1\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{9x^2}{16} - \frac{3}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{3x}{4}$	-1	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$, and since there are no poles then

$$
\begin{aligned}
d &= \alpha_\infty^- \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= (-)[\sqrt{r}]_\infty \\
&= 0 + (-)\left(\frac{3x}{4}\right) \\
&= -\frac{3x}{4} \\
&= -\frac{3x}{4}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{3x}{4}\right)(0) + \left(\left(-\frac{3}{4}\right) + \left(-\frac{3x}{4}\right)^2 - \left(\frac{9x^2}{16} - \frac{3}{4}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int -\frac{3x}{4} dx} \\
&= e^{-\frac{3x^2}{8}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{5x}{2}\, dx} \\
&= z_1 e^{-\frac{5x^2}{8}} \\
&= z_1 \left(e^{-\frac{5x^2}{8}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = e^{-x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{5x}{2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{5x^2}{4}}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{i\sqrt{\pi}\,\sqrt{3}\,\operatorname{erf}\left(\frac{i\sqrt{3}\,x}{2}\right)}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-x^2}\right) + c_2 \left(\mathrm{e}^{-x^2} \left(-\frac{i\sqrt{\pi}\,\sqrt{3}\,\operatorname{erf}\left(\frac{i\sqrt{3}\,x}{2}\right)}{3}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2\frac{d^2}{dx^2}y(x) + 5x\left(\frac{d}{dx}y(x)\right) + (2x^2+4)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = (-x^2-2)\,y(x) - \frac{5x\left(\frac{d}{dx}y(x)\right)}{2}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{5x\left(\frac{d}{dx}y(x)\right)}{2} + (x^2+2)\,y(x) = 0$

- Multiply by denominators

  $2\frac{d^2}{dx^2}y(x) + 5x\left(\frac{d}{dx}y(x)\right) + (2x^2+4)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ○ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ○ Shift index using $k->k-m$

  $x^m \cdot y(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ○ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

- Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

  $\frac{d^2}{dx^2}y(x) = \sum\limits_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$
- Shift index using $k- > k+2$

  $\frac{d^2}{dx^2}y(x) = \sum\limits_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $4a_2 + 4a_0 + (12a_3 + 9a_1)\, x + \left(\sum\limits_{k=2}^{\infty} (2a_{k+2}(k+2)\,(k+1) + a_k(5k+4) + 2a_{k-2})\, x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[4a_2 + 4a_0 = 0, 12a_3 + 9a_1 = 0]$
- Solve for the dependent coefficient(s)

  $\left\{a_2 = -a_0, a_3 = -\frac{3a_1}{4}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $(2k^2 + 6k + 4)\, a_{k+2} + 5a_k k + 4a_k + 2a_{k-2} = 0$
- Shift index using $k- > k+2$

  $\left(2(k+2)^2 + 6k + 16\right) a_{k+4} + 5a_{k+2}(k+2) + 4a_{k+2} + 2a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum\limits_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{5ka_{k+2}+2a_k+14a_{k+2}}{2(k^2+7k+12)}, a_2 = -a_0, a_3 = -\frac{3a_1}{4}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 22

```
dsolve(2*diff(diff(y(x),x),x)+5*diff(y(x),x)*x+(2*x^2+4)*y(x) = 0,y(x),singsol=all)
```

$$y = e^{-x^2}\left(c_1 + \operatorname{erf}\left(\frac{i\sqrt{3}\, x}{2}\right) c_2\right)$$

Mathematica DSolve solution

Solving time : 0.106 (sec)
Leaf size : 42

```
DSolve[{2*D[y[x],{x,2}]+5*x*D[y[x],x]+(4+2*x^2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to \frac{1}{3}e^{-x^2}\left(\sqrt{3\pi}c_2\operatorname{erfi}\left(\frac{\sqrt{3}x}{2}\right)+3c_1\right)$$

2.1.80 Problem 82

Internal problem ID [9252]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 82
Date solved : `Monday, January 27, 2025 at 06:00:33 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + 4xy' + \left(4x^2 + 2\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.054 (sec)

Writing the ode as

$$y'' + 4xy' + \left(4x^2 + 2\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4x \\ C &= 4x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.144: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x}{1}\, dx} \\ &= z_1 e^{-x^2} \\ &= z_1 \left(\mathrm{e}^{-x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x^2}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x^2}\right) + c_2\left(\mathrm{e}^{-x^2}(x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + 4x\left(\frac{d}{dx}y(x)\right) + (4x^2+2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

- ☐ Rewrite ODE with series expansions
  - ○ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  - ○ Shift index using $k- >k-m$

    $x^m \cdot y(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  - ○ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  - ○ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - ○ Shift index using $k- >k+2$

    $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 + 2a_0 + (6a_3 + 6a_1)\,x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 2a_k(2k+1) + 4a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 + 2a_0 = 0, 6a_3 + 6a_1 = 0]$

- Solve for the dependent coefficient(s)

  $\{a_2 = -a_0, a_3 = -a_1\}$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\,a_{k+2} + 4a_k k + 2a_k + 4a_{k-2} = 0$

- Shift index using $k->k+2$

  $$\left((k+2)^2+3k+8\right)a_{k+4}+4a_{k+2}(k+2)+2a_{k+2}+4a_k=0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y(x)=\sum_{k=0}^{\infty}a_k x^k, a_{k+4}=-\frac{2(2ka_{k+2}+2a_k+5a_{k+2})}{k^2+7k+12}, a_2=-a_0, a_3=-a_1\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 16

```
dsolve(diff(diff(y(x),x),x)+4*diff(y(x),x)*x+(4*x^2+2)*y(x) = 0,y(x),singsol=all)
```

$$y=\mathrm{e}^{-x^2}(c_2x+c_1)$$

Mathematica DSolve solution

Solving time : 0.03 (sec)
Leaf size : 20

```
DSolve[{D[y[x],{x,2}]+4*x*D[y[x],x]+(2+4*x^2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x)\to e^{-x^2}(c_2x+c_1)$$

2.1.81 Problem 83

Internal problem ID [9253]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 83
Date solved : `Monday, January 27, 2025 at 06:00:34 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + 4xy' + \left(4x^2 + 2\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.061 (sec)

Writing the ode as

$$y'' + 4xy' + \left(4x^2 + 2\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4x \\ C &= 4x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.146: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x}{1}\,dx} \\ &= z_1 e^{-x^2} \\ &= z_1\left(\mathrm{e}^{-x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x^2}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x^2}\right) + c_2\left(\mathrm{e}^{-x^2}(x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2}y(x) + 4x\left(\frac{d}{dx}y(x)\right) + (4x^2+2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ○ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$
  $x^m \cdot y(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$
  ○ Shift index using $k- >k-m$
  $x^m \cdot y(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$
  ○ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion
  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$
  ○ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion
  $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$
  ○ Shift index using $k- >k+2$
  $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions
$2a_2 + 2a_0 + (6a_3 + 6a_1)\,x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 2a_k(2k+1) + 4a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0
  $[2a_2 + 2a_0 = 0, 6a_3 + 6a_1 = 0]$
- Solve for the dependent coefficient(s)
  $\{a_2 = -a_0, a_3 = -a_1\}$
- Each term in the series must be 0, giving the recursion relation
  $(k^2 + 3k + 2)\,a_{k+2} + 4a_k k + 2a_k + 4a_{k-2} = 0$

- Shift index using $k->k+2$

  $$\left((k+2)^2+3k+8\right)a_{k+4}+4a_{k+2}(k+2)+2a_{k+2}+4a_k=0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y(x)=\sum_{k=0}^{\infty}a_k x^k, a_{k+4}=-\frac{2(2ka_{k+2}+2a_k+5a_{k+2})}{k^2+7k+12}, a_2=-a_0, a_3=-a_1\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 16

```
dsolve(diff(diff(y(x),x),x)+4*diff(y(x),x)*x+(4*x^2+2)*y(x) = 0,y(x),singsol=all)
```

$$y=e^{-x^2}(c_2x+c_1)$$

Mathematica DSolve solution

Solving time : 0.022 (sec)
Leaf size : 20

```
DSolve[{D[y[x],{x,2}]+4*x*D[y[x],x]+(2+4*x^2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x)\to e^{-x^2}(c_2x+c_1)$$

2.1.82 Problem 84

Internal problem ID [9254]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 84
Date solved : `Monday, January 27, 2025 at 06:00:34 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2\left(x^2+x+1\right)y''+x\left(11x^2+11x+9\right)y'+\left(7x^2+10x+6\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 1.072 (sec)

Writing the ode as

$$\left(2x^4+2x^3+2x^2\right)y''+\left(11x^3+11x^2+9x\right)y'+\left(7x^2+10x+6\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^4+2x^3+2x^2 \\ B &= 11x^3+11x^2+9x \\ C &= 7x^2+10x+6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{21x^4+18x^3+27x^2-2x-3}{16\left(x^3+x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 21x^4+18x^3+27x^2-2x-3 \\ t &= 16\left(x^3+x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{21x^4+18x^3+27x^2-2x-3}{16\left(x^3+x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.148: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^3 + x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ of order 2. There is a pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2} + \frac{1}{4x} + \frac{-\frac{5}{24} + \frac{i\sqrt{3}}{24}}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{5}{24} - \frac{i\sqrt{3}}{24}}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{1}{8} - \frac{43i\sqrt{3}}{72}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{-\frac{1}{8} + \frac{43i\sqrt{3}}{72}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{24} + \frac{i\sqrt{3}}{24}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{6+6i\sqrt{3}}}{12}
\end{aligned}
$$

For the pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{24} - \frac{i\sqrt{3}}{24}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{6-6i\sqrt{3}}}{12}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{21x^4 + 18x^3 + 27x^2 - 2x - 3}{16\left(x^3 + x^2 + x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{21}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{21x^4 + 18x^3 + 27x^2 - 2x - 3}{16\left(x^3 + x^2 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
$-\frac{1}{2} + \frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}$	$\frac{1}{2} - \frac{\sqrt{6+6i\sqrt{3}}}{12}$
$-\frac{1}{2} - \frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}$	$\frac{1}{2} - \frac{\sqrt{6-6i\sqrt{3}}}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying

$\alpha_\infty^+ = \frac{7}{4}$ then

$$\begin{aligned}d &= \alpha_\infty^+ - \left(\alpha_{c1}^- + \alpha_{c2}^+ + \alpha_{c3}^+\right)\\ &= \frac{7}{4} - \left(\frac{7}{4}\right)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^+}{x-c_3}\right) + (+)[\sqrt{r}]_\infty\\ &= \frac{1}{4x} + \frac{\frac{1}{2}+\frac{\sqrt{6+6i\sqrt{3}}}{12}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2}+\frac{\sqrt{6-6i\sqrt{3}}}{12}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}} + (0)\\ &= \frac{1}{4x} + \frac{\frac{1}{2}+\frac{\sqrt{6+6i\sqrt{3}}}{12}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2}+\frac{\sqrt{6-6i\sqrt{3}}}{12}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}}\\ &= \frac{7x^2+3x+1}{4x\,(x^2+x+1)}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{4x} + \frac{\frac{1}{2}+\frac{\sqrt{6+6i\sqrt{3}}}{12}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2}+\frac{\sqrt{6-6i\sqrt{3}}}{12}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}}\right)(0) + \left(\left(-\frac{1}{4x^2} - \frac{\frac{1}{2}+\frac{\sqrt{6+6i\sqrt{3}}}{12}}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^2} - \frac{\frac{1}{2}+\frac{\sqrt{6-6i\sqrt{3}}}{12}}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2}\right) + \left(\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\,dx}\\ &= e^{\int\left(\frac{1}{4x} + \frac{\frac{1}{2}+\frac{\sqrt{6+6i\sqrt{3}}}{12}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2}+\frac{\sqrt{6-6i\sqrt{3}}}{12}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}}\right)dx}\\ &= 2\left(x^2+x+1\right)^{3/4}\sqrt{2}\,x^{1/4}\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{11x^3+11x^2+9x}{2x^4+2x^3+2x^2}\,dx}\\ &= z_1 e^{-\frac{\ln\left(x^2+x+1\right)}{4} - \frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6} - \frac{9\ln(x)}{4}}\\ &= z_1\left(\frac{\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}}{\left(x^2+x+1\right)^{1/4}x^{9/4}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{2\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}\sqrt{2}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{11x^3+11x^2+9x}{2x^4+2x^3+2x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{\mathrm{e}^{-\frac{\ln(x^2+x+1)}{2} - \frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3} - \frac{9\ln(x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{\ln(x^2+x+1)}{2} - \frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3} - \frac{9\ln(x)}{2}}\,x^4\,\mathrm{e}^{\frac{2\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{8x^2+8x+8}\,dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{2\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}\sqrt{2}}{x^2}\right) \\
&\quad + c_2 \left(\frac{2\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}\sqrt{2}}{x^2}\left(\int \frac{\mathrm{e}^{-\frac{\ln(x^2+x+1)}{2} - \frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3} - \frac{9\ln(x)}{2}}\,x^4\,\mathrm{e}^{\frac{2\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{8x^2+8x+8}\,dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$2x^2(x^2+x+1)\left(\frac{d^2}{dx^2}y(x)\right) + x(11x^2+11x+9)\left(\frac{d}{dx}y(x)\right) + (7x^2+10x+6)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{(7x^2+10x+6)y(x)}{2x^2(x^2+x+1)} - \frac{(11x^2+11x+9)\left(\frac{d}{dx}y(x)\right)}{2x(x^2+x+1)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) + \frac{(11x^2+11x+9)\left(\frac{d}{dx}y(x)\right)}{2x(x^2+x+1)} + \frac{(7x^2+10x+6)y(x)}{2x^2(x^2+x+1)} = 0$
- ☐ Check to see if x_0 is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{11x^2+11x+9}{2x(x^2+x+1)}, P_3(x) = \frac{7x^2+10x+6}{2x^2(x^2+x+1)}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

$\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{9}{2}$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 3$

- ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x^2(x^2+x+1)\left(\frac{d^2}{dx^2}y(x)\right) + x(11x^2+11x+9)\left(\frac{d}{dx}y(x)\right) + (7x^2+10x+6)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ◦ Shift index using $k->k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ◦ Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- ◦ Shift index using $k->k+2-m$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)\,(3+2r)\,x^r + \left(a_1(3+r)\,(5+2r) + a_0(5+2r)\,(2+r)\right)x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+2)\,(2k+\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-2, -\frac{3}{2}\right\}$

- Each term must be 0

  $a_1(3+r)\,(5+2r) + a_0(5+2r)\,(2+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = -\frac{(2+r)a_0}{3+r}$

- Each term in the series must be 0, giving the recursion relation

  $2((a_k + a_{k-2} + a_{k-1})\,k + (a_k + a_{k-2} + a_{k-1})\,r + 2a_k - a_{k-2} + a_{k-1})\left(k + r + \frac{3}{2}\right) = 0$

- Shift index using $k->k+2$

  $2((a_{k+2} + a_k + a_{k+1})\,(k+2) + (a_{k+2} + a_k + a_{k+1})\,r + 2a_{k+2} - a_k + a_{k+1})\left(k + \frac{7}{2} + r\right) = 0$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{ka_k + ka_{k+1} + ra_k + ra_{k+1} + a_k + 3a_{k+1}}{k+4+r}$$

- Recursion relation for $r = -2$

$$a_{k+2} = -\frac{ka_k + ka_{k+1} - a_k + a_{k+1}}{k+2}$$

- Solution for $r = -2$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{ka_k + ka_{k+1} - a_k + a_{k+1}}{k+2}, a_1 = 0\right]$$

- Recursion relation for $r = -\frac{3}{2}$

$$a_{k+2} = -\frac{ka_k + ka_{k+1} - \frac{1}{2}a_k + \frac{3}{2}a_{k+1}}{k+\frac{5}{2}}$$

- Solution for $r = -\frac{3}{2}$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}, a_{k+2} = -\frac{ka_k + ka_{k+1} - \frac{1}{2}a_k + \frac{3}{2}a_{k+1}}{k+\frac{5}{2}}, a_1 = -\frac{a_0}{3}\right]$$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{3}{2}}\right), a_{k+2} = -\frac{ka_k + ka_{k+1} - a_k + a_{k+1}}{k+2}, a_1 = 0, b_{k+2} = -\frac{kb_k + kb_{k+1} - \frac{1}{2}b_k +}{k+\frac{5}{2}}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Mo
      <- Heun successful: received ODE is equivalent to the  HeunG  ODE, case  a <> 0, e
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `2.651` (sec)
Leaf size : 231

```
dsolve(2*x^2*(x^2+x+1)*diff(diff(y(x),x),x)+x*(11*x^2+11*x+9)*diff(y(x),x)+(7*x^2+10*x
```

$$y = \frac{\left(2x+i\sqrt{3}+1\right)^{\frac{5\sqrt{3}+3i}{6\sqrt{3}+6i}} \left(-2x+i\sqrt{3}-1\right)^{\frac{64i\sqrt{3}+2368}{\left(\sqrt{3}+i\right)^{3}\left(i-\sqrt{3}\right)^{4}\left(13\sqrt{3}+9i\right)}} \mathrm{e}^{-\frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}} \left(\mathrm{HeunG}\left(\frac{\sqrt{3}+i}{i-\sqrt{3}}, 0, 0, \frac{5}{2}, \frac{1}{2}\right.\right.}{x^{5/2}\left(x^{2}+x+1\right)}$$

Mathematica DSolve solution

Solving time : `0.455` (sec)
Leaf size : 135

```
DSolve[{2*x^2*(1+x+x^2)*D[y[x],{x,2}]+x*(9+11*x+11*x^2)*D[y[x],x]+(6+10*x+7*x^2)*y[x]==0,{}}
```

$$y(x) \to \exp\left(\int_1^x \frac{K[1](7K[1]+3)+1}{4K[1]\left(K[1]^2+K[1]+1\right)}dK[1] - \frac{1}{2}\int_1^x \left(\frac{K[2]+1}{K[2]^2+K[2]+1} + \frac{9}{2K[2]}\right)dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{7K[1]^2+3K[1]+1}{4K[1]\left(K[1]^2+K[1]+1\right)}dK[1]\right)dK[3] + c_1\right)$$

2.1.83 Problem 85

Internal problem ID [9255]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 85
Date solved : `Monday, January 27, 2025 at 06:00:36 PM`
CAS classification :
`[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, '_with_symmetry_[0,F(x)]']`

Solve

$$3x^2y'' + 2x\left(-2x^2 + x + 1\right)y' + \left(-8x^2 + 2x\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.359 (sec)

Writing the ode as

$$3x^2y'' + \left(-4x^3 + 2x^2 + 2x\right)y' + \left(-8x^2 + 2x\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^2 \\ B &= -4x^3 + 2x^2 + 2x \\ C &= -8x^2 + 2x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^4 - 4x^3 + 15x^2 - 4x - 2}{9x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^4 - 4x^3 + 15x^2 - 4x - 2 \\ t &= 9x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4 - 4x^3 + 15x^2 - 4x - 2}{9x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.150: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 9x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{4x^2}{9} - \frac{4x}{9} + \frac{5}{3} - \frac{2}{9x^2} - \frac{4}{9x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{2}{9}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{2}{3} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{3} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}
$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r} \approx \frac{2x}{3} - \frac{1}{3} + \frac{7}{6x} + \frac{1}{4x^2} - \frac{17}{16x^3} - \frac{31}{32x^4} + \frac{85}{64x^5} + \frac{353}{128x^6} + \dots \qquad (9)
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a = \frac{2}{3}
$$

From Eq. (9) the sum up to $v = 1$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= -\frac{1}{3} + \frac{2x}{3} \qquad (10)
\end{aligned}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{9} - \frac{4}{9}x + \frac{4}{9}x^2
$$

This shows that the coefficient of 1 in the above is $\frac{1}{9}$. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{4x^4 - 4x^3 + 15x^2 - 4x - 2}{9x^2} \\
&= Q + \frac{R}{9x^2} \\
&= \left(\frac{4}{9}x^2 - \frac{4}{9}x + \frac{5}{3}\right) + \left(\frac{-4x - 2}{9x^2}\right) \\
&= \frac{4x^2}{9} - \frac{4x}{9} + \frac{5}{3} + \frac{-4x - 2}{9x^2}
\end{aligned}
$$

We see that the coefficient of the term x in the quotient is $\frac{5}{3}$. Now b can be found.

$$
\begin{aligned}
b &= \left(\frac{5}{3}\right) - \left(\frac{1}{9}\right) \\
&= \frac{14}{9}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= -\frac{1}{3} + \frac{2x}{3} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{14}{9}}{\frac{2}{3}} - 1\right) &&= \frac{2}{3} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{14}{9}}{\frac{2}{3}} - 1\right) &&= -\frac{5}{3}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^4 - 4x^3 + 15x^2 - 4x - 2}{9x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{2}{3}$	$\frac{1}{3}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$-\frac{1}{3} + \frac{2x}{3}$	$\frac{2}{3}$	$-\frac{5}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{2}{3}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+\right) \\
&= \frac{2}{3} - \left(\frac{2}{3}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{2}{3x} + \left(-\frac{1}{3} + \frac{2x}{3} \right) \\
&= \frac{2}{3x} - \frac{1}{3} + \frac{2x}{3} \\
&= \frac{2}{3x} - \frac{1}{3} + \frac{2x}{3}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{2}{3x} - \frac{1}{3} + \frac{2x}{3}\right)(0) + \left(\left(-\frac{2}{3x^2} + \frac{2}{3}\right) + \left(\frac{2}{3x} - \frac{1}{3} + \frac{2x}{3}\right)^2 - \left(\frac{4x^4 - 4x^3 + 15x^2 - 4x - 2}{9x^2}\right)\right) = 0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(\frac{2}{3x} - \frac{1}{3} + \frac{2x}{3}\right) dx} \\
&= x^{2/3} \mathrm{e}^{\frac{x(x-1)}{3}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-4x^3 + 2x^2 + 2x}{3x^2}\, dx} \\
&= z_1 e^{\frac{x^2}{3} - \frac{x}{3} - \frac{\ln(x)}{3}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{x(x-1)}{3}}}{x^{1/3}} \right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^{1/3} \mathrm{e}^{\frac{2x(x-1)}{3}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^3 + 2x^2 + 2x}{3x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{2x^2}{3} - \frac{2x}{3} - \frac{2\ln(x)}{3}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{2x^2}{3} - \frac{2x}{3} - \frac{2\ln(x)}{3}} \mathrm{e}^{-\frac{4x(x-1)}{3}}}{x^{2/3}}\, dx \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{1/3} \mathrm{e}^{\frac{2x(x-1)}{3}} \right) + c_2 \left(x^{1/3} \mathrm{e}^{\frac{2x(x-1)}{3}} \left(\int \frac{\mathrm{e}^{\frac{2x^2}{3} - \frac{2x}{3} - \frac{2\ln(x)}{3}} \mathrm{e}^{-\frac{4x(x-1)}{3}}}{x^{2/3}}\, dx \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$3x^2\left(\frac{d^2}{dx^2}y(x)\right)+2x(-2x^2+x+1)\left(\frac{d}{dx}y(x)\right)+(-8x^2+2x)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{2(4x-1)y(x)}{3x}+\frac{2(2x^2-x-1)\left(\frac{d}{dx}y(x)\right)}{3x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)-\frac{2(2x^2-x-1)\left(\frac{d}{dx}y(x)\right)}{3x}-\frac{2(4x-1)y(x)}{3x}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{2(2x^2-x-1)}{3x},P_3(x)=-\frac{2(4x-1)}{3x}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{2}{3}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $3\left(\frac{d^2}{dx^2}y(x)\right)x+(-4x^2+2x+2)\left(\frac{d}{dx}y(x)\right)+(2-8x)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  ◦ Shift index using $k->k-m$

  $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..2$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-1}$

  ◦ Shift index using $k->k+1$

  $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$a_0r(-1+3r)\,x^{-1+r}+(a_1(1+r)\,(2+3r)+2a_0(1+r))\,x^r+\left(\sum_{k=1}^{\infty}(a_{k+1}(k+1+r)\,(3k+2+3r)+\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-1+3r)=0$
- Values of r that satisfy the indicial equation
  $r\in\left\{0,\frac{1}{3}\right\}$
- Each term must be 0
  $a_1(1+r)\,(2+3r)+2a_0(1+r)=0$
- Each term in the series must be 0, giving the recursion relation
  $(k+1+r)\,(3ka_{k+1}+3ra_{k+1}+2a_k-4a_{k-1}+2a_{k+1})=0$
- Shift index using $k->k+1$
  $(k+r+2)\,(3(k+1)\,a_{k+2}+3ra_{k+2}+2a_{k+1}-4a_k+2a_{k+2})=0$
- Recursion relation that defines series solution to ODE
  $a_{k+2}=\frac{2(-a_{k+1}+2a_k)}{3k+5+3r}$
- Recursion relation for $r=0$
  $a_{k+2}=\frac{2(-a_{k+1}+2a_k)}{3k+5}$
- Solution for $r=0$
  $\left[y(x)=\sum_{k=0}^{\infty}a_kx^k,a_{k+2}=\frac{2(-a_{k+1}+2a_k)}{3k+5},2a_1+2a_0=0\right]$
- Recursion relation for $r=\frac{1}{3}$
  $a_{k+2}=\frac{2(-a_{k+1}+2a_k)}{3k+6}$
- Solution for $r=\frac{1}{3}$
  $\left[y(x)=\sum_{k=0}^{\infty}a_kx^{k+\frac{1}{3}},a_{k+2}=\frac{2(-a_{k+1}+2a_k)}{3k+6},4a_1+\frac{8a_0}{3}=0\right]$
- Combine solutions and rename parameters
  $\left[y(x)=\left(\sum_{k=0}^{\infty}a_kx^k\right)+\left(\sum_{k=0}^{\infty}b_kx^{k+\frac{1}{3}}\right),a_{k+2}=\frac{2(-a_{k+1}+2a_k)}{3k+5},2a_1+2a_0=0,b_{k+2}=\frac{2(-b_{k+1}+2b_k)}{3k+6},4b_1+\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
```

```
    -> Mathieu
       -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
    -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @
    <- Heun successful: received ODE is equivalent to the  HeunB  ODE, case  c = 0
 <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.280 (sec)
Leaf size : 38

```
dsolve(3*x^2*diff(diff(y(x),x),x)+2*x*(-2*x^2+x+1)*diff(y(x),x)+(-8*x^2+2*x)*y(x) = 0,
```

$$y = c_1 x^{1/3} \mathrm{e}^{\frac{2(x-1)x}{3}} + c_2 \,\mathrm{HeunB}\left(-\frac{1}{3}, \frac{\sqrt{6}}{3}, -\frac{7}{3}, \frac{4\sqrt{6}}{9}, -\frac{\sqrt{6}\,x}{3}\right)$$

Mathematica DSolve solution

Solving time : 0.943 (sec)
Leaf size : 53

```
DSolve[{3*x^2*D[y[x],{x,2}]+2*x*(1+x-2*x^2)*D[y[x],x]+(2*x-8*x^2)*y[x]==0,{}},y[x],x,Include
```

$$y(x) \to e^{\frac{2}{3}(x-1)x} \sqrt[3]{x} \left(c_2 \int_1^x \frac{e^{-\frac{2}{3}(K[1]-1)K[1]}}{K[1]^{4/3}} dK[1] + c_1\right)$$

2.1.84 Problem 86

Internal problem ID [9256]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 86
Date solved : `Monday, January 27, 2025 at 06:00:37 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$12x^2(1+x)\,y'' + x\left(3x^2+35x+11\right)y' - \left(-5x^2-10x+1\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.338 (sec)

Writing the ode as

$$\left(12x^3+12x^2\right)y'' + \left(3x^3+35x^2+11x\right)y' + \left(5x^2+10x-1\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 12x^3+12x^2 \\ B &= 3x^3+35x^2+11x \\ C &= 5x^2+10x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9x^4-30x^3-197x^2-190x-95}{576\left(x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9x^4-30x^3-197x^2-190x-95 \\ t &= 576\left(x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{9x^4-30x^3-197x^2-190x-95}{576\left(x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.152: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 576(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{64} - \frac{7}{64\,(1+x)^2} - \frac{95}{576x^2} - \frac{1}{12\,(1+x)}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{95}{576}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{19}{24} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{24} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{8} - \frac{1}{3x} - \frac{29}{24x^2} - \frac{193}{72x^3} - \frac{3017}{216x^4} - \frac{40009}{648x^5} - \frac{642029}{1944x^6} - \frac{10350493}{5832x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{8}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{8} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{64}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{9x^4 - 30x^3 - 197x^2 - 190x - 95}{576x^4 + 1152x^3 + 576x^2} \\ &= Q + \frac{R}{576x^4 + 1152x^3 + 576x^2} \\ &= \left(\frac{1}{64}\right) + \left(\frac{-48x^3 - 206x^2 - 190x - 95}{576x^4 + 1152x^3 + 576x^2}\right) \\ &= \frac{1}{64} + \frac{-48x^3 - 206x^2 - 190x - 95}{576x^4 + 1152x^3 + 576x^2} \end{aligned}$$

Since the degree of t is 4, then we see that the coefficient of the term x^3 in the remainder R is -48. Dividing this by leading coefficient in t which is 576 gives $-\frac{1}{12}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{12}\right) - (0) \\ &= -\frac{1}{12} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{8} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{12}}{\frac{1}{8}} - 0\right) &&= -\frac{1}{3} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{12}}{\frac{1}{8}} - 0\right) &&= \frac{1}{3}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{9x^4 - 30x^3 - 197x^2 - 190x - 95}{576\left(x^2+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-1	2	0	$\frac{7}{8}$	$\frac{1}{8}$
0	2	0	$\frac{19}{24}$	$\frac{5}{24}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{8}$	$-\frac{1}{3}$	$\frac{1}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{3}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-}\right) \\
&= \frac{1}{3} - \left(\frac{1}{3}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{8+8x} + \frac{5}{24x} + (-)\left(\frac{1}{8}\right) \\
&= \frac{1}{8+8x} + \frac{5}{24x} - \frac{1}{8} \\
&= \frac{1}{8+8x} + \frac{5}{24x} - \frac{1}{8}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{1}{8+8x}+\frac{5}{24x}-\frac{1}{8}\right)(0)+\left(\left(-\frac{1}{8\left(1+x\right)^{2}}-\frac{5}{24x^{2}}\right)+\left(\frac{1}{8+8x}+\frac{5}{24x}-\frac{1}{8}\right)^{2}-\left(\frac{9x^{4}-30x^{3}-19}{576\left(x\right.}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\,dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{8+8x}+\frac{5}{24x}-\frac{1}{8}\right)dx} \\
&= x^{5/24}(1+x)^{1/8}\,\mathrm{e}^{-\frac{x}{8}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x^3+35x^2+11x}{12x^3+12x^2}\,dx} \\
&= z_1 e^{-\frac{x}{8}-\frac{11\ln(x)}{24}-\frac{7\ln(1+x)}{8}} \\
&= z_1\left(\frac{\mathrm{e}^{-\frac{x}{8}}}{x^{11/24}\left(1+x\right)^{7/8}}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\frac{\mathrm{e}^{-\frac{x}{4}}}{x^{1/4}\left(1+x\right)^{3/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1\int \frac{e^{\int -\frac{3x^3+35x^2+11x}{12x^3+12x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1\int \frac{e^{-\frac{x}{4}-\frac{11\ln(x)}{12}-\frac{7\ln(1+x)}{4}}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(\int \mathrm{e}^{-\frac{x}{4}-\frac{11\ln(x)}{12}-\frac{7\ln(1+x)}{4}}\sqrt{x}\left(1+x\right)^{3/2}\mathrm{e}^{\frac{x}{2}}\,dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\mathrm{e}^{-\frac{x}{4}}}{x^{1/4}\left(1+x\right)^{3/4}}\right)+c_2\left(\frac{\mathrm{e}^{-\frac{x}{4}}}{x^{1/4}\left(1+x\right)^{3/4}}\left(\int \mathrm{e}^{-\frac{x}{4}-\frac{11\ln(x)}{12}-\frac{7\ln(1+x)}{4}}\sqrt{x}\left(1+x\right)^{3/2}\mathrm{e}^{\frac{x}{2}}\,dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$12x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(3x^2+35x+11)\left(\frac{d}{dx}y(x)\right)-(-5x^2-10x+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{(5x^2+10x-1)y(x)}{12(x+1)x^2}-\frac{(3x^2+35x+11)\left(\frac{d}{dx}y(x)\right)}{12x(x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)+\frac{(3x^2+35x+11)\left(\frac{d}{dx}y(x)\right)}{12x(x+1)}+\frac{(5x^2+10x-1)y(x)}{12(x+1)x^2}=0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

$\left[P_2(x)=\frac{3x^2+35x+11}{12x(x+1)},P_3(x)=\frac{5x^2+10x-1}{12(x+1)x^2}\right]$

- ○ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

$\left.((x+1)\cdot P_2(x))\right|_{x=-1}=\frac{7}{4}$

- ○ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

$\left.((x+1)^2\cdot P_3(x))\right|_{x=-1}=0$

- ○ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1$

- Multiply by denominators

$12x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(3x^2+35x+11)\left(\frac{d}{dx}y(x)\right)+(5x^2+10x-1)\,y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(12u^3-24u^2+12u)\left(\frac{d^2}{du^2}y(u)\right)+(3u^3+26u^2-50u+21)\left(\frac{d}{du}y(u)\right)+(5u^2-6)\,y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot y(u)$ to series expansion for $m=0..2$

$u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

- ○ Shift index using $k->k-m$

$u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

- ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..3$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ○ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

- ○ Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$3a_0r(3+4r)\, u^{-1+r} + (3a_1(1+r)(7+4r) - 2a_0(3+4r)(1+3r))\, u^r + (3a_2(2+r)(11+4r) - 2a_1($$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $3r(3+4r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{0, -\frac{3}{4}\right\}$
- The coefficients of each power of u must be 0
  $[3a_1(1+r)(7+4r) - 2a_0(3+4r)(1+3r) = 0, 3a_2(2+r)(11+4r) - 2a_1(7+4r)(4+3r) + 2a_0r($
- Solve for the dependent coefficient(s)
  $\left\{a_1 = \frac{2a_0(12r^2+13r+3)}{3(4r^2+11r+7)}, a_2 = \frac{2a_0(54r^3+135r^2+101r+24)}{9(4r^3+23r^2+41r+22)}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $12(-2a_k + a_{k-1} + a_{k+1})\, k^2 + (24(-2a_k + a_{k-1} + a_{k+1})\, r - 26a_k + 3a_{k-2} - 10a_{k-1} + 33a_{k+1})\, k + 12$
- Shift index using $k- > k+2$
  $12(-2a_{k+2} + a_{k+1} + a_{k+3})\,(k+2)^2 + (24(-2a_{k+2} + a_{k+1} + a_{k+3})\, r - 26a_{k+2} + 3a_k - 10a_{k+1} + 33a_k$
- Recursion relation that defines series solution to ODE
  $a_{k+3} = -\frac{12k^2a_{k+1} - 24k^2a_{k+2} + 24kra_{k+1} - 48kra_{k+2} + 12r^2a_{k+1} - 24r^2a_{k+2} + 3ka_k + 38ka_{k+1} - 122ka_{k+2} + 3ra_k + 38ra_{k+1} - 122ra_{k+}}{3(4k^2+8kr+4r^2+27k+27r+45)}$
- Recursion relation for $r = 0$
  $a_{k+3} = -\frac{12k^2a_{k+1} - 24k^2a_{k+2} + 3ka_k + 38ka_{k+1} - 122ka_{k+2} + 5a_k + 26a_{k+1} - 154a_{k+2}}{3(4k^2+27k+45)}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+3} = -\frac{12k^2a_{k+1} - 24k^2a_{k+2} + 3ka_k + 38ka_{k+1} - 122ka_{k+2} + 5a_k + 26a_{k+1} - 154a_{k+2}}{3(4k^2+27k+45)}, a_1 = \frac{2a_0}{7}, a_2 = \right.$
- Revert the change of variables $u = x+1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+3} = -\frac{12k^2a_{k+1} - 24k^2a_{k+2} + 3ka_k + 38ka_{k+1} - 122ka_{k+2} + 5a_k + 26a_{k+1} - 154a_{k+2}}{3(4k^2+27k+45)}, a_1 = \frac{2a_0}{7}, \right.$
- Recursion relation for $r = -\frac{3}{4}$
  $a_{k+3} = -\frac{12k^2a_{k+1} - 24k^2a_{k+2} + 3ka_k + 20ka_{k+1} - 86ka_{k+2} + \frac{11}{4}a_k + \frac{17}{4}a_{k+1} - 76a_{k+2}}{3(4k^2+21k+27)}$
- Solution for $r = -\frac{3}{4}$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{3}{4}}, a_{k+3} = -\frac{12k^2a_{k+1} - 24k^2a_{k+2} + 3ka_k + 20ka_{k+1} - 86ka_{k+2} + \frac{11}{4}a_k + \frac{17}{4}a_{k+1} - 76a_{k+2}}{3(4k^2+21k+27)}, a_1 = 0, a_2 = \right.$
- Revert the change of variables $u = x+1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^{k-\frac{3}{4}}, a_{k+3} = -\frac{12k^2a_{k+1} - 24k^2a_{k+2} + 3ka_k + 20ka_{k+1} - 86ka_{k+2} + \frac{11}{4}a_k + \frac{17}{4}a_{k+1} - 76a_{k+2}}{3(4k^2+21k+27)}, a_1 = 0, \right.$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k(x+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k(x+1)^{k-\frac{3}{4}}\right), a_{k+3} = -\frac{12k^2a_{k+1} - 24k^2a_{k+2} + 3ka_k + 38ka_{k+1} - 122ka_{k+2} + 5}{3(4k^2+27k+45)}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @
      <- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0,
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.401 (sec)
Leaf size : 43

```
dsolve(12*x^2*(x+1)*diff(diff(y(x),x),x)+x*(3*x^2+35*x+11)*diff(y(x),x)-(-5*x^2-10*x+1
```

$$y = \frac{e^{-\frac{x}{4}} \left(\operatorname{HeunC}\left(\frac{1}{4}, \frac{7}{12}, -\frac{3}{4}, -\frac{1}{12}, \frac{1}{2}, -x\right) x^{7/12} c_2 + \operatorname{HeunC}\left(\frac{1}{4}, -\frac{7}{12}, -\frac{3}{4}, -\frac{1}{12}, \frac{1}{2}, -x\right) c_1\right)}{\left(x+1\right)^{3/4} x^{1/4}}$$

Mathematica DSolve solution

Solving time : 0.37 (sec)
Leaf size : 118

```
DSolve[{12*x^2*(1+x)*D[y[x],{x,2}]+x*(11+35*x+3*x^2)*D[y[x],x]-(1-10*x-5*x^2)*y[x]==0,{}},y[
```

$$y(x) \to \exp\left(\int_1^x \frac{1}{24}\left(\frac{3}{K[1]+1} - 3 + \frac{5}{K[1]}\right) dK[1] - \frac{1}{2}\int_1^x \frac{1}{12}\left(\frac{21}{K[2]+1} + 3 + \frac{11}{K[2]}\right) dK[2]\right) \left(c_2 \int_1^x \exp\left(-2\int_1^{K[3]} \frac{-3K[1]^2 + 5K[1] + 5}{24K[1]^2 + 24K[1]} dK[1]\right) dK[3] + c_1\right)$$

2.1.85 Problem 87

Internal problem ID [9257]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 87
Date solved : `Monday, January 27, 2025 at 06:00:37 PM`
CAS classification :
`[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, '_with_symmetry_[0,F(x)]']`

Solve

$$x^2\left(10x^2+x+5\right)y''+x\left(48x^2+3x+4\right)y'+\left(36x^2+x\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 1.019 (sec)

Writing the ode as

$$\left(10x^4+x^3+5x^2\right)y''+\left(48x^3+3x^2+4x\right)y'+\left(36x^2+x\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=10x^4+x^3+5x^2\\ B&=48x^3+3x^2+4x\\ C&=36x^2+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-96x^4-16x^3-97x^2-12x-24}{4\left(10x^3+x^2+5x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=-96x^4-16x^3-97x^2-12x-24\\ t&=4\left(10x^3+x^2+5x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-96x^4 - 16x^3 - 97x^2 - 12x - 24}{4\left(10x^3 + x^2 + 5x\right)^2} \right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.154: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(10x^3 + x^2 + 5x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{20} + \frac{i\sqrt{199}}{20}$ of order 2. There is a pole at $x = -\frac{1}{20} - \frac{i\sqrt{199}}{20}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$\begin{aligned} r = &-\frac{6}{25x^2} - \frac{3}{125x} + \frac{-\frac{1}{19900} - \frac{i\sqrt{199}}{1990}}{\left(x + \frac{1}{20} - \frac{i\sqrt{199}}{20}\right)^2} \\ &+ \frac{-\frac{1}{19900} + \frac{i\sqrt{199}}{1990}}{\left(x + \frac{1}{20} + \frac{i\sqrt{199}}{20}\right)^2} + \frac{\frac{3}{250} - \frac{647i\sqrt{199}}{9900250}}{x + \frac{1}{20} - \frac{i\sqrt{199}}{20}} + \frac{\frac{3}{250} + \frac{647i\sqrt{199}}{9900250}}{x + \frac{1}{20} + \frac{i\sqrt{199}}{20}} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{6}{25}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{5} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{2}{5} \end{aligned}$$

For the pole at $x = -\frac{1}{20} + \frac{i\sqrt{199}}{20}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{20}-\frac{i\sqrt{199}}{20}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{19900} - \frac{i\sqrt{199}}{1990}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{989826 - 1990i\sqrt{199}}}{1990} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{989826 - 1990i\sqrt{199}}}{1990}
\end{aligned}
$$

For the pole at $x = -\frac{1}{20} - \frac{i\sqrt{199}}{20}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{20}+\frac{i\sqrt{199}}{20}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{19900} + \frac{i\sqrt{199}}{1990}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{989826 + 1990i\sqrt{199}}}{1990} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{989826 + 1990i\sqrt{199}}}{1990}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{-96x^4 - 16x^3 - 97x^2 - 12x - 24}{4\left(10x^3 + x^2 + 5x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{6}{25}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{5} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{2}{5}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-96x^4 - 16x^3 - 97x^2 - 12x - 24}{4\left(10x^3 + x^2 + 5x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{5}$	$\frac{2}{5}$
$-\frac{1}{20} + \frac{i\sqrt{199}}{20}$	2	0	$\frac{1}{2} + \frac{\sqrt{989826-1990i\sqrt{199}}}{1990}$	$\frac{1}{2} - \frac{\sqrt{989826-1990i\sqrt{199}}}{1990}$
$-\frac{1}{20} - \frac{i\sqrt{199}}{20}$	2	0	$\frac{1}{2} + \frac{\sqrt{989826+1990i\sqrt{199}}}{1990}$	$\frac{1}{2} - \frac{\sqrt{989826+1990i\sqrt{199}}}{1990}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{5}$	$\frac{2}{5}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying

$\alpha_\infty^+ = \frac{3}{5}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\ &= \frac{3}{5} - \left(\frac{3}{5}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\ &= \frac{3}{5x} + \frac{\frac{1}{2} - \frac{\sqrt{989826-1990i\sqrt{199}}}{1990}}{x + \frac{1}{20} - \frac{i\sqrt{199}}{20}} + \frac{\frac{1}{2} - \frac{\sqrt{989826+1990i\sqrt{199}}}{1990}}{x + \frac{1}{20} + \frac{i\sqrt{199}}{20}} + (0) \\ &= \frac{3}{5x} + \frac{\frac{1}{2} - \frac{\sqrt{989826-1990i\sqrt{199}}}{1990}}{x + \frac{1}{20} - \frac{i\sqrt{199}}{20}} + \frac{\frac{1}{2} - \frac{\sqrt{989826+1990i\sqrt{199}}}{1990}}{x + \frac{1}{20} + \frac{i\sqrt{199}}{20}} \\ &= \frac{12x^2 + x + 6}{20x^3 + 2x^2 + 10x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{5x} + \frac{\frac{1}{2} - \frac{\sqrt{989826-1990i\sqrt{199}}}{1990}}{x + \frac{1}{20} - \frac{i\sqrt{199}}{20}} + \frac{\frac{1}{2} - \frac{\sqrt{989826+1990i\sqrt{199}}}{1990}}{x + \frac{1}{20} + \frac{i\sqrt{199}}{20}}\right)(0) + \left(\left(-\frac{3}{5x^2} - \frac{\frac{1}{2} - \frac{\sqrt{989826-1990i\sqrt{199}}}{1990}}{\left(x + \frac{1}{20} - \frac{i\sqrt{199}}{20}\right)^2} - \right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{3}{5x} + \frac{\frac{1}{2} - \frac{\sqrt{989826-1990i\sqrt{199}}}{1990}}{x + \frac{1}{20} - \frac{i\sqrt{199}}{20}} + \frac{\frac{1}{2} - \frac{\sqrt{989826+1990i\sqrt{199}}}{1990}}{x + \frac{1}{20} + \frac{i\sqrt{199}}{20}}\right) dx} \\ &= x^{3/5} e^{-\frac{\sqrt{199}\arctan\left(\frac{(20x+1)\sqrt{199}}{199}\right)}{995}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{48x^3+3x^2+4x}{10x^4+x^3+5x^2}\, dx} \\ &= z_1 e^{-\ln(10x^2+x+5) - \frac{\sqrt{199}\arctan\left(\frac{(20x+1)\sqrt{199}}{199}\right)}{995} - \frac{2\ln(x)}{5}} \\ &= z_1 \left(\frac{e^{-\frac{\sqrt{199}\arctan\left(\frac{(20x+1)\sqrt{199}}{199}\right)}{995}}}{(10x^2 + x + 5)\, x^{2/5}} \right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/5} e^{-\frac{2\sqrt{199}\, \arctan\left(\frac{(20x+1)\sqrt{199}}{199}\right)}{995}}}{10x^2 + x + 5}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{48x^3+3x^2+4x}{10x^4+x^3+5x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2\ln(10x^2+x+5) - \frac{2\sqrt{199}\, \arctan\left(\frac{(20x+1)\sqrt{199}}{199}\right)}{995} - \frac{4\ln(x)}{5}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{e^{-2\ln(10x^2+x+5) - \frac{2\sqrt{199}\, \arctan\left(\frac{(20x+1)\sqrt{199}}{199}\right)}{995} - \frac{4\ln(x)}{5}} (10x^2+x+5)^2 e^{\frac{4\sqrt{199}\, \arctan\left(\frac{(20x+1)\sqrt{199}}{199}\right)}{995}}}{x^{2/5}} dx \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{1/5} e^{-\frac{2\sqrt{199}\, \arctan\left(\frac{(20x+1)\sqrt{199}}{199}\right)}{995}}}{10x^2+x+5} \right) \\
&\quad + c_2 \left(\frac{x^{1/5} e^{-\frac{2\sqrt{199}\, \arctan\left(\frac{(20x+1)\sqrt{199}}{199}\right)}{995}}}{10x^2+x+5} \right) \left(\int \frac{e^{-2\ln(10x^2+x+5) - \frac{2\sqrt{199}\, \arctan\left(\frac{(20x+1)\sqrt{199}}{199}\right)}{995} - \frac{4\ln(x)}{5}} (10x^2+x+5)^2 e^{\frac{4\sqrt{199}\, \mathrm{arc}}{}}}{x^{2/5}} \right.
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(10x^2+x+5)\left(\frac{d^2}{dx^2}y(x)\right) + x(48x^2+3x+4)\left(\frac{d}{dx}y(x)\right) + (36x^2+x)\, y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(36x+1)y(x)}{x(10x^2+x+5)} - \frac{(48x^2+3x+4)\left(\frac{d}{dx}y(x)\right)}{x(10x^2+x+5)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(48x^2+3x+4)\left(\frac{d}{dx}y(x)\right)}{x(10x^2+x+5)} + \frac{(36x+1)y(x)}{x(10x^2+x+5)} = 0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{48x^2+3x+4}{x(10x^2+x+5)}, P_3(x) = \frac{36x+1}{x(10x^2+x+5)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$(x \cdot P_2(x))\bigg|_{x=0} = \frac{4}{5}$

- ∘ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

- ∘ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x(10x^2 + x + 5)\left(\frac{d^2}{dx^2}y(x)\right) + (48x^2 + 3x + 4)\left(\frac{d}{dx}y(x)\right) + (36x + 1)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - ∘ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - ∘ Shift index using $k- >k - m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - ∘ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..2$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$

  - ∘ Shift index using $k- >k + 1 - m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, x^{k+r}$

  - ∘ Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 1..3$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k (k + r)(k + r - 1)\, x^{k+r-2+m}$

  - ∘ Shift index using $k- >k + 2 - m$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k + 2 - m + r)(k + 1 - m + r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-1 + 5r)\, x^{-1+r} + \left(a_1(1 + r)(4 + 5r) + a_0(1 + r)^2\right) x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k + r + 1)(5k + 4 + 5r\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1 + 5r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1}{5}\right\}$

- Each term must be 0

  $a_1(1 + r)(4 + 5r) + a_0(1 + r)^2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k + r + 1)\left((a_k + 10a_{k-1} + 5a_{k+1})\, k + (a_k + 10a_{k-1} + 5a_{k+1})\, r + a_k + 8a_{k-1} + 4a_{k+1}\right) = 0$

- Shift index using $k- >k + 1$

  $(k + r + 2)\left((a_{k+1} + 10a_k + 5a_{k+2})(k + 1) + (a_{k+1} + 10a_k + 5a_{k+2})\, r + a_{k+1} + 8a_k + 4a_{k+2}\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{10ka_k + ka_{k+1} + 10ra_k + ra_{k+1} + 18a_k + 2a_{k+1}}{5k + 5r + 9}$

- Recursion relation for $r = 0$

$a_{k+2} = -\frac{10ka_k+ka_{k+1}+18a_k+2a_{k+1}}{5k+9}$

- Solution for $r = 0$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{10ka_k+ka_{k+1}+18a_k+2a_{k+1}}{5k+9}, 4a_1 + a_0 = 0\right]$$

- Recursion relation for $r = \frac{1}{5}$

$a_{k+2} = -\frac{10ka_k+ka_{k+1}+20a_k+\frac{11}{5}a_{k+1}}{5k+10}$

- Solution for $r = \frac{1}{5}$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{5}}, a_{k+2} = -\frac{10ka_k+ka_{k+1}+20a_k+\frac{11}{5}a_{k+1}}{5k+10}, 6a_1 + \frac{36a_0}{25} = 0\right]$$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{5}}\right), a_{k+2} = -\frac{10ka_k+ka_{k+1}+18a_k+2a_{k+1}}{5k+9}, 4a_1 + a_0 = 0, b_{k+2} = -\frac{10kb_k+\cdots}{}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Mo
      <- Heun successful: received ODE is equivalent to the  HeunG  ODE, case  a <> 0, e
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 2.530 (sec)
Leaf size : 162

```
dsolve(x^2*(10*x^2+x+5)*diff(diff(y(x),x),x)+x*(48*x^2+3*x+4)*diff(y(x),x)+(36*x^2+x)*y(
```

$$y = \frac{e^{-\frac{\sqrt{199}\arctan\left(\frac{(20x+1)\sqrt{199}}{199}\right)}{995}}\left(i\sqrt{199}-20x-1\right)^{\frac{i\sqrt{199}}{1990}}\left(i\sqrt{199}+20x+1\right)^{-\frac{i\sqrt{199}}{1990}}\left(\operatorname{HeunG}\left(\frac{\sqrt{199}+i}{i-\sqrt{199}}, 0, 0, \frac{1}{5}, \frac{6}{5}, -\frac{i\sqrt{199}}{995}\cdots\right.\right.}{10x^2+x+5}$$

Mathematica DSolve solution

Solving time : `1.139` (sec)
Leaf size : 132

```
DSolve[{x^2*(5+x+10*x^2)*D[y[x],{x,2}]+x*(4+3*x+48*x^2)*D[y[x],x]+(x+36*x^2)*y[x]==0,{}},y[x
```

$$y(x) \to \exp\left(\int_1^x \left(\frac{3}{5K[1]} - \frac{1}{10\left(10K[1]^2 + K[1] + 5\right)}\right) dK[1] - \frac{1}{2}\int_1^x \frac{48K[2]^2 + 3K[2] + 4}{10K[2]^3 + K[2]^2 + 5K[2]} dK[2]\right) \left(c_2 \int_1^x \exp\left(-2\int_1^{K[3]} \left(\frac{3}{5K[1]} - \frac{1}{10\left(10K[1]^2 + K[1] + 5\right)}\right) \right.\right. \left.\left. + c_1\right)$$

2.1.86 Problem 88

Internal problem ID [9258]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 88
Date solved : `Monday, January 27, 2025 at 06:00:39 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$18x^2(1+x)\,y''+3x\left(x^2+11x+5\right)y'-\left(-5x^2-2x+1\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.417 (sec)

Writing the ode as

$$\left(18x^3+18x^2\right)y''+\left(3x^3+33x^2+15x\right)y'+\left(5x^2+2x-1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 18x^3+18x^2 \\ B &= 3x^3+33x^2+15x \\ C &= 5x^2+2x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{x^4-18x^3-45x^2-18x-27}{144\left(x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4-18x^3-45x^2-18x-27 \\ t &= 144\left(x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x^4-18x^3-45x^2-18x-27}{144\left(x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.156: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 144(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{144} - \frac{35}{144\,(1+x)^2} - \frac{3}{16x^2} + \frac{1}{4x} - \frac{7}{18\,(1+x)}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{12} - \frac{5}{6x} - \frac{53}{12x^2} - \frac{523}{12x^3} - \frac{6659}{12x^4} - \frac{94267}{12x^5} - \frac{1432421}{12x^6} - \frac{22802941}{12x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{12}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{12} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{144}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^4 - 18x^3 - 45x^2 - 18x - 27}{144x^4 + 288x^3 + 144x^2} \\ &= Q + \frac{R}{144x^4 + 288x^3 + 144x^2} \\ &= \left(\frac{1}{144}\right) + \left(\frac{-20x^3 - 46x^2 - 18x - 27}{144x^4 + 288x^3 + 144x^2}\right) \\ &= \frac{1}{144} + \frac{-20x^3 - 46x^2 - 18x - 27}{144x^4 + 288x^3 + 144x^2} \end{aligned}$$

Since the degree of t is 4, then we see that the coefficient of the term x^3 in the remainder R is -20. Dividing this by leading coefficient in t which is 144 gives $-\frac{5}{36}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{5}{36}\right) - (0) \\ &= -\frac{5}{36} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{12} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{5}{36}}{\frac{1}{12}} - 0\right) &= -\frac{5}{6} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{-\frac{5}{36}}{\frac{1}{12}} - 0\right) = \frac{5}{6}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^4 - 18x^3 - 45x^2 - 18x - 27}{144\left(x^2 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{7}{12}$	$\frac{5}{12}$
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{12}$	$-\frac{5}{6}$	$\frac{5}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{5}{6}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= \frac{5}{6} - \left(\frac{5}{6}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{7}{12\left(1 + x\right)} + \frac{1}{4x} + (-)\left(\frac{1}{12}\right) \\
&= \frac{7}{12\left(1 + x\right)} + \frac{1}{4x} - \frac{1}{12} \\
&= \frac{7}{12 + 12x} + \frac{1}{4x} - \frac{1}{12}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{7}{12\left(1+x\right)} + \frac{1}{4x} - \frac{1}{12}\right)(0) + \left(\left(-\frac{7}{12\left(1+x\right)^{2}} - \frac{1}{4x^{2}}\right) + \left(\frac{7}{12\left(1+x\right)} + \frac{1}{4x} - \frac{1}{12}\right)^{2} - \left(\frac{x^{4} - 18x^{3}}{1}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(\frac{7}{12(1+x)} + \frac{1}{4x} - \frac{1}{12}\right) dx} \\
&= x^{1/4} (1+x)^{7/12}\, \mathrm{e}^{-\frac{x}{12}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x^3+33x^2+15x}{18x^3+18x^2}\, dx} \\
&= z_1 e^{-\frac{x}{12} - \frac{5\ln(x)}{12} - \frac{5\ln(1+x)}{12}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{12}}}{x^{5/12}\left(1+x\right)^{5/12}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(1+x\right)^{1/6} \mathrm{e}^{-\frac{x}{6}}}{x^{1/6}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^3+33x^2+15x}{18x^3+18x^2}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{x}{6} - \frac{5\ln(x)}{6} - \frac{5\ln(1+x)}{6}}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{x}{6} - \frac{5\ln(x)}{6} - \frac{5\ln(1+x)}{6}} x^{1/3} \mathrm{e}^{\frac{x}{3}}}{\left(1+x\right)^{1/3}} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\left(1+x\right)^{1/6} \mathrm{e}^{-\frac{x}{6}}}{x^{1/6}}\right) + c_2 \left(\frac{\left(1+x\right)^{1/6} \mathrm{e}^{-\frac{x}{6}}}{x^{1/6}} \left(\int \frac{\mathrm{e}^{-\frac{x}{6} - \frac{5\ln(x)}{6} - \frac{5\ln(1+x)}{6}} x^{1/3} \mathrm{e}^{\frac{x}{3}}}{\left(1+x\right)^{1/3}} dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$18x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)+3x(x^2+11x+5)\left(\frac{d}{dx}y(x)\right)-(-5x^2-2x+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(5x^2+2x-1)y(x)}{18(x+1)x^2}-\frac{(x^2+11x+5)\left(\frac{d}{dx}y(x)\right)}{6x(x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{(x^2+11x+5)\left(\frac{d}{dx}y(x)\right)}{6x(x+1)}+\frac{(5x^2+2x-1)y(x)}{18(x+1)x^2}=0$

□ Check to see if x_0 is a regular singular point

  ○ Define functions

  $\left[P_2(x)=\frac{x^2+11x+5}{6x(x+1)},P_3(x)=\frac{5x^2+2x-1}{18(x+1)x^2}\right]$

  ○ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left.((x+1)\cdot P_2(x))\right|_{x=-1}=\frac{5}{6}$

  ○ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left.((x+1)^2\cdot P_3(x))\right|_{x=-1}=0$

  ○ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $18x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)+3x(x^2+11x+5)\left(\frac{d}{dx}y(x)\right)+(5x^2+2x-1)\,y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(18u^3-36u^2+18u)\left(\frac{d^2}{du^2}y(u)\right)+(3u^3+24u^2-42u+15)\left(\frac{d}{du}y(u)\right)+(5u^2-8u+2)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ○ Convert $u^m\cdot y(u)$ to series expansion for $m=0..2$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

  ○ Shift index using $k->k-m$

  $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

  ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..3$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  ○ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  ○ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  ○ Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$3a_0r(-1+6r)\,u^{-1+r} + (3a_1(1+r)(5+6r) - 2a_0(1+3r)(-1+6r))\,u^r + (3a_2(2+r)(11+6r) - 2$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $3r(-1+6r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{0, \frac{1}{6}\right\}$
- The coefficients of each power of u must be 0
  $[3a_1(1+r)(5+6r) - 2a_0(1+3r)(-1+6r) = 0, 3a_2(2+r)(11+6r) - 2a_1(4+3r)(5+6r) + 2a_0$
- Solve for the dependent coefficient(s)
  $\left\{a_1 = \frac{2a_0(18r^2+3r-1)}{3(6r^2+11r+5)}, a_2 = \frac{2a_0(81r^3+126r^2+21r+4)}{9(6r^3+29r^2+45r+22)}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $18(-2a_k + a_{k-1} + a_{k+1})\,k^2 + 3(12(-2a_k + a_{k-1} + a_{k+1})\,r - 2a_k + a_{k-2} - 10a_{k-1} + 11a_{k+1})\,k + 18($
- Shift index using $k- >k+2$
  $18(-2a_{k+2} + a_{k+1} + a_{k+3})\,(k+2)^2 + 3(12(-2a_{k+2} + a_{k+1} + a_{k+3})\,r - 2a_{k+2} + a_k - 10a_{k+1} + 11a_{k+}$
- Recursion relation that defines series solution to ODE
  $a_{k+3} = -\frac{18k^2a_{k+1}-36k^2a_{k+2}+36kra_{k+1}-72kra_{k+2}+18r^2a_{k+1}-36r^2a_{k+2}+3ka_k+42ka_{k+1}-150ka_{k+2}+3ra_k+42ra_{k+1}-150ra_{k+}}{3(6k^2+12kr+6r^2+35k+35r+51)}$
- Recursion relation for $r = 0$
  $a_{k+3} = -\frac{18k^2a_{k+1}-36k^2a_{k+2}+3ka_k+42ka_{k+1}-150ka_{k+2}+5a_k+16a_{k+1}-154a_{k+2}}{3(6k^2+35k+51)}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+3} = -\frac{18k^2a_{k+1}-36k^2a_{k+2}+3ka_k+42ka_{k+1}-150ka_{k+2}+5a_k+16a_{k+1}-154a_{k+2}}{3(6k^2+35k+51)}, a_1 = -\frac{2a_0}{15}, a_2 =\right.$
- Revert the change of variables $u = x+1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+3} = -\frac{18k^2a_{k+1}-36k^2a_{k+2}+3ka_k+42ka_{k+1}-150ka_{k+2}+5a_k+16a_{k+1}-154a_{k+2}}{3(6k^2+35k+51)}, a_1 = -\frac{2a}{15}\right.$
- Recursion relation for $r = \frac{1}{6}$
  $a_{k+3} = -\frac{18k^2a_{k+1}-36k^2a_{k+2}+3ka_k+48ka_{k+1}-162ka_{k+2}+\frac{11}{2}a_k+\frac{47}{2}a_{k+1}-180a_{k+2}}{3(6k^2+37k+57)}$
- Solution for $r = \frac{1}{6}$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{6}}, a_{k+3} = -\frac{18k^2a_{k+1}-36k^2a_{k+2}+3ka_k+48ka_{k+1}-162ka_{k+2}+\frac{11}{2}a_k+\frac{47}{2}a_{k+1}-180a_{k+2}}{3(6k^2+37k+57)}, a_1 = 0, a_2 =\right.$
- Revert the change of variables $u = x+1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^{k+\frac{1}{6}}, a_{k+3} = -\frac{18k^2a_{k+1}-36k^2a_{k+2}+3ka_k+48ka_{k+1}-162ka_{k+2}+\frac{11}{2}a_k+\frac{47}{2}a_{k+1}-180a_{k+2}}{3(6k^2+37k+57)}, a_1 =\right.$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k(x+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k(x+1)^{k+\frac{1}{6}}\right), a_{k+3} = -\frac{18k^2a_{k+1}-36k^2a_{k+2}+3ka_k+42ka_{k+1}-150ka_{k+2}+5}{3(6k^2+35k+51)}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @
      <- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0,
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.383 (sec)
Leaf size : 38

```
dsolve(18*x^2*(x+1)*diff(diff(y(x),x),x)+3*x*(x^2+11*x+5)*diff(y(x),x)-(-5*x^2-2*x+1)*
```

$$y = \frac{e^{-\frac{x}{6}} \left(\sqrt{x}\, \operatorname{HeunC}\left(\frac{1}{6}, \frac{1}{2}, -\frac{1}{6}, -\frac{5}{36}, \frac{1}{4}, -x\right) c_2 + \operatorname{HeunC}\left(\frac{1}{6}, -\frac{1}{2}, -\frac{1}{6}, -\frac{5}{36}, \frac{1}{4}, -x\right) c_1\right)}{x^{1/6}}$$

Mathematica DSolve solution

Solving time : 0.368 (sec)
Leaf size : 118

```
DSolve[{18*x^2*(1+x)*D[y[x],{x,2}]+3*x*(5+11*x+x^2)*D[y[x],x]-(1-2*x-5*x^2)*y[x]==0,{}},y[x]
```

$$y(x) \to \exp\left(\int_1^x \frac{1}{12}\left(\frac{7}{K[1]+1} - 1 + \frac{3}{K[1]}\right) dK[1] - \frac{1}{2}\int_1^x \frac{1}{6}\left(\frac{5}{K[2]+1} + 1 + \frac{5}{K[2]}\right) dK[2]\right)\left(c_2 \int_1^x \exp\left(-2\int_1^{K[3]} \frac{-K[1]^2 + 9K[1] + 3}{12K[1]^2 + 12K[1]} dK[1]\right) dK[3] + c_1\right)$$

2.1.87 Problem 89

Internal problem ID [9259]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 89
Date solved : `Monday, January 27, 2025 at 06:00:40 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2y'' + x(3+2x)\,y' - (1-x)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.248 (sec)

Writing the ode as

$$2x^2y'' + \left(2x^2+3x\right)y' + (x-1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= 2x^2+3x \\ C &= x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2+4x+5}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2+4x+5 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2+4x+5}{16x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.158: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{5}{16x^2} + \frac{1}{4x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{4x} + \frac{1}{4x^2} - \frac{1}{8x^3} + \frac{1}{16x^5} - \frac{3}{64x^6} - \frac{1}{128x^7} + \frac{11}{256x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 + 4x + 5}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{4x + 5}{16x^2}\right) \\ &= \frac{1}{4} + \frac{4x + 5}{16x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 16 gives $\frac{1}{4}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{4}\right) - (0) \\ &= \frac{1}{4} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{4}}{\frac{1}{2}} - 0\right) &&= \frac{1}{4} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{4}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^2 + 4x + 5}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{4} - \left(-\frac{1}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{4x} + (-)\left(\frac{1}{2}\right) \\ &= -\frac{1}{4x} - \frac{1}{2} \\ &= -\frac{1}{4x} - \frac{1}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{4x} - \frac{1}{2}\right)(0) + \left(\left(\frac{1}{4x^2}\right) + \left(-\frac{1}{4x} - \frac{1}{2}\right)^2 - \left(\frac{4x^2+4x+5}{16x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{4x} - \frac{1}{2}\right) dx} \\ &= \frac{e^{-\frac{x}{2}}}{x^{1/4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^2+3x}{2x^2}\,dx} \\
&= z_1 e^{-\frac{x}{2}-\frac{3\ln(x)}{4}} \\
&= z_1 \left(\frac{e^{-\frac{x}{2}}}{x^{3/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{e^{-x}}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2+3x}{2x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x-\frac{3\ln(x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\sqrt{x}\,e^{x} - \frac{\sqrt{\pi}\,\operatorname{erfi}\left(\sqrt{x}\right)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{-x}}{x}\right) + c_2 \left(\frac{e^{-x}}{x}\left(\sqrt{x}\,e^{x} - \frac{\sqrt{\pi}\,\operatorname{erfi}\left(\sqrt{x}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(2x+3)\left(\frac{d}{dx}y(x)\right) - (1-x)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(x-1)y(x)}{2x^2} - \frac{(2x+3)\left(\frac{d}{dx}y(x)\right)}{2x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(2x+3)\left(\frac{d}{dx}y(x)\right)}{2x} + \frac{(x-1)y(x)}{2x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2x+3}{2x}, P_3(x) = \frac{x-1}{2x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = \frac{3}{2}$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{1}{2}$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- • Multiply by denominators

  $2x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(2x+3)\left(\frac{d}{dx}y(x)\right) + (x-1)\,y(x) = 0$

- • Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - ◦ Shift index using $k- >k - m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - ◦ Shift index using $k- >k + 1 - m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - ◦ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-1+2r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(2k+2r-1) + a_{k-1}(2k+2r-1)\right) x^{k+r}\right) = 0$

- • $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-1+2r) = 0$

- • Values of r that satisfy the indicial equation

  $r \in \left\{-1, \frac{1}{2}\right\}$

- • Each term in the series must be 0, giving the recursion relation

  $2\left(k+r-\frac{1}{2}\right)\left(a_k(k+r+1) + a_{k-1}\right) = 0$

- • Shift index using $k- >k + 1$

  $2\left(k+\frac{1}{2}+r\right)\left(a_{k+1}(k+2+r) + a_k\right) = 0$

- • Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{k+2+r}$

- • Recursion relation for $r = -1$

  $a_{k+1} = -\frac{a_k}{k+1}$

- • Solution for $r = -1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = -\frac{a_k}{k+1}\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = -\frac{a_k}{k+\frac{5}{2}}$

- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{a_k}{k+\frac{5}{2}}\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = -\frac{a_k}{k+1}, b_{k+1} = -\frac{b_k}{k+\frac{5}{2}}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Whittaker successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returnin
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 1.849 (sec)
Leaf size : 52

```
dsolve(2*x^2*diff(diff(y(x),x),x)+x*(2*x+3)*diff(y(x),x)-(1-x)*y(x) = 0,y(x),singsol=all
```

$$y = -\frac{3\left(2c_1(-x)^{3/2} + \mathrm{e}^{-x}\left(xc_1\sqrt{\pi}\,\mathrm{erf}\left(\sqrt{-x}\right) - \frac{4c_2\sqrt{x}\,\sqrt{-x}}{3}\right)\right)}{4\sqrt{-x}\,x^{3/2}}$$

Mathematica DSolve solution

Solving time : 0.03 (sec)
Leaf size : 33

```
DSolve[{2*x^2*D[y[x],{x,2}]+x*(3+2*x)*D[y[x],x]-(1-x)*y[x]==0,{}},y[x],x,IncludeSingularSolu
```

$$y(x) \to \frac{e^{-x}\left(c_2 x^{3/2} L_{-\frac{3}{2}}^{\frac{3}{2}}(x) + c_1\right)}{x}$$

2.1.88 Problem 90

Internal problem ID [9260]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 90
Date solved : `Monday, January 27, 2025 at 06:00:40 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2y'' + x(5+x)\,y' - (2-3x)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.268 (sec)

Writing the ode as

$$2x^2y'' + \left(x^2+5x\right)y' + (3x-2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= x^2+5x \\ C &= 3x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2-14x+21}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2-14x+21 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2-14x+21}{16x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.160: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16} + \frac{21}{16x^2} - \frac{7}{8x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{4} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \qquad (8) \end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{4} - \frac{7}{4x} - \frac{7}{2x^2} - \frac{49}{2x^3} - \frac{196}{x^4} - \frac{1715}{x^5} - \frac{31899}{2x^6} - \frac{309729}{2x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{4}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{16}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 14x + 21}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{1}{16}\right) + \left(\frac{-14x + 21}{16x^2}\right) \\ &= \frac{1}{16} + \frac{-14x + 21}{16x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -14. Dividing this by leading coefficient in t which is 16 gives $-\frac{7}{8}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{7}{8}\right) - (0) \\ &= -\frac{7}{8}\end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{4} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{7}{8}}{\frac{1}{4}} - 0\right) &&= -\frac{7}{4} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{7}{8}}{\frac{1}{4}} - 0\right) &&= \frac{7}{4}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 14x + 21}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{4}$	$-\frac{7}{4}$	$\frac{7}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{7}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^+) \\ &= \frac{7}{4} - \left(\frac{7}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{7}{4x} + (-)\left(\frac{1}{4}\right) \\ &= \frac{7}{4x} - \frac{1}{4} \\ &= -\frac{x-7}{4x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{7}{4x} - \frac{1}{4}\right)(0) + \left(\left(-\frac{7}{4x^2}\right) + \left(\frac{7}{4x} - \frac{1}{4}\right)^2 - \left(\frac{x^2 - 14x + 21}{16x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{7}{4x} - \frac{1}{4}\right) dx} \\ &= x^{7/4} e^{-\frac{x}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^2+5x}{2x^2}\,dx} \\ &= z_1 e^{-\frac{x}{4}-\frac{5\ln(x)}{4}} \\ &= z_1\left(\frac{\mathrm{e}^{-\frac{x}{4}}}{x^{5/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}\,\mathrm{e}^{-\frac{x}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+5x}{2x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\frac{x}{2}-\frac{5\ln(x)}{2}}}{(y_1)^2}\,dx \\ &= y_1\left(-\frac{2\,\mathrm{e}^{\frac{x}{2}}}{5x^{5/2}} - \frac{2\,\mathrm{e}^{\frac{x}{2}}}{15x^{3/2}} - \frac{2\,\mathrm{e}^{\frac{x}{2}}}{15\sqrt{x}} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,\sqrt{x}}{2}\right)}{15}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1y_1 + c_2y_2 \\ &= c_1\left(\sqrt{x}\,\mathrm{e}^{-\frac{x}{2}}\right) + c_2\left(\sqrt{x}\,\mathrm{e}^{-\frac{x}{2}}\left(-\frac{2\,\mathrm{e}^{\frac{x}{2}}}{5x^{5/2}} - \frac{2\,\mathrm{e}^{\frac{x}{2}}}{15x^{3/2}} - \frac{2\,\mathrm{e}^{\frac{x}{2}}}{15\sqrt{x}} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,\sqrt{x}}{2}\right)}{15}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(5+x)\left(\frac{d}{dx}y(x)\right) - (-3x+2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(3x-2)y(x)}{2x^2} - \frac{(5+x)\left(\frac{d}{dx}y(x)\right)}{2x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(5+x)\left(\frac{d}{dx}y(x)\right)}{2x} + \frac{(3x-2)y(x)}{2x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x)=\frac{5+x}{2x}, P_3(x)=\frac{3x-2}{2x^2}\right]$

- $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=\frac{5}{2}$

- $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=-1$

- $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2x^2\left(\frac{d^2}{dx^2}y(x)\right)+x(5+x)\left(\frac{d}{dx}y(x)\right)+(3x-2)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

  $x^m\cdot y(x)=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m\cdot y(x)=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)\,(-1+2r)\,x^r+\left(\sum_{k=1}^{\infty}\left(a_k(k+r+2)\,(2k+2r-1)+a_{k-1}(k+r+2)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(-1+2r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{-2,\frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(\left(k+r-\frac{1}{2}\right)a_k+\frac{a_{k-1}}{2}\right)(k+r+2)=0$

- Shift index using $k->k+1$

  $2\left(\left(k+\frac{1}{2}+r\right)a_{k+1}+\frac{a_k}{2}\right)(k+r+3)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{a_k}{2k+1+2r}$

- Recursion relation for $r=-2$

  $a_{k+1}=-\frac{a_k}{2k-3}$

- Solution for $r=-2$

$$\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+1}=-\frac{a_k}{2k-3}\right]$$

- Recursion relation for $r=\frac{1}{2}$

  $a_{k+1}=-\frac{a_k}{2k+2}$

- Solution for $r=\frac{1}{2}$

$$\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1}=-\frac{a_k}{2k+2}\right]$$

- Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^{k-2}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1}=-\frac{a_k}{2k-3}, b_{k+1}=-\frac{b_k}{2k+2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 52

```
dsolve(2*x^2*diff(diff(y(x),x),x)+x*(x+5)*diff(y(x),x)-(2-3*x)*y(x) = 0,y(x),singsol=all
```

$$y=\frac{i\mathrm{e}^{-\frac{x}{2}} x^{5/2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,\sqrt{x}}{2}\right)\sqrt{\pi}\,c_2+c_1 x^{5/2}\mathrm{e}^{-\frac{x}{2}}+2c_2(x^2+x+3)}{x^2}$$

Mathematica DSolve solution

Solving time : 0.381 (sec)
Leaf size : 94

```
DSolve[{2*x^2*D[y[x],{x,2}]+x*(5+x)*D[y[x],x]-(2-3*x)*y[x]==0,{}},y[x],x,IncludeSingularSolu
```

$$y(x) \to \frac{e^{-\frac{x}{2}-\frac{5}{4}}\left(15c_1x^{5/2}-2c_2e^{x/2}x^2-6c_2e^{x/2}-2c_2e^{x/2}x+\sqrt{2}c_2(-x)^{5/2}\Gamma\left(\frac{1}{2},-\frac{x}{2}\right)\right)}{15x^2}$$

2.1.89 Problem 91

Internal problem ID [9261]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 91
Date solved : `Monday, January 27, 2025 at 06:00:41 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$3x^2y'' + x(1+x)\,y' - y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.427 (sec)

Writing the ode as

$$3x^2y'' + \left(x^2+x\right)y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^2 \\ B &= x^2+x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2+2x+7}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2+2x+7 \\ t &= 36x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2+2x+7}{36x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.162: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{36} + \frac{7}{36x^2} + \frac{1}{18x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{6} + \frac{1}{6x} + \frac{1}{2x^2} - \frac{1}{2x^3} - \frac{1}{4x^4} + \frac{7}{4x^5} - \frac{7}{4x^6} - \frac{17}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{6}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{36}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 + 2x + 7}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{1}{36}\right) + \left(\frac{2x + 7}{36x^2}\right) \\ &= \frac{1}{36} + \frac{2x + 7}{36x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 2. Dividing this by leading coefficient in t which is 36 gives $\frac{1}{18}$. Now b can be found.

$$\begin{aligned}b &= \left(\frac{1}{18}\right) - (0) \\ &= \frac{1}{18}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{6} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{1}{18}}{\frac{1}{6}} - 0\right) &= \frac{1}{6} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{1}{18}}{\frac{1}{6}} - 0\right) &= -\frac{1}{6}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 2x + 7}{36x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{6}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{6} - \left(-\frac{1}{6}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{6x} + (-)\left(\frac{1}{6}\right) \\ &= -\frac{1}{6x} - \frac{1}{6} \\ &= -\frac{1+x}{6x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered} (0) + 2\left(-\frac{1}{6x} - \frac{1}{6}\right)(0) + \left(\left(\frac{1}{6x^2}\right) + \left(-\frac{1}{6x} - \frac{1}{6}\right)^2 - \left(\frac{x^2+2x+7}{36x^2}\right) \right) = 0 \\ 0 = 0 \end{gathered}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(-\frac{1}{6x} - \frac{1}{6}\right) dx} \\ &= \frac{e^{-\frac{x}{6}}}{x^{1/6}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2+x}{3x^2}\,dx} \\
&= z_1 e^{-\frac{x}{6}-\frac{\ln(x)}{6}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{6}}}{x^{1/6}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-\frac{x}{3}}}{x^{1/3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+x}{3x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x}{3}-\frac{\ln(x)}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \mathrm{e}^{-\frac{x}{3}-\frac{\ln(x)}{3}} x^{2/3} \mathrm{e}^{\frac{2x}{3}}\,dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{-\frac{x}{3}}}{x^{1/3}}\right) + c_2 \left(\frac{\mathrm{e}^{-\frac{x}{3}}}{x^{1/3}} \left(\int \mathrm{e}^{-\frac{x}{3}-\frac{\ln(x)}{3}} x^{2/3} \mathrm{e}^{\frac{2x}{3}}\,dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$3x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(x+1)\left(\frac{d}{dx}y(x)\right) - y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{y(x)}{3x^2} - \frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{3x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{3x} - \frac{y(x)}{3x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{x+1}{3x}, P_3(x) = -\frac{1}{3x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$\left(x\cdot P_2(x)\right)\bigg|_{x=0} = \frac{1}{3}$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0} = -\frac{1}{3}$

- ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $3x^2\left(\frac{d^2}{dx^2}y(x)\right)+x(x+1)\left(\frac{d}{dx}y(x)\right)-y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ◦ Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+3r)\,(-1+r)\,x^r+\left(\sum_{k=1}^{\infty}\left(a_k(3k+3r+1)\,(k+r-1)+a_{k-1}(k+r-1)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+3r)\,(-1+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{1,-\frac{1}{3}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $3\left(\left(k+r+\frac{1}{3}\right)a_k+\frac{a_{k-1}}{3}\right)(k+r-1)=0$

- Shift index using $k->k+1$

  $3\left(\left(k+\frac{4}{3}+r\right)a_{k+1}+\frac{a_k}{3}\right)(k+r)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{a_k}{3k+4+3r}$

- Recursion relation for $r=1$

  $a_{k+1}=-\frac{a_k}{3k+7}$

- Solution for $r=1$

  $\left[y(x)=\sum_{k=0}^{\infty}a_k x^{k+1}, a_{k+1}=-\frac{a_k}{3k+7}\right]$

- Recursion relation for $r=-\frac{1}{3}$

  $a_{k+1}=-\frac{a_k}{3k+3}$

- Solution for $r=-\frac{1}{3}$

  $\left[y(x)=\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{3}}, a_{k+1}=-\frac{a_k}{3k+3}\right]$

- Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty}a_k x^{k+1}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k-\frac{1}{3}}\right), a_{k+1}=-\frac{a_k}{3k+7}, b_{k+1}=-\frac{b_k}{3k+3}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Whittaker successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returnin
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.085 (sec)
Leaf size : 30

```
dsolve(3*x^2*diff(diff(y(x),x),x)+x*(x+1)*diff(y(x),x)-y(x) = 0,y(x),singsol=all)
```

$$y=\frac{\mathrm{e}^{-\frac{x}{6}}\left(x^{1/6}\,\mathrm{WhittakerM}\left(-\frac{1}{6},\frac{2}{3},\frac{x}{3}\right)c_1+\mathrm{e}^{-\frac{x}{6}}c_2\right)}{x^{1/3}}$$

Mathematica DSolve solution

Solving time : 0.075 (sec)
Leaf size : 56

```
DSolve[{3*x^2*D[y[x],{x,2}]+x*(1+x)*D[y[x],x]-y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tru
```

$$y(x)\to\frac{e^{\frac{1}{3}(-x-1)}\left(c_2x^{2/3}-3\sqrt[3]{3}ec_1(-x)^{2/3}\Gamma\left(\frac{4}{3},-\frac{x}{3}\right)\right)}{x}$$

2.1.90 Problem 92

Internal problem ID [9262]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 92
Date solved : `Monday, January 27, 2025 at 06:00:42 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2y'' - xy' + (1-2x)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.169 (sec)

Writing the ode as

$$2x^2y'' - xy' + (1-2x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= -x \\ C &= 1-2x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3+16x}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3+16x \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3+16x}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.164: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2} + \frac{1}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1, \; e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(1 - (1)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{1}{(x-(0))}\right) \\ &= \frac{1}{2x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2x} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2x} + \frac{1-16x}{16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{1+4\sqrt{x}}{4x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= e^{\int \omega \, dx} \\ &= e^{\int \frac{1+4\sqrt{x}}{4x} dx} \\ &= x^{1/4} e^{2\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{2x^2}\,dx} \\
&= z_1 e^{\frac{\ln(x)}{4}} \\
&= z_1\left(x^{1/4}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}\,\mathrm{e}^{2\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{2x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{\ln(x)}{2}}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{\mathrm{e}^{-4\sqrt{x}}}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x}\,\mathrm{e}^{2\sqrt{x}}\right) + c_2\left(\sqrt{x}\,\mathrm{e}^{2\sqrt{x}}\left(-\frac{\mathrm{e}^{-4\sqrt{x}}}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$2x^2\left(\frac{d^2}{dx^2}y(x)\right) - x\left(\frac{d}{dx}y(x)\right) + (-2x+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = \frac{(2x-1)y(x)}{2x^2} + \frac{\frac{d}{dx}y(x)}{2x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) - \frac{\frac{d}{dx}y(x)}{2x} - \frac{(2x-1)y(x)}{2x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{1}{2x}, P_3(x) = -\frac{2x-1}{2x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -\frac{1}{2}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{1}{2}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x^2\left(\frac{d^2}{dx^2}y(x)\right) - x\left(\frac{d}{dx}y(x)\right) + (-2x+1)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- Rewrite ODE with series expansions

  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k-m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

  - Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)\,(k+r-1) - 2a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)\,(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{1, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k+r-\frac{1}{2}\right)(k+r-1)\, a_k - 2a_{k-1} = 0$

- Shift index using $k- >k+1$

  $2\left(k+\frac{1}{2}+r\right)(k+r)\, a_{k+1} - 2a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k}{(2k+1+2r)(k+r)}$

- Recursion relation for $r = 1$

  $a_{k+1} = \frac{2a_k}{(2k+3)(k+1)}$

- Solution for $r = 1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = \frac{2a_k}{(2k+3)(k+1)}\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = \frac{2a_k}{(2k+2)\left(k+\frac{1}{2}\right)}$

- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{2a_k}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$

- Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^{k+1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1}=\frac{2a_k}{(2k+3)(k+1)}, b_{k+1}=\frac{2b_k}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 25

```
dsolve(2*x^2*diff(diff(y(x),x),x)-diff(y(x),x)*x+(1-2*x)*y(x) = 0,y(x),singsol=all)
```

$$y=\sqrt{x}\left(c_1 \sinh\left(2\sqrt{x}\right)+c_2 \cosh\left(2\sqrt{x}\right)\right)$$

Mathematica DSolve solution

Solving time : 0.053 (sec)
Leaf size : 41

```
DSolve[{2*x^2*D[y[x],{x,2}]-x*D[y[x],x]+(1-2*x)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x)\to \frac{1}{2}e^{-2\sqrt{x}}\sqrt{x}\left(2c_1e^{4\sqrt{x}}-c_2\right)$$

2.1.91 Problem 93

Internal problem ID [9263]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 93
Date solved : `Monday, January 27, 2025 at 06:00:42 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$3x^2y'' + x(1+x)\,y' - (1+3x)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.584 (sec)

Writing the ode as

$$3x^2y'' + \left(x^2+x\right)y' + (-3x-1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^2 \\ B &= x^2+x \\ C &= -3x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2+38x+7}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2+38x+7 \\ t &= 36x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2+38x+7}{36x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.166: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{36} + \frac{7}{36x^2} + \frac{19}{18x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{6} + \frac{19}{6x} - \frac{59}{2x^2} + \frac{1121}{2x^3} - \frac{53041}{4x^4} + \frac{1404613}{4x^5} - \frac{39845827}{4x^6} + \frac{1184064097}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{6} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{36}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 38x + 7}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{1}{36}\right) + \left(\frac{38x + 7}{36x^2}\right) \\ &= \frac{1}{36} + \frac{38x + 7}{36x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 38. Dividing this by leading coefficient in t which is 36 gives $\frac{19}{18}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{19}{18}\right) - (0) \\ &= \frac{19}{18} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{6} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{19}{18}}{\frac{1}{6}} - 0\right) &&= \frac{19}{6} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{19}{18}}{\frac{1}{6}} - 0\right) &&= -\frac{19}{6} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 38x + 7}{36x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{6}$	$\frac{19}{6}$	$-\frac{19}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{19}{6}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^+) \\ &= \frac{19}{6} - \left(\frac{7}{6}\right) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{7}{6x} + \left(\frac{1}{6}\right) \\ &= \frac{7}{6x} + \frac{1}{6} \\ &= \frac{7+x}{6x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered} (2) + 2\left(\frac{7}{6x} + \frac{1}{6}\right)(2x + a_1) + \left(\left(-\frac{7}{6x^2}\right) + \left(\frac{7}{6x} + \frac{1}{6}\right)^2 - \left(\frac{x^2 + 38x + 7}{36x^2}\right)\right) = 0 \\ \frac{(-a_1 + 20)\, x - 2a_0 + 7a_1}{3x} = 0 \end{gathered}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 70, a_1 = 20\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + 20x + 70$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2+20x+70\right) \mathrm{e}^{\int \left(\frac{7}{6x}+\frac{1}{6}\right)dx} \\
&= \left(x^2+20x+70\right) \mathrm{e}^{\frac{x}{6}+\frac{7\ln(x)}{6}} \\
&= \left(x^2+20x+70\right) x^{7/6} \mathrm{e}^{\frac{x}{6}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2+x}{3x^2}\, dx} \\
&= z_1 e^{-\frac{x}{6}-\frac{\ln(x)}{6}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{6}}}{x^{1/6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \left(x^2+20x+70\right) x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+x}{3x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{x}{3}-\frac{\ln(x)}{3}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{x}{3}-\frac{\ln(x)}{3}}}{\left(x^2+20x+70\right)^2 x^2} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\left(x^2+20x+70\right) x\right) + c_2\left(\left(x^2+20x+70\right) x \left(\int \frac{\mathrm{e}^{-\frac{x}{3}-\frac{\ln(x)}{3}}}{\left(x^2+20x+70\right)^2 x^2} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$3x^2\left(\frac{d^2}{dx^2}y(x)\right)+x(x+1)\left(\frac{d}{dx}y(x)\right)-(3x+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{(3x+1)y(x)}{3x^2}-\frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{3x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{3x}-\frac{(3x+1)y(x)}{3x^2}=0$

☐ Check to see if $x_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{x+1}{3x}, P_3(x)=-\frac{3x+1}{3x^2}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{1}{3}$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=-\frac{1}{3}$

  - $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $3x^2\left(\frac{d^2}{dx^2}y(x)\right)+x(x+1)\left(\frac{d}{dx}y(x)\right)+(-3x-1)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

    $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+3r)\,(-1+r)\,x^r+\left(\sum_{k=1}^{\infty}\left(a_k(3k+3r+1)\,(k+r-1)+a_{k-1}(k-4+r)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(1+3r)(-1+r) = 0$

- Values of r that satisfy the indicial equation
  $r \in \left\{1, -\frac{1}{3}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $3\left(k+r+\frac{1}{3}\right)(k+r-1)\, a_k + a_{k-1}(k-4+r) = 0$
- Shift index using $k->k+1$
  $3\left(k+\frac{4}{3}+r\right)(k+r)\, a_{k+1} + a_k(k+r-3) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = -\frac{a_k(k+r-3)}{(3k+4+3r)(k+r)}$
- Recursion relation for $r=1$; series terminates at $k=2$
  $a_{k+1} = -\frac{a_k(k-2)}{(3k+7)(k+1)}$
- Apply recursion relation for $k=0$
  $a_1 = \frac{2a_0}{7}$
- Apply recursion relation for $k=1$
  $a_2 = \frac{a_1}{20}$
- Express in terms of a_0
  $a_2 = \frac{a_0}{70}$
- Terminating series solution of the ODE for $r=1$. Use reduction of order to find the second li
  $y(x) = a_0 \cdot \left(1+\frac{2}{7}x+\frac{1}{70}x^2\right)$
- Recursion relation for $r=-\frac{1}{3}$
  $a_{k+1} = -\frac{a_k\left(k-\frac{10}{3}\right)}{(3k+3)\left(k-\frac{1}{3}\right)}$
- Solution for $r=-\frac{1}{3}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}, a_{k+1} = -\frac{a_k\left(k-\frac{10}{3}\right)}{(3k+3)\left(k-\frac{1}{3}\right)}\right]$
- Combine solutions and rename parameters
  $\left[y(x) = a_0 \cdot \left(1+\frac{2}{7}x+\frac{1}{70}x^2\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{3}}\right), b_{k+1} = -\frac{b_k\left(k-\frac{10}{3}\right)}{(3k+3)\left(k-\frac{1}{3}\right)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric sol
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
```

```
   <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   -> Trying to convert hypergeometric functions to elementary form...
   <- elementary form for at least one hypergeometric solution is achieved - returnin
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.049 (sec)
Leaf size : 41

```
dsolve(3*x^2*diff(diff(y(x),x),x)+x*(x+1)*diff(y(x),x)-(3*x+1)*y(x) = 0,y(x),singsol=all
```

$$y = \frac{c_2 \mathrm{e}^{-\frac{x}{3}} \text{hypergeom}\left([3], \left[-\frac{1}{3}\right], \frac{x}{3}\right) + 70c_1\left(x^{4/3} + \frac{2x^{7/3}}{7} + \frac{x^{10/3}}{70}\right)}{x^{1/3}}$$

Mathematica DSolve solution

Solving time : 2.192 (sec)
Leaf size : 60

```
DSolve[{3*x^2*D[y[x],{x,2}]+x*(1+x)*D[y[x],x]-(1+3*x)*y[x]==0,{}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to ex\left(x^2 + 20x + 70\right)\left(c_2 \int_1^x \frac{e^{-\frac{K[1]}{3}-\frac{7}{3}}}{K[1]^{7/3}\left(K[1]^2 + 20K[1] + 70\right)^2} dK[1] + c_1\right)$$

2.1.92 Problem 94

Internal problem ID [9264]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 94
Date solved : `Monday, January 27, 2025 at 06:00:44 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(3+x)\,y''+x(1+5x)\,y'+(1+x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.289 (sec)

Writing the ode as

$$\left(2x^3+6x^2\right)y''+\left(5x^2+x\right)y'+(1+x)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=2x^3+6x^2\\ B&=5x^2+x \\ C&=1+x \end{aligned}\tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned}\tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-3x^2-30x-35}{16\left(x^2+3x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=-3x^2-30x-35\\ t&=16\left(x^2+3x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-3x^2-30x-35}{16\left(x^2+3x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.168: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 + 3x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -3$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{108x} + \frac{5}{108\,(3 + x)} - \frac{35}{144x^2} + \frac{7}{36\,(3 + x)^2}$$

For the pole at $x = -3$ let b be the coefficient of $\frac{1}{(3+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{6} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{5}{12} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 - 30x - 35}{16\left(x^2 + 3x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x^2 - 30x - 35}{16\left(x^2 + 3x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-3	2	0	$\frac{7}{6}$	$-\frac{1}{6}$
0	2	0	$\frac{7}{12}$	$\frac{5}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{1}{4} - \left(\frac{1}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{6\,(3+x)} + \frac{5}{12x} + (-)\,(0) \\
&= -\frac{1}{6\,(3+x)} + \frac{5}{12x} \\
&= \frac{x+5}{4x\,(3+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{6(3+x)} + \frac{5}{12x}\right)(0) + \left(\left(\frac{1}{6(3+x)^2} - \frac{5}{12x^2}\right) + \left(-\frac{1}{6(3+x)} + \frac{5}{12x}\right)^2 - \left(\frac{-3x^2 - 30x - 35}{16(x^2+3x)^2}\right)\right)$$

$$0 =$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(-\frac{1}{6(3+x)} + \frac{5}{12x}\right) dx} \\
&= \frac{x^{5/12}}{(3+x)^{1/6}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{5x^2+x}{2x^3+6x^2}\, dx} \\
&= z_1 e^{-\frac{7\ln(3+x)}{6} - \frac{\ln(x)}{12}} \\
&= z_1 \left(\frac{1}{(3+x)^{7/6} x^{1/12}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/3}}{(3+x)^{4/3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5x^2+x}{2x^3+6x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{7\ln(3+x)}{3} - \frac{\ln(x)}{6}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{e^{-\frac{7\ln(3+x)}{3} - \frac{\ln(x)}{6}} (3+x)^{8/3}}{x^{2/3}} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{1/3}}{(3+x)^{4/3}}\right) + c_2 \left(\frac{x^{1/3}}{(3+x)^{4/3}} \left(\int \frac{e^{-\frac{7\ln(3+x)}{3} - \frac{\ln(x)}{6}} (3+x)^{8/3}}{x^{2/3}} dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2(x+3)\left(\frac{d^2}{dx^2}y(x)\right)+x(5x+1)\left(\frac{d}{dx}y(x)\right)+(x+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(x+1)y(x)}{2x^2(x+3)}-\frac{(5x+1)\left(\frac{d}{dx}y(x)\right)}{2x(x+3)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{(5x+1)\left(\frac{d}{dx}y(x)\right)}{2x(x+3)}+\frac{(x+1)y(x)}{2x^2(x+3)}=0$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x)=\frac{5x+1}{2x(x+3)},P_3(x)=\frac{x+1}{2x^2(x+3)}\right]$

  - $(x+3)\cdot P_2(x)$ is analytic at $x=-3$

    $\left((x+3)\cdot P_2(x)\right)\Big|_{x=-3}=\frac{7}{3}$

  - $(x+3)^2\cdot P_3(x)$ is analytic at $x=-3$

    $\left((x+3)^2\cdot P_3(x)\right)\Big|_{x=-3}=0$

  - $x=-3$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-3$

- Multiply by denominators

  $2x^2(x+3)\left(\frac{d^2}{dx^2}y(x)\right)+x(5x+1)\left(\frac{d}{dx}y(x)\right)+(x+1)\,y(x)=0$

- Change variables using $x=u-3$ so that the regular singular point is at $u=0$

  $(2u^3-12u^2+18u)\left(\frac{d^2}{du^2}y(u)\right)+(5u^2-29u+42)\left(\frac{d}{du}y(u)\right)+(u-2)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

    $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

  - Shift index using $k->k-m$

    $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$6a_0r(4+3r)\, u^{-1+r} + (6a_1(1+r)(7+3r) - a_0(12r^2+17r+2))\, u^r + \left(\sum_{k=1}^{\infty} (6a_{k+1}(k+r+1)(3k+\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $$6r(4+3r) = 0$$
- Values of r that satisfy the indicial equation
  $$r \in \left\{0, -\frac{4}{3}\right\}$$
- Each term must be 0
  $$6a_1(1+r)(7+3r) - a_0(12r^2+17r+2) = 0$$
- Each term in the series must be 0, giving the recursion relation
  $$2(-6a_k + a_{k-1} + 9a_{k+1})\, k^2 + (4(-6a_k + a_{k-1} + 9a_{k+1})\, r - 17a_k - a_{k-1} + 60a_{k+1})\, k + 2(-6a_k + a_{k-}$$
- Shift index using $k- >k+1$
  $$2(-6a_{k+1} + a_k + 9a_{k+2})\,(k+1)^2 + (4(-6a_{k+1} + a_k + 9a_{k+2})\, r - 17a_{k+1} - a_k + 60a_{k+2})\,(k+1) + 2$$
- Recursion relation that defines series solution to ODE
  $$a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} + 4kra_k - 24kra_{k+1} + 2r^2a_k - 12r^2a_{k+1} + 3ka_k - 41ka_{k+1} + 3ra_k - 41ra_{k+1} + a_k - 31a_{k+1}}{6(3k^2+6kr+3r^2+16k+16r+20)}$$
- Recursion relation for $r = 0$
  $$a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} + 3ka_k - 41ka_{k+1} + a_k - 31a_{k+1}}{6(3k^2+16k+20)}$$
- Solution for $r = 0$
  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} + 3ka_k - 41ka_{k+1} + a_k - 31a_{k+1}}{6(3k^2+16k+20)}, 42a_1 - 2a_0 = 0\right]$$
- Revert the change of variables $u = x + 3$
  $$\left[y(x) = \sum_{k=0}^{\infty} a_k (x+3)^k, a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} + 3ka_k - 41ka_{k+1} + a_k - 31a_{k+1}}{6(3k^2+16k+20)}, 42a_1 - 2a_0 = 0\right]$$
- Recursion relation for $r = -\frac{4}{3}$
  $$a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} - \frac{7}{3}ka_k - 9ka_{k+1} + \frac{5}{9}a_k + \frac{7}{3}a_{k+1}}{6(3k^2+8k+4)}$$
- Solution for $r = -\frac{4}{3}$
  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{4}{3}}, a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} - \frac{7}{3}ka_k - 9ka_{k+1} + \frac{5}{9}a_k + \frac{7}{3}a_{k+1}}{6(3k^2+8k+4)}, -6a_1 - \frac{2a_0}{3} = 0\right]$$
- Revert the change of variables $u = x + 3$
  $$\left[y(x) = \sum_{k=0}^{\infty} a_k (x+3)^{k-\frac{4}{3}}, a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} - \frac{7}{3}ka_k - 9ka_{k+1} + \frac{5}{9}a_k + \frac{7}{3}a_{k+1}}{6(3k^2+8k+4)}, -6a_1 - \frac{2a_0}{3} = 0\right]$$
- Combine solutions and rename parameters
  $$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k (x+3)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+3)^{k-\frac{4}{3}}\right), a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} + 3ka_k - 41ka_{k+1} + a_k - 31a_{k+1}}{6(3k^2+16k+20)}, 42\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
```

```
Group is reducible, not completely reducible
Solution has integrals. Trying a special function solution free of integrals...
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful
   -> Trying to convert hypergeometric functions to elementary form...
   <- elementary form for at least one hypergeometric solution is achieved - return
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.075 (sec)
Leaf size : 36

```
dsolve(2*x^2*(x+3)*diff(diff(y(x),x),x)+x*(5*x+1)*diff(y(x),x)+(x+1)*y(x) = 0,y(x),si
```

$$y = c_1\sqrt{x}\,\text{hypergeom}\left(\left[1, \frac{3}{2}\right], \left[\frac{7}{6}\right], -\frac{x}{3}\right) + \frac{c_2 x^{1/3}}{(x+3)\left(1+\frac{x}{3}\right)^{1/3}}$$

Mathematica DSolve solution

Solving time : 0.256 (sec)
Leaf size : 108

```
DSolve[{2*x^2*(3+x)*D[y[x],{x,2}]+x*(1+5*x)*D[y[x],x]+(1+x)*y[x]==0,{}},y[x],x,IncludeSingul
```

$$y(x) \to \exp\left(\int_1^x \frac{K[1]+5}{4K[1]^2+12K[1]}dK[1] - \frac{1}{2}\int_1^x \frac{5K[2]+1}{2K[2]^2+6K[2]}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{K[1]+5}{4K[1]^2+12K[1]}dK[1]\right)dK[3] + c_1\right)$$

2.1.93 Problem 95

Internal problem ID [9265]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 95
Date solved : `Monday, January 27, 2025 at 06:00:44 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(4+x)\,y''-x(1-3x)\,y'+y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.520 (sec)

Writing the ode as

$$x^2(4+x)\,y''+\left(3x^2-x\right)y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(4+x) \\ B &= 3x^2-x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{3x^2-6x-7}{4\left(x^2+4x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3x^2-6x-7 \\ t &= 4\left(x^2+4x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{3x^2-6x-7}{4\left(x^2+4x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.170: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + 4x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -4$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{128\,(4 + x)} - \frac{5}{128x} - \frac{7}{64x^2} + \frac{65}{64\,(4 + x)^2}$$

For the pole at $x = -4$ let b be the coefficient of $\frac{1}{(4+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{65}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{13}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{5}{8} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{8} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x^2 - 6x - 7}{4\left(x^2 + 4x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3x^2 - 6x - 7}{4\left(x^2 + 4x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-4	2	0	$\frac{13}{8}$	$-\frac{5}{8}$
0	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{5}{8\,(4+x)} + \frac{1}{8x} + (-)\,(0) \\
&= -\frac{5}{8\,(4+x)} + \frac{1}{8x} \\
&= -\frac{x-1}{2x\,(4+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{5}{8\,(4+x)} + \frac{1}{8x}\right)(0) + \left(\left(\frac{5}{8\,(4+x)^2} - \frac{1}{8x^2}\right) + \left(-\frac{5}{8\,(4+x)} + \frac{1}{8x}\right)^2 - \left(\frac{3x^2 - 6x - 7}{4\,(x^2+4x)^2}\right)\right) =$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(-\frac{5}{8(4+x)} + \frac{1}{8x}\right) dx} \\ &= \frac{x^{1/8}}{(4+x)^{5/8}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3x^2 - x}{x^2(4+x)}\, dx} \\ &= z_1 e^{\frac{\ln(x)}{8} - \frac{13\ln(4+x)}{8}} \\ &= z_1 \left(\frac{x^{1/8}}{(4+x)^{13/8}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/4}}{(4+x)^{9/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3x^2 - x}{x^2(4+x)}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\frac{\ln(x)}{4} - \frac{13\ln(4+x)}{4}}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{\frac{\ln(x)}{4} - \frac{13\ln(4+x)}{4}}(4+x)^{9/2}}{\sqrt{x}}\, dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{x^{1/4}}{(4+x)^{9/4}}\right) + c_2 \left(\frac{x^{1/4}}{(4+x)^{9/4}}\left(\int \frac{\mathrm{e}^{\frac{\ln(x)}{4} - \frac{13\ln(4+x)}{4}}(4+x)^{9/2}}{\sqrt{x}}\, dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x+4)\left(\frac{d^2}{dx^2}y(x)\right) - x(1-3x)\left(\frac{d}{dx}y(x)\right) + y(x) = 0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x) = -\frac{y(x)}{x^2(x+4)} - \frac{(3x-1)\left(\frac{d}{dx}y(x)\right)}{x(x+4)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x) + \frac{(3x-1)\left(\frac{d}{dx}y(x)\right)}{x(x+4)} + \frac{y(x)}{x^2(x+4)} = 0$

□ Check to see if x_0 is a regular singular point

- Define functions

$\left[P_2(x) = \frac{3x-1}{x(x+4)}, P_3(x) = \frac{1}{x^2(x+4)}\right]$

- $(x+4)\cdot P_2(x)$ is analytic at $x=-4$

$\left((x+4)\cdot P_2(x)\right)\bigg|_{x=-4} = \frac{13}{4}$

- $(x+4)^2\cdot P_3(x)$ is analytic at $x=-4$

$\left((x+4)^2\cdot P_3(x)\right)\bigg|_{x=-4} = 0$

- $x=-4$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = -4$

- Multiply by denominators

$x^2(x+4)\left(\frac{d^2}{dx^2}y(x)\right) + x(3x-1)\left(\frac{d}{dx}y(x)\right) + y(x) = 0$

- Change variables using $x = u-4$ so that the regular singular point is at $u=0$

$(u^3-8u^2+16u)\left(\frac{d^2}{du^2}y(u)\right) + (3u^2-25u+52)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k->k+1-m$

$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k->k+2-m$

$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

Rewrite ODE with series expansions

$4a_0r(9+4r)\,u^{-1+r} + (4a_1(1+r)(13+4r) - a_0(8r^2+17r-1))\,u^r + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+1+r)(4k+\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$4r(9+4r)=0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{9}{4}\right\}$$

- Each term must be 0

$$4a_1(1+r)(13+4r) - a_0(8r^2+17r-1) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(-8a_k + a_{k-1} + 16a_{k+1})k^2 + (2(-8a_k + a_{k-1} + 16a_{k+1})r - 17a_k + 68a_{k+1})k + (-8a_k + a_{k-1} +$$

- Shift index using $k- >k+1$

$$(-8a_{k+1} + a_k + 16a_{k+2})(k+1)^2 + (2(-8a_{k+1} + a_k + 16a_{k+2})r - 17a_{k+1} + 68a_{k+2})(k+1) + (-$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k^2a_k - 8k^2a_{k+1} + 2kra_k - 16kra_{k+1} + r^2a_k - 8r^2a_{k+1} + 2ka_k - 33ka_{k+1} + 2ra_k - 33ra_{k+1} - 24a_{k+1}}{4(4k^2+8kr+4r^2+25k+25r+34)}$$

- Recursion relation for $r=0$

$$a_{k+2} = -\frac{k^2a_k - 8k^2a_{k+1} + 2ka_k - 33ka_{k+1} - 24a_{k+1}}{4(4k^2+25k+34)}$$

- Solution for $r=0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2a_k - 8k^2a_{k+1} + 2ka_k - 33ka_{k+1} - 24a_{k+1}}{4(4k^2+25k+34)}, 52a_1 + a_0 = 0\right]$$

- Revert the change of variables $u = x+4$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k (x+4)^k, a_{k+2} = -\frac{k^2a_k - 8k^2a_{k+1} + 2ka_k - 33ka_{k+1} - 24a_{k+1}}{4(4k^2+25k+34)}, 52a_1 + a_0 = 0\right]$$

- Recursion relation for $r = -\frac{9}{4}$

$$a_{k+2} = -\frac{k^2a_k - 8k^2a_{k+1} - \frac{5}{2}ka_k + 3ka_{k+1} + \frac{9}{16}a_k + \frac{39}{4}a_{k+1}}{4(4k^2+7k-2)}$$

- Solution for $r = -\frac{9}{4}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{9}{4}}, a_{k+2} = -\frac{k^2a_k - 8k^2a_{k+1} - \frac{5}{2}ka_k + 3ka_{k+1} + \frac{9}{16}a_k + \frac{39}{4}a_{k+1}}{4(4k^2+7k-2)}, -20a_1 - \frac{5a_0}{4} = 0\right]$$

- Revert the change of variables $u = x+4$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k (x+4)^{k-\frac{9}{4}}, a_{k+2} = -\frac{k^2a_k - 8k^2a_{k+1} - \frac{5}{2}ka_k + 3ka_{k+1} + \frac{9}{16}a_k + \frac{39}{4}a_{k+1}}{4(4k^2+7k-2)}, -20a_1 - \frac{5a_0}{4} = 0\right]$$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k (x+4)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+4)^{k-\frac{9}{4}}\right), a_{k+2} = -\frac{k^2a_k - 8k^2a_{k+1} + 2ka_k - 33ka_{k+1} - 24a_{k+1}}{4(4k^2+25k+34)}, 52a_1$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
```

```
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returnin
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.112 (sec)
Leaf size : 27

```
dsolve(x^2*(x+4)*diff(diff(y(x),x),x)-x*(-3*x+1)*diff(y(x),x)+y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_1 x^{1/4}}{(x+4)^{9/4}} + c_2 \,\mathrm{hypergeom}\left([1,3]\,,\left[\frac{7}{4}\right],-\frac{x}{4}\right) x$$

Mathematica DSolve solution

Solving time : 0.219 (sec)
Leaf size : 109

```
DSolve[{x^2*(4+x)*D[y[x],{x,2}]-x*(1-3*x)*D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \exp\left(\int_1^x \frac{1-K[1]}{2K[1]^2+8K[1]}dK[1] - \frac{1}{2}\int_1^x \frac{3K[2]-1}{K[2](K[2]+4)}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{1-K[1]}{2K[1]^2+8K[1]}dK[1]\right)dK[3] + c_1\right)$$

2.1.94 Problem 96

Internal problem ID [9266]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 96
Date solved : `Monday, January 27, 2025 at 06:00:45 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2y'' + 5xy' + (1+x)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.228 (sec)

Writing the ode as

$$2x^2y'' + 5xy' + (1+x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= 5x \\ C &= 1+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3-8x}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3-8x \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3-8x}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.172: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{2x} - \frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1, \; e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(1 - (1)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{1}{(x-(0))}\right) \\ &= \frac{1}{2x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right) p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right) p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2x} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2x} + \frac{1+8x}{16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{1 + 2\sqrt{2}\,\sqrt{-x}}{4x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \frac{1+2\sqrt{2}\,\sqrt{-x}}{4x} dx} \\ &= x^{1/4} \mathrm{e}^{\sqrt{2}\,\sqrt{-x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{5x}{2x^2}\,dx} \\
&= z_1 e^{-\frac{5\ln(x)}{4}} \\
&= z_1 \left(\frac{1}{x^{5/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{\sqrt{2}\,\sqrt{-x}}}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5x}{2x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{5\ln(x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\sqrt{2}\,\sqrt{-x}\left(1-\mathrm{e}^{-2\sqrt{2}\,\sqrt{-x}}\right)}{2\sqrt{x}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{\sqrt{2}\,\sqrt{-x}}}{x}\right) + c_2 \left(\frac{\mathrm{e}^{\sqrt{2}\,\sqrt{-x}}}{x}\left(-\frac{\sqrt{2}\,\sqrt{-x}\left(1-\mathrm{e}^{-2\sqrt{2}\,\sqrt{-x}}\right)}{2\sqrt{x}}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2\left(\frac{d^2}{dx^2}y(x)\right) + 5x\left(\frac{d}{dx}y(x)\right) + (x+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(x+1)y(x)}{2x^2} - \frac{5\left(\frac{d}{dx}y(x)\right)}{2x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{5\left(\frac{d}{dx}y(x)\right)}{2x} + \frac{(x+1)y(x)}{2x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = \frac{5}{2x}, P_3(x) = \frac{x+1}{2x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{5}{2}$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{1}{2}$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x^2\left(\frac{d^2}{dx^2}y(x)\right) + 5x\left(\frac{d}{dx}y(x)\right) + (x+1)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - ◦ Shift index using $k- >k - m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - ◦ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r}$

  - ◦ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(1+2r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(2k+2r+1) + a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-1, -\frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2(k+r+1)\left(k+r+\frac{1}{2}\right)a_k + a_{k-1} = 0$

- Shift index using $k- >k + 1$

  $2(k+2+r)\left(k+\frac{3}{2}+r\right)a_{k+1} + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{(k+2+r)(2k+3+2r)}$

- Recursion relation for $r = -1$

  $a_{k+1} = -\frac{a_k}{(k+1)(2k+1)}$

- Solution for $r = -1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = -\frac{a_k}{(k+1)(2k+1)}\right]$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+1} = -\frac{a_k}{\left(k+\frac{3}{2}\right)(2k+2)}$

- Solution for $r = -\frac{1}{2}$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = -\frac{a_k}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{2}}\right), a_{k+1} = -\frac{a_k}{(k+1)(2k+1)}, b_{k+1} = -\frac{b_k}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 29

```
dsolve(2*x^2*diff(diff(y(x),x),x)+5*diff(y(x),x)*x+(x+1)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_1 \sin\left(\sqrt{x}\,\sqrt{2}\right) + c_2 \cos\left(\sqrt{x}\,\sqrt{2}\right)}{x}$$

Mathematica DSolve solution

Solving time : 0.085 (sec)
Leaf size : 60

```
DSolve[{2*x^2*D[y[x],{x,2}]+5*x*D[y[x],x]+(1+x)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to \frac{2c_1 e^{i\sqrt{2}\sqrt{x}} + i\sqrt{2}c_2 e^{-i\sqrt{2}\sqrt{x}}}{2x}$$

2.1.95 Problem 97

Internal problem ID [9267]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 97
Date solved : `Monday, January 27, 2025 at 06:00:46 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$6x^2y'' + x(10-x)\,y' - (2+x)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.375 (sec)

Writing the ode as

$$6x^2y'' + \left(-x^2+10x\right)y' + (-x-2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 6x^2 \\ B &= -x^2+10x \\ C &= -x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2+4x+28}{144x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2+4x+28 \\ t &= 144x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2+4x+28}{144x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.174: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 2\\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 144x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{144} + \frac{1}{36x} + \frac{7}{36x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{0} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{12} + \frac{1}{6x} + \frac{1}{x^2} - \frac{2}{x^3} - \frac{2}{x^4} + \frac{28}{x^5} - \frac{56}{x^6} - \frac{272}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{12}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{12}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{144}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 + 4x + 28}{144x^2} \\ &= Q + \frac{R}{144x^2} \\ &= \left(\frac{1}{144}\right) + \left(\frac{4x + 28}{144x^2}\right) \\ &= \frac{1}{144} + \frac{4x + 28}{144x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 144 gives $\frac{1}{36}$. Now b can be found.

$$\begin{aligned}b &= \left(\frac{1}{36}\right) - (0) \\ &= \frac{1}{36}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{12} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{36}}{\frac{1}{12}} - 0\right) &&= \frac{1}{6} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{36}}{\frac{1}{12}} - 0\right) &&= -\frac{1}{6}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 4x + 28}{144x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{12}$	$\frac{1}{6}$	$-\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{1}{6}$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= -\frac{1}{6} - \left(-\frac{1}{6}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{6x} + (-)\left(\frac{1}{12}\right) \\ &= -\frac{1}{6x} - \frac{1}{12} \\ &= -\frac{2+x}{12x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{6x} - \frac{1}{12}\right)(0) + \left(\left(\frac{1}{6x^2}\right) + \left(-\frac{1}{6x} - \frac{1}{12}\right)^2 - \left(\frac{x^2+4x+28}{144x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{6x} - \frac{1}{12}\right) dx} \\ &= \frac{e^{-\frac{x}{12}}}{x^{1/6}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2+10x}{6x^2}\,dx} \\
&= z_1 e^{\frac{x}{12}-\frac{5\ln(x)}{6}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{x}{12}}}{x^{5/6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+10x}{6x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{x}{6}-\frac{5\ln(x)}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \mathrm{e}^{\frac{x}{6}-\frac{5\ln(x)}{3}} x^2 dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x}\left(\int \mathrm{e}^{\frac{x}{6}-\frac{5\ln(x)}{3}} x^2 dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$6x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(10-x)\left(\frac{d}{dx}y(x)\right) - (x+2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{(x+2)y(x)}{6x^2} + \frac{(-10+x)\left(\frac{d}{dx}y(x)\right)}{6x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) - \frac{(-10+x)\left(\frac{d}{dx}y(x)\right)}{6x} - \frac{(x+2)y(x)}{6x^2} = 0$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{-10+x}{6x}, P_3(x) = -\frac{x+2}{6x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

$(x \cdot P_2(x))\bigg|_{x=0} = \frac{5}{3}$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{1}{3}$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- • Multiply by denominators

  $6x^2\left(\frac{d^2}{dx^2}y(x)\right) - x(-10 + x)\left(\frac{d}{dx}y(x)\right) + (-x - 2)\,y(x) = 0$

- • Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - ○ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - ○ Shift index using $k- >k - m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - ○ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k + r)\,x^{k+r-1+m}$

  - ○ Shift index using $k- >k + 1 - m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\,x^{k+r}$

  - ○ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $2a_0(1 + r)\,(-1 + 3r)\,x^r + \left(\sum_{k=1}^{\infty} (2a_k(k + r + 1)\,(3k + 3r - 1) - a_{k-1}(k + r))\,x^{k+r}\right) = 0$

- • $a_0$cannot be 0 by assumption, giving the indicial equation

  $2(1 + r)\,(-1 + 3r) = 0$

- • Values of r that satisfy the indicial equation

  $r \in \left\{-1, \frac{1}{3}\right\}$

- • Each term in the series must be 0, giving the recursion relation

  $6(k + r + 1)\left(k - \frac{1}{3} + r\right)a_k - a_{k-1}(k + r) = 0$

- • Shift index using $k- >k + 1$

  $6(k + 2 + r)\left(k + \frac{2}{3} + r\right)a_{k+1} - a_k(k + r + 1) = 0$

- • Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r+1)}{2(k+2+r)(3k+2+3r)}$

- • Recursion relation for $r = -1$

  $a_{k+1} = \frac{a_k k}{2(k+1)(3k-1)}$

- • Solution for $r = -1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = \frac{a_k k}{2(k+1)(3k-1)}\right]$

- • Recursion relation for $r = \frac{1}{3}$

$$a_{k+1} = \frac{a_k \left(k+\frac{4}{3}\right)}{2\left(k+\frac{7}{3}\right)\left(3k+3\right)}$$

- Solution for $r = \frac{1}{3}$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+1} = \frac{a_k \left(k+\frac{4}{3}\right)}{2\left(k+\frac{7}{3}\right)\left(3k+3\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+1} = \frac{a_k k}{2(k+1)(3k-1)}, b_{k+1} = \frac{b_k \left(k+\frac{4}{3}\right)}{2\left(k+\frac{7}{3}\right)\left(3k+3\right)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Whittaker successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - return
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.066 (sec)
Leaf size : 27

```
dsolve(6*x^2*diff(diff(y(x),x),x)+x*(10-x)*diff(y(x),x)-(x+2)*y(x) = 0,y(x),singsol=al
```

$$y = \frac{c_2 x^{5/6} + c_1 \operatorname{WhittakerM}\left(-\frac{1}{6}, \frac{2}{3}, \frac{x}{6}\right) e^{\frac{x}{12}} x}{x^{11/6}}$$

Mathematica DSolve solution

Solving time : 0.032 (sec)
Leaf size : 38

```
DSolve[{6*x^2*D[y[x],{x,2}]+x*(10-x)*D[y[x],x]-(2+x)*y[x]==0,{}},y[x],x,IncludeSingularSolut
```

$$y(x) \to c_2 \sqrt[3]{x} L_{-\frac{4}{3}}^{\frac{4}{3}}\left(\frac{x}{6}\right) + \frac{6\sqrt[3]{6}c_1}{x}$$

2.1.96 Problem 98

Internal problem ID [9268]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 98
Date solved : `Monday, January 27, 2025 at 06:00:47 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(3+4x)\,y''+x(11+4x)\,y'-(3+4x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.316 (sec)

Writing the ode as

$$\left(4x^3+3x^2\right)y''+\left(4x^2+11x\right)y'+(-3-4x)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^3+3x^2 \\ B &= 4x^2+11x \\ C &= -3-4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{48x^2+8x+91}{4\left(4x^2+3x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 48x^2+8x+91 \\ t &= 4\left(4x^2+3x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{48x^2+8x+91}{4\left(4x^2+3x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.176: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(4x^2 + 3x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{3}{4}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{28}{9\left(x + \frac{3}{4}\right)^2} + \frac{176}{27\left(x + \frac{3}{4}\right)} - \frac{176}{27x} + \frac{91}{36x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{91}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{13}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{7}{6} \end{aligned}$$

For the pole at $x = -\frac{3}{4}$ let b be the coefficient of $\frac{1}{\left(x+\frac{3}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{28}{9}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{3} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{4}{3} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{48x^2 + 8x + 91}{4\left(4x^2 + 3x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{48x^2 + 8x + 91}{4\left(4x^2 + 3x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{13}{6}$	$-\frac{7}{6}$
$-\frac{3}{4}$	2	0	$\frac{7}{3}$	$-\frac{4}{3}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{5}{2}\right) \\
&= 2
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{7}{6x} - \frac{4}{3\left(x+\frac{3}{4}\right)} + (-)(0) \\
&= -\frac{7}{6x} - \frac{4}{3\left(x+\frac{3}{4}\right)} \\
&= \frac{-7-20x}{8x^2+6x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(-\frac{7}{6x} - \frac{4}{3\left(x+\frac{3}{4}\right)}\right)(2x + a_1) + \left(\left(\frac{7}{6x^2} + \frac{4}{3\left(x+\frac{3}{4}\right)^2}\right) + \left(-\frac{7}{6x} - \frac{4}{3\left(x+\frac{3}{4}\right)}\right)^2 - \left(\frac{48x^2 + 8x}{4\left(4x^2 + 3\right.}\right.\right.$$

$$\frac{12a_1 x - 8x + 32a_0 -}{x(3+4x)}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{7}{48}, a_1 = \frac{2}{3}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + \frac{2}{3}x + \frac{7}{48}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 + \frac{2}{3}x + \frac{7}{48}\right) e^{\int \left(-\frac{7}{6x} - \frac{4}{3\left(x+\frac{3}{4}\right)}\right) dx} \\
&= \left(x^2 + \frac{2}{3}x + \frac{7}{48}\right) e^{-\frac{7\ln(x)}{6} - \frac{4\ln(3+4x)}{3}} \\
&= \frac{x^2 + \frac{2}{3}x + \frac{7}{48}}{x^{7/6}(3+4x)^{4/3}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x^2+11x}{4x^3+3x^2} \, dx} \\
&= z_1 e^{-\frac{11\ln(x)}{6} + \frac{4\ln(3+4x)}{3}} \\
&= z_1 \left(\frac{(3+4x)^{4/3}}{x^{11/6}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2 + \frac{2}{3}x + \frac{7}{48}}{x^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x^2+11x}{4x^3+3x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{11\ln(x)}{3}+\frac{8\ln(3+4x)}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{e^{-\frac{11\ln(x)}{3}+\frac{8\ln(3+4x)}{3}}x^6}{\left(x^2+\frac{2}{3}x+\frac{7}{48}\right)^2}\,dx \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^2+\frac{2}{3}x+\frac{7}{48}}{x^3} \right) + c_2 \left(\frac{x^2+\frac{2}{3}x+\frac{7}{48}}{x^3} \left(\int \frac{e^{-\frac{11\ln(x)}{3}+\frac{8\ln(3+4x)}{3}}x^6}{\left(x^2+\frac{2}{3}x+\frac{7}{48}\right)^2}\,dx \right) \right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2(3+4x)\left(\frac{d^2}{dx^2}y(x)\right)+x(11+4x)\left(\frac{d}{dx}y(x)\right)-(3+4x)\,y(x)=0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x)=\frac{y(x)}{x^2}-\frac{(11+4x)\left(\frac{d}{dx}y(x)\right)}{x(3+4x)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x)+\frac{(11+4x)\left(\frac{d}{dx}y(x)\right)}{x(3+4x)}-\frac{y(x)}{x^2}=0$
- □ Check to see if x_0 is a regular singular point
  - Define functions
    $\left[P_2(x)=\frac{11+4x}{x(3+4x)},P_3(x)=-\frac{1}{x^2}\right]$
  - $x\cdot P_2(x)$ is analytic at $x=0$
    $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{11}{3}$
  - $x^2\cdot P_3(x)$ is analytic at $x=0$
    $\left.(x^2\cdot P_3(x))\right|_{x=0}=-1$
  - $x=0$is a regular singular point
    Check to see if x_0 is a regular singular point
    $x_0=0$
- Multiply by denominators
  $x^2(3+4x)\left(\frac{d^2}{dx^2}y(x)\right)+x(11+4x)\left(\frac{d}{dx}y(x)\right)+(-3-4x)\,y(x)=0$
- Assume series solution for $y(x)$
  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$
- □ Rewrite ODE with series expansions
  - Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

$x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..3$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(3+r)\,(-1+3r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+3)\,(3k+3r-1) + 4a_{k-1}(k+r)\,(k-2+r)\right) x^{k+r}\right) =$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(3+r)\,(-1+3r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-3, \frac{1}{3}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $3\left(k-\frac{1}{3}+r\right)(k+r+3)\, a_k + 4a_{k-1}(k+r)\,(k-2+r) = 0$

- Shift index using $k- >k+1$

  $3\left(k+\frac{2}{3}+r\right)(k+4+r)\, a_{k+1} + 4a_k(k+r+1)\,(k+r-1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{4a_k(k+r+1)(k+r-1)}{(3k+2+3r)(k+4+r)}$

- Recursion relation for $r = -3$; series terminates at $k = 2$

  $a_{k+1} = -\frac{4a_k(k-2)(k-4)}{(3k-7)(k+1)}$

- Apply recursion relation for $k = 0$

  $a_1 = \frac{32a_0}{7}$

- Apply recursion relation for $k = 1$

  $a_2 = \frac{3a_1}{2}$

- Express in terms of a_0

  $a_2 = \frac{48a_0}{7}$

- Terminating series solution of the ODE for $r = -3$. Use reduction of order to find the second

  $y(x) = a_0 \cdot \left(\frac{48}{7}x^2 + \frac{32}{7}x + 1\right)$

- Recursion relation for $r = \frac{1}{3}$

  $a_{k+1} = -\frac{4a_k\left(k+\frac{4}{3}\right)\left(k-\frac{2}{3}\right)}{(3k+3)\left(k+\frac{13}{3}\right)}$

- Solution for $r = \frac{1}{3}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+1} = -\frac{4a_k\left(k+\frac{4}{3}\right)\left(k-\frac{2}{3}\right)}{(3k+3)\left(k+\frac{13}{3}\right)}\right]$

- Combine solutions and rename parameters

$$\left[y(x) = a_0 \cdot \left(\frac{48}{7}x^2 + \frac{32}{7}x + 1\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), b_{k+1} = -\frac{4b_k\left(k+\frac{4}{3}\right)\left(k-\frac{2}{3}\right)}{(3k+3)\left(k+\frac{13}{3}\right)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returnin
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.113 (sec)
Leaf size : 41

```
dsolve(x^2*(4*x+3)*diff(diff(y(x),x),x)+x*(11+4*x)*diff(y(x),x)-(4*x+3)*y(x) = 0,y(x),si
```

$$y = \frac{c_1 (48x^2 + 32x + 7)}{x^3} + c_2 \,\text{hypergeom}\left([3, 5], \left[\frac{13}{3}\right], -\frac{4x}{3}\right) (4x + 3)^{11/3} x^{1/3}$$

Mathematica DSolve solution

Solving time : 0.809 (sec)
Leaf size : 143

```
DSolve[{x^2*(3+4*x)*D[y[x],{x,2}]+x*(11+4*x)*D[y[x],x]-(3+4*x)*y[x]==0,{}},y[x],x,IncludeSin
```

$$y(x) \to \frac{1}{48}\left(48x^2+32x+7\right)\exp\left(\int_1^x -\frac{20K[1]+7}{8K[1]^2+6K[1]}dK[1] - \frac{1}{2}\int_1^x \frac{4K[2]+11}{4K[2]^2+3K[2]}dK[2]\right)\left(c_2\int_1^x \frac{2304\exp\left(-2\int_1^{K[3]} -\frac{20K[1]+7}{8K[1]^2+6K[1]}dK[1]\right)}{\left(48K[3]^2+32K[3]+7\right)^2}dK[3] + c_1\right)$$

2.1.97 Problem 99

Internal problem ID [9269]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 99
Date solved : `Monday, January 27, 2025 at 06:00:47 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(2+3x)\,y''+x(4+11x)\,y'-(1-x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.186 (sec)

Writing the ode as

$$\left(6x^3+4x^2\right)y''+\left(11x^2+4x\right)y'+(x-1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 6x^3+4x^2 \\ B &= 11x^2+4x \\ C &= x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-35}{16\,(2+3x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -35 \\ t &= 16(2+3x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{35}{16\,(2+3x)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.178: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(2+3x)^2$. There is a pole at $x = -\frac{2}{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{35}{144\left(x + \frac{2}{3}\right)^2}$$

For the pole at $x = -\frac{2}{3}$ let b be the coefficient of $\frac{1}{\left(x+\frac{2}{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{35}{16\,(2+3x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{35}{144}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{7}{12} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{5}{12}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=-\frac{35}{16\left(2+3x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$-\frac{2}{3}$	2	0	$\frac{7}{12}$	$\frac{5}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{7}{12}$	$\frac{5}{12}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{5}{12}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}\right) \\
&= \frac{5}{12}-\left(\frac{5}{12}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^{-}}{x-c_1}\right)+(-)[\sqrt{r}]_\infty \\
&= \frac{5}{12\left(x+\frac{2}{3}\right)}+(-)\left(0\right) \\
&= \frac{5}{12\left(x+\frac{2}{3}\right)} \\
&= \frac{5}{8+12x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}
$$

Let

$$
p(x)=1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{5}{12\left(x+\frac{2}{3}\right)}\right)(0)+\left(\left(-\frac{5}{12\left(x+\frac{2}{3}\right)^{2}}\right)+\left(\frac{5}{12\left(x+\frac{2}{3}\right)}\right)^{2}-\left(-\frac{35}{16\left(2+3x\right)^{2}}\right)\right)=0$$

$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{5}{12\left(x+\frac{2}{3}\right)}dx} \\
&= (2+3x)^{5/12}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{11x^2+4x}{6x^3+4x^2}\,dx} \\
&= z_1 e^{-\frac{5\ln(2+3x)}{12}-\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{1}{(2+3x)^{5/12}\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\frac{1}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1\int \frac{e^{\int -\frac{11x^2+4x}{6x^3+4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1\int \frac{e^{-\frac{5\ln(2+3x)}{6}-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(2\,\mathrm{e}^{-\frac{5\ln(2+3x)}{6}-\ln(x)}x(2+3x)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1y_1+c_2y_2 \\
&= c_1\left(\frac{1}{\sqrt{x}}\right)+c_2\left(\frac{1}{\sqrt{x}}\left(2\,\mathrm{e}^{-\frac{5\ln(2+3x)}{6}-\ln(x)}x(2+3x)\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2(3x+2)\left(\frac{d^2}{dx^2}y(x)\right)+x(4+11x)\left(\frac{d}{dx}y(x)\right)-(1-x)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{(x-1)y(x)}{2(3x+2)x^2}-\frac{(4+11x)\left(\frac{d}{dx}y(x)\right)}{2x(3x+2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x)+\frac{(4+11x)\left(\frac{d}{dx}y(x)\right)}{2x(3x+2)}+\frac{(x-1)y(x)}{2(3x+2)x^2}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=\frac{4+11x}{2x(3x+2)},P_3(x)=\frac{x-1}{2(3x+2)x^2}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

$\left(x\cdot P_2(x)\right)\Big|_{x=0}=1$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=-\frac{1}{4}$

- ◦ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$2x^2(3x+2)\left(\frac{d^2}{dx^2}y(x)\right)+x(4+11x)\left(\frac{d}{dx}y(x)\right)+(x-1)\,y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

$x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ◦ Shift index using $k->k-m$

$x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ◦ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..3$

$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- ◦ Shift index using $k->k+2-m$

$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

Rewrite ODE with series expansions

$$a_0(1+2r)(-1+2r)x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)(2k+2r-1) + a_{k-1}(2k+2r-1)(3k-2+3r\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(1+2r)(-1+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $4\left(k+r-\frac{1}{2}\right)\left(\left(\frac{3k}{2}+\frac{3r}{2}-1\right)a_{k-1}+a_k\left(k+r+\frac{1}{2}\right)\right) = 0$
- Shift index using $k->k+1$
  $4\left(k+r+\frac{1}{2}\right)\left(\left(\frac{3k}{2}+\frac{1}{2}+\frac{3r}{2}\right)a_k+a_{k+1}\left(k+\frac{3}{2}+r\right)\right) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = -\frac{(3k+3r+1)a_k}{2k+3+2r}$
- Recursion relation for $r = -\frac{1}{2}$
  $a_{k+1} = -\frac{\left(3k-\frac{1}{2}\right)a_k}{2k+2}$
- Solution for $r = -\frac{1}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = -\frac{\left(3k-\frac{1}{2}\right)a_k}{2k+2}\right]$
- Recursion relation for $r = \frac{1}{2}$
  $a_{k+1} = -\frac{\left(3k+\frac{5}{2}\right)a_k}{2k+4}$
- Solution for $r = \frac{1}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{\left(3k+\frac{5}{2}\right)a_k}{2k+4}\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = -\frac{\left(3k-\frac{1}{2}\right)a_k}{2k+2}, b_{k+1} = -\frac{\left(3k+\frac{5}{2}\right)b_k}{2k+4}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.025` (sec)
Leaf size : 19

```
dsolve(2*x^2*(2+3*x)*diff(diff(y(x),x),x)+x*(4+11*x)*diff(y(x),x)-(1-x)*y(x) = 0,y(x),
```

$$y = \frac{c_2(2+3x)^{1/6}+c_1}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.323 (sec)
Leaf size : 69

```
DSolve[{2*x^2*(2+3*x)*D[y[x],{x,2}]+x*(4+11*x)*D[y[x],x]-(1-x)*y[x]==0,{}},y[x],x,IncludeSingul
```

$$y(x) \to \sqrt[6]{2}(3x+2)^{5/12}\left(c_2\sqrt[6]{3x+2}+2^{2/3}c_1\right)\exp\left(-\frac{1}{2}\int_1^x\left(\frac{5}{6K[1]+4}+\frac{1}{K[1]}\right)dK[1]\right)$$

2.1.98 Problem 100

Internal problem ID [9270]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 100
Date solved : Monday, January 27, 2025 at 06:00:48 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$x^2(2+x)\,y''+5x(1-x)\,y'-(2-8x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.839 (sec)

Writing the ode as

$$x^2(2+x)\,y''+\left(-5x^2+5x\right)y'+(8x-2)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(2+x) \\ B &= -5x^2+5x \\ C &= 8x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{3x^2-126x+21}{4\left(x^2+2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3x^2-126x+21 \\ t &= 4\left(x^2+2x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{3x^2-126x+21}{4\left(x^2+2x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.180: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{147}{16\,(2+x)} + \frac{21}{16x^2} + \frac{285}{16\,(2+x)^2} - \frac{147}{16x}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(2+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{285}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{19}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{15}{4} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x^2 - 126x + 21}{4\left(x^2 + 2x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3x^2 - 126x + 21}{4\left(x^2 + 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{19}{4}$	$-\frac{15}{4}$
0	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{9}{2}\right) \\
&= 4
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{15}{4(2+x)} - \frac{3}{4x} + (-)(0) \\
&= -\frac{15}{4(2+x)} - \frac{3}{4x} \\
&= -\frac{3(3x+1)}{2x(2+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(12x^2 + 6xa_3 + 2a_2\right) + 2\left(-\frac{15}{4\,(2+x)} - \frac{3}{4x}\right)\left(4x^3 + 3x^2 a_3 + 2a_2 x + a_1\right) + \left(\left(\frac{15}{4\,(2+x)^2} + \frac{3}{4x^2}\right) + \left(-\frac{}{4\,(2}\right.\right.$$
$$\frac{3(4+a_3)\,x^3 + (8a_2 + 3a_3}{}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{1}{40}, a_1 = \frac{1}{5}, a_2 = \frac{3}{2}, a_3 = -4\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 - 4x^3 + \frac{3}{2}x^2 + \frac{1}{5}x + \frac{1}{40}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \left(x^4 - 4x^3 + \frac{3}{2}x^2 + \frac{1}{5}x + \frac{1}{40}\right) \mathrm{e}^{\int \left(-\frac{15}{4(2+x)} - \frac{3}{4x}\right) dx} \\
&= \left(x^4 - 4x^3 + \frac{3}{2}x^2 + \frac{1}{5}x + \frac{1}{40}\right) \mathrm{e}^{-\frac{3\ln(x)}{4} - \frac{15\ln(2+x)}{4}} \\
&= \frac{40x^4 - 160x^3 + 60x^2 + 8x + 1}{40x^{3/4}\,(2+x)^{15/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-5x^2+5x}{x^2(2+x)}\, dx} \\
&= z_1 e^{-\frac{5\ln(x)}{4} + \frac{15\ln(2+x)}{4}} \\
&= z_1 \left(\frac{(2+x)^{15/4}}{x^{5/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{40x^4 - 160x^3 + 60x^2 + 8x + 1}{40x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-5x^2+5x}{x^2(2+x)}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{5\ln(x)}{2}+\frac{15\ln(2+x)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{10\sqrt{2+x}\, x^{5/2}\left(8x^5\sqrt{x(2+x)} + 4200\ln\left(\frac{x+\sqrt{x(2+x)}}{x}\right)x^4 - 4200\ln\left(\frac{\sqrt{x(2+x)}-x}{x}\right)x^4 + 328x^4\sqrt{x(2\right.}{} \right.
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{40x^4 - 160x^3 + 60x^2 + 8x + 1}{40x^2}\right) \\
&\quad + c_2 \left(\frac{40x^4 - 160x^3 + 60x^2 + 8x + 1}{40x^2} \left(\frac{10\sqrt{2+x}\, x^{5/2}\left(8x^5\sqrt{x(2+x)} + 4200\ln\left(\frac{x+\sqrt{x(2+x)}}{x}\right)x^4 - 420\right.}{} \right.\right.
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x+2)\left(\frac{d^2}{dx^2}y(x)\right) + 5x(1-x)\left(\frac{d}{dx}y(x)\right) - (2-8x)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{2(4x-1)y(x)}{(x+2)x^2} + \frac{5(x-1)\left(\frac{d}{dx}y(x)\right)}{x(x+2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) - \frac{5(x-1)\left(\frac{d}{dx}y(x)\right)}{x(x+2)} + \frac{2(4x-1)y(x)}{(x+2)x^2} = 0$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{5(x-1)}{x(x+2)}, P_3(x) = \frac{2(4x-1)}{(x+2)x^2}\right]$

  - $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

    $\left((x+2)\cdot P_2(x)\right)\Big|_{x=-2} = -\frac{15}{2}$

  - $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

    $\left((x+2)^2\cdot P_3(x)\right)\Big|_{x=-2} = 0$

  - $x=-2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -2$

- Multiply by denominators

  $x^2(x+2)\left(\frac{d^2}{dx^2}y(x)\right) - 5x(x-1)\left(\frac{d}{dx}y(x)\right) + (8x-2)\,y(x) = 0$

- Change variables using $x = u - 2$ so that the regular singular point is at $u = 0$
  $(u^3 - 4u^2 + 4u)\left(\frac{d^2}{du^2}y(u)\right) + (-5u^2 + 25u - 30)\left(\frac{d}{du}y(u)\right) + (8u - 18)\,y(u) = 0$
- Assume series solution for $y(u)$
  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$
  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$
- ○ Shift index using $k- >k - m$
  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
- ○ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$
  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k + r)\, u^{k+r-1+m}$
- ○ Shift index using $k- >k + 1 - m$
  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, u^{k+r}$
- ○ Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$
  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, u^{k+r-2+m}$
- ○ Shift index using $k- >k + 2 - m$
  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k + 2 - m + r)\,(k + 1 - m + r)\, u^{k+r}$

  Rewrite ODE with series expansions
  $2a_0 r(-17 + 2r)\, u^{-1+r} + (2a_1(1 + r)\,(-15 + 2r) - a_0(4r^2 - 29r + 18))\, u^r + \left(\sum_{k=1}^{\infty} (2a_{k+1}(k + 1 + r)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $2r(-17 + 2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, \frac{17}{2}\}$
- Each term must be 0
  $2a_1(1 + r)\,(-15 + 2r) - a_0(4r^2 - 29r + 18) = 0$
- Each term in the series must be 0, giving the recursion relation
  $(-4a_k + a_{k-1} + 4a_{k+1})\, k^2 + ((-8a_k + 2a_{k-1} + 8a_{k+1})\, r + 29a_k - 8a_{k-1} - 26a_{k+1})\, k + (-4a_k + a_{k-}$
- Shift index using $k- >k + 1$
  $(-4a_{k+1} + a_k + 4a_{k+2})\,(k + 1)^2 + ((-8a_{k+1} + 2a_k + 8a_{k+2})\, r + 29a_{k+1} - 8a_k - 26a_{k+2})\,(k + 1) + ($
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} + 2kra_k - 8kra_{k+1} + r^2 a_k - 4r^2 a_{k+1} - 6ka_k + 21ka_{k+1} - 6ra_k + 21ra_{k+1} + 8a_k + 7a_{k+1}}{2(2k^2 + 4kr + 2r^2 - 9k - 9r - 26)}$
- Recursion relation for $r = 0$
  $a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} - 6ka_k + 21ka_{k+1} + 8a_k + 7a_{k+1}}{2(2k^2 - 9k - 26)}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} - 6ka_k + 21ka_{k+1} + 8a_k + 7a_{k+1}}{2(2k^2 - 9k - 26)}, -30a_1 - 18a_0 = 0\right]$
- Revert the change of variables $u = x + 2$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x + 2)^k, a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} - 6ka_k + 21ka_{k+1} + 8a_k + 7a_{k+1}}{2(2k^2 - 9k - 26)}, -30a_1 - 18a_0 = 0\right]$
- Recursion relation for $r = \frac{17}{2}$

$$a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} + 11k a_k - 47k a_{k+1} + \frac{117}{4} a_k - \frac{207}{2} a_{k+1}}{2(2k^2+25k+42)}$$

- Solution for $r = \frac{17}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{17}{2}}, a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} + 11k a_k - 47k a_{k+1} + \frac{117}{4} a_k - \frac{207}{2} a_{k+1}}{2(2k^2+25k+42)}, 38a_1 - \frac{121a_0}{2} = 0\right]$$

- Revert the change of variables $u = x + 2$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k (x+2)^{k+\frac{17}{2}}, a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} + 11k a_k - 47k a_{k+1} + \frac{117}{4} a_k - \frac{207}{2} a_{k+1}}{2(2k^2+25k+42)}, 38a_1 - \frac{121a_0}{2} = 0\right]$$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k (x+2)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+2)^{k+\frac{17}{2}}\right), a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} - 6k a_k + 21k a_{k+1} + 8a_k + 7a_{k+1}}{2(2k^2-9k-26)}, \right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - return
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.113 (sec)
Leaf size : 113

```
dsolve(x^2*(x+2)*diff(diff(y(x),x),x)+5*x*(1-x)*diff(y(x),x)-(-8*x+2)*y(x) = 0,y(x),si
```

$$y = \frac{c_1(40x^4 - 160x^3 + 60x^2 + 8x + 1)}{x^2} + \frac{4\left(1050\left(x^4 - 4x^3 + \frac{3}{2}x^2 + \frac{1}{5}x + \frac{1}{40}\right)x^{3/2}\operatorname{arcsinh}\left(\frac{\sqrt{x}\sqrt{2}}{2}\right) + \sqrt{x+2}\,x^2\left(x^5 + 41x^4 - \frac{6987}{4}x^3 + \frac{13367}{4}x^2 - \right.\right.}{105\,(x+2)^{3/4}\,x^{7/2}}$$

Mathematica DSolve solution

Solving time : 0.773 (sec)
Leaf size : 163

```
DSolve[{x^2*(2+x)*D[y[x],{x,2}]+5*x*(1-x)*D[y[x],x]-(2-8*x)*y[x]==0,{}},y[x],x,IncludeSingular
```

$$y(x) \to \frac{1}{40}\left(40x^4 - 160x^3 + 60x^2 + 8x + 1\right)\exp\left(\int_1^x -\frac{9K[1]+3}{2K[1]^2+4K[1]}dK[1] - \frac{1}{2}\int_1^x \frac{5-5K[2]}{K[2]^2+2K[2]}dK[2]\right)\left(c_2\int_1^x \frac{1600\exp\left(-2\int_1^{K[3]} -\frac{9K[1]+3}{2K[1]^2+4K[1]}dK[1]\right)}{\left(40K[3]^4-160K[3]^3+60K[3]^2+8K[3]+1\right)^2}dK[3] + c_1\right)$$

2.1.99 Problem 101

Internal problem ID [9271]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 101
Date solved : `Monday, January 27, 2025 at 06:00:49 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$8x^2\left(-x^2+1\right)y''+2x\left(-13x^2+1\right)y'+\left(-9x^2+1\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.335 (sec)

Writing the ode as

$$\left(-8x^4+8x^2\right)y''+\left(-26x^3+2x\right)y'+\left(-9x^2+1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= -8x^4+8x^2\\ B &= -26x^3+2x\\ C &= -9x^2+1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-7x^4-26x^2-15}{64\left(x^3-x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s &= -7x^4-26x^2-15\\ t &= 64\left(x^3-x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-7x^4-26x^2-15}{64\left(x^3-x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.182: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64(x^3 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{15}{64x^2} + \frac{1}{4x - 4} - \frac{3}{16\,(x + 1)^2} - \frac{3}{16\,(x - 1)^2} - \frac{1}{4\,(x + 1)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{15}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{3}{8} \end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{-7x^4 - 26x^2 - 15}{64\,(x^3 - x)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{7}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-7x^4 - 26x^2 - 15}{64\,(x^3 - x)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{8}$	$\frac{3}{8}$
1	2	0	$\frac{3}{4}$	$\frac{1}{4}$
-1	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{8}$	$\frac{1}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{8}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{7}{8} - \left(\frac{7}{8}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{3}{8x} + \frac{1}{4x-4} + \frac{1}{4x+4} + (0) \\
&= \frac{3}{8x} + \frac{1}{4x-4} + \frac{1}{4x+4} \\
&= \frac{7x^2-3}{8x^3-8x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{8x} + \frac{1}{4x-4} + \frac{1}{4x+4}\right)(0) + \left(\left(-\frac{3}{8x^2} - \frac{1}{4(x-1)^2} - \frac{1}{4(x+1)^2}\right) + \left(\frac{3}{8x} + \frac{1}{4x-4} + \frac{1}{4x+4}\right)^2\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{3}{8x} + \frac{1}{4x-4} + \frac{1}{4x+4}\right) dx} \\
&= x^{3/8}(x+1)^{1/4}(x-1)^{1/4}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-26x^3+2x}{-8x^4+8x^2}\, dx} \\
&= z_1 e^{-\frac{\ln(x)}{8} - \frac{3\ln(x+1)}{4} - \frac{3\ln(x-1)}{4}} \\
&= z_1 \left(\frac{1}{x^{1/8}(x+1)^{3/4}(x-1)^{3/4}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{x^{1/4}(x^2-1)^{1/4}}{(x+1)^{3/4}(x-1)^{3/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-26x^3+2x}{-8x^4+8x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x)}{4} - \frac{3\ln(x+1)}{2} - \frac{3\ln(x-1)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{e^{-\frac{\ln(x)}{4} - \frac{3\ln(x+1)}{2} - \frac{3\ln(x-1)}{2}}(x+1)^{3/2}(x-1)^{3/2}}{\sqrt{x}\,\sqrt{x^2-1}}\, dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^{1/4}(x^2-1)^{1/4}}{(x+1)^{3/4}(x-1)^{3/4}}\right) + c_2\left(\frac{x^{1/4}(x^2-1)^{1/4}}{(x+1)^{3/4}(x-1)^{3/4}}\left(\int \frac{e^{-\frac{\ln(x)}{4}-\frac{3\ln(x+1)}{2}-\frac{3\ln(x-1)}{2}}(x+1)^{3/2}(x-1)^{3/2}}{\sqrt{x}\sqrt{x^2-1}} dx
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$8x^2(-x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)+2x(-13x^2+1)\left(\frac{d}{dx}y(x)\right)+(-9x^2+1)y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(9x^2-1)y(x)}{8(x^2-1)x^2}-\frac{(13x^2-1)\left(\frac{d}{dx}y(x)\right)}{4x(x^2-1)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{(13x^2-1)\left(\frac{d}{dx}y(x)\right)}{4x(x^2-1)}+\frac{(9x^2-1)y(x)}{8(x^2-1)x^2}=0$

☐ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=\frac{13x^2-1}{4x(x^2-1)}, P_3(x)=\frac{9x^2-1}{8(x^2-1)x^2}\right]$
- ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=\frac{3}{2}$
- ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$
- ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $8(x^2-1)x^2\left(\frac{d^2}{dx^2}y(x)\right)+2x(13x^2-1)\left(\frac{d}{dx}y(x)\right)+(9x^2-1)y(x)=0$
- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(8u^4-32u^3+40u^2-16u)\left(\frac{d^2}{du^2}y(u)\right)+(26u^3-78u^2+76u-24)\left(\frac{d}{du}y(u)\right)+(9u^2-18u+8)y$
- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot y(u)$ to series expansion for $m=0..2$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty} a_k u^{k+r+m}$
- ◦ Shift index using $k->k-m$

  $u^m\cdot y(u)=\sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..3$

$$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..4$

$$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$-8a_0r(1+2r)\,u^{-1+r}+(-8a_1(1+r)\,(3+2r)+4a_0(1+2r)\,(2+5r))\,u^r+(-8a_2(2+r)\,(5+2r)+$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-8r(1+2r)=0$
- Values of r that satisfy the indicial equation
  $r\in\left\{0,-\frac{1}{2}\right\}$
- The coefficients of each power of u must be 0
  $[-8a_1(1+r)\,(3+2r)+4a_0(1+2r)\,(2+5r)=0,-8a_2(2+r)\,(5+2r)+4a_1(3+2r)\,(7+5r)-2a$
- Solve for the dependent coefficient(s)
  $\left\{a_1=\frac{a_0(10r^2+9r+2)}{2(2r^2+5r+3)},a_2=\frac{a_0(34r^3+76r^2+41r+5)}{4(2r^3+11r^2+19r+10)}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $8(5a_k+a_{k-2}-4a_{k-1}-2a_{k+1})\,k^2+2(8(5a_k+a_{k-2}-4a_{k-1}-2a_{k+1})\,r+18a_k-7a_{k-2}+9a_{k-1}-2$
- Shift index using $k->k+2$
  $8(5a_{k+2}+a_k-4a_{k+1}-2a_{k+3})\,(k+2)^2+2(8(5a_{k+2}+a_k-4a_{k+1}-2a_{k+3})\,r+18a_{k+2}-7a_k+9a_k$
- Recursion relation that defines series solution to ODE
  $a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+16kra_k-64kra_{k+1}+80kra_{k+2}+8r^2a_k-32r^2a_{k+1}+40r^2a_{k+2}+18ka_k-110ka_{k+1}+196ka_{k+2}+}{8(2k^2+4kr+2r^2+13k+13r+21)}$
- Recursion relation for $r=0$
  $a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+18ka_k-110ka_{k+1}+196ka_{k+2}+9a_k-96a_{k+1}+240a_{k+2}}{8(2k^2+13k+21)}$
- Solution for $r=0$
  $\left[y(u)=\sum_{k=0}^{\infty}a_ku^k,a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+18ka_k-110ka_{k+1}+196ka_{k+2}+9a_k-96a_{k+1}+240a_{k+2}}{8(2k^2+13k+21)},a_1=\frac{a_0}{3},a\right.$
- Revert the change of variables $u=x+1$
  $\left[y(x)=\sum_{k=0}^{\infty}a_k(x+1)^k,a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+18ka_k-110ka_{k+1}+196ka_{k+2}+9a_k-96a_{k+1}+240a_{k+2}}{8(2k^2+13k+21)},a_1=\right.$
- Recursion relation for $r=-\frac{1}{2}$
  $a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+10ka_k-78ka_{k+1}+156ka_{k+2}+2a_k-49a_{k+1}+152a_{k+2}}{8(2k^2+11k+15)}$
- Solution for $r=-\frac{1}{2}$
  $\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k-\frac{1}{2}},a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+10ka_k-78ka_{k+1}+156ka_{k+2}+2a_k-49a_{k+1}+152a_{k+2}}{8(2k^2+11k+15)},a_1=0,a\right.$
- Revert the change of variables $u=x+1$
  $\left[y(x)=\sum_{k=0}^{\infty}a_k(x+1)^{k-\frac{1}{2}},a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+10ka_k-78ka_{k+1}+156ka_{k+2}+2a_k-49a_{k+1}+152a_{k+2}}{8(2k^2+11k+15)},a_1\right.$
- Combine solutions and rename parameters
  $\left[y(x)=\left(\sum_{k=0}^{\infty}a_k(x+1)^k\right)+\left(\sum_{k=0}^{\infty}b_k(x+1)^{k-\frac{1}{2}}\right),a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+18ka_k-110ka_{k+1}+196ka}{8(2k^2+13k+21)}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special functio
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.111 (sec)
Leaf size : 34

```
dsolve(8*x^2*(-x^2+1)*diff(diff(y(x),x),x)+2*x*(-13*x^2+1)*diff(y(x),x)+(-9*x^2+1)*y(x
```

$$y = \frac{x^{1/4}\left(\text{LegendreQ}\left(-\frac{1}{8}, \frac{1}{8}, \sqrt{-x^2+1}\right) c_2 x^{1/8} + c_1\right)}{\sqrt{x^2-1}}$$

Mathematica DSolve solution

Solving time : 0.308 (sec)
Leaf size : 118

```
DSolve[{8*x^2*(1-x^2)*D[y[x],{x,2}]+2*x*(1-13*x^2)*D[y[x],x]+(1-9*x^2)*y[x]==0,{}},y[x],x,In
```

$$y(x) \to \exp\left(\int_1^x \frac{3-7K[1]^2}{8K[1]-8K[1]^3}dK[1] - \frac{1}{2}\int_1^x \frac{1-13K[2]^2}{4K[2]-4K[2]^3}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{3-7K[1]^2}{8K[1]-8K[1]^3}dK[1]\right)dK[3] + c_1\right)$$

2.1.100 Problem 102

Internal problem ID [9272]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 102
Date solved : `Monday, January 27, 2025 at 06:00:50 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(x^2+1\right)y''-2x\left(-x^2+2\right)y'+4y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.357 (sec)

Writing the ode as

$$\left(x^4+x^2\right)y''+\left(2x^3-4x\right)y'+4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=x^4+x^2\\ B&=2x^3-4x\\ C&=4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-x^2+2}{\left(x^3+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=-x^2+2\\ t&=\left(x^3+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-x^2+2}{\left(x^3+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.184: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x - i)^2} + \frac{3}{4(x + i)^2} + \frac{7i}{4(x - i)} - \frac{7i}{4(x + i)} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-x^2+2}{(x^3+x)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{2}{x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)} + (-)(0) \\
&= \frac{2}{x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)} \\
&= \frac{x^2+2}{x^3+x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{2}{x}-\frac{1}{2(x-i)}-\frac{1}{2(x+i)}\right)(0)+\left(\left(-\frac{2}{x^2}+\frac{1}{2(x-i)^2}+\frac{1}{2(x+i)^2}\right)+\left(\frac{2}{x}-\frac{1}{2(x-i)}-\frac{1}{2(x+\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\,dx}\\
&= \mathrm{e}^{\int \left(\frac{2}{x}-\frac{1}{2(x-i)}-\frac{1}{2(x+i)}\right)dx}\\
&= \frac{x^2}{\sqrt{x^2+1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^3-4x}{x^4+x^2}\,dx}\\
&= z_1 e^{-\frac{3\ln\left(x^2+1\right)}{2}+2\ln(x)}\\
&= z_1\left(\frac{x^2}{\left(x^2+1\right)^{3/2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\frac{x^4}{\left(x^2+1\right)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1\int \frac{e^{\int -\frac{2x^3-4x}{x^4+x^2}\,dx}}{\left(y_1\right)^2}\,dx\\
&= y_1\int \frac{e^{-3\ln(x^2+1)+4\ln(x)}}{\left(y_1\right)^2}\,dx\\
&= y_1\left(-\frac{\left(3x^2+1\right)\left(x^2+1\right)^3\mathrm{e}^{-3\ln(x^2+1)+4\ln(x)}}{3x^7}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1y_1+c_2y_2\\
&= c_1\left(\frac{x^4}{\left(x^2+1\right)^2}\right)+c_2\left(\frac{x^4}{\left(x^2+1\right)^2}\left(-\frac{\left(3x^2+1\right)\left(x^2+1\right)^3\mathrm{e}^{-3\ln(x^2+1)+4\ln(x)}}{3x^7}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) - 2x(-x^2+2)\left(\frac{d}{dx}y(x)\right) + 4y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{4y(x)}{x^2(x^2+1)} - \frac{2(x^2-2)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{2(x^2-2)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)} + \frac{4y(x)}{x^2(x^2+1)} = 0$

☐ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{2(x^2-2)}{x(x^2+1)}, P_3(x) = \frac{4}{x^2(x^2+1)}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = -4$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 4$

- ◦ $x = 0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

  $x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + 2x(x^2-2)\left(\frac{d}{dx}y(x)\right) + 4y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- ◦ Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$

- ◦ Shift index using $k->k+2-m$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0(-1+r)(-4+r)\, x^r + a_1 r(-3+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)(k+r-4) + a_{k-2}(k-2+r)(k\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)(-4+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{1, 4\}$

- Each term must be 0
  $a_1 r(-3+r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-1)\left(a_k(k+r-4) + a_{k-2}(k-2+r)\right) = 0$
- Shift index using $k->k+2$
  $(k+r+1)\left(a_{k+2}(k-2+r) + a_k(k+r)\right) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{a_k(k+r)}{k-2+r}$
- Recursion relation for $r = 1$
  $a_{k+2} = -\frac{a_k(k+1)}{k-1}$
- Solution for $r = 1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k(k+1)}{k-1}, a_1 = 0\right]$
- Recursion relation for $r = 4$
  $a_{k+2} = -\frac{a_k(k+4)}{k+2}$
- Solution for $r = 4$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+4}, a_{k+2} = -\frac{a_k(k+4)}{k+2}, a_1 = 0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{4+k}\right), a_{k+2} = -\frac{a_k(k+1)}{k-1}, a_1 = 0, b_{k+2} = -\frac{b_k(4+k)}{k+2}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 26

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)-2*x*(-x^2+2)*diff(y(x),x)+4*y(x) = 0,y(x),sing
```

$$y = \frac{x(c_1 x^3 + 3c_2 x^2 + c_2)}{(x^2+1)^2}$$

Mathematica DSolve solution

Solving time : 0.207 (sec)
Leaf size : 101

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]-2*x*(2-x^2)*D[y[x],x]+4*y[x]==0,{}},y[x],x,IncludeSingularSol
```

$$y(x) \to \exp\left(\int_1^x \frac{K[1]^2+2}{K[1]^3+K[1]}dK[1] - \frac{1}{2}\int_1^x \frac{2(K[2]^2-2)}{K[2]^3+K[2]}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{K[1]^2+2}{K[1]^3+K[1]}dK[1]\right)dK[3] + c_1\right)$$

2.1.101 Problem 103

Internal problem ID [9273]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 103
Date solved : `Monday, January 27, 2025 at 06:00:51 PM`
CAS classification : `[[_2nd_order, _exact, _linear, _homogeneous]]`

Solve

$$x\left(x^{2}+3\right) y''+\left(-x^{2}+2\right) y'-8 x y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.277 (sec)

Writing the ode as

$$\left(x^{3}+3 x\right) y''+\left(-x^{2}+2\right) y'-8 x y=0 \tag{1}$$

$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^{3}+3 x \\ B &= -x^{2}+2 \\ C &= -8 x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2 A} d x}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2 A B'-2 B A'+B^{2}-4 A C}{4 A^{2}} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{35 x^{4}+74 x^{2}-8}{4\left(x^{3}+3 x\right)^{2}} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 35 x^{4}+74 x^{2}-8 \\ t &= 4\left(x^{3}+3 x\right)^{2} \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{35 x^{4}+74 x^{2}-8}{4\left(x^{3}+3 x\right)^{2}}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.186: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3+3x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i\sqrt{3}$ of order 2. There is a pole at $x = -i\sqrt{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{85}{144\left(x - i\sqrt{3}\right)^2} + \frac{85}{144\left(x + i\sqrt{3}\right)^2} - \frac{187 i\sqrt{3}}{144\left(x - i\sqrt{3}\right)} + \frac{187 i\sqrt{3}}{144\left(x + i\sqrt{3}\right)} - \frac{2}{9x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{2}{9}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{2}{3} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{3} \end{aligned}$$

For the pole at $x = i\sqrt{3}$ let b be the coefficient of $\frac{1}{\left(x-i\sqrt{3}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{85}{144}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{17}{12}\\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{12}
\end{aligned}
$$

For the pole at $x = -i\sqrt{3}$ let b be the coefficient of $\frac{1}{\left(x+i\sqrt{3}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{85}{144}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{17}{12}\\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{12}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{35x^4 + 74x^2 - 8}{4\left(x^3+3x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{35}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2}\\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{35x^4 + 74x^2 - 8}{4\left(x^3+3x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{2}{3}$	$\frac{1}{3}$
$i\sqrt{3}$	2	0	$\frac{17}{12}$	$-\frac{5}{12}$
$-i\sqrt{3}$	2	0	$\frac{17}{12}$	$-\frac{5}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{7}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{+} + \alpha_{c_3}^{+}\right)\\
&= \frac{7}{2} - \left(\frac{7}{2}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2} \right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{+}}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{2}{3x} + \frac{17}{12\left(x-i\sqrt{3}\right)} + \frac{17}{12\left(x+i\sqrt{3}\right)} + (0) \\
&= \frac{2}{3x} + \frac{17}{12\left(x-i\sqrt{3}\right)} + \frac{17}{12\left(x+i\sqrt{3}\right)} \\
&= \frac{2}{3x} + \frac{17x}{6x^2+18}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{2}{3x} + \frac{17}{12\left(x-i\sqrt{3}\right)} + \frac{17}{12\left(x+i\sqrt{3}\right)} \right)(0) + \left(\left(-\frac{2}{3x^2} - \frac{17}{12\left(x-i\sqrt{3}\right)^2} - \frac{17}{12\left(x+i\sqrt{3}\right)^2} \right) + \left(\frac{2}{3x}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{2}{3x} + \frac{17}{12\left(x-i\sqrt{3}\right)} + \frac{17}{12\left(x+i\sqrt{3}\right)} \right) dx} \\
&= \left(x^2+3\right)^{17/12} x^{2/3}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2+2}{x^3+3x}\, dx} \\
&= z_1 e^{\frac{5\ln\left(x^2+3\right)}{12} - \frac{\ln(x)}{3}} \\
&= z_1 \left(\frac{\left(x^2+3\right)^{5/12}}{x^{1/3}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = \left(x^2+3\right)^{11/6} x^{1/3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+2}{x^3+3x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{5\ln(x^2+3)}{6}-\frac{2\ln(x)}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{x^{1/3}(8x^4+44x^2+55)\,\mathrm{e}^{\frac{5\ln(x^2+3)}{6}-\frac{2\ln(x)}{3}}}{55\,(x^2+3)^{8/3}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\left(x^2+3\right)^{11/6} x^{1/3}\right) + c_2\left(\left(x^2+3\right)^{11/6} x^{1/3}\left(-\frac{x^{1/3}(8x^4+44x^2+55)\,\mathrm{e}^{\frac{5\ln(x^2+3)}{6}-\frac{2\ln(x)}{3}}}{55\,(x^2+3)^{8/3}}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x(x^2+3)\left(\frac{d^2}{dx^2}y(x)\right)+(-x^2+2)\left(\frac{d}{dx}y(x)\right)-8xy(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{8y(x)}{x^2+3}+\frac{(x^2-2)\left(\frac{d}{dx}y(x)\right)}{x(x^2+3)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)-\frac{(x^2-2)\left(\frac{d}{dx}y(x)\right)}{x(x^2+3)}-\frac{8y(x)}{x^2+3}=0$

☐ Check to see if x_0 is a regular singular point

- ∘ Define functions

  $\left[P_2(x)=-\frac{x^2-2}{x(x^2+3)}, P_3(x)=-\frac{8}{x^2+3}\right]$

- ∘ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{2}{3}$

- ∘ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

- ∘ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x(x^2+3)\left(\frac{d^2}{dx^2}y(x)\right)+(-x^2+2)\left(\frac{d}{dx}y(x)\right)-8xy(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x \cdot y(x)$ to series expansion

$$x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

- Shift index using $k->k-1$

$$x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..2$

$$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 1..3$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-1+3r)\, x^{-1+r} + a_1(1+r)(2+3r)\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+r+1)(3k+2+3r) + a_{k-1}(k+r+1$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-1+3r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{0, \frac{1}{3}\right\}$
- Each term must be 0
  $a_1(1+r)(2+3r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $(k+r+1)\left(a_{k-1}(k-5+r) + 3\left(k+r+\frac{2}{3}\right)a_{k+1}\right) = 0$
- Shift index using $k->k+1$
  $(k+r+2)\left(a_k(k+r-4) + 3\left(k+\frac{5}{3}+r\right)a_{k+2}\right) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{a_k(k+r-4)}{3k+5+3r}$
- Recursion relation for $r = 0$; series terminates at $k = 4$
  $a_{k+2} = -\frac{a_k(k-4)}{3k+5}$
- Solution for $r = 0$
  $$\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(k-4)}{3k+5}, 2a_1 = 0\right]$$
- Recursion relation for $r = \frac{1}{3}$
  $a_{k+2} = -\frac{a_k\left(k-\frac{11}{3}\right)}{3k+6}$
- Solution for $r = \frac{1}{3}$
  $$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -\frac{a_k\left(k-\frac{11}{3}\right)}{3k+6}, 4a_1 = 0\right]$$
- Combine solutions and rename parameters
  $$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = -\frac{a_k(-4+k)}{3k+5}, 2a_1 = 0, b_{k+2} = -\frac{b_k\left(k-\frac{11}{3}\right)}{3k+6}, 4b_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 32

```
dsolve(x*(x^2+3)*diff(diff(y(x),x),x)+(-x^2+2)*diff(y(x),x)-8*x*y(x) = 0,y(x),singsol=
```

$$y = c_1 (x^2+3)^{11/6} x^{1/3} + \frac{c_2(8x^4+44x^2+55)}{8}$$

Mathematica DSolve solution

Solving time : 0.227 (sec)
Leaf size : 116

```
DSolve[{x*(3+x^2)*D[y[x],{x,2}]+(2-x^2)*D[y[x],x]-8*x*y[x]==0,{}},y[x],x,IncludeSingularSolu
```

$$y(x) \to \exp\left(\int_1^x \frac{7K[1]^2+4}{2K[1]^3+6K[1]}dK[1] - \frac{1}{2}\int_1^x \frac{2-K[2]^2}{K[2]^3+3K[2]}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{7K[1]^2+4}{2K[1]^3+6K[1]}dK[1]\right)dK[3] + c_1\right)$$

2.1.102 Problem 104

Internal problem ID [9274]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 104
Date solved : `Monday, January 27, 2025 at 06:00:52 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2\left(-x^2+1\right)y''+x\left(-19x^2+7\right)y'-\left(14x^2+1\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.314 (sec)

Writing the ode as

$$\left(-4x^4+4x^2\right)y''+\left(-19x^3+7x\right)y'+\left(-14x^2-1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -4x^4+4x^2 \\ B &= -19x^3+7x \\ C &= -14x^2-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-15x^4-42x^2+9}{64\left(x^3-x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -15x^4-42x^2+9 \\ t &= 64\left(x^3-x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-15x^4-42x^2+9}{64\left(x^3-x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.188: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64(x^3 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16\,(x-1)^2} - \frac{3}{16\,(x+1)^2} + \frac{9}{64x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{9}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{8} \end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-15x^4 - 42x^2 + 9}{64\,(x^3 - x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{15}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{8} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{8}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-15x^4 - 42x^2 + 9}{64\,(x^3 - x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{9}{8}$	$-\frac{1}{8}$
1	2	0	$\frac{3}{4}$	$\frac{1}{4}$
-1	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{8}$	$\frac{3}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{3}{8}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{3}{8} - \left(\frac{3}{8}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{8x} + \frac{1}{4x-4} + \frac{1}{4x+4} + (-)(0) \\ &= -\frac{1}{8x} + \frac{1}{4x-4} + \frac{1}{4x+4} \\ &= \frac{3x^2+1}{8x^3-8x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{8x} + \frac{1}{4x-4} + \frac{1}{4x+4}\right)(0) + \left(\left(\frac{1}{8x^2} - \frac{1}{4(x-1)^2} - \frac{1}{4(x+1)^2}\right) + \left(-\frac{1}{8x} + \frac{1}{4x-4} + \frac{1}{4x+}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{8x} + \frac{1}{4x-4} + \frac{1}{4x+4}\right) dx} \\ &= \frac{(x-1)^{1/4}(x+1)^{1/4}}{x^{1/8}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-19x^3+7x}{-4x^4+4x^2}\, dx} \\ &= z_1 e^{-\frac{3\ln(x-1)}{4} - \frac{3\ln(x+1)}{4} - \frac{7\ln(x)}{8}} \\ &= z_1\left(\frac{1}{(x-1)^{3/4}(x+1)^{3/4}x^{7/8}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2-1)^{1/4}}{(x-1)^{3/4}(x+1)^{3/4}x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-19x^3+7x}{-4x^4+4x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{3\ln(x-1)}{2} - \frac{3\ln(x+1)}{2} - \frac{7\ln(x)}{4}}}{(y_1)^2}\, dx \\ &= y_1\left(\int \frac{\mathrm{e}^{-\frac{3\ln(x-1)}{2} - \frac{3\ln(x+1)}{2} - \frac{7\ln(x)}{4}}(x-1)^{3/2}(x+1)^{3/2}x^2}{\sqrt{x^2-1}} dx\right)\end{aligned}$$

Therefore the solution is

$$y = c_1y_1 + c_2y_2$$

$$= c_1\left(\frac{(x^2-1)^{1/4}}{(x-1)^{3/4}(x+1)^{3/4}x}\right) + c_2\left(\frac{(x^2-1)^{1/4}}{(x-1)^{3/4}(x+1)^{3/4}x}\left(\int \frac{e^{-\frac{3\ln(x-1)}{2}-\frac{3\ln(x+1)}{2}-\frac{7\ln(x)}{4}}(x-1)^{3/2}(x+1)^{3/2}}{\sqrt{x^2-1}}\right.\right.$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2(-x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + x(-19x^2+7)\left(\frac{d}{dx}y(x)\right) - (14x^2+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(14x^2+1)y(x)}{4(x^2-1)x^2} - \frac{(19x^2-7)\left(\frac{d}{dx}y(x)\right)}{4x(x^2-1)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(19x^2-7)\left(\frac{d}{dx}y(x)\right)}{4x(x^2-1)} + \frac{(14x^2+1)y(x)}{4(x^2-1)x^2} = 0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{19x^2-7}{4x(x^2-1)}, P_3(x) = \frac{14x^2+1}{4(x^2-1)x^2}\right]$
- ○ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1} = \frac{3}{2}$
- ○ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$
- ○ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $4(x^2-1)\,x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(19x^2-7)\left(\frac{d}{dx}y(x)\right) + (14x^2+1)\,y(x) = 0$
- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(4u^4-16u^3+20u^2-8u)\left(\frac{d^2}{du^2}y(u)\right) + (19u^3-57u^2+50u-12)\left(\frac{d}{du}y(u)\right) + (14u^2-28u+15)\,y(\imath$
- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot y(u)$ to series expansion for $m = 0..2$

  $u^m\cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$
- ○ Shift index using $k->k-m$

  $u^m\cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
- ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..3$

$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..4$

$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

Rewrite ODE with series expansions

$-4a_0r(1+2r)\, u^{-1+r} + (-4a_1(1+r)\,(3+2r) + 5a_0(4r^2+6r+3))\, u^r + (-4a_2(2+r)\,(5+2r) +$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-4r(1+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{0, -\frac{1}{2}\right\}$
- The coefficients of each power of u must be 0
  $[-4a_1(1+r)\,(3+2r) + 5a_0(4r^2+6r+3) = 0, -4a_2(2+r)\,(5+2r) + 5a_1(4r^2+14r+13) - a_0$
- Solve for the dependent coefficient(s)
  $\left\{a_1 = \frac{5a_0(4r^2+6r+3)}{4(2r^2+5r+3)}, a_2 = \frac{a_0(272r^4+1352r^3+2464r^2+1948r+639)}{16(4r^4+28r^3+71r^2+77r+30)}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $4(5a_k + a_{k-2} - 4a_{k-1} - 2a_{k+1})\, k^2 + (8(5a_k + a_{k-2} - 4a_{k-1} - 2a_{k+1})\, r + 30a_k - a_{k-2} - 9a_{k-1} -$
- Shift index using $k- >k+2$
  $4(5a_{k+2} + a_k - 4a_{k+1} - 2a_{k+3})\,(k+2)^2 + (8(5a_{k+2} + a_k - 4a_{k+1} - 2a_{k+3})\, r + 30a_{k+2} - a_k - 9a_k$
- Recursion relation that defines series solution to ODE
  $a_{k+3} = \frac{4k^2a_k-16k^2a_{k+1}+20k^2a_{k+2}+8kra_k-32kra_{k+1}+40kra_{k+2}+4r^2a_k-16r^2a_{k+1}+20r^2a_{k+2}+15ka_k-73ka_{k+1}+110ka_{k+2}}{4(2k^2+4kr+2r^2+13k+13r+21)}$
- Recursion relation for $r = 0$
  $a_{k+3} = \frac{4k^2a_k-16k^2a_{k+1}+20k^2a_{k+2}+15ka_k-73ka_{k+1}+110ka_{k+2}+14a_k-85a_{k+1}+155a_{k+2}}{4(2k^2+13k+21)}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+3} = \frac{4k^2a_k-16k^2a_{k+1}+20k^2a_{k+2}+15ka_k-73ka_{k+1}+110ka_{k+2}+14a_k-85a_{k+1}+155a_{k+2}}{4(2k^2+13k+21)}, a_1 = \frac{5c}{4}\right.$
- Revert the change of variables $u = x+1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+3} = \frac{4k^2a_k-16k^2a_{k+1}+20k^2a_{k+2}+15ka_k-73ka_{k+1}+110ka_{k+2}+14a_k-85a_{k+1}+155a_{k+2}}{4(2k^2+13k+21)}, c\right.$
- Recursion relation for $r = -\frac{1}{2}$
  $a_{k+3} = \frac{4k^2a_k-16k^2a_{k+1}+20k^2a_{k+2}+11ka_k-57ka_{k+1}+90ka_{k+2}+\frac{15}{2}a_k-\frac{105}{2}a_{k+1}+105a_{k+2}}{4(2k^2+11k+15)}$
- Solution for $r = -\frac{1}{2}$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{1}{2}}, a_{k+3} = \frac{4k^2a_k-16k^2a_{k+1}+20k^2a_{k+2}+11ka_k-57ka_{k+1}+90ka_{k+2}+\frac{15}{2}a_k-\frac{105}{2}a_{k+1}+105a_{k+2}}{4(2k^2+11k+15)}, a_1 =\right.$
- Revert the change of variables $u = x+1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^{k-\frac{1}{2}}, a_{k+3} = \frac{4k^2a_k-16k^2a_{k+1}+20k^2a_{k+2}+11ka_k-57ka_{k+1}+90ka_{k+2}+\frac{15}{2}a_k-\frac{105}{2}a_{k+1}+105a_{k+}}{4(2k^2+11k+15)}\right.$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k(x+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k(x+1)^{k-\frac{1}{2}}\right), a_{k+3} = \frac{4k^2a_k-16k^2a_{k+1}+20k^2a_{k+2}+15ka_k-73ka_{k+1}+11}{4(2k^2+13k+21)}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.101 (sec)
Leaf size : 44

```
dsolve(4*x^2*(-x^2+1)*diff(diff(y(x),x),x)+x*(-19*x^2+7)*diff(y(x),x)-(14*x^2+1)*y(x) =
```

$$y = \frac{c_1 \operatorname{LegendreP}\left(-\frac{3}{8}, \frac{5}{8}, \sqrt{-x^2+1}\right) + c_2 \operatorname{LegendreQ}\left(-\frac{3}{8}, \frac{5}{8}, \sqrt{-x^2+1}\right)}{x^{3/8}\sqrt{x^2-1}}$$

Mathematica DSolve solution

Solving time : 0.308 (sec)
Leaf size : 120

```
DSolve[{4*x^2*(1-x^2)*D[y[x],{x,2}]+x*(7-19*x^2)*D[y[x],x]-(1+14*x^2)*y[x]==0,{}},y[x],x,Inclu
```

$$y(x) \to \exp\left(\int_1^x -\frac{3K[1]^2+1}{8K[1]-8K[1]^3}dK[1] - \frac{1}{2}\int_1^x \frac{7-19K[2]^2}{4K[2]-4K[2]^3}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} -\frac{3K[1]^2+1}{8K[1]-8K[1]^3}dK[1]\right)dK[3]+c_1\right)$$

2.1.103 Problem 105

Internal problem ID [9275]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 105
Date solved : Monday, January 27, 2025 at 06:00:52 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$3x^2\left(-x^2+2\right)y''+x\left(-11x^2+1\right)y'+\left(-5x^2+1\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.359 (sec)

Writing the ode as

$$\left(-3x^4+6x^2\right)y''+\left(-11x^3+x\right)y'+\left(-5x^2+1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -3x^4+6x^2 \\ B &= -11x^3+x \\ C &= -5x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-5x^4-4x^2-35}{36\left(x^3-2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -5x^4-4x^2-35 \\ t &= 36\left(x^3-2x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-5x^4-4x^2-35}{36\left(x^3-2x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.190: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^3 - 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \sqrt{2}$ of order 2. There is a pole at $x = -\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{64\left(x-\sqrt{2}\right)^2} - \frac{7}{64\left(x+\sqrt{2}\right)^2} + \frac{31\sqrt{2}}{384\left(x-\sqrt{2}\right)} - \frac{31\sqrt{2}}{384\left(x+\sqrt{2}\right)} - \frac{35}{144x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12} \end{aligned}$$

For the pole at $x=\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x-\sqrt{2}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{7}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{7}{8} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{8}
\end{aligned}
$$

For the pole at $x=-\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x+\sqrt{2}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{7}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{7}{8} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{8}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{-5x^4-4x^2-35}{36\left(x^3-2x\right)^2}
$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{5}{36}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{5}{6} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{6}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-5x^4-4x^2-35}{36\left(x^3-2x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{12}$	$\frac{5}{12}$
$\sqrt{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$
$-\sqrt{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{5}{6}$	$\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=\frac{5}{6}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{-} + \alpha_{c_3}^{-}\right) \\
&= \frac{5}{6} - \left(\frac{5}{6}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{7}{12x} + \frac{1}{8x-8\sqrt{2}} + \frac{1}{8x+8\sqrt{2}} + (0) \\
&= \frac{7}{12x} + \frac{1}{8x-8\sqrt{2}} + \frac{1}{8x+8\sqrt{2}} \\
&= \frac{5x^2-7}{6x^3-12x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{7}{12x} + \frac{1}{8x-8\sqrt{2}} + \frac{1}{8x+8\sqrt{2}}\right)(0) + \left(\left(-\frac{7}{12x^2} - \frac{1}{8\left(x-\sqrt{2}\right)^2} - \frac{1}{8\left(x+\sqrt{2}\right)^2}\right) + \left(\frac{7}{12x} + \frac{1}{8x-}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(\frac{7}{12x} + \frac{1}{8x-8\sqrt{2}} + \frac{1}{8x+8\sqrt{2}}\right) dx} \\
&= \left(x-\sqrt{2}\right)^{1/8} \left(x+\sqrt{2}\right)^{1/8} x^{7/12}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-11x^3+x}{-3x^4+6x^2}\, dx} \\
&= z_1 e^{-\frac{7\ln\left(x^2-2\right)}{8} - \frac{\ln(x)}{12}} \\
&= z_1 \left(\frac{1}{\left(x^2-2\right)^{7/8} x^{1/12}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{\left(x^2-2\right)^{3/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-11x^3+x}{-3x^4+6x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{7\ln(x^2-2)}{4} - \frac{\ln(x)}{6}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{e^{-\frac{7\ln(x^2-2)}{4} - \frac{\ln(x)}{6}} (x^2-2)^{3/2}}{x} dx \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x}}{(x^2-2)^{3/4}} \right) + c_2 \left(\frac{\sqrt{x}}{(x^2-2)^{3/4}} \left(\int \frac{e^{-\frac{7\ln(x^2-2)}{4} - \frac{\ln(x)}{6}} (x^2-2)^{3/2}}{x} dx \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$3x^2(-x^2+2)\left(\frac{d^2}{dx^2}y(x)\right) + x(-11x^2+1)\left(\frac{d}{dx}y(x)\right) + (-5x^2+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(5x^2-1)y(x)}{3x^2(x^2-2)} - \frac{(11x^2-1)\left(\frac{d}{dx}y(x)\right)}{3x(x^2-2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(11x^2-1)\left(\frac{d}{dx}y(x)\right)}{3x(x^2-2)} + \frac{(5x^2-1)y(x)}{3x^2(x^2-2)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{11x^2-1}{3x(x^2-2)}, P_3(x) = \frac{5x^2-1}{3x^2(x^2-2)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = \frac{1}{6}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{1}{6}$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $3x^2(x^2-2)\left(\frac{d^2}{dx^2}y(x)\right) + x(11x^2-1)\left(\frac{d}{dx}y(x)\right) + (5x^2-1)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- Shift index using $k- > k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x^m \cdot \left(\frac{d}{dx} y(x)\right)$ to series expansion for $m = 1..3$

  $x^m \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$
- Shift index using $k- > k + 1 - m$

  $x^m \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$
- Convert $x^m \cdot \left(\frac{d^2}{dx^2} y(x)\right)$ to series expansion for $m = 2..4$

  $x^m \cdot \left(\frac{d^2}{dx^2} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-2+m}$
- Shift index using $k- > k + 2 - m$

  $x^m \cdot \left(\frac{d^2}{dx^2} y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0(-1+3r)(-1+2r)\, x^r - a_1(2+3r)(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (-a_k(3k+3r-1)(2k+2r-1) + a_k$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-(-1+3r)(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2}, \frac{1}{3}\right\}$
- Each term must be 0

  $-a_1(2+3r)(1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $-6\left(\frac{(-k-r+1)a_{k-2}}{2} + \left(k+r-\frac{1}{2}\right) a_k\right)\left(k-\frac{1}{3}+r\right) = 0$
- Shift index using $k- > k + 2$

  $-6\left(\frac{(-k-1-r)a_k}{2} + \left(k+\frac{3}{2}+r\right) a_{k+2}\right)\left(k+\frac{5}{3}+r\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{(k+r+1)a_k}{2k+3+2r}$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = \frac{\left(k+\frac{3}{2}\right)a_k}{2k+4}$
- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{\left(k+\frac{3}{2}\right)a_k}{2k+4}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{3}$

  $a_{k+2} = \frac{\left(k+\frac{4}{3}\right)a_k}{2k+\frac{11}{3}}$
- Solution for $r = \frac{1}{3}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = \frac{\left(k+\frac{4}{3}\right)a_k}{2k+\frac{11}{3}}, a_1 = 0\right]$
- Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{3}}\right), a_{k+2}=\frac{\left(k+\frac{3}{2}\right)a_k}{2k+4}, a_1=0, b_{k+2}=\frac{\left(k+\frac{4}{3}\right)b_k}{2k+\frac{11}{3}}, b_1=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - return
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.082 (sec)
Leaf size : 35

```
dsolve(3*x^2*(-x^2+2)*diff(diff(y(x),x),x)+x*(-11*x^2+1)*diff(y(x),x)+(-5*x^2+1)*y(x)
```

$$y=\frac{c_1\sqrt{x}}{\left(-2x^2+4\right)^{3/4}}+c_2 x^{1/3}\,\mathrm{hypergeom}\left(\left[\frac{2}{3},1\right],\left[\frac{11}{12}\right],\frac{x^2}{2}\right)$$

Mathematica DSolve solution

Solving time : 0.298 (sec)
Leaf size : 118

```
DSolve[{3*x^2*(2-x^2)*D[y[x],{x,2}]+x*(1-11*x^2)*D[y[x],x]+(1-5*x^2)*y[x]==0,{}},y[x],x,Includ
```

$$y(x) \to \exp\left(\int_1^x \frac{7-5K[1]^2}{12K[1]-6K[1]^3}dK[1] - \frac{1}{2}\int_1^x \frac{1-11K[2]^2}{6K[2]-3K[2]^3}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{7-5K[1]^2}{12K[1]-6K[1]^3}dK[1]\right)dK[3] + c_1\right)$$

2.1.104 Problem 106

Internal problem ID [9276]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 106
Date solved : `Monday, January 27, 2025 at 06:00:53 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2\left(x^2+2\right)y''-x\left(-7x^2+12\right)y'+\left(3x^2+7\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.340 (sec)

Writing the ode as

$$\left(2x^4+4x^2\right)y''+\left(7x^3-12x\right)y'+\left(3x^2+7\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^4+4x^2 \\ B &= 7x^3-12x \\ C &= 3x^2+7 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{-3x^4-72x^2+128}{16\left(x^3+2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3x^4-72x^2+128 \\ t &= 16\left(x^3+2x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-3x^4-72x^2+128}{16\left(x^3+2x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.192: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 6 - 4 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^3 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i\sqrt{2}$ of order 2. There is a pole at $x = -i\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2} + \frac{65}{64\left(x - i\sqrt{2}\right)^2} + \frac{65}{64\left(x + i\sqrt{2}\right)^2} + \frac{135i\sqrt{2}}{128\left(x - i\sqrt{2}\right)} - \frac{135i\sqrt{2}}{128\left(x + i\sqrt{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}$$

For the pole at $x = i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x-i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{65}{64}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{13}{8}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{8}\end{aligned}$$

For the pole at $x = -i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x+i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{65}{64}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{13}{8}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{8}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^4 - 72x^2 + 128}{16\left(x^3 + 2x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4}\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x^4 - 72x^2 + 128}{16\left(x^3 + 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1
$i\sqrt{2}$	2	0	$\frac{13}{8}$	$-\frac{5}{8}$
$-i\sqrt{2}$	2	0	$\frac{13}{8}$	$-\frac{5}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{4}$ then

$$\begin{aligned}d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right)\\ &= \frac{3}{4} - \left(\frac{3}{4}\right)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{2}{x} - \frac{5}{8\left(x-i\sqrt{2}\right)} - \frac{5}{8\left(x+i\sqrt{2}\right)} + (0) \\ &= \frac{2}{x} - \frac{5}{8\left(x-i\sqrt{2}\right)} - \frac{5}{8\left(x+i\sqrt{2}\right)} \\ &= \frac{2}{x} - \frac{5x}{4x^2+8}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{2}{x} - \frac{5}{8\left(x-i\sqrt{2}\right)} - \frac{5}{8\left(x+i\sqrt{2}\right)}\right)(0) + \left(\left(-\frac{2}{x^2} + \frac{5}{8\left(x-i\sqrt{2}\right)^2} + \frac{5}{8\left(x+i\sqrt{2}\right)^2}\right) + \left(\frac{2}{x} - \frac{}{8\left(x-\right.}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= pe^{\int \omega\, dx} \\ &= e^{\int \left(\frac{2}{x} - \frac{5}{8\left(x-i\sqrt{2}\right)} - \frac{5}{8\left(x+i\sqrt{2}\right)}\right) dx} \\ &= \frac{x^2}{\left(x^2+2\right)^{5/8}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{7x^3-12x}{2x^4+4x^2}\, dx} \\ &= z_1 e^{-\frac{13\ln\left(x^2+2\right)}{8} + \frac{3\ln(x)}{2}} \\ &= z_1 \left(\frac{x^{3/2}}{\left(x^2+2\right)^{13/8}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{7/2}}{\left(x^2+2\right)^{9/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x^3-12x}{2x^4+4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{13\ln(x^2+2)}{4}+3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{e^{-\frac{13\ln(x^2+2)}{4}+3\ln(x)}(x^2+2)^{9/2}}{x^7}\,dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^{7/2}}{(x^2+2)^{9/4}}\right) + c_2\left(\frac{x^{7/2}}{(x^2+2)^{9/4}}\left(\int \frac{e^{-\frac{13\ln(x^2+2)}{4}+3\ln(x)}(x^2+2)^{9/2}}{x^7}\,dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2(x^2+2)\left(\frac{d^2}{dx^2}y(x)\right) - x(-7x^2+12)\left(\frac{d}{dx}y(x)\right) + (3x^2+7)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(3x^2+7)y(x)}{2(x^2+2)x^2} - \frac{(7x^2-12)\left(\frac{d}{dx}y(x)\right)}{2x(x^2+2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(7x^2-12)\left(\frac{d}{dx}y(x)\right)}{2x(x^2+2)} + \frac{(3x^2+7)y(x)}{2(x^2+2)x^2} = 0$

- ☐ Check to see if x_0 is a regular singular point
  - ◦ Define functions

    $\left[P_2(x) = \frac{7x^2-12}{2x(x^2+2)}, P_3(x) = \frac{3x^2+7}{2(x^2+2)x^2}\right]$
  - ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -3$
  - ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{7}{4}$
  - ◦ $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $2x^2(x^2+2)\left(\frac{d^2}{dx^2}y(x)\right) + x(7x^2-12)\left(\frac{d}{dx}y(x)\right) + (3x^2+7)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- > k - m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
  - Shift index using $k- > k + 1 - m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$
  - Shift index using $k- > k + 2 - m$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)\,(-7+2r)\, x^r + a_1(1+2r)\,(-5+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r-1)\,(2k+2r-7) + a_{k-}\right.$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)\,(-7+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2}, \frac{7}{2}\right\}$
- Each term must be 0

  $a_1(1+2r)\,(-5+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r-\frac{1}{2}\right)\left(\frac{a_{k-2}(k+r-1)}{2} + a_k\left(k+r-\frac{7}{2}\right)\right) = 0$
- Shift index using $k- > k + 2$

  $4\left(k+\frac{3}{2}+r\right)\left(\frac{a_k(k+r+1)}{2} + a_{k+2}\left(k-\frac{3}{2}+r\right)\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r+1)}{2k-3+2r}$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{a_k\left(k+\frac{3}{2}\right)}{2k-2}$
- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{a_k\left(k+\frac{3}{2}\right)}{2k-2}, a_1 = 0\right]$
- Recursion relation for $r = \frac{7}{2}$

  $a_{k+2} = -\frac{a_k\left(k+\frac{9}{2}\right)}{2k+4}$
- Solution for $r = \frac{7}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{7}{2}}, a_{k+2} = -\frac{a_k\left(k+\frac{9}{2}\right)}{2k+4}, a_1 = 0\right]$

- Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{7}{2}}\right), a_{k+2}=-\frac{a_k\left(k+\frac{3}{2}\right)}{2k-2}, a_1=0, b_{k+2}=-\frac{b_k\left(k+\frac{9}{2}\right)}{2k+4}, b_1=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - return
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.085 (sec)
Leaf size : 35

```
dsolve(2*x^2*(x^2+2)*diff(diff(y(x),x),x)-x*(-7*x^2+12)*diff(y(x),x)+(3*x^2+7)*y(x) =
```

$$y=\frac{c_1 x^{7/2}}{(2x^2+4)^{9/4}}+c_2\sqrt{x}\,\mathrm{hypergeom}\left(\left[\frac{3}{4},1\right],\left[-\frac{1}{2}\right],-\frac{x^2}{2}\right)$$

Mathematica DSolve solution

Solving time : 0.316 (sec)
Leaf size : 117

```
DSolve[{2*x^2*(2+x^2)*D[y[x],{x,2}]-x*(12-7*x^2)*D[y[x],x]+(7+3*x^2)*y[x]==0,{}},y[x],x,Incl
```

$$y(x)\to\exp\left(\int_1^x\frac{3K[1]^2+16}{4K[1]^3+8K[1]}dK[1]-\frac{1}{2}\int_1^x\left(\frac{13K[2]}{2\left(K[2]^2+2\right)}-\frac{3}{K[2]}\right)dK[2]\right)\left(c_2\int_1^x\exp\left(-2\int_1^{K[3]}\frac{3K[1]^2+16}{4K[1]^3+8K[1]}dK[1]\right)dK[3]+c_1\right)$$

2.1.105 Problem 107

Internal problem ID [9277]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 107
Date solved : `Monday, January 27, 2025 at 06:00:54 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2\left(x^2+2\right)y''+x\left(7x^2+4\right)y'-\left(-3x^2+1\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.312 (sec)

Writing the ode as

$$\left(2x^4+4x^2\right)y''+\left(7x^3+4x\right)y'+\left(3x^2-1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^4+4x^2 \\ B &= 7x^3+4x \\ C &= 3x^2-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{-3x^2+24}{16\left(x^2+2\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3x^2+24 \\ t &= 16\left(x^2+2\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-3x^2+24}{16\left(x^2+2\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.194: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2+2)^2$. There is a pole at $x = i\sqrt{2}$ of order 2. There is a pole at $x = -i\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{15}{64\left(x - i\sqrt{2}\right)^2} - \frac{15}{64\left(x + i\sqrt{2}\right)^2} - \frac{9i\sqrt{2}}{128\left(x - i\sqrt{2}\right)} + \frac{9i\sqrt{2}}{128\left(x + i\sqrt{2}\right)}$$

For the pole at $x = i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x - i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{15}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{8} \end{aligned}$$

For the pole at $x = -i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x + i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{15}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{8} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 + 24}{16\left(x^2 + 2\right)^2}$$

Since the gcd$(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x^2 + 24}{16\left(x^2 + 2\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$i\sqrt{2}$	2	0	$\frac{5}{8}$	$\frac{3}{8}$
$-i\sqrt{2}$	2	0	$\frac{5}{8}$	$\frac{3}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{3}{4} - \left(\frac{3}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{3}{8\left(x - i\sqrt{2}\right)} + \frac{3}{8\left(x + i\sqrt{2}\right)} + (0) \\
&= \frac{3}{8\left(x - i\sqrt{2}\right)} + \frac{3}{8\left(x + i\sqrt{2}\right)} \\
&= \frac{3x}{4x^2 + 8}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{8\left(x - i\sqrt{2}\right)} + \frac{3}{8\left(x + i\sqrt{2}\right)}\right)(0) + \left(\left(-\frac{3}{8\left(x - i\sqrt{2}\right)^2} - \frac{3}{8\left(x + i\sqrt{2}\right)^2}\right) + \left(\frac{3}{8\left(x - i\sqrt{2}\right)} + \frac{}{8(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(\frac{3}{8\left(x-i\sqrt{2}\right)} + \frac{3}{8\left(x+i\sqrt{2}\right)}\right) dx} \\ &= \left(-x^2 - 2\right)^{3/8} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{7x^3+4x}{2x^4+4x^2}\, dx} \\ &= z_1 e^{-\frac{5\ln\left(x^2+2\right)}{8} - \frac{\ln(x)}{2}} \\ &= z_1\left(\frac{1}{\left(x^2+2\right)^{5/8}\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(-1)^{3/8}}{\left(x^2+2\right)^{1/4}\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{7x^3+4x}{2x^4+4x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{5\ln\left(x^2+2\right)}{4} - \ln(x)}}{(y_1)^2}\, dx \\ &= y_1\left(\int -\mathrm{e}^{-\frac{5\ln\left(x^2+2\right)}{4} - \ln(x)}\sqrt{x^2+2}\, x(-1)^{1/4}\, dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{(-1)^{3/8}}{\left(x^2+2\right)^{1/4}\sqrt{x}}\right) + c_2\left(\frac{(-1)^{3/8}}{\left(x^2+2\right)^{1/4}\sqrt{x}}\left(\int -\mathrm{e}^{-\frac{5\ln\left(x^2+2\right)}{4} - \ln(x)}\sqrt{x^2+2}\, x(-1)^{1/4}\, dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2(x^2+2)\left(\frac{d^2}{dx^2}y(x)\right)+x(7x^2+4)\left(\frac{d}{dx}y(x)\right)-(-3x^2+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(3x^2-1)y(x)}{2(x^2+2)x^2}-\frac{(7x^2+4)\left(\frac{d}{dx}y(x)\right)}{2x(x^2+2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{(7x^2+4)\left(\frac{d}{dx}y(x)\right)}{2x(x^2+2)}+\frac{(3x^2-1)y(x)}{2(x^2+2)x^2}=0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{7x^2+4}{2x(x^2+2)},P_3(x)=\frac{3x^2-1}{2(x^2+2)x^2}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=1$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=-\frac{1}{4}$

  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2x^2(x^2+2)\left(\frac{d^2}{dx^2}y(x)\right)+x(7x^2+4)\left(\frac{d}{dx}y(x)\right)+(3x^2-1)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

    $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..4$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$a_0(1+2r)(-1+2r)x^r + a_1(3+2r)(1+2r)x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)(2k+2r-1) + a_{k-2}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(1+2r)(-1+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0
  $a_1(3+2r)(1+2r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $4\left(\frac{a_{k-2}(k+r-1)}{2} + a_k\left(k+r+\frac{1}{2}\right)\right)\left(k+r-\frac{1}{2}\right) = 0$
- Shift index using $k->k+2$
  $4\left(\frac{a_k(k+r+1)}{2} + a_{k+2}\left(k+\frac{5}{2}+r\right)\right)\left(k+\frac{3}{2}+r\right) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{a_k(k+r+1)}{2k+5+2r}$
- Recursion relation for $r = -\frac{1}{2}$
  $a_{k+2} = -\frac{a_k\left(k+\frac{1}{2}\right)}{2k+4}$
- Solution for $r = -\frac{1}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{a_k\left(k+\frac{1}{2}\right)}{2k+4}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{2}$
  $a_{k+2} = -\frac{a_k\left(k+\frac{3}{2}\right)}{2k+6}$
- Solution for $r = \frac{1}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{a_k\left(k+\frac{3}{2}\right)}{2k+6}, a_1 = 0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{a_k\left(k+\frac{1}{2}\right)}{2k+4}, a_1 = 0, b_{k+2} = -\frac{b_k\left(k+\frac{3}{2}\right)}{2k+6}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Legendre successful
```

```
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.058 (sec)
Leaf size : 35

```
dsolve(2*x^2*(x^2+2)*diff(diff(y(x),x),x)+x*(7*x^2+4)*diff(y(x),x)-(-3*x^2+1)*y(x) = 0,y
```

$$y = \frac{c_2 \operatorname{LegendreQ}\left(-\frac{1}{4}, \frac{1}{4}, \frac{i\sqrt{2}\,x}{2}\right)(x^2+2)^{1/8} + c_1}{(x^2+2)^{1/4}\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.243 (sec)
Leaf size : 95

```
DSolve[{2*x^2*(2+x^2)*D[y[x],{x,2}]+x*(4+7*x^2)*D[y[x],x]-(1-3*x^2)*y[x]==0,{}},y[x],x,IncludeS
```

$$y(x) \to \frac{\left(c_2 \sqrt[8]{x^2+2}\,\mathrm{Gamma}\left(\frac{3}{4}\right) Q_{-\frac{1}{4}}^{\frac{1}{4}}\left(\frac{ix}{\sqrt{2}}\right) + 2^{3/8} c_1\right)\exp\left(\int_1^x -\frac{3K[1]^2+4}{4K[1]^3+8K[1]}dK[1]\right)}{\sqrt[8]{x^2+2}\,\mathrm{Gamma}\left(\frac{3}{4}\right)}$$

2.1.106 Problem 108

Internal problem ID [9278]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 108
Date solved : `Monday, January 27, 2025 at 06:00:55 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2\left(2x^2+1\right)y''+5x\left(6x^2+1\right)y'-\left(-40x^2+2\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.517 (sec)

Writing the ode as

$$\left(4x^4+2x^2\right)y''+\left(30x^3+5x\right)y'+\left(40x^2-2\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^4+2x^2 \\ B &= 30x^3+5x \\ C &= 40x^2-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{20x^4+12x^2+21}{16\left(2x^3+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 20x^4+12x^2+21 \\ t &= 16\left(2x^3+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{20x^4+12x^2+21}{16\left(2x^3+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.196: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(2x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{i\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{i\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{21}{16x^2} + \frac{5}{16\left(x - \frac{i\sqrt{2}}{2}\right)^2} + \frac{5}{16\left(x + \frac{i\sqrt{2}}{2}\right)^2} + \frac{13i\sqrt{2}}{16\left(x - \frac{i\sqrt{2}}{2}\right)} - \frac{13i\sqrt{2}}{16\left(x + \frac{i\sqrt{2}}{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{4} \end{aligned}$$

For the pole at $x = \frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}\end{aligned}$$

For the pole at $x = -\frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{20x^4 + 12x^2 + 21}{16\,(2x^3 + x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4}\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{20x^4 + 12x^2 + 21}{16\,(2x^3 + x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
$\frac{i\sqrt{2}}{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-\frac{i\sqrt{2}}{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{4}$ then

$$\begin{aligned}d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right)\\ &= \frac{5}{4} - \left(\frac{5}{4}\right)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{7}{4x} - \frac{1}{4\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{1}{4\left(x+\frac{i\sqrt{2}}{2}\right)} + (0) \\
&= \frac{7}{4x} - \frac{1}{4\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{1}{4\left(x+\frac{i\sqrt{2}}{2}\right)} \\
&= \frac{10x^2+7}{8x^3+4x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{7}{4x} - \frac{1}{4\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{1}{4\left(x+\frac{i\sqrt{2}}{2}\right)} \right)(0) + \left(\left(-\frac{7}{4x^2} + \frac{1}{4\left(x-\frac{i\sqrt{2}}{2}\right)^2} + \frac{1}{4\left(x+\frac{i\sqrt{2}}{2}\right)^2} \right) + \left(\frac{7}{4x} - \frac{}{4} \right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{7}{4x} - \frac{1}{4\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{1}{4\left(x+\frac{i\sqrt{2}}{2}\right)} \right) dx} \\
&= \frac{2^{3/4} x^{7/4}}{2\left(2x^2+1\right)^{1/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{30x^3+5x}{4x^4+2x^2} \, dx} \\
&= z_1 e^{-\frac{5\ln\left(x\left(2x^2+1\right)\right)}{4}} \\
&= z_1 \left(\frac{1}{\left(2x^3+x\right)^{5/4}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{2^{3/4} x^{3/4}}{2\left(2x^2+1\right)^{5/4} \left(2x^3+x\right)^{1/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{30x^3+5x}{4x^4+2x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{5\ln(2x^3+x)}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{(2x^2+1)^{5/2}\sqrt{2}}{(2x^3+x)^2\, x^{3/2}} dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{2^{3/4}x^{3/4}}{2(2x^2+1)^{5/4}(2x^3+x)^{1/4}}\right) + c_2 \left(\frac{2^{3/4}x^{3/4}}{2(2x^2+1)^{5/4}(2x^3+x)^{1/4}} \left(\int \frac{(2x^2+1)^{5/2}\sqrt{2}}{(2x^3+x)^2\, x^{3/2}} dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$2x^2(2x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + 5x(6x^2+1)\left(\frac{d}{dx}y(x)\right) - (-40x^2+2)\, y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{(20x^2-1)y(x)}{x^2(2x^2+1)} - \frac{5(6x^2+1)\left(\frac{d}{dx}y(x)\right)}{2x(2x^2+1)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x) + \frac{5(6x^2+1)\left(\frac{d}{dx}y(x)\right)}{2x(2x^2+1)} + \frac{(20x^2-1)y(x)}{x^2(2x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{5(6x^2+1)}{2x(2x^2+1)}, P_3(x) = \frac{20x^2-1}{x^2(2x^2+1)}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{5}{2}$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -1$
  - $x = 0$is a regular singular point
    Check to see if x_0 is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $2x^2(2x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + 5x(6x^2+1)\left(\frac{d}{dx}y(x)\right) + (40x^2-2)\, y(x) = 0$
- Assume series solution for $y(x)$

$y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - ∘ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - ∘ Shift index using $k->k-m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - ∘ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
  - ∘ Shift index using $k->k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - ∘ Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$
  - ∘ Shift index using $k->k+2-m$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)(-1+2r)\, x^r + a_1(3+r)(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+2)(2k+2r-1) + 2a_{k-2}(k+r\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-2, \frac{1}{2}\right\}$
- Each term must be 0

  $a_1(3+r)(1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $2(k+r+2)\left(a_{k-2}(2k+1+2r) + \left(k+r-\frac{1}{2}\right) a_k\right) = 0$
- Shift index using $k->k+2$

  $2(k+r+4)\left(a_k(2k+2r+5) + \left(k+\frac{3}{2}+r\right) a_{k+2}\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2a_k(2k+2r+5)}{2k+3+2r}$
- Recursion relation for $r = -2$

  $a_{k+2} = -\frac{2a_k(2k+1)}{2k-1}$
- Solution for $r = -2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{2a_k(2k+1)}{2k-1}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{2a_k(2k+6)}{2k+4}$
- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{2a_k(2k+6)}{2k+4}, a_1 = 0\right]$

- Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty}a_k x^{k-2}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{2}}\right), a_{k+2}=-\frac{2a_k(2k+1)}{2k-1}, a_1=0, b_{k+2}=-\frac{2b_k(2k+6)}{2k+4}, b_1=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - return
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.085 (sec)
Leaf size : 35

```
dsolve(2*x^2*(2*x^2+1)*diff(diff(y(x),x),x)+5*x*(6*x^2+1)*diff(y(x),x)-(-40*x^2+2)*y(x
```

$$y=\frac{c_1\sqrt{x}}{(2x^2+1)^{3/2}}+\frac{c_2\,\mathrm{hypergeom}\left(\left[\frac{1}{4},1\right],\left[-\frac{1}{4}\right],-2x^2\right)}{x^2}$$

Mathematica DSolve solution

Solving time : `0.295` (sec)
Leaf size : 118

```
DSolve[{2*x^2*(1+2*x^2)*D[y[x],{x,2}]+5*x*(1+6*x^2)*D[y[x],x]-(2-40*x^2)*y[x]==0,{}},y[x],x,Inc
```

$$y(x) \to \exp\left(\int_1^x \frac{10K[1]^2+7}{8K[1]^3+4K[1]}dK[1] - \frac{1}{2}\int_1^x \frac{30K[2]^2+5}{4K[2]^3+2K[2]}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{10K[1]^2+7}{8K[1]^3+4K[1]}dK[1]\right)dK[3] + c_1\right)$$

2.1.107 Problem 109

Internal problem ID [9279]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 109
Date solved : `Monday, January 27, 2025 at 06:00:56 PM`
CAS classification : `[[_2nd_order, _exact, _linear, _homogeneous]]`

Solve

$$x(x^2+1)\,y''+(7x^2+4)\,y'+8xy=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.331 (sec)

Writing the ode as

$$(x^3+x)\,y''+(7x^2+4)\,y'+8xy=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= x^3+x\\ B &= 7x^2+4\\ C &= 8x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{3x^4+14x^2+8}{4\,(x^3+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s &= 3x^4+14x^2+8\\ t &= 4(x^3+x)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{3x^4+14x^2+8}{4\,(x^3+x)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.198: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3+x)^2$. There is a pole at $x=0$ of order 2. There is a pole at $x=i$ of order 2. There is a pole at $x=-i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2} - \frac{3}{16(x-i)^2} - \frac{3}{16(x+i)^2} + \frac{7i}{16(x-i)} - \frac{7i}{16(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4}\\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{3x^4 + 14x^2 + 8}{4\left(x^3 + x\right)^2}
$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3x^4 + 14x^2 + 8}{4\left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1
i	2	0	$\frac{3}{4}$	$\frac{1}{4}$
$-i$	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right)\\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + \frac{1}{4x-4i} + \frac{1}{4x+4i} + (-)\,(0) \\
&= -\frac{1}{x} + \frac{1}{4x-4i} + \frac{1}{4x+4i} \\
&= -\frac{1}{x} + \frac{x}{2x^2+2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{x} + \frac{1}{4x-4i} + \frac{1}{4x+4i}\right)(0) + \left(\left(\frac{1}{x^2} - \frac{1}{4\,(x-i)^2} - \frac{1}{4\,(x+i)^2}\right) + \left(-\frac{1}{x} + \frac{1}{4x-4i} + \frac{1}{4x+4i}\right)^2\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{1}{x} + \frac{1}{4x-4i} + \frac{1}{4x+4i}\right) dx} \\
&= \frac{\left(x^2+1\right)^{1/4}}{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7x^2+4}{x^3+x}\, dx} \\
&= z_1 e^{-\frac{3\ln\left(x^2+1\right)}{4} - 2\ln(x)} \\
&= z_1 \left(\frac{1}{\left(x^2+1\right)^{3/4} x^2}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{\sqrt{x^2+1}\, x^3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x^2+4}{x^3+x}dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{3\ln(x^2+1)}{2}-4\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{x^5}{\sqrt{x^2+1}} - \frac{x^3}{3\sqrt{x^2+1}} + \frac{4x^7}{\sqrt{x^2+1}} + \frac{8x^9}{3\sqrt{x^2+1}} + \frac{x\sqrt{x^2+1}}{2} - \frac{\operatorname{arcsinh}(x)}{2} \right. \\
&\quad \left. + \frac{\sqrt{x^2+1}\, x^3}{3} - \frac{4x^5\sqrt{x^2+1}}{3} - \frac{8x^7\sqrt{x^2+1}}{3} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{\sqrt{x^2+1}\, x^3} \right) + c_2 \left(\frac{1}{\sqrt{x^2+1}\, x^3} \left(\frac{x^5}{\sqrt{x^2+1}} - \frac{x^3}{3\sqrt{x^2+1}} + \frac{4x^7}{\sqrt{x^2+1}} + \frac{8x^9}{3\sqrt{x^2+1}} \right. \right. \\
&\quad \left. \left. + \frac{x\sqrt{x^2+1}}{2} - \frac{\operatorname{arcsinh}(x)}{2} + \frac{\sqrt{x^2+1}\, x^3}{3} - \frac{4x^5\sqrt{x^2+1}}{3} - \frac{8x^7\sqrt{x^2+1}}{3} \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + (7x^2+4)\left(\frac{d}{dx}y(x)\right) + 8xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{8y(x)}{x^2+1} - \frac{(7x^2+4)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(7x^2+4)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)} + \frac{8y(x)}{x^2+1} = 0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{7x^2+4}{x(x^2+1)}, P_3(x) = \frac{8}{x^2+1}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 4$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$
  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

$$x(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)+(7x^2+4)\left(\frac{d}{dx}y(x)\right)+8xy(x)=0$$

- Assume series solution for $y(x)$

$$y(x)=\sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- ∘ Convert $x\cdot y(x)$ to series expansion

$$x\cdot y(x)=\sum_{k=0}^{\infty} a_k x^{k+r+1}$$

- ∘ Shift index using $k->k-1$

$$x\cdot y(x)=\sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

- ∘ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..2$

$$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

- ∘ Shift index using $k->k+1-m$

$$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- ∘ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=1..3$

$$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

- ∘ Shift index using $k->k+2-m$

$$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$a_0r(3+r)\,x^{-1+r}+a_1(1+r)\,(4+r)\,x^r+\left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+r+1)\,(k+r+4)+a_{k-1}(k+r+3)\,(k+\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(3+r)=0$
- Values of r that satisfy the indicial equation
  $r\in\{-3,0\}$
- Each term must be 0
  $a_1(1+r)\,(4+r)=0$
- Each term in the series must be 0, giving the recursion relation
  $(k+r+1)\,(a_{k+1}(k+r+4)+a_{k-1}(k+r+3))=0$
- Shift index using $k->k+1$
  $(k+r+2)\,(a_{k+2}(k+5+r)+a_k(k+r+4))=0$
- Recursion relation that defines series solution to ODE
  $a_{k+2}=-\frac{a_k(k+r+4)}{k+5+r}$
- Recursion relation for $r=-3$
  $a_{k+2}=-\frac{a_k(k+1)}{k+2}$
- Solution for $r=-3$
  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2}=-\frac{a_k(k+1)}{k+2}, -2a_1=0\right]$
- Recursion relation for $r=0$
  $a_{k+2}=-\frac{a_k(k+4)}{k+5}$
- Solution for $r=0$
  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=-\frac{a_k(k+4)}{k+5}, 4a_1=0\right]$

- Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty}a_k x^{k-3}\right)+\left(\sum_{k=0}^{\infty}b_k x^k\right), a_{k+2}=-\frac{a_k(k+1)}{k+2}, -2a_1=0, b_{k+2}=-\frac{b_k(4+k)}{5+k}, 4b_1=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.037 (sec)
Leaf size : 32

```
dsolve(x*(x^2+1)*diff(diff(y(x),x),x)+(7*x^2+4)*diff(y(x),x)+8*x*y(x) = 0,y(x),singsol
```

$$y=\frac{-\sqrt{x^2+1}\,c_2 x+\operatorname{arcsinh}(x)\,c_2+c_1}{\sqrt{x^2+1}\,x^3}$$

Mathematica DSolve solution

Solving time : 0.22 (sec)
Leaf size : 108

```
DSolve[{x*(1+x^2)*D[y[x],{x,2}]+(4+7*x^2)*D[y[x],x]+8*x*y[x]==0,{}},y[x],x,IncludeSingularSo
```

$$y(x)\to\exp\left(\int_1^x-\frac{K[1]^2+2}{2\left(K[1]^3+K[1]\right)}dK[1]-\frac{1}{2}\int_1^x\frac{7K[2]^2+4}{K[2]^3+K[2]}dK[2]\right)\left(c_2\int_1^x\exp\left(-2\int_1^{K[3]}-\frac{K[1]^2+2}{2\left(K[1]^3+K[1]\right)}dK[1]\right)dK[3]+c_1\right)$$

2.1.108 Problem 110

Internal problem ID [9280]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 110
Date solved : `Monday, January 27, 2025 at 06:00:56 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2\left(x^2+1\right)y''+x\left(8x^2+3\right)y'-\left(-4x^2+3\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.312 (sec)

Writing the ode as

$$\left(2x^4+2x^2\right)y''+\left(8x^3+3x\right)y'+\left(4x^2-3\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=2x^4+2x^2\\ B&=8x^3+3x\\ C&=4x^2-3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{36x^2+21}{16\left(x^3+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=36x^2+21\\ t&=16\left(x^3+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{36x^2+21}{16\left(x^3+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.200: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{15}{64\,(x-i)^2} - \frac{15}{64\,(x+i)^2} + \frac{27i}{64\,(x-i)} - \frac{27i}{64\,(x+i)} + \frac{21}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{15}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{8} \end{aligned}$$

For the pole at $x=-i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{15}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{8}\\
\alpha_c^- &= \frac{1}{2}-\sqrt{1+4b}=\frac{3}{8}
\end{aligned}
$$

Since the order of r at ∞ is $4>2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^+ &= 0\\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{36x^2+21}{16\left(x^3+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
i	2	0	$\frac{5}{8}$	$\frac{3}{8}$
$-i$	2	0	$\frac{5}{8}$	$\frac{3}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=0$ then

$$
\begin{aligned}
d &= \alpha_\infty^+-\left(\alpha_{c_1}^-+\alpha_{c_2}^-+\alpha_{c_3}^-\right)\\
&= 0-(0)\\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^-}{x-c_1}\right)+\left((-)[\sqrt{r}]_{c_2}+\frac{\alpha_{c_2}^-}{x-c_2}\right)+\left((-)[\sqrt{r}]_{c_3}+\frac{\alpha_{c_3}^-}{x-c_3}\right)+(+)[\sqrt{r}]_\infty\\
&= -\frac{3}{4x}+\frac{3}{8\left(x-i\right)}+\frac{3}{8\left(x+i\right)}+(0)\\
&= -\frac{3}{4x}+\frac{3}{8\left(x-i\right)}+\frac{3}{8\left(x+i\right)}\\
&= -\frac{3}{4x\left(x^2+1\right)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{4x} + \frac{3}{8(x-i)} + \frac{3}{8(x+i)}\right)(0) + \left(\left(\frac{3}{4x^2} - \frac{3}{8(x-i)^2} - \frac{3}{8(x+i)^2}\right) + \left(-\frac{3}{4x} + \frac{3}{8(x-i)} + \frac{}{8}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{3}{4x} + \frac{3}{8(x-i)} + \frac{3}{8(x+i)}\right) dx} \\ &= \frac{(x^2+1)^{3/8}}{x^{3/4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{8x^3+3x}{2x^4+2x^2}\, dx} \\ &= z_1 e^{-\frac{3\ln(x)}{4} - \frac{5\ln(x^2+1)}{8}} \\ &= z_1 \left(\frac{1}{x^{3/4}(x^2+1)^{5/8}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^{3/2}(x^2+1)^{1/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{8x^3+3x}{2x^4+2x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{3\ln(x)}{2} - \frac{5\ln(x^2+1)}{4}}}{(y_1)^2}\, dx \\ &= y_1 \left(\int e^{-\frac{3\ln(x)}{2} - \frac{5\ln(x^2+1)}{4}} x^3 \sqrt{x^2+1}\, dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{1}{x^{3/2}(x^2+1)^{1/4}}\right) + c_2 \left(\frac{1}{x^{3/2}(x^2+1)^{1/4}} \left(\int e^{-\frac{3\ln(x)}{2} - \frac{5\ln(x^2+1)}{4}} x^3 \sqrt{x^2+1}\, dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(8x^2+3)\left(\frac{d}{dx}y(x)\right)-(-4x^2+3)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(4x^2-3)y(x)}{2x^2(x^2+1)}-\frac{(8x^2+3)\left(\frac{d}{dx}y(x)\right)}{2x(x^2+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d^2}{dx^2}y(x)+\frac{(8x^2+3)\left(\frac{d}{dx}y(x)\right)}{2x(x^2+1)}+\frac{(4x^2-3)y(x)}{2x^2(x^2+1)}=0$

- □ Check to see if x_0 is a regular singular point
  - ∘ Define functions

    $\left[P_2(x)=\frac{8x^2+3}{2x(x^2+1)}, P_3(x)=\frac{4x^2-3}{2x^2(x^2+1)}\right]$

  - ∘ $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\bigg|_{x=0}=\frac{3}{2}$

  - ∘ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\bigg|_{x=0}=-\frac{3}{2}$

  - ∘ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(8x^2+3)\left(\frac{d}{dx}y(x)\right)+(4x^2-3)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - ∘ Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

    $x^m\cdot y(x)=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - ∘ Shift index using $k->k-m$

    $x^m\cdot y(x)=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - ∘ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - ∘ Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - ∘ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..4$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,x^{k+r-2+m}$

  - ∘ Shift index using $k->k+2-m$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$a_0(2r+3)(-1+r)x^r + a_1(5+2r)rx^{1+r} + \left(\sum_{k=2}^{\infty}\left(a_k(2k+2r+3)(k+r-1)+2a_{k-2}(k+r)(k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $$(2r+3)(-1+r) = 0$$
- Values of r that satisfy the indicial equation
  $$r \in \left\{1, -\frac{3}{2}\right\}$$
- Each term must be 0
  $$a_1(5+2r)r = 0$$
- Solve for the dependent coefficient(s)
  $$a_1 = 0$$
- Each term in the series must be 0, giving the recursion relation
  $$2\left(\left(k+r+\frac{3}{2}\right)a_k + a_{k-2}(k+r)\right)(k+r-1) = 0$$
- Shift index using $k->k+2$
  $$2\left(\left(k+\frac{7}{2}+r\right)a_{k+2} + a_k(k+r+2)\right)(k+r+1) = 0$$
- Recursion relation that defines series solution to ODE
  $$a_{k+2} = -\frac{2a_k(k+r+2)}{2k+7+2r}$$
- Recursion relation for $r=1$
  $$a_{k+2} = -\frac{2a_k(k+3)}{2k+9}$$
- Solution for $r=1$
  $$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{2a_k(k+3)}{2k+9}, a_1 = 0\right]$$
- Recursion relation for $r=-\frac{3}{2}$
  $$a_{k+2} = -\frac{2a_k\left(k+\frac{1}{2}\right)}{2k+4}$$
- Solution for $r=-\frac{3}{2}$
  $$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}, a_{k+2} = -\frac{2a_k\left(k+\frac{1}{2}\right)}{2k+4}, a_1 = 0\right]$$
- Combine solutions and rename parameters
  $$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{3}{2}}\right), a_{k+2} = -\frac{2a_k(k+3)}{2k+9}, a_1 = 0, b_{k+2} = -\frac{2b_k\left(k+\frac{1}{2}\right)}{2k+4}, b_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
```

```
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returnin
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.100 (sec)
Leaf size : 31

```
dsolve(2*x^2*(x^2+1)*diff(diff(y(x),x),x)+x*(8*x^2+3)*diff(y(x),x)-(-4*x^2+3)*y(x) = 0,y
```

$$y = c_1 x \,\mathrm{hypergeom}\left(\left[1, \frac{3}{2}\right], \left[\frac{9}{4}\right], -x^2\right) + \frac{c_2}{\left(x^2+1\right)^{1/4} x^{3/2}}$$

Mathematica DSolve solution

Solving time : 0.188 (sec)
Leaf size : 99

```
DSolve[{2*x^2*(1+x^2)*D[y[x],{x,2}]+x*(3+8*x^2)*D[y[x],x]-(3-4*x^2)*y[x]==0,{}},y[x],x,IncludeS
```

$$y(x) \to \exp\left(\int_1^x -\frac{3}{4\left(K[1]^3+K[1]\right)}dK[1] - \frac{1}{2}\int_1^x \frac{8K[2]^2+3}{2\left(K[2]^3+K[2]\right)}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} -\frac{3}{4\left(K[1]^3+K[1]\right)}dK[1]\right)dK[3] + c_1\right)$$

2.1.109 Problem 111

Internal problem ID [9281]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 111
Date solved : `Monday, January 27, 2025 at 06:00:57 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$9x^2y'' + 3x\left(x^2+3\right)y' - \left(-5x^2+1\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.277 (sec)

Writing the ode as

$$9x^2y'' + \left(3x^3+9x\right)y' + \left(5x^2-1\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^2 \\ B &= 3x^3+9x \\ C &= 5x^2-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4-8x^2-5}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4-8x^2-5 \\ t &= 36x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4-8x^2-5}{36x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.202: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{36} - \frac{2}{9} - \frac{5}{36x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{6} - \frac{2}{3x} - \frac{7}{4x^3} - \frac{7}{x^5} - \frac{595}{16x^7} - \frac{889}{4x^9} - \frac{45647}{32x^{11}} - \frac{76811}{8x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{6}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{36}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^4 - 8x^2 - 5}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{x^2}{36} - \frac{2}{9}\right) + \left(-\frac{5}{36x^2}\right) \\ &= \frac{x^2}{36} - \frac{2}{9} - \frac{5}{36x^2}\end{aligned}$$

We see that the coefficient of the term x in the quotient is $-\frac{2}{9}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{2}{9}\right) - (0) \\ &= -\frac{2}{9}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{x}{6} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{2}{9}}{\frac{1}{6}} - 1\right) &&= -\frac{7}{6} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{2}{9}}{\frac{1}{6}} - 1\right) &&= \frac{1}{6}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 8x^2 - 5}{36x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{6}$	$-\frac{7}{6}$	$\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{6}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{6} - \left(\frac{1}{6}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{6x} + (-)\left(\frac{x}{6}\right) \\ &= \frac{1}{6x} - \frac{x}{6} \\ &= \frac{1}{6x} - \frac{x}{6} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{6x} - \frac{x}{6}\right)(0) + \left(\left(-\frac{1}{6x^2} - \frac{1}{6}\right) + \left(\frac{1}{6x} - \frac{x}{6}\right)^2 - \left(\frac{x^4 - 8x^2 - 5}{36x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{1}{6x} - \frac{x}{6}\right) dx} \\ &= x^{1/6} e^{-\frac{x^2}{12}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x^3+9x}{9x^2}\,dx} \\
&= z_1 e^{-\frac{x^2}{12}-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{e^{-\frac{x^2}{12}}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{e^{-\frac{x^2}{6}}}{x^{1/3}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^3+9x}{9x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{6}-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\int e^{-\frac{x^2}{6}-\ln(x)} x^{2/3} e^{\frac{x^2}{3}}\,dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{-\frac{x^2}{6}}}{x^{1/3}}\right) + c_2 \left(\frac{e^{-\frac{x^2}{6}}}{x^{1/3}}\left(\int e^{-\frac{x^2}{6}-\ln(x)} x^{2/3} e^{\frac{x^2}{3}}\,dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$9x^2\left(\frac{d^2}{dx^2}y(x)\right) + 3x(x^2+3)\left(\frac{d}{dx}y(x)\right) - (-5x^2+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(5x^2-1)y(x)}{9x^2} - \frac{(x^2+3)\left(\frac{d}{dx}y(x)\right)}{3x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lir

  $\frac{d^2}{dx^2}y(x) + \frac{(x^2+3)\left(\frac{d}{dx}y(x)\right)}{3x} + \frac{(5x^2-1)y(x)}{9x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = \frac{x^2+3}{3x}, P_3(x) = \frac{5x^2-1}{9x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{1}{9}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $9x^2\left(\frac{d^2}{dx^2}y(x)\right) + 3x(x^2+3)\left(\frac{d}{dx}y(x)\right) + (5x^2-1)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k-m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+3r)\,(-1+3r)\,x^r + a_1(4+3r)\,(2+3r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r+1)\,(3k+3r-1) + a_{k-2}(3k\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+3r)\,(-1+3r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{3}, \frac{1}{3}\right\}$

- Each term must be 0

  $a_1(4+3r)\,(2+3r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(3k+3r-1)\,(3a_kk + 3a_kr + a_k + a_{k-2}) = 0$

- Shift index using $k- >k+2$

  $(3k+3r+5)\,(3a_{k+2}(k+2) + 3a_{k+2}r + a_{k+2} + a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{3k+7+3r}$

- Recursion relation for $r=-\frac{1}{3}$

  $a_{k+2}=-\frac{a_k}{3k+6}$
- Solution for $r=-\frac{1}{3}$

  $$\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}, a_{k+2}=-\frac{a_k}{3k+6}, a_1=0\right]$$
- Recursion relation for $r=\frac{1}{3}$

  $a_{k+2}=-\frac{a_k}{3k+8}$
- Solution for $r=\frac{1}{3}$

  $$\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2}=-\frac{a_k}{3k+8}, a_1=0\right]$$
- Combine solutions and rename parameters

  $$\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2}=-\frac{a_k}{3k+6}, a_1=0, b_{k+2}=-\frac{b_k}{3k+8}, b_1=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Whittaker successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - return
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.041` (sec)
Leaf size : 37

```
dsolve(9*x^2*diff(diff(y(x),x),x)+3*x*(x^2+3)*diff(y(x),x)-(-5*x^2+1)*y(x) = 0,y(x),si
```

$$y=\frac{e^{-\frac{x^2}{12}}\left(e^{-\frac{x^2}{12}} c_2 x+\operatorname{WhittakerM}\left(\frac{1}{3}, \frac{1}{6}, \frac{x^2}{6}\right) x^{1/3} c_1\right)}{x^{4/3}}$$

Mathematica DSolve solution

Solving time : 0.456 (sec)
Leaf size : 70

```
DSolve[{9*x^2*D[y[x],{x,2}]+3*x*(3+x^2)*D[y[x],x]-(1-5*x^2)*y[x]==0,{}},y[x],x,IncludeSingularS
```

$$y(x) \to \frac{e^{-\frac{x^2}{6}-\frac{1}{6}}\left(2\sqrt[3]{e}c_1x^{4/3}+\sqrt[3]{6}c_2(-x^2)^{2/3}\,\Gamma\left(\frac{1}{3},-\frac{x^2}{6}\right)\right)}{2x^{5/3}}$$

2.1.110 Problem 112

Internal problem ID [9282]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 112
Date solved : `Monday, January 27, 2025 at 06:00:58 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$6x^2y'' + x(6x^2+1)\,y' + (9x^2+1)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.336 (sec)

Writing the ode as

$$6x^2y'' + (6x^3+x)\,y' + (9x^2+1)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 6x^2 \\ B &= 6x^3 + x \\ C &= 9x^2 + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{36x^4 - 132x^2 - 35}{144x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 36x^4 - 132x^2 - 35 \\ t &= 144x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{36x^4 - 132x^2 - 35}{144x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.204: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 144x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{4} - \frac{11}{12} - \frac{35}{144x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{11}{12x} - \frac{13}{12x^3} - \frac{143}{72x^5} - \frac{130}{27x^7} - \frac{17017}{1296x^9} - \frac{597961}{15552x^{11}} - \frac{11016863}{93312x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{36x^4 - 132x^2 - 35}{144x^2} \\ &= Q + \frac{R}{144x^2} \\ &= \left(\frac{x^2}{4} - \frac{11}{12}\right) + \left(-\frac{35}{144x^2}\right) \\ &= \frac{x^2}{4} - \frac{11}{12} - \frac{35}{144x^2}\end{aligned}$$

We see that the coefficient of the term x in the quotient is $-\frac{11}{12}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{11}{12}\right) - (0) \\ &= -\frac{11}{12}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{11}{12}}{\frac{1}{2}} - 1\right) &= -\frac{17}{12} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{11}{12}}{\frac{1}{2}} - 1\right) &= \frac{5}{12}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{36x^4 - 132x^2 - 35}{144x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{12}$	$\frac{5}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	$-\frac{17}{12}$	$\frac{5}{12}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{5}{12}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{5}{12} - \left(\frac{5}{12}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{5}{12x} + (-)\left(\frac{x}{2}\right) \\ &= \frac{5}{12x} - \frac{x}{2} \\ &= \frac{5}{12x} - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{5}{12x} - \frac{x}{2}\right)(0) + \left(\left(-\frac{5}{12x^2} - \frac{1}{2}\right) + \left(\frac{5}{12x} - \frac{x}{2}\right)^2 - \left(\frac{36x^4 - 132x^2 - 35}{144x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{5}{12x} - \frac{x}{2}\right) dx} \\ &= x^{5/12} e^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{6x^3+x}{6x^2}\,dx} \\
&= z_1 e^{-\frac{x^2}{4}-\frac{\ln(x)}{12}} \\
&= z_1\left(\frac{\mathrm{e}^{-\frac{x^2}{4}}}{x^{1/12}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^{1/3}\mathrm{e}^{-\frac{x^2}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{6x^3+x}{6x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}-\frac{\ln(x)}{6}}}{(y_1)^2}\,dx \\
&= y_1\left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}-\frac{\ln(x)}{6}}\,\mathrm{e}^{x^2}}{x^{2/3}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^{1/3}\mathrm{e}^{-\frac{x^2}{2}}\right) + c_2\left(x^{1/3}\mathrm{e}^{-\frac{x^2}{2}}\left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}-\frac{\ln(x)}{6}}\,\mathrm{e}^{x^2}}{x^{2/3}}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$6x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(6x^2+1)\left(\frac{d}{dx}y(x)\right) + (9x^2+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(9x^2+1)y(x)}{6x^2} - \frac{(6x^2+1)\left(\frac{d}{dx}y(x)\right)}{6x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(6x^2+1)\left(\frac{d}{dx}y(x)\right)}{6x} + \frac{(9x^2+1)y(x)}{6x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

$\left[P_2(x) = \frac{6x^2+1}{6x}, P_3(x) = \frac{9x^2+1}{6x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = \frac{1}{6}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{1}{6}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $6x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(6x^2+1)\left(\frac{d}{dx}y(x)\right) + (9x^2+1)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k-m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+3r)(-1+2r)\,x^r + a_1(2+3r)(1+2r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(3k+3r-1)(2k+2r-1) + 3a_{k-2}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+3r)(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2}, \frac{1}{3}\right\}$

- Each term must be 0

  $a_1(2+3r)(1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $6\left(\left(k - \frac{1}{3} + r\right)a_k + a_{k-2}\right)\left(k + r - \frac{1}{2}\right) = 0$

- Shift index using $k- >k+2$

  $6\left(\left(k + \frac{5}{3} + r\right)a_{k+2} + a_k\right)\left(k + \frac{3}{2} + r\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{3a_k}{3k+5+3r}$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{3a_k}{3k+\frac{13}{2}}$
- Solution for $r = \frac{1}{2}$

  $$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{3a_k}{3k+\frac{13}{2}}, a_1 = 0\right]$$
- Recursion relation for $r = \frac{1}{3}$

  $a_{k+2} = -\frac{3a_k}{3k+6}$
- Solution for $r = \frac{1}{3}$

  $$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -\frac{3a_k}{3k+6}, a_1 = 0\right]$$
- Combine solutions and rename parameters

  $$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = -\frac{3a_k}{3k+\frac{13}{2}}, a_1 = 0, b_{k+2} = -\frac{3b_k}{3k+6}, b_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Whittaker successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - return
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.076 (sec)
Leaf size : 36

```
dsolve(6*x^2*diff(diff(y(x),x),x)+x*(6*x^2+1)*diff(y(x),x)+(9*x^2+1)*y(x) = 0,y(x),si
```

$$y = \frac{e^{-\frac{x^2}{4}}\left(e^{-\frac{x^2}{4}} x^{11/12} c_2 + \text{WhittakerM}\left(\frac{11}{24}, \frac{1}{24}, \frac{x^2}{2}\right) c_1\right)}{x^{7/12}}$$

Mathematica DSolve solution

Solving time : `0.413` (sec)
Leaf size : 61

```
DSolve[{6*x^2*D[y[x],{x,2}]+x*(1+6*x^2)*D[y[x],x]+(1+9*x^2)*y[x]==0,{}},y[x],x,IncludeSingular
```

$$y(x) \to \frac{e^{-\frac{x^2}{2}} \left(2c_1 x^{11/6} + \sqrt[12]{2} c_2 (-x^2)^{11/12} \, \Gamma\left(\frac{1}{12}, -\frac{x^2}{2}\right)\right)}{2x^{3/2}}$$

2.1.111 Problem 113

Internal problem ID [9283]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 113
Date solved : `Monday, January 27, 2025 at 06:00:59 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$9x^2\left(x^2+1\right)y''+3x\left(13x^2+3\right)y'-\left(-25x^2+1\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.329 (sec)

Writing the ode as

$$\left(9x^4+9x^2\right)y''+\left(39x^3+9x\right)y'+\left(25x^2-1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=9x^4+9x^2\\ B&=39x^3+9x \\ C&=25x^2-1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{-9x^4+6x^2-5}{36\left(x^3+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=-9x^4+6x^2-5\\ t&=36\left(x^3+x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-9x^4+6x^2-5}{36\left(x^3+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.206: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{36\,(x-i)^2} - \frac{5}{36\,(x+i)^2} - \frac{i}{12\,(x-i)} + \frac{i}{12x+12i} - \frac{5}{36x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-9x^4 + 6x^2 - 5}{36\left(x^3 + x\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-9x^4 + 6x^2 - 5}{36\left(x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{6}$	$\frac{1}{6}$
i	2	0	$\frac{5}{6}$	$\frac{1}{6}$
$-i$	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{6x} + \frac{1}{6x-6i} + \frac{1}{6x+6i} + (-)(0) \\
&= \frac{1}{6x} + \frac{1}{6x-6i} + \frac{1}{6x+6i} \\
&= \frac{1}{6x} + \frac{x}{3x^2+3}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{6x} + \frac{1}{6x-6i} + \frac{1}{6x+6i}\right)(0) + \left(\left(-\frac{1}{6x^2} - \frac{1}{6(x-i)^2} - \frac{1}{6(x+i)^2}\right) + \left(\frac{1}{6x} + \frac{1}{6x-6i} + \frac{1}{6x+6i}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{6x} + \frac{1}{6x-6i} + \frac{1}{6x+6i}\right) dx} \\
&= (x^2+1)^{1/6} (-x)^{1/6}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{39x^3+9x}{9x^4+9x^2}\, dx} \\
&= z_1 e^{-\frac{\ln(x)}{2} - \frac{5\ln(x^2+1)}{6}} \\
&= z_1 \left(\frac{1}{\sqrt{x}\,(x^2+1)^{5/6}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(-x)^{1/6}}{\sqrt{x}\,(x^2+1)^{2/3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{39x^3+9x}{9x^4+9x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\ln(x) - \frac{5\ln(x^2+1)}{3}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\ln(x) - \frac{5\ln(x^2+1)}{3}}\, x(x^2+1)^{4/3}}{(-x)^{1/3}}\, dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(-x)^{1/6}}{\sqrt{x}\,(x^2+1)^{2/3}} \right) + c_2 \left(\frac{(-x)^{1/6}}{\sqrt{x}\,(x^2+1)^{2/3}} \left(\int \frac{e^{-\ln(x) - \frac{5\ln(x^2+1)}{3}} x(x^2+1)^{4/3}}{(-x)^{1/3}} dx \right) \right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$9x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + 3x(13x^2+3)\left(\frac{d}{dx}y(x)\right) - (-25x^2+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x) = -\frac{(25x^2-1)y(x)}{9x^2(x^2+1)} - \frac{(13x^2+3)\left(\frac{d}{dx}y(x)\right)}{3x(x^2+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x) + \frac{(13x^2+3)\left(\frac{d}{dx}y(x)\right)}{3x(x^2+1)} + \frac{(25x^2-1)y(x)}{9x^2(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

$\left[P_2(x) = \frac{13x^2+3}{3x(x^2+1)}, P_3(x) = \frac{25x^2-1}{9x^2(x^2+1)}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

$\left.(x \cdot P_2(x))\right|_{x=0} = 1$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{9}$

- ○ $x = 0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

$9x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + 3x(13x^2+3)\left(\frac{d}{dx}y(x)\right) + (25x^2-1)\,y(x) = 0$

- Assume series solution for $y(x)$

$y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

$x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ○ Shift index using $k- >k - m$

$x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ○ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- ○ Shift index using $k- >k + 1 - m$

$$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+3r)\,(-1+3r)\, x^r + a_1(4+3r)\,(2+3r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r+1)\,(3k+3r-1) + a_{k-2}(3k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(1+3r)\,(-1+3r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{-\frac{1}{3}, \frac{1}{3}\right\}$
- Each term must be 0
  $a_1(4+3r)\,(2+3r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $9\left(\left(k-\frac{1}{3}+r\right)a_{k-2} + \left(k+r+\frac{1}{3}\right)a_k\right)\left(k-\frac{1}{3}+r\right) = 0$
- Shift index using $k->k+2$
  $9\left(\left(k+\frac{5}{3}+r\right)a_k + \left(k+\frac{7}{3}+r\right)a_{k+2}\right)\left(k+\frac{5}{3}+r\right) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{(3k+3r+5)a_k}{3k+7+3r}$
- Recursion relation for $r = -\frac{1}{3}$
  $a_{k+2} = -\frac{(3k+4)a_k}{3k+6}$
- Solution for $r = -\frac{1}{3}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}, a_{k+2} = -\frac{(3k+4)a_k}{3k+6}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{3}$
  $a_{k+2} = -\frac{(3k+6)a_k}{3k+8}$
- Solution for $r = \frac{1}{3}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -\frac{(3k+6)a_k}{3k+8}, a_1 = 0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = -\frac{(3k+4)a_k}{3k+6}, a_1 = 0, b_{k+2} = -\frac{(3k+6)b_k}{3k+8}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returr
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.149 (sec)
Leaf size : 33

```
dsolve(9*x^2*(x^2+1)*diff(diff(y(x),x),x)+3*x*(13*x^2+3)*diff(y(x),x)-(-25*x^2+1)*y(x)
```

$$y = \frac{c_1}{\left(x^2+1\right)^{2/3} x^{1/3}} + c_2 x^{1/3} \operatorname{hypergeom}\left(\left[1,1\right], \left[\frac{4}{3}\right], -x^2\right)$$

Mathematica DSolve solution

Solving time : 0.239 (sec)
Leaf size : 113

```
DSolve[{9*x^2*(1+x^2)*D[y[x],{x,2}]+3*x*(3+13*x^2)*D[y[x],x]-(1-25*x^2)*y[x]==0,{}},y[x],x,I
```

$$y(x) \to \exp\left(\int_1^x \frac{3K[1]^2+1}{6\left(K[1]^3+K[1]\right)} dK[1] - \frac{1}{2}\int_1^x \left(\frac{10K[2]}{3\left(K[2]^2+1\right)} + \frac{1}{K[2]}\right) dK[2]\right) \left(c_2 \int_1^x \exp\left(-2\int_1^{K[3]} \frac{3K[1]^2+1}{6\left(K[1]^3+K[1]\right)} dK[1]\right) dK[3] + c_1\right)$$

2.1.112 Problem 114

Internal problem ID [9284]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 114
Date solved : `Monday, January 27, 2025 at 06:00:59 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2\left(x^2+1\right)y''+4x\left(6x^2+1\right)y'-\left(-25x^2+1\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.282 (sec)

Writing the ode as

$$\left(4x^4+4x^2\right)y''+\left(24x^3+4x\right)y'+\left(25x^2-1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=4x^4+4x^2\\B&=24x^3+4x\\C&=25x^2-1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-x^2-6}{4\left(x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=-x^2-6\\t&=4\left(x^2+1\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-x^2-6}{4\left(x^2+1\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.208: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16\,(x-i)^2} + \frac{5}{16\,(x+i)^2} + \frac{7i}{16\,(x-i)} - \frac{7i}{16\,(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 6}{4\left(x^2+1\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 - 6}{4\left(x^2+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-i$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right)\\ &= \frac{1}{2} - \left(-\frac{1}{2}\right)\\ &= 1\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (-)[\sqrt{r}]_\infty\\ &= -\frac{1}{4(x-i)} - \frac{1}{4(x+i)} + (-)(0)\\ &= -\frac{1}{4(x-i)} - \frac{1}{4(x+i)}\\ &= -\frac{x}{2x^2+2}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{4(x-i)} - \frac{1}{4(x+i)}\right)(1) + \left(\left(\frac{1}{4(x-i)^2} + \frac{1}{4(x+i)^2}\right) + \left(-\frac{1}{4(x-i)} - \frac{1}{4(x+i)}\right)^2 - \left(\frac{-}{4(\cdots} \right.\right.$$

$$\frac{(x^2+1 \cdots}{(-x+i)^2 (\cdots}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (x)\, e^{\int \left(-\frac{1}{4(x-i)} - \frac{1}{4(x+i)}\right) dx} \\
&= (x) \frac{1}{\left((-x+i)(x+i)\right)^{1/4}} \\
&= \frac{x}{\left(-x^2-1\right)^{1/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{24x^3+4x}{4x^4+4x^2}\, dx} \\
&= z_1 e^{-\frac{\ln(x)}{2} - \frac{5\ln\left(x^2+1\right)}{4}} \\
&= z_1 \left(\frac{1}{\sqrt{x}\left(x^2+1\right)^{5/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(\frac{1}{2} - \frac{i}{2}\right)\sqrt{x}\,\sqrt{2}}{\left(x^2+1\right)^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{24x^3+4x}{4x^4+4x^2}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-\ln(x) - \frac{5\ln\left(x^2+1\right)}{2}}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(i\left(-\frac{\left(x^2+1\right)^{3/2}}{x} + x\sqrt{x^2+1} + \operatorname{arcsinh}(x)\right)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\left(\frac{1}{2} - \frac{i}{2}\right) \sqrt{x}\, \sqrt{2}}{\left(x^2+1\right)^{3/2}} \right) \\ &\quad + c_2 \left(\frac{\left(\frac{1}{2} - \frac{i}{2}\right) \sqrt{x}\, \sqrt{2}}{\left(x^2+1\right)^{3/2}} \left(i \left(-\frac{\left(x^2+1\right)^{3/2}}{x} + x\sqrt{x^2+1} + \operatorname{arcsinh}(x) \right) \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + 4x(6x^2+1)\left(\frac{d}{dx}y(x)\right) - (-25x^2+1)\, y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(25x^2-1)y(x)}{4x^2(x^2+1)} - \frac{(6x^2+1)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(6x^2+1)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)} + \frac{(25x^2-1)y(x)}{4x^2(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{6x^2+1}{x(x^2+1)}, P_3(x) = \frac{25x^2-1}{4x^2(x^2+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$

  ◦ $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + 4x(6x^2+1)\left(\frac{d}{dx}y(x)\right) + (25x^2-1)\, y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k-m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

$$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) + a_{k-2}$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(1+2r)\,(-1+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0
  $a_1(3+2r)\,(1+2r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $4\left(k+r+\frac{1}{2}\right)\left(\left(k+r+\frac{1}{2}\right)a_{k-2} + \left(k+r-\frac{1}{2}\right)a_k\right) = 0$
- Shift index using $k- >k+2$
  $4\left(k+\frac{5}{2}+r\right)\left(\left(k+\frac{5}{2}+r\right)a_k + \left(k+\frac{3}{2}+r\right)a_{k+2}\right) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{(2k+2r+5)a_k}{2k+3+2r}$
- Recursion relation for $r = -\frac{1}{2}$
  $a_{k+2} = -\frac{(2k+4)a_k}{2k+2}$
- Solution for $r = -\frac{1}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{(2k+4)a_k}{2k+2}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{2}$
  $a_{k+2} = -\frac{(2k+6)a_k}{2k+4}$
- Solution for $r = \frac{1}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{(2k+6)a_k}{2k+4}, a_1 = 0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{(2k+4)a_k}{2k+2}, a_1 = 0, b_{k+2} = -\frac{b_k(2k+6)}{2k+4}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.044 (sec)
Leaf size : 34

```
dsolve(4*x^2*(x^2+1)*diff(diff(y(x),x),x)+4*x*(6*x^2+1)*diff(y(x),x)-(-25*x^2+1)*y(x) =
```

$$y = \frac{-\sqrt{x^2+1}\, c_2 + x(\operatorname{arcsinh}(x)\, c_2 + c_1)}{\sqrt{x}\, (x^2+1)^{3/2}}$$

Mathematica DSolve solution

Solving time : 0.454 (sec)
Leaf size : 70

```
DSolve[{4*x^2*(1+x^2)*D[y[x],{x,2}]+4*x*(1+6*x^2)*D[y[x],x]-(1-25*x^2)*y[x]==0,{}},y[x],x,Inclu
```

$$y(x) \to \frac{\left(c_2 x \operatorname{arcsinh}(x) - c_2\sqrt{x^2+1} + c_1 x\right) \exp\left(-\frac{1}{2}\int_1^x \frac{6K[1]^2+1}{K[1]^3+K[1]} dK[1]\right)}{\sqrt[4]{x^2+1}}$$

2.1.113 Problem 115

Internal problem ID [9285]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 115
Date solved : `Monday, January 27, 2025 at 06:01:00 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$8x^2\left(2x^2+1\right)y''+2x\left(34x^2+5\right)y'-\left(-30x^2+1\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.478 (sec)

Writing the ode as

$$\left(16x^4+8x^2\right)y''+\left(68x^3+10x\right)y'+\left(30x^2-1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 16x^4+8x^2 \\ B &= 68x^3+10x \\ C &= 30x^2-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{132x^4+148x^2-7}{64\left(2x^3+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 132x^4+148x^2-7 \\ t &= 64\left(2x^3+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{132x^4+148x^2-7}{64\left(2x^3+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.210: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64(2x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{i\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{i\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16\left(x - \frac{i\sqrt{2}}{2}\right)^2} - \frac{3}{16\left(x + \frac{i\sqrt{2}}{2}\right)^2} - \frac{i\sqrt{2}}{2\left(x - \frac{i\sqrt{2}}{2}\right)} + \frac{i\sqrt{2}}{2x + i\sqrt{2}} - \frac{7}{64x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{8} \end{aligned}$$

For the pole at $x = \frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4}\\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

For the pole at $x = -\frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4}\\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{132x^4 + 148x^2 - 7}{64\left(2x^3 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{33}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{8}\\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{8}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{132x^4 + 148x^2 - 7}{64\left(2x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{8}$	$\frac{1}{8}$
$\frac{i\sqrt{2}}{2}$	2	0	$\frac{3}{4}$	$\frac{1}{4}$
$-\frac{i\sqrt{2}}{2}$	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{11}{8}$	$-\frac{3}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{11}{8}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right)\\
&= \frac{11}{8} - \left(\frac{11}{8}\right)\\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{7}{8x} + \frac{1}{4x-2i\sqrt{2}} + \frac{1}{4x+2i\sqrt{2}} + (0) \\
&= \frac{7}{8x} + \frac{1}{4x-2i\sqrt{2}} + \frac{1}{4x+2i\sqrt{2}} \\
&= \frac{22x^2+7}{16x^3+8x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{7}{8x} + \frac{1}{4x-2i\sqrt{2}} + \frac{1}{4x+2i\sqrt{2}} \right)(0) + \left(\left(-\frac{7}{8x^2} - \frac{1}{4\left(x - \frac{i\sqrt{2}}{2}\right)^2} - \frac{1}{4\left(x + \frac{i\sqrt{2}}{2}\right)^2} \right) + \left(\frac{7}{8x} + \frac{1}{4x -}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(\frac{7}{8x} + \frac{1}{4x-2i\sqrt{2}} + \frac{1}{4x+2i\sqrt{2}} \right) dx} \\
&= 2^{1/4} \left(2x^2+1 \right)^{1/4} x^{7/8}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{68x^3+10x}{16x^4+8x^2}\, dx} \\
&= z_1 e^{-\frac{5\ln(x)}{8} - \frac{3\ln\left(2x^2+1\right)}{4}} \\
&= z_1 \left(\frac{1}{x^{5/8} \left(2x^2+1\right)^{3/4}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/4} 2^{1/4}}{\sqrt{2x^2+1}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{68x^3+10x}{16x^4+8x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{5\ln(x)}{4}-\frac{3\ln(2x^2+1)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{5\ln(x)}{4}-\frac{3\ln(2x^2+1)}{2}}(2x^2+1)\sqrt{2}}{2\sqrt{x}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^{1/4}2^{1/4}}{\sqrt{2x^2+1}}\right) + c_2\left(\frac{x^{1/4}2^{1/4}}{\sqrt{2x^2+1}}\left(\int \frac{\mathrm{e}^{-\frac{5\ln(x)}{4}-\frac{3\ln(2x^2+1)}{2}}(2x^2+1)\sqrt{2}}{2\sqrt{x}}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$8x^2(2x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)+2x(34x^2+5)\left(\frac{d}{dx}y(x)\right)-(-30x^2+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x)=-\frac{(30x^2-1)y(x)}{8x^2(2x^2+1)}-\frac{(34x^2+5)\left(\frac{d}{dx}y(x)\right)}{4x(2x^2+1)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x)+\frac{(34x^2+5)\left(\frac{d}{dx}y(x)\right)}{4x(2x^2+1)}+\frac{(30x^2-1)y(x)}{8x^2(2x^2+1)}=0$

☐ Check to see if x_0 is a regular singular point
  - Define functions
    $\left[P_2(x)=\frac{34x^2+5}{4x(2x^2+1)}, P_3(x)=\frac{30x^2-1}{8x^2(2x^2+1)}\right]$
  - $x\cdot P_2(x)$ is analytic at $x=0$
    $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{5}{4}$
  - $x^2\cdot P_3(x)$ is analytic at $x=0$
    $\left.(x^2\cdot P_3(x))\right|_{x=0}=-\frac{1}{8}$
  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point
  $x_0=0$

- Multiply by denominators
  $8x^2(2x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)+2x(34x^2+5)\left(\frac{d}{dx}y(x)\right)+(30x^2-1)\,y(x)=0$
- Assume series solution for $y(x)$
  $y(x)=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

$$x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- > k - m$

$$x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx} y(x)\right)$ to series expansion for $m = 1..3$

$$x^m \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- > k + 1 - m$

$$x^m \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2} y(x)\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d^2}{dx^2} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- > k + 2 - m$

$$x^m \cdot \left(\frac{d^2}{dx^2} y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+2r)(-1+4r)\, x^r + a_1(3+2r)(3+4r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+1)(4k+4r-1) + 2a_{k-2}(2\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
$(1+2r)(-1+4r) = 0$
- Values of r that satisfy the indicial equation
$r \in \left\{-\frac{1}{2}, \frac{1}{4}\right\}$
- Each term must be 0
$a_1(3+2r)(3+4r) = 0$
- Solve for the dependent coefficient(s)
$a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
$8\left(\left(2k+2r-\frac{5}{2}\right) a_{k-2} + a_k\left(k+r-\frac{1}{4}\right)\right)\left(k+r+\frac{1}{2}\right) = 0$
- Shift index using $k- > k + 2$
$8\left(\left(2k+\frac{3}{2}+2r\right) a_k + a_{k+2}\left(k+\frac{7}{4}+r\right)\right)\left(k+\frac{5}{2}+r\right) = 0$
- Recursion relation that defines series solution to ODE
$a_{k+2} = -\frac{2(4k+4r+3)a_k}{4k+7+4r}$
- Recursion relation for $r = -\frac{1}{2}$
$a_{k+2} = -\frac{2(4k+1)a_k}{4k+5}$
- Solution for $r = -\frac{1}{2}$
$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{2(4k+1)a_k}{4k+5}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{4}$
$a_{k+2} = -\frac{2(4k+4)a_k}{4k+8}$
- Solution for $r = \frac{1}{4}$
$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+2} = -\frac{2(4k+4)a_k}{4k+8}, a_1 = 0\right]$
- Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{4}}\right), a_{k+2}=-\frac{2(4k+1)a_k}{4k+5}, a_1=0, b_{k+2}=-\frac{2(4k+4)b_k}{4k+8}, b_1=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special functio
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.099 (sec)
Leaf size : 46

```
dsolve(8*x^2*(2*x^2+1)*diff(diff(y(x),x),x)+2*x*(34*x^2+5)*diff(y(x),x)-(-30*x^2+1)*y(
```

$$y=\frac{c_1 \operatorname{LegendreP}\left(\frac{3}{8}, \frac{3}{8}, \sqrt{2x^2+1}\right)+c_2 \operatorname{LegendreQ}\left(\frac{3}{8}, \frac{3}{8}, \sqrt{2x^2+1}\right)}{\sqrt{2x^2+1}\, x^{1/8}}$$

Mathematica DSolve solution

Solving time : 0.359 (sec)
Leaf size : 118

```
DSolve[{8*x^2*(1+2*x^2)*D[y[x],{x,2}]+2*x*(5+34*x^2)*D[y[x],x]-(1-30*x^2)*y[x]==0,{}},y[x],x,I
```

$$y(x)\to \exp \left(\int_1^x \left(\frac{K[1]}{2K[1]^2+1}+\frac{7}{8K[1]}\right)dK[1] -\frac{1}{2}\int_1^x \frac{34K[2]^2+5}{8K[2]^3+4K[2]}dK[2]\right)\left(c_2\int_1^x \exp \left(-2\int_1^{K[3]} \left(\frac{K[1]}{2K[1]^2+1}+\frac{7}{8K[1]}\right)dK[1]\right)dK[3] +c_1\right)$$

2.1.114 Problem 116

Internal problem ID [9286]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 116
Date solved : `Monday, January 27, 2025 at 06:01:01 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(1+x)\,y''-x(1-3x)\,y'+y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.153 (sec)

Writing the ode as

$$\left(2x^3+2x^2\right)y''+\left(3x^2-x\right)y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^3+2x^2 \\ B &= 3x^2-x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-3}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{3}{16x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.212: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{3}{16x^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=-\frac{3}{16x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=\frac{1}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{4}-\left(\frac{1}{4}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^-}{x-c_1}\right)+(-)[\sqrt{r}]_\infty \\
&= \frac{1}{4x}+(-)\,(0) \\
&= \frac{1}{4x} \\
&= \frac{1}{4x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}
$$

Let

$$
p(x)=1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{1}{4x}\right)(0)+\left(\left(-\frac{1}{4x^2}\right)+\left(\frac{1}{4x}\right)^2-\left(-\frac{3}{16x^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \frac{1}{4x} dx} \\ &= x^{1/4} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3x^2-x}{2x^3+2x^2}\, dx} \\ &= z_1 e^{\frac{\ln(x)}{4}-\ln(1+x)} \\ &= z_1\left(\frac{x^{1/4}}{1+x}\right) \end{aligned}$$

Which simplifies to

$$y_1=\frac{\sqrt{x}}{1+x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1\int \frac{e^{\int -\frac{3x^2-x}{2x^3+2x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1\int \frac{e^{\frac{\ln(x)}{2}-2\ln(1+x)}}{(y_1)^2}\, dx \\ &= y_1\left(2\,\mathrm{e}^{\frac{\ln(x)}{2}-2\ln(1+x)}(1+x)^2\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\sqrt{x}}{1+x}\right)+c_2\left(\frac{\sqrt{x}}{1+x}\left(2\,\mathrm{e}^{\frac{\ln(x)}{2}-2\ln(1+x)}(1+x)^2\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right) - x(1-3x)\left(\frac{d}{dx}y(x)\right) + y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{y(x)}{2(x+1)x^2} - \frac{(3x-1)\left(\frac{d}{dx}y(x)\right)}{2x(x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(3x-1)\left(\frac{d}{dx}y(x)\right)}{2x(x+1)} + \frac{y(x)}{2(x+1)x^2} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{3x-1}{2x(x+1)}, P_3(x) = \frac{1}{2(x+1)x^2}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1} = 2$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$

  ◦ $x=-1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -1$

- Multiply by denominators

  $2x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right) + x(3x-1)\left(\frac{d}{dx}y(x)\right) + y(x) = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(2u^3-4u^2+2u)\left(\frac{d^2}{du^2}y(u)\right) + (3u^2-7u+4)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  ◦ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

  ◦ Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $2a_0r(1+r)\,u^{-1+r} + \left(2a_1(1+r)(2+r) - a_0(1+r)(-1+4r)\right)u^r + \left(\sum_{k=1}^{\infty} (2a_{k+1}(k+r+1)(k+\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$2r(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $2a_1(1+r)(2+r) - a_0(1+r)(-1+4r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $(-4a_k + 2a_{k-1} + 2a_{k+1})k^2 + ((-8a_k + 4a_{k-1} + 4a_{k+1})r - 3a_k - 3a_{k-1} + 6a_{k+1})k + (-4a_k + 2a_{k-}$

- Shift index using $k- > k+1$

  $(-4a_{k+1} + 2a_k + 2a_{k+2})(k+1)^2 + ((-8a_{k+1} + 4a_k + 4a_{k+2})r - 3a_{k+1} - 3a_k + 6a_{k+2})(k+1) + (-$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} + 4kra_k - 8kra_{k+1} + 2r^2a_k - 4r^2a_{k+1} + ka_k - 11ka_{k+1} + ra_k - 11ra_{k+1} - 6a_{k+1}}{2(k^2 + 2kr + r^2 + 5k + 5r + 6)}$

- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - 3ka_k - 3ka_{k+1} + a_k + a_{k+1}}{2(k^2 + 3k + 2)}$

- Solution for $r = -1$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-1}, a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - 3ka_k - 3ka_{k+1} + a_k + a_{k+1}}{2(k^2 + 3k + 2)}, 0 = 0\right]$

- Revert the change of variables $u = x + 1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x+1)^{k-1}, a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - 3ka_k - 3ka_{k+1} + a_k + a_{k+1}}{2(k^2 + 3k + 2)}, 0 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} + ka_k - 11ka_{k+1} - 6a_{k+1}}{2(k^2 + 5k + 6)}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} + ka_k - 11ka_{k+1} - 6a_{k+1}}{2(k^2 + 5k + 6)}, 4a_1 + a_0 = 0\right]$

- Revert the change of variables $u = x + 1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x+1)^k, a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} + ka_k - 11ka_{k+1} - 6a_{k+1}}{2(k^2 + 5k + 6)}, 4a_1 + a_0 = 0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k (x+1)^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k (x+1)^k\right), a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - 3ka_k - 3ka_{k+1} + a_k + a_{k+1}}{2(k^2 + 3k + 2)}, 0 = 0,$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.023 (sec)
Leaf size : 19

```
dsolve(2*x^2*(x+1)*diff(diff(y(x),x),x)-x*(-3*x+1)*diff(y(x),x)+y(x) = 0,y(x),singsol=
```

$$y = \frac{c_2\sqrt{x} + c_1 x}{x + 1}$$

Mathematica DSolve solution

Solving time : 0.234 (sec)
Leaf size : 53

```
DSolve[{2*x^2*(1+x)*D[y[x],{x,2}]-x*(1-3*x)*D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolu
```

$$y(x) \to \sqrt[4]{x}(2c_2\sqrt{x} + c_1) \exp\left(-\frac{1}{2}\int_1^x \left(\frac{2}{K[1]+1} - \frac{1}{2K[1]}\right) dK[1]\right)$$

2.1.115 Problem 117

Internal problem ID [9287]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 117
Date solved : `Monday, January 27, 2025 at 06:01:02 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$6x^2\left(2x^2+1\right)y''+x\left(50x^2+1\right)y'+\left(30x^2+1\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.164 (sec)

Writing the ode as

$$\left(12x^4+6x^2\right)y''+\left(50x^3+x\right)y'+\left(30x^2+1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 12x^4+6x^2 \\ B &= 50x^3+x \\ C &= 30x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-35}{144x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -35 \\ t &= 144x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{35}{144x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.214: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 144x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{35}{144x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{35}{144x^2}$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{35}{144}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{7}{12}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{5}{12}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=-\frac{35}{144x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{12}$	$\frac{5}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{7}{12}$	$\frac{5}{12}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{5}{12}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}\right)\\
&= \frac{5}{12}-\left(\frac{5}{12}\right)\\
&= 0
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^{-}}{x-c_1}\right)+(-)[\sqrt{r}]_\infty\\
&= \frac{5}{12x}+(-)\,(0)\\
&= \frac{5}{12x}\\
&= \frac{5}{12x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{5}{12x}\right)(0)+\left(\left(-\frac{5}{12x^2}\right)+\left(\frac{5}{12x}\right)^2-\left(-\frac{35}{144x^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}z_1(x)&=pe^{\int \omega\,dx}\\&=e^{\int \frac{5}{12x}dx}\\&=x^{5/12}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1&=z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\&=z_1e^{-\int \frac{1}{2}\frac{50x^3+x}{12x^4+6x^2}\,dx}\\&=z_1e^{-\ln(2x^2+1)-\frac{\ln(x)}{12}}\\&=z_1\left(\frac{1}{(2x^2+1)\,x^{1/12}}\right)\end{aligned}$$

Which simplifies to

$$y_1=\frac{x^{1/3}}{2x^2+1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2&=y_1\int \frac{e^{\int -\frac{50x^3+x}{12x^4+6x^2}\,dx}}{(y_1)^2}\,dx\\&=y_1\int \frac{e^{-2\ln(2x^2+1)-\frac{\ln(x)}{6}}}{(y_1)^2}\,dx\\&=y_1\left(6x^{1/3}e^{-2\ln(2x^2+1)-\frac{\ln(x)}{6}}\left(2x^2+1\right)^2\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y&=c_1y_1+c_2y_2\\&=c_1\left(\frac{x^{1/3}}{2x^2+1}\right)+c_2\left(\frac{x^{1/3}}{2x^2+1}\left(6x^{1/3}e^{-2\ln(2x^2+1)-\frac{\ln(x)}{6}}\left(2x^2+1\right)^2\right)\right)\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$6x^2(2x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(50x^2+1)\left(\frac{d}{dx}y(x)\right)+(30x^2+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(30x^2+1)y(x)}{6x^2(2x^2+1)}-\frac{(50x^2+1)\left(\frac{d}{dx}y(x)\right)}{6x(2x^2+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{(50x^2+1)\left(\frac{d}{dx}y(x)\right)}{6x(2x^2+1)}+\frac{(30x^2+1)y(x)}{6x^2(2x^2+1)}=0$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x)=\frac{50x^2+1}{6x(2x^2+1)},P_3(x)=\frac{30x^2+1}{6x^2(2x^2+1)}\right]$
  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{1}{6}$
  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=\frac{1}{6}$
  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $6x^2(2x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(50x^2+1)\left(\frac{d}{dx}y(x)\right)+(30x^2+1)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

    $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$
  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..4$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$
  - Shift index using $k->k+2-m$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$a_0(-1+3r)(-1+2r)x^r + a_1(2+3r)(1+2r)x^{1+r} + \left(\sum_{k=2}^{\infty}(a_k(3k+3r-1)(2k+2r-1)+2a$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+3r)(-1+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{\frac{1}{2}, \frac{1}{3}\right\}$
- Each term must be 0
  $a_1(2+3r)(1+2r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $(3k+3r-1)(2k+2r-1)(a_k+2a_{k-2}) = 0$
- Shift index using $k- >k+2$
  $(3k+3r+5)(2k+2r+3)(a_{k+2}+2a_k) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -2a_k$
- Recursion relation for $r = \frac{1}{2}$
  $a_{k+2} = -2a_k$
- Solution for $r = \frac{1}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -2a_k, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{3}$
  $a_{k+2} = -2a_k$
- Solution for $r = \frac{1}{3}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -2a_k, a_1 = 0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = -2a_k, a_1 = 0, b_{k+2} = -2b_k, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.035 (sec)
Leaf size : 24

```
dsolve(6*x^2*(2*x^2+1)*diff(diff(y(x),x),x)+x*(50*x^2+1)*diff(y(x),x)+(30*x^2+1)*y(x) =
```

$$y = \frac{x^{1/3}\left(c_1 x^{1/6} + c_2\right)}{2x^2 + 1}$$

Mathematica DSolve solution

Solving time : 0.277 (sec)
Leaf size : 58

```
DSolve[{6*x^2*(1+2*x^2)*D[y[x],{x,2}]+x*(1+50*x^2)*D[y[x],x]+(1+30*x^2)*y[x]==0,{}},y[x],x,Inc
```

$$y(x) \to x^{5/12}\left(6c_2\sqrt[6]{x} + c_1\right)\exp\left(-\frac{1}{2}\int_1^x \frac{50K[1]^2 + 1}{12K[1]^3 + 6K[1]}dK[1]\right)$$

2.1.116 Problem 118

Internal problem ID [9288]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 118
Date solved : `Monday, January 27, 2025 at 06:01:02 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$28x^2(1-3x)\,y''-7x(5+9x)\,y'+7(2+9x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.157 (sec)

Writing the ode as

$$\left(-84x^3+28x^2\right)y''+\left(-63x^2-35x\right)y'+(63x+14)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -84x^3+28x^2\\ B &= -63x^2-35x\\ C &= 63x+14 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{33}{64x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 33\\ t &= 64x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{33}{64x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.216: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{33}{64x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{33}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{8} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{33}{64x^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{33}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{11}{8} \\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=-\frac{3}{8}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{33}{64x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{11}{8}$	$-\frac{3}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{11}{8}$	$-\frac{3}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=-\frac{3}{8}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{3}{8}-\left(-\frac{3}{8}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^-}{x-c_1}\right)+(-)[\sqrt{r}]_\infty \\
&= -\frac{3}{8x}+(-)(0) \\
&= -\frac{3}{8x} \\
&= -\frac{3}{8x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}
$$

Let

$$
p(x)=1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{8x}\right)(0) + \left(\left(\frac{3}{8x^2}\right) + \left(-\frac{3}{8x}\right)^2 - \left(\frac{33}{64x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int -\frac{3}{8x} dx} \\ &= \frac{1}{x^{3/8}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-63x^2 - 35x}{-84x^3 + 28x^2} \, dx} \\ &= z_1 e^{\frac{5 \ln(x)}{8} - \ln(-1+3x)} \\ &= z_1 \left(\frac{x^{5/8}}{-1+3x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/4}}{-1+3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-63x^2 - 35x}{-84x^3 + 28x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{5 \ln(x)}{4} - 2\ln(-1+3x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{4\sqrt{x}\, e^{\frac{5 \ln(x)}{4} - 2\ln(-1+3x)} (-1+3x)^2}{7}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{x^{1/4}}{-1+3x}\right) + c_2 \left(\frac{x^{1/4}}{-1+3x} \left(\frac{4\sqrt{x}\, e^{\frac{5 \ln(x)}{4} - 2\ln(-1+3x)} (-1+3x)^2}{7}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$28x^2(1-3x)\left(\frac{d^2}{dx^2}y(x)\right)-7x(5+9x)\left(\frac{d}{dx}y(x)\right)+7(2+9x)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{(2+9x)y(x)}{4(3x-1)x^2}-\frac{(5+9x)\left(\frac{d}{dx}y(x)\right)}{4x(3x-1)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{(5+9x)\left(\frac{d}{dx}y(x)\right)}{4x(3x-1)}-\frac{(2+9x)y(x)}{4(3x-1)x^2}=0$

□ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x)=\frac{5+9x}{4x(3x-1)},P_3(x)=-\frac{2+9x}{4(3x-1)x^2}\right]$
- $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=-\frac{5}{4}$
- $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=\frac{1}{2}$
- $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4(3x-1)\,x^2\left(\frac{d^2}{dx^2}y(x)\right)+x(5+9x)\left(\frac{d}{dx}y(x)\right)+(-9x-2)\,y(x)=0$
- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_k x^{k+r+m}$
- Shift index using $k->k-m$

  $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$
- Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$
- Shift index using $k->k+1-m$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$
- Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..3$

  $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$
- Shift index using $k->k+2-m$

  $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$-a_0(-1+4r)(-2+r)x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(4k+4r-1)(k+r-2) + 3a_{k-1}(4k+4r-1)(k+r-2)\right)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-(-1+4r)(-2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{2, \frac{1}{4}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $-4(a_k - 3a_{k-1})\left(k+r-\frac{1}{4}\right)(k+r-2) = 0$
- Shift index using $k->k+1$
  $-4(a_{k+1} - 3a_k)\left(k+\frac{3}{4}+r\right)(k+r-1) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = 3a_k$
- Recursion relation for $r = 2$
  $a_{k+1} = 3a_k$
- Solution for $r = 2$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = 3a_k\right]$
- Recursion relation for $r = \frac{1}{4}$
  $a_{k+1} = 3a_k$
- Solution for $r = \frac{1}{4}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+1} = 3a_k\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{4}}\right), a_{k+1} = 3a_k, b_{k+1} = 3b_k\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.023 (sec)
Leaf size : 23

```
dsolve(28*x^2*(-3*x+1)*diff(diff(y(x),x),x)-7*x*(5+9*x)*diff(y(x),x)+7*(2+9*x)*y(x) = 0,
```

$$y = \frac{c_1 x^2 + c_2 x^{1/4}}{3x - 1}$$

Mathematica DSolve solution

Solving time : `0.279` (sec)
Leaf size : 60

```
DSolve[{28*x^2*(1-3*x)*D[y[x],{x,2}]-7*x*(5+9*x)*D[y[x],x]+7*(2+9*x)*y[x]==0,{}},y[x],x,Incl
```

$$y(x) \rightarrow \frac{\left(4c_2x^{7/4}+7c_1\right)\exp\left(-\frac{1}{2}\int_1^x\left(\frac{6}{3K[1]-1}-\frac{5}{4K[1]}\right)dK[1]\right)}{7x^{3/8}}$$

2.1.117 Problem 119

Internal problem ID [9289]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 119
Date solved : `Monday, January 27, 2025 at 06:01:03 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$8x^2\left(-x^2+2\right)y''+2x\left(-21x^2+10\right)y'-\left(35x^2+2\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.154 (sec)

Writing the ode as

$$\left(-8x^4+16x^2\right)y''+\left(-42x^3+20x\right)y'+\left(-35x^2-2\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -8x^4+16x^2 \\ B &= -42x^3+20x \\ C &= -35x^2-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-7}{64x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -7 \\ t &= 64x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{7}{64x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.218: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{64x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{7}{64x^2}$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{7}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{7}{8} \\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{8}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=-\frac{7}{64x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{8}$	$\frac{1}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=\frac{1}{8}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{8}-\left(\frac{1}{8}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^-}{x-c_1}\right)+(-)[\sqrt{r}]_\infty \\
&= \frac{1}{8x}+(-)(0) \\
&= \frac{1}{8x} \\
&= \frac{1}{8x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}
$$

Let

$$
p(x)=1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{8x}\right)(0) + \left(\left(-\frac{1}{8x^2}\right) + \left(\frac{1}{8x}\right)^2 - \left(-\frac{7}{64x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \frac{1}{8x} dx} \\ &= x^{1/8} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-42x^3+20x}{-8x^4+16x^2} dx} \\ &= z_1 e^{-\ln(x^2-2)-\frac{5\ln(x)}{8}} \\ &= z_1 \left(\frac{1}{(x^2-2)\, x^{5/8}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{(x^2-2)\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-42x^3+20x}{-8x^4+16x^2} dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-2\ln(x^2-2)-\frac{5\ln(x)}{4}}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{4x^2 e^{-2\ln(x^2-2)-\frac{5\ln(x)}{4}} (x^2-2)^2}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{1}{(x^2-2)\sqrt{x}}\right) + c_2\left(\frac{1}{(x^2-2)\sqrt{x}}\left(\frac{4x^2 e^{-2\ln(x^2-2)-\frac{5\ln(x)}{4}} (x^2-2)^2}{3}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$8x^2(-x^2+2)\left(\frac{d^2}{dx^2}y(x)\right)+2x(-21x^2+10)\left(\frac{d}{dx}y(x)\right)-(35x^2+2)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{(35x^2+2)y(x)}{8x^2(x^2-2)}-\frac{(21x^2-10)\left(\frac{d}{dx}y(x)\right)}{4x(x^2-2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x)+\frac{(21x^2-10)\left(\frac{d}{dx}y(x)\right)}{4x(x^2-2)}+\frac{(35x^2+2)y(x)}{8x^2(x^2-2)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{21x^2-10}{4x(x^2-2)}, P_3(x)=\frac{35x^2+2}{8x^2(x^2-2)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{5}{4}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=-\frac{1}{8}$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

$8x^2(x^2-2)\left(\frac{d^2}{dx^2}y(x)\right)+2x(21x^2-10)\left(\frac{d}{dx}y(x)\right)+(35x^2+2)\,y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

  ◦ Shift index using $k->k-m$

  $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..4$

  $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  ◦ Shift index using $k->k+2-m$

  $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$-2a_0(1+2r)(-1+4r)x^r - 2a_1(3+2r)(3+4r)x^{1+r} + \left(\sum_{k=2}^{\infty}(-2a_k(2k+2r+1)(4k+4r-1)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-2(1+2r)(-1+4r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{-\frac{1}{2}, \frac{1}{4}\right\}$
- Each term must be 0
  $-2a_1(3+2r)(3+4r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $-(2k+2r+1)(4k+4r-1)(2a_k - a_{k-2}) = 0$
- Shift index using $k- >k+2$
  $-(2k+2r+5)(4k+4r+7)(2a_{k+2} - a_k) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{a_k}{2}$
- Recursion relation for $r = -\frac{1}{2}$
  $a_{k+2} = \frac{a_k}{2}$
- Solution for $r = -\frac{1}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = \frac{a_k}{2}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{4}$
  $a_{k+2} = \frac{a_k}{2}$
- Solution for $r = \frac{1}{4}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+2} = \frac{a_k}{2}, a_1 = 0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{4}}\right), a_{k+2} = \frac{a_k}{2}, a_1 = 0, b_{k+2} = \frac{b_k}{2}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.041 (sec)
Leaf size : 22

```
dsolve(8*x^2*(-x^2+2)*diff(diff(y(x),x),x)+2*x*(-21*x^2+10)*diff(y(x),x)-(35*x^2+2)*y(x)
```

$$y = \frac{c_2 x^{3/4} + c_1}{(x^2 - 2)\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.27 (sec)
Leaf size : 62

```
DSolve[{8*x^2*(2-x^2)*D[y[x],{x,2}]+2*x*(10-21*x^2)*D[y[x],x]-(2+35*x^2)*y[x]==0,{}},y[x],x,In
```

$$y(x) \to \frac{1}{3}\sqrt[8]{x}\left(4c_2 x^{3/4} + 3c_1\right)\exp\left(-\frac{1}{2}\int_1^x \left(\frac{4K[1]}{K[1]^2 - 2} + \frac{5}{4K[1]}\right) dK[1]\right)$$

2.1.118 Problem 120

Internal problem ID [9290]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 120
Date solved : `Monday, January 27, 2025 at 06:01:03 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2\left(x^2+3x+1\right)y''-4x\left(-3x^2-3x+1\right)y'+3\left(x^2-x+1\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.100 (sec)

Writing the ode as

$$\left(4x^4+12x^3+4x^2\right)y''+\left(12x^3+12x^2-4x\right)y'+\left(3x^2-3x+3\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^4+12x^3+4x^2 \\ B &= 12x^3+12x^2-4x \\ C &= 3x^2-3x+3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s=0$$
$$t=1$$

Therefore eq. (4) becomes

$$z''(x)=0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.220: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{12x^3+12x^2-4x}{4x^4+12x^3+4x^2}\, dx} \\ &= z_1 e^{-\ln(x^2+3x+1)+\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{\sqrt{x}}{x^2+3x+1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{x^2+3x+1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{12x^3+12x^2-4x}{4x^4+12x^3+4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x^2+3x+1)+\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\sqrt{x}}{x^2+3x+1}\right) + c_2\left(\frac{\sqrt{x}}{x^2+3x+1}(x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2(x^2+3x+1)\left(\frac{d^2}{dx^2}y(x)\right) - 4x(-3x^2-3x+1)\left(\frac{d}{dx}y(x)\right) + 3(x^2-x+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{3(x^2-x+1)y(x)}{4x^2(x^2+3x+1)} - \frac{(3x^2+3x-1)\left(\frac{d}{dx}y(x)\right)}{x(x^2+3x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(3x^2+3x-1)\left(\frac{d}{dx}y(x)\right)}{x(x^2+3x+1)} + \frac{3(x^2-x+1)y(x)}{4x^2(x^2+3x+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{3x^2+3x-1}{x(x^2+3x+1)}, P_3(x) = \frac{3(x^2-x+1)}{4x^2(x^2+3x+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{3}{4}$

  ◦ $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4x^2(x^2+3x+1)\left(\frac{d^2}{dx^2}y(x)\right) + 4x(3x^2+3x-1)\left(\frac{d}{dx}y(x)\right) + (3x^2-3x+3)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y(x)=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
- Shift index using $k->k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..4$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$
- Shift index using $k->k+2-m$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)\,(-3+2r)\, x^r+\left(a_1(1+2r)\,(-1+2r)+3a_0(1+2r)\,(-1+2r)\right) x^{1+r}+\left(\sum_{k=2}^{\infty} (a_k(2k+2\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)\,(-3+2r)=0$
- Values of r that satisfy the indicial equation

  $r\in\left\{\frac{1}{2},\frac{3}{2}\right\}$
- Each term must be 0

  $a_1(1+2r)\,(-1+2r)+3a_0(1+2r)\,(-1+2r)=0$
- Solve for the dependent coefficient(s)

  $a_1=-3a_0$
- Each term in the series must be 0, giving the recursion relation

  $(2k+2r-1)\,(2k+2r-3)\,(a_k+3a_{k-1}+a_{k-2})=0$
- Shift index using $k->k+2$

  $(2k+2r+3)\,(2k+2r+1)\,(a_{k+2}+3a_{k+1}+a_k)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-3a_{k+1}-a_k$
- Recursion relation for $r=\frac{1}{2}$

  $a_{k+2}=-3a_{k+1}-a_k$
- Solution for $r=\frac{1}{2}$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2}=-3a_{k+1}-a_k, a_1=-3a_0\right]$
- Recursion relation for $r=\frac{3}{2}$

  $a_{k+2}=-3a_{k+1}-a_k$
- Solution for $r=\frac{3}{2}$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2}=-3a_{k+1}-a_k, a_1=-3a_0\right]$
- Combine solutions and rename parameters

  $\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+2}=-3a_{k+1}-a_k, a_1=-3a_0, b_{k+2}=-3b_{k+1}-b_k, b_1=\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.038 (sec)
Leaf size : 23

```
dsolve(4*x^2*(x^2+3*x+1)*diff(diff(y(x),x),x)-4*x*(-3*x^2-3*x+1)*diff(y(x),x)+3*(x^2-x
```

$$y = \frac{\sqrt{x}\,(c_2 x + c_1)}{x^2 + 3x + 1}$$

Mathematica DSolve solution

Solving time : 0.182 (sec)
Leaf size : 52

```
DSolve[{4*x^2*(1+3*x+x^2)*D[y[x],{x,2}]-4*x*(1-3*x-3*x^2)*D[y[x],x]+3*(1-x+x^2)*y[x]==0,{}},
```

$$y(x) \to (c_2 x + c_1) \exp\left(-\frac{1}{2}\int_1^x \frac{3K[1](K[1]+1)-1}{K[1](K[1](K[1]+3)+1)}dK[1]\right)$$

2.1.119 Problem 121

Internal problem ID [9291]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 121
Date solved : `Monday, January 27, 2025 at 06:01:04 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$3x^2(1+x)^2 y'' - x\left(-11x^2 - 10x + 1\right) y' + \left(5x^2 + 1\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.148 (sec)

Writing the ode as

$$3x^2(1+x)^2 y'' + \left(11x^3 + 10x^2 - x\right) y' + \left(5x^2 + 1\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^2(1+x)^2 \\ B &= 11x^3 + 10x^2 - x \\ C &= 5x^2 + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -5 \\ t &= 36x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{5}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.222: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{36x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{6} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{5}{36x^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{5}{36}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{6} \\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{6}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=-\frac{5}{36x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{6}$	$\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=\frac{1}{6}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{6}-\left(\frac{1}{6}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^-}{x-c_1}\right)+(-)[\sqrt{r}]_\infty \\
&= \frac{1}{6x}+(-)\,(0) \\
&= \frac{1}{6x} \\
&= \frac{1}{6x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}
$$

Let

$$
p(x)=1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{1}{6x}\right)(0)+\left(\left(-\frac{1}{6x^2}\right)+\left(\frac{1}{6x}\right)^2-\left(-\frac{5}{36x^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}z_1(x)&=pe^{\int \omega\,dx}\\&=\mathrm{e}^{\int \frac{1}{6x}dx}\\&=x^{1/6}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1&=z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\&=z_1e^{-\int \frac{1}{2}\frac{11x^3+10x^2-x}{3x^2(1+x)^2}\,dx}\\&=z_1e^{-2\ln(1+x)+\frac{\ln(x)}{6}}\\&=z_1\left(\frac{x^{1/6}}{(1+x)^2}\right)\end{aligned}$$

Which simplifies to

$$y_1=\frac{x^{1/3}}{(1+x)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2&=y_1\int \frac{e^{\int -\frac{11x^3+10x^2-x}{3x^2(1+x)^2}\,dx}}{(y_1)^2}\,dx\\&=y_1\int \frac{e^{-4\ln(1+x)+\frac{\ln(x)}{3}}}{(y_1)^2}\,dx\\&=y_1\left(\frac{3x^{1/3}\mathrm{e}^{-4\ln(1+x)+\frac{\ln(x)}{3}}(1+x)^4}{2}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y&=c_1y_1+c_2y_2\\&=c_1\left(\frac{x^{1/3}}{(1+x)^2}\right)+c_2\left(\frac{x^{1/3}}{(1+x)^2}\left(\frac{3x^{1/3}\mathrm{e}^{-4\ln(1+x)+\frac{\ln(x)}{3}}(1+x)^4}{2}\right)\right)\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$3x^2(x+1)^2\left(\frac{d^2}{dx^2}y(x)\right)-x(-11x^2-10x+1)\left(\frac{d}{dx}y(x)\right)+(5x^2+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(5x^2+1)y(x)}{3x^2(x+1)^2}-\frac{\left(\frac{d}{dx}y(x)\right)(11x-1)}{3x(x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{\left(\frac{d}{dx}y(x)\right)(11x-1)}{3x(x+1)}+\frac{(5x^2+1)y(x)}{3x^2(x+1)^2}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{11x-1}{3x(x+1)},P_3(x)=\frac{5x^2+1}{3x^2(x+1)^2}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left.((x+1)\cdot P_2(x))\right|_{x=-1}=4$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left.((x+1)^2\cdot P_3(x))\right|_{x=-1}=2$

  ◦ $x=-1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=-1$

- Multiply by denominators

  $3x^2(x+1)^2\left(\frac{d^2}{dx^2}y(x)\right)+x(x+1)(11x-1)\left(\frac{d}{dx}y(x)\right)+(5x^2+1)\,y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(3u^4-6u^3+3u^2)\left(\frac{d^2}{du^2}y(u)\right)+(11u^3-23u^2+12u)\left(\frac{d}{du}y(u)\right)+(5u^2-10u+6)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot y(u)$ to series expansion for $m=0..2$

    $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

  ◦ Shift index using $k->k-m$

    $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=1..3$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  ◦ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=2..4$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

  ◦ Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$3a_0(2+r)(1+r)\,u^r + (3a_1(3+r)(2+r) - a_0(2+r)(5+6r))\,u^{1+r} + \left(\sum_{k=2}^{\infty} (3a_k(k+r+2)(k+$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $3(2+r)(1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-2, -1\}$
- Each term must be 0

  $3a_1(3+r)(2+r) - a_0(2+r)(5+6r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = \frac{a_0(5+6r)}{3(3+r)}$
- Each term in the series must be 0, giving the recursion relation

  $3(a_k + a_{k-2} - 2a_{k-1})\,k^2 + (6(a_k + a_{k-2} - 2a_{k-1})\,r + 9a_k - 4a_{k-2} - 5a_{k-1})\,k + 3(a_k + a_{k-2} - 2a$
- Shift index using $k- >k+2$

  $3(a_{k+2} + a_k - 2a_{k+1})\,(k+2)^2 + (6(a_{k+2} + a_k - 2a_{k+1})\,r + 9a_{k+2} - 4a_k - 5a_{k+1})\,(k+2) + 3(a_{k+}$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{3k^2a_k - 6k^2a_{k+1} + 6kra_k - 12kra_{k+1} + 3r^2a_k - 6r^2a_{k+1} + 8ka_k - 29ka_{k+1} + 8ra_k - 29ra_{k+1} + 5a_k - 33a_{k+1}}{3(k^2+2kr+r^2+7k+7r+12)}$
- Recursion relation for $r = -2$

  $a_{k+2} = -\frac{3k^2a_k - 6k^2a_{k+1} - 4ka_k - 5ka_{k+1} + a_k + a_{k+1}}{3(k^2+3k+2)}$
- Solution for $r = -2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-2}, a_{k+2} = -\frac{3k^2a_k - 6k^2a_{k+1} - 4ka_k - 5ka_{k+1} + a_k + a_{k+1}}{3(k^2+3k+2)}, a_1 = -\frac{7a_0}{3}\right]$
- Revert the change of variables $u = x+1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^{k-2}, a_{k+2} = -\frac{3k^2a_k - 6k^2a_{k+1} - 4ka_k - 5ka_{k+1} + a_k + a_{k+1}}{3(k^2+3k+2)}, a_1 = -\frac{7a_0}{3}\right]$
- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{3k^2a_k - 6k^2a_{k+1} + 2ka_k - 17ka_{k+1} - 10a_{k+1}}{3(k^2+5k+6)}$
- Solution for $r = -1$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-1}, a_{k+2} = -\frac{3k^2a_k - 6k^2a_{k+1} + 2ka_k - 17ka_{k+1} - 10a_{k+1}}{3(k^2+5k+6)}, a_1 = -\frac{a_0}{6}\right]$
- Revert the change of variables $u = x+1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^{k-1}, a_{k+2} = -\frac{3k^2a_k - 6k^2a_{k+1} + 2ka_k - 17ka_{k+1} - 10a_{k+1}}{3(k^2+5k+6)}, a_1 = -\frac{a_0}{6}\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k(x+1)^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k(x+1)^{k-1}\right), a_{k+2} = -\frac{3k^2a_k - 6k^2a_{k+1} - 4ka_k - 5ka_{k+1} + a_k + a_{k+1}}{3(k^2+3k+2)}, a\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
```

```
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.019 (sec)
Leaf size : 19

```
dsolve(3*x^2*(x+1)^2*diff(diff(y(x),x),x)-x*(-11*x^2-10*x+1)*diff(y(x),x)+(5*x^2+1)*y(x)
```

$$y = \frac{x^{1/3} c_2 + c_1 x}{(x+1)^2}$$

Mathematica DSolve solution

Solving time : 0.242 (sec)
Leaf size : 58

```
DSolve[{3*x^2*(1+x)^2*D[y[x],{x,2}]-x*(1-10*x-11*x^2)*D[y[x],x]+(1+5*x^2)*y[x]==0,{}},y[x],x,I
```

$$y(x) \to \frac{1}{2} \sqrt[6]{x} (3 c_2 x^{2/3} + 2 c_1) \exp \left(-\frac{1}{2} \int_1^x \left(\frac{4}{K[1]+1} - \frac{1}{3K[1]} \right) dK[1] \right)$$

2.1.120 Problem 122

Internal problem ID [9292]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 122
Date solved : `Monday, January 27, 2025 at 06:01:04 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2\left(x^2+2x+3\right)y''-x\left(-15x^2-14x+3\right)y'+\left(7x^2+3\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.171 (sec)

Writing the ode as

$$\left(4x^4+8x^3+12x^2\right)y''+\left(15x^3+14x^2-3x\right)y'+\left(7x^2+3\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=4x^4+8x^3+12x^2\\B&=15x^3+14x^2-3x\\C&=7x^2+3\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-7}{64x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=-7\\t&=64x^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{7}{64x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.224: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{64x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{8} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{7}{64x^2}$$

Since the gcd$(s,t) = 1$. This gives $b = -\frac{7}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{7}{64x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{8}$	$\frac{1}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{8}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{8} - \left(\frac{1}{8}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{8x} + (-)(0) \\
&= \frac{1}{8x} \\
&= \frac{1}{8x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{1}{8x}\right)(0)+\left(\left(-\frac{1}{8x^2}\right)+\left(\frac{1}{8x}\right)^2-\left(-\frac{7}{64x^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}z_1(x)&=pe^{\int \omega\,dx}\\&=\mathrm{e}^{\int \frac{1}{8x}dx}\\&=x^{1/8}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1&=z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\&=z_1e^{-\int \frac{1}{2}\frac{15x^3+14x^2-3x}{4x^4+8x^3+12x^2}\,dx}\\&=z_1e^{\frac{\ln(x)}{8}-\ln(x^2+2x+3)}\\&=z_1\left(\frac{x^{1/8}}{x^2+2x+3}\right)\end{aligned}$$

Which simplifies to

$$y_1=\frac{x^{1/4}}{x^2+2x+3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2&=y_1\int \frac{e^{\int -\frac{15x^3+14x^2-3x}{4x^4+8x^3+12x^2}\,dx}}{(y_1)^2}\,dx\\&=y_1\int \frac{e^{\frac{\ln(x)}{4}-2\ln(x^2+2x+3)}}{(y_1)^2}\,dx\\&=y_1\left(\frac{4\sqrt{x}\,\mathrm{e}^{\frac{\ln(x)}{4}-2\ln(x^2+2x+3)}\,(x^2+2x+3)^2}{3}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y&=c_1y_1+c_2y_2\\&=c_1\left(\frac{x^{1/4}}{x^2+2x+3}\right)+c_2\left(\frac{x^{1/4}}{x^2+2x+3}\left(\frac{4\sqrt{x}\,\mathrm{e}^{\frac{\ln(x)}{4}-2\ln(x^2+2x+3)}\,(x^2+2x+3)^2}{3}\right)\right)\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2(x^2+2x+3)\left(\frac{d^2}{dx^2}y(x)\right) - x(-15x^2-14x+3)\left(\frac{d}{dx}y(x)\right) + (7x^2+3)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(7x^2+3)y(x)}{4x^2(x^2+2x+3)} - \frac{(15x^2+14x-3)\left(\frac{d}{dx}y(x)\right)}{4x(x^2+2x+3)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(15x^2+14x-3)\left(\frac{d}{dx}y(x)\right)}{4x(x^2+2x+3)} + \frac{(7x^2+3)y(x)}{4x^2(x^2+2x+3)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{15x^2+14x-3}{4x(x^2+2x+3)}, P_3(x) = \frac{7x^2+3}{4x^2(x^2+2x+3)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x=0$

  $(x \cdot P_2(x))\bigg|_{x=0} = -\frac{1}{4}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x=0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{1}{4}$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2(x^2+2x+3)\left(\frac{d^2}{dx^2}y(x)\right) + x(15x^2+14x-3)\left(\frac{d}{dx}y(x)\right) + (7x^2+3)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  ◦ Shift index using $k- >k+2-m$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$3a_0(-1+4r)(-1+r)x^r + (3a_1(3+4r)r + 2a_0r(3+4r))x^{1+r} + \left(\sum_{k=2}^{\infty} (3a_k(4k+4r-1)(k+r-\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $3(-1+4r)(-1+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{1, \frac{1}{4}\}$
- Each term must be 0
  $3a_1(3+4r)r + 2a_0r(3+4r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = -\frac{2a_0}{3}$
- Each term in the series must be 0, giving the recursion relation
  $(4k+4r-1)(k+r-1)(3a_k + 2a_{k-1} + a_{k-2}) = 0$
- Shift index using $k->k+2$
  $(4k+4r+7)(k+r+1)(3a_{k+2} + 2a_{k+1} + a_k) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{2a_{k+1}}{3} - \frac{a_k}{3}$
- Recursion relation for $r = 1$
  $a_{k+2} = -\frac{2a_{k+1}}{3} - \frac{a_k}{3}$
- Solution for $r = 1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{2a_{k+1}}{3} - \frac{a_k}{3}, a_1 = -\frac{2a_0}{3}\right]$
- Recursion relation for $r = \frac{1}{4}$
  $a_{k+2} = -\frac{2a_{k+1}}{3} - \frac{a_k}{3}$
- Solution for $r = \frac{1}{4}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+2} = -\frac{2a_{k+1}}{3} - \frac{a_k}{3}, a_1 = -\frac{2a_0}{3}\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{4}}\right), a_{k+2} = -\frac{2a_{k+1}}{3} - \frac{a_k}{3}, a_1 = -\frac{2a_0}{3}, b_{k+2} = -\frac{2b_{k+1}}{3} - \frac{b_k}{3}, b_1 = -\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.024 (sec)
Leaf size : 24

```
dsolve(4*x^2*(x^2+2*x+3)*diff(diff(y(x),x),x)-x*(-15*x^2-14*x+3)*diff(y(x),x)+(7*x^2+3
```

$$y = \frac{c_2 x^{1/4} + c_1 x}{x^2 + 2x + 3}$$

Mathematica DSolve solution

Solving time : 0.268 (sec)
Leaf size : 67

```
DSolve[{4*x^2*(3+2*x+x^2)*D[y[x],{x,2}]-x*(3-14*x-15*x^2)*D[y[x],x]+(3+7*x^2)*y[x]==0,{}},y[
```

$$y(x) \to \frac{1}{3}\sqrt[8]{x}\left(4c_2 x^{3/4} + 3c_1\right)\exp\left(-\frac{1}{2}\int_1^x \left(\frac{4(K[1]+1)}{K[1](K[1]+2)+3} - \frac{1}{4K[1]}\right) dK[1]\right)$$

2.1.121 Problem 123

Internal problem ID [9293]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 123
Date solved : `Monday, January 27, 2025 at 06:01:05 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(x^2-2x+1)\,y''-x(3+x)\,y'+(4+x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.301 (sec)

Writing the ode as

$$x^2(x-1)^2\,y''+\left(-x^2-3x\right)y'+(4+x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(x-1)^2\\ B &= -x^2-3x\\ C &= 4+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t}\\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{7x^2+10x-1}{4x^2\left(x-1\right)^4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 7x^2+10x-1\\ t &= 4x^2(x-1)^4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{7x^2+10x-1}{4x^2\left(x-1\right)^4}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.226: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2(x-1)^4$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{(x-1)^3} - \frac{3}{2(x-1)} + \frac{4}{(x-1)^4} + \frac{3}{2x} + \frac{7}{4(x-1)^2} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$
\begin{aligned}
\alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a}+v\right)\\
\alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a}+v\right)
\end{aligned}
$$

The partial fraction decomposition of r is

$$
r = -\frac{2}{(x-1)^3} - \frac{3}{2(x-1)} + \frac{4}{(x-1)^4} + \frac{3}{2x} + \frac{7}{4(x-1)^2} - \frac{1}{4x^2}
$$

There is pole in r at $x=1$ of order 4, hence $v=2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c=1$ gives

$$
[\sqrt{r}]_c \approx \frac{2}{(x-1)^2} - \frac{1}{2(x-1)} + \frac{21}{32} - \frac{9x}{32} + \frac{53(x-1)^2}{256} - \frac{149(x-1)^3}{1024} + \dots \tag{2B}
$$

Using eq. (1B), taking the sum up to $v=2$ the above becomes

$$
[\sqrt{r}]_c = \frac{2}{(x-1)^2} \tag{3B}
$$

The above shows that the coefficient of $\frac{1}{(x-1)^2}$ is

$$
a = 2
$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c=1$. This term becomes $\frac{1}{(x-1)^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be -2. Therefore

$$
\begin{aligned}
b &= (-2) - (0)\\
&= -2
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= \frac{2}{(x-1)^2}\\
\alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a}+v\right) = \frac{1}{2}\left(\frac{-2}{2}+2\right) = \frac{1}{2}\\
\alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a}+v\right) = \frac{1}{2}\left(-\frac{-2}{2}+2\right) = \frac{3}{2}
\end{aligned}
$$

Since the order of r at ∞ is $4>2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^+ &= 0\\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{7x^2+10x-1}{4x^2(x-1)^4}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
1	4	$\frac{2}{(x-1)^2}$	$\frac{1}{2}$	$\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2} + (-)\,(0) \\ &= \frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2} \\ &= \frac{2x^2+x+1}{2x\,(x-1)^2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{4}{(x-1)^3} - \frac{1}{2\,(x-1)^2}\right) + \left(\frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\,dx} \\ &= e^{\int \left(\frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2}\right)dx} \\ &= \sqrt{x}\,\sqrt{x-1}\,e^{-\frac{2}{x-1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2-3x}{x^2(x-1)^2}\,dx} \\
&= z_1 e^{\frac{3\ln(x)}{2}-\frac{2}{x-1}-\frac{3\ln(x-1)}{2}} \\
&= z_1 \left(\frac{x^{3/2}\mathrm{e}^{-\frac{2}{x-1}}}{(x-1)^{3/2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^{3/2}\mathrm{e}^{-\frac{4}{x-1}}\sqrt{x(x-1)}}{(x-1)^{3/2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-3x}{x^2(x-1)^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3\ln(x)-\frac{4}{x-1}-3\ln(x-1)}}{(y_1)^2}\,dx \\
&= y_1 \left(\mathrm{e}^{-4}\,\mathrm{Ei}_1\left(-\frac{4}{x-1}-4\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^{3/2}\mathrm{e}^{-\frac{4}{x-1}}\sqrt{x(x-1)}}{(x-1)^{3/2}}\right) + c_2\left(\frac{x^{3/2}\mathrm{e}^{-\frac{4}{x-1}}\sqrt{x(x-1)}}{(x-1)^{3/2}}\left(\mathrm{e}^{-4}\,\mathrm{Ei}_1\left(-\frac{4}{x-1}-4\right)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x^2-2x+1)\left(\frac{d^2}{dx^2}y(x)\right) - x(x+3)\left(\frac{d}{dx}y(x)\right) + (x+4)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(x+4)y(x)}{x^2(x^2-2x+1)} + \frac{(x+3)\left(\frac{d}{dx}y(x)\right)}{x(x^2-2x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) - \frac{(x+3)\left(\frac{d}{dx}y(x)\right)}{x(x^2-2x+1)} + \frac{(x+4)y(x)}{x^2(x^2-2x+1)} = 0$

- □ Check to see if x_0 is a regular singular point
  - ○ Define functions

$\left[P_2(x) = -\frac{x+3}{x(x^2-2x+1)}, P_3(x) = \frac{x+4}{x^2(x^2-2x+1)}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = -3$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 4$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2-2x+1)\left(\frac{d^2}{dx^2}y(x)\right) - x(x+3)\left(\frac{d}{dx}y(x)\right) + (x+4)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k- >k + 2 - m$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)^2\,x^r + \left(a_1(-1+r)^2 - a_0(1+2r)\,(-1+r)\right)x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-2)^2 - a_{k-1}(2k -\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 2$

- Each term must be 0

  $a_1(-1+r)^2 - a_0(1+2r)\,(-1+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = \frac{a_0(1+2r)}{-1+r}$

- Each term in the series must be 0, giving the recursion relation

  $((a_k + a_{k-2} - 2a_{k-1})\,k + (a_k + a_{k-2} - 2a_{k-1})\,r - 2a_k - 3a_{k-2} + a_{k-1})\,(k+r-2) = 0$

- Shift index using $k- >k + 2$

$$((a_{k+2}+a_k-2a_{k+1})(k+2)+(a_{k+2}+a_k-2a_{k+1})r-2a_{k+2}-3a_k+a_{k+1})(k+r)=0$$

- Recursion relation that defines series solution to ODE
$$a_{k+2}=-\frac{ka_k-2ka_{k+1}+ra_k-2ra_{k+1}-a_k-3a_{k+1}}{k+r}$$
- Recursion relation for $r=2$
$$a_{k+2}=-\frac{ka_k-2ka_{k+1}+a_k-7a_{k+1}}{k+2}$$
- Solution for $r=2$
$$\left[y(x)=\sum_{k=0}^{\infty}a_kx^{k+2}, a_{k+2}=-\frac{ka_k-2ka_{k+1}+a_k-7a_{k+1}}{k+2}, a_1=5a_0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 45

```
dsolve(x^2*(x^2-2*x+1)*diff(diff(y(x),x),x)-x*(x+3)*diff(y(x),x)+(x+4)*y(x) = 0,y(x),sin
```

$$y=\frac{x^2\left(\mathrm{Ei}_1\left(-\frac{4x}{x-1}\right)\mathrm{e}^{-\frac{4x}{x-1}}c_2+\mathrm{e}^{-\frac{4}{x-1}}c_1\right)}{x-1}$$

Mathematica DSolve solution

Solving time : 0.305 (sec)
Leaf size : 116

```
DSolve[{x^2*(1-2*x+x^2)*D[y[x],{x,2}]-x*(3+x)*D[y[x],x]+(4+x)*y[x]==0,{}},y[x],x,IncludeSingula
```

$$y(x)\to \exp\left(\int_1^x \frac{2K[1]^2+K[1]+1}{2(K[1]-1)^2K[1]}dK[1]-\frac{1}{2}\int_1^x -\frac{K[2]+3}{(K[2]-1)^2K[2]}dK[2]\right)\left(c_2\int_1^x\exp\left(-2\int_1^{K[3]}\frac{2K[1]^2+K[1]+1}{2(K[1]-1)^2K[1]}dK[1]\right)dK[3]+c_1\right)$$

2.1.122 Problem 124

Internal problem ID [9294]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 124
Date solved : `Monday, January 27, 2025 at 06:01:06 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(2+x)\,y''+5x^2y'+(1+x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.207 (sec)

Writing the ode as

$$\left(2x^3+4x^2\right)y''+5x^2y'+(1+x)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^3+4x^2 \\ B &= 5x^2 \\ C &= 1+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-3x^2-24x-16}{16\left(x^2+2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3x^2-24x-16 \\ t &= 16\left(x^2+2x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-3x^2-24x-16}{16\left(x^2+2x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.228: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16\,(2+x)^2} - \frac{1}{8x} - \frac{1}{4x^2} + \frac{1}{16+8x}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(2+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 - 24x - 16}{16\left(x^2 + 2x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x^2 - 24x - 16}{16\left(x^2 + 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= \frac{1}{4} - \left(\frac{1}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{4\,(2+x)} + \frac{1}{2x} + (-)\,(0) \\
&= -\frac{1}{4\,(2+x)} + \frac{1}{2x} \\
&= \frac{x+4}{4x\,(2+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{4(2+x)} + \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{4(2+x)^2} - \frac{1}{2x^2}\right) + \left(-\frac{1}{4(2+x)} + \frac{1}{2x}\right)^2 - \left(\frac{-3x^2 - 24x - 16}{16(x^2+2x)^2}\right)\right) = 0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(-\frac{1}{4(2+x)} + \frac{1}{2x}\right) dx} \\ &= \frac{\sqrt{x}}{(2+x)^{1/4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{5x^2}{2x^3+4x^2}\, dx} \\ &= z_1 e^{-\frac{5\ln(2+x)}{4}} \\ &= z_1 \left(\frac{1}{(2+x)^{5/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{(2+x)^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{5x^2}{2x^3+4x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{5\ln(2+x)}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(2\sqrt{2+x} - 2\sqrt{2}\, \operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\sqrt{x}}{(2+x)^{3/2}}\right) + c_2 \left(\frac{\sqrt{x}}{(2+x)^{3/2}} \left(2\sqrt{2+x} - 2\sqrt{2}\, \operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2(x+2)\left(\frac{d^2}{dx^2}y(x)\right)+5x^2\left(\frac{d}{dx}y(x)\right)+(x+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(x+1)y(x)}{2(x+2)x^2}-\frac{5\left(\frac{d}{dx}y(x)\right)}{2(x+2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{5\left(\frac{d}{dx}y(x)\right)}{2(x+2)}+\frac{(x+1)y(x)}{2(x+2)x^2}=0$

- □ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{5}{2(x+2)},P_3(x)=\frac{x+1}{2(x+2)x^2}\right]$

  - $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

    $\left((x+2)\cdot P_2(x)\right)\Big|_{x=-2}=\frac{5}{2}$

  - $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

    $\left((x+2)^2\cdot P_3(x)\right)\Big|_{x=-2}=0$

  - $x=-2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-2$

- Multiply by denominators

  $2x^2(x+2)\left(\frac{d^2}{dx^2}y(x)\right)+5x^2\left(\frac{d}{dx}y(x)\right)+(x+1)\,y(x)=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

  $(2u^3-8u^2+8u)\left(\frac{d^2}{du^2}y(u)\right)+(5u^2-20u+20)\left(\frac{d}{du}y(u)\right)+(u-1)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

    $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

  - Shift index using $k->k-m$

    $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$4a_0r(3+2r)\, u^{-1+r} + (4a_1(1+r)(5+2r) - a_0(8r^2+12r+1))\, u^r + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+r+1)(2k+5$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $4r(3+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, -\frac{3}{2}\}$
- Each term must be 0
  $4a_1(1+r)(5+2r) - a_0(8r^2+12r+1) = 0$
- Each term in the series must be 0, giving the recursion relation
  $2(-4a_k + a_{k-1} + 4a_{k+1})\, k^2 + (4(-4a_k + a_{k-1} + 4a_{k+1})\, r - 12a_k - a_{k-1} + 28a_{k+1})\, k + 2(-4a_k + a_{k-}$
- Shift index using $k- >k+1$
  $2(-4a_{k+1} + a_k + 4a_{k+2})(k+1)^2 + (4(-4a_{k+1} + a_k + 4a_{k+2})\, r - 12a_{k+1} - a_k + 28a_{k+2})(k+1) + 2$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 4kra_k - 16kra_{k+1} + 2r^2a_k - 8r^2a_{k+1} + 3ka_k - 28ka_{k+1} + 3ra_k - 28ra_{k+1} + a_k - 21a_{k+1}}{4(2k^2+4kr+2r^2+11k+11r+14)}$
- Recursion relation for $r = 0$
  $a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 3ka_k - 28ka_{k+1} + a_k - 21a_{k+1}}{4(2k^2+11k+14)}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 3ka_k - 28ka_{k+1} + a_k - 21a_{k+1}}{4(2k^2+11k+14)}, 20a_1 - a_0 = 0\right]$
- Revert the change of variables $u = x + 2$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x+2)^k, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 3ka_k - 28ka_{k+1} + a_k - 21a_{k+1}}{4(2k^2+11k+14)}, 20a_1 - a_0 = 0\right]$
- Recursion relation for $r = -\frac{3}{2}$
  $a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} - 3ka_k - 4ka_{k+1} + a_k + 3a_{k+1}}{4(2k^2+5k+2)}$
- Solution for $r = -\frac{3}{2}$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{3}{2}}, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} - 3ka_k - 4ka_{k+1} + a_k + 3a_{k+1}}{4(2k^2+5k+2)}, -4a_1 - a_0 = 0\right]$
- Revert the change of variables $u = x + 2$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x+2)^{k-\frac{3}{2}}, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} - 3ka_k - 4ka_{k+1} + a_k + 3a_{k+1}}{4(2k^2+5k+2)}, -4a_1 - a_0 = 0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k (x+2)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+2)^{k-\frac{3}{2}}\right), a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 3ka_k - 28ka_{k+1} + a_k - 21a_{k+1}}{4(2k^2+11k+14)}, 20a\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
```

```
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.040 (sec)
Leaf size : 39

```
dsolve(2*x^2*(x+2)*diff(diff(y(x),x),x)+5*diff(y(x),x)*x^2+(x+1)*y(x) = 0,y(x),singsol
```

$$y = \frac{\sqrt{x}\left(\sqrt{x+2}\,\sqrt{2}\,c_2 - 2\,\operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)c_2 + c_1\right)}{(x+2)^{3/2}}$$

Mathematica DSolve solution

Solving time : 0.39 (sec)
Leaf size : 83

```
DSolve[{2*x^2*(2+x)*D[y[x],{x,2}]+5*x^2*D[y[x],x]+(1+x)*y[x]==0,{}},y[x],x,IncludeSingularSo
```

$$y(x) \to \frac{\exp\left(\int_1^x \frac{K[1]+4}{4K[1]^2+8K[1]}dK[1]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[2]}\frac{K[1]+4}{4K[1]^2+8K[1]}dK[1]\right)dK[2]+c_1\right)}{(x+2)^{5/4}}$$

2.1.123 Problem 125

Internal problem ID [9295]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 125
Date solved : `Monday, January 27, 2025 at 06:01:06 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(-x^2+2\right)y''-2x\left(2x^2+1\right)y'+\left(-2x^2+2\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.372 (sec)

Writing the ode as

$$\left(-x^4+2x^2\right)y''+\left(-4x^3-2x\right)y'+\left(-2x^2+2\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-x^4+2x^2\\B&=-4x^3-2x\\C&=-2x^2+2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{3x^2-1}{\left(x^3-2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=3x^2-1\\t&=\left(x^3-2x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{3x^2-1}{\left(x^3-2x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.230: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^3 - 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \sqrt{2}$ of order 2. There is a pole at $x = -\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{5}{16 (x - \sqrt{2})^2} + \frac{5}{16 (x + \sqrt{2})^2} - \frac{3\sqrt{2}}{32 (x - \sqrt{2})} + \frac{3\sqrt{2}}{32 (x + \sqrt{2})}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = \sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x-\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

For the pole at $x = -\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x+\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3x^2 - 1}{\left(x^3 - 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$\sqrt{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-\sqrt{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= 0 - (0) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + (+)[\sqrt{r}]_{\infty} \\ &= \frac{1}{2x} - \frac{1}{4\left(x-\sqrt{2}\right)} - \frac{1}{4\left(x+\sqrt{2}\right)} + (0) \\ &= \frac{1}{2x} - \frac{1}{4\left(x-\sqrt{2}\right)} - \frac{1}{4\left(x+\sqrt{2}\right)} \\ &= -\frac{1}{x^3-2x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{4\left(x-\sqrt{2}\right)} - \frac{1}{4\left(x+\sqrt{2}\right)}\right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{1}{4\left(x-\sqrt{2}\right)^2} + \frac{1}{4\left(x+\sqrt{2}\right)^2}\right) + \left(\frac{1}{2x} - \frac{}{4(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{1}{4\left(x-\sqrt{2}\right)} - \frac{1}{4\left(x+\sqrt{2}\right)}\right) dx} \\ &= \frac{\sqrt{x}}{\left(x-\sqrt{2}\right)^{1/4}\left(x+\sqrt{2}\right)^{1/4}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x^3-2x}{-x^4+2x^2}\, dx} \\ &= z_1 e^{-\frac{5\ln\left(x^2-2\right)}{4} + \frac{\ln(x)}{2}} \\ &= z_1\left(\frac{\sqrt{x}}{\left(x^2-2\right)^{5/4}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x}{\left(x^2-2\right)^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^3-2x}{-x^4+2x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{5\ln(x^2-2)}{2}+\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\sqrt{x^2-2} + \sqrt{2}\,\arctan\left(\frac{\sqrt{2}}{\sqrt{x^2-2}}\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x}{(x^2-2)^{3/2}}\right) + c_2 \left(\frac{x}{(x^2-2)^{3/2}}\left(\sqrt{x^2-2} + \sqrt{2}\,\arctan\left(\frac{\sqrt{2}}{\sqrt{x^2-2}}\right)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2(-x^2+2)\left(\frac{d^2}{dx^2}y(x)\right) - 2x(2x^2+1)\left(\frac{d}{dx}y(x)\right) + (-2x^2+2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{2(x^2-1)y(x)}{x^2(x^2-2)} - \frac{2(2x^2+1)\left(\frac{d}{dx}y(x)\right)}{x(x^2-2)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) + \frac{2(2x^2+1)\left(\frac{d}{dx}y(x)\right)}{x(x^2-2)} + \frac{2(x^2-1)y(x)}{x^2(x^2-2)} = 0$

□ Check to see if x_0 is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{2(2x^2+1)}{x(x^2-2)}, P_3(x) = \frac{2(x^2-1)}{x^2(x^2-2)}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = -1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 1$
  - $x = 0$is a regular singular point
    Check to see if x_0 is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $x^2(x^2-2)\left(\frac{d^2}{dx^2}y(x)\right) + 2x(2x^2+1)\left(\frac{d}{dx}y(x)\right) + (2x^2-2)\,y(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- >k + 2 - m$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0(-1+r)^2 x^r - 2a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(-2a_k(k+r-1)^2 + a_{k-2}(k+r)\,(k+r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2(-1+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 1$

- Each term must be 0

  $-2a_1 r^2 = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $-2a_k(k+r-1)^2 + a_{k-2}(k+r)\,(k+r-1) = 0$

- Shift index using $k- >k + 2$

  $-2a_{k+2}(k+r+1)^2 + a_k(k+r+2)\,(k+r+1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k(k+r+2)}{2(k+r+1)}$

- Recursion relation for $r = 1$

  $a_{k+2} = \frac{a_k(k+3)}{2(k+2)}$

- Solution for $r = 1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = \frac{a_k(k+3)}{2(k+2)}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.036 (sec)
Leaf size : 42

```
dsolve(x^2*(-x^2+2)*diff(diff(y(x),x),x)-2*x*(2*x^2+1)*diff(y(x),x)+(-2*x^2+2)*y(x) = 0,
```

$$y = \frac{x\left(\sqrt{2}\, c_2 \sqrt{x^2-2} + 2\arctan\left(\frac{\sqrt{2}}{\sqrt{x^2-2}}\right) c_2 + c_1\right)}{\left(x^2-2\right)^{3/2}}$$

Mathematica DSolve solution

Solving time : 0.252 (sec)
Leaf size : 97

```
DSolve[{x^2*(2-x^2)*D[y[x],{x,2}]-2*x*(1+2*x^2)*D[y[x],x]+(2-2*x^2)*y[x]==0,{}},y[x],x,Include
```

$$y(x) \to \exp\left(\int_1^x \frac{1}{2K[1]-K[1]^3}dK[1] - \frac{1}{2}\int_1^x \left(\frac{5K[2]}{K[2]^2-2} - \frac{1}{K[2]}\right) dK[2]\right)\left(c_2 \int_1^x \exp\left(-2\int_1^{K[3]} \frac{1}{2K[1]-K[1]^3}dK[1]\right) dK[3] + c_1\right)$$

2.1.124 Problem 126

Internal problem ID [9296]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 126
Date solved : `Monday, January 27, 2025 at 06:01:07 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x(5-x)\,y' + (9-4x)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.428 (sec)

Writing the ode as

$$x^2y'' + \left(x^2-5x\right)y' + (9-4x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2-5x \\ C &= 9-4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2+6x-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2+6x-1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2+6x-1}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.232: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{3}{2x} - \frac{5}{2x^2} + \frac{15}{2x^3} - \frac{115}{4x^4} + \frac{495}{4x^5} - \frac{2285}{4x^6} + \frac{11055}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 6x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{6x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{6x - 1}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 6. Dividing this by leading coefficient in t which is 4 gives $\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{3}{2}\right) - (0) \\ &= \frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{3}{2}}{\frac{1}{2}} - 0\right) &= \frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{3}{2}}{\frac{1}{2}} - 0\right) &= -\frac{3}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 6x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{3}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - (\alpha_{c_1}^+) \\
&= \frac{3}{2} - \left(\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2x} + \frac{1}{2} \\
&= \frac{1+x}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2x} + \frac{1}{2}\right)(1) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} + \frac{1}{2}\right)^2 - \left(\frac{x^2+6x-1}{4x^2}\right)\right) &= 0 \\
\frac{1-a_0}{x} &= 0
\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 1 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (1+x)\, e^{\int \left(\frac{1}{2x}+\frac{1}{2}\right)dx} \\ &= (1+x)\, e^{\frac{x}{2}+\frac{\ln(x)}{2}} \\ &= (1+x)\sqrt{x}\, e^{\frac{x}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^2-5x}{x^2}\, dx} \\ &= z_1 e^{-\frac{x}{2}+\frac{5\ln(x)}{2}} \\ &= z_1 \left(x^{5/2} e^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^3(1+x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2-5x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-x+5\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\operatorname{Ei}_1(x) - \frac{e^{-x}}{-1-x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(x^3(1+x)\right) + c_2\left(x^3(1+x)\left(-\operatorname{Ei}_1(x) - \frac{e^{-x}}{-1-x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right) - x(5-x)\left(\frac{d}{dx}y(x)\right) + (9-4x)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = \frac{(-9+4x)y(x)}{x^2} - \frac{(x-5)\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x) + \frac{(x-5)\left(\frac{d}{dx}y(x)\right)}{x} - \frac{(-9+4x)y(x)}{x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{x-5}{x}, P_3(x) = -\frac{-9+4x}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -5$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 9$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(x-5)\left(\frac{d}{dx}y(x)\right) + (9-4x)\,y(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$
    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k-m$
    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$
    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k- >k+1-m$
    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion
    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,x^{k+r}$
    Rewrite ODE with series expansions
    $a_0(-3+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-3)^2 + a_{k-1}(k-5+r)\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-3+r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 3$
- Each term in the series must be 0, giving the recursion relation
  $a_k(k+r-3)^2 + a_{k-1}(k-5+r) = 0$
- Shift index using $k- >k+1$
  $a_{k+1}(k-2+r)^2 + a_k(k+r-4) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = -\frac{a_k(k+r-4)}{(k-2+r)^2}$
- Recursion relation for $r = 3$; series terminates at $k = 1$

$a_{k+1} = -\frac{a_k(k-1)}{(k+1)^2}$

- Apply recursion relation for $k = 0$

  $a_1 = a_0$

- Terminating series solution of the ODE for $r = 3$. Use reduction of order to find the second li

  $y(x) = a_0 \cdot (x + 1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 27

```
dsolve(x^2*diff(diff(y(x),x),x)-x*(-x+5)*diff(y(x),x)+(9-4*x)*y(x) = 0,y(x),singsol=al
```

$$y = \left(-c_2 \mathrm{e}^{-x} + \left(\mathrm{Ei}_1\left(x\right) c_2 + c_1\right)\left(x + 1\right)\right) x^3$$

Mathematica DSolve solution

Solving time : 0.708 (sec)
Leaf size : 72

```
DSolve[{x^2*D[y[x],{x,2}]-x*(5-x)*D[y[x],x]+(9-4*x)*y[x]==0,{}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to \sqrt{x}(x+1)\left(c_2 \int_1^x \frac{e^{-K[2]-1}}{K[2](K[2]+1)^2} dK[2] + c_1\right) \exp\left(\frac{1}{2}\left(-\int_1^x \left(1 - \frac{5}{K[1]}\right) dK[1] + x + 1\right)\right)$$

2.1.125 Problem 127

Internal problem ID [9297]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 127
Date solved : `Monday, January 27, 2025 at 06:01:08 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2\left(x^2+x+1\right)y''+12x^2(1+x)\,y'+\left(3x^2+3x+1\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.780 (sec)

Writing the ode as

$$\left(4x^4+4x^3+4x^2\right)y''+\left(12x^3+12x^2\right)y'+\left(3x^2+3x+1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=4x^4+4x^3+4x^2\\ B&=12x^3+12x^2\\ C&=3x^2+3x+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{2x^2-4x-1}{4\left(x^3+x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=2x^2-4x-1\\ t&=4\left(x^3+x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{2x^2-4x-1}{4\left(x^3+x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.234: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ of order 2. There is a pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{2x} - \frac{1}{4x^2} + \frac{-\frac{3}{8} - \frac{i\sqrt{3}}{8}}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{3}{8} + \frac{i\sqrt{3}}{8}}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2} + \frac{\frac{1}{4} - \frac{5i\sqrt{3}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{4} + \frac{5i\sqrt{3}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{8} - \frac{i\sqrt{3}}{8}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{-2-2i\sqrt{3}}}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{-2-2i\sqrt{3}}}{4}
\end{aligned}$$

For the pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{8} + \frac{i\sqrt{3}}{8}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{-2+2i\sqrt{3}}}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{-2+2i\sqrt{3}}}{4}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2x^2-4x-1}{4\left(x^3+x^2+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-\frac{1}{2}+\frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2}+\frac{\sqrt{-2-2i\sqrt{3}}}{4}$	$\frac{1}{2}-\frac{\sqrt{-2-2i\sqrt{3}}}{4}$
$-\frac{1}{2}-\frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2}+\frac{\sqrt{-2+2i\sqrt{3}}}{4}$	$\frac{1}{2}-\frac{\sqrt{-2+2i\sqrt{3}}}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x - c_3}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + \frac{\frac{1}{2} - \frac{\sqrt{-2-2i\sqrt{3}}}{4}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} - \frac{\sqrt{-2+2i\sqrt{3}}}{4}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} + (-)(0) \\
&= \frac{1}{2x} + \frac{\frac{1}{2} - \frac{\sqrt{-2-2i\sqrt{3}}}{4}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} - \frac{\sqrt{-2+2i\sqrt{3}}}{4}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} \\
&= \frac{2x^2 + 1}{2x(x^2 + x + 1)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x} + \frac{\frac{1}{2} - \frac{\sqrt{-2-2i\sqrt{3}}}{4}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} - \frac{\sqrt{-2+2i\sqrt{3}}}{4}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{\frac{1}{2} - \frac{\sqrt{-2-2i\sqrt{3}}}{4}}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} - \frac{\frac{1}{2} - \frac{\sqrt{-2+2i\sqrt{3}}}{4}}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2}\right) -
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} + \frac{\frac{1}{2} - \frac{\sqrt{-2-2i\sqrt{3}}}{4}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} - \frac{\sqrt{-2+2i\sqrt{3}}}{4}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}}\right) dx} \\
&= (x^2 + x + 1)^{1/4} \sqrt{x}\,\sqrt{2}\, \mathrm{e}^{-\frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{12x^3 + 12x^2}{4x^4 + 4x^3 + 4x^2}\, dx} \\
&= z_1 e^{-\frac{3\ln(x^2 + x + 1)}{4} - \frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{2}}}{(x^2 + x + 1)^{3/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{-\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)} \sqrt{x}\,\sqrt{2}}{\sqrt{x^2 + x + 1}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{12x^3+12x^2}{4x^4+4x^3+4x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{3\ln\left(x^2+x+1\right)}{2}-\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{3\ln\left(x^2+x+1\right)}{2}-\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}\left(x^2+x+1\right)\mathrm{e}^{2\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}}{2x} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{-\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}\sqrt{x}\,\sqrt{2}}{\sqrt{x^2+x+1}}\right) \\
&\quad + c_2 \left(\frac{\mathrm{e}^{-\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}\sqrt{x}\,\sqrt{2}}{\sqrt{x^2+x+1}}\left(\int \frac{\mathrm{e}^{-\frac{3\ln\left(x^2+x+1\right)}{2}-\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}\left(x^2+x+1\right)\mathrm{e}^{2\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}}{2x} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2(x^2+x+1)\left(\frac{d^2}{dx^2}y(x)\right)+12x^2(x+1)\left(\frac{d}{dx}y(x)\right)+(3x^2+3x+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(3x^2+3x+1)y(x)}{4x^2(x^2+x+1)}-\frac{3(x+1)\left(\frac{d}{dx}y(x)\right)}{x^2+x+1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{3(x+1)\left(\frac{d}{dx}y(x)\right)}{x^2+x+1}+\frac{(3x^2+3x+1)y(x)}{4x^2(x^2+x+1)}=0$

- ☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{3(x+1)}{x^2+x+1}, P_3(x)=\frac{3x^2+3x+1}{4x^2(x^2+x+1)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=0$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=\frac{1}{4}$

  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

$$4x^2(x^2+x+1)\left(\frac{d^2}{dx^2}y(x)\right)+12x^2(x+1)\left(\frac{d}{dx}y(x)\right)+(3x^2+3x+1)\,y(x)=0$$

- Assume series solution for $y(x)$

$$y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

    $$x^m\cdot y(x)=\sum_{k=0}^{\infty}a_k x^{k+r+m}$$
  - Shift index using $k->k-m$

    $$x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$$
  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=2..3$

    $$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$$
  - Shift index using $k->k+1-m$

    $$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$$
  - Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..4$

    $$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$
  - Shift index using $k->k+2-m$

    $$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+2r)^2x^r+\left(a_1(1+2r)^2+a_0(3+2r)\,(1+2r)\right)x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_k(2k+2r-1)^2+a_{k-1}(2k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1+2r)^2=0$$
- Values of r that satisfy the indicial equation

  $$r=\frac{1}{2}$$
- Each term must be 0

  $$a_1(1+2r)^2+a_0(3+2r)\,(1+2r)=0$$
- Solve for the dependent coefficient(s)

  $$a_1=-\frac{(3+2r)a_0}{1+2r}$$
- Each term in the series must be 0, giving the recursion relation

  $$4\left((a_k+a_{k-2}+a_{k-1})\,k+(a_k+a_{k-2}+a_{k-1})\,r-\frac{a_k}{2}-\frac{3a_{k-2}}{2}+\frac{a_{k-1}}{2}\right)\left(k+r-\frac{1}{2}\right)=0$$
- Shift index using $k->k+2$

  $$4\left((a_{k+2}+a_k+a_{k+1})\,(k+2)+(a_{k+2}+a_k+a_{k+1})\,r-\frac{a_{k+2}}{2}-\frac{3a_k}{2}+\frac{a_{k+1}}{2}\right)\left(k+\frac{3}{2}+r\right)=0$$
- Recursion relation that defines series solution to ODE

  $$a_{k+2}=-\frac{2ka_k+2ka_{k+1}+2ra_k+2ra_{k+1}+a_k+5a_{k+1}}{2k+2r+3}$$
- Recursion relation for $r=\frac{1}{2}$

  $$a_{k+2}=-\frac{2ka_k+2ka_{k+1}+2a_k+6a_{k+1}}{2k+4}$$
- Solution for $r=\frac{1}{2}$

  $$\left[y(x)=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}},a_{k+2}=-\frac{2ka_k+2ka_{k+1}+2a_k+6a_{k+1}}{2k+4},a_1=-2a_0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 2.609 (sec)
Leaf size : 143

```
dsolve(4*x^2*(x^2+x+1)*diff(diff(y(x),x),x)+12*x^2*(x+1)*diff(y(x),x)+(3*x^2+3*x+1)*y(x)
```

$$y = \frac{e^{-\frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{2}}\sqrt{-2x+i\sqrt{3}-1}\,\sqrt{x}\left(c_1\left(\frac{-2ix+\sqrt{3}-i}{\sqrt{3}+2ix+i}\right)^{\frac{1}{4}-\frac{i\sqrt{3}}{4}}+c_2\left(\frac{-2ix+\sqrt{3}-i}{\sqrt{3}+2ix+i}\right)^{\frac{3}{4}+\frac{i\sqrt{3}}{4}}\operatorname{hypergeom}\left(\left[1,\frac{1}{2}+\right.\right.\right.}{\left(x^2+x+1\right)^{3/4}}$$

Mathematica DSolve solution

Solving time : 0.512 (sec)
Leaf size : 120

```
DSolve[{4*x^2*(1+x+x^2)*D[y[x],{x,2}]+12*x^2*(1+x)*D[y[x],x]+(1+3*x+3*x^2)*y[x]==0,{}},y[x],
```

$$y(x) \to \exp\left(\int_1^x \frac{2K[1]^2+1}{2K[1](K[1]^2+K[1]+1)}dK[1] - \frac{1}{2}\int_1^x \frac{3(K[2]+1)}{K[2]^2+K[2]+1}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{2K[1]^2+1}{2K[1](K[1]^2+K[1]+1)}dK[1]\right)dK[3] + c_1\right)$$

2.1.126 Problem 128

Internal problem ID [9298]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 128
Date solved : `Monday, January 27, 2025 at 06:01:09 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(x^2+x+1\right)y''-x\left(-2x^2-4x+1\right)y'+y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.926 (sec)

Writing the ode as

$$x^2\left(x^2+x+1\right)y''+\left(2x^3+4x^2-x\right)y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=x^2\left(x^2+x+1\right)\\ B&=2x^3+4x^2-x \\ C&=1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{10x^2-8x-1}{4\left(x^3+x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=10x^2-8x-1\\ t&=4\left(x^3+x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{10x^2-8x-1}{4\left(x^3+x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.236: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ of order 2. There is a pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{-\frac{29}{24} - \frac{7i\sqrt{3}}{24}}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{29}{24} + \frac{7i\sqrt{3}}{24}}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2} + \frac{\frac{3}{4} - \frac{41i\sqrt{3}}{36}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{3}{4} + \frac{41i\sqrt{3}}{36}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} - \frac{3}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x=-\frac{1}{2}+\frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{29}{24}-\frac{7i\sqrt{3}}{24}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{1}{2}+\frac{\sqrt{-138-42i\sqrt{3}}}{12} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{2}-\frac{\sqrt{-138-42i\sqrt{3}}}{12}
\end{aligned}$$

For the pole at $x=-\frac{1}{2}-\frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{29}{24}+\frac{7i\sqrt{3}}{24}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{1}{2}+\frac{\sqrt{-138+42i\sqrt{3}}}{12} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{2}-\frac{\sqrt{-138+42i\sqrt{3}}}{12}
\end{aligned}$$

Since the order of r at ∞ is $4>2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{10x^2-8x-1}{4\left(x^3+x^2+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-\frac{1}{2}+\frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2}+\frac{\sqrt{-138-42i\sqrt{3}}}{12}$	$\frac{1}{2}-\frac{\sqrt{-138-42i\sqrt{3}}}{12}$
$-\frac{1}{2}-\frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2}+\frac{\sqrt{-138+42i\sqrt{3}}}{12}$	$\frac{1}{2}-\frac{\sqrt{-138+42i\sqrt{3}}}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=1$ then

$$\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{+}+\alpha_{c_2}^{-}+\alpha_{c_3}^{-}\right) \\
&= 1-(1) \\
&= 0
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + \frac{\frac{1}{2} - \frac{\sqrt{-138-42i\sqrt{3}}}{12}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} - \frac{\sqrt{-138+42i\sqrt{3}}}{12}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}} + (-)\,(0) \\
&= \frac{1}{2x} + \frac{\frac{1}{2} - \frac{\sqrt{-138-42i\sqrt{3}}}{12}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} - \frac{\sqrt{-138+42i\sqrt{3}}}{12}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}} \\
&= \frac{2x^2-2x+1}{2x\,(x^2+x+1)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x} + \frac{\frac{1}{2} - \frac{\sqrt{-138-42i\sqrt{3}}}{12}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} - \frac{\sqrt{-138+42i\sqrt{3}}}{12}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{\frac{1}{2} - \frac{\sqrt{-138-42i\sqrt{3}}}{12}}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^2} - \frac{\frac{1}{2} - \frac{\sqrt{-138+42i}}{12}}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right.}\right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} + \frac{\frac{1}{2} - \frac{\sqrt{-138-42i\sqrt{3}}}{12}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} - \frac{\sqrt{-138+42i\sqrt{3}}}{12}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}}\right) dx} \\
&= \sqrt{x}\,\left(x^2+x+1\right)^{1/4} \sqrt{2}\, \mathrm{e}^{-\frac{7\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^3+4x^2-x}{x^2\left(x^2+x+1\right)}\, dx} \\
&= z_1 e^{\frac{\ln(x)}{2} - \frac{3\ln\left(x^2+x+1\right)}{4} - \frac{7\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}} \\
&= z_1 \left(\frac{\sqrt{x}\,\mathrm{e}^{-\frac{7\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}}{\left(x^2+x+1\right)^{3/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x\,\mathrm{e}^{-\frac{7\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}\sqrt{2}}{\sqrt{x^2+x+1}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3+4x^2-x}{x^2(x^2+x+1)}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)-\frac{3\ln(x^2+x+1)}{2}-\frac{7\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{e^{\ln(x)-\frac{3\ln(x^2+x+1)}{2}-\frac{7\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}(x^2+x+1)\,e^{\frac{14\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{2x^2}\,dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x\,e^{-\frac{7\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}\sqrt{2}}{\sqrt{x^2+x+1}}\right) \\
&\quad + c_2 \left(\frac{x\,e^{-\frac{7\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}\sqrt{2}}{\sqrt{x^2+x+1}}\left(\int \frac{e^{\ln(x)-\frac{3\ln(x^2+x+1)}{2}-\frac{7\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}(x^2+x+1)\,e^{\frac{14\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{2x^2}\,dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x^2+x+1)\left(\frac{d^2}{dx^2}y(x)\right) - x(-2x^2-4x+1)\left(\frac{d}{dx}y(x)\right) + y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{y(x)}{x^2(x^2+x+1)} - \frac{(2x^2+4x-1)\left(\frac{d}{dx}y(x)\right)}{x(x^2+x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d^2}{dx^2}y(x) + \frac{(2x^2+4x-1)\left(\frac{d}{dx}y(x)\right)}{x(x^2+x+1)} + \frac{y(x)}{x^2(x^2+x+1)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{2x^2+4x-1}{x(x^2+x+1)}, P_3(x) = \frac{1}{x^2(x^2+x+1)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 1$

  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

$$x^2(x^2+x+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(2x^2+4x-1)\left(\frac{d}{dx}y(x)\right)+y(x)=0$$

- Assume series solution for $y(x)$

$$y(x)=\sum_{k=0}^{\infty} a_k x^{k+r}$$

- □ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

    $$x^m \cdot \left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$
  - ◦ Shift index using $k->k+1-m$

    $$x^m \cdot \left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$
  - ◦ Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

    $$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$
  - ◦ Shift index using $k->k+2-m$

    $$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+r)^2\,x^r+\left(a_1r^2+a_0r(3+r)\right)x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_k(k+r-1)^2+a_{k-1}(k+r-1)\,(k+2+r)\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1+r)^2=0$$
- Values of r that satisfy the indicial equation

  $$r=1$$
- Each term must be 0

  $$a_1r^2+a_0r(3+r)=0$$
- Solve for the dependent coefficient(s)

  $$a_1=-\frac{(3+r)a_0}{r}$$
- Each term in the series must be 0, giving the recursion relation

  $$\left((a_k+a_{k-2}+a_{k-1})\,k+(a_k+a_{k-2}+a_{k-1})\,r-a_k-2a_{k-2}+2a_{k-1}\right)(k+r-1)=0$$
- Shift index using $k->k+2$

  $$\left((a_{k+2}+a_k+a_{k+1})\,(k+2)+(a_{k+2}+a_k+a_{k+1})\,r-a_{k+2}-2a_k+2a_{k+1}\right)(k+r+1)=0$$
- Recursion relation that defines series solution to ODE

  $$a_{k+2}=-\frac{ka_k+ka_{k+1}+ra_k+ra_{k+1}+4a_{k+1}}{k+r+1}$$
- Recursion relation for $r=1$

  $$a_{k+2}=-\frac{ka_k+ka_{k+1}+a_k+5a_{k+1}}{k+2}$$
- Solution for $r=1$

  $$\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2}=-\frac{ka_k+ka_{k+1}+a_k+5a_{k+1}}{k+2}, a_1=-4a_0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.240 (sec)
Leaf size : 147

```
dsolve(x^2*(x^2+x+1)*diff(diff(y(x),x),x)-x*(-2*x^2-4*x+1)*diff(y(x),x)+y(x) = 0,y(x),si
```

$$y = \frac{e^{-\frac{7\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}} x \left(c_2 \left(-2x+i\sqrt{3}-1\right)^{-\frac{1}{4}-\frac{7i\sqrt{3}}{12}} \left(2x+i\sqrt{3}+1\right)^{\frac{3}{4}+\frac{7i\sqrt{3}}{12}} \operatorname{hypergeom}\left(\left[1, \frac{1}{2}+\frac{7i\sqrt{3}}{6}\right], \left[\frac{3}{2}+\frac{7}{}\right.\right.\right.}{\left(x^2+x+1\right)^{3/4}}$$

Mathematica DSolve solution

Solving time : `0.594` (sec)
Leaf size : 130

```
DSolve[{x^2*(1+x+x^2)*D[y[x],{x,2}]-x*(1-4*x-2*x^2)*D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSing
```

$$y(x)\to \exp\left(\int_1^x \frac{1}{2}\left(\frac{K[1]-3}{K[1]^2+K[1]+1}+\frac{1}{K[1]}\right)dK[1]-\frac{1}{2}\int_1^x\left(\frac{3K[2]+5}{K[2]^2+K[2]+1}-\frac{1}{K[2]}\right)dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]}\frac{2K[1]^2-2K[1]+1}{2K[1]\left(K[1]^2+K[1]+1\right)}dK[1]\right)dK[3]+c_1\right)$$

2.1.127 Problem 129

Internal problem ID [9299]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 129
Date solved : `Monday, January 27, 2025 at 06:01:11 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$9x^2y'' + 3x\left(-2x^2 + 3x + 5\right) y' + \left(-14x^2 + 12x + 1\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.506 (sec)

Writing the ode as

$$9x^2y'' + \left(-6x^3 + 9x^2 + 15x\right) y' + \left(-14x^2 + 12x + 1\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^2 \\ B &= -6x^3 + 9x^2 + 15x \\ C &= -14x^2 + 12x + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^4 - 12x^3 + 33x^2 - 18x - 9}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^4 - 12x^3 + 33x^2 - 18x - 9 \\ t &= 36x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4 - 12x^3 + 33x^2 - 18x - 9}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.238: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{9} - \frac{x}{3} + \frac{11}{12} - \frac{1}{4x^2} - \frac{1}{2x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{3} - \frac{1}{2} + \frac{1}{x} + \frac{3}{4x^2} - \frac{3}{4x^3} - \frac{27}{8x^4} - \frac{117}{32x^5} + \frac{405}{64x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{3}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= -\frac{1}{2} + \frac{x}{3} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4} - \frac{1}{3}x + \frac{1}{9}x^2$$

This shows that the coefficient of 1 in the above is $\frac{1}{4}$. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^4 - 12x^3 + 33x^2 - 18x - 9}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{1}{9}x^2 - \frac{1}{3}x + \frac{11}{12}\right) + \left(\frac{-18x - 9}{36x^2}\right) \\ &= \frac{x^2}{9} - \frac{x}{3} + \frac{11}{12} + \frac{-18x - 9}{36x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is $\frac{11}{12}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{11}{12}\right) - \left(\frac{1}{4}\right) \\ &= \frac{2}{3} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= -\frac{1}{2} + \frac{x}{3} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{2}{3}}{\frac{1}{3}} - 1\right) &&= \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{2}{3}}{\frac{1}{3}} - 1\right) &&= -\frac{3}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^4 - 12x^3 + 33x^2 - 18x - 9}{36x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$-\frac{1}{2}+\frac{x}{3}$	$\frac{1}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^+) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + \left(-\frac{1}{2} + \frac{x}{3}\right) \\ &= \frac{1}{2x} - \frac{1}{2} + \frac{x}{3} \\ &= \frac{1}{2x} - \frac{1}{2} + \frac{x}{3} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{2} + \frac{x}{3}\right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{1}{3}\right) + \left(\frac{1}{2x} - \frac{1}{2} + \frac{x}{3}\right)^2 - \left(\frac{4x^4 - 12x^3 + 33x^2 - 18x - 9}{36x^2}\right)\right) = 0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{1}{2x} - \frac{1}{2} + \frac{x}{3}\right) dx} \\ &= \sqrt{x}\, e^{\frac{x(x-3)}{6}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-6x^3+9x^2+15x}{9x^2}\, dx} \\
&= z_1 e^{\frac{x^2}{6}-\frac{x}{2}-\frac{5\ln(x)}{6}} \\
&= z_1\left(\frac{e^{\frac{x(x-3)}{6}}}{x^{5/6}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{\frac{x(x-3)}{3}}}{x^{1/3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-6x^3+9x^2+15x}{9x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{3}-x-\frac{5\ln(x)}{3}}}{(y_1)^2}\, dx \\
&= y_1\left(\int e^{\frac{x^2}{3}-x-\frac{5\ln(x)}{3}} x^{2/3} e^{-\frac{2x(x-3)}{3}}\, dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{e^{\frac{x(x-3)}{3}}}{x^{1/3}}\right) + c_2\left(\frac{e^{\frac{x(x-3)}{3}}}{x^{1/3}}\left(\int e^{\frac{x^2}{3}-x-\frac{5\ln(x)}{3}} x^{2/3} e^{-\frac{2x(x-3)}{3}}\, dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$9x^2\left(\frac{d^2}{dx^2}y(x)\right) + 3x(-2x^2+3x+5)\left(\frac{d}{dx}y(x)\right) + (-14x^2+12x+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{(14x^2-12x-1)y(x)}{9x^2} + \frac{(2x^2-3x-5)\left(\frac{d}{dx}y(x)\right)}{3x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) - \frac{(2x^2-3x-5)\left(\frac{d}{dx}y(x)\right)}{3x} - \frac{(14x^2-12x-1)y(x)}{9x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

$\left[P_2(x) = -\frac{2x^2-3x-5}{3x}, P_3(x) = -\frac{14x^2-12x-1}{9x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{5}{3}$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{1}{9}$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $9x^2\left(\frac{d^2}{dx^2}y(x)\right) - 3x(2x^2 - 3x - 5)\left(\frac{d}{dx}y(x)\right) + (-14x^2 + 12x + 1)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - ◦ Shift index using $k- >k - m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k + r)\,x^{k+r-1+m}$

  - ◦ Shift index using $k- >k + 1 - m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\,x^{k+r}$

  - ◦ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1 + 3r)^2\,x^r + \left(a_1(4 + 3r)^2 + 3a_0(4 + 3r)\right)x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k + 3r + 1)^2 + 3a_{k-1}(3k + 3r + 1\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1 + 3r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = -\frac{1}{3}$

- Each term must be 0

  $a_1(4 + 3r)^2 + 3a_0(4 + 3r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = -\frac{3a_0}{4+3r}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(3k + 3r + 1)^2 + (3k + 3r + 1)\,(-2a_{k-2} + 3a_{k-1}) = 0$

- Shift index using $k- >k + 2$

  $a_{k+2}(3k + 3r + 7)^2 + (3k + 3r + 7)\,(-2a_k + 3a_{k+1}) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+2} = \frac{2a_k - 3a_{k+1}}{3k+3r+7}$

- Recursion relation for $r = -\frac{1}{3}$

$a_{k+2} = \frac{2a_k - 3a_{k+1}}{3k+6}$

- Solution for $r = -\frac{1}{3}$

$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}, a_{k+2} = \frac{2a_k - 3a_{k+1}}{3k+6}, a_1 = -a_0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Mo
      <- Heun successful: received ODE is equivalent to the  HeunB  ODE, case  c = 0
   Special function solution also has integrals. Returning default Liouvillian solution.
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.369 (sec)
Leaf size : 32

```
dsolve(9*x^2*diff(diff(y(x),x),x)+3*x*(-2*x^2+3*x+5)*diff(y(x),x)+(-14*x^2+12*x+1)*y(x)
```

$$y = \frac{e^{\frac{x(x-3)}{3}} \left(\left(\int \frac{e^{-\frac{x(x-3)}{3}}}{x} dx\right) c_2 + c_1\right)}{x^{1/3}}$$

Mathematica DSolve solution

Solving time : `0.476` (sec)
Leaf size : 52

```
DSolve[{9*x^2*D[y[x],{x,2}]+3*x*(5+3*x-2*x^2)*D[y[x],x]+(1+12*x-14*x^2)*y[x]==0,{}},y[x],x,I
```

$$
y(x) \to \frac{e^{\frac{1}{3}(x-3)x}\left(c_2 \int_1^x \frac{e^{K[1]-\frac{K[1]^2}{3}}}{K[1]}dK[1] + c_1\right)}{\sqrt[3]{x}}
$$

2.1.128 Problem 130

Internal problem ID [9300]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 130
Date solved : `Monday, January 27, 2025 at 06:01:11 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+2x)\,y''+x\left(3x^2+14x+5\right)y'+\left(12x^2+18x+4\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.617 (sec)

Writing the ode as

$$\left(2x^3+x^2\right)y''+\left(3x^3+14x^2+5x\right)y'+\left(12x^2+18x+4\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=2x^3+x^2\\ B&=3x^3+14x^2+5x\\ C&=12x^2+18x+4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{9x^4-12x^3-16x^2-4x-1}{4\left(2x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=9x^4-12x^3-16x^2-4x-1\\ t&=4\left(2x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{9x^4-12x^3-16x^2-4x-1}{4\left(2x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.240: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{16} - \frac{15}{64\left(x + \frac{1}{2}\right)^2} - \frac{21}{16\left(x + \frac{1}{2}\right)} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x + \frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{15}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{3}{8} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3}{4} - \frac{7}{8x} - \frac{19}{48x^2} - \frac{151}{288x^3} - \frac{139}{192x^4} - \frac{11383}{10368x^5} - \frac{38729}{20736x^6} - \frac{1212655}{373248x^7} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{3}{4} \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9}{16}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{9x^4 - 12x^3 - 16x^2 - 4x - 1}{16x^4 + 16x^3 + 4x^2} \\
&= Q + \frac{R}{16x^4 + 16x^3 + 4x^2} \\
&= \left(\frac{9}{16}\right) + \left(\frac{-21x^3 - \frac{73}{4}x^2 - 4x - 1}{16x^4 + 16x^3 + 4x^2}\right) \\
&= \frac{9}{16} + \frac{-21x^3 - \frac{73}{4}x^2 - 4x - 1}{16x^4 + 16x^3 + 4x^2}
\end{aligned}$$

Since the degree of t is 4, then we see that the coefficient of the term x^3 in the remainder R is -21. Dividing this by leading coefficient in t which is 16 gives $-\frac{21}{16}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{21}{16}\right) - (0) \\
&= -\frac{21}{16}
\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{3}{4} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{21}{16}}{\frac{3}{4}} - 0\right) &= -\frac{7}{8} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{21}{16}}{\frac{3}{4}} - 0\right) &= \frac{7}{8}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{9x^4 - 12x^3 - 16x^2 - 4x - 1}{4\left(2x^2 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-\frac{1}{2}$	2	0	$\frac{5}{8}$	$\frac{3}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{3}{4}$	$-\frac{7}{8}$	$\frac{7}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{7}{8}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{-}\right) \\
&= \frac{7}{8} - \left(\frac{7}{8}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c) [\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty) [\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+) [\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + \left((-) [\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2} \right) + (-) [\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \frac{3}{8\left(x + \frac{1}{2}\right)} + (-)\left(\frac{3}{4}\right) \\
&= \frac{1}{2x} + \frac{3}{8\left(x + \frac{1}{2}\right)} - \frac{3}{4} \\
&= \frac{-3x^2 + 2x + 1}{4x^2 + 2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{1}{2x}+\frac{3}{8\left(x+\frac{1}{2}\right)}-\frac{3}{4}\right)(0)+\left(\left(-\frac{1}{2x^2}-\frac{3}{8\left(x+\frac{1}{2}\right)^2}\right)+\left(\frac{1}{2x}+\frac{3}{8\left(x+\frac{1}{2}\right)}-\frac{3}{4}\right)^2-\left(\frac{9x^4-12x^3 -}{4(2x}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}z_1(x)&=pe^{\int\omega\,dx}\\&=\mathrm{e}^{\int\left(\frac{1}{2x}+\frac{3}{8\left(x+\frac{1}{2}\right)}-\frac{3}{4}\right)dx}\\&=(1+2x)^{3/8}\sqrt{x}\,\mathrm{e}^{-\frac{3x}{4}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1&=z_1e^{\int-\frac{1}{2}\frac{B}{A}\,dx}\\&=z_1e^{-\int\frac{1}{2}\frac{3x^3+14x^2+5x}{2x^3+x^2}\,dx}\\&=z_1e^{-\frac{3x}{4}-\frac{5\ln(1+2x)}{8}-\frac{5\ln(x)}{2}}\\&=z_1\left(\frac{\mathrm{e}^{-\frac{3x}{4}}}{(1+2x)^{5/8}x^{5/2}}\right)\end{aligned}$$

Which simplifies to

$$y_1=\frac{\mathrm{e}^{-\frac{3x}{2}}}{(1+2x)^{1/4}x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int\frac{e^{\int-\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2&=y_1\int\frac{e^{\int-\frac{3x^3+14x^2+5x}{2x^3+x^2}\,dx}}{(y_1)^2}\,dx\\&=y_1\int\frac{e^{-\frac{3x}{2}-\frac{5\ln(1+2x)}{4}-5\ln(x)}}{(y_1)^2}\,dx\\&=y_1\left(\int\mathrm{e}^{-\frac{3x}{2}-\frac{5\ln(1+2x)}{4}-5\ln(x)}\sqrt{1+2x}\,x^4\mathrm{e}^{3x}dx\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y&=c_1y_1+c_2y_2\\&=c_1\left(\frac{\mathrm{e}^{-\frac{3x}{2}}}{(1+2x)^{1/4}x^2}\right)+c_2\left(\frac{\mathrm{e}^{-\frac{3x}{2}}}{(1+2x)^{1/4}x^2}\left(\int\mathrm{e}^{-\frac{3x}{2}-\frac{5\ln(1+2x)}{4}-5\ln(x)}\sqrt{1+2x}\,x^4\mathrm{e}^{3x}dx\right)\right)\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(2x+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(3x^2+14x+5)\left(\frac{d}{dx}y(x)\right)+(12x^2+18x+4)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{2(6x^2+9x+2)y(x)}{x^2(2x+1)}-\frac{(3x^2+14x+5)\left(\frac{d}{dx}y(x)\right)}{x(2x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{(3x^2+14x+5)\left(\frac{d}{dx}y(x)\right)}{x(2x+1)}+\frac{2(6x^2+9x+2)y(x)}{x^2(2x+1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{3x^2+14x+5}{x(2x+1)},P_3(x)=\frac{2(6x^2+9x+2)}{x^2(2x+1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=5$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=4$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(2x+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(3x^2+14x+5)\left(\frac{d}{dx}y(x)\right)+(12x^2+18x+4)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  ◦ Shift index using $k->k-m$

  $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..3$

  $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  ◦ Shift index using $k->k+2-m$

  $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$a_0(2+r)^2 x^r + \left(a_1(3+r)^2 + 2a_0(3+r)^2\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+2)^2 + 2a_{k-1}(k+r+2)^2 + 3a_{k-2}\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(2+r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = -2$
- Each term must be 0
  $a_1(3+r)^2 + 2a_0(3+r)^2 = 0$
- Solve for the dependent coefficient(s)
  $a_1 = -2a_0$
- Each term in the series must be 0, giving the recursion relation
  $((2k+2r+4)\, a_{k-1} + a_k(k+r+2) + 3a_{k-2})\,(k+r+2) = 0$
- Shift index using $k- >k+2$
  $((2k+8+2r)\, a_{k+1} + a_{k+2}(k+r+4) + 3a_k)\,(k+r+4) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{2ka_{k+1}+2ra_{k+1}+3a_k+8a_{k+1}}{k+r+4}$
- Recursion relation for $r = -2$
  $a_{k+2} = -\frac{2ka_{k+1}+3a_k+4a_{k+1}}{k+2}$
- Solution for $r = -2$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{2ka_{k+1}+3a_k+4a_{k+1}}{k+2}, a_1 = -2a_0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Mo
      <- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `1.970` (sec)
Leaf size : 53

```
dsolve(x^2*(2*x+1)*diff(diff(y(x),x),x)+x*(3*x^2+14*x+5)*diff(y(x),x)+(12*x^2+18*x+4)*
```

$$y = \frac{e^{-\frac{3x}{2}} \left((2x+1)^{1/4} \operatorname{HeunC}\left(-\frac{3}{4}, \frac{1}{4}, 0, \frac{21}{32}, -\frac{5}{32}, 2x+1\right) c_2 + \operatorname{HeunC}\left(-\frac{3}{4}, -\frac{1}{4}, 0, \frac{21}{32}, -\frac{5}{32}, 2x+1\right) c_1 \right)}{(2x+1)^{1/4} x^2}$$

Mathematica DSolve solution

Solving time : `0.337` (sec)
Leaf size : 120

```
DSolve[{x^2*(1+2*x)*D[y[x],{x,2}]+x*(5+14*x+3*x^2)*D[y[x],x]+(4+18*x+12*x^2)*y[x]==0,{}},y[x
```

$$y(x) \to \exp\left(\int_1^x \frac{1}{4} \left(\frac{3}{2K[1]+1} - 3 + \frac{2}{K[1]} \right) dK[1] - \frac{1}{2} \int_1^x \left(\frac{5}{4K[2]+2} + \frac{3}{2} + \frac{5}{K[2]} \right) dK[2] \right) \left(c_2 \int_1^x \exp\left(-2 \int_1^{K[3]} \frac{-3K[1]^2 + 2K[1] + 1}{4K[1]^2 + 2K[1]} dK[1] \right) dK[3] + c_1 \right)$$

2.1.129 Problem 131

Internal problem ID [9301]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 131
Date solved : `Monday, January 27, 2025 at 06:01:13 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$16x^2y'' + 4x\left(2x^2 + x + 6\right) y' + \left(18x^2 + 5x + 1\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.469 (sec)

Writing the ode as

$$16x^2y'' + \left(8x^3 + 4x^2 + 24x\right) y' + \left(18x^2 + 5x + 1\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 16x^2 \\ B &= 8x^3 + 4x^2 + 24x \\ C &= 18x^2 + 5x + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^4 + 4x^3 - 31x^2 - 8x - 16}{64x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^4 + 4x^3 - 31x^2 - 8x - 16 \\ t &= 64x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4 + 4x^3 - 31x^2 - 8x - 16}{64x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.242: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{16} + \frac{x}{16} - \frac{31}{64} - \frac{1}{4x^2} - \frac{1}{8x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{4} + \frac{1}{8} - \frac{1}{x} + \frac{1}{4x^2} - \frac{21}{8x^3} + \frac{37}{16x^4} - \frac{377}{32x^5} + \frac{1137}{64x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{1}{8} + \frac{x}{4}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{64} + \frac{1}{16}x + \frac{1}{16}x^2$$

This shows that the coefficient of 1 in the above is $\frac{1}{64}$. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^4 + 4x^3 - 31x^2 - 8x - 16}{64x^2} \\ &= Q + \frac{R}{64x^2} \\ &= \left(\frac{1}{16}x^2 + \frac{1}{16}x - \frac{31}{64}\right) + \left(\frac{-8x - 16}{64x^2}\right) \\ &= \frac{x^2}{16} + \frac{x}{16} - \frac{31}{64} + \frac{-8x - 16}{64x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is $-\frac{31}{64}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{31}{64}\right) - \left(\frac{1}{64}\right) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{8} + \frac{x}{4} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{4}} - 1\right) &= -\frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{4}} - 1\right) &= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^4 + 4x^3 - 31x^2 - 8x - 16}{64x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{1}{8}+\frac{x}{4}$	$-\frac{3}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\left(\frac{1}{8} + \frac{x}{4}\right) \\
&= \frac{1}{2x} - \frac{1}{8} - \frac{x}{4} \\
&= \frac{1}{2x} - \frac{1}{8} - \frac{x}{4}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered}
(0) + 2\left(\frac{1}{2x} - \frac{1}{8} - \frac{x}{4}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{4}\right) + \left(\frac{1}{2x} - \frac{1}{8} - \frac{x}{4}\right)^2 - \left(\frac{4x^4 + 4x^3 - 31x^2 - 8x - 16}{64x^2}\right)\right) = 0 \\
0 = 0
\end{gathered}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{2x} - \frac{1}{8} - \frac{x}{4}\right) dx} \\
&= \sqrt{x}\, e^{-\frac{x(x+1)}{8}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{8x^3+4x^2+24x}{16x^2}\,dx} \\
&= z_1 e^{-\frac{x^2}{8}-\frac{x}{8}-\frac{3\ln(x)}{4}} \\
&= z_1\left(\frac{\mathrm{e}^{-\frac{x(x+1)}{8}}}{x^{3/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-\frac{x(x+1)}{4}}}{x^{1/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{8x^3+4x^2+24x}{16x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{4}-\frac{x}{4}-\frac{3\ln(x)}{2}}}{(y_1)^2}\,dx \\
&= y_1\left(\int \mathrm{e}^{-\frac{x^2}{4}-\frac{x}{4}-\frac{3\ln(x)}{2}}\sqrt{x}\,\mathrm{e}^{\frac{x(x+1)}{2}}\,dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\mathrm{e}^{-\frac{x(x+1)}{4}}}{x^{1/4}}\right) + c_2\left(\frac{\mathrm{e}^{-\frac{x(x+1)}{4}}}{x^{1/4}}\left(\int \mathrm{e}^{-\frac{x^2}{4}-\frac{x}{4}-\frac{3\ln(x)}{2}}\sqrt{x}\,\mathrm{e}^{\frac{x(x+1)}{2}}\,dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$16x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x(2x^2+x+6)\left(\frac{d}{dx}y(x)\right) + (18x^2+5x+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(18x^2+5x+1)y(x)}{16x^2} - \frac{(2x^2+x+6)\left(\frac{d}{dx}y(x)\right)}{4x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(2x^2+x+6)\left(\frac{d}{dx}y(x)\right)}{4x} + \frac{(18x^2+5x+1)y(x)}{16x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

$\left[P_2(x) = \frac{2x^2+x+6}{4x}, P_3(x) = \frac{18x^2+5x+1}{16x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = \frac{3}{2}$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{1}{16}$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $16x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x(2x^2 + x + 6)\left(\frac{d}{dx}y(x)\right) + (18x^2 + 5x + 1)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - ◦ Shift index using $k->k-m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - ◦ Shift index using $k->k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - ◦ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+4r)^2\,x^r + \left(a_1(5+4r)^2 + a_0(5+4r)\right)x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(4k+4r+1)^2 + a_{k-1}(4k+4r+1)\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+4r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = -\frac{1}{4}$

- Each term must be 0

  $a_1(5+4r)^2 + a_0(5+4r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = -\frac{a_0}{5+4r}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k+4r+1)^2 + (4k+4r+1)\,(2a_{k-2} + a_{k-1}) = 0$

- Shift index using $k->k+2$

  $a_{k+2}(4k+4r+9)^2 + (4k+4r+9)\,(2a_k + a_{k+1}) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+2} = -\frac{2a_k + a_{k+1}}{4k+4r+9}$

- Recursion relation for $r = -\frac{1}{4}$

$a_{k+2} = -\frac{2a_k + a_{k+1}}{4k+8}$

- Solution for $r = -\frac{1}{4}$

$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{4}}, a_{k+2} = -\frac{2a_k + a_{k+1}}{4k+8}, a_1 = -\frac{a_0}{4}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Mo
      <- Heun successful: received ODE is equivalent to the  HeunB  ODE, case  c = 0
   Special function solution also has integrals. Returning default Liouvillian solution.
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.400 (sec)
Leaf size : 32

```
dsolve(16*x^2*diff(diff(y(x),x),x)+4*x*(2*x^2+x+6)*diff(y(x),x)+(18*x^2+5*x+1)*y(x) = 0,
```

$$y = \frac{e^{-\frac{x(x+1)}{4}}\left(\left(\int \frac{e^{\frac{x(x+1)}{4}}}{x}dx\right)c_2 + c_1\right)}{x^{1/4}}$$

Mathematica DSolve solution

Solving time : `0.488` (sec)
Leaf size : 57

```
DSolve[{16*x^2*D[y[x],{x,2}]+4*x*(6+x+2*x^2)*D[y[x],x]+(1+5*x+18*x^2)*y[x]==0,{}},y[x],x,Inc
```

$$y(x) \to \frac{e^{\frac{1}{4}(-x^2-x-3)} \left(c_2 \int_1^x \frac{e^{\frac{1}{4}K[1](K[1]+1)}}{K[1]} dK[1] + c_1\right)}{\sqrt[4]{x}}$$

2.1.130 Problem 132

Internal problem ID [9302]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 132
Date solved : `Monday, January 27, 2025 at 06:01:13 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$9x^2(1+x)\,y''+3x\left(-x^2+11x+5\right)y'+\left(-7x^2+16x+1\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.346 (sec)

Writing the ode as

$$\left(9x^3+9x^2\right)y''+\left(-3x^3+33x^2+15x\right)y'+\left(-7x^2+16x+1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^3+9x^2\\ B &= -3x^3+33x^2+15x\\ C &= -7x^2+16x+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t}\\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{x^4+6x^3+3x^2-18x-9}{36\left(x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4+6x^3+3x^2-18x-9\\ t &= 36\left(x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x^4+6x^3+3x^2-18x-9}{36\left(x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.244: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{36} + \frac{1}{9 + 9x} + \frac{7}{36\,(1 + x)^2} - \frac{1}{4x^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{6} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{6} + \frac{1}{3x} - \frac{5}{6x^2} + \frac{5}{6x^3} - \frac{7}{3x^4} + \frac{41}{6x^5} - \frac{149}{6x^6} + \frac{277}{3x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{6} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{36}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^4 + 6x^3 + 3x^2 - 18x - 9}{36x^4 + 72x^3 + 36x^2} \\ &= Q + \frac{R}{36x^4 + 72x^3 + 36x^2} \\ &= \left(\frac{1}{36}\right) + \left(\frac{4x^3 + 2x^2 - 18x - 9}{36x^4 + 72x^3 + 36x^2}\right) \\ &= \frac{1}{36} + \frac{4x^3 + 2x^2 - 18x - 9}{36x^4 + 72x^3 + 36x^2} \end{aligned}$$

Since the degree of t is 4, then we see that the coefficient of the term x^3 in the remainder R is 4. Dividing this by leading coefficient in t which is 36 gives $\frac{1}{9}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{9}\right) - (0) \\ &= \frac{1}{9} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{6} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{1}{9}}{\frac{1}{6}} - 0\right) &= \frac{1}{3} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{\frac{1}{9}}{\frac{1}{6}} - 0\right) = -\frac{1}{3}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^4 + 6x^3 + 3x^2 - 18x - 9}{36\,(x^2 + x)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{7}{6}$	$-\frac{1}{6}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{1}{3}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= \frac{1}{3} - \left(\frac{1}{3}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{6\,(1 + x)} + \frac{1}{2x} + \left(\frac{1}{6}\right) \\
&= -\frac{1}{6\,(1 + x)} + \frac{1}{2x} + \frac{1}{6} \\
&= -\frac{1}{6 + 6x} + \frac{1}{2x} + \frac{1}{6}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{6\left(1+x\right)}+\frac{1}{2x}+\frac{1}{6}\right)(0)+\left(\left(\frac{1}{6\left(1+x\right)^{2}}-\frac{1}{2x^{2}}\right)+\left(-\frac{1}{6\left(1+x\right)}+\frac{1}{2x}+\frac{1}{6}\right)^{2}-\left(\frac{x^{4}+6x^{3}+3x}{36\left(x^{2}\right.}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega \,dx}\\
&= \mathrm{e}^{\int \left(-\frac{1}{6(1+x)}+\frac{1}{2x}+\frac{1}{6}\right)dx}\\
&= \frac{\sqrt{x}\,\mathrm{e}^{\frac{x}{6}}}{\left(1+x\right)^{1/6}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\
&= z_1 e^{-\int \frac{1}{2}\frac{-3x^3+33x^2+15x}{9x^3+9x^2}\,dx}\\
&= z_1 e^{\frac{x}{6}-\frac{5\ln(x)}{6}-\frac{7\ln(1+x)}{6}}\\
&= z_1\left(\frac{\mathrm{e}^{\frac{x}{6}}}{x^{5/6}\left(1+x\right)^{7/6}}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\frac{\mathrm{e}^{\frac{x}{3}}}{x^{1/3}\left(1+x\right)^{4/3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1\int \frac{e^{\int -\frac{-3x^3+33x^2+15x}{9x^3+9x^2}\,dx}}{\left(y_1\right)^2}\,dx\\
&= y_1\int \frac{e^{\frac{x}{3}-\frac{5\ln(x)}{3}-\frac{7\ln(1+x)}{3}}}{\left(y_1\right)^2}\,dx\\
&= y_1\left(\int \mathrm{e}^{\frac{x}{3}-\frac{5\ln(x)}{3}-\frac{7\ln(1+x)}{3}}x^{2/3}\left(1+x\right)^{8/3}\mathrm{e}^{-\frac{2x}{3}}\,dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1y_1+c_2y_2\\
&= c_1\left(\frac{\mathrm{e}^{\frac{x}{3}}}{x^{1/3}\left(1+x\right)^{4/3}}\right)+c_2\left(\frac{\mathrm{e}^{\frac{x}{3}}}{x^{1/3}\left(1+x\right)^{4/3}}\left(\int \mathrm{e}^{\frac{x}{3}-\frac{5\ln(x)}{3}-\frac{7\ln(1+x)}{3}}x^{2/3}\left(1+x\right)^{8/3}\mathrm{e}^{-\frac{2x}{3}}\,dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$9x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)+3x(-x^2+11x+5)\left(\frac{d}{dx}y(x)\right)+(-7x^2+16x+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=\frac{(7x^2-16x-1)y(x)}{9x^2(x+1)}+\frac{(x^2-11x-5)\left(\frac{d}{dx}y(x)\right)}{3x(x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)-\frac{(x^2-11x-5)\left(\frac{d}{dx}y(x)\right)}{3x(x+1)}-\frac{(7x^2-16x-1)y(x)}{9x^2(x+1)}=0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

$\left[P_2(x)=-\frac{x^2-11x-5}{3x(x+1)},P_3(x)=-\frac{7x^2-16x-1}{9x^2(x+1)}\right]$

- ○ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

$\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=\frac{7}{3}$

- ○ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

$\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

- ○ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1$

- Multiply by denominators

$9x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)-3(x^2-11x-5)\,x\left(\frac{d}{dx}y(x)\right)+(-7x^2+16x+1)\,y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(9u^3-18u^2+9u)\left(\frac{d^2}{du^2}y(u)\right)+(-3u^3+42u^2-60u+21)\left(\frac{d}{du}y(u)\right)+(-7u^2+30u-22)\,y(u)=$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot y(u)$ to series expansion for $m=0..2$

$u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

- ○ Shift index using $k->k-m$

$u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

- ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..3$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ○ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- ○ Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$3a_0 r(4+3r)\, u^{-1+r} + (3a_1(1+r)(7+3r) - 2a_0(9r^2+21r+11))\, u^r + (3a_2(2+r)(10+3r) - 2a_1($$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $3r(4+3r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, -\frac{4}{3}\}$
- The coefficients of each power of u must be 0
  $[3a_1(1+r)(7+3r) - 2a_0(9r^2+21r+11) = 0, 3a_2(2+r)(10+3r) - 2a_1(9r^2+39r+41) + 3a_0(2$
- Solve for the dependent coefficient(s)
  $\left\{a_1 = \frac{2a_0(9r^2+21r+11)}{3(3r^2+10r+7)}, a_2 = \frac{a_0(243r^4+1593r^3+3699r^2+3567r+1174)}{9(9r^4+78r^3+241r^2+312r+140)}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $9(-2a_k + a_{k-1} + a_{k+1})\, k^2 + 3(6(-2a_k + a_{k-1} + a_{k+1})\, r - 14a_k - a_{k-2} + 5a_{k-1} + 10a_{k+1})\, k + 9(-2a$
- Shift index using $k- >k+2$
  $9(-2a_{k+2} + a_{k+1} + a_{k+3})\,(k+2)^2 + 3(6(-2a_{k+2} + a_{k+1} + a_{k+3})\, r - 14a_{k+2} - a_k + 5a_{k+1} + 10a_{k+3})$
- Recursion relation that defines series solution to ODE
  $a_{k+3} = -\frac{9k^2a_{k+1} - 18k^2a_{k+2} + 18kra_{k+1} - 36kra_{k+2} + 9r^2a_{k+1} - 18r^2a_{k+2} - 3ka_k + 51ka_{k+1} - 114ka_{k+2} - 3ra_k + 51ra_{k+1} - 114ra_{k+2}}{3(3k^2+6kr+3r^2+22k+22r+39)}$
- Recursion relation for $r = 0$
  $a_{k+3} = -\frac{9k^2a_{k+1} - 18k^2a_{k+2} - 3ka_k + 51ka_{k+1} - 114ka_{k+2} - 7a_k + 72a_{k+1} - 178a_{k+2}}{3(3k^2+22k+39)}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+3} = -\frac{9k^2a_{k+1} - 18k^2a_{k+2} - 3ka_k + 51ka_{k+1} - 114ka_{k+2} - 7a_k + 72a_{k+1} - 178a_{k+2}}{3(3k^2+22k+39)}, a_1 = \frac{22a_0}{21}, a_2 = \right.$
- Revert the change of variables $u = x+1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+3} = -\frac{9k^2a_{k+1} - 18k^2a_{k+2} - 3ka_k + 51ka_{k+1} - 114ka_{k+2} - 7a_k + 72a_{k+1} - 178a_{k+2}}{3(3k^2+22k+39)}, a_1 = \frac{22a_0}{21}, \right.$
- Recursion relation for $r = -\frac{4}{3}$
  $a_{k+3} = -\frac{9k^2a_{k+1} - 18k^2a_{k+2} - 3ka_k + 27ka_{k+1} - 66ka_{k+2} - 3a_k + 20a_{k+1} - 58a_{k+2}}{3(3k^2+14k+15)}$
- Solution for $r = -\frac{4}{3}$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{4}{3}}, a_{k+3} = -\frac{9k^2a_{k+1} - 18k^2a_{k+2} - 3ka_k + 27ka_{k+1} - 66ka_{k+2} - 3a_k + 20a_{k+1} - 58a_{k+2}}{3(3k^2+14k+15)}, a_1 = \frac{2a_0}{3}, a_2 = \right.$
- Revert the change of variables $u = x+1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^{k-\frac{4}{3}}, a_{k+3} = -\frac{9k^2a_{k+1} - 18k^2a_{k+2} - 3ka_k + 27ka_{k+1} - 66ka_{k+2} - 3a_k + 20a_{k+1} - 58a_{k+2}}{3(3k^2+14k+15)}, a_1 = \frac{2a_0}{3}, \right.$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k(x+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k(x+1)^{k-\frac{4}{3}}\right), a_{k+3} = -\frac{9k^2a_{k+1} - 18k^2a_{k+2} - 3ka_k + 51ka_{k+1} - 114ka_{k+2} - 7a}{3(3k^2+22k+39)}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @
      <- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0,
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.388 (sec)
Leaf size : 36

```
dsolve(9*x^2*(x+1)*diff(diff(y(x),x),x)+3*x*(-x^2+11*x+5)*diff(y(x),x)+(-7*x^2+16*x+1)
```

$$y = \frac{\frac{c_1 \operatorname{HeunC}\left(-\frac{1}{3},-\frac{4}{3},0,-\frac{1}{9},\frac{11}{18},x+1\right)}{(x+1)^{4/3}} + c_2 \operatorname{HeunC}\left(-\frac{1}{3},\frac{4}{3},0,-\frac{1}{9},\frac{11}{18},x+1\right)}{x^{1/3}}$$

Mathematica DSolve solution

Solving time : 0.346 (sec)
Leaf size : 120

```
DSolve[{9*x^2*(1+x)*D[y[x],{x,2}]+3*x*(5+11*x-x^2)*D[y[x],x]+(1+16*x-7*x^2)*y[x]==0,{}},y[x]
```

$$y(x) \to \exp\left(\int_1^x \frac{K[1]^2+3K[1]+3}{6K[1]^2+6K[1]}dK[1] - \frac{1}{2}\int_1^x \frac{1}{3}\left(\frac{7}{K[2]+1} - 1 + \frac{5}{K[2]}\right)dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{K[1]^2+3K[1]+3}{6K[1]^2+6K[1]}dK[1]\right)dK[3] + c_1\right)$$

2.1.131 Problem 133

Internal problem ID [9303]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 133
Date solved : `Monday, January 27, 2025 at 06:01:14 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$36x^2(1-2x)\,y''+24x(1-9x)\,y'+(1-70x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.310 (sec)

Writing the ode as

$$\left(-72x^3+36x^2\right)y''+\left(-216x^2+24x\right)y'+(1-70x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -72x^3+36x^2 \\ B &= -216x^2+24x \\ C &= 1-70x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{-32x^2+48x-9}{36\left(2x^2-x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -32x^2+48x-9 \\ t &= 36\left(2x^2-x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-32x^2+48x-9}{36\left(2x^2-x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.246: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(2x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{1}{3x} + \frac{7}{36\left(x - \frac{1}{2}\right)^2} - \frac{1}{3\left(x - \frac{1}{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{6}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-32x^2+48x-9}{36\left(2x^2-x\right)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{2}{9}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b} = \frac{2}{3} \\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{3}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-32x^2+48x-9}{36\left(2x^2-x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{2}{3}$	$\frac{1}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{3}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= \frac{1}{3} - \left(\frac{1}{3}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} - \frac{1}{6\left(x-\frac{1}{2}\right)} + (-)(0) \\
&= \frac{1}{2x} - \frac{1}{6\left(x-\frac{1}{2}\right)} \\
&= \frac{-3+4x}{12x^2-6x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+(\omega'+\omega^2-r)\,p=0 \tag{1A}$$

Let

$$p(x)=1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{1}{2x}-\frac{1}{6\left(x-\frac{1}{2}\right)}\right)(0)+\left(\left(-\frac{1}{2x^2}+\frac{1}{6\left(x-\frac{1}{2}\right)^2}\right)+\left(\frac{1}{2x}-\frac{1}{6\left(x-\frac{1}{2}\right)}\right)^2-\left(\frac{-32x^2+48x-9}{36\left(2x^2-x\right)^2}\right.\right.$$

$$0=$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x)&=pe^{\int\omega\,dx}\\
&=\mathrm{e}^{\int\left(\frac{1}{2x}-\frac{1}{6\left(x-\frac{1}{2}\right)}\right)dx}\\
&=\frac{\sqrt{x}}{(-1+2x)^{1/6}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1&=z_1e^{\int-\frac{1}{2}\frac{B}{A}\,dx}\\
&=z_1e^{-\int\frac{1}{2}\frac{-216x^2+24x}{-72x^3+36x^2}\,dx}\\
&=z_1e^{-\frac{\ln(x)}{3}-\frac{7\ln(-1+2x)}{6}}\\
&=z_1\left(\frac{1}{x^{1/3}(-1+2x)^{7/6}}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\frac{x^{1/6}}{(-1+2x)^{4/3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int\frac{e^{\int-\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2&=y_1\int\frac{e^{\int-\frac{-216x^2+24x}{-72x^3+36x^2}\,dx}}{(y_1)^2}\,dx\\
&=y_1\int\frac{e^{-\frac{2\ln(x)}{3}-\frac{7\ln(-1+2x)}{3}}}{(y_1)^2}\,dx\\
&=y_1\Bigg(3(-1+2x)^{1/3}\\
&\quad-\ln\left((-1+2x)^{1/3}+1\right)+\frac{\ln\left((-1+2x)^{2/3}-(-1+2x)^{1/3}+1\right)}{2}-\sqrt{3}\arctan\left(\left(\frac{-1+2(-1+2x)^{1/3}}{3}\right.\right.
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^{1/6}}{(-1+2x)^{4/3}}\right) \\
&\quad + c_2\left(\frac{x^{1/6}}{(-1+2x)^{4/3}}\left(3(-1+2x)^{1/3} - \ln\left((-1+2x)^{1/3}+1\right) + \frac{\ln\left((-1+2x)^{2/3}-(-1+2x)^{1/3}+1\right)}{2} - \sqrt{3}\ \right.\right.
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$36x^2(-2x+1)\left(\frac{d^2}{dx^2}y(x)\right) + 24x(1-9x)\left(\frac{d}{dx}y(x)\right) + (1-70x)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(-1+70x)y(x)}{36x^2(2x-1)} - \frac{2(-1+9x)\left(\frac{d}{dx}y(x)\right)}{3x(2x-1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{2(-1+9x)\left(\frac{d}{dx}y(x)\right)}{3x(2x-1)} + \frac{(-1+70x)y(x)}{36x^2(2x-1)} = 0$

- ☐ Check to see if x_0 is a regular singular point
  - ○ Define functions

    $\left[P_2(x) = \frac{2(-1+9x)}{3x(2x-1)}, P_3(x) = \frac{-1+70x}{36x^2(2x-1)}\right]$
  - ○ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{2}{3}$
  - ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{1}{36}$
  - ○ $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $36x^2(2x-1)\left(\frac{d^2}{dx^2}y(x)\right) + 24x(-1+9x)\left(\frac{d}{dx}y(x)\right) + (-1+70x)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - ○ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - ○ Shift index using $k- >k-m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - ○ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..3$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0(-1+6r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(6k+6r-1)^2 + 2a_{k-1}(6k+1+6r)\,(6k+6r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-(-1+6r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = \frac{1}{6}$

- Each term in the series must be 0, giving the recursion relation

  $-36\left(\left(-2k-2r-\frac{1}{3}\right) a_{k-1} + a_k\left(k+r-\frac{1}{6}\right)\right)\left(k+r-\frac{1}{6}\right) = 0$

- Shift index using $k->k+1$

  $-36\left(\left(-2k-\frac{7}{3}-2r\right) a_k + a_{k+1}\left(k+\frac{5}{6}+r\right)\right)\left(k+\frac{5}{6}+r\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2(6k+6r+7)a_k}{6k+6r+5}$

- Recursion relation for $r = \frac{1}{6}$

  $a_{k+1} = \frac{2(6k+8)a_k}{6k+6}$

- Solution for $r = \frac{1}{6}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{6}}, a_{k+1} = \frac{2(6k+8)a_k}{6k+6}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
```

```
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
         -> solution has integrals; searching for one without integrals...
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
            <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
         <- hypergeometric solution without integrals succesful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returnin
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.150 (sec)
Leaf size : 93

```
dsolve(36*x^2*(1-2*x)*diff(diff(y(x),x),x)+24*x*(1-9*x)*diff(y(x),x)+(1-70*x)*y(x) = 0,y
```

$$y = \frac{x^{1/6}\left(2\sqrt{3}\arctan\left(\frac{\sqrt{3}\,(-1+2x)^{1/3}}{-2+(-1+2x)^{1/3}}\right)c_2 - 2\ln\left(1+(-1+2x)^{1/3}\right)c_2 + \ln\left(1-(-1+2x)^{1/3}+(-1+2x)^{2/3}\right)}{3\,(-1+2x)^{4/3}}$$

Mathematica DSolve solution

Solving time : 0.296 (sec)
Leaf size : 112

```
DSolve[{36*x^2*(1-2*x)*D[y[x],{x,2}]+24*x*(1-9*x)*D[y[x],x]+(1-70*x)*y[x]==0,{}},y[x],x,Include
```

$$y(x) \to \exp\left(\int_1^x \frac{3-4K[1]}{6K[1]-12K[1]^2}dK[1] - \frac{1}{2}\int_1^x \frac{2-18K[2]}{3K[2]-6K[2]^2}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{3-4K[1]}{6K[1]-12K[1]^2}dK[1]\right)dK[3] + c_1\right)$$

2.1.132 Problem 134

Internal problem ID [9304]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 134
Date solved : `Monday, January 27, 2025 at 06:01:15 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+x)\,y''-x(3-x)\,y'+4y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.220 (sec)

Writing the ode as

$$x^2(1+x)\,y''+\left(x^2-3x\right)y'+4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(1+x) \\ B &= x^2-3x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-x^2-10x-1}{4\left(x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2-10x-1 \\ t &= 4\left(x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-x^2-10x-1}{4\left(x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.248: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4-2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+x)^2$. There is a pole at $x=0$ of order 2. There is a pole at $x=-1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1,2,4,6,12]$$

Attempting to find a solution using case $n=1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{2}{1+x} + \frac{2}{(1+x)^2} - \frac{2}{x}$$

For the pole at $x=-1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b=2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

For the pole at $x=0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 10x - 1}{4(x^2 + x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 - 10x - 1}{4(x^2 + x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	2	-1
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^- + \alpha_{c_2}^+) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{1+x} + \frac{1}{2x} + (-)(0) \\
&= -\frac{1}{1+x} + \frac{1}{2x} \\
&= -\frac{x-1}{2x(1+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{1+x} + \frac{1}{2x}\right)(1) + \left(\left(\frac{1}{(1+x)^2} - \frac{1}{2x^2}\right) + \left(-\frac{1}{1+x} + \frac{1}{2x}\right)^2 - \left(\frac{-x^2 - 10x - 1}{4\,(x^2+x)^2}\right)\right) = 0$$
$$\frac{1 + a_0}{x\,(1+x)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (x-1)\, \mathrm{e}^{\int \left(-\frac{1}{1+x} + \frac{1}{2x}\right) dx} \\
&= (x-1)\, \mathrm{e}^{\frac{\ln(x)}{2} - \ln(1+x)} \\
&= \frac{(x-1)\sqrt{x}}{1+x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2-3x}{x^2(1+x)}\, dx} \\
&= z_1 e^{\frac{3\ln(x)}{2} - 2\ln(1+x)} \\
&= z_1 \left(\frac{x^{3/2}}{(1+x)^2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2(x-1)}{(1+x)^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2-3x}{x^2(1+x)}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{3\ln(x) - 4\ln(1+x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{4}{x-1} + \ln(x)\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^2(x-1)}{(1+x)^3} \right) + c_2 \left(\frac{x^2(x-1)}{(1+x)^3} \left(-\frac{4}{x-1} + \ln(x) \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right) - x(-x+3)\left(\frac{d}{dx}y(x)\right) + 4y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{4y(x)}{(x+1)x^2} - \frac{(x-3)\left(\frac{d}{dx}y(x)\right)}{x(x+1)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(x-3)\left(\frac{d}{dx}y(x)\right)}{x(x+1)} + \frac{4y(x)}{(x+1)x^2} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{x-3}{x(x+1)}, P_3(x) = \frac{4}{(x+1)x^2}\right]$
  - $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((x+1) \cdot P_2(x)\right)\Big|_{x=-1} = 4$
  - $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $\left((x+1)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$
  - $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$
- Multiply by denominators

  $x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right) + x(x-3)\left(\frac{d}{dx}y(x)\right) + 4y(x) = 0$
- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(u^3 - 2u^2 + u)\left(\frac{d^2}{du^2}y(u)\right) + (u^2 - 5u + 4)\left(\frac{d}{du}y(u)\right) + 4y(u) = 0$
- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$
  - Shift index using $k- >k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$
  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(3+r)\, u^{-1+r} + \left(a_1(1+r)(4+r) - a_0(2r^2+3r-4)\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)(k+4+r) - \right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(3+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-3, 0\}$$

- Each term must be 0

$$a_1(1+r)(4+r) - a_0(2r^2+3r-4) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)(k+4+r) - a_k(2k^2+4kr+2r^2+3k+3r-4) + a_{k-1}(k+r-1)^2 = 0$$

- Shift index using $k->k+1$

$$a_{k+2}(k+2+r)(k+5+r) - a_{k+1}\left(2(k+1)^2+4(k+1)\, r+2r^2+3k-1+3r\right) + a_k(k+r)^2 = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k^2a_k-2k^2a_{k+1}+2kra_k-4kra_{k+1}+r^2a_k-2r^2a_{k+1}-7ka_{k+1}-7ra_{k+1}-a_{k+1}}{(k+2+r)(k+5+r)}$$

- Recursion relation for $r = -3$

$$a_{k+2} = -\frac{k^2a_k-2k^2a_{k+1}-6ka_k+5ka_{k+1}+9a_k+2a_{k+1}}{(k-1)(k+2)}$$

- Series not valid for $r = -3$, division by 0 in the recursion relation at $k = 1$

$$a_{k+2} = -\frac{k^2a_k-2k^2a_{k+1}-6ka_k+5ka_{k+1}+9a_k+2a_{k+1}}{(k-1)(k+2)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k^2a_k-2k^2a_{k+1}-7ka_{k+1}-a_{k+1}}{(k+2)(k+5)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2a_k-2k^2a_{k+1}-7ka_{k+1}-a_{k+1}}{(k+2)(k+5)}, 4a_1+4a_0 = 0\right]$$

- Revert the change of variables $u = x+1$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+2} = -\frac{k^2a_k-2k^2a_{k+1}-7ka_{k+1}-a_{k+1}}{(k+2)(k+5)}, 4a_1+4a_0 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 30

```
dsolve(x^2*(x+1)*diff(diff(y(x),x),x)-x*(-x+3)*diff(y(x),x)+4*y(x) = 0,y(x),singsol=al
```

$$y = \frac{x^2(c_2(x-1)\ln(x) + c_1 x - c_1 - 4c_2)}{(x+1)^3}$$

Mathematica DSolve solution

Solving time : 0.459 (sec)
Leaf size : 111

```
DSolve[{x^2*(1+x)*D[y[x],{x,2}]-x*(3-x)*D[y[x],x]+4*y[x]==0,{}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to (x-1)\exp\left(\int_1^x \left(\frac{1}{2K[1]} - \frac{1}{K[1]+1}\right) dK[1] - \frac{1}{2}\int_1^x \frac{K[2]-3}{K[2](K[2]+1)} dK[2]\right)\left(c_2\int_1^x \frac{\exp\left(-2\int_1^{K[3]}\left(\frac{1}{2K[1]} - \frac{1}{K[1]+1}\right) dK[1]\right)}{(K[3]-1)^2} dK[3] + c_1\right)$$

2.1.133 Problem 135

Internal problem ID [9305]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 135
Date solved : `Monday, January 27, 2025 at 06:01:16 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1-2x)\,y''-x(5-4x)\,y'+(9-4x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.216 (sec)

Writing the ode as

$$\left(-2x^3+x^2\right)y''+\left(4x^2-5x\right)y'+(9-4x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=-2x^3+x^2\\ B&=4x^2-5x\\ C&=9-4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{8x-1}{4\left(2x^2-x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=8x-1\\ t&=4\left(2x^2-x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{8x-1}{4\left(2x^2-x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.250: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 1 \\
&= 3
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 3 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 3 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4\left(x - \frac{1}{2}\right)^2} - \frac{1}{x - \frac{1}{2}} + \frac{1}{x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $3 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{8x - 1}{4\left(2x^2 - x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
3	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= 0 - (0) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} - \frac{1}{2\left(x - \frac{1}{2}\right)} + (0) \\
&= \frac{1}{2x} - \frac{1}{2\left(x - \frac{1}{2}\right)} \\
&= -\frac{1}{2x\left(-1 + 2x\right)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{1}{2x}-\frac{1}{2\left(x-\frac{1}{2}\right)}\right)(0)+\left(\left(-\frac{1}{2x^2}+\frac{1}{2\left(x-\frac{1}{2}\right)^2}\right)+\left(\frac{1}{2x}-\frac{1}{2\left(x-\frac{1}{2}\right)}\right)^2-\left(\frac{8x-1}{4\left(2x^2-x\right)^2}\right)\right)=$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\,dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x}-\frac{1}{2\left(x-\frac{1}{2}\right)}\right)dx} \\
&= \frac{\sqrt{x}}{\sqrt{-1+2x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x^2-5x}{-2x^3+x^2}\,dx} \\
&= z_1 e^{-\frac{3\ln(-1+2x)}{2}+\frac{5\ln(x)}{2}} \\
&= z_1\left(\frac{x^{5/2}}{(-1+2x)^{3/2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\frac{x^3}{(-1+2x)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1\int \frac{e^{\int -\frac{4x^2-5x}{-2x^3+x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1\int \frac{e^{-3\ln(-1+2x)+5\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(2x-\ln(x))
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1y_1+c_2y_2 \\
&= c_1\left(\frac{x^3}{(-1+2x)^2}\right)+c_2\left(\frac{x^3}{(-1+2x)^2}(2x-\ln(x))\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(-2x+1)\left(\frac{d^2}{dx^2}y(x)\right)-x(5-4x)\left(\frac{d}{dx}y(x)\right)+(9-4x)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(-9+4x)y(x)}{x^2(2x-1)}+\frac{(-5+4x)\left(\frac{d}{dx}y(x)\right)}{x(2x-1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)-\frac{(-5+4x)\left(\frac{d}{dx}y(x)\right)}{x(2x-1)}+\frac{(-9+4x)y(x)}{x^2(2x-1)}=0$

- □ Check to see if x_0 is a regular singular point
  - ◦ Define functions

    $\left[P_2(x)=-\frac{-5+4x}{x(2x-1)},P_3(x)=\frac{-9+4x}{x^2(2x-1)}\right]$
  - ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=-5$
  - ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=9$
  - ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(2x-1)\left(\frac{d^2}{dx^2}y(x)\right)-x(-5+4x)\left(\frac{d}{dx}y(x)\right)+(-9+4x)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

- □ Rewrite ODE with series expansions
  - ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

    $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$
  - ◦ Shift index using $k->k-m$

    $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$
  - ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$
  - ◦ Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - ◦ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..3$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$
  - ◦ Shift index using $k->k+2-m$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$-a_0(-3+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r-3)^2 + 2a_{k-1}(k+r-2)(k+r-3)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-(-3+r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 3$
- Each term in the series must be 0, giving the recursion relation
  $-a_k(k+r-3)^2 + 2a_{k-1}(k+r-2)(k+r-3) = 0$
- Shift index using $k- >k+1$
  $-a_{k+1}(k+r-2)^2 + 2a_k(k+r-1)(k+r-2) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{2a_k(k+r-1)}{k+r-2}$
- Recursion relation for $r = 3$
  $a_{k+1} = \frac{2a_k(k+2)}{k+1}$
- Solution for $r = 3$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = \frac{2a_k(k+2)}{k+1}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.019` (sec)
Leaf size : 26

```
dsolve(x^2*(1-2*x)*diff(diff(y(x),x),x)-x*(5-4*x)*diff(y(x),x)+(9-4*x)*y(x) = 0,y(x),s
```

$$y = \frac{x^3(2c_2x - c_2\ln(x) + c_1)}{(-1+2x)^2}$$

Mathematica DSolve solution

Solving time : `0.212` (sec)
Leaf size : 95

```
DSolve[{x^2*(1-2*x)*D[y[x],{x,2}]-x*(5-4*x)*D[y[x],x]+(9-4*x)*y[x]==0,{}},y[x],x,IncludeSingula
```

$$y(x) \to \exp\left(\int_1^x \frac{1}{2K[1]-4K[1]^2}dK[1] - \frac{1}{2}\int_1^x \left(\frac{6}{2K[2]-1} - \frac{5}{K[2]}\right)dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{1}{2K[1]-4K[1]^2}dK[1]\right)dK[3] + c_1\right)$$

2.1.134 Problem 136

Internal problem ID [9306]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 136
Date solved : `Monday, January 27, 2025 at 06:01:16 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(2+x)\,y''+x^2y'+(1-x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.202 (sec)

Writing the ode as

$$\left(2x^3+4x^2\right)y''+x^2y'+(1-x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x^3+4x^2\\B&=x^2\\C&=1-x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{5x^2+8x-16}{16\left(x^2+2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=5x^2+8x-16\\t&=16\left(x^2+2x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{5x^2+8x-16}{16\left(x^2+2x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.252: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{3}{8x} - \frac{3}{8(2+x)} - \frac{3}{16(2+x)^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(2+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{5x^2 + 8x - 16}{16\left(x^2 + 2x\right)^2}$$

Since the gcd$(s,t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{5x^2 + 8x - 16}{16\left(x^2 + 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{3}{4}$	$\frac{1}{4}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\
&= \frac{5}{4} - \left(\frac{5}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{3}{4\left(2 + x\right)} + \frac{1}{2x} + (0) \\
&= \frac{3}{4\left(2 + x\right)} + \frac{1}{2x} \\
&= \frac{5x + 4}{4x\left(2 + x\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{4(2+x)} + \frac{1}{2x}\right)(0) + \left(\left(-\frac{3}{4(2+x)^2} - \frac{1}{2x^2}\right) + \left(\frac{3}{4(2+x)} + \frac{1}{2x}\right)^2 - \left(\frac{5x^2+8x-16}{16(x^2+2x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{3}{4(2+x)} + \frac{1}{2x}\right) dx} \\ &= (2+x)^{3/4} \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^2}{2x^3+4x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(2+x)}{4}} \\ &= z_1 \left(\frac{1}{(2+x)^{1/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{2+x}\,\sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2}{2x^3+4x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{\ln(2+x)}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right)}{2} + \frac{1}{\sqrt{2+x}}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\sqrt{2+x}\,\sqrt{x}\right) + c_2\left(\sqrt{2+x}\,\sqrt{x}\left(-\frac{\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right)}{2} + \frac{1}{\sqrt{2+x}}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2(x+2)\left(\frac{d^2}{dx^2}y(x)\right)+x^2\left(\frac{d}{dx}y(x)\right)+(1-x)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{(x-1)y(x)}{2(x+2)x^2}-\frac{\frac{d}{dx}y(x)}{2(x+2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{\frac{d}{dx}y(x)}{2(x+2)}-\frac{(x-1)y(x)}{2(x+2)x^2}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=\frac{1}{2(x+2)},P_3(x)=-\frac{x-1}{2(x+2)x^2}\right]$

- ○ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

  $\left((x+2)\cdot P_2(x)\right)\bigg|_{x=-2}=\frac{1}{2}$

- ○ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

  $\left((x+2)^2\cdot P_3(x)\right)\bigg|_{x=-2}=0$

- ○ $x=-2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-2$

- Multiply by denominators

  $2x^2(x+2)\left(\frac{d^2}{dx^2}y(x)\right)+x^2\left(\frac{d}{dx}y(x)\right)+(1-x)\,y(x)=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

  $(2u^3-8u^2+8u)\left(\frac{d^2}{du^2}y(u)\right)+(u^2-4u+4)\left(\frac{d}{du}y(u)\right)+(3-u)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

- ○ Shift index using $k->k-m$

  $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

- ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ○ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- ○ Shift index using $k->k+2-m$

$$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$4a_0r(-1+2r)\,u^{-1+r}+\left(4a_1(1+r)\,(1+2r)-a_0(8r^2-4r-3)\right)u^r+\left(\sum_{k=1}^{\infty}(4a_{k+1}(k+1+r)\,(2k+$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$4r(-1+2r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,\frac{1}{2}\right\}$$

- Each term must be 0

$$4a_1(1+r)\,(1+2r)-a_0(8r^2-4r-3)=0$$

- Each term in the series must be 0, giving the recursion relation

$$2(-4a_k+a_{k-1}+4a_{k+1})\,k^2+(4(-4a_k+a_{k-1}+4a_{k+1})\,r+4a_k-5a_{k-1}+12a_{k+1})\,k+2(-4a_k+a_{k-}$$

- Shift index using $k->k+1$

$$2(-4a_{k+1}+a_k+4a_{k+2})\,(k+1)^2+(4(-4a_{k+1}+a_k+4a_{k+2})\,r+4a_{k+1}-5a_k+12a_{k+2})\,(k+1)+2$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{2k^2a_k-8k^2a_{k+1}+4kra_k-16kra_{k+1}+2r^2a_k-8r^2a_{k+1}-ka_k-12ka_{k+1}-ra_k-12ra_{k+1}-a_k-a_{k+1}}{4(2k^2+4kr+2r^2+7k+7r+6)}$$

- Recursion relation for $r=0$

$$a_{k+2}=-\frac{2k^2a_k-8k^2a_{k+1}-ka_k-12ka_{k+1}-a_k-a_{k+1}}{4(2k^2+7k+6)}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty}a_ku^k,a_{k+2}=-\frac{2k^2a_k-8k^2a_{k+1}-ka_k-12ka_{k+1}-a_k-a_{k+1}}{4(2k^2+7k+6)},4a_1+3a_0=0\right]$$

- Revert the change of variables $u=x+2$

$$\left[y(x)=\sum_{k=0}^{\infty}a_k(x+2)^k,a_{k+2}=-\frac{2k^2a_k-8k^2a_{k+1}-ka_k-12ka_{k+1}-a_k-a_{k+1}}{4(2k^2+7k+6)},4a_1+3a_0=0\right]$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+2}=-\frac{2k^2a_k-8k^2a_{k+1}+ka_k-20ka_{k+1}-a_k-9a_{k+1}}{4(2k^2+9k+10)}$$

- Solution for $r=\frac{1}{2}$

$$\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k+\frac{1}{2}},a_{k+2}=-\frac{2k^2a_k-8k^2a_{k+1}+ka_k-20ka_{k+1}-a_k-9a_{k+1}}{4(2k^2+9k+10)},12a_1+3a_0=0\right]$$

- Revert the change of variables $u=x+2$

$$\left[y(x)=\sum_{k=0}^{\infty}a_k(x+2)^{k+\frac{1}{2}},a_{k+2}=-\frac{2k^2a_k-8k^2a_{k+1}+ka_k-20ka_{k+1}-a_k-9a_{k+1}}{4(2k^2+9k+10)},12a_1+3a_0=0\right]$$

- Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty}a_k(x+2)^k\right)+\left(\sum_{k=0}^{\infty}b_k(x+2)^{k+\frac{1}{2}}\right),a_{k+2}=-\frac{2k^2a_k-8k^2a_{k+1}-ka_k-12ka_{k+1}-a_k-a_{k+1}}{4(2k^2+7k+6)},4a_1+$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
```

```
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.025 (sec)
Leaf size : 50

```
dsolve(2*x^2*(x+2)*diff(diff(y(x),x),x)+diff(y(x),x)*x^2+(1-x)*y(x) = 0,y(x),singsol=a
```

$$y = c_1\sqrt{x\left(x+2\right)}+\frac{c_2\left(\left(x+2\right)\operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)-\sqrt{2}\,\sqrt{x+2}\right)\sqrt{x}}{\sqrt{x+2}}$$

Mathematica DSolve solution

Solving time : 0.508 (sec)
Leaf size : 92

```
DSolve[{2*x^2*(2+x)*D[y[x],{x,2}]+x^2*D[y[x],x]+(1-x)*y[x]==0,{}},y[x],x,IncludeSingularSolu
```

$$y(x)\to \frac{\exp\left(\int_1^x \frac{5K[1]+4}{4K[1]^2+8K[1]}dK[1]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[2]} \frac{5K[1]+4}{4K[1]^2+8K[1]}dK[1]\right)dK[2]+c_1\right)}{\sqrt[4]{2}\sqrt[4]{x+2}}$$

2.1.135 Problem 137

Internal problem ID [9307]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 137
Date solved : `Monday, January 27, 2025 at 06:01:17 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(1+x)\,y''-x(6-x)\,y'+(8-x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.206 (sec)

Writing the ode as

$$\left(2x^3+2x^2\right)y''+\left(x^2-6x\right)y'+(8-x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^3+2x^2 \\ B &= x^2-6x \\ C &= 8-x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{5x^2-20x-4}{16\left(x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 5x^2-20x-4 \\ t &= 16\left(x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{5x^2-20x-4}{16\left(x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.254: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4\,(1+x)} + \frac{21}{16\,(1+x)^2} - \frac{1}{4x^2} - \frac{3}{4x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{5x^2 - 20x - 4}{16\left(x^2 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{5x^2 - 20x - 4}{16\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= -\frac{1}{4} - \left(-\frac{1}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{4(1+x)} + \frac{1}{2x} + (-)(0) \\
&= -\frac{3}{4(1+x)} + \frac{1}{2x} \\
&= -\frac{x-2}{4x(1+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{4(1+x)} + \frac{1}{2x}\right)(0) + \left(\left(\frac{3}{4(1+x)^2} - \frac{1}{2x^2}\right) + \left(-\frac{3}{4(1+x)} + \frac{1}{2x}\right)^2 - \left(\frac{5x^2 - 20x - 4}{16(x^2+x)^2}\right)\right) = 0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(-\frac{3}{4(1+x)} + \frac{1}{2x}\right) dx} \\
&= \frac{\sqrt{x}}{(1+x)^{3/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x^2 - 6x}{2x^3 + 2x^2}\, dx} \\
&= z_1 e^{-\frac{7\ln(1+x)}{4} + \frac{3\ln(x)}{2}} \\
&= z_1 \left(\frac{x^{3/2}}{(1+x)^{7/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2}{(1+x)^{5/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2 - 6x}{2x^3 + 2x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{7\ln(1+x)}{2} + 3\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{2(1+x)^{3/2}}{3} + 2\sqrt{1+x} + \ln\left(\sqrt{1+x} - 1\right) - \ln\left(1 + \sqrt{1+x}\right)\right)
\end{aligned}$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{x^2}{(1+x)^{5/2}}\right)$$
$$+ c_2 \left(\frac{x^2}{(1+x)^{5/2}} \left(\frac{2(1+x)^{3/2}}{3} + 2\sqrt{1+x} + \ln\left(\sqrt{1+x}-1\right) - \ln\left(1+\sqrt{1+x}\right)\right)\right)$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right) - x(-x+6)\left(\frac{d}{dx}y(x)\right) + (8-x)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{(-8+x)y(x)}{2x^2(x+1)} - \frac{(-6+x)\left(\frac{d}{dx}y(x)\right)}{2x(x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(-6+x)\left(\frac{d}{dx}y(x)\right)}{2x(x+1)} - \frac{(-8+x)y(x)}{2x^2(x+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{-6+x}{2x(x+1)}, P_3(x) = -\frac{-8+x}{2x^2(x+1)}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1} = \frac{7}{2}$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $2x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right) + x(-6+x)\left(\frac{d}{dx}y(x)\right) + (8-x)\,y(x) = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(2u^3-4u^2+2u)\left(\frac{d^2}{du^2}y(u)\right) + (u^2-8u+7)\left(\frac{d}{du}y(u)\right) + (9-u)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot y(u)$ to series expansion for $m = 0..1$

  $u^m\cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

  ◦ Shift index using $k->k-m$

  $u^m\cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(5+2r)\, u^{-1+r} + \left(a_1(1+r)\,(7+2r) - a_0(4r^2+4r-9)\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)\,(2k+7+\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(5+2r)=0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, -\frac{5}{2}\right\}$

- Each term must be 0

  $a_1(1+r)\,(7+2r) - a_0(4r^2+4r-9) = 0$

- Each term in the series must be 0, giving the recursion relation

  $(-4a_k + 2a_{k-1} + 2a_{k+1})\, k^2 + ((-8a_k + 4a_{k-1} + 4a_{k+1})\, r - 4a_k - 5a_{k-1} + 9a_{k+1})\, k + (-4a_k + 2a$

- Shift index using $k->k+1$

  $(-4a_{k+1} + 2a_k + 2a_{k+2})\,(k+1)^2 + ((-8a_{k+1} + 4a_k + 4a_{k+2})\, r - 4a_{k+1} - 5a_k + 9a_{k+2})\,(k+1) +$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} + 4kra_k - 8kra_{k+1} + 2r^2a_k - 4r^2a_{k+1} - ka_k - 12ka_{k+1} - ra_k - 12ra_{k+1} - a_k + a_{k+1}}{2k^2 + 4kr + 2r^2 + 13k + 13r + 18}$

- Recursion relation for $r=0$

  $a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - ka_k - 12ka_{k+1} - a_k + a_{k+1}}{2k^2 + 13k + 18}$

- Solution for $r=0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - ka_k - 12ka_{k+1} - a_k + a_{k+1}}{2k^2 + 13k + 18}, 7a_1 + 9a_0 = 0\right]$

- Revert the change of variables $u = x+1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - ka_k - 12ka_{k+1} - a_k + a_{k+1}}{2k^2 + 13k + 18}, 7a_1 + 9a_0 = 0\right]$

- Recursion relation for $r = -\frac{5}{2}$

  $a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - 11ka_k + 8ka_{k+1} + 14a_k + 6a_{k+1}}{2k^2 + 3k - 2}$

- Solution for $r = -\frac{5}{2}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{5}{2}}, a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - 11ka_k + 8ka_{k+1} + 14a_k + 6a_{k+1}}{2k^2 + 3k - 2}, -3a_1 - 6a_0 = 0\right]$

- Revert the change of variables $u = x+1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^{k-\frac{5}{2}}, a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - 11ka_k + 8ka_{k+1} + 14a_k + 6a_{k+1}}{2k^2 + 3k - 2}, -3a_1 - 6a_0 = 0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k(x+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k(x+1)^{k-\frac{5}{2}}\right), a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - ka_k - 12ka_{k+1} - a_k + a_{k+1}}{2k^2 + 13k + 18}, 7a_1\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.030` (sec)
Leaf size : 50

```
dsolve(2*x^2*(x+1)*diff(diff(y(x),x),x)-x*(-x+6)*diff(y(x),x)+(8-x)*y(x) = 0,y(x),singso
```

$$y = \frac{2x^2\left(\frac{3\ln(\sqrt{x+1}-1)c_2}{2}-\frac{3\ln(\sqrt{x+1}+1)c_2}{2}+(x+4)\,c_2\sqrt{x+1}+\frac{3c_1}{2}\right)}{3\,(x+1)^{5/2}}$$

Mathematica DSolve solution

Solving time : `0.244` (sec)
Leaf size : 109

```
DSolve[{2*x^2*(1+x)*D[y[x],{x,2}]-x*(6-x)*D[y[x],x]+(8-x)*y[x]==0,{}},y[x],x,IncludeSingularSo
```

$$y(x) \to \exp\left(\int_1^x \frac{2-K[1]}{4K[1]^2+4K[1]}dK[1]-\frac{1}{2}\int_1^x\left(\frac{7}{2(K[2]+1)}-\frac{3}{K[2]}\right)dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]}\frac{2-K[1]}{4K[1]^2+4K[1]}dK[1]\right)dK[3]+c_1\right)$$

2.1.136 Problem 138

Internal problem ID [9308]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 138
Date solved : `Monday, January 27, 2025 at 06:01:17 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+2x)\,y''+x(5+9x)\,y'+(4+3x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.227 (sec)

Writing the ode as

$$\left(2x^3+x^2\right)y''+\left(9x^2+5x\right)y'+(4+3x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=2x^3+x^2\\ B&=9x^2+5x\\ C&=4+3x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{21x^2+6x-1}{4\left(2x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=21x^2+6x-1\\ t&=4\left(2x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{21x^2+6x-1}{4\left(2x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.256: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{5}{2x} + \frac{5}{16\left(x + \frac{1}{2}\right)^2} - \frac{5}{2\left(x + \frac{1}{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{21x^2 + 6x - 1}{4\left(2x^2 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{21x^2 + 6x - 1}{4\left(2x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-\frac{1}{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\
&= \frac{7}{4} - \left(\frac{7}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \frac{5}{4\left(x+\frac{1}{2}\right)} + (0) \\
&= \frac{1}{2x} + \frac{5}{4\left(x+\frac{1}{2}\right)} \\
&= \frac{1+7x}{4x^2+2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{5}{4\left(x+\frac{1}{2}\right)}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{5}{4\left(x+\frac{1}{2}\right)^2}\right) + \left(\frac{1}{2x} + \frac{5}{4\left(x+\frac{1}{2}\right)}\right)^2 - \left(\frac{21x^2+6x-1}{4\left(2x^2+x\right)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{2x} + \frac{5}{4\left(x+\frac{1}{2}\right)}\right) dx} \\ &= \sqrt{x}\,(1+2x)^{5/4} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{9x^2+5x}{2x^3+x^2} \, dx} \\ &= z_1 e^{-\frac{5\ln(x)}{2} + \frac{\ln(1+2x)}{4}} \\ &= z_1 \left(\frac{(1+2x)^{1/4}}{x^{5/2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(1+2x)^{3/2}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{9x^2+5x}{2x^3+x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-5\ln(x) + \frac{\ln(1+2x)}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(-\ln\left(\sqrt{1+2x}+1\right) + \frac{2}{3(1+2x)^{3/2}} + \frac{2}{\sqrt{1+2x}} + \ln\left(\sqrt{1+2x}-1\right)\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{(1+2x)^{3/2}}{x^2}\right) \\
&\quad + c_2\left(\frac{(1+2x)^{3/2}}{x^2}\left(-\ln\left(\sqrt{1+2x}+1\right)+\frac{2}{3\,(1+2x)^{3/2}}+\frac{2}{\sqrt{1+2x}}+\ln\left(\sqrt{1+2x}-1\right)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(2x+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(5+9x)\left(\frac{d}{dx}y(x)\right)+(3x+4)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(3x+4)y(x)}{x^2(2x+1)}-\frac{(5+9x)\left(\frac{d}{dx}y(x)\right)}{x(2x+1)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{(5+9x)\left(\frac{d}{dx}y(x)\right)}{x(2x+1)}+\frac{(3x+4)y(x)}{x^2(2x+1)}=0$

□ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x)=\frac{5+9x}{x(2x+1)}, P_3(x)=\frac{3x+4}{x^2(2x+1)}\right]$
  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=5$
  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=4$
  - $x=0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=0$
- Multiply by denominators

  $x^2(2x+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(5+9x)\left(\frac{d}{dx}y(x)\right)+(3x+4)\,y(x)=0$
- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions
  - Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

    $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$
  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

$$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..3$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(2+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+2)^2 + a_{k-1}(k+r+2)\,(2k-1+2r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(2+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = -2$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r+2)\left(a_k(k+r+2) + a_{k-1}(2k-1+2r)\right) = 0$$

- Shift index using $k- >k+1$

$$(k+r+3)\left(a_{k+1}(k+r+3) + a_k(2k+2r+1)\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k(2k+2r+1)}{k+r+3}$$

- Recursion relation for $r = -2$

$$a_{k+1} = -\frac{a_k(2k-3)}{k+1}$$

- Solution for $r = -2$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+1} = -\frac{a_k(2k-3)}{k+1}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.031` (sec)
Leaf size : 73

```
dsolve(x^2*(2*x+1)*diff(diff(y(x),x),x)+x*(5+9*x)*diff(diff(y(x),x)+(3*x+4)*y(x) = 0,y(x),sin
```

$$y = \frac{c_2\left(x+\frac{1}{2}\right)^2 \ln\left(\sqrt{2x+1}-1\right) - c_2\left(x+\frac{1}{2}\right)^2 \ln\left(\sqrt{2x+1}+1\right) + c_2\left(x+\frac{2}{3}\right)\sqrt{2x+1} + 4\left(x+\frac{1}{2}\right)^2 c_1}{x^2\sqrt{2x+1}}$$

Mathematica DSolve solution

Solving time : 0.216 (sec)
Leaf size : 110

```
DSolve[{x^2*(1+2*x)*D[y[x],{x,2}]+x*(5+9*x)*D[y[x],x]+(4+3*x)*y[x]==0,{}},y[x],x,IncludeSing
```

$$y(x) \to \exp\left(\int_1^x \frac{7K[1]+1}{4K[1]^2+2K[1]}dK[1] - \frac{1}{2}\int_1^x \frac{9K[2]+5}{2K[2]^2+K[2]}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{7K[1]+1}{4K[1]^2+2K[1]}dK[1]\right)dK[3] + c_1\right)$$

2.1.137 Problem 139

Internal problem ID [9309]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 139
Date solved : `Monday, January 27, 2025 at 06:01:18 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1-2x)\,y''-x(5+4x)\,y'+(9+4x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.251 (sec)

Writing the ode as

$$\left(-2x^3+x^2\right)y''+\left(-4x^2-5x\right)y'+(9+4x)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -2x^3+x^2 \\ B &= -4x^2-5x \\ C &= 9+4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{32x^2+56x-1}{4\left(2x^2-x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 32x^2+56x-1 \\ t &= 4\left(2x^2-x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{32x^2+56x-1}{4\left(2x^2-x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.258: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{13}{x} - \frac{1}{4x^2} + \frac{35}{4\left(x - \frac{1}{2}\right)^2} - \frac{13}{x - \frac{1}{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{5}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{32x^2 + 56x - 1}{4\left(2x^2 - x\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{32x^2 + 56x - 1}{4\left(2x^2 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= -1 - (-2) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} - \frac{5}{2\left(x - \frac{1}{2}\right)} + (-)(0) \\
&= \frac{1}{2x} - \frac{5}{2\left(x - \frac{1}{2}\right)} \\
&= \frac{-1 - 8x}{4x^2 - 2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{5}{2\left(x-\frac{1}{2}\right)}\right)(1) + \left(\left(-\frac{1}{2x^2} + \frac{5}{2\left(x-\frac{1}{2}\right)^2}\right) + \left(\frac{1}{2x} - \frac{5}{2\left(x-\frac{1}{2}\right)}\right)^2 - \left(\frac{32x^2+56x-1}{4\left(2x^2-x\right)^2}\right)\right)$$
$$\frac{-1+8a_0}{x\left(-1+2x\right)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{1}{8}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{1}{8}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{1}{8}\right) e^{\int \left(\frac{1}{2x} - \frac{5}{2\left(x-\frac{1}{2}\right)}\right) dx} \\
&= \left(x + \frac{1}{8}\right) e^{-\frac{5\ln(-1+2x)}{2} + \frac{\ln(x)}{2}} \\
&= \frac{\left(x+\frac{1}{8}\right)\sqrt{x}}{\left(-1+2x\right)^{5/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x^2-5x}{-2x^3+x^2} \, dx} \\
&= z_1 e^{-\frac{7\ln(-1+2x)}{2} + \frac{5\ln(x)}{2}} \\
&= z_1 \left(\frac{x^{5/2}}{\left(-1+2x\right)^{7/2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^3\left(x+\frac{1}{8}\right)}{\left(-1+2x\right)^6}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^2-5x}{-2x^3+x^2} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{-7\ln(-1+2x)+5\ln(x)}}{\left(y_1\right)^2} \, dx \\
&= y_1 \left(\frac{32x^3}{3} - 44x^2 + \frac{203x}{2} - \frac{3125}{16\left(1+8x\right)} - 64\ln\left(x\right)\right)
\end{aligned}$$

Therefore the solution is

$$y = c_1y_1 + c_2y_2$$
$$= c_1\left(\frac{x^3\left(x+\frac{1}{8}\right)}{\left(-1+2x\right)^6}\right) + c_2\left(\frac{x^3\left(x+\frac{1}{8}\right)}{\left(-1+2x\right)^6}\left(\frac{32x^3}{3} - 44x^2 + \frac{203x}{2} - \frac{3125}{16\left(1+8x\right)} - 64\ln\left(x\right)\right)\right)$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2(-2x+1)\left(\frac{d^2}{dx^2}y(x)\right) - x(5+4x)\left(\frac{d}{dx}y(x)\right) + (4x+9)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = \frac{(4x+9)y(x)}{x^2(2x-1)} - \frac{(5+4x)\left(\frac{d}{dx}y(x)\right)}{x(2x-1)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) + \frac{(5+4x)\left(\frac{d}{dx}y(x)\right)}{x(2x-1)} - \frac{(4x+9)y(x)}{x^2(2x-1)} = 0$

□ Check to see if x_0 is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{5+4x}{x(2x-1)}, P_3(x) = -\frac{4x+9}{x^2(2x-1)}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -5$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 9$
  - $x = 0$is a regular singular point
    Check to see if x_0 is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $x^2(2x-1)\left(\frac{d^2}{dx^2}y(x)\right) + x(5+4x)\left(\frac{d}{dx}y(x)\right) + (-4x-9)\,y(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$
    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k-m$
    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$
    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k- >k+1-m$

$$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..3$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- > k+2-m$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0(-3+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r-3)^2 + 2a_{k-1}(k+1+r)\,(k-2+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-(-3+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 3$$

- Each term in the series must be 0, giving the recursion relation

$$-a_k(k+r-3)^2 + 2a_{k-1}(k+1+r)\,(k-2+r) = 0$$

- Shift index using $k- > k+1$

$$-a_{k+1}(k-2+r)^2 + 2a_k(k+r+2)\,(k+r-1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{2a_k(k+r+2)(k+r-1)}{(k-2+r)^2}$$

- Recursion relation for $r = 3$

$$a_{k+1} = \frac{2a_k(k+5)(k+2)}{(k+1)^2}$$

- Solution for $r = 3$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = \frac{2a_k(k+5)(k+2)}{(k+1)^2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 54

```
dsolve(x^2*(1-2*x)*diff(diff(y(x),x),x)-x*(5+4*x)*diff(y(x),x)+(4*x+9)*y(x) = 0,y(x),s
```

$$y = \frac{x^3\left(-6c_2\left(x+\frac{1}{8}\right)\ln(x) + c_2x^4 - 4c_2x^3 + 9c_2x^2 + \left(8c_1 + \frac{609c_2}{512}\right)x + c_1 - \frac{9375c_2}{4096}\right)}{(-1+2x)^6}$$

Mathematica DSolve solution

Solving time : 0.489 (sec)
Leaf size : 129

```
DSolve[{x^2*(1-2*x)*D[y[x],{x,2}]-x*(5+4*x)*D[y[x],x]+(9+4*x)*y[x]==0,{}},y[x],x,IncludeSingula
```

$$y(x) \to \frac{1}{8}(8x+1)\exp\left(\int_1^x \frac{8K[1]+1}{2K[1]-4K[1]^2}dK[1] - \frac{1}{2}\int_1^x -\frac{4K[2]+5}{K[2]-2K[2]^2}dK[2]\right)\left(c_2\int_1^x \frac{64\exp\left(-2\int_1^{K[3]}\frac{8K[1]+1}{2K[1]-4K[1]^2}dK[1]\right)}{(8K[3]+1)^2}dK[3]+c_1\right)$$

2.1.138 Problem 140

Internal problem ID [9310]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 140
Date solved : Monday, January 27, 2025 at 06:01:19 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$x^2(1-x)\,y''+x(7+x)\,y'+(9-x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.260 (sec)

Writing the ode as

$$\left(-x^3+x^2\right)y''+\left(x^2+7x\right)y'+(9-x)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^3+x^2 \\ B &= x^2+7x \\ C &= 9-x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-x^2+82x-1}{4\left(x^2-x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2+82x-1 \\ t &= 4\left(x^2-x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-x^2+82x-1}{4\left(x^2-x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.260: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{20}{-1+x} + \frac{20}{x} + \frac{20}{(-1+x)^2} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 20$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 5 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -4 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 + 82x - 1}{4\left(x^2 - x\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 + 82x - 1}{4\left(x^2 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
1	2	0	5	-4

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= \frac{1}{2} - \left(-\frac{7}{2}\right) \\
&= 4
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} - \frac{4}{-1 + x} + (-)(0) \\
&= \frac{1}{2x} - \frac{4}{-1 + x} \\
&= -\frac{1 + 7x}{2x(-1 + x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(12x^2 + 6xa_3 + 2a_2) + 2\left(\frac{1}{2x} - \frac{4}{-1+x}\right)(4x^3 + 3x^2a_3 + 2a_2x + a_1) + \left(\left(-\frac{1}{2x^2} + \frac{4}{(-1+x)^2}\right) + \left(\frac{1}{2x} - \right.\right.$$

$$\frac{(a_3 - 16)\, x^3 + (4a_2 - 9a_3)\, x^2}{x\,(-1}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1, a_1 = 16, a_2 = 36, a_3 = 16\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 + 16x^3 + 36x^2 + 16x + 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\, dx} \\
&= (x^4 + 16x^3 + 36x^2 + 16x + 1)\, e^{\int \left(\frac{1}{2x} - \frac{4}{-1+x}\right) dx} \\
&= (x^4 + 16x^3 + 36x^2 + 16x + 1)\, e^{-4\ln(-1+x) + \frac{\ln(x)}{2}} \\
&= \frac{(x^4 + 16x^3 + 36x^2 + 16x + 1)\sqrt{x}}{(-1+x)^4}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2+7x}{-x^3+x^2}\, dx} \\
&= z_1 e^{4\ln(-1+x) - \frac{7\ln(x)}{2}} \\
&= z_1 \left(\frac{(-1+x)^4}{x^{7/2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^4 + 16x^3 + 36x^2 + 16x + 1}{x^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+7x}{-x^3+x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{8\ln(-1+x) - 7\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\ln(x) - \frac{20\left(-2x^3 - \frac{15}{2}x^2 - \frac{14}{3}x - \frac{5}{12}\right)}{x^4 + 16x^3 + 36x^2 + 16x + 1}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^4+16x^3+36x^2+16x+1}{x^3}\right) \\
&\quad + c_2\left(\frac{x^4+16x^3+36x^2+16x+1}{x^3}\left(\ln(x) - \frac{20\left(-2x^3-\frac{15}{2}x^2-\frac{14}{3}x-\frac{5}{12}\right)}{x^4+16x^3+36x^2+16x+1}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(1-x)\left(\frac{d^2}{dx^2}y(x)\right)+x(7+x)\left(\frac{d}{dx}y(x)\right)+(9-x)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(x-9)y(x)}{x^2(x-1)}+\frac{(7+x)\left(\frac{d}{dx}y(x)\right)}{x(x-1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)-\frac{(7+x)\left(\frac{d}{dx}y(x)\right)}{x(x-1)}+\frac{(x-9)y(x)}{x^2(x-1)}=0$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x)=-\frac{7+x}{x(x-1)}, P_3(x)=\frac{x-9}{x^2(x-1)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=7$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=9$

  - $x=0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=0$

- Multiply by denominators

  $x^2(x-1)\left(\frac{d^2}{dx^2}y(x)\right)-x(7+x)\left(\frac{d}{dx}y(x)\right)+(x-9)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

    $x^m\cdot y(x)=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m\cdot y(x)=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..3$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$
- Shift index using $k->k+2-m$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0(3+r)^2\,x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r+3)^2 + a_{k-1}(k-2+r)^2\right)x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-(3+r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = -3$
- Each term in the series must be 0, giving the recursion relation

  $-a_k(k+r+3)^2 + a_{k-1}(k-2+r)^2 = 0$
- Shift index using $k->k+1$

  $-a_{k+1}(k+4+r)^2 + a_k(k+r-1)^2 = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r-1)^2}{(k+4+r)^2}$
- Recursion relation for $r = -3$; series terminates at $k = 4$

  $a_{k+1} = \frac{a_k(k-4)^2}{(k+1)^2}$
- Apply recursion relation for $k = 0$

  $a_1 = 16a_0$
- Apply recursion relation for $k = 1$

  $a_2 = \frac{9a_1}{4}$
- Express in terms of a_0

  $a_2 = 36a_0$
- Apply recursion relation for $k = 2$

  $a_3 = \frac{4a_2}{9}$
- Express in terms of a_0

  $a_3 = 16a_0$
- Apply recursion relation for $k = 3$

  $a_4 = \frac{a_3}{16}$
- Express in terms of a_0

  $a_4 = a_0$
- Terminating series solution of the ODE for $r = -3$. Use reduction of order to find the second lin

  $y(x) = a_0 \cdot (x^4 + 16x^3 + 36x^2 + 16x + 1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 72

```
dsolve(x^2*(1-x)*diff(diff(y(x),x),x)+x*(7+x)*diff(y(x),x)+(9-x)*y(x) = 0,y(x),singsol
```

$$y = \frac{3c_2(x^4+16x^3+36x^2+16x+1)\ln(x)+c_1x^4+(16c_1+120c_2)x^3+(36c_1+450c_2)x^2+(16c_1+280c_2)}{x^3}$$

Mathematica DSolve solution

Solving time : 0.751 (sec)
Leaf size : 145

```
DSolve[{x^2*(1-x)*D[y[x],{x,2}]+x*(7+x)*D[y[x],x]+(9-x)*y[x]==0,{}},y[x],x,IncludeSingularSo
```

$$y(x)\to (x^4+16x^3+36x^2+16x+1)\exp\left(\int_1^x\left(\frac{1}{2K[1]}-\frac{4}{K[1]-1}\right)dK[1] -\frac{1}{2}\int_1^x\frac{K[2]+7}{K[2]-K[2]^2}dK[2]\right)\left(c_2\int_1^x\frac{\exp\left(-2\int_1^{K[3]}\left(\frac{1}{2K[1]}-\frac{4}{K[1]-1}\right)dK[1]\right)}{(K[3]^4+16K[3]^3+36K[3]^2+16K[3]+1)^2}dK[3] + c_1\right)$$

2.1.139 Problem 141

Internal problem ID [9311]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 141
Date solved : `Monday, January 27, 2025 at 06:01:19 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x(-x^2+1)\,y' + (x^2+1)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.253 (sec)

Writing the ode as

$$x^2y'' + (x^3-x)\,y' + (x^2+1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^3 - x \\ C &= x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 4x^2 - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4 - 4x^2 - 1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 4x^2 - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.262: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{4} - 1 - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{1}{x} - \frac{5}{4x^3} - \frac{5}{2x^5} - \frac{105}{16x^7} - \frac{155}{8x^9} - \frac{1965}{32x^{11}} - \frac{3265}{16x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $([\sqrt{r}]_\infty)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$([\sqrt{r}]_\infty)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^4 - 4x^2 - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{x^2}{4} - 1\right) + \left(-\frac{1}{4x^2}\right) \\ &= \frac{x^2}{4} - 1 - \frac{1}{4x^2}\end{aligned}$$

We see that the coefficient of the term x in the quotient is -1. Now b can be found.

$$\begin{aligned}b &= (-1) - (0) \\ &= -1\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 1\right) &&= -\frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 1\right) &&= \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 4x^2 - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^+) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\left(\frac{x}{2}\right) \\ &= \frac{1}{2x} - \frac{x}{2} \\ &= \frac{1}{2x} - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2x} - \frac{x}{2}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{2}\right) + \left(\frac{1}{2x} - \frac{x}{2}\right)^2 - \left(\frac{x^4 - 4x^2 - 1}{4x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{1}{2x} - \frac{x}{2}\right) dx} \\ &= \sqrt{x}\, e^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^3-x}{x^2}\,dx} \\
&= z_1 e^{-\frac{x^2}{4}+\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\,\mathrm{e}^{-\frac{x^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{-\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^3-x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}+\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{\mathrm{Ei}_1\left(-\frac{x^2}{2}\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x\,\mathrm{e}^{-\frac{x^2}{2}}\right) + c_2\left(x\,\mathrm{e}^{-\frac{x^2}{2}}\left(-\frac{\mathrm{Ei}_1\left(-\frac{x^2}{2}\right)}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) - x(-x^2+1)\left(\frac{d}{dx}y(x)\right) + (x^2+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(x^2+1)y(x)}{x^2} - \frac{(x^2-1)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(x^2-1)\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(x^2+1)y(x)}{x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - ○ Define functions

$\left[P_2(x) = \frac{x^2-1}{x}, P_3(x) = \frac{x^2+1}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = -1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 1$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(x^2-1)\left(\frac{d}{dx}y(x)\right) + (x^2+1)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)^2\,x^r + a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)^2 + a_{k-2}(k+r-1)\right)x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 1$

- Each term must be 0

  $a_1 r^2 = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\,(a_k(k+r-1) + a_{k-2}) = 0$

- Shift index using $k- >k + 2$

  $(k+r+1)\,(a_{k+2}(k+r+1) + a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{k+r+1}$

- Recursion relation for $r = 1$
  $a_{k+2} = -\frac{a_k}{k+2}$
- Solution for $r = 1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{k+2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 23

```
dsolve(x^2*diff(diff(y(x),x),x)-x*(-x^2+1)*diff(y(x),x)+(x^2+1)*y(x) = 0,y(x),singsol=al
```

$$y = x\,\mathrm{e}^{-\frac{x^2}{2}}\left(c_1 + c_2\,\mathrm{Ei}_1\left(-\frac{x^2}{2}\right)\right)$$

Mathematica DSolve solution

Solving time : 0.071 (sec)
Leaf size : 35

```
DSolve[{x^2*D[y[x],{x,2}]-x*(1-x^2)*D[y[x],x]+(1+x^2)*y[x]==0,{}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to \frac{1}{2}e^{-\frac{x^2}{2}}x\left(c_1\,\mathrm{ExpIntegralEi}\left(\frac{x^2}{2}\right) + 2c_2\right)$$

2.1.140 Problem 142

Internal problem ID [9312]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 142
Date solved : `Monday, January 27, 2025 at 06:01:20 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(x^2+1)\,y''-3x(-x^2+1)\,y'+4y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.320 (sec)

Writing the ode as

$$(x^4+x^2)\,y''+(3x^3-3x)\,y'+4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2\\ B &= 3x^3-3x\\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t}\\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{3x^4-10x^2-1}{4\,(x^3+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3x^4-10x^2-1\\ t &= 4(x^3+x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{3x^4-10x^2-1}{4\,(x^3+x)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.264: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{3}{4\,(x-i)^2} + \frac{3}{4\,(x+i)^2} + \frac{i}{4x-4i} - \frac{i}{4\,(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x=-i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{2}\\
\alpha_c^- &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty=0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r=\frac{s}{t}=\frac{3x^4-10x^2-1}{4\left(x^3+x\right)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=\frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{2}\\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{3x^4-10x^2-1}{4\left(x^3+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=-\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right)\\
&= -\frac{1}{2}-\left(-\frac{1}{2}\right)\\
&= 0
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)} + (-)(0) \\
&= \frac{1}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)} \\
&= \frac{1}{2x} - \frac{x}{x^2+1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)}\right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{1}{2(x-i)^2} + \frac{1}{2(x+i)^2}\right) + \left(\frac{1}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x+
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)}\right) dx} \\
&= \frac{\sqrt{x}}{\sqrt{x^2+1}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x^3-3x}{x^4+x^2}\, dx} \\
&= z_1 e^{-\frac{3\ln(x^2+1)}{2} + \frac{3\ln(x)}{2}} \\
&= z_1 \left(\frac{x^{3/2}}{(x^2+1)^{3/2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^2}{(x^2+1)^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^3-3x}{x^4+x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-3\ln(x^2+1)+3\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{x^2}{2} + \ln(x)\right)
\end{aligned}
$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2\\ &= c_1\left(\frac{x^2}{(x^2+1)^2}\right) + c_2\left(\frac{x^2}{(x^2+1)^2}\left(\frac{x^2}{2} + \ln(x)\right)\right)\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) - 3x(-x^2+1)\left(\frac{d}{dx}y(x)\right) + 4y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{4y(x)}{x^2(x^2+1)} - \frac{3(x^2-1)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{3(x^2-1)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)} + \frac{4y(x)}{x^2(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{3(x^2-1)}{x(x^2+1)}, P_3(x) = \frac{4}{x^2(x^2+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -3$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 4$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + 3x(x^2-1)\left(\frac{d}{dx}y(x)\right) + 4y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  ◦ Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$

  ◦ Shift index using $k- >k+2-m$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-2+r)^2\,x^r + a_1(-1+r)^2\,x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-2)^2 + a_{k-2}(k+r-2)\,(k+r)\right)x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 2$
- Each term must be 0

  $a_1(-1+r)^2 = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)\,(a_k(k+r-2) + a_{k-2}(k+r)) = 0$
- Shift index using $k- >k+2$

  $(k+r)\,(a_{k+2}(k+r) + a_k(k+r+2)) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r+2)}{k+r}$
- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{a_k(k+4)}{k+2}$
- Solution for $r = 2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{a_k(k+4)}{k+2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 27

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)-3*x*(-x^2+1)*diff(y(x),x)+4*y(x) = 0,y(x),singso
```

$$y = \frac{x^2\left(c_1 + c_2\left(\frac{x^2}{2} + \ln(x)\right)\right)}{(x^2+1)^2}$$

Mathematica DSolve solution

Solving time : 0.236 (sec)
Leaf size : 107

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]-3*x*(1-x^2)*D[y[x],x]+4*y[x]==0,{}},y[x],x,IncludeSingular
```

$$y(x) \to \exp\left(\int_1^x -\frac{K[1]^2-1}{2\left(K[1]^3+K[1]\right)}dK[1] - \frac{1}{2}\int_1^x \frac{3(K[2]^2-1)}{K[2]^3+K[2]}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} -\frac{K[1]^2-1}{2\left(K[1]^3+K[1]\right)}dK[1]\right)dK[3]+c_1\right)$$

2.1.141 Problem 143

Internal problem ID [9313]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 143
Date solved : `Monday, January 27, 2025 at 06:01:21 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' + 2x^3y' + (3x^2+1)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.236 (sec)

Writing the ode as

$$4x^2y'' + 2x^3y' + (3x^2+1)\, y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= 2x^3 \\ C &= 3x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 8x^2 - 4}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4 - 8x^2 - 4 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 8x^2 - 4}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.266: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{16} - \frac{1}{2} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{4} - \frac{1}{x} - \frac{5}{2x^3} - \frac{10}{x^5} - \frac{105}{2x^7} - \frac{310}{x^9} - \frac{1965}{x^{11}} - \frac{13060}{x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{4} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{16}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^4 - 8x^2 - 4}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{x^2}{16} - \frac{1}{2}\right) + \left(-\frac{1}{4x^2}\right) \\ &= \frac{x^2}{16} - \frac{1}{2} - \frac{1}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{4} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{4}} - 1\right) &&= -\frac{3}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{4}} - 1\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 8x^2 - 4}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{4}$	$-\frac{3}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\left(\frac{x}{4}\right) \\ &= \frac{1}{2x} - \frac{x}{4} \\ &= \frac{1}{2x} - \frac{x}{4} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2x} - \frac{x}{4}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{4}\right) + \left(\frac{1}{2x} - \frac{x}{4}\right)^2 - \left(\frac{x^4 - 8x^2 - 4}{16x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{1}{2x} - \frac{x}{4}\right) dx} \\ &= \sqrt{x}\, e^{-\frac{x^2}{8}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x^3}{4x^2}\,dx} \\ &= z_1 e^{-\frac{x^2}{8}} \\ &= z_1\left(\mathrm{e}^{-\frac{x^2}{8}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^2}{4}}\sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3}{4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{4}}}{(y_1)^2}\,dx \\ &= y_1\left(-\frac{\mathrm{Ei}_1\left(-\frac{x^2}{4}\right)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-\frac{x^2}{4}}\sqrt{x}\right) + c_2\left(\mathrm{e}^{-\frac{x^2}{4}}\sqrt{x}\left(-\frac{\mathrm{Ei}_1\left(-\frac{x^2}{4}\right)}{2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2\left(\frac{d^2}{dx^2}y(x)\right) + 2\left(\frac{d}{dx}y(x)\right)x^3 + (3x^2+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(3x^2+1)y(x)}{4x^2} - \frac{x\left(\frac{d}{dx}y(x)\right)}{2}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{x\left(\frac{d}{dx}y(x)\right)}{2} + \frac{(3x^2+1)y(x)}{4x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - ○ Define functions

$\left[P_2(x) = \frac{x}{2}, P_3(x) = \frac{3x^2+1}{4x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 0$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{1}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) + 2\left(\frac{d}{dx}y(x)\right)x^3 + (3x^2+1)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^3 \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x^3 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r+2}$

- Shift index using $k- >k - 2$

  $x^3 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=2}^{\infty} a_{k-2}(k-2+r)\,x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)^2 x^r + a_1(1+2r)^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r-1)^2 + a_{k-2}(2k+2r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = \frac{1}{2}$

- Each term must be 0

  $a_1(1+2r)^2 = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(2k+2r-1)^2 + a_{k-2}(2k+2r-1) = 0$

- Shift index using $k- >k + 2$

  $a_{k+2}(2k+2r+3)^2 + a_k(2k+2r+3) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{2k+2r+3}$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{a_k}{2k+4}$
- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{a_k}{2k+4}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 25

```
dsolve(4*x^2*diff(diff(y(x),x),x)+2*x^3*diff(y(x),x)+(3*x^2+1)*y(x) = 0,y(x),singsol=all
```

$$y = \sqrt{x}\, e^{-\frac{x^2}{4}} \left(c_1 + c_2 \operatorname{Ei}_1\left(-\frac{x^2}{4}\right)\right)$$

Mathematica DSolve solution

Solving time : 0.193 (sec)
Leaf size : 44

```
DSolve[{4*x^2*D[y[x],{x,2}]+2*x^3*D[y[x],x]+(1+3*x^2)*y[x]==0,{}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to \frac{1}{2} e^{-\frac{x^2}{4}-\frac{1}{2}} \sqrt{x} \left(c_2 \operatorname{ExpIntegralEi}\left(\frac{x^2}{4}\right) + 2ec_1\right)$$

2.1.142 Problem 144

Internal problem ID [9314]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 144
Date solved : `Monday, January 27, 2025 at 06:01:21 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(x^2+1\right) y''-x\left(-2x^2+1\right) y'+y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.241 (sec)

Writing the ode as

$$\left(x^4+x^2\right) y''+\left(2x^3-x\right) y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2 \\ B &= 2x^3-x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{2x^2-1}{4\left(x^3+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2x^2-1 \\ t &= 4\left(x^3+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{2x^2-1}{4\left(x^3+x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.268: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16\,(x-i)^2} - \frac{3}{16\,(x+i)^2} - \frac{5i}{16\,(x-i)} + \frac{5i}{16\,(x+i)} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{2x^2 - 1}{4\left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
i	2	0	$\frac{3}{4}$	$\frac{1}{4}$
$-i$	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x - c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \frac{1}{4x - 4i} + \frac{1}{4x + 4i} + (-)(0) \\
&= \frac{1}{2x} + \frac{1}{4x - 4i} + \frac{1}{4x + 4i} \\
&= \frac{1}{2x} + \frac{x}{2x^2 + 2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{1}{4x-4i} + \frac{1}{4x+4i}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{4\,(x-i)^2} - \frac{1}{4\,(x+i)^2}\right) + \left(\frac{1}{2x} + \frac{1}{4x-4i} + \frac{1}{4x+4i}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{1}{2x} + \frac{1}{4x-4i} + \frac{1}{4x+4i}\right) dx} \\ &= \sqrt{x}\,(x^2+1)^{1/4} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x^3-x}{x^4+x^2}\, dx} \\ &= z_1 e^{\frac{\ln(x)}{2} - \frac{3\ln(x^2+1)}{4}} \\ &= z_1 \left(\frac{\sqrt{x}}{(x^2+1)^{3/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x}{\sqrt{x^2+1}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3-x}{x^4+x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\ln(x) - \frac{3\ln(x^2+1)}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(-\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{x}{\sqrt{x^2+1}}\right) + c_2\left(\frac{x}{\sqrt{x^2+1}}\left(-\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)-x(-2x^2+1)\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{y(x)}{x^2(x^2+1)}-\frac{(2x^2-1)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)+\frac{(2x^2-1)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}+\frac{y(x)}{x^2(x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=\frac{2x^2-1}{x(x^2+1)}, P_3(x)=\frac{1}{x^2(x^2+1)}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

$(x\cdot P_2(x))\Big|_{x=0}=-1$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$(x^2\cdot P_3(x))\Big|_{x=0}=1$

- ◦ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(2x^2-1)\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ◦ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..4$

$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- ◦ Shift index using $k->k+2-m$

$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

Rewrite ODE with series expansions

$a_0(-1+r)^2x^r+a_1r^2x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_k(k+r-1)^2+a_{k-2}(k-2+r)\,(k+r-1)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(-1+r)^2=0$

- Values of r that satisfy the indicial equation

$r = 1$

- Each term must be 0

  $a_1 r^2 = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $(k + r - 1)(a_k(k + r - 1) + a_{k-2}(k - 2 + r)) = 0$
- Shift index using $k- >k + 2$

  $(k + r + 1)(a_{k+2}(k + r + 1) + a_k(k + r)) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r)}{k+r+1}$
- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{a_k(k+1)}{k+2}$
- Solution for $r = 1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k(k+1)}{k+2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.022` (sec)
Leaf size : 25

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)-x*(-2*x^2+1)*diff(y(x),x)+y(x) = 0,y(x),singsol=
```

$$y = \frac{x\left(\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) c_2 + c_1\right)}{\sqrt{x^2 + 1}}$$

Mathematica DSolve solution

Solving time : 0.21 (sec)
Leaf size : 112

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]-x*(1-2*x^2)*D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSo
```

$$y(x) \to \exp\left(\int_1^x \frac{2K[1]^2+1}{2\left(K[1]^3+K[1]\right)}dK[1] - \frac{1}{2}\int_1^x \frac{2K[2]^2-1}{K[2]^3+K[2]}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{2K[1]^2+1}{2\left(K[1]^3+K[1]\right)}dK[1]\right)dK[3] + c_1\right)$$

2.1.143 Problem 145

Internal problem ID [9315]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 145
Date solved : Monday, January 27, 2025 at 06:01:22 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$2x^2\left(x^2+2\right)y''+7x^3y'+\left(3x^2+1\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.303 (sec)

Writing the ode as

$$\left(2x^4+4x^2\right)y''+7x^3y'+\left(3x^2+1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x^4+4x^2\\B&=7x^3\\C&=3x^2+1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-3x^4-16}{16\left(x^3+2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=-3x^4-16\\t&=16\left(x^3+2x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-3x^4-16}{16\left(x^3+2x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.270: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^3 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i\sqrt{2}$ of order 2. There is a pole at $x = -i\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{64\left(x - i\sqrt{2}\right)^2} - \frac{7}{64\left(x + i\sqrt{2}\right)^2} - \frac{9i\sqrt{2}}{128\left(x - i\sqrt{2}\right)} + \frac{9i\sqrt{2}}{128\left(x + i\sqrt{2}\right)} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x-i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}
\end{aligned}
$$

For the pole at $x = -i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x+i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{-3x^4 - 16}{16\,(x^3 + 2x)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-3x^4 - 16}{16\,(x^3 + 2x)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$i\sqrt{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$
$-i\sqrt{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{3}{4} - \left(\frac{3}{4}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x - c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \frac{1}{8x - 8i\sqrt{2}} + \frac{1}{8x + 8i\sqrt{2}} + (0) \\
&= \frac{1}{2x} + \frac{1}{8x - 8i\sqrt{2}} + \frac{1}{8x + 8i\sqrt{2}} \\
&= \frac{1}{2x} + \frac{x}{4x^2 + 8}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{1}{8x - 8i\sqrt{2}} + \frac{1}{8x + 8i\sqrt{2}}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{8\left(x - i\sqrt{2}\right)^2} - \frac{1}{8\left(x + i\sqrt{2}\right)^2}\right) + \left(\frac{1}{2x} + \frac{}{8x}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{2x} + \frac{1}{8x - 8i\sqrt{2}} + \frac{1}{8x + 8i\sqrt{2}}\right) dx} \\
&= \left(x^2 + 2\right)^{1/8} \sqrt{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{7x^3}{2x^4 + 4x^2} \, dx} \\
&= z_1 e^{-\frac{7 \ln\left(x^2 + 2\right)}{8}} \\
&= z_1 \left(\frac{1}{\left(x^2 + 2\right)^{7/8}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{\left(x^2 + 2\right)^{3/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x^3}{2x^4+4x^2} dx}}{(y_1)^2} dx \\
&= y_1 \int \frac{e^{-\frac{7\ln(x^2+2)}{4}}}{(y_1)^2} dx \\
&= y_1 \left(\int \frac{1}{(x^2+2)^{1/4} x} dx \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x}}{(x^2+2)^{3/4}} \right) + c_2 \left(\frac{\sqrt{x}}{(x^2+2)^{3/4}} \left(\int \frac{1}{(x^2+2)^{1/4} x} dx \right) \right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2(x^2+2)\left(\frac{d^2}{dx^2}y(x)\right) + 7\left(\frac{d}{dx}y(x)\right)x^3 + (3x^2+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(3x^2+1)y(x)}{2(x^2+2)x^2} - \frac{7\left(\frac{d}{dx}y(x)\right)x}{2(x^2+2)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{7\left(\frac{d}{dx}y(x)\right)x}{2(x^2+2)} + \frac{(3x^2+1)y(x)}{2(x^2+2)x^2} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{7x}{2(x^2+2)}, P_3(x) = \frac{3x^2+1}{2(x^2+2)x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 0$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{1}{4}$
  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $2x^2(x^2+2)\left(\frac{d^2}{dx^2}y(x)\right) + 7\left(\frac{d}{dx}y(x)\right)x^3 + (3x^2+1)\,y(x) = 0$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

$x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^3 \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x^3 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r+2}$

- Shift index using $k- >k-2$

  $x^3 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=2}^{\infty} a_{k-2}(k-2+r)\, x^{k+r}$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)^2\, x^r + a_1(1+2r)^2\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r-1)^2 + a_{k-2}(2k+2r-1)\,(k+r-1)\right) x\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = \frac{1}{2}$

- Each term must be 0

  $a_1(1+2r)^2 = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r-\frac{1}{2}\right)\left(\frac{a_{k-2}(k+r-1)}{2} + \left(k+r-\frac{1}{2}\right) a_k\right) = 0$

- Shift index using $k- >k+2$

  $4\left(k+\frac{3}{2}+r\right)\left(\frac{a_k(k+r+1)}{2} + \left(k+\frac{3}{2}+r\right) a_{k+2}\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r+1)}{2k+2r+3}$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{a_k\left(k+\frac{3}{2}\right)}{2k+4}$

- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{a_k\left(k+\frac{3}{2}\right)}{2k+4}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
         -> solution has integrals; searching for one without integrals...
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
            <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
         <- hypergeometric solution without integrals succesful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returnin
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.219 (sec)
Leaf size : 81

```
dsolve(2*x^2*(x^2+2)*diff(diff(y(x),x),x)+7*x^3*diff(y(x),x)+(3*x^2+1)*y(x) = 0,y(x),sin
```

$$y = \frac{\sqrt{x}\left(2^{3/4}c_1 + 2\arctan\left(\frac{\sqrt{2}\,(2x^2+4)^{1/4}}{2}\right)c_2 + \ln\left(-\sqrt{2}\,(2x^2+4)^{1/4}+2\right)c_2 - \ln\left(\sqrt{2}\,(2x^2+4)^{1/4}+2\right)c_2\right)2}{2\,(x^2+2)^{3/4}}$$

Mathematica DSolve solution

Solving time : 0.411 (sec)
Leaf size : 93

```
DSolve[{2*x^2*(2+x^2)*D[y[x],{x,2}]+7*x^3*D[y[x],x]+(1+3*x^2)*y[x]==0,{}},y[x],x,IncludeSingula
```

$$y(x) \to \frac{\exp\left(\int_1^x \frac{3K[1]^2+4}{4K[1]^3+8K[1]}dK[1]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[2]} \frac{3K[1]^2+4}{4K[1]^3+8K[1]}dK[1]\right)dK[2]+c_1\right)}{(x^2+2)^{7/8}}$$

2.1.144 Problem 146

Internal problem ID [9316]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 146
Date solved : `Monday, January 27, 2025 at 06:01:23 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(x^2+1\right)y''-x\left(-4x^2+1\right)y'+\left(2x^2+1\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.277 (sec)

Writing the ode as

$$\left(x^4+x^2\right)y''+\left(4x^3-x\right)y'+\left(2x^2+1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2 \\ B &= 4x^3-x \\ C &= 2x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-6x^2-1}{4\left(x^3+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -6x^2-1 \\ t &= 4\left(x^3+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-6x^2-1}{4\left(x^3+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.272: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16\,(x-i)^2} + \frac{5}{16\,(x+i)^2} + \frac{3i}{16\,(x-i)} - \frac{3i}{16\,(x+i)} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

For the pole at $x=-i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{5}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{4}\\ \alpha_c^- &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{4}\end{aligned}$$

Since the order of r at ∞ is $4>2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= 0\\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{-6x^2-1}{4\left(x^3+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
i	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-i$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=0$ then

$$\begin{aligned}d &= \alpha_\infty^+-\left(\alpha_{c_1}^++\alpha_{c_2}^-+\alpha_{c_3}^-\right)\\ &= 0-(0)\\ &= 0\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^+}{x-c_1}\right)+\left((-)[\sqrt{r}]_{c_2}+\frac{\alpha_{c_2}^-}{x-c_2}\right)+\left((-)[\sqrt{r}]_{c_3}+\frac{\alpha_{c_3}^-}{x-c_3}\right)+(+)[\sqrt{r}]_\infty\\ &= \frac{1}{2x}-\frac{1}{4\left(x-i\right)}-\frac{1}{4\left(x+i\right)}+(0)\\ &= \frac{1}{2x}-\frac{1}{4\left(x-i\right)}-\frac{1}{4\left(x+i\right)}\\ &= \frac{1}{2x^3+2x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{4(x-i)} - \frac{1}{4(x+i)}\right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{1}{4(x-i)^2} + \frac{1}{4(x+i)^2}\right) + \left(\frac{1}{2x} - \frac{1}{4(x-i)} - \frac{1}{4(x+\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{1}{2x} - \frac{1}{4(x-i)} - \frac{1}{4(x+i)}\right) dx} \\ &= \frac{\sqrt{x}}{\left(x^2+1\right)^{1/4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{4x^3 - x}{x^4 + x^2} \, dx} \\ &= z_1 e^{-\frac{5\ln\left(x^2+1\right)}{4} + \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{\sqrt{x}}{\left(x^2+1\right)^{5/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x}{\left(x^2+1\right)^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x^3 - x}{x^4 + x^2} \, dx}}{\left(y_1\right)^2} \, dx \\ &= y_1 \int \frac{e^{-\frac{5\ln\left(x^2+1\right)}{2} + \ln(x)}}{\left(y_1\right)^2} \, dx \\ &= y_1 \left(\sqrt{x^2+1} - \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{x}{\left(x^2+1\right)^{3/2}}\right) + c_2 \left(\frac{x}{\left(x^2+1\right)^{3/2}} \left(\sqrt{x^2+1} - \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) - x(-4x^2+1)\left(\frac{d}{dx}y(x)\right) + (2x^2+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(2x^2+1)y(x)}{x^2(x^2+1)} - \frac{(4x^2-1)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(4x^2-1)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)} + \frac{(2x^2+1)y(x)}{x^2(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{4x^2-1}{x(x^2+1)}, P_3(x) = \frac{2x^2+1}{x^2(x^2+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 1$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + x(4x^2-1)\left(\frac{d}{dx}y(x)\right) + (2x^2+1)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  ◦ Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

  ◦ Shift index using $k- >k+2-m$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

$$a_0(-1+r)^2 x^r + a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)^2 + a_{k-2}(k+r)(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 1$
- Each term must be 0
  $a_1 r^2 = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-1)\left(a_k(k+r-1) + a_{k-2}(k+r)\right) = 0$
- Shift index using $k->k+2$
  $(k+r+1)\left(a_{k+2}(k+r+1) + a_k(k+r+2)\right) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{a_k(k+r+2)}{k+r+1}$
- Recursion relation for $r = 1$
  $a_{k+2} = -\frac{a_k(k+3)}{k+2}$
- Solution for $r = 1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k(k+3)}{k+2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.056` (sec)
Leaf size : 35

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)-x*(-4*x^2+1)*diff(y(x),x)+(2*x^2+1)*y(x) = 0,y(x
```

$$y = \frac{x\left(\sqrt{x^2+1}\, c_2 - \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) c_2 + c_1\right)}{(x^2+1)^{3/2}}$$

Mathematica DSolve solution

Solving time : 0.157 (sec)
Leaf size : 96

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]-x*(1-4*x^2)*D[y[x],x]+(1+2*x^2)*y[x]==0,{}},y[x],x,Include
```

$$y(x) \to \exp\left(\int_1^x \frac{1}{2K[1]^3+2K[1]}dK[1] - \frac{1}{2}\int_1^x \frac{4K[2]^2-1}{K[2]^3+K[2]}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{1}{2K[1]^3+2K[1]}dK[1]\right)dK[3] + c_1\right)$$

2.1.145 Problem 147

Internal problem ID [9317]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 147
Date solved : `Monday, January 27, 2025 at 06:01:24 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2\left(x^2+4\right)y''+3x\left(3x^2+8\right)y'+\left(-9x^2+1\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.327 (sec)

Writing the ode as

$$\left(4x^4+16x^2\right)y''+\left(9x^3+24x\right)y'+\left(-9x^2+1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^4+16x^2 \\ B &= 9x^3+24x \\ C &= -9x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{153x^4+704x^2-256}{64\left(x^3+4x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 153x^4+704x^2-256 \\ t &= 64\left(x^3+4x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{153x^4+704x^2-256}{64\left(x^3+4x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.274: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64(x^3+4x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 2i$ of order 2. There is a pole at $x = -2i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{39}{256\,(x-2i)^2} - \frac{39}{256\,(x+2i)^2} - \frac{377i}{512\,(x-2i)} + \frac{377i}{512\,(x+2i)} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = 2i$ let b be the coefficient of $\frac{1}{(x-2i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{39}{256}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{13}{16} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{16} \end{aligned}$$

For the pole at $x = -2i$ let b be the coefficient of $\frac{1}{(x+2i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{39}{256}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{13}{16}\\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{16}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{153x^4 + 704x^2 - 256}{64\,(x^3+4x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{153}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{17}{8}\\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{9}{8}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{153x^4 + 704x^2 - 256}{64\,(x^3+4x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$2i$	2	0	$\frac{13}{16}$	$\frac{3}{16}$
$-2i$	2	0	$\frac{13}{16}$	$\frac{3}{16}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{17}{8}$	$-\frac{9}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{17}{8}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+ + \alpha_{c_3}^+\right)\\
&= \frac{17}{8} - \left(\frac{17}{8}\right)\\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{+}}{x-c_3}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + \frac{13}{16\,(x-2i)} + \frac{13}{16\,(x+2i)} + (0) \\
&= \frac{1}{2x} + \frac{13}{16\,(x-2i)} + \frac{13}{16\,(x+2i)} \\
&= \frac{1}{2x} + \frac{13x}{8x^2+32}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{13}{16\,(x-2i)} + \frac{13}{16\,(x+2i)}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{13}{16\,(x-2i)^2} - \frac{13}{16\,(x+2i)^2}\right) + \left(\frac{1}{2x} + \frac{1}{16\,(x}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} + \frac{13}{16(x-2i)} + \frac{13}{16(x+2i)}\right) dx} \\
&= \left(x^2+4\right)^{13/16} \sqrt{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{9x^3+24x}{4x^4+16x^2}\, dx} \\
&= z_1 e^{-\frac{3\ln(x)}{4} - \frac{3\ln\left(x^2+4\right)}{16}} \\
&= z_1 \left(\frac{1}{x^{3/4}\left(x^2+4\right)^{3/16}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(x^2+4\right)^{5/8}}{x^{1/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{9x^3+24x}{4x^4+16x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{\mathrm{e}^{-\frac{3\ln(x)}{2} - \frac{3\ln\left(x^2+4\right)}{8}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{3\ln(x)}{2} - \frac{3\ln\left(x^2+4\right)}{8}}\sqrt{x}}{\left(x^2+4\right)^{5/4}} dx\right)
\end{aligned}$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{(x^2+4)^{5/8}}{x^{1/4}}\right) + c_2 \left(\frac{(x^2+4)^{5/8}}{x^{1/4}} \left(\int \frac{e^{-\frac{3\ln(x)}{2} - \frac{3\ln(x^2+4)}{8}} \sqrt{x}}{(x^2+4)^{5/4}} dx\right)\right)$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$4x^2(x^2+4)\left(\frac{d^2}{dx^2}y(x)\right) + 3x(3x^2+8)\left(\frac{d}{dx}y(x)\right) + (-9x^2+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = \frac{(9x^2-1)y(x)}{4x^2(x^2+4)} - \frac{3(3x^2+8)\left(\frac{d}{dx}y(x)\right)}{4x(x^2+4)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) + \frac{3(3x^2+8)\left(\frac{d}{dx}y(x)\right)}{4x(x^2+4)} - \frac{(9x^2-1)y(x)}{4x^2(x^2+4)} = 0$

□ Check to see if x_0 is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{3(3x^2+8)}{4x(x^2+4)}, P_3(x) = -\frac{9x^2-1}{4x^2(x^2+4)}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $(x \cdot P_2(x))\bigg|_{x=0} = \frac{3}{2}$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{1}{16}$
  - $x = 0$is a regular singular point
    Check to see if x_0 is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $4x^2(x^2+4)\left(\frac{d^2}{dx^2}y(x)\right) + 3x(3x^2+8)\left(\frac{d}{dx}y(x)\right) + (-9x^2+1)\,y(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$
    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k-m$
    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$
    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k- >k+1-m$

$$x^m \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- ◦ Convert $x^m \cdot \left(\frac{d^2}{dx^2} y(x)\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d^2}{dx^2} y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- ◦ Shift index using $k- >k+2-m$

$$x^m \cdot \left(\frac{d^2}{dx^2} y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+4r)^2\, x^r + a_1(5+4r)^2\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(4k+4r+1)^2 + a_{k-2}(4k+4r+1)\,(k-3+r)\right) x^{k+}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+4r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = -\frac{1}{4}$$

- Each term must be 0

$$a_1(5+4r)^2 = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$16\left(\frac{a_{k-2}(k-3+r)}{4} + a_k\left(k+r+\frac{1}{4}\right)\right)\left(k+r+\frac{1}{4}\right) = 0$$

- Shift index using $k- >k+2$

$$16\left(\frac{a_k(k+r-1)}{4} + a_{k+2}\left(k+\frac{9}{4}+r\right)\right)\left(k+\frac{9}{4}+r\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k(k+r-1)}{4k+4r+9}$$

- Recursion relation for $r = -\frac{1}{4}$

$$a_{k+2} = -\frac{a_k\left(k-\frac{5}{4}\right)}{4k+8}$$

- Solution for $r = -\frac{1}{4}$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{4}}, a_{k+2} = -\frac{a_k\left(k-\frac{5}{4}\right)}{4k+8}, a_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
```

```
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.114 (sec)
Leaf size : 66

```
dsolve(4*x^2*(x^2+4)*diff(diff(y(x),x),x)+3*x*(3*x^2+8)*diff(y(x),x)+(-9*x^2+1)*y(x) = 0
```

$$y = \frac{c_2(x^2+4)^{5/8}\left(x^2 \,\text{hypergeom}\left(\left[1,1,\frac{13}{8}\right],[2,2],-\frac{x^2}{4}\right)-\frac{32\gamma}{5}+\frac{64\ln(2)}{5}-\frac{64\ln(x)}{5}-\frac{32\Psi\left(\frac{5}{8}\right)}{5}\right)2^{3/4}+c_1(x^2+4)^{5/}}{x^{1/4}}$$

Mathematica DSolve solution

Solving time : 0.273 (sec)
Leaf size : 118

```
DSolve[{4*x^2*(4+x^2)*D[y[x],{x,2}]+3*x*(8+3*x^2)*D[y[x],x]+(1-9*x^2)*y[x]==0,{}},y[x],x,Inclu
```

$$y(x)\to \exp\left(\int_1^x \frac{17K[1]^2+16}{8K[1]^3+32K[1]}dK[1] -\frac{1}{2}\int_1^x \frac{9K[2]^2+24}{4K[2]^3+16K[2]}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]}\frac{17K[1]^2+16}{8K[1]^3+32K[1]}dK[1]\right)dK[3] + c_1\right)$$

2.1.146 Problem 148

Internal problem ID [9318]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 148
Date solved : `Monday, January 27, 2025 at 06:01:24 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$3x^2\left(x^2+3\right)y''+x\left(11x^2+3\right)y'+\left(5x^2+1\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.318 (sec)

Writing the ode as

$$\left(3x^4+9x^2\right)y''+\left(11x^3+3x\right)y'+\left(5x^2+1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=3x^4+9x^2\\ B&=11x^3+3x\\ C&=5x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-5x^4+18x^2-81}{36\left(x^3+3x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=-5x^4+18x^2-81\\ t&=36\left(x^3+3x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-5x^4+18x^2-81}{36\left(x^3+3x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.276: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^3 + 3x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i\sqrt{3}$ of order 2. There is a pole at $x = -i\sqrt{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} - \frac{5}{36\left(x - i\sqrt{3}\right)^2} - \frac{5}{36\left(x + i\sqrt{3}\right)^2} - \frac{7i\sqrt{3}}{108\left(x - i\sqrt{3}\right)} + \frac{7i\sqrt{3}}{108\left(x + i\sqrt{3}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = i\sqrt{3}$ let b be the coefficient of $\frac{1}{\left(x-i\sqrt{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6}\\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}
\end{aligned}$$

For the pole at $x = -i\sqrt{3}$ let b be the coefficient of $\frac{1}{\left(x+i\sqrt{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6}\\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-5x^4 + 18x^2 - 81}{36\left(x^3 + 3x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{5}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6}\\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-5x^4 + 18x^2 - 81}{36\left(x^3 + 3x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$i\sqrt{3}$	2	0	$\frac{5}{6}$	$\frac{1}{6}$
$-i\sqrt{3}$	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{6}$	$\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{6}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right)\\
&= \frac{5}{6} - \left(\frac{5}{6}\right)\\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \frac{1}{6x-6i\sqrt{3}} + \frac{1}{6x+6i\sqrt{3}} + (0) \\
&= \frac{1}{2x} + \frac{1}{6x-6i\sqrt{3}} + \frac{1}{6x+6i\sqrt{3}} \\
&= \frac{1}{2x} + \frac{x}{3x^2+9}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{1}{6x-6i\sqrt{3}} + \frac{1}{6x+6i\sqrt{3}}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{6\left(x-i\sqrt{3}\right)^2} - \frac{1}{6\left(x+i\sqrt{3}\right)^2}\right) + \left(\frac{1}{2x} + \frac{1}{6x-\cdots}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(\frac{1}{2x} + \frac{1}{6x-6i\sqrt{3}} + \frac{1}{6x+6i\sqrt{3}}\right)dx} \\
&= \sqrt{x}\left(x^2+3\right)^{1/6}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{11x^3+3x}{3x^4+9x^2}\, dx} \\
&= z_1 e^{-\frac{\ln(x)}{6} - \frac{5\ln\left(x^2+3\right)}{6}} \\
&= z_1\left(\frac{1}{x^{1/6}\left(x^2+3\right)^{5/6}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/3}}{\left(x^2+3\right)^{2/3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{11x^3+3x}{3x^4+9x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x)}{3}-\frac{5\ln(x^2+3)}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{\ln(x)}{3}-\frac{5\ln(x^2+3)}{3}}(x^2+3)^{4/3}}{x^{2/3}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{1/3}}{(x^2+3)^{2/3}}\right) + c_2 \left(\frac{x^{1/3}}{(x^2+3)^{2/3}}\left(\int \frac{\mathrm{e}^{-\frac{\ln(x)}{3}-\frac{5\ln(x^2+3)}{3}}(x^2+3)^{4/3}}{x^{2/3}}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$3x^2(x^2+3)\left(\frac{d^2}{dx^2}y(x)\right) + x(11x^2+3)\left(\frac{d}{dx}y(x)\right) + (5x^2+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(5x^2+1)y(x)}{3x^2(x^2+3)} - \frac{(11x^2+3)\left(\frac{d}{dx}y(x)\right)}{3x(x^2+3)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(11x^2+3)\left(\frac{d}{dx}y(x)\right)}{3x(x^2+3)} + \frac{(5x^2+1)y(x)}{3x^2(x^2+3)} = 0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{11x^2+3}{3x(x^2+3)}, P_3(x) = \frac{5x^2+1}{3x^2(x^2+3)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = \frac{1}{3}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{1}{9}$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $3x^2(x^2+3)\left(\frac{d^2}{dx^2}y(x)\right) + x(11x^2+3)\left(\frac{d}{dx}y(x)\right) + (5x^2+1)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
- Shift index using $k- >k + 1 - m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$
- Shift index using $k- >k + 2 - m$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+3r)^2 x^r + a_1(2+3r)^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r-1)^2 + a_{k-2}(3k+3r-1)\,(k+r-1)\right) x^{k+r}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+3r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = \frac{1}{3}$
- Each term must be 0

  $a_1(2+3r)^2 = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $9\left(\frac{a_{k-2}(k+r-1)}{3} + \left(k - \frac{1}{3} + r\right) a_k\right)\left(k - \frac{1}{3} + r\right) = 0$
- Shift index using $k- >k + 2$

  $9\left(\frac{a_k(k+r+1)}{3} + \left(k + \frac{5}{3} + r\right) a_{k+2}\right)\left(k + \frac{5}{3} + r\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r+1)}{3k+3r+5}$
- Recursion relation for $r = \frac{1}{3}$

  $a_{k+2} = -\frac{a_k\left(k+\frac{4}{3}\right)}{3k+6}$
- Solution for $r = \frac{1}{3}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -\frac{a_k\left(k+\frac{4}{3}\right)}{3k+6}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
         -> solution has integrals; searching for one without integrals...
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
            <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
         <- hypergeometric solution without integrals succesful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - return
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.220 (sec)
Leaf size : 102

```
dsolve(3*x^2*(x^2+3)*diff(diff(y(x),x),x)+x*(11*x^2+3)*diff(y(x),x)+(5*x^2+1)*y(x) = 0
```

$$y = \frac{x^{1/3}\left(2\sqrt{3}\arctan\left(\frac{(9x^2+27)^{1/3}\sqrt{3}}{6+(9x^2+27)^{1/3}}\right)c_2 + 3\,3^{1/3}c_1 + 2\ln\left(3-(9x^2+27)^{1/3}\right)c_2 - \ln\left((9x^2+27)^{2/3} + 3(9x\right.\right.}{9\,(x^2+3)^{2/3}}$$

Mathematica DSolve solution

Solving time : 0.106 (sec)
Leaf size : 57

```
DSolve[{3*x^2*(3+x^2)*D[y[x],x]+x*(3+11*x^2)*D[y[x],x]+(1+5*x^2)*y[x]==0,{}},y[x],x,IncludeS
```

$$y(x) \to c_1 \exp\left(\int_1^x -\frac{5K[1]^2+1}{3K[1]^4+11K[1]^3+9K[1]^2+3K[1]}dK[1]\right)$$
$$y(x) \to 0$$

2.1.147 Problem 149

Internal problem ID [9319]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 149
Date solved : `Monday, January 27, 2025 at 06:01:25 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$9x^2y'' - 3x(-2x^2+7)\,y' + (2x^2+25)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.274 (sec)

Writing the ode as

$$9x^2y'' + (6x^3-21x)\,y' + (2x^2+25)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^2 \\ B &= 6x^3 - 21x \\ C &= 2x^2 + 25 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^4 - 24x^2 - 9}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^4 - 24x^2 - 9 \\ t &= 36x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4 - 24x^2 - 9}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.278: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{9} - \frac{2}{3} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{3} - \frac{1}{x} - \frac{15}{8x^3} - \frac{45}{8x^5} - \frac{2835}{128x^7} - \frac{12555}{128x^9} - \frac{477495}{1024x^{11}} - \frac{2380185}{1024x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{3}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{3} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{9}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^4 - 24x^2 - 9}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{x^2}{9} - \frac{2}{3}\right) + \left(-\frac{1}{4x^2}\right) \\ &= \frac{x^2}{9} - \frac{2}{3} - \frac{1}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is $-\frac{2}{3}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{2}{3}\right) - (0) \\ &= -\frac{2}{3} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{3} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{2}{3}}{\frac{1}{3}} - 1\right) &&= -\frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{2}{3}}{\frac{1}{3}} - 1\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^4 - 24x^2 - 9}{36x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{3}$	$-\frac{3}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^+) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\left(\frac{x}{3}\right) \\
&= \frac{1}{2x} - \frac{x}{3} \\
&= \frac{1}{2x} - \frac{x}{3}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2x} - \frac{x}{3}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{3}\right) + \left(\frac{1}{2x} - \frac{x}{3}\right)^2 - \left(\frac{4x^4 - 24x^2 - 9}{36x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{2x} - \frac{x}{3}\right) dx} \\
&= \sqrt{x}\, e^{-\frac{x^2}{6}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{6x^3-21x}{9x^2}\,dx} \\
&= z_1 e^{-\frac{x^2}{6}+\frac{7\ln(x)}{6}} \\
&= z_1 \left(x^{7/6} \mathrm{e}^{-\frac{x^2}{6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^{5/3} \mathrm{e}^{-\frac{x^2}{3}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{6x^3-21x}{9x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{3}+\frac{7\ln(x)}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\mathrm{Ei}_1\left(-\frac{x^2}{3}\right)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{5/3} \mathrm{e}^{-\frac{x^2}{3}}\right) + c_2 \left(x^{5/3} \mathrm{e}^{-\frac{x^2}{3}} \left(-\frac{\mathrm{Ei}_1\left(-\frac{x^2}{3}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$9x^2 \left(\frac{d^2}{dx^2} y(x)\right) - 3x(-2x^2+7)\left(\frac{d}{dx} y(x)\right) + (2x^2+25)\, y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2} y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2} y(x) = -\frac{(2x^2+25)y(x)}{9x^2} - \frac{(2x^2-7)\left(\frac{d}{dx} y(x)\right)}{3x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2} y(x) + \frac{(2x^2-7)\left(\frac{d}{dx} y(x)\right)}{3x} + \frac{(2x^2+25)y(x)}{9x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = \frac{2x^2-7}{3x}, P_3(x) = \frac{2x^2+25}{9x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = -\frac{7}{3}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{25}{9}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $9x^2\left(\frac{d^2}{dx^2}y(x)\right) + 3x(2x^2-7)\left(\frac{d}{dx}y(x)\right) + (2x^2+25)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-5+3r)^2 x^r + a_1(-2+3r)^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r-5)^2 + 2a_{k-2}(3k+3r-5)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-5+3r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = \frac{5}{3}$

- Each term must be 0

  $a_1(-2+3r)^2 = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(3k+3r-5)^2 + 2a_{k-2}(3k+3r-5) = 0$

- Shift index using $k- >k + 2$

  $a_{k+2}(3k+3r+1)^2 + 2a_k(3k+3r+1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2a_k}{3k+3r+1}$

- Recursion relation for $r = \frac{5}{3}$

  $a_{k+2} = -\frac{2a_k}{3k+6}$
- Solution for $r = \frac{5}{3}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{3}}, a_{k+2} = -\frac{2a_k}{3k+6}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 25

```
dsolve(9*x^2*diff(diff(y(x),x),x)-3*x*(-2*x^2+7)*diff(y(x),x)+(2*x^2+25)*y(x) = 0,y(x),s
```

$$y = x^{5/3} \mathrm{e}^{-\frac{x^2}{3}} \left(c_1 + c_2 \operatorname{Ei}_1\left(-\frac{x^2}{3}\right)\right)$$

Mathematica DSolve solution

Solving time : 0.124 (sec)
Leaf size : 39

```
DSolve[{9*x^2*D[y[x],{x,2}]-3*x*(7-2*x^2)*D[y[x],x]+(25+2*x^2)*y[x]==0,{}},y[x],x,IncludeSingul
```

$$y(x) \to \frac{1}{2} e^{-\frac{x^2}{3}} x^{5/3} \left(c_2 \operatorname{ExpIntegralEi}\left(\frac{x^2}{3}\right) + 2c_1\right)$$

2.1.148 Problem 150

Internal problem ID [9320]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 150
Date solved : `Monday, January 27, 2025 at 06:01:26 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x\left(-x^2+1\right)y' + \left(x^2+1\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.243 (sec)

Writing the ode as

$$x^2y'' + \left(x^3-x\right)y' + \left(x^2+1\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^3 - x \\ C &= x^2 + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 4x^2 - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4 - 4x^2 - 1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 4x^2 - 1}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.280: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{4} - 1 - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{1}{x} - \frac{5}{4x^3} - \frac{5}{2x^5} - \frac{105}{16x^7} - \frac{155}{8x^9} - \frac{1965}{32x^{11}} - \frac{3265}{16x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^4 - 4x^2 - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{x^2}{4} - 1\right) + \left(-\frac{1}{4x^2}\right) \\ &= \frac{x^2}{4} - 1 - \frac{1}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is -1. Now b can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 1\right) &&= -\frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 1\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 4x^2 - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\left(\frac{x}{2}\right) \\ &= \frac{1}{2x} - \frac{x}{2} \\ &= \frac{1}{2x} - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2x} - \frac{x}{2}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{2}\right) + \left(\frac{1}{2x} - \frac{x}{2}\right)^2 - \left(\frac{x^4 - 4x^2 - 1}{4x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{1}{2x} - \frac{x}{2}\right) dx} \\ &= \sqrt{x}\, e^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^3-x}{x^2}\,dx} \\ &= z_1 e^{-\frac{x^2}{4}+\frac{\ln(x)}{2}} \\ &= z_1\left(\sqrt{x}\,\mathrm{e}^{-\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{-\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^3-x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{2}+\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(-\frac{\mathrm{Ei}_1\left(-\frac{x^2}{2}\right)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(x\,\mathrm{e}^{-\frac{x^2}{2}}\right) + c_2\left(x\,\mathrm{e}^{-\frac{x^2}{2}}\left(-\frac{\mathrm{Ei}_1\left(-\frac{x^2}{2}\right)}{2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) - x(-x^2+1)\left(\frac{d}{dx}y(x)\right) + (x^2+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(x^2+1)y(x)}{x^2} - \frac{(x^2-1)\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(x^2-1)\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(x^2+1)y(x)}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = \frac{x^2-1}{x}, P_3(x) = \frac{x^2+1}{x^2}\right]$

- ∘ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -1$

- ∘ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 1$

- ∘ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(x^2-1)\left(\frac{d}{dx}y(x)\right) + (x^2+1)\, y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ∘ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ∘ Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ∘ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- ∘ Shift index using $k- >k + 1 - m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- ∘ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)^2\, x^r + a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)^2 + a_{k-2}(k+r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 1$

- Each term must be 0

  $a_1 r^2 = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\,(a_k(k+r-1) + a_{k-2}) = 0$

- Shift index using $k- >k + 2$

  $(k+r+1)\,(a_{k+2}(k+r+1) + a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{k+r+1}$

- Recursion relation for $r = 1$
  $a_{k+2} = -\frac{a_k}{k+2}$
- Solution for $r = 1$
  $$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{k+2}, a_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 23

```
dsolve(x^2*diff(diff(y(x),x),x)-x*(-x^2+1)*diff(y(x),x)+(x^2+1)*y(x) = 0,y(x),singsol=
```

$$y = x\,e^{-\frac{x^2}{2}} \left(c_1 + c_2 \operatorname{Ei}_1\left(-\frac{x^2}{2}\right)\right)$$

Mathematica DSolve solution

Solving time : 0.025 (sec)
Leaf size : 35

```
DSolve[{x^2*D[y[x],{x,2}]-x*(1-x^2)*D[y[x],x]+(1+x^2)*y[x]==0,{}},y[x],x,IncludeSingularSolu
```

$$y(x) \to \frac{1}{2} e^{-\frac{x^2}{2}} x \left(c_1 \operatorname{ExpIntegralEi}\left(\frac{x^2}{2}\right) + 2c_2\right)$$

2.1.149 Problem 151

Internal problem ID [9321]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 151
Date solved : `Monday, January 27, 2025 at 06:01:26 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1-2x)\,y''+3xy'+(1+4x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.221 (sec)

Writing the ode as

$$\left(-2x^3+x^2\right)y''+3xy'+(1+4x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -2x^3+x^2 \\ B &= 3x \\ C &= 1+4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{32x^2+16x-1}{4\left(2x^2-x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 32x^2+16x-1 \\ t &= 4\left(2x^2-x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{32x^2+16x-1}{4\left(2x^2-x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.282: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{x} - \frac{1}{4x^2} + \frac{15}{4\left(x - \frac{1}{2}\right)^2} - \frac{3}{x - \frac{1}{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{32x^2 + 16x - 1}{4\left(2x^2 - x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{32x^2 + 16x - 1}{4\left(2x^2 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} - \frac{3}{2\left(x-\frac{1}{2}\right)} + (-)\,(0) \\
&= \frac{1}{2x} - \frac{3}{2\left(x-\frac{1}{2}\right)} \\
&= \frac{-1-4x}{4x^2-2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{3}{2\left(x - \frac{1}{2}\right)}\right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{3}{2\left(x - \frac{1}{2}\right)^2}\right) + \left(\frac{1}{2x} - \frac{3}{2\left(x - \frac{1}{2}\right)}\right)^2 - \left(\frac{32x^2 + 16x - 1}{4\left(2x^2 - x\right)^2}\right)\right)$$

$$0 = ($$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{3}{2\left(x - \frac{1}{2}\right)}\right) dx} \\ &= \frac{\sqrt{x}}{(-1 + 2x)^{3/2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{3x}{-2x^3 + x^2} \, dx} \\ &= z_1 e^{\frac{3\ln(-1+2x)}{2} - \frac{3\ln(x)}{2}} \\ &= z_1 \left(\frac{(-1 + 2x)^{3/2}}{x^{3/2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{-2x^3 + x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{3\ln(-1+2x) - 3\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{8x^3}{3} + \frac{1}{2} + 6x - 6x^2 - \ln(x)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x}\left(\frac{8x^3}{3} + \frac{1}{2} + 6x - 6x^2 - \ln(x)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(-2x+1)\left(\frac{d^2}{dx^2}y(x)\right)+3x\left(\frac{d}{dx}y(x)\right)+(4x+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{(4x+1)y(x)}{x^2(2x-1)}+\frac{3\left(\frac{d}{dx}y(x)\right)}{x(2x-1)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)-\frac{3\left(\frac{d}{dx}y(x)\right)}{x(2x-1)}-\frac{(4x+1)y(x)}{x^2(2x-1)}=0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=-\frac{3}{x(2x-1)},P_3(x)=-\frac{4x+1}{x^2(2x-1)}\right]$
  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\Bigg|_{x=0}=3$
  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\Bigg|_{x=0}=1$
  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$
- Multiply by denominators

  $x^2(2x-1)\left(\frac{d^2}{dx^2}y(x)\right)-3x\left(\frac{d}{dx}y(x)\right)+(-4x-1)\,y(x)=0$
- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

    $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_k x^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$
  - Convert $x\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r}$
  - Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..3$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$
  - Shift index using $k->k+2-m$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0(1+r)^2x^r+\left(\sum_{k=1}^{\infty}\left(-a_k(k+r+1)^2+2a_{k-1}(k+r)\,(k-3+r)\right)x^{k+r}\right)=0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

$-(1+r)^2 = 0$

- Values of r that satisfy the indicial equation
  $r = -1$
- Each term in the series must be 0, giving the recursion relation
  $-a_k(k+r+1)^2 + 2a_{k-1}(k+r)(k-3+r) = 0$
- Shift index using $k- >k+1$
  $-a_{k+1}(k+2+r)^2 + 2a_k(k+r+1)(k+r-2) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{2a_k(k+r+1)(k+r-2)}{(k+2+r)^2}$
- Recursion relation for $r = -1$; series terminates at $k = 3$
  $a_{k+1} = \frac{2a_k k(k-3)}{(k+1)^2}$
- Apply recursion relation for $k = 0$
  $a_1 = 0$
- Apply recursion relation for $k = 1$
  $a_2 = -a_1$
- Express in terms of a_0
  $a_2 = 0$
- Apply recursion relation for $k = 2$
  $a_3 = -\frac{4a_2}{9}$
- Express in terms of a_0
  $a_3 = 0$
- Terminating series solution of the ODE for $r = -1$. Use reduction of order to find the second
  $y(x) = a_0 \cdot 0$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 31

```
dsolve(x^2*(1-2*x)*diff(diff(y(x),x),x)+3*diff(y(x),x)*x+(4*x+1)*y(x) = 0,y(x),singsol
```

$$y = \frac{3c_2 \ln(x) + (-8x^3 + 18x^2 - 18x) c_2 + c_1}{x}$$

Mathematica DSolve solution

Solving time : `0.242` (sec)
Leaf size : 105

```
DSolve[{x^2*(1-2*x)*D[y[x],{x,2}]+3*x*D[y[x],x]+(1+4*x)*y[x]==0,{}},y[x],x,IncludeSingularSolu
```

$$y(x) \to \exp\left(\int_1^x \frac{4K[1]+1}{2K[1]-4K[1]^2}dK[1] - \frac{1}{2}\int_1^x \frac{3}{K[2]-2K[2]^2}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{4K[1]+1}{2K[1]-4K[1]^2}dK[1]\right)dK[3] + c_1\right)$$

2.1.150 Problem 152

Internal problem ID [9322]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 152
Date solved : `Monday, January 27, 2025 at 06:01:27 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x(1+x)\, y''+(1-x)\, y'+y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.220 (sec)

Writing the ode as

$$\left(x^2+x\right) y''+(1-x)\, y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+x \\ B &= 1-x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{-x^2-10x-1}{4\left(x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2-10x-1 \\ t &= 4\left(x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-x^2-10x-1}{4\left(x^2+x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.284: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{2}{(1 + x)^2} + \frac{2}{1 + x} - \frac{2}{x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 10x - 1}{4\left(x^2 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 - 10x - 1}{4\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	2	-1
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{1+x} + \frac{1}{2x} + (-)(0) \\
&= -\frac{1}{1+x} + \frac{1}{2x} \\
&= -\frac{x-1}{2x(1+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{1+x} + \frac{1}{2x}\right)(1) + \left(\left(\frac{1}{(1+x)^2} - \frac{1}{2x^2}\right) + \left(-\frac{1}{1+x} + \frac{1}{2x}\right)^2 - \left(\frac{-x^2 - 10x - 1}{4\,(x^2+x)^2}\right)\right) = 0$$

$$\frac{1 + a_0}{x\,(1+x)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (x-1)\, e^{\int \left(-\frac{1}{1+x} + \frac{1}{2x}\right) dx} \\
&= (x-1)\, e^{\frac{\ln(x)}{2} - \ln(1+x)} \\
&= \frac{(x-1)\sqrt{x}}{1+x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1-x}{x^2+x}\, dx} \\
&= z_1 e^{-\frac{\ln(x)}{2} + \ln(1+x)} \\
&= z_1 \left(\frac{1+x}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x - 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1-x}{x^2+x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\ln(x) + 2\ln(1+x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{4}{x-1} + \ln(x)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(x-1) + c_2\left(x - 1\left(-\frac{4}{x-1} + \ln(x)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x(x+1)\left(\frac{d^2}{dx^2}y(x)\right) + (1-x)\left(\frac{d}{dx}y(x)\right) + y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{y(x)}{x(x+1)} + \frac{(x-1)\left(\frac{d}{dx}y(x)\right)}{x(x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) - \frac{(x-1)\left(\frac{d}{dx}y(x)\right)}{x(x+1)} + \frac{y(x)}{x(x+1)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{x-1}{x(x+1)}, P_3(x) = \frac{1}{x(x+1)}\right]$

  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1} = -2$

  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1} = 0$

  - $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $x(x+1)\left(\frac{d^2}{dx^2}y(x)\right) + (1-x)\left(\frac{d}{dx}y(x)\right) + y(x) = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(u^2-u)\left(\frac{d^2}{du^2}y(u)\right) + (2-u)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$-a_0 r(-3+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(k-2+r) + a_k(k+r-1)^2\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-r(-3+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 3\}$
- Each term in the series must be 0, giving the recursion relation
  $-a_{k+1}(k+1+r)\,(k-2+r) + a_k(k+r-1)^2 = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_k(k+r-1)^2}{(k+1+r)(k-2+r)}$
- Recursion relation for $r = 0$; series terminates at $k = 1$
  $a_{k+1} = \frac{a_k(k-1)^2}{(k+1)(k-2)}$
- Apply recursion relation for $k = 0$
  $a_1 = -\frac{a_0}{2}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second linea
  $y(u) = a_0 \cdot \left(1 - \frac{u}{2}\right)$
- Revert the change of variables $u = x + 1$
  $\left[y(x) = a_0\left(-\frac{x}{2} + \frac{1}{2}\right)\right]$
- Recursion relation for $r = 3$
  $a_{k+1} = \frac{a_k(k+2)^2}{(k+4)(k+1)}$
- Solution for $r = 3$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+3}, a_{k+1} = \frac{a_k(k+2)^2}{(k+4)(k+1)}\right]$
- Revert the change of variables $u = x + 1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^{k+3}, a_{k+1} = \frac{a_k(k+2)^2}{(k+4)(k+1)}\right]$
- Combine solutions and rename parameters
  $\left[y(x) = a_0\left(-\frac{x}{2} + \frac{1}{2}\right) + \left(\sum_{k=0}^{\infty} b_k(x+1)^{k+3}\right), b_{k+1} = \frac{b_k(k+2)^2}{(4+k)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 20

```
dsolve(x*(x+1)*diff(diff(y(x),x),x)+(1-x)*diff(y(x),x)+y(x) = 0,y(x),singsol=all)
```

$$y = c_2(x-1)\ln(x) - 4c_2 + c_1(x-1)$$

Mathematica DSolve solution

Solving time : 0.451 (sec)
Leaf size : 112

```
DSolve[{x*(1+x)*D[y[x],{x,2}]+(1-x)*D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to (x-1)\exp\left(\int_1^x \left(\frac{1}{2K[1]} - \frac{1}{K[1]+1}\right) dK[1] - \frac{1}{2}\int_1^x \left(\frac{1}{K[2]} - \frac{2}{K[2]+1}\right) dK[2]\right)\left(c_2\int_1^x \frac{\exp\left(-2\int_1^{K[3]} \frac{1-K[1]}{2K[1]^2+2K[1]} dK[1]\right)}{(K[3]-1)^2} dK[3] + c_1\right)$$

2.1.151 Problem 153

Internal problem ID [9323]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 153
Date solved : `Monday, January 27, 2025 at 06:01:28 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1-x)\,y''-x(3-5x)\,y'+(4-5x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.216 (sec)

Writing the ode as

$$\left(-x^3+x^2\right)y''+\left(5x^2-3x\right)y'+(4-5x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-x^3+x^2\\B&=5x^2-3x\\C&=4-5x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{15x^2-6x-1}{4\left(x^2-x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=15x^2-6x-1\\t&=4\left(x^2-x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{15x^2-6x-1}{4\left(x^2-x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.286: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{x} - \frac{1}{4x^2} + \frac{2}{(-1+x)^2} + \frac{2}{-1+x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15x^2 - 6x - 1}{4\left(x^2 - x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15x^2 - 6x - 1}{4\left(x^2 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
1	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\
&= \frac{5}{2} - \left(\frac{5}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \frac{2}{-1+x} + (0) \\
&= \frac{1}{2x} + \frac{2}{-1+x} \\
&= \frac{-1+5x}{2x\,(-1+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{2}{-1+x}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{2}{(-1+x)^2}\right) + \left(\frac{1}{2x} + \frac{2}{-1+x}\right)^2 - \left(\frac{15x^2 - 6x - 1}{4(x^2 - x)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{1}{2x} + \frac{2}{-1+x}\right) dx} \\ &= (-1+x)^2 \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{5x^2 - 3x}{-x^3 + x^2}\, dx} \\ &= z_1 e^{\ln(-1+x) + \frac{3\ln(x)}{2}} \\ &= z_1 \left((-1+x)\, x^{3/2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (-1+x)^3 x^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{5x^2 - 3x}{-x^3 + x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{2\ln(-1+x) + 3\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{1}{3(-1+x)^3} - \frac{1}{-1+x} + \frac{1}{2(-1+x)^2} - \ln(-1+x) + \ln(x)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left((-1+x)^3 x^2\right) \\ &\quad + c_2\left((-1+x)^3 x^2 \left(-\frac{1}{3(-1+x)^3} - \frac{1}{-1+x} + \frac{1}{2(-1+x)^2} - \ln(-1+x) + \ln(x)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(1-x)\left(\frac{d^2}{dx^2}y(x)\right)-x(3-5x)\left(\frac{d}{dx}y(x)\right)+(4-5x)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(-4+5x)y(x)}{x^2(x-1)}+\frac{(-3+5x)\left(\frac{d}{dx}y(x)\right)}{x(x-1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)-\frac{(-3+5x)\left(\frac{d}{dx}y(x)\right)}{x(x-1)}+\frac{(-4+5x)y(x)}{x^2(x-1)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=-\frac{-3+5x}{x(x-1)},P_3(x)=\frac{-4+5x}{x^2(x-1)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=-3$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=4$

  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(x-1)\left(\frac{d^2}{dx^2}y(x)\right)-x(-3+5x)\left(\frac{d}{dx}y(x)\right)+(-4+5x)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

    $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..3$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$-a_0(-2+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r-2)^2 + a_{k-1}(k+r-2)(k-6+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-(-2+r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 2$
- Each term in the series must be 0, giving the recursion relation
  $-a_k(k+r-2)^2 + a_{k-1}(k+r-2)(k-6+r) = 0$
- Shift index using $k- >k+1$
  $-a_{k+1}(k+r-1)^2 + a_k(k+r-1)(k+r-5) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_k(k+r-5)}{k+r-1}$
- Recursion relation for $r = 2$; series terminates at $k = 3$
  $a_{k+1} = \frac{a_k(k-3)}{k+1}$
- Apply recursion relation for $k = 0$
  $a_1 = -3a_0$
- Apply recursion relation for $k = 1$
  $a_2 = -a_1$
- Express in terms of a_0
  $a_2 = 3a_0$
- Apply recursion relation for $k = 2$
  $a_3 = -\frac{a_2}{3}$
- Express in terms of a_0
  $a_3 = -a_0$
- Terminating series solution of the ODE for $r = 2$. Use reduction of order to find the second li
  $y(x) = a_0 \cdot (-x^3 + 3x^2 - 3x + 1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 47

```
dsolve(x^2*(1-x)*diff(diff(y(x),x),x)-x*(3-5*x)*diff(y(x),x)+(4-5*x)*y(x) = 0,y(x),si
```

$$y = x^2\left(c_1(x-1)^3 + c_2\left(-(x-1)^3 \ln(x-1) + (x-1)^3 \ln(x) - x^2 + \frac{5x}{2} - \frac{11}{6}\right)\right)$$

Mathematica DSolve solution

Solving time : 0.255 (sec)
Leaf size : 104

```
DSolve[{x^2*(1-x)*D[y[x],{x,2}]-x*(3-5*x)*D[y[x],x]+(4-5*x)*y[x]==0,{}},y[x],x,IncludeSingularS
```

$$y(x) \to \exp\left(\int_1^x \left(\frac{1}{2K[1]} + \frac{2}{K[1]-1}\right) dK[1] - \frac{1}{2}\int_1^x \left(-\frac{3}{K[2]} - \frac{2}{K[2]-1}\right) dK[2]\right) \left(c_2 \int_1^x \exp\left(-2\int_1^{K[3]} \frac{1-5K[1]}{2K[1]-2K[1]^2} dK[1]\right) dK[3] + c_1\right)$$

2.1.152 Problem 154

Internal problem ID [9324]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 154
Date solved : `Monday, January 27, 2025 at 06:01:28 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(x^2+1\right) y''-x\left(9x^2+1\right) y'+\left(25x^2+1\right) y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.303 (sec)

Writing the ode as

$$\left(x^4+x^2\right) y''+\left(-9x^3-x\right) y'+\left(25x^2+1\right) y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2 \\ B &= -9x^3-x \\ C &= 25x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{-x^4-98x^2-1}{4\left(x^3+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^4-98x^2-1 \\ t &= 4\left(x^3+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-x^4-98x^2-1}{4\left(x^3+x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.288: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{6}{(x-i)^2} + \frac{6}{(x+i)^2} + \frac{6i}{x-i} - \frac{6i}{x+i}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^4 - 98x^2 - 1}{4(x^3 + x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^4 - 98x^2 - 1}{4(x^3 + x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
i	2	0	3	-2
$-i$	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\ &= \frac{1}{2} - \left(-\frac{7}{2}\right) \\ &= 4\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} - \frac{2}{x-i} - \frac{2}{x+i} + (-)\,(0) \\
&= \frac{1}{2x} - \frac{2}{x-i} - \frac{2}{x+i} \\
&= \frac{1}{2x} - \frac{4x}{x^2+1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\left(12x^2 + 6xa_3 + 2a_2\right) + 2\left(\frac{1}{2x} - \frac{2}{x-i} - \frac{2}{x+i}\right)\left(4x^3 + 3x^2 a_3 + 2a_2 x + a_1\right) + \left(\left(-\frac{1}{2x^2} + \frac{2}{(x-i)^2} + \frac{2}{(x+i}\right.\right.
$$

$$
\left(x^4 a_3 + 4(4 + a\right.
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 1, a_1 = 0, a_2 = -4, a_3 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^4 - 4x^2 + 1
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^4 - 4x^2 + 1\right) e^{\int \left(\frac{1}{2x} - \frac{2}{x-i} - \frac{2}{x+i}\right) dx} \\
&= \left(x^4 - 4x^2 + 1\right) e^{\frac{\ln(x)}{2} - 2\ln(x^2+1)} \\
&= \frac{\left(x^4 - 4x^2 + 1\right)\sqrt{x}}{\left(x^2+1\right)^2}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-9x^3 - x}{x^4 + x^2} \, dx} \\
&= z_1 e^{\frac{\ln(x)}{2} + 2\ln(x^2+1)} \\
&= z_1 \left(\sqrt{x}\left(x^2+1\right)^2\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^5 - 4x^3 + x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-9x^3-x}{x^4+x^2} dx}}{(y_1)^2} dx \\
&= y_1 \int \frac{e^{\ln(x)+4\ln(x^2+1)}}{(y_1)^2} dx \\
&= y_1 \left(\ln(x) + \frac{-6x^2+3}{x^4-4x^2+1} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 (x^5 - 4x^3 + x) + c_2 \left(x^5 - 4x^3 + x \left(\ln(x) + \frac{-6x^2+3}{x^4-4x^2+1} \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) - x(9x^2+1)\left(\frac{d}{dx}y(x)\right) + (25x^2+1)\, y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(25x^2+1)y(x)}{x^2(x^2+1)} + \frac{(9x^2+1)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) - \frac{(9x^2+1)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)} + \frac{(25x^2+1)y(x)}{x^2(x^2+1)} = 0$

- □ Check to see if x_0 is a regular singular point
  - ◦ Define functions

    $\left[P_2(x) = -\frac{9x^2+1}{x(x^2+1)}, P_3(x) = \frac{25x^2+1}{x^2(x^2+1)}\right]$

  - ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -1$

  - ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 1$

  - ◦ $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) - x(9x^2+1)\left(\frac{d}{dx}y(x)\right) + (25x^2+1)\, y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

$x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)^2 x^r + a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)^2 + a_{k-2}(k-7+r)^2\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 1$
- Each term must be 0

  $a_1 r^2 = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)^2 + a_{k-2}(k-7+r)^2 = 0$
- Shift index using $k- >k+2$

  $a_{k+2}(k+1+r)^2 + a_k(k+r-5)^2 = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r-5)^2}{(k+1+r)^2}$
- Recursion relation for $r = 1$; series terminates at $k = 4$

  $a_{k+2} = -\frac{a_k(k-4)^2}{(k+2)^2}$
- Solution for $r = 1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k(k-4)^2}{(k+2)^2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 41

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)-x*(9*x^2+1)*diff(y(x),x)+(25*x^2+1)*y(x) = 0,y
```

$$y = x\left(c_2\left(x^4-4x^2+1\right)\ln\left(x\right)+c_1x^4+\left(-4c_1-6c_2\right)x^2+c_1+3c_2\right)$$

Mathematica DSolve solution

Solving time : 5.271 (sec)
Leaf size : 138

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]-x*(1+9*x^2)*D[y[x],x]+(1+25*x^2)*y[x]==0,{}},y[x],x,Includ
```

$$y(x)\to\left(x^4-4x^2+1\right)\exp\left(\int_1^x\frac{1-7K[1]^2}{2\left(K[1]^3+K[1]\right)}dK[1]-\frac{1}{2}\int_1^x-\frac{9K[2]^2+1}{K[2]^3+K[2]}dK[2]\right)\left(c_2\int_1^x\frac{\exp\left(-2\int_1^{K[3]}\frac{1-7K[1]^2}{2(K[1]^3+K[1])}dK[1]\right)}{\left(K[3]^4-4K[3]^2+1\right)^2}dK[3]+c_1\right)$$

2.1.153 Problem 155

Internal problem ID [9325]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 155
Date solved : `Monday, January 27, 2025 at 06:01:29 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$9x^2y'' + 3x\left(-x^2+1\right)y' + \left(7x^2+1\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 1.111 (sec)

Writing the ode as

$$9x^2y'' + \left(-3x^3+3x\right)y' + \left(7x^2+1\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^2 \\ B &= -3x^3+3x \\ C &= 7x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 36x^2 - 9}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4 - 36x^2 - 9 \\ t &= 36x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 36x^2 - 9}{36x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.290: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{36} - 1 - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{6} - \frac{3}{x} - \frac{111}{4x^3} - \frac{999}{2x^5} - \frac{180819}{16x^7} - \frac{2292705}{8x^9} - \frac{249239511}{32x^{11}} - \frac{3548540907}{16x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{6} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{36}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^4 - 36x^2 - 9}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{x^2}{36} - 1\right) + \left(-\frac{1}{4x^2}\right) \\ &= \frac{x^2}{36} - 1 - \frac{1}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is -1. Now b can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{6} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{\frac{1}{6}} - 1\right) &&= -\frac{7}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{\frac{1}{6}} - 1\right) &&= \frac{5}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 36x^2 - 9}{36x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{6}$	$-\frac{7}{2}$	$\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{5}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{5}{2} - \left(\frac{1}{2}\right) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\left(\frac{x}{6}\right) \\ &= \frac{1}{2x} - \frac{x}{6} \\ &= \frac{1}{2x} - \frac{x}{6} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{1}{2x} - \frac{x}{6}\right)(2x + a_1) + \left(\left(-\frac{1}{2x^2} - \frac{1}{6}\right) + \left(\frac{1}{2x} - \frac{x}{6}\right)^2 - \left(\frac{x^4 - 36x^2 - 9}{36x^2}\right) \right) = 0$$

$$\frac{x^2 a_1 + 2(6 + a_0)\, x + 3a_1}{3x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -6, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 6$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2-6\right) e^{\int \left(\frac{1}{2x}-\frac{x}{6}\right) dx} \\
&= \left(x^2-6\right) e^{-\frac{x^2}{12}+\frac{\ln(x)}{2}} \\
&= \left(x^2-6\right) \sqrt{x}\, e^{-\frac{x^2}{12}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3x^3+3x}{9x^2} \, dx} \\
&= z_1 e^{\frac{x^2}{12}-\frac{\ln(x)}{6}} \\
&= z_1 \left(\frac{e^{\frac{x^2}{12}}}{x^{1/6}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^{1/3}\left(x^2-6\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x^3+3x}{9x^2} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{6}-\frac{\ln(x)}{3}}}{\left(y_1\right)^2} \, dx \\
&= y_1 \left(\int \frac{e^{\frac{x^2}{6}-\frac{\ln(x)}{3}}}{x^{2/3}\left(x^2-6\right)^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^{1/3}\left(x^2-6\right)\right) + c_2\left(x^{1/3}\left(x^2-6\right)\left(\int \frac{e^{\frac{x^2}{6}-\frac{\ln(x)}{3}}}{x^{2/3}\left(x^2-6\right)^2} dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$9x^2\left(\frac{d^2}{dx^2}y(x)\right)+3x(-x^2+1)\left(\frac{d}{dx}y(x)\right)+(7x^2+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{(7x^2+1)y(x)}{9x^2}+\frac{(x^2-1)\left(\frac{d}{dx}y(x)\right)}{3x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)-\frac{(x^2-1)\left(\frac{d}{dx}y(x)\right)}{3x}+\frac{(7x^2+1)y(x)}{9x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

- ◦ Define functions

$\left[P_2(x)=-\frac{x^2-1}{3x},P_3(x)=\frac{7x^2+1}{9x^2}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

$\left.(x\cdot P_2(x))\right|_{x=0}=\frac{1}{3}$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left.(x^2\cdot P_3(x))\right|_{x=0}=\frac{1}{9}$

- ◦ $x=0$is a regular singular point

Check to see if $x_0=0$ is a regular singular point

$x_0=0$

- Multiply by denominators

$9x^2\left(\frac{d^2}{dx^2}y(x)\right)-3x(x^2-1)\left(\frac{d}{dx}y(x)\right)+(7x^2+1)\,y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

$x^m\cdot y(x)=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

- ◦ Shift index using $k->k-m$

$x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ◦ Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

$x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

Rewrite ODE with series expansions

$a_0(-1+3r)^2\,x^r+a_1(2+3r)^2\,x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_k(3k+3r-1)^2-a_{k-2}(3k-13+3r)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(-1+3r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = \frac{1}{3}$

- Each term must be 0

  $a_1(2+3r)^2 = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(3k+3r-1)^2 + (-3k+13-3r)\, a_{k-2} = 0$

- Shift index using $k->k+2$

  $a_{k+2}(3k+5+3r)^2 + a_k(-3k-3r+7) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k(3k+3r-7)}{(3k+5+3r)^2}$

- Recursion relation for $r = \frac{1}{3}$; series terminates at $k = 2$

  $a_{k+2} = \frac{a_k(3k-6)}{(3k+6)^2}$

- Solution for $r = \frac{1}{3}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = \frac{a_k(3k-6)}{(3k+6)^2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solut
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special fu
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.030 (sec)
Leaf size : 19

```
dsolve(9*x^2*diff(diff(y(x),x),x)+3*x*(-x^2+1)*diff(y(x),x)+(7*x^2+1)*y(x) = 0,y(x),si
```

$$y = -\frac{x^{1/3}(x^2-6)(c_1-c_2)}{6}$$

Mathematica DSolve solution

Solving time : 0.597 (sec)
Leaf size : 59

```
DSolve[{9*x^2*D[y[x],{x,2}]+3*x*(1-x^2)*D[y[x],x]+(1+7*x^2)*y[x]==0,{}},y[x],x,IncludeSingul
```

$$y(x) \to \sqrt[3]{e}\sqrt[3]{x}(x^2-6)\left(c_2\int_1^x \frac{e^{\frac{K[1]^2}{6}-1}}{K[1](K[1]^2-6)^2}dK[1]+c_1\right)$$

2.1.154 Problem 156

Internal problem ID [9326]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 156
Date solved : `Monday, January 27, 2025 at 06:01:31 PM`
CAS classification : `[[_2nd_order, _exact, _linear, _homogeneous]]`

Solve

$$x\left(x^{2}+1\right) y''+\left(-x^{2}+1\right) y'-8xy=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.348 (sec)

Writing the ode as

$$\left(x^{3}+x\right) y''+\left(-x^{2}+1\right) y'-8xy=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^{3}+x \\ B &= -x^{2}+1 \\ C &= -8x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^{2}-4AC}{4A^{2}} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{35x^{4}+22x^{2}-1}{4\left(x^{3}+x\right)^{2}} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 35x^{4}+22x^{2}-1 \\ t &= 4\left(x^{3}+x\right)^{2} \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{35x^{4}+22x^{2}-1}{4\left(x^{3}+x\right)^{2}}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.292: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4\,(x-i)^2} + \frac{3}{4\,(x+i)^2} - \frac{15i}{4\,(x-i)} + \frac{15i}{4\,(x+i)} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{35x^4 + 22x^2 - 1}{4\left(x^3 + x\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{35}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2}\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{35x^4 + 22x^2 - 1}{4\left(x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{2}$ then

$$\begin{aligned}d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+ + \alpha_{c_3}^+\right)\\ &= \frac{7}{2} - \left(\frac{7}{2}\right)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{+}}{x-c_3}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)} + (0) \\
&= \frac{1}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)} \\
&= \frac{1}{2x} + \frac{3x}{x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{3}{2(x-i)^2} - \frac{3}{2(x+i)^2}\right) + \left(\frac{1}{2x} + \frac{3}{2(x-i)} + \frac{}{2(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)}\right) dx} \\
&= (x^2+1)^{3/2} \sqrt{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2+1}{x^3+x}\, dx} \\
&= z_1 e^{\frac{\ln(x^2+1)}{2} - \frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\sqrt{x^2+1}}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = (x^2+1)^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+1}{x^3+x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\ln(x^2+1) - \ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{1}{2x^2+2} + \frac{1}{4(x^2+1)^2} - \frac{\ln(x^2+1)}{2} + \ln(x)\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\left(x^2+1\right)^2\right) + c_2\left(\left(x^2+1\right)^2\left(\frac{1}{2x^2+2} + \frac{1}{4\left(x^2+1\right)^2} - \frac{\ln\left(x^2+1\right)}{2} + \ln\left(x\right)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + (-x^2+1)\left(\frac{d}{dx}y(x)\right) - 8xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{8y(x)}{x^2+1} + \frac{(x^2-1)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) - \frac{(x^2-1)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)} - \frac{8y(x)}{x^2+1} = 0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{x^2-1}{x(x^2+1)}, P_3(x) = -\frac{8}{x^2+1}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + (-x^2+1)\left(\frac{d}{dx}y(x)\right) - 8xy(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y(x)$ to series expansion

    $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

  ◦ Shift index using $k- >k-1$

    $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

  ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..2$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

$$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 1..3$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- > k+2-m$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0r^2x^{-1+r} + a_1(1+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)^2 + a_{k-1}(k+r+1)\,(k-5+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 0$
- Each term must be 0

  $a_1(1+r)^2 = 0$
- Each term in the series must be 0, giving the recursion relation

  $((a_{k-1} + a_{k+1})\, k - 5a_{k-1} + a_{k+1})\,(k+1) = 0$
- Shift index using $k- > k+1$

  $((a_k + a_{k+2})\,(k+1) - 5a_k + a_{k+2})\,(k+2) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k-4)}{k+2}$
- Recursion relation for $r = 0$; series terminates at $k = 4$

  $a_{k+2} = -\frac{a_k(k-4)}{k+2}$
- Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(k-4)}{k+2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 48

```
dsolve(x*(x^2+1)*diff(diff(y(x),x),x)+(-x^2+1)*diff(y(x),x)-8*x*y(x) = 0,y(x),singsol=al
```

$$y = c_1 \left(x^2+1\right)^2 + c_2 \left(-\frac{\left(x^2+1\right)^2 \ln\left(x^2+1\right)}{2} + \left(x^2+1\right)^2 \ln\left(x\right) + \frac{x^2}{2} + \frac{3}{4}\right)$$

Mathematica DSolve solution

Solving time : 0.225 (sec)
Leaf size : 112

```
DSolve[{x*(1+x^2)*D[y[x],{x,2}]+(1-x^2)*D[y[x],x]-8*x*y[x]==0,{}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to \exp\left(\int_1^x \frac{7K[1]^2+1}{2\left(K[1]^3+K[1]\right)}dK[1] - \frac{1}{2}\int_1^x \frac{1-K[2]^2}{K[2]^3+K[2]}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{7K[1]^2+1}{2\left(K[1]^3+K[1]\right)}dK[1]\right)dK[3] + c_1\right)$$

2.1.155 Problem 157

Internal problem ID [9327]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 157
Date solved : `Monday, January 27, 2025 at 06:01:31 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' + 2x\left(-x^2+4\right)y' + \left(7x^2+1\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.675 (sec)

Writing the ode as

$$4x^2y'' + \left(-2x^3+8x\right)y' + \left(7x^2+1\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -2x^3+8x \\ C &= 7x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 40x^2 - 4}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4 - 40x^2 - 4 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 40x^2 - 4}{16x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.294: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{16} - \frac{5}{2} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{4} - \frac{5}{x} - \frac{101}{2x^3} - \frac{1010}{x^5} - \frac{50601}{2x^7} - \frac{710030}{x^9} - \frac{21351501}{x^{11}} - \frac{672670100}{x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{4}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{16}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^4 - 40x^2 - 4}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{x^2}{16} - \frac{5}{2}\right) + \left(-\frac{1}{4x^2}\right) \\ &= \frac{x^2}{16} - \frac{5}{2} - \frac{1}{4x^2}\end{aligned}$$

We see that the coefficient of the term x in the quotient is $-\frac{5}{2}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{5}{2}\right) - (0) \\ &= -\frac{5}{2}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{x}{4} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{4}} - 1\right) &&= -\frac{11}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{4}} - 1\right) &&= \frac{9}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 40x^2 - 4}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{4}$	$-\frac{11}{2}$	$\frac{9}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{9}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{9}{2} - \left(\frac{1}{2}\right) \\ &= 4 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\left(\frac{x}{4}\right) \\ &= \frac{1}{2x} - \frac{x}{4} \\ &= \frac{1}{2x} - \frac{x}{4} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(12x^2 + 6xa_3 + 2a_2\right) + 2\left(\frac{1}{2x} - \frac{x}{4}\right)\left(4x^3 + 3a_3x^2 + 2a_2x + a_1\right) + \left(\left(-\frac{1}{2x^2} - \frac{1}{4}\right) + \left(\frac{1}{2x} - \frac{x}{4}\right)^2 - \left(\frac{x^4 - 4}{16}\right.\right.$$
$$\frac{x^4a_3 + 2(16 + a_2)\,x^3 + 3(a_1 + 6a_3)\,x^2 + 4(a_0 + 2a_2)\,x}{2x}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 32, a_1 = 0, a_2 = -16, a_3 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 - 16x^2 + 32$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^4 - 16x^2 + 32\right) e^{\int \left(\frac{1}{2x} - \frac{x}{4}\right) dx} \\
&= \left(x^4 - 16x^2 + 32\right) e^{-\frac{x^2}{8} + \frac{\ln(x)}{2}} \\
&= \left(x^4 - 16x^2 + 32\right) \sqrt{x}\, e^{-\frac{x^2}{8}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-2x^3+8x}{4x^2} \, dx} \\
&= z_1 e^{\frac{x^2}{8} - \ln(x)} \\
&= z_1 \left(\frac{e^{\frac{x^2}{8}}}{x} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^4 - 16x^2 + 32}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^3+8x}{4x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{4} - 2\ln(x)}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{e^{\frac{x^2}{4} - 2\ln(x)} x}{\left(x^4 - 16x^2 + 32\right)^2} dx \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^4 - 16x^2 + 32}{\sqrt{x}} \right) + c_2 \left(\frac{x^4 - 16x^2 + 32}{\sqrt{x}} \left(\int \frac{e^{\frac{x^2}{4} - 2\ln(x)} x}{\left(x^4 - 16x^2 + 32\right)^2} dx \right) \right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2\left(\frac{d^2}{dx^2}y(x)\right)+2x(-x^2+4)\left(\frac{d}{dx}y(x)\right)+(7x^2+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{(7x^2+1)y(x)}{4x^2}+\frac{(x^2-4)\left(\frac{d}{dx}y(x)\right)}{2x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x)-\frac{(x^2-4)\left(\frac{d}{dx}y(x)\right)}{2x}+\frac{(7x^2+1)y(x)}{4x^2}=0$

☐ Check to see if $x_0=0$ is a regular singular point

- Define functions

$\left[P_2(x)=-\frac{x^2-4}{2x},P_3(x)=\frac{7x^2+1}{4x^2}\right]$

- $x\cdot P_2(x)$ is analytic at $x=0$

$\left.(x\cdot P_2(x))\right|_{x=0}=2$

- $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left.(x^2\cdot P_3(x))\right|_{x=0}=\frac{1}{4}$

- $x=0$is a regular singular point

Check to see if $x_0=0$ is a regular singular point

$x_0=0$

- Multiply by denominators

$4x^2\left(\frac{d^2}{dx^2}y(x)\right)-2x(x^2-4)\left(\frac{d}{dx}y(x)\right)+(7x^2+1)\,y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

$x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- Shift index using $k->k-m$

$x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

$x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

Rewrite ODE with series expansions

$a_0(1+2r)^2\,x^r+a_1(3+2r)^2\,x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_k(2k+2r+1)^2-a_{k-2}(2k-11+2r)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(1+2r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = -\frac{1}{2}$

- Each term must be 0

  $a_1(3+2r)^2 = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(2k+2r+1)^2 + (-2k+11-2r)\, a_{k-2} = 0$

- Shift index using $k->k+2$

  $a_{k+2}(2k+5+2r)^2 + a_k(-2k-2r+7) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k(2k+2r-7)}{(2k+5+2r)^2}$

- Recursion relation for $r = -\frac{1}{2}$; series terminates at $k = 4$

  $a_{k+2} = \frac{a_k(2k-8)}{(2k+4)^2}$

- Solution for $r = -\frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = \frac{a_k(2k-8)}{(2k+4)^2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric sol
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.095 (sec)
Leaf size : 24

```
dsolve(4*x^2*diff(diff(y(x),x),x)+2*x*(-x^2+4)*diff(y(x),x)+(7*x^2+1)*y(x) = 0,y(x),sing
```

$$y = \frac{(x^4 - 16x^2 + 32)(c_1 + 2c_2)}{32\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.765 (sec)
Leaf size : 70

```
DSolve[{4*x^2*D[y[x],{x,2}]+2*x*(4-x^2)*D[y[x],x]+(1+7*x^2)*y[x]==0,{}},y[x],x,IncludeSingular
```

$$y(x) \to \frac{\sqrt{e}(x^4 - 16x^2 + 32)\left(c_2 \int_1^x \frac{e^{\frac{K[1]^2}{4}-1}}{K[1](K[1]^4 - 16K[1]^2 + 32)^2} dK[1] + c_1\right)}{\sqrt{x}}$$

2.1.156 Problem 158

Internal problem ID [9328]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 158
Date solved : `Monday, January 27, 2025 at 06:01:32 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2(1+x)\,y''+8x^2y'+(1+x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.150 (sec)

Writing the ode as

$$\left(4x^3+4x^2\right)y''+8x^2y'+(1+x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^3+4x^2 \\ B &= 8x^2 \\ C &= 1+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{1}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.296: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2}\\ \alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=-\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=\frac{1}{2}$ then

$$\begin{aligned}d &= \alpha_\infty^- - (\alpha_{c_1}^-)\\ &= \frac{1}{2}-\left(\frac{1}{2}\right)\\ &= 0\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^-}{x-c_1}\right)+(-)[\sqrt{r}]_\infty\\ &= \frac{1}{2x}+(-)(0)\\ &= \frac{1}{2x}\\ &= \frac{1}{2x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{1}{2x}\right)(0)+\left(\left(-\frac{1}{2x^2}\right)+\left(\frac{1}{2x}\right)^2-\left(-\frac{1}{4x^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{8x^2}{4x^3+4x^2} \, dx} \\ &= z_1 e^{-\ln(1+x)} \\ &= z_1\left(\frac{1}{1+x}\right) \end{aligned}$$

Which simplifies to

$$y_1=\frac{\sqrt{x}}{1+x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1\int \frac{e^{\int -\frac{8x^2}{4x^3+4x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1\int \frac{e^{-2\ln(1+x)}}{(y_1)^2} \, dx \\ &= y_1(\ln(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\sqrt{x}}{1+x}\right)+c_2\left(\frac{\sqrt{x}}{1+x}(\ln(x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)+8x^2\left(\frac{d}{dx}y(x)\right)+(x+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{y(x)}{4x^2}-\frac{2\left(\frac{d}{dx}y(x)\right)}{x+1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)+\frac{2\left(\frac{d}{dx}y(x)\right)}{x+1}+\frac{y(x)}{4x^2}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=\frac{2}{x+1},P_3(x)=\frac{1}{4x^2}\right]$

- ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=2$

- ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

- ◦ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1$

- Multiply by denominators

$4x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)+8x^2\left(\frac{d}{dx}y(x)\right)+(x+1)\,y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(4u^3-8u^2+4u)\left(\frac{d^2}{du^2}y(u)\right)+(8u^2-16u+8)\left(\frac{d}{du}y(u)\right)+uy(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u\cdot y(u)$ to series expansion

  $u\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+1}$

- ◦ Shift index using $k->k-1$

  $u\cdot y(u)=\sum_{k=1}^{\infty}a_{k-1}u^{k+r}$

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ◦ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- ◦ Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$4a_0r(1+r)\,u^{-1+r} + (4a_1(1+r)(2+r) - 8a_0r(1+r))\,u^r + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+r+1)(k+2+r) - 8\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $4r(1+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-1, 0\}$
- Each term must be 0
  $4a_1(1+r)(2+r) - 8a_0r(1+r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k-1}(2k-1+2r)^2 - 8(k+r+1)\left(\left(-\frac{k}{2} - \frac{r}{2} - 1\right)a_{k+1} + a_k(k+r)\right) = 0$
- Shift index using $k- >k+1$
  $a_k(2k+2r+1)^2 - 8(k+2+r)\left(\left(-\frac{k}{2} - \frac{3}{2} - \frac{r}{2}\right)a_{k+2} + a_{k+1}(k+r+1)\right) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 8kra_k - 16kra_{k+1} + 4r^2a_k - 8r^2a_{k+1} + 4ka_k - 24ka_{k+1} + 4ra_k - 24ra_{k+1} + a_k - 16a_{k+1}}{4(k+2+r)(k+3+r)}$
- Recursion relation for $r = -1$
  $a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} - 4ka_k - 8ka_{k+1} + a_k}{4(k+1)(k+2)}$
- Solution for $r = -1$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-1}, a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} - 4ka_k - 8ka_{k+1} + a_k}{4(k+1)(k+2)}, 0 = 0\right]$
- Revert the change of variables $u = x+1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x+1)^{k-1}, a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} - 4ka_k - 8ka_{k+1} + a_k}{4(k+1)(k+2)}, 0 = 0\right]$
- Recursion relation for $r = 0$
  $a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 4ka_k - 24ka_{k+1} + a_k - 16a_{k+1}}{4(k+2)(k+3)}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 4ka_k - 24ka_{k+1} + a_k - 16a_{k+1}}{4(k+2)(k+3)}, 8a_1 = 0\right]$
- Revert the change of variables $u = x+1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x+1)^k, a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 4ka_k - 24ka_{k+1} + a_k - 16a_{k+1}}{4(k+2)(k+3)}, 8a_1 = 0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k (x+1)^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k (x+1)^k\right), a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} - 4ka_k - 8ka_{k+1} + a_k}{4(k+1)(k+2)}, 0 = 0, b_{k+2}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
```

```
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.040 (sec)
Leaf size : 19

```
dsolve(4*x^2*(x+1)*diff(diff(y(x),x),x)+8*diff(y(x),x)*x^2+(x+1)*y(x) = 0,y(x),singsol
```

$$y = \frac{\sqrt{x}\,(c_2 \ln(x) + c_1)}{x+1}$$

Mathematica DSolve solution

Solving time : 0.038 (sec)
Leaf size : 24

```
DSolve[{4*x^2*(1+x)*D[y[x],{x,2}]+8*x^2*D[y[x],x]+(1+x)*y[x]==0,{}},y[x],x,IncludeSingularSo
```

$$y(x) \to \frac{\sqrt{x}(c_2 \log(x) + c_1)}{x+1}$$

2.1.157 Problem 159

Internal problem ID [9329]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 159
Date solved : Monday, January 27, 2025 at 06:01:33 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$9x^2(3+x)\,y''+3x(3+7x)\,y'+(3+4x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.211 (sec)

Writing the ode as

$$\left(9x^3+27x^2\right)y''+\left(21x^2+9x\right)y'+(3+4x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^3+27x^2\\ B &= 21x^2+9x\\ C &= 3+4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t}\\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1\\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{1}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.298: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2}\\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=-\frac{1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right)\\
&= \frac{1}{2}-\left(\frac{1}{2}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^-}{x-c_1}\right)+(-)[\sqrt{r}]_\infty\\
&= \frac{1}{2x}+(-)(0)\\
&= \frac{1}{2x}\\
&= \frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}
$$

Let

$$
p(x)=1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{1}{2x}\right)(0)+\left(\left(-\frac{1}{2x^2}\right)+\left(\frac{1}{2x}\right)^2-\left(-\frac{1}{4x^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{21x^2+9x}{9x^3+27x^2}\, dx} \\ &= z_1 e^{-\ln(3+x)-\frac{\ln(x)}{6}} \\ &= z_1\left(\frac{1}{(3+x)\,x^{1/6}}\right) \end{aligned}$$

Which simplifies to

$$y_1=\frac{x^{1/3}}{3+x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1\int \frac{e^{\int -\frac{21x^2+9x}{9x^3+27x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1\int \frac{e^{-2\ln(3+x)-\frac{\ln(x)}{3}}}{(y_1)^2}\, dx \\ &= y_1\left(\frac{x^2}{9+3x}+\frac{2x}{3+x}+\frac{3}{3+x}-\frac{2\ln(3+x)\,x}{3}-2\ln(3+x)+\ln(x)\right. \\ &\qquad \left.+\frac{2\ln(3+x)\,(3+x)}{3}-\frac{x}{3}-2\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1y_1+c_2y_2 \\ &= c_1\left(\frac{x^{1/3}}{3+x}\right) \\ &\quad +c_2\left(\frac{x^{1/3}}{3+x}\left(\frac{x^2}{9+3x}+\frac{2x}{3+x}+\frac{3}{3+x}-\frac{2\ln(3+x)\,x}{3}-2\ln(3+x)+\ln(x)+\frac{2\ln(3+x)\,(3+x)}{3}-\frac{x}{3}-2\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$9x^2(x+3)\left(\frac{d^2}{dx^2}y(x)\right)+3x(7x+3)\left(\frac{d}{dx}y(x)\right)+(3+4x)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(3+4x)y(x)}{9x^2(x+3)}-\frac{(7x+3)\left(\frac{d}{dx}y(x)\right)}{3x(x+3)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{(7x+3)\left(\frac{d}{dx}y(x)\right)}{3x(x+3)}+\frac{(3+4x)y(x)}{9x^2(x+3)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=\frac{7x+3}{3x(x+3)},P_3(x)=\frac{3+4x}{9x^2(x+3)}\right]$

- ◦ $(x+3)\cdot P_2(x)$ is analytic at $x=-3$

  $\left.((x+3)\cdot P_2(x))\right|_{x=-3}=2$

- ◦ $(x+3)^2\cdot P_3(x)$ is analytic at $x=-3$

  $\left.((x+3)^2\cdot P_3(x))\right|_{x=-3}=0$

- ◦ $x=-3$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-3$

- Multiply by denominators

  $9x^2(x+3)\left(\frac{d^2}{dx^2}y(x)\right)+3x(7x+3)\left(\frac{d}{dx}y(x)\right)+(3+4x)\,y(x)=0$

- Change variables using $x=u-3$ so that the regular singular point is at $u=0$

  $(9u^3-54u^2+81u)\left(\frac{d^2}{du^2}y(u)\right)+(21u^2-117u+162)\left(\frac{d}{du}y(u)\right)+(-9+4u)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

- ◦ Shift index using $k->k-m$

  $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ◦ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- ◦ Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$81a_0r(1+r)\,u^{-1+r} + (81a_1(1+r)(2+r) - 9a_0(1+r)(1+6r))\,u^r + \left(\sum_{k=1}^{\infty}(81a_{k+1}(k+r+1)(k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $81r(1+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-1, 0\}$
- Each term must be 0
  $81a_1(1+r)(2+r) - 9a_0(1+r)(1+6r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $81a_{k+1}(k+r+1)(k+2+r) - 54(k+r+1)\,a_k\left(k+r+\frac{1}{6}\right) + a_{k-1}(3k-1+3r)^2 = 0$
- Shift index using $k- >k+1$
  $81a_{k+2}(k+2+r)(k+3+r) - 54(k+2+r)\,a_{k+1}\left(k+\frac{7}{6}+r\right) + a_k(3k+3r+2)^2 = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{9k^2a_k-54k^2a_{k+1}+18kra_k-108kra_{k+1}+9r^2a_k-54r^2a_{k+1}+12ka_k-171ka_{k+1}+12ra_k-171ra_{k+1}+4a_k-126a_{k+1}}{81(k+2+r)(k+3+r)}$
- Recursion relation for $r = -1$
  $a_{k+2} = -\frac{9k^2a_k-54k^2a_{k+1}-6ka_k-63ka_{k+1}+a_k-9a_{k+1}}{81(k+1)(k+2)}$
- Solution for $r = -1$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-1}, a_{k+2} = -\frac{9k^2a_k-54k^2a_{k+1}-6ka_k-63ka_{k+1}+a_k-9a_{k+1}}{81(k+1)(k+2)}, 0 = 0\right]$
- Revert the change of variables $u = x+3$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+3)^{k-1}, a_{k+2} = -\frac{9k^2a_k-54k^2a_{k+1}-6ka_k-63ka_{k+1}+a_k-9a_{k+1}}{81(k+1)(k+2)}, 0 = 0\right]$
- Recursion relation for $r = 0$
  $a_{k+2} = -\frac{9k^2a_k-54k^2a_{k+1}+12ka_k-171ka_{k+1}+4a_k-126a_{k+1}}{81(k+2)(k+3)}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{9k^2a_k-54k^2a_{k+1}+12ka_k-171ka_{k+1}+4a_k-126a_{k+1}}{81(k+2)(k+3)}, 162a_1 - 9a_0 = 0\right]$
- Revert the change of variables $u = x+3$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+3)^k, a_{k+2} = -\frac{9k^2a_k-54k^2a_{k+1}+12ka_k-171ka_{k+1}+4a_k-126a_{k+1}}{81(k+2)(k+3)}, 162a_1 - 9a_0 = 0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k(x+3)^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k(x+3)^k\right), a_{k+2} = -\frac{9k^2a_k-54k^2a_{k+1}-6ka_k-63ka_{k+1}+a_k-9a_{k+1}}{81(k+1)(k+2)}, 0\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
```

```
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.030 (sec)
Leaf size : 19

```
dsolve(9*x^2*(x+3)*diff(diff(y(x),x),x)+3*x*(3+7*x)*diff(y(x),x)+(4*x+3)*y(x) = 0,y(x),s
```

$$y = \frac{x^{1/3}(c_2 \ln(x) + c_1)}{x + 3}$$

Mathematica DSolve solution

Solving time : 0.303 (sec)
Leaf size : 49

```
DSolve[{9*x^2*(3+x)*D[y[x],{x,2}]+3*x*(3+7*x)*D[y[x],x]+(3+4*x)*y[x]==0,{}},y[x],x,IncludeSing
```

$$y(x) \to \sqrt{x}(c_2 \log(x) + c_1) \exp\left(-\frac{1}{2}\int_1^x \left(\frac{2}{K[1]+3} + \frac{1}{3K[1]}\right) dK[1]\right)$$

2.1.158 Problem 160

Internal problem ID [9330]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 160
Date solved : `Monday, January 27, 2025 at 06:01:34 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(-x^2+2\right) y''-x\left(3x^2+2\right) y'+\left(-x^2+2\right) y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.168 (sec)

Writing the ode as

$$\left(-x^4+2x^2\right) y''+\left(-3x^3-2x\right) y'+\left(-x^2+2\right) y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^4+2x^2 \\ B &= -3x^3-2x \\ C &= -x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.300: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=-\frac{1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}\right)\\
&= \frac{1}{2}-\left(\frac{1}{2}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^{-}}{x-c_1}\right)+(-)[\sqrt{r}]_\infty\\
&= \frac{1}{2x}+(-)\,(0)\\
&= \frac{1}{2x}\\
&= \frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}
$$

Let

$$
p(x)=1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{1}{2x}\right)(0)+\left(\left(-\frac{1}{2x^2}\right)+\left(\frac{1}{2x}\right)^2-\left(-\frac{1}{4x^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}z_1(x)&=pe^{\int\omega\,dx}\\&=\mathrm{e}^{\int\frac{1}{2x}dx}\\&=\sqrt{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1&=z_1e^{\int-\frac{1}{2}\frac{B}{A}\,dx}\\&=z_1e^{-\int\frac{1}{2}\frac{-3x^3-2x}{-x^4+2x^2}\,dx}\\&=z_1e^{-\ln(x^2-2)+\frac{\ln(x)}{2}}\\&=z_1\left(\frac{\sqrt{x}}{x^2-2}\right)\end{aligned}$$

Which simplifies to

$$y_1=\frac{x}{x^2-2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int\frac{e^{\int-\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2&=y_1\int\frac{e^{\int-\frac{-3x^3-2x}{-x^4+2x^2}\,dx}}{(y_1)^2}\,dx\\&=y_1\int\frac{e^{-2\ln(x^2-2)+\ln(x)}}{(y_1)^2}\,dx\\&=y_1(\ln(x))\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y&=c_1y_1+c_2y_2\\&=c_1\left(\frac{x}{x^2-2}\right)+c_2\left(\frac{x}{x^2-2}(\ln(x))\right)\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(-x^2+2)\left(\frac{d^2}{dx^2}y(x)\right)-x(3x^2+2)\left(\frac{d}{dx}y(x)\right)+(-x^2+2)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{y(x)}{x^2}-\frac{(3x^2+2)\left(\frac{d}{dx}y(x)\right)}{x(x^2-2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{(3x^2+2)\left(\frac{d}{dx}y(x)\right)}{x(x^2-2)}+\frac{y(x)}{x^2}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{3x^2+2}{x(x^2-2)},P_3(x)=\frac{1}{x^2}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\Big|_{x=0}=-1$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\Big|_{x=0}=1$

  - $x=0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=0$

- Multiply by denominators

  $x^2(x^2-2)\left(\frac{d^2}{dx^2}y(x)\right)+x(3x^2+2)\left(\frac{d}{dx}y(x)\right)+(x^2-2)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

    $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..4$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

    Rewrite ODE with series expansions

$$-2a_0(-1+r)^2 x^r - 2a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(-2a_k(k+r-1)^2 + a_{k-2}(k+r-1)^2\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-2(-1+r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 1$
- Each term must be 0
  $-2a_1 r^2 = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $-2\left(a_k - \frac{a_{k-2}}{2}\right)(k+r-1)^2 = 0$
- Shift index using $k- >k+2$
  $-2\left(a_{k+2} - \frac{a_k}{2}\right)(k+r+1)^2 = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{a_k}{2}$
- Recursion relation for $r = 1$
  $a_{k+2} = \frac{a_k}{2}$
- Solution for $r = 1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = \frac{a_k}{2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 19

```
dsolve(x^2*(-x^2+2)*diff(diff(y(x),x),x)-x*(3*x^2+2)*diff(y(x),x)+(-x^2+2)*y(x) = 0,y(x)
```

$$y = \frac{x(c_2 \ln(x) + c_1)}{x^2 - 2}$$

Mathematica DSolve solution

Solving time : 0.267 (sec)
Leaf size : 51

```
DSolve[{x^2*(2-x^2)*D[y[x],{x,2}]-x*(2+3*x^2)*D[y[x],x]+(2-x^2)*y[x]==0,{}},y[x],x,IncludeSi
```

$$y(x) \to \sqrt{x}(c_2 \log(x) + c_1) \exp\left(-\frac{1}{2}\int_1^x \left(\frac{4K[1]}{K[1]^2-2} - \frac{1}{K[1]}\right) dK[1]\right)$$

2.1.159 Problem 161

Internal problem ID [9331]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 161
Date solved : `Monday, January 27, 2025 at 06:01:34 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$16x^2\left(x^2+1\right)y''+8x\left(9x^2+1\right)y'+\left(49x^2+1\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.217 (sec)

Writing the ode as

$$\left(16x^4+16x^2\right)y''+\left(72x^3+8x\right)y'+\left(49x^2+1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 16x^4+16x^2 \\ B &= 72x^3+8x \\ C &= 49x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{1}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.302: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the gcd$(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{1}{2x}\right)(0)+\left(\left(-\frac{1}{2x^2}\right)+\left(\frac{1}{2x}\right)^2-\left(-\frac{1}{4x^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}z_1(x)&=pe^{\int \omega\,dx}\\&=\mathrm{e}^{\int \frac{1}{2x}dx}\\&=\sqrt{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1&=z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\&=z_1e^{-\int \frac{1}{2}\frac{72x^3+8x}{16x^4+16x^2}\,dx}\\&=z_1e^{-\frac{\ln(x)}{4}-\ln(x^2+1)}\\&=z_1\left(\frac{1}{x^{1/4}\,(x^2+1)}\right)\end{aligned}$$

Which simplifies to

$$y_1=\frac{x^{1/4}}{x^2+1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2&=y_1\int \frac{e^{\int -\frac{72x^3+8x}{16x^4+16x^2}\,dx}}{(y_1)^2}\,dx\\&=y_1\int \frac{e^{-\frac{\ln(x)}{2}-2\ln(x^2+1)}}{(y_1)^2}\,dx\\&=y_1\left(\ln(x)-\ln\left(x^2+1\right)x^2-\ln\left(x^2+1\right)+\frac{x^4}{2x^2+2}+\frac{x^2}{x^2+1}+\frac{1}{2x^2+2}\right.\\&\qquad\left.+\ln\left(x^2+1\right)\left(x^2+1\right)-\frac{x^2}{2}-1\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y&=c_1y_1+c_2y_2\\&=c_1\left(\frac{x^{1/4}}{x^2+1}\right)\\&\quad+c_2\left(\frac{x^{1/4}}{x^2+1}\left(\ln(x)-\ln\left(x^2+1\right)x^2-\ln\left(x^2+1\right)+\frac{x^4}{2x^2+2}+\frac{x^2}{x^2+1}+\frac{1}{2x^2+2}+\ln\left(x^2+1\right)\left(x^2+1\right)-\frac{x^2}{2}\right.\right.\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$16x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)+8x(9x^2+1)\left(\frac{d}{dx}y(x)\right)+(49x^2+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(49x^2+1)y(x)}{16x^2(x^2+1)}-\frac{(9x^2+1)\left(\frac{d}{dx}y(x)\right)}{2x(x^2+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d^2}{dx^2}y(x)+\frac{(9x^2+1)\left(\frac{d}{dx}y(x)\right)}{2x(x^2+1)}+\frac{(49x^2+1)y(x)}{16x^2(x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{9x^2+1}{2x(x^2+1)}, P_3(x)=\frac{49x^2+1}{16x^2(x^2+1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=\frac{1}{2}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=\frac{1}{16}$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $16x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)+8x(9x^2+1)\left(\frac{d}{dx}y(x)\right)+(49x^2+1)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  ◦ Shift index using $k->k-m$

  $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..4$

  $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  ◦ Shift index using $k->k+2-m$

  $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$a_0(-1+4r)^2 x^r + a_1(3+4r)^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(4k+4r-1)^2 + a_{k-2}(4k+4r-1)^2\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+4r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = \frac{1}{4}$
- Each term must be 0
  $a_1(3+4r)^2 = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $(4k+4r-1)^2 (a_k + a_{k-2}) = 0$
- Shift index using $k- >k+2$
  $(4k+4r+7)^2 (a_{k+2} + a_k) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -a_k$
- Recursion relation for $r = \frac{1}{4}$
  $a_{k+2} = -a_k$
- Solution for $r = \frac{1}{4}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+2} = -a_k, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.042 (sec)
Leaf size : 21

```
dsolve(16*x^2*(x^2+1)*diff(diff(y(x),x),x)+8*x*(9*x^2+1)*diff(y(x),x)+(49*x^2+1)*y(x)
```

$$y = \frac{x^{1/4}(c_2 \ln(x) + c_1)}{x^2+1}$$

Mathematica DSolve solution

Solving time : 0.244 (sec)
Leaf size : 53

```
DSolve[{16*x^2*(1+x^2)*D[y[x],{x,2}]+8*x*(1+9*x^2)*D[y[x],x]+(1+49*x^2)*y[x]==0,{}},y[x],x,Inc
```

$$y(x) \to \sqrt{x}(c_2 \log(x) + c_1) \exp\left(-\frac{1}{2}\int_1^x \frac{9K[1]^2 + 1}{2\left(K[1]^3 + K[1]\right)} dK[1]\right)$$

2.1.160 Problem 162

Internal problem ID [9332]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 162
Date solved : `Monday, January 27, 2025 at 06:01:35 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(4+3x)\,y''-x(4-3x)\,y'+4y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.161 (sec)

Writing the ode as

$$\left(3x^3+4x^2\right)y''+\left(3x^2-4x\right)y'+4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=3x^3+4x^2\\B&=3x^2-4x\\C&=4\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=-1\\t&=4x^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{1}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.304: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\,(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0)+2\left(\frac{1}{2x}\right)(0)+\left(\left(-\frac{1}{2x^2}\right)+\left(\frac{1}{2x}\right)^2-\left(-\frac{1}{4x^2}\right)\right) &= 0\\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$
\begin{aligned}
z_1(x) &= pe^{\int \omega \, dx}\\
&= \mathrm{e}^{\int \frac{1}{2x} dx}\\
&= \sqrt{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}\\
&= z_1 e^{-\int \frac{1}{2}\frac{3x^2-4x}{3x^3+4x^2} \, dx}\\
&= z_1 e^{\frac{\ln(x)}{2}-\ln(4+3x)}\\
&= z_1 \left(\frac{\sqrt{x}}{4+3x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x}{4+3x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^2-4x}{3x^3+4x^2} \, dx}}{(y_1)^2} \, dx\\
&= y_1 \int \frac{e^{\ln(x)-2\ln(4+3x)}}{(y_1)^2} \, dx\\
&= y_1 (\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2\\
&= c_1\left(\frac{x}{4+3x}\right) + c_2\left(\frac{x}{4+3x}(\ln(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(3x+4)\left(\frac{d^2}{dx^2}y(x)\right)-x(4-3x)\left(\frac{d}{dx}y(x)\right)+4y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{4y(x)}{x^2(3x+4)}-\frac{(3x-4)\left(\frac{d}{dx}y(x)\right)}{x(3x+4)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)+\frac{(3x-4)\left(\frac{d}{dx}y(x)\right)}{x(3x+4)}+\frac{4y(x)}{x^2(3x+4)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=\frac{3x-4}{x(3x+4)},P_3(x)=\frac{4}{x^2(3x+4)}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

$(x\cdot P_2(x))\Big|_{x=0}=-1$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$(x^2\cdot P_3(x))\Big|_{x=0}=1$

- ◦ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$x^2(3x+4)\left(\frac{d^2}{dx^2}y(x)\right)+x(3x-4)\left(\frac{d}{dx}y(x)\right)+4y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ◦ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..3$

$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- ◦ Shift index using $k->k+2-m$

$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

Rewrite ODE with series expansions

$4a_0(-1+r)^2\,x^r+\left(\sum_{k=1}^{\infty}\left(4a_k(k+r-1)^2+3a_{k-1}(k+r-1)^2\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$4(-1+r)^2=0$

- Values of r that satisfy the indicial equation

$r = 1$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)^2 (4a_k + 3a_{k-1}) = 0$

- Shift index using $k- >k+1$

  $(k+r)^2 (4a_{k+1} + 3a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{3a_k}{4}$

- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{3a_k}{4}$

- Solution for $r = 1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{3a_k}{4}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 19

```
dsolve(x^2*(3*x+4)*diff(diff(y(x),x),x)-x*(4-3*x)*diff(y(x),x)+4*y(x) = 0,y(x),singsol=a
```

$$y = \frac{x(c_2 \ln(x) + c_1)}{3x + 4}$$

Mathematica DSolve solution

Solving time : `0.251` (sec)
Leaf size : 49

```
DSolve[{x^2*(4+3*x)*D[y[x],{x,2}]-x*(4-3*x)*D[y[x],x]+4*y[x]==0,{}},y[x],x,IncludeSingularSolut
```

$$y(x) \to \sqrt{x}(c_2 \log(x) + c_1) \exp\left(-\frac{1}{2} \int_1^x \left(\frac{6}{3K[1]+4} - \frac{1}{K[1]}\right) dK[1]\right)$$

2.1.161 Problem 163

Internal problem ID [9333]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 163
Date solved : `Monday, January 27, 2025 at 06:01:35 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2\left(x^2+3x+1\right)y''+8x^2(3+2x)\,y'+\left(9x^2+3x+1\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.156 (sec)

Writing the ode as

$$\left(4x^4+12x^3+4x^2\right)y''+\left(16x^3+24x^2\right)y'+\left(9x^2+3x+1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=4x^4+12x^3+4x^2\\B&=16x^3+24x^2\\C&=9x^2+3x+1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=-1\\t&=4x^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{1}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.306: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2}\\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=-\frac{1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right)\\
&= \frac{1}{2}-\left(\frac{1}{2}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^-}{x-c_1}\right)+(-)[\sqrt{r}]_\infty\\
&= \frac{1}{2x}+(-)\,(0)\\
&= \frac{1}{2x}\\
&= \frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}
$$

Let

$$
p(x)=1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{1}{2x}\right)(0)+\left(\left(-\frac{1}{2x^2}\right)+\left(\frac{1}{2x}\right)^2-\left(-\frac{1}{4x^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}z_1(x)&=pe^{\int \omega\,dx}\\&=\mathrm{e}^{\int \frac{1}{2x}dx}\\&=\sqrt{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1&=z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\&=z_1e^{-\int \frac{1}{2}\frac{16x^3+24x^2}{4x^4+12x^3+4x^2}\,dx}\\&=z_1e^{-\ln(x^2+3x+1)}\\&=z_1\left(\frac{1}{x^2+3x+1}\right)\end{aligned}$$

Which simplifies to

$$y_1=\frac{\sqrt{x}}{x^2+3x+1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2&=y_1\int \frac{e^{\int -\frac{16x^3+24x^2}{4x^4+12x^3+4x^2}\,dx}}{(y_1)^2}\,dx\\&=y_1\int \frac{e^{-2\ln(x^2+3x+1)}}{(y_1)^2}\,dx\\&=y_1(\ln(x))\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y&=c_1y_1+c_2y_2\\&=c_1\left(\frac{\sqrt{x}}{x^2+3x+1}\right)+c_2\left(\frac{\sqrt{x}}{x^2+3x+1}(\ln(x))\right)\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2(x^2+3x+1)\left(\frac{d^2}{dx^2}y(x)\right)+8x^2(2x+3)\left(\frac{d}{dx}y(x)\right)+(9x^2+3x+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(9x^2+3x+1)y(x)}{4x^2(x^2+3x+1)}-\frac{2(2x+3)\left(\frac{d}{dx}y(x)\right)}{x^2+3x+1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{2(2x+3)\left(\frac{d}{dx}y(x)\right)}{x^2+3x+1}+\frac{(9x^2+3x+1)y(x)}{4x^2(x^2+3x+1)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{2(2x+3)}{x^2+3x+1},P_3(x)=\frac{9x^2+3x+1}{4x^2(x^2+3x+1)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\bigg|_{x=0}=0$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\bigg|_{x=0}=\frac{1}{4}$

  - $x=0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=0$

- Multiply by denominators

  $4x^2(x^2+3x+1)\left(\frac{d^2}{dx^2}y(x)\right)+8x^2(2x+3)\left(\frac{d}{dx}y(x)\right)+(9x^2+3x+1)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

    $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=2..3$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..4$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$a_0(-1+2r)^2 x^r + \left(a_1(1+2r)^2 + 3a_0(1+2r)^2\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r-1)^2 + 3a_{k-1}(2k+2r-1\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+2r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = \frac{1}{2}$
- Each term must be 0
  $a_1(1+2r)^2 + 3a_0(1+2r)^2 = 0$
- Solve for the dependent coefficient(s)
  $a_1 = -3a_0$
- Each term in the series must be 0, giving the recursion relation
  $(2k+2r-1)^2 (a_k + 3a_{k-1} + a_{k-2}) = 0$
- Shift index using $k- >k+2$
  $(2k+2r+3)^2 (a_{k+2} + 3a_{k+1} + a_k) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -3a_{k+1} - a_k$
- Recursion relation for $r = \frac{1}{2}$
  $a_{k+2} = -3a_{k+1} - a_k$
- Solution for $r = \frac{1}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -3a_{k+1} - a_k, a_1 = -3a_0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.040` (sec)
Leaf size : 24

```
dsolve(4*x^2*(x^2+3*x+1)*diff(diff(y(x),x),x)+8*x^2*(2*x+3)*diff(y(x),x)+(9*x^2+3*x+1)*y
```

$$y = \frac{\sqrt{x}\,(c_2 \ln(x) + c_1)}{x^2 + 3x + 1}$$

Mathematica DSolve solution

Solving time : 0.058 (sec)
Leaf size : 29

```
DSolve[{4*x^2*(1+3*x+x^2)*D[y[x],{x,2}]+8*x^2*(3+2*x)*D[y[x],x]+(1+3*x+9*x^2)*y[x]==0,{}},y[
```

$$y(x) \to \frac{\sqrt{x}(c_2 \log(x) + c_1)}{x^2 + 3x + 1}$$

2.1.162 Problem 164

Internal problem ID [9334]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 164
Date solved : `Monday, January 27, 2025 at 06:01:36 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1-x)^2 y'' - x\left(-3x^2+2x+1\right) y' + \left(x^2+1\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.159 (sec)

Writing the ode as

$$x^2(-1+x)^2 y'' + \left(3x^3-2x^2-x\right) y' + \left(x^2+1\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(-1+x)^2 \\ B &= 3x^3-2x^2-x \\ C &= x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.308: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2}\\ \alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=-\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$\begin{aligned}d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}\right)\\ &= \frac{1}{2}-\left(\frac{1}{2}\right)\\ &= 0\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^{-}}{x-c_1}\right)+(-)[\sqrt{r}]_\infty\\ &= \frac{1}{2x}+(-)\,(0)\\ &= \frac{1}{2x}\\ &= \frac{1}{2x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{1}{2x}\right)(0)+\left(\left(-\frac{1}{2x^2}\right)+\left(\frac{1}{2x}\right)^2-\left(-\frac{1}{4x^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}z_1(x)&=pe^{\int \omega\,dx}\\&=\mathrm{e}^{\int \frac{1}{2x}dx}\\&=\sqrt{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1&=z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\&=z_1e^{-\int \frac{1}{2}\frac{3x^3-2x^2-x}{x^2(-1+x)^2}\,dx}\\&=z_1e^{-2\ln(-1+x)+\frac{\ln(x)}{2}}\\&=z_1\left(\frac{\sqrt{x}}{(-1+x)^2}\right)\end{aligned}$$

Which simplifies to

$$y_1=\frac{x}{(-1+x)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2&=y_1\int \frac{e^{\int -\frac{3x^3-2x^2-x}{x^2(-1+x)^2}\,dx}}{(y_1)^2}\,dx\\&=y_1\int \frac{e^{-4\ln(-1+x)+\ln(x)}}{(y_1)^2}\,dx\\&=y_1(\ln(x))\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y&=c_1y_1+c_2y_2\\&=c_1\left(\frac{x}{(-1+x)^2}\right)+c_2\left(\frac{x}{(-1+x)^2}(\ln(x))\right)\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(1-x)^2\left(\frac{d^2}{dx^2}y(x)\right)-x(-3x^2+2x+1)\left(\frac{d}{dx}y(x)\right)+(x^2+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(x^2+1)y(x)}{x^2(x-1)^2}-\frac{\left(\frac{d}{dx}y(x)\right)(3x+1)}{x(x-1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{\left(\frac{d}{dx}y(x)\right)(3x+1)}{x(x-1)}+\frac{(x^2+1)y(x)}{x^2(x-1)^2}=0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=\frac{3x+1}{x(x-1)},P_3(x)=\frac{x^2+1}{x^2(x-1)^2}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=-1$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=1$

- ○ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(x-1)^2\left(\frac{d^2}{dx^2}y(x)\right)+x(x-1)(3x+1)\left(\frac{d}{dx}y(x)\right)+(x^2+1)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ○ Shift index using $k->k-m$

  $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ○ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ○ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..4$

  $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- ○ Shift index using $k->k+2-m$

  $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

Rewrite ODE with series expansions

$$a_0(-1+r)^2 x^r + (-2a_0r^2 + a_1r^2) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)^2 - 2a_{k-1}(k+r-1)^2 + a_{k-2}(k+r\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 1$
- Each term must be 0
  $-2a_0r^2 + a_1r^2 = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 2a_0$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-1)^2 (a_k - 2a_{k-1} + a_{k-2}) = 0$
- Shift index using $k->k+2$
  $(k+r+1)^2 (a_{k+2} - 2a_{k+1} + a_k) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = 2a_{k+1} - a_k$
- Recursion relation for $r = 1$
  $a_{k+2} = 2a_{k+1} - a_k$
- Solution for $r = 1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = 2a_{k+1} - a_k, a_1 = 2a_0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 17

```
dsolve(x^2*(1-x)^2*diff(diff(y(x),x),x)-x*(-3*x^2+2*x+1)*diff(y(x),x)+(x^2+1)*y(x) = 0
```

$$y = \frac{x(c_2 \ln(x) + c_1)}{(x-1)^2}$$

Mathematica DSolve solution

Solving time : 0.283 (sec)
Leaf size : 47

```
DSolve[{x^2*(1-x)^2*D[y[x],{x,2}]-x*(1+2*x-3*x^2)*D[y[x],x]+(1+x^2)*y[x]==0,{}},y[x],x,IncludeS
```

$$y(x) \rightarrow \sqrt{x}(c_2 \log(x) + c_1) \exp\left(-\frac{1}{2} \int_1^x \left(\frac{4}{K[1]-1} - \frac{1}{K[1]}\right) dK[1]\right)$$

2.1.163 Problem 165

Internal problem ID [9335]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 165
Date solved : `Monday, January 27, 2025 at 06:01:37 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$9x^2\left(x^2+x+1\right)y''+3x\left(13x^2+7x+1\right)y'+\left(25x^2+4x+1\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.267 (sec)

Writing the ode as

$$\left(9x^4+9x^3+9x^2\right)y''+\left(39x^3+21x^2+3x\right)y'+\left(25x^2+4x+1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^4+9x^3+9x^2 \\ B &= 39x^3+21x^2+3x \\ C &= 25x^2+4x+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{1}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.310: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the gcd$(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\,(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{1}{2x}\right)(0)+\left(\left(-\frac{1}{2x^2}\right)+\left(\frac{1}{2x}\right)^2-\left(-\frac{1}{4x^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\,dx} \\ &= \mathrm{e}^{\int \frac{1}{2x}dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{39x^3+21x^2+3x}{9x^4+9x^3+9x^2}\,dx} \\ &= z_1 e^{-\ln(x^2+x+1)-\frac{\ln(x)}{6}} \\ &= z_1\left(\frac{1}{(x^2+x+1)\,x^{1/6}}\right) \end{aligned}$$

Which simplifies to

$$y_1=\frac{x^{1/3}}{x^2+x+1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1\int \frac{e^{\int -\frac{39x^3+21x^2+3x}{9x^4+9x^3+9x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1\int \frac{e^{-2\ln(x^2+x+1)-\frac{\ln(x)}{3}}}{(y_1)^2}\,dx \\ &= y_1\left(2x-\frac{19}{24}+(x-1)^2-\frac{x^5}{3(x^2+x+1)}+\frac{x}{3x^2+3x+3}-\frac{x^4}{3(x^2+x+1)}\right. \\ &\qquad \left.-\frac{x^3}{3(x^2+x+1)}+\frac{x^2}{3x^2+3x+3}+\frac{1}{3x^2+3x+3}+\frac{x^3}{3}-x^2+\ln(x)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1y_1+c_2y_2 \\ &= c_1\left(\frac{x^{1/3}}{x^2+x+1}\right) \\ &\quad +c_2\left(\frac{x^{1/3}}{x^2+x+1}\left(2x-\frac{19}{24}+(x-1)^2-\frac{x^5}{3(x^2+x+1)}+\frac{x}{3x^2+3x+3}-\frac{x^4}{3(x^2+x+1)}-\frac{x^3}{3(x^2+x+1)}+\frac{}{3\ldots}\right.\right. \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$9x^2(x^2+x+1)\left(\frac{d^2}{dx^2}y(x)\right)+3x(13x^2+7x+1)\left(\frac{d}{dx}y(x)\right)+(25x^2+4x+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(25x^2+4x+1)y(x)}{9x^2(x^2+x+1)}-\frac{(13x^2+7x+1)\left(\frac{d}{dx}y(x)\right)}{3x(x^2+x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{(13x^2+7x+1)\left(\frac{d}{dx}y(x)\right)}{3x(x^2+x+1)}+\frac{(25x^2+4x+1)y(x)}{9x^2(x^2+x+1)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=\frac{13x^2+7x+1}{3x(x^2+x+1)},P_3(x)=\frac{25x^2+4x+1}{9x^2(x^2+x+1)}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=\frac{1}{3}$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=\frac{1}{9}$

- ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $9x^2(x^2+x+1)\left(\frac{d^2}{dx^2}y(x)\right)+3x(13x^2+7x+1)\left(\frac{d}{dx}y(x)\right)+(25x^2+4x+1)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ◦ Shift index using $k->k-m$

  $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ◦ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..4$

  $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- ◦ Shift index using $k->k+2-m$

  $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$a_0(-1+3r)^2 x^r + \left(a_1(2+3r)^2 + a_0(2+3r)^2\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r-1)^2 + a_{k-1}(3k+3r-1)^2\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+3r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = \frac{1}{3}$
- Each term must be 0
  $a_1(2+3r)^2 + a_0(2+3r)^2 = 0$
- Solve for the dependent coefficient(s)
  $a_1 = -a_0$
- Each term in the series must be 0, giving the recursion relation
  $(3k+3r-1)^2 (a_k + a_{k-1} + a_{k-2}) = 0$
- Shift index using $k- >k+2$
  $(3k+3r+5)^2 (a_{k+2} + a_{k+1} + a_k) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -a_{k+1} - a_k$
- Recursion relation for $r = \frac{1}{3}$
  $a_{k+2} = -a_{k+1} - a_k$
- Solution for $r = \frac{1}{3}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -a_{k+1} - a_k, a_1 = -a_0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.030` (sec)
Leaf size : 22

```
dsolve(9*x^2*(x^2+x+1)*diff(diff(y(x),x),x)+3*x*(13*x^2+7*x+1)*diff(y(x),x)+(25*x^2+4*x+
```

$$y = \frac{x^{1/3}(c_2 \ln(x) + c_1)}{x^2 + x + 1}$$

Mathematica DSolve solution

Solving time : `0.294` (sec)
Leaf size : 58

```
DSolve[{9*x^2*(1+x+x^2)*D[y[x],{x,2}]+3*x*(1+7*x+13*x^2)*D[y[x],x]+(1+4*x+25*x^2)*y[x]==0,{}
```

$$y(x)\to \sqrt{x}(c_2\log(x)+c_1)\exp\left(-\frac{1}{2}\int_1^x\left(\frac{4K[1]+2}{K[1]^2+K[1]+1}+\frac{1}{3K[1]}\right)dK[1]\right)$$

2.1.164 Problem 166

Internal problem ID [9336]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 166
Date solved : `Monday, January 27, 2025 at 06:01:37 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(2+x)\,y''-x(4-7x)\,y'-(5-3x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.219 (sec)

Writing the ode as

$$\left(2x^3+4x^2\right)y''+\left(7x^2-4x\right)y'+(3x-5)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^3+4x^2 \\ B &= 7x^2-4x \\ C &= 3x-5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-3x^2-32x+128}{16\left(x^2+2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3x^2-32x+128 \\ t &= 16\left(x^2+2x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-3x^2-32x+128}{16\left(x^2+2x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.312: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{45}{16\,(2+x)^2} - \frac{5}{2x} + \frac{2}{x^2} + \frac{5}{2\,(2+x)}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(2+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{45}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{4} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 - 32x + 128}{16\left(x^2 + 2x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x^2 - 32x + 128}{16\left(x^2 + 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{9}{4}$	$-\frac{5}{4}$
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= \frac{3}{4} - \left(\frac{3}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{5}{4\left(2+x\right)} + \frac{2}{x} + (0) \\ &= -\frac{5}{4\left(2+x\right)} + \frac{2}{x} \\ &= \frac{3x+16}{4x\left(2+x\right)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{5}{4(2+x)} + \frac{2}{x}\right)(0) + \left(\left(\frac{5}{4(2+x)^2} - \frac{2}{x^2}\right) + \left(-\frac{5}{4(2+x)} + \frac{2}{x}\right)^2 - \left(\frac{-3x^2 - 32x + 128}{16(x^2+2x)^2}\right)\right) =$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{5}{4(2+x)} + \frac{2}{x}\right) dx} \\
&= \frac{x^2}{(2+x)^{5/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7x^2-4x}{2x^3+4x^2}\, dx} \\
&= z_1 e^{-\frac{9\ln(2+x)}{4} + \frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\sqrt{x}}{(2+x)^{9/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{5/2}}{(2+x)^{7/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x^2-4x}{2x^3+4x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{9\ln(2+x)}{2} + \ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{-\frac{11(2+x)^{5/2}}{8} + \frac{10(2+x)^{3/2}}{3} - \frac{5\sqrt{2+x}}{2}}{x^3} - \frac{5\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right)}{16}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{5/2}}{(2+x)^{7/2}}\right) \\
&\quad + c_2 \left(\frac{x^{5/2}}{(2+x)^{7/2}} \left(\frac{-\frac{11(2+x)^{5/2}}{8} + \frac{10(2+x)^{3/2}}{3} - \frac{5\sqrt{2+x}}{2}}{x^3} - \frac{5\sqrt{2}\,\mathrm{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right)}{16}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2(x+2)\left(\frac{d^2}{dx^2}y(x)\right) - x(4-7x)\left(\frac{d}{dx}y(x)\right) - (5-3x)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(3x-5)y(x)}{2(x+2)x^2} - \frac{(-4+7x)\left(\frac{d}{dx}y(x)\right)}{2x(x+2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(-4+7x)\left(\frac{d}{dx}y(x)\right)}{2x(x+2)} + \frac{(3x-5)y(x)}{2(x+2)x^2} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{-4+7x}{2x(x+2)}, P_3(x) = \frac{3x-5}{2(x+2)x^2}\right]$

  ◦ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

    $\left((x+2)\cdot P_2(x)\right)\bigg|_{x=-2} = \frac{9}{2}$

  ◦ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

    $\left((x+2)^2\cdot P_3(x)\right)\bigg|_{x=-2} = 0$

  ◦ $x=-2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -2$

- Multiply by denominators

  $2x^2(x+2)\left(\frac{d^2}{dx^2}y(x)\right) + x(-4+7x)\left(\frac{d}{dx}y(x)\right) + (3x-5)\,y(x) = 0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

  $(2u^3 - 8u^2 + 8u)\left(\frac{d^2}{du^2}y(u)\right) + (7u^2 - 32u + 36)\left(\frac{d}{du}y(u)\right) + (3u-11)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

    $u^m\cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

  ◦ Shift index using $k->k-m$

    $u^m\cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$4a_0r(7+2r)\, u^{-1+r} + (4a_1(1+r)\,(9+2r) - a_0(8r^2+24r+11))\, u^r + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+r+1)\,(2k\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $4r(7+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{0, -\frac{7}{2}\right\}$
- Each term must be 0
  $4a_1(1+r)\,(9+2r) - a_0(8r^2+24r+11) = 0$
- Each term in the series must be 0, giving the recursion relation
  $2(-4a_k + a_{k-1} + 4a_{k+1})\, k^2 + (4(-4a_k + a_{k-1} + 4a_{k+1})\, r - 24a_k + a_{k-1} + 44a_{k+1})\, k + 2(-4a_k +$
- Shift index using $k->k+1$
  $2(-4a_{k+1} + a_k + 4a_{k+2})\,(k+1)^2 + (4(-4a_{k+1} + a_k + 4a_{k+2})\, r - 24a_{k+1} + a_k + 44a_{k+2})\,(k+1)$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 4kra_k - 16kra_{k+1} + 2r^2a_k - 8r^2a_{k+1} + 5ka_k - 40ka_{k+1} + 5ra_k - 40ra_{k+1} + 3a_k - 43a_{k+1}}{4(2k^2+4kr+2r^2+15k+15r+22)}$
- Recursion relation for $r = 0$
  $a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 5ka_k - 40ka_{k+1} + 3a_k - 43a_{k+1}}{4(2k^2+15k+22)}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 5ka_k - 40ka_{k+1} + 3a_k - 43a_{k+1}}{4(2k^2+15k+22)}, 36a_1 - 11a_0 = 0\right]$
- Revert the change of variables $u = x+2$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+2)^k, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 5ka_k - 40ka_{k+1} + 3a_k - 43a_{k+1}}{4(2k^2+15k+22)}, 36a_1 - 11a_0 = 0\right]$
- Recursion relation for $r = -\frac{7}{2}$
  $a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} - 9ka_k + 16ka_{k+1} + 10a_k - a_{k+1}}{4(2k^2+k-6)}$
- Solution for $r = -\frac{7}{2}$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{7}{2}}, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} - 9ka_k + 16ka_{k+1} + 10a_k - a_{k+1}}{4(2k^2+k-6)}, -20a_1 - 25a_0 = 0\right]$
- Revert the change of variables $u = x+2$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+2)^{k-\frac{7}{2}}, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} - 9ka_k + 16ka_{k+1} + 10a_k - a_{k+1}}{4(2k^2+k-6)}, -20a_1 - 25a_0 = 0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k(x+2)^k\right) + \left(\sum_{k=0}^{\infty} b_k(x+2)^{k-\frac{7}{2}}\right), a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 5ka_k - 40ka_{k+1} + 3a_k - 43a_{k+1}}{4(2k^2+15k+22)},\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.042 (sec)
Leaf size : 55

```
dsolve(2*x^2*(x+2)*diff(diff(y(x),x),x)-x*(4-7*x)*diff(y(x),x)-(5-3*x)*y(x) = 0,y(x),sin
```

$$y = \frac{15 \operatorname{arctanh}\left(\frac{\sqrt{2}\sqrt{x+2}}{2}\right) c_2 x^3 + 33 c_2 \sqrt{2}\left(x^2 + \frac{52}{33}x + \frac{32}{33}\right)\sqrt{x+2} + c_1 x^3}{\sqrt{x}\,(x+2)^{7/2}}$$

Mathematica DSolve solution

Solving time : 0.264 (sec)
Leaf size : 106

```
DSolve[{2*x^2*(2+x)*D[y[x],{x,2}]-x*(4-7*x)*D[y[x],x]-(5-3*x)*y[x]==0,{}},y[x],x,IncludeSingula
```

$$y(x) \to \exp\left(\int_1^x \left(\frac{2}{K[1]} - \frac{5}{4(K[1]+2)}\right) dK[1] - \frac{1}{2}\int_1^x \left(\frac{9}{2(K[2]+2)} - \frac{1}{K[2]}\right) dK[2]\right)\left(c_2 \int_1^x \exp\left(-2\int_1^{K[3]} \frac{3K[1]+16}{4K[1]^2+8K[1]} dK[1]\right) dK[3] + c_1\right)$$

2.1.165 Problem 167

Internal problem ID [9337]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 167
Date solved : `Monday, January 27, 2025 at 06:01:38 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1-2x)\,y''+x(8-9x)\,y'+(6-3x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.208 (sec)

Writing the ode as

$$\left(-2x^3+x^2\right)y''+\left(-9x^2+8x\right)y'+(6-3x)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -2x^3+x^2 \\ B &= -9x^2+8x \\ C &= 6-3x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{21x^2-20x+24}{4\left(2x^2-x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 21x^2-20x+24 \\ t &= 4\left(2x^2-x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{21x^2-20x+24}{4\left(2x^2-x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.314: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{x^2} + \frac{19}{x} + \frac{77}{16\left(x - \frac{1}{2}\right)^2} - \frac{19}{x - \frac{1}{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{77}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{7}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{21x^2 - 20x + 24}{4\left(2x^2 - x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{21x^2 - 20x + 24}{4\left(2x^2 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	3	-2
$\frac{1}{2}$	2	0	$\frac{11}{4}$	$-\frac{7}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= \frac{7}{4} - \left(\frac{3}{4}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{2}{x} + \frac{11}{4\left(x-\frac{1}{2}\right)} + (0) \\
&= -\frac{2}{x} + \frac{11}{4\left(x-\frac{1}{2}\right)} \\
&= \frac{4+3x}{4x^2 - 2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=x+a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{2}{x}+\frac{11}{4\left(x-\frac{1}{2}\right)}\right)(1)+\left(\left(\frac{2}{x^2}-\frac{11}{4\left(x-\frac{1}{2}\right)^2}\right)+\left(-\frac{2}{x}+\frac{11}{4\left(x-\frac{1}{2}\right)}\right)^2-\left(\frac{21x^2-20x+24}{4\left(2x^2-x\right)^2}\right)\right)=\frac{4-3a_0}{x\left(-1+2x\right)}=0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0=\frac{4}{3}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x)=x+\frac{4}{3}$$

Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x)&=pe^{\int \omega\, dx}\\
&=\left(x+\frac{4}{3}\right)\mathrm{e}^{\int\left(-\frac{2}{x}+\frac{11}{4\left(x-\frac{1}{2}\right)}\right)dx}\\
&=\left(x+\frac{4}{3}\right)\mathrm{e}^{\frac{11\ln(-1+2x)}{4}-2\ln(x)}\\
&=\frac{\left(x+\frac{4}{3}\right)\left(-1+2x\right)^{11/4}}{x^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1&=z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\
&=z_1e^{-\int \frac{1}{2}\frac{-9x^2+8x}{-2x^3+x^2}\,dx}\\
&=z_1e^{\frac{7\ln(-1+2x)}{4}-4\ln(x)}\\
&=z_1\left(\frac{\left(-1+2x\right)^{7/4}}{x^4}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\frac{\left(-1+2x\right)^{9/2}\left(4+3x\right)}{3x^6}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int\frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-9x^2+8x}{-2x^3+x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{7\ln(-1+2x)}{2}-8\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{(231x^3-198x^2+66x-8)\, x^8 \mathrm{e}^{\frac{7\ln(-1+2x)}{2}-8\ln(x)}}{385\,(4+3x)\,(-1+2x)^8} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(-1+2x)^{9/2}\,(4+3x)}{3x^6} \right) \\
&\quad + c_2 \left(\frac{(-1+2x)^{9/2}\,(4+3x)}{3x^6} \left(-\frac{(231x^3-198x^2+66x-8)\, x^8 \mathrm{e}^{\frac{7\ln(-1+2x)}{2}-8\ln(x)}}{385\,(4+3x)\,(-1+2x)^8} \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(-2x+1)\left(\frac{d^2}{dx^2}y(x)\right) + x(8-9x)\left(\frac{d}{dx}y(x)\right) + (6-3x)\, y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{3(x-2)y(x)}{x^2(2x-1)} - \frac{(-8+9x)\left(\frac{d}{dx}y(x)\right)}{x(2x-1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(-8+9x)\left(\frac{d}{dx}y(x)\right)}{x(2x-1)} + \frac{3(x-2)y(x)}{x^2(2x-1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{-8+9x}{x(2x-1)}, P_3(x) = \frac{3(x-2)}{x^2(2x-1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 8$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 6$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(2x-1)\left(\frac{d^2}{dx^2}y(x)\right) + x(-8+9x)\left(\frac{d}{dx}y(x)\right) + (3x-6)\, y(x) = 0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - ○ Convert $x^m \cdot y(x)$ to series expansion for $m=0..1$

    $x^m \cdot y(x)=\sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - ○ Shift index using $k->k-m$

    $x^m \cdot y(x)=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - ○ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
  - ○ Shift index using $k->k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - ○ Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..3$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$
  - ○ Shift index using $k->k+2-m$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0(6+r)\,(1+r)\, x^r+\left(\sum_{k=1}^{\infty}\left(-a_k(k+r+6)\,(k+r+1)+a_{k-1}(k+2+r)\,(2k-1+2r)\right) x^{k+r}\right)=$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-(6+r)\,(1+r)=0$
- Values of r that satisfy the indicial equation

  $r\in\{-6,-1\}$
- Each term in the series must be 0, giving the recursion relation

  $2(k+2+r)\left(k+r-\frac{1}{2}\right)a_{k-1}-a_k(k+r+6)\,(k+r+1)=0$
- Shift index using $k->k+1$

  $2(k+r+3)\left(k+\frac{1}{2}+r\right)a_k-a_{k+1}(k+7+r)\,(k+2+r)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{(k+r+3)(2k+2r+1)a_k}{(k+7+r)(k+2+r)}$
- Recursion relation for $r=-6$; series terminates at $k=3$

  $a_{k+1}=\frac{(k-3)(2k-11)a_k}{(k+1)(k-4)}$
- Apply recursion relation for $k=0$

  $a_1=-\frac{33a_0}{4}$
- Apply recursion relation for $k=1$

  $a_2=-3a_1$
- Express in terms of a_0

  $a_2=\frac{99a_0}{4}$
- Apply recursion relation for $k=2$

  $a_3=-\frac{7a_2}{6}$
- Express in terms of a_0

  $a_3=-\frac{231a_0}{8}$
- Terminating series solution of the ODE for $r=-6$. Use reduction of order to find the second lin

  $y(x)=a_0\cdot\left(-\frac{231}{8}x^3+\frac{99}{4}x^2-\frac{33}{4}x+1\right)$

- Recursion relation for $r = -1$

  $a_{k+1} = \frac{(k+2)(2k-1)a_k}{(k+6)(k+1)}$
- Solution for $r = -1$

  $$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = \frac{(k+2)(2k-1)a_k}{(k+6)(k+1)}\right]$$
- Combine solutions and rename parameters

  $$\left[y(x) = a_0 \cdot \left(-\frac{231}{8}x^3 + \frac{99}{4}x^2 - \frac{33}{4}x + 1\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), b_{k+1} = \frac{(k+2)(2k-1)b_k}{(k+6)(k+1)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.019` (sec)
Leaf size : 43

```
dsolve(x^2*(1-2*x)*diff(diff(y(x),x),x)+x*(8-9*x)*diff(y(x),x)+(6-3*x)*y(x) = 0,y(x),s
```

$$y = \frac{48c_1\left(-\frac{1}{2}+x\right)^4\left(x+\frac{4}{3}\right)\sqrt{-1+2x}+231c_2\left(x^3-\frac{6}{7}x^2+\frac{2}{7}x-\frac{8}{231}\right)}{x^6}$$

Mathematica DSolve solution

Solving time : `0.512` (sec)
Leaf size : 130

```
DSolve[{x^2*(1-2*x)*D[y[x],{x,2}]+x*(8-9*x)*D[y[x],x]+(6-3*x)*y[x]==0,{}},y[x],x,IncludeSing
```

$$y(x) \to \frac{1}{3}(3x+4)\exp\left(\int_1^x -\frac{3K[1]+4}{2K[1]-4K[1]^2}dK[1] - \frac{1}{2}\int_1^x \frac{8-9K[2]}{K[2]-2K[2]^2}dK[2]\right)\left(c_2\int_1^x \frac{9\exp\left(-2\int_1^{K[3]} -\frac{3K[1]+4}{2K[1]-4K[1]^2}dK[1]\right)}{(3K[3]+4)^2}dK[3] + c_1\right)$$

2.1.166 Problem 168

Internal problem ID [9338]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 168
Date solved : `Monday, January 27, 2025 at 06:01:38 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2 (x^2+1) y''+x(10x^2+3) y'-(-14x^2+15) y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.323 (sec)

Writing the ode as

$$(x^4+x^2) y''+(10x^3+3x) y'+(14x^2-15) y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=x^4+x^2\\ B&=10x^3+3x\\ C&=14x^2-15 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{24x^4+66x^2+63}{4\left(x^3+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=24x^4+66x^2+63\\ t&=4\left(x^3+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{24x^4+66x^2+63}{4\left(x^3+x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.316: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{21}{16 (x - i)^2} + \frac{21}{16 (x + i)^2} + \frac{99i}{16 (x - i)} - \frac{99i}{16 (x + i)} + \frac{63}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{63}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{9}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{7}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{4} \end{aligned}$$

For the pole at $x=-i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{21}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{7}{4} \\
\alpha_c^- &= \frac{1}{2}-\sqrt{1+4b}=-\frac{3}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty=0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r=\frac{s}{t}=\frac{24x^4+66x^2+63}{4\left(x^3+x\right)^2}
$$

Since the $\gcd(s,t)=1$. This gives $b=6$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=3 \\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=-2
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{24x^4+66x^2+63}{4\left(x^3+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{9}{2}$	$-\frac{7}{2}$
i	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
$-i$	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	3	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=3$ then

$$
\begin{aligned}
d &= \alpha_\infty^+-\left(\alpha_{c_1}^++\alpha_{c_2}^-+\alpha_{c_3}^-\right) \\
&= 3-(3) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{9}{2x} - \frac{3}{4\,(x-i)} - \frac{3}{4\,(x+i)} + (0) \\
&= \frac{9}{2x} - \frac{3}{4\,(x-i)} - \frac{3}{4\,(x+i)} \\
&= \frac{9}{2x} - \frac{3x}{2x^2+2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{9}{2x} - \frac{3}{4\,(x-i)} - \frac{3}{4\,(x+i)}\right)(0) + \left(\left(-\frac{9}{2x^2} + \frac{3}{4\,(x-i)^2} + \frac{3}{4\,(x+i)^2}\right) + \left(\frac{9}{2x} - \frac{3}{4\,(x-i)} - 4(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{9}{2x} - \frac{3}{4(x-i)} - \frac{3}{4(x+i)}\right) dx} \\
&= \frac{x^{9/2}}{\left(x^2+1\right)^{3/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{10x^3+3x}{x^4+x^2} \, dx} \\
&= z_1 e^{-\frac{3\ln(x)}{2} - \frac{7\ln\left(x^2+1\right)}{4}} \\
&= z_1 \left(\frac{1}{x^{3/2}\left(x^2+1\right)^{7/4}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{x^3}{\left(x^2+1\right)^{5/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{10x^3+3x}{x^4+x^2} dx}}{(y_1)^2} dx \\
&= y_1 \int \frac{e^{-3\ln(x)-\frac{7\ln(x^2+1)}{2}}}{(y_1)^2} dx \\
&= y_1 \left(-\frac{(x^2+1)^{5/2}}{8x^8} + \frac{(x^2+1)^{5/2}}{16x^6} - \frac{(x^2+1)^{5/2}}{64x^4} - \frac{(x^2+1)^{5/2}}{128x^2} + \frac{(x^2+1)^{3/2}}{128} + \frac{3\sqrt{x^2+1}}{128} \right. \\
&\quad \left. - \frac{3\,\mathrm{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)}{128} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^3}{(x^2+1)^{5/2}} \right) \\
&\quad + c_2 \left(\frac{x^3}{(x^2+1)^{5/2}} \left(-\frac{(x^2+1)^{5/2}}{8x^8} + \frac{(x^2+1)^{5/2}}{16x^6} - \frac{(x^2+1)^{5/2}}{64x^4} - \frac{(x^2+1)^{5/2}}{128x^2} + \frac{(x^2+1)^{3/2}}{128} + \frac{3\sqrt{x^2+1}}{128} - \frac{3\,\mathrm{ar}}{} \right. \right.
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + x(10x^2+3)\left(\frac{d}{dx}y(x)\right) - (-14x^2+15)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(14x^2-15)y(x)}{x^2(x^2+1)} - \frac{(10x^2+3)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(10x^2+3)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)} + \frac{(14x^2-15)y(x)}{x^2(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{10x^2+3}{x(x^2+1)}, P_3(x) = \frac{14x^2-15}{x^2(x^2+1)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 3$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -15$

  - $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators
$x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(10x^2+3)\left(\frac{d}{dx}y(x)\right)+(14x^2-15)\,y(x)=0$
- Assume series solution for $y(x)$
$y(x)=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions
  - Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$
  $x^m\cdot y(x)=\sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k->k-m$
  $x^m\cdot y(x)=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$
  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k->k+1-m$
  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..4$
  $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$
  - Shift index using $k->k+2-m$
  $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions
  $a_0(5+r)\,(-3+r)\,x^r+a_1(6+r)\,(-2+r)\,x^{1+r}+\left(\sum_{k=2}^{\infty}(a_k(k+r+5)\,(k+r-3)+a_{k-2}(k+r-$

- $a_0$cannot be 0 by assumption, giving the indicial equation
$(5+r)\,(-3+r)=0$
- Values of r that satisfy the indicial equation
$r\in\{-5,3\}$
- Each term must be 0
$a_1(6+r)\,(-2+r)=0$
- Solve for the dependent coefficient(s)
$a_1=0$
- Each term in the series must be 0, giving the recursion relation
$(k+r+5)\,(a_k(k+r-3)+a_{k-2}(k+r))=0$
- Shift index using $k->k+2$
$(k+r+7)\,(a_{k+2}(k+r-1)+a_k(k+r+2))=0$
- Recursion relation that defines series solution to ODE
$a_{k+2}=-\frac{a_k(k+r+2)}{k+r-1}$
- Recursion relation for $r=-5$
$a_{k+2}=-\frac{a_k(k-3)}{k-6}$
- Series not valid for $r=-5$, division by 0 in the recursion relation at $k=6$
$a_{k+2}=-\frac{a_k(k-3)}{k-6}$
- Recursion relation for $r=3$
$a_{k+2}=-\frac{a_k(k+5)}{k+2}$
- Solution for $r=3$

$$\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2}=-\frac{a_k(k+5)}{k+2}, a_1=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `1.696` (sec)
Leaf size : 59

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)+x*(10*x^2+3)*diff(y(x),x)-(-14*x^2+15)*y(x) = 0,
```

$$y=\frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) c_2 x^8-c_2\left(x^4-\frac{8}{3} x^2-\frac{8}{3}\right)\left(x^2+2\right) \sqrt{x^2+1}+c_1 x^8}{\left(x^2+1\right)^{5/2} x^5}$$

Mathematica DSolve solution

Solving time : `0.26` (sec)
Leaf size : 112

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]+x*(3+10*x^2)*D[y[x],x]-(15-14*x^2)*y[x]==0,{}},y[x],x,Includ
```

$$y(x)\to \exp\left(\int_1^x \frac{6K[1]^2+9}{2\left(K[1]^3+K[1]\right)}dK[1]-\frac{1}{2}\int_1^x \frac{10K[2]^2+3}{K[2]^3+K[2]}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{6K[1]^2+9}{2\left(K[1]^3+K[1]\right)}dK[1]\right)dK[3]+c_1\right)$$

2.1.167 Problem 169

Internal problem ID [9339]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 169
Date solved : `Monday, January 27, 2025 at 06:01:39 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(-2x^2+1\right)y''+x\left(-13x^2+7\right)y'-14x^2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.294 (sec)

Writing the ode as

$$\left(-2x^4+x^2\right)y''+\left(-13x^3+7x\right)y'-14x^2y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-2x^4+x^2\\ B&=-13x^3+7x\\ C&=-14x^2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{5x^4-68x^2+35}{4\left(2x^3-x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=5x^4-68x^2+35\\ t&=4\left(2x^3-x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{5x^4-68x^2+35}{4\left(2x^3-x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.318: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^3 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{64\left(x - \frac{\sqrt{2}}{2}\right)^2} + \frac{9}{64\left(x + \frac{\sqrt{2}}{2}\right)^2} - \frac{279\sqrt{2}}{64\left(x - \frac{\sqrt{2}}{2}\right)} + \frac{279\sqrt{2}}{64\left(x + \frac{\sqrt{2}}{2}\right)} + \frac{35}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2} \end{aligned}$$

For the pole at $x = \frac{\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{9}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{8}
\end{aligned}
$$

For the pole at $x = -\frac{\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{9}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{8}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{5x^4 - 68x^2 + 35}{4\left(2x^3 - x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{5x^4 - 68x^2 + 35}{4\left(2x^3 - x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$
$\frac{\sqrt{2}}{2}$	2	0	$\frac{9}{8}$	$-\frac{1}{8}$
$-\frac{\sqrt{2}}{2}$	2	0	$\frac{9}{8}$	$-\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\
&= -\frac{1}{4} - \left(-\frac{1}{4}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^+}{x-c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{5}{2x} + \frac{9}{8\left(x-\frac{\sqrt{2}}{2}\right)} + \frac{9}{8\left(x+\frac{\sqrt{2}}{2}\right)} + (-)(0) \\
&= -\frac{5}{2x} + \frac{9}{8\left(x-\frac{\sqrt{2}}{2}\right)} + \frac{9}{8\left(x+\frac{\sqrt{2}}{2}\right)} \\
&= \frac{-x^2+5}{4x^3-2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{5}{2x} + \frac{9}{8\left(x-\frac{\sqrt{2}}{2}\right)} + \frac{9}{8\left(x+\frac{\sqrt{2}}{2}\right)} \right)(0) + \left(\left(\frac{5}{2x^2} - \frac{9}{8\left(x-\frac{\sqrt{2}}{2}\right)^2} - \frac{9}{8\left(x+\frac{\sqrt{2}}{2}\right)^2} \right) + \left(-\frac{5}{2x} + \frac{}{8} \right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(-\frac{5}{2x} + \frac{9}{8\left(x-\frac{\sqrt{2}}{2}\right)} + \frac{9}{8\left(x+\frac{\sqrt{2}}{2}\right)} \right) dx} \\
&= \frac{\left(2x+\sqrt{2}\right)^{9/8} \left(2x-\sqrt{2}\right)^{9/8}}{x^{5/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-13x^3+7x}{-2x^4+x^2}\, dx} \\
&= z_1 e^{\frac{\ln\left(2x^2-1\right)}{8} - \frac{7\ln(x)}{2}} \\
&= z_1 \left(\frac{\left(2x^2-1\right)^{1/8}}{x^{7/2}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{2\left(2x^2-1\right)^{5/4} 2^{1/8}}{x^6}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-13x^3+7x}{-2x^4+x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{\ln(2x^2-1)}{4}-7\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{(5x^4-20x^2+8)\,x^7 \mathrm{e}^{\frac{\ln(2x^2-1)}{4}-7\ln(x)} 2^{3/4}}{120\,(2x^2-1)^{3/2}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{2(2x^2-1)^{5/4}\,2^{1/8}}{x^6}\right) \\
&\quad + c_2 \left(\frac{2(2x^2-1)^{5/4}\,2^{1/8}}{x^6}\left(\frac{(5x^4-20x^2+8)\,x^7 \mathrm{e}^{\frac{\ln(2x^2-1)}{4}-7\ln(x)} 2^{3/4}}{120\,(2x^2-1)^{3/2}}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2(-2x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + x(-13x^2+7)\left(\frac{d}{dx}y(x)\right) - 14x^2 y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{14y(x)}{2x^2-1} - \frac{(13x^2-7)\left(\frac{d}{dx}y(x)\right)}{x(2x^2-1)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x) + \frac{(13x^2-7)\left(\frac{d}{dx}y(x)\right)}{x(2x^2-1)} + \frac{14y(x)}{2x^2-1} = 0$

□ Check to see if x_0 is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{13x^2-7}{x(2x^2-1)}, P_3(x) = \frac{14}{2x^2-1}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = 7$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$
  - $x = 0$is a regular singular point
    Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

$x(2x^2-1)\left(\frac{d^2}{dx^2}y(x)\right)+(13x^2-7)\left(\frac{d}{dx}y(x)\right)+14xy(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y(x)$ to series expansion

  $x \cdot y(x)=\sum_{k=0}^{\infty}a_k x^{k+r+1}$

  ◦ Shift index using $k->k-1$

  $x \cdot y(x)=\sum_{k=1}^{\infty}a_{k-1}x^{k+r}$

  ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..2$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=1..3$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  ◦ Shift index using $k->k+2-m$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0r(6+r)\,x^{-1+r}-a_1(1+r)\,(7+r)\,x^r+\left(\sum_{k=1}^{\infty}(-a_{k+1}(k+r+1)\,(k+7+r)+a_{k-1}(2k+5+2r)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-r(6+r)=0$

- Values of r that satisfy the indicial equation

$r\in\{-6,0\}$

- Each term must be 0

$-a_1(1+r)\,(7+r)=0$

- Each term in the series must be 0, giving the recursion relation

$2(k+r+1)\left(\left(k+r+\frac{5}{2}\right)a_{k-1}-\frac{a_{k+1}(k+7+r)}{2}\right)=0$

- Shift index using $k->k+1$

$2(k+r+2)\left(\left(k+\frac{7}{2}+r\right)a_k-\frac{a_{k+2}(k+8+r)}{2}\right)=0$

- Recursion relation that defines series solution to ODE

$a_{k+2}=\frac{(2k+2r+7)a_k}{k+8+r}$

- Recursion relation for $r=-6$

$a_{k+2}=\frac{(2k-5)a_k}{k+2}$

- Solution for $r=-6$

$\left[y(x)=\sum_{k=0}^{\infty}a_k x^{k-6}, a_{k+2}=\frac{(2k-5)a_k}{k+2}, 5a_1=0\right]$

- Recursion relation for $r=0$

$a_{k+2}=\frac{(2k+7)a_k}{k+8}$

- Solution for $r = 0$
$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{(2k+7)a_k}{k+8}, -7a_1 = 0\right]$$
- Combine solutions and rename parameters
$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-6}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = \frac{(2k-5)a_k}{k+2}, 5a_1 = 0, b_{k+2} = \frac{(2k+7)b_k}{k+8}, -7b_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.021` (sec)
Leaf size : 35

```
dsolve(x^2*(-2*x^2+1)*diff(diff(y(x),x),x)+x*(-13*x^2+7)*diff(y(x),x)-14*x^2*y(x) = 0,
```

$$y = \frac{c_1 \left(2x^2-1\right)^{5/4} + 5c_2 x^4 - 20c_2 x^2 + 8c_2}{x^6}$$

Mathematica DSolve solution

Solving time : `0.236` (sec)
Leaf size : 116

```
DSolve[{x^2*(1-2*x^2)*D[y[x],{x,2}]+x*(7-13*x^2)*D[y[x],x]-14*x^2*y[x]==0,{}},y[x],x,Include
```

$$y(x) \to \exp\left(\int_1^x \frac{5-K[1]^2}{4K[1]^3-2K[1]}dK[1] - \frac{1}{2}\int_1^x \frac{7-13K[2]^2}{K[2]-2K[2]^3}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{5-K[1]^2}{4K[1]^3-2K[1]}dK[1]\right)dK[3] + c_1\right)$$

2.1.168 Problem 170

Internal problem ID [9340]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 170
Date solved : `Monday, January 27, 2025 at 06:01:40 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2(1+x)\,y''+4x(1+2x)\,y'-(1+3x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.215 (sec)

Writing the ode as

$$\left(4x^3+4x^2\right)y''+\left(8x^2+4x\right)y'+(-3x-1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=4x^3+4x^2\\ B&=8x^2+4x\\ C&=-3x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{3x+4}{4x\left(1+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=3x+4\\ t&=4x(1+x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{3x+4}{4x\left(1+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.320: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x(1+x)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{1+x} + \frac{1}{x} - \frac{1}{4\,(1+x)^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x+4}{4x(1+x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3x+4}{4x(1+x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= \frac{3}{2} - \left(\frac{3}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{x} + \frac{1}{2+2x} + (0) \\
&= \frac{1}{x} + \frac{1}{2+2x} \\
&= \frac{1}{x} + \frac{1}{2+2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x} + \frac{1}{2+2x}\right)(0) + \left(\left(-\frac{1}{x^2} - \frac{1}{2(1+x)^2}\right) + \left(\frac{1}{x} + \frac{1}{2+2x}\right)^2 - \left(\frac{3x+4}{4x(1+x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega \, dx} \\ &= e^{\int \left(\frac{1}{x} + \frac{1}{2+2x}\right) dx} \\ &= \sqrt{1+x}\, x \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{8x^2+4x}{4x^3+4x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x(1+x))}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x(1+x)}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{1+x}\, x}{\sqrt{x(1+x)}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{8x^2+4x}{4x^3+4x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\ln(x(1+x))}}{(y_1)^2}\, dx \\ &= y_1 \left(\ln(1+x) - \frac{1}{x} - \ln(x)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\sqrt{1+x}\, x}{\sqrt{x(1+x)}}\right) + c_2 \left(\frac{\sqrt{1+x}\, x}{\sqrt{x(1+x)}}\left(\ln(1+x) - \frac{1}{x} - \ln(x)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)+4x(2x+1)\left(\frac{d}{dx}y(x)\right)-(3x+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{(3x+1)y(x)}{4x^2(x+1)}-\frac{(2x+1)\left(\frac{d}{dx}y(x)\right)}{x(x+1)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{(2x+1)\left(\frac{d}{dx}y(x)\right)}{x(x+1)}-\frac{(3x+1)y(x)}{4x^2(x+1)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=\frac{2x+1}{x(x+1)},P_3(x)=-\frac{3x+1}{4x^2(x+1)}\right]$
- ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=1$
- ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$
- ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $4x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)+4x(2x+1)\left(\frac{d}{dx}y(x)\right)+(-3x-1)\,y(x)=0$
- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(4u^3-8u^2+4u)\left(\frac{d^2}{du^2}y(u)\right)+(8u^2-12u+4)\left(\frac{d}{du}y(u)\right)+(-3u+2)\,y(u)=0$
- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$
- ◦ Shift index using $k->k-m$

  $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$
- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$
- ◦ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$
- ◦ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$
- ◦ Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$4a_0r^2u^{-1+r} + \left(4a_1(1+r)^2 - 2a_0(4r^2+2r-1)\right)u^r + \left(\sum_{k=1}^{\infty}\left(4a_{k+1}(k+1+r)^2 - 2a_k(4k^2+8kr\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $4r^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 0$
- Each term must be 0
  $4a_1(1+r)^2 - 2a_0(4r^2+2r-1) = 0$
- Each term in the series must be 0, giving the recursion relation
  $(4k^2-4k-3)\,a_{k-1} + (-8k^2-4k+2)\,a_k + 4a_{k+1}(k+1)^2 = 0$
- Shift index using $k- >k+1$
  $\left(4(k+1)^2-4k-7\right)a_k + \left(-8(k+1)^2-4k-2\right)a_{k+1} + 4a_{k+2}(k+2)^2 = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{4k^2a_k-8k^2a_{k+1}+4ka_k-20ka_{k+1}-3a_k-10a_{k+1}}{4(k+2)^2}$
- Recursion relation for $r = 0$
  $a_{k+2} = -\frac{4k^2a_k-8k^2a_{k+1}+4ka_k-20ka_{k+1}-3a_k-10a_{k+1}}{4(k+2)^2}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_ku^k, a_{k+2} = -\frac{4k^2a_k-8k^2a_{k+1}+4ka_k-20ka_{k+1}-3a_k-10a_{k+1}}{4(k+2)^2}, 4a_1+2a_0 = 0\right]$
- Revert the change of variables $u = x+1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+2} = -\frac{4k^2a_k-8k^2a_{k+1}+4ka_k-20ka_{k+1}-3a_k-10a_{k+1}}{4(k+2)^2}, 4a_1+2a_0 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.022` (sec)
Leaf size : 28

```
dsolve(4*x^2*(x+1)*diff(diff(y(x),x),x)+4*x*(2*x+1)*diff(y(x),x)-(3*x+1)*y(x) = 0,y(x)
```

$$y = \frac{\ln(x+1)\,c_2x - \ln(x)\,c_2x + c_1x - c_2}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.252 (sec)
Leaf size : 96

```
DSolve[{4*x^2*(1+x)*D[y[x],{x,2}]+4*x*(1+2*x)*D[y[x],x]-(1+3*x)*y[x]==0,{}},y[x],x,IncludeSingu
```

$$y(x) \to \exp\left(\int_1^x \left(\frac{1}{2K[1]+2}+\frac{1}{K[1]}\right) dK[1]-\frac{1}{2}\int_1^x \left(\frac{1}{K[2]+1}+\frac{1}{K[2]}\right) dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{3K[1]+2}{2K[1]^2+2K[1]} dK[1]\right) dK[3]+c_1\right)$$

2.1.169 Problem 171

Internal problem ID [9341]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 171
Date solved : `Monday, January 27, 2025 at 06:01:41 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(2+3x)\,y''+x(4+21x)\,y'-(1-9x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.220 (sec)

Writing the ode as

$$\left(6x^3+4x^2\right)y''+\left(21x^2+4x\right)y'+(9x-1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=6x^3+4x^2\\ B&=21x^2+4x\\ C&=9x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-27x-48}{16x\,(2+3x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=-27x-48\\ t&=16x(2+3x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-27x-48}{16x\,(2+3x)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.322: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x(2+3x)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = -\frac{2}{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{4x} + \frac{5}{16\left(x + \frac{2}{3}\right)^2} + \frac{3}{4\left(x + \frac{2}{3}\right)}$$

For the pole at $x = -\frac{2}{3}$ let b be the coefficient of $\frac{1}{\left(x+\frac{2}{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-27x - 48}{16x(2+3x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-27x - 48}{16x(2+3x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1
$-\frac{2}{3}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - (\alpha_{c_1}^- + \alpha_{c_2}^-) \\
&= \frac{3}{4} - \left(\frac{3}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{x} - \frac{1}{4\left(x+\frac{2}{3}\right)} + (0) \\
&= \frac{1}{x} - \frac{1}{4\left(x+\frac{2}{3}\right)} \\
&= \frac{8+9x}{12x^2+8x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x} - \frac{1}{4\left(x+\frac{2}{3}\right)}\right)(0) + \left(\left(-\frac{1}{x^2} + \frac{1}{4\left(x+\frac{2}{3}\right)^2}\right) + \left(\frac{1}{x} - \frac{1}{4\left(x+\frac{2}{3}\right)}\right)^2 - \left(\frac{-27x-48}{16x\,(2+3x)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{x} - \frac{1}{4\left(x+\frac{2}{3}\right)}\right) dx} \\ &= \frac{x}{(2+3x)^{1/4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{21x^2+4x}{6x^3+4x^2}\, dx} \\ &= z_1 e^{-\frac{5\ln(2+3x)}{4} - \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{(2+3x)^{5/4}\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{(2+3x)^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{21x^2+4x}{6x^3+4x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{5\ln(2+3x)}{2} - \ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{\sqrt{2+3x}}{x} - \frac{3\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2+3x}\,\sqrt{2}}{2}\right)}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\sqrt{x}}{(2+3x)^{3/2}}\right) + c_2\left(\frac{\sqrt{x}}{(2+3x)^{3/2}}\left(-\frac{\sqrt{2+3x}}{x} - \frac{3\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2+3x}\,\sqrt{2}}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2(3x+2)\left(\frac{d^2}{dx^2}y(x)\right) + x(4+21x)\left(\frac{d}{dx}y(x)\right) - (1-9x)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(-1+9x)y(x)}{2(3x+2)x^2} - \frac{(4+21x)\left(\frac{d}{dx}y(x)\right)}{2x(3x+2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(4+21x)\left(\frac{d}{dx}y(x)\right)}{2x(3x+2)} + \frac{(-1+9x)y(x)}{2(3x+2)x^2} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{4+21x}{2x(3x+2)}, P_3(x) = \frac{-1+9x}{2(3x+2)x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x=0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$

  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $2x^2(3x+2)\left(\frac{d^2}{dx^2}y(x)\right) + x(4+21x)\left(\frac{d}{dx}y(x)\right) + (-1+9x)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k-m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..3$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$
- Shift index using $k->k+2-m$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)(-1+2r)\, x^r + \left(\sum_{k=1}^{\infty} (a_k(2k+2r+1)(2k+2r-1) + 3a_{k-1}(2k+2r+1)(k+r))\, x^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $4\left(\left(k+r-\frac{1}{2}\right)a_k + \frac{3a_{k-1}(k+r)}{2}\right)\left(k+r+\frac{1}{2}\right) = 0$
- Shift index using $k->k+1$

  $4\left(\left(k+r+\frac{1}{2}\right)a_{k+1} + \frac{3a_k(k+r+1)}{2}\right)\left(k+\frac{3}{2}+r\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{3a_k(k+r+1)}{2k+2r+1}$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+1} = -\frac{3a_k\left(k+\frac{1}{2}\right)}{2k}$
- Solution for $r = -\frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = -\frac{3a_k\left(k+\frac{1}{2}\right)}{2k}\right]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = -\frac{3a_k\left(k+\frac{3}{2}\right)}{2k+2}$
- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{3a_k\left(k+\frac{3}{2}\right)}{2k+2}\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = -\frac{3a_k\left(k+\frac{1}{2}\right)}{2k}, b_{k+1} = -\frac{3b_k\left(k+\frac{3}{2}\right)}{2k+2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.040 (sec)
Leaf size : 48

```
dsolve(2*x^2*(2+3*x)*diff(diff(y(x),x),x)+x*(4+21*x)*diff(y(x),x)-(1-9*x)*y(x) = 0,y(x
```

$$y = \frac{c_1 x + \sqrt{2}\,\sqrt{2+3x}\,c_2 + 3\,\operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{2+3x}}{2}\right) c_2 x}{(2+3x)^{3/2}\,\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.312 (sec)
Leaf size : 102

```
DSolve[{2*x^2*(2+3*x)*D[y[x],{x,2}]+x*(4+21*x)*D[y[x],x]-(1-9*x)*y[x]==0,{}},y[x],x,IncludeS
```

$$y(x) \to \exp\left(\int_1^x \left(\frac{1}{K[1]} - \frac{3}{12K[1]+8}\right) dK[1] - \frac{1}{2}\int_1^x \left(\frac{15}{6K[2]+4} + \frac{1}{K[2]}\right) dK[2]\right)\left(c_2 \int_1^x \exp\left(-2\int_1^{K[3]} \frac{9K[1]+8}{12K[1]^2+8K[1]} dK[1]\right) dK[3] + c_1\right)$$

2.1.170 Problem 172

Internal problem ID [9342]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 172
Date solved : Monday, January 27, 2025 at 06:01:41 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$x^2y'' + x(2+x)\,y' - (2-3x)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.233 (sec)

Writing the ode as

$$x^2y'' + \left(x^2+2x\right)y' + (3x-2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2+2x \\ C &= 3x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2-8x+8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2-8x+8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2-8x+8}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.324: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{2}{x} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{2}{x} - \frac{2}{x^2} - \frac{8}{x^3} - \frac{36}{x^4} - \frac{176}{x^5} - \frac{912}{x^6} - \frac{4928}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 8x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-8x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-8x + 8}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -8. Dividing this by leading coefficient in t which is 4 gives -2. Now b can be found.

$$\begin{aligned} b &= (-2) - (0) \\ &= -2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-2}{\frac{1}{2}} - 0\right) &= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-2}{\frac{1}{2}} - 0\right) &= 2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 8x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	-2	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= 2 - (2) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{2}{x} + (-)\left(\frac{1}{2}\right) \\ &= \frac{2}{x} - \frac{1}{2} \\ &= -\frac{x-4}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{2}{x} - \frac{1}{2}\right)(0) + \left(\left(-\frac{2}{x^2}\right) + \left(\frac{2}{x} - \frac{1}{2}\right)^2 - \left(\frac{x^2-8x+8}{4x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{2}{x} - \frac{1}{2}\right) dx} \\ &= x^2 e^{-\frac{x}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2+2x}{x^2}\,dx} \\
&= z_1 e^{-\frac{x}{2}-\ln(x)} \\
&= z_1 \left(\frac{e^{-\frac{x}{2}}}{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x\,e^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+2x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{e^x}{3x^3} - \frac{e^x}{6x^2} - \frac{e^x}{6x} - \frac{\mathrm{Ei}_1(-x)}{6}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x\,e^{-x}\right) + c_2 \left(x\,e^{-x}\left(-\frac{e^x}{3x^3} - \frac{e^x}{6x^2} - \frac{e^x}{6x} - \frac{\mathrm{Ei}_1(-x)}{6}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(x+2)\left(\frac{d}{dx}y(x)\right) - (-3x+2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(3x-2)y(x)}{x^2} - \frac{(x+2)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(x+2)\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(3x-2)y(x)}{x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{x+2}{x}, P_3(x) = \frac{3x-2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

$(x \cdot P_2(x))\bigg|_{x=0} = 2$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = -2$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(x+2)\left(\frac{d}{dx}y(x)\right) + (3x-2)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ○ Shift index using $k- > k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ○ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- ○ Shift index using $k- > k + 1 - m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ○ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)\,(-1+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+2)\,(k+r-1) + a_{k-1}(k+r+2)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 1\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r+2)\,(a_k(k+r-1) + a_{k-1}) = 0$

- Shift index using $k- > k + 1$

  $(k+r+3)\,(a_{k+1}(k+r) + a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{k+r}$

- Recursion relation for $r = -2$

  $a_{k+1} = -\frac{a_k}{k-2}$

- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 2$

  $a_{k+1} = -\frac{a_k}{k-2}$

- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{a_k}{k+1}$

- Solution for $r = 1$

$$\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1}=-\frac{a_k}{k+1}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 40

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(x+2)*diff(y(x),x)-(2-3*x)*y(x) = 0,y(x),singsol=all)
```

$$y=\frac{\mathrm{Ei}_1(-x)\,\mathrm{e}^{-x} c_2 x^3+\mathrm{e}^{-x} c_1 x^3+c_2(x^2+x+2)}{x^2}$$

Mathematica DSolve solution

Solving time : 0.265 (sec)
Leaf size : 36

```
DSolve[{x^2*D[y[x],{x,2}]+x*(2+x)*D[y[x],x]-(2-3*x)*y[x]==0,{}},y[x],x,IncludeSingularSolutions
```

$$y(x)\to e^{-x-1}x\left(c_2\int_1^x \frac{e^{K[1]}}{K[1]^4}dK[1]+c_1\right)$$

2.1.171 Problem 173

Internal problem ID [9343]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 173
Date solved : `Monday, January 27, 2025 at 06:01:42 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2(1+x)\,y''+4x(3+8x)\,y'-(5-49x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.226 (sec)

Writing the ode as

$$\left(4x^3+4x^2\right)y''+\left(32x^2+12x\right)y'+(49x-5)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=4x^3+4x^2\\B&=32x^2+12x\\C&=49x-5\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-x^2-8x+8}{4\left(x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=-x^2-8x+8\\t&=4\left(x^2+x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-x^2-8x+8}{4\left(x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.326: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{6}{x} + \frac{15}{4\,(1+x)^2} + \frac{6}{1+x} + \frac{2}{x^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 8x + 8}{4\left(x^2 + x\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 - 8x + 8}{4\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{2(1 + x)} + \frac{2}{x} + (-)(0) \\
&= -\frac{3}{2(1 + x)} + \frac{2}{x} \\
&= \frac{x + 4}{2x(1 + x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{2(1+x)} + \frac{2}{x}\right)(0) + \left(\left(\frac{3}{2(1+x)^2} - \frac{2}{x^2}\right) + \left(-\frac{3}{2(1+x)} + \frac{2}{x}\right)^2 - \left(\frac{-x^2 - 8x + 8}{4(x^2+x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(-\frac{3}{2(1+x)} + \frac{2}{x}\right) dx} \\ &= \frac{x^2}{(1+x)^{3/2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{32x^2+12x}{4x^3+4x^2}\, dx} \\ &= z_1 e^{-\frac{5\ln(1+x)}{2} - \frac{3\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{(1+x)^{5/2}\, x^{3/2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{(1+x)^4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{32x^2+12x}{4x^3+4x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-5\ln(1+x) - 3\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(\ln(x) - \frac{1}{3x^3} - \frac{3}{2x^2} - \frac{3}{x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\sqrt{x}}{(1+x)^4}\right) + c_2 \left(\frac{\sqrt{x}}{(1+x)^4} \left(\ln(x) - \frac{1}{3x^3} - \frac{3}{2x^2} - \frac{3}{x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)+4x(3+8x)\left(\frac{d}{dx}y(x)\right)-(5-49x)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(-5+49x)y(x)}{4(x+1)x^2}-\frac{(3+8x)\left(\frac{d}{dx}y(x)\right)}{x(x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{(3+8x)\left(\frac{d}{dx}y(x)\right)}{x(x+1)}+\frac{(-5+49x)y(x)}{4(x+1)x^2}=0$

☐ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=\frac{3+8x}{x(x+1)},P_3(x)=\frac{-5+49x}{4(x+1)x^2}\right]$

- ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=5$

- ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

- ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $4x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)+4x(3+8x)\left(\frac{d}{dx}y(x)\right)+(-5+49x)\,y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(4u^3-8u^2+4u)\left(\frac{d^2}{du^2}y(u)\right)+(32u^2-52u+20)\left(\frac{d}{du}y(u)\right)+(-54+49u)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

- ◦ Shift index using $k->k-m$

  $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ◦ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- ◦ Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$4a_0 r(4+r)\, u^{-1+r} + (4a_1(1+r)(5+r) - 2a_0(4r^2+22r+27))\, u^r + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+1+r)(k+5$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $4r(4+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-4, 0\}$
- Each term must be 0
  $4a_1(1+r)(5+r) - 2a_0(4r^2+22r+27) = 0$
- Each term in the series must be 0, giving the recursion relation
  $4a_{k+1}(k+1+r)(k+5+r) - 2a_k(4k^2+8kr+4r^2+22k+22r+27) + a_{k-1}(2k+5+2r)^2 = 0$
- Shift index using $k- >k+1$
  $4a_{k+2}(k+2+r)(k+6+r) - 2a_{k+1}\left(4(k+1)^2+8(k+1)\,r+4r^2+22k+49+22r\right) + a_k(2k+2r$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 8kra_k - 16kra_{k+1} + 4r^2a_k - 8r^2a_{k+1} + 28ka_k - 60ka_{k+1} + 28ra_k - 60ra_{k+1} + 49a_k - 106a_{k+1}}{4(k+2+r)(k+6+r)}$
- Recursion relation for $r = -4$
  $a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} - 4ka_k + 4ka_{k+1} + a_k + 6a_{k+1}}{4(k-2)(k+2)}$
- Series not valid for $r = -4$, division by 0 in the recursion relation at $k = 2$
  $a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} - 4ka_k + 4ka_{k+1} + a_k + 6a_{k+1}}{4(k-2)(k+2)}$
- Recursion relation for $r = 0$
  $a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 28ka_k - 60ka_{k+1} + 49a_k - 106a_{k+1}}{4(k+2)(k+6)}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 28ka_k - 60ka_{k+1} + 49a_k - 106a_{k+1}}{4(k+2)(k+6)}, 20a_1 - 54a_0 = 0\right]$
- Revert the change of variables $u = x+1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 28ka_k - 60ka_{k+1} + 49a_k - 106a_{k+1}}{4(k+2)(k+6)}, 20a_1 - 54a_0 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.038 (sec)
Leaf size : 40

```
dsolve(4*x^2*(x+1)*diff(diff(y(x),x),x)+4*x*(3+8*x)*diff(y(x),x)-(5-49*x)*y(x) = 0,y(x
```

$$y = \frac{c_1 x^3 + 6\ln(x) c_2 x^3 - 18 c_2 x^2 - 9 c_2 x - 2 c_2}{(x+1)^4 x^{5/2}}$$

Mathematica DSolve solution

Solving time : 0.208 (sec)
Leaf size : 104

```
DSolve[{4*x^2*(1+x)*D[y[x],{x,2}]+4*x*(3+8*x)*D[y[x],x]-(5-49*x)*y[x]==0,{}},y[x],x,IncludeS
```

$$y(x) \to \exp\left(\int_1^x \frac{K[1]+4}{2K[1]^2+2K[1]}dK[1] - \frac{1}{2}\int_1^x \frac{8K[2]+3}{K[2]^2+K[2]}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{K[1]+4}{2K[1]^2+2K[1]}dK[1]\right)dK[3] + c_1\right)$$

2.1.172 Problem 174

Internal problem ID [9344]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 174
Date solved : `Monday, January 27, 2025 at 06:01:42 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+x)\,y''-x(3+10x)\,y'+30xy=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.241 (sec)

Writing the ode as

$$x^2(1+x)\,y''+\left(-10x^2-3x\right)y'+30xy=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2(1+x)\\B&=-10x^2-3x\\C&=30x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-48x+15}{4\left(x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=-48x+15\\t&=4\left(x^2+x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-48x+15}{4\left(x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.328: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 3 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 3 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{39}{2x} + \frac{15}{4x^2} + \frac{39}{2\,(1+x)} + \frac{63}{4\,(1+x)^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{63}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{7}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is $3 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-48x + 15}{4\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{9}{2}$	$-\frac{7}{2}$
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
3	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 0 - (-1) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{7}{2(1+x)} + \frac{5}{2x} + (0) \\
&= -\frac{7}{2(1+x)} + \frac{5}{2x} \\
&= -\frac{2x - 5}{2x(1+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{7}{2(1+x)}+\frac{5}{2x}\right)(1)+\left(\left(\frac{7}{2(1+x)^2}-\frac{5}{2x^2}\right)+\left(-\frac{7}{2(1+x)}+\frac{5}{2x}\right)^2-\left(\frac{-48x+15}{4(x^2+x)^2}\right)\right)=0$$

$$\frac{5+2a_0}{x(1+x)}=0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0=-\frac{5}{2}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x)=x-\frac{5}{2}$$

Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x)&=pe^{\int \omega\,dx}\\
&=\left(x-\frac{5}{2}\right)\mathrm{e}^{\int\left(-\frac{7}{2(1+x)}+\frac{5}{2x}\right)dx}\\
&=\left(x-\frac{5}{2}\right)\mathrm{e}^{-\frac{7\ln(1+x)}{2}+\frac{5\ln(x)}{2}}\\
&=\frac{\left(x-\frac{5}{2}\right)x^{5/2}}{(1+x)^{7/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1&=z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\
&=z_1e^{-\int \frac{1}{2}\frac{-10x^2-3x}{x^2(1+x)}\,dx}\\
&=z_1e^{\frac{7\ln(1+x)}{2}+\frac{3\ln(x)}{2}}\\
&=z_1\left((1+x)^{7/2}x^{3/2}\right)
\end{aligned}$$

Which simplifies to

$$y_1=x^5-\frac{5}{2}x^4$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int\frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2&=y_1\int\frac{e^{\int -\frac{-10x^2-3x}{x^2(1+x)}\,dx}}{(y_1)^2}\,dx\\
&=y_1\int\frac{e^{7\ln(1+x)+3\ln(x)}}{(y_1)^2}\,dx\\
&=y_1\left(x-\frac{823543}{6250(2x-5)}-\frac{1}{25x^4}-\frac{52}{125x^3}-\frac{1354}{625x^2}-\frac{27708}{3125x}+12\ln(x)\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^5 - \frac{5}{2}x^4\right) \\
&\quad + c_2\left(x^5 - \frac{5}{2}x^4\left(x - \frac{823543}{6250\,(2x-5)} - \frac{1}{25x^4} - \frac{52}{125x^3} - \frac{1354}{625x^2} - \frac{27708}{3125x} + 12\ln(x)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right) - x(10x+3)\left(\frac{d}{dx}y(x)\right) + 30xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{30y(x)}{x(x+1)} + \frac{(10x+3)\left(\frac{d}{dx}y(x)\right)}{x(x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) - \frac{(10x+3)\left(\frac{d}{dx}y(x)\right)}{x(x+1)} + \frac{30y(x)}{x(x+1)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{10x+3}{x(x+1)}, P_3(x) = \frac{30}{x(x+1)}\right]$

  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1} = -7$

  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$

  - $x=-1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -1$

- Multiply by denominators

  $x(x+1)\left(\frac{d^2}{dx^2}y(x)\right) + (-10x-3)\left(\frac{d}{dx}y(x)\right) + 30y(x) = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(u^2-u)\left(\frac{d^2}{du^2}y(u)\right) + (-10u+7)\left(\frac{d}{du}y(u)\right) + 30y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

$u^{m}\cdot\left(\frac{d^{2}}{d u^{2}}y(u)\right)=\sum_{k=0}^{\infty}a_{k}(k+r)\left(k+r-1\right)u^{k+r-2+m}$

- ∘ Shift index using $k->k+2-m$

  $u^{m}\cdot\left(\frac{d^{2}}{d u^{2}}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\left(k+1-m+r\right)u^{k+r}$

  Rewrite ODE with series expansions

  $-a_{0}r(-8+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-a_{k+1}(k+1+r)\left(k-7+r\right)+a_{k}(k+r-5)\left(k+r-6\right)\right)u^{k+r}\right)=$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-8+r)=0$
- Values of r that satisfy the indicial equation

  $r\in\{0,8\}$
- Each term in the series must be 0, giving the recursion relation

  $-a_{k+1}(k+1+r)\left(k-7+r\right)+a_{k}(k+r-5)\left(k+r-6\right)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_{k}(k+r-5)(k+r-6)}{(k+1+r)(k-7+r)}$
- Recursion relation for $r=0$; series terminates at $k=5$

  $a_{k+1}=\frac{a_{k}(k-5)(k-6)}{(k+1)(k-7)}$
- Apply recursion relation for $k=0$

  $a_{1}=-\frac{30a_{0}}{7}$
- Apply recursion relation for $k=1$

  $a_{2}=-\frac{5a_{1}}{3}$
- Express in terms of a_0

  $a_{2}=\frac{50a_{0}}{7}$
- Apply recursion relation for $k=2$

  $a_{3}=-\frac{4a_{2}}{5}$
- Express in terms of a_0

  $a_{3}=-\frac{40a_{0}}{7}$
- Apply recursion relation for $k=3$

  $a_{4}=-\frac{3a_{3}}{8}$
- Express in terms of a_0

  $a_{4}=\frac{15a_{0}}{7}$
- Apply recursion relation for $k=4$

  $a_{5}=-\frac{2a_{4}}{15}$
- Express in terms of a_0

  $a_{5}=-\frac{2a_{0}}{7}$
- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li

  $y(u)=a_{0}\cdot\left(1-\frac{30}{7}u+\frac{50}{7}u^{2}-\frac{40}{7}u^{3}+\frac{15}{7}u^{4}-\frac{2}{7}u^{5}\right)$
- Revert the change of variables $u=x+1$

  $\left[y(x)=a_{0}\left(\frac{5}{7}x^{4}-\frac{2}{7}x^{5}\right)\right]$
- Recursion relation for $r=8$

  $a_{k+1}=\frac{a_{k}(k+3)(k+2)}{(k+9)(k+1)}$
- Solution for $r=8$

  $\left[y(u)=\sum_{k=0}^{\infty}a_{k}u^{k+8},a_{k+1}=\frac{a_{k}(k+3)(k+2)}{(k+9)(k+1)}\right]$
- Revert the change of variables $u=x+1$

  $\left[y(x)=\sum_{k=0}^{\infty}a_{k}(x+1)^{k+8},a_{k+1}=\frac{a_{k}(k+3)(k+2)}{(k+9)(k+1)}\right]$

- Combine solutions and rename parameters

$$\left[y(x) = a_0\left(\frac{5}{7}x^4 - \frac{2}{7}x^5\right) + \left(\sum_{k=0}^{\infty} b_k(x+1)^{k+8}\right), b_{k+1} = \frac{b_k(k+3)(k+2)}{(k+9)(k+1)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 65

```
dsolve(x^2*(x+1)*diff(diff(y(x),x),x)-x*(3+10*x)*diff(y(x),x)+30*x*y(x) = 0,y(x),singsol
```

$$y = 3c_2x^4\left(x-\frac{5}{2}\right)\ln(x) + \frac{c_2x^6}{4} + \frac{(16c_1 - 5c_2)x^5}{8} + \frac{(-80c_1 - 299c_2)x^4}{16} + 5c_2x^3 + \frac{5c_2x^2}{4} + \frac{c_2x}{4} + \frac{c_2}{40}$$

Mathematica DSolve solution

Solving time : 0.518 (sec)
Leaf size : 125

```
DSolve[{x^2*(1+x)*D[y[x],{x,2}]-x*(3+10*x)*D[y[x],x]+30*x*y[x]==0,{}},y[x],x,IncludeSingularSol
```

$$y(x) \to \frac{1}{2}(2x-5)\exp\left(\int_1^x \frac{5-2K[1]}{2K[1]^2+2K[1]}dK[1] - \frac{1}{2}\int_1^x \left(-\frac{7}{K[2]+1} - \frac{3}{K[2]}\right)dK[2]\right)\left(c_2\int_1^x \frac{4\exp\left(-2\int_1^{K[3]}\frac{5-2K[1]}{2K[1]^2+2K[1]}dK[1]\right)}{(5-2K[3])^2}dK[3] + c_1\right)$$

2.1.173 Problem 175

Internal problem ID [9345]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 175
Date solved : `Monday, January 27, 2025 at 06:01:43 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x(1+x)\,y' - 3(3+x)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.285 (sec)

Writing the ode as

$$x^2y'' + \left(x^2+x\right)y' + (-3x-9)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2+x \\ C &= -3x-9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2+14x+35}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2+14x+35 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2+14x+35}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.330: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{7}{2x} + \frac{35}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{7}{2x} - \frac{7}{2x^2} + \frac{49}{2x^3} - \frac{735}{4x^4} + \frac{5831}{4x^5} - \frac{48363}{4x^6} + \frac{415373}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 14x + 35}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{14x + 35}{4x^2}\right) \\ &= \frac{1}{4} + \frac{14x + 35}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 14. Dividing this by leading coefficient in t which is 4 gives $\frac{7}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{7}{2}\right) - (0) \\ &= \frac{7}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{\frac{7}{2}}{\frac{1}{2}} - 0\right) = \frac{7}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{\frac{7}{2}}{\frac{1}{2}} - 0\right) = -\frac{7}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 14x + 35}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{7}{2}$	$-\frac{7}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+\right) \\
&= \frac{7}{2} - \left(\frac{7}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{7}{2x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} + \frac{7}{2x} \\
&= \frac{x+7}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2} + \frac{7}{2x}\right)(0) + \left(\left(-\frac{7}{2x^2}\right) + \left(\frac{1}{2} + \frac{7}{2x}\right)^2 - \left(\frac{x^2+14x+35}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{2} + \frac{7}{2x}\right) dx} \\
&= x^{7/2} e^{\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2+x}{x^2}\,dx} \\
&= z_1 e^{-\frac{x}{2}-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{e^{-\frac{x}{2}}}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^3$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{e^{-x}}{6x^6}+\frac{e^{-x}}{30x^5}-\frac{e^{-x}}{120x^4}+\frac{e^{-x}}{360x^3}-\frac{e^{-x}}{720x^2}+\frac{e^{-x}}{720x}-\frac{\mathrm{Ei}_1(x)}{720}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x^3) + c_2\left(x^3\left(-\frac{e^{-x}}{6x^6}+\frac{e^{-x}}{30x^5}-\frac{e^{-x}}{120x^4}+\frac{e^{-x}}{360x^3}-\frac{e^{-x}}{720x^2}+\frac{e^{-x}}{720x}-\frac{\mathrm{Ei}_1(x)}{720}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right)+x(x+1)\left(\frac{d}{dx}y(x)\right)-3(x+3)\,y(x)=0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x)=\frac{3(x+3)y(x)}{x^2}-\frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x)+\frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{x}-\frac{3(x+3)y(x)}{x^2}=0$
- □ Check to see if $x_0=0$ is a regular singular point
  - ○ Define functions
    $\left[P_2(x)=\frac{x+1}{x}, P_3(x)=-\frac{3(x+3)}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1$
- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -9$
- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(x+1)\left(\frac{d}{dx}y(x)\right) + (-3x-9)\,y(x) = 0$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
- Shift index using $k- >k + 1 - m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(3+r)\,(-3+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+3)\,(k+r-3) + a_{k-1}(k-4+r)\right)x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(3+r)\,(-3+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-3, 3\}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+3)\,(k+r-3) + a_{k-1}(k-4+r) = 0$
- Shift index using $k- >k + 1$

  $a_{k+1}(k+4+r)\,(k-2+r) + a_k(k+r-3) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r-3)}{(k+4+r)(k-2+r)}$
- Recursion relation for $r = -3$; series terminates at $k = 6$

  $a_{k+1} = -\frac{a_k(k-6)}{(k+1)(k-5)}$
- Series not valid for $r = -3$, division by 0 in the recursion relation at $k = 5$

  $a_{k+1} = -\frac{a_k(k-6)}{(k+1)(k-5)}$
- Recursion relation for $r = 3$

$a_{k+1} = -\frac{a_k k}{(k+7)(k+1)}$

- Solution for $r = 3$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = -\frac{a_k k}{(k+7)(k+1)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 50

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(x+1)*diff(y(x),x)-3*(x+3)*y(x) = 0,y(x),singsol=all
```

$$y = \frac{-c_2(x^5 - x^4 + 2x^3 - 6x^2 + 24x - 120) e^{-x} + x^6(\mathrm{Ei}_1(x) c_2 + c_1)}{x^3}$$

Mathematica DSolve solution

Solving time : 0.367 (sec)
Leaf size : 40

```
DSolve[{x^2*D[y[x],{x,2}]+x*(1+x)*D[y[x],x]-3*(3+x)*y[x]==0,{}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to e^{7/2} x^3 \left(c_2 \int_1^x \frac{e^{-K[1]-7}}{K[1]^7} dK[1] + c_1\right)$$

2.1.174 Problem 176

Internal problem ID [9346]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 176
Date solved : Monday, January 27, 2025 at 06:01:44 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$x^2(1+2x)\,y''+x(9+13x)\,y'+(7+5x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.243 (sec)

Writing the ode as

$$\left(2x^3+x^2\right)y''+\left(13x^2+9x\right)y'+(7+5x)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=2x^3+x^2\\ B&=13x^2+9x \\ C&=7+5x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{77x^2+86x+35}{4\left(2x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=77x^2+86x+35\\ t&=4\left(2x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{77x^2+86x+35}{4\left(2x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.332: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{45}{16\left(x + \frac{1}{2}\right)^2} + \frac{27}{2\left(x + \frac{1}{2}\right)} - \frac{27}{2x} + \frac{35}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{5}{2} \end{aligned}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{45}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{9}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{5}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{77x^2 + 86x + 35}{4\left(2x^2 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{77}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{7}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{77x^2 + 86x + 35}{4\left(2x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$
$-\frac{1}{2}$	2	0	$\frac{9}{4}$	$-\frac{5}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{11}{4}$	$-\frac{7}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{7}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= -\frac{7}{4} - \left(-\frac{15}{4}\right) \\
&= 2
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{5}{2x} - \frac{5}{4\left(x+\frac{1}{2}\right)} + (-)\left(0\right) \\
&= -\frac{5}{2x} - \frac{5}{4\left(x+\frac{1}{2}\right)} \\
&= \frac{-5-15x}{4x^2+2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(-\frac{5}{2x} - \frac{5}{4\left(x+\frac{1}{2}\right)}\right)(2x + a_1) + \left(\left(\frac{5}{2x^2} + \frac{5}{4\left(x+\frac{1}{2}\right)^2}\right) + \left(-\frac{5}{2x} - \frac{5}{4\left(x+\frac{1}{2}\right)}\right)^2 - \left(\frac{77x^2 + 86x}{4\left(2x^2 + \right.}\right.\right.$$

$$\frac{(11a_1 - 8)\, x + 26a_0}{2x^2 + x}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{20}{143}, a_1 = \frac{8}{11}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + \frac{8}{11}x + \frac{20}{143}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 + \frac{8}{11}x + \frac{20}{143}\right) e^{\int \left(-\frac{5}{2x} - \frac{5}{4\left(x+\frac{1}{2}\right)}\right) dx} \\
&= \left(x^2 + \frac{8}{11}x + \frac{20}{143}\right) e^{-\frac{5\ln(1+2x)}{4} - \frac{5\ln(x)}{2}} \\
&= \frac{x^2 + \frac{8}{11}x + \frac{20}{143}}{(1+2x)^{5/4}\, x^{5/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{13x^2+9x}{2x^3+x^2}\, dx} \\
&= z_1 e^{\frac{5\ln(1+2x)}{4} - \frac{9\ln(x)}{2}} \\
&= z_1 \left(\frac{(1+2x)^{5/4}}{x^{9/2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2 + \frac{8}{11}x + \frac{20}{143}}{x^7}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{13x^2+9x}{2x^3+x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{5\ln(1+2x)}{2}-9\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{143(1+2x)\left(35x^3-45x^2+36x-20\right)x^9 \mathrm{e}^{\frac{5\ln(1+2x)}{2}-9\ln(x)}}{315\left(143x^2+104x+20\right)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^2+\frac{8}{11}x+\frac{20}{143}}{x^7}\right) \\
&\quad + c_2 \left(\frac{x^2+\frac{8}{11}x+\frac{20}{143}}{x^7}\left(\frac{143(1+2x)\left(35x^3-45x^2+36x-20\right)x^9 \mathrm{e}^{\frac{5\ln(1+2x)}{2}-9\ln(x)}}{315\left(143x^2+104x+20\right)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(2x+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(9+13x)\left(\frac{d}{dx}y(x)\right)+(7+5x)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(7+5x)y(x)}{x^2(2x+1)}-\frac{(9+13x)\left(\frac{d}{dx}y(x)\right)}{x(2x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{(9+13x)\left(\frac{d}{dx}y(x)\right)}{x(2x+1)}+\frac{(7+5x)y(x)}{x^2(2x+1)}=0$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x)=\frac{9+13x}{x(2x+1)}, P_3(x)=\frac{7+5x}{x^2(2x+1)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\Big|_{x=0}=9$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=7$

  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(2x+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(9+13x)\left(\frac{d}{dx}y(x)\right)+(7+5x)\,y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - ○ Convert $x^m \cdot y(x)$ to series expansion for $m=0..1$

    $x^m \cdot y(x)=\sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - ○ Shift index using $k->k-m$

    $x^m \cdot y(x)=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - ○ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
  - ○ Shift index using $k->k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - ○ Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..3$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$
  - ○ Shift index using $k->k+2-m$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(7+r)\,(1+r)\,x^r+\left(\sum_{k=1}^{\infty}\left(a_k(k+r+7)\,(k+r+1)+a_{k-1}(k+4+r)\,(2k-1+2r)\right)x^{k+r}\right)=$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(7+r)\,(1+r)=0$
- Values of r that satisfy the indicial equation

  $r\in\{-7,-1\}$
- Each term in the series must be 0, giving the recursion relation

  $2(k+4+r)\left(k+r-\frac{1}{2}\right)a_{k-1}+a_k(k+r+7)\,(k+r+1)=0$
- Shift index using $k->k+1$

  $2(k+r+5)\left(k+\frac{1}{2}+r\right)a_k+a_{k+1}(k+8+r)\,(k+2+r)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{(k+r+5)(2k+2r+1)a_k}{(k+8+r)(k+2+r)}$
- Recursion relation for $r=-7$; series terminates at $k=2$

  $a_{k+1}=-\frac{(k-2)(2k-13)a_k}{(k+1)(k-5)}$
- Apply recursion relation for $k=0$

  $a_1=\frac{26a_0}{5}$
- Apply recursion relation for $k=1$

  $a_2=\frac{11a_1}{8}$
- Express in terms of a_0

  $a_2=\frac{143a_0}{20}$
- Terminating series solution of the ODE for $r=-7$. Use reduction of order to find the second

  $y(x)=a_0\cdot\left(\frac{143}{20}x^2+\frac{26}{5}x+1\right)$
- Recursion relation for $r=-1$

  $a_{k+1}=-\frac{(k+4)(2k-1)a_k}{(k+7)(k+1)}$
- Solution for $r=-1$

$$\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1}=-\frac{(k+4)(2k-1)a_k}{(k+7)(k+1)}\right]$$

- Combine solutions and rename parameters

$$\left[y(x)=a_0\cdot\left(\frac{143}{20}x^2+\frac{26}{5}x+1\right)+\left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), b_{k+1}=-\frac{(4+k)(2k-1)b_k}{(k+7)(k+1)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.021 (sec)
Leaf size : 50

```
dsolve(x^2*(2*x+1)*diff(diff(y(x),x),x)+x*(9+13*x)*diff(y(x),x)+(7+5*x)*y(x) = 0,y(x),si
```

$$y=\frac{280c_2\left(x+\frac{1}{2}\right)^3\left(x^3-\frac{9}{7}x^2+\frac{36}{35}x-\frac{4}{7}\right)\sqrt{2x+1}+143c_1x^2+104c_1x+20c_1}{x^7}$$

Mathematica DSolve solution

Solving time : 0.948 (sec)
Leaf size : 141

```
DSolve[{x^2*(1+2*x)*D[y[x],{x,2}]+x*(9+13*x)*D[y[x],x]+(7+5*x)*y[x]==0,{}},y[x],x,IncludeSingul
```

$$y(x)\to\frac{1}{143}(143x^2+104x+20)\exp\left(\int_1^x-\frac{15K[1]+5}{4K[1]^2+2K[1]}dK[1]-\frac{1}{2}\int_1^x\frac{13K[2]+9}{2K[2]^2+K[2]}dK[2]\right)\left(c_2\int_1^x\frac{20449\exp\left(-2\int_1^{K[3]}-\frac{15K[1]+5}{4K[1]^2+2K[1]}dK[1]\right)}{(143K[3]^2+104K[3]+20)^2}dK[3]+c_1\right)$$

2.1.175 Problem 177

Internal problem ID [9347]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 177
Date solved : `Monday, January 27, 2025 at 06:01:44 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2(1+2x)\,y''-2x(4-x)\,y'-(7+5x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.200 (sec)

Writing the ode as

$$\left(8x^3+4x^2\right)y''+\left(2x^2-8x\right)y'+(-5x-7)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 8x^3+4x^2\\ B &= 2x^2-8x\\ C &= -5x-7 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t}\\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{33x^2+132x+60}{16\left(2x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 33x^2+132x+60\\ t &= 16\left(2x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{33x^2+132x+60}{16\left(2x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.334: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(2x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{27}{4x} + \frac{15}{4x^2} + \frac{9}{64\left(x + \frac{1}{2}\right)^2} + \frac{27}{4\left(x + \frac{1}{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{2} \end{aligned}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{9}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{9}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{8} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{33x^2 + 132x + 60}{16\left(2x^2 + x\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{33}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{8} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{8}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{33x^2 + 132x + 60}{16\left(2x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
$-\frac{1}{2}$	2	0	$\frac{9}{8}$	$-\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{11}{8}$	$-\frac{3}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{3}{8}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= -\frac{3}{8} - \left(-\frac{3}{8}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{2x} + \frac{9}{8\left(x+\frac{1}{2}\right)} + (-)(0) \\
&= -\frac{3}{2x} + \frac{9}{8\left(x+\frac{1}{2}\right)} \\
&= -\frac{3(x+2)}{4x(1+2x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{3}{2x}+\frac{9}{8\left(x+\frac{1}{2}\right)}\right)(0)+\left(\left(\frac{3}{2x^2}-\frac{9}{8\left(x+\frac{1}{2}\right)^2}\right)+\left(-\frac{3}{2x}+\frac{9}{8\left(x+\frac{1}{2}\right)}\right)^2-\left(\frac{33x^2+132x+60}{16\left(2x^2+x\right)^2}\right)\right.$$
$$0=$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x)&=pe^{\int \omega\,dx}\\
&=\mathrm{e}^{\int\left(-\frac{3}{2x}+\frac{9}{8\left(x+\frac{1}{2}\right)}\right)dx}\\
&=\frac{(1+2x)^{9/8}}{x^{3/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1&=z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\
&=z_1e^{-\int \frac{1}{2}\frac{2x^2-8x}{8x^3+4x^2}\,dx}\\
&=z_1e^{\ln(x)-\frac{9\ln(1+2x)}{8}}\\
&=z_1\left(\frac{x}{(1+2x)^{9/8}}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\frac{1}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2&=y_1\int \frac{e^{\int -\frac{2x^2-8x}{8x^3+4x^2}\,dx}}{(y_1)^2}\,dx\\
&=y_1\int \frac{e^{2\ln(x)-\frac{9\ln(1+2x)}{4}}}{(y_1)^2}\,dx\\
&=y_1\left(\frac{2(1+2x)\left(5x^3-10x^2-40x-16\right)\mathrm{e}^{2\ln(x)-\frac{9\ln(1+2x)}{4}}}{35x^2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y&=c_1y_1+c_2y_2\\
&=c_1\left(\frac{1}{\sqrt{x}}\right)+c_2\left(\frac{1}{\sqrt{x}}\left(\frac{2(1+2x)\left(5x^3-10x^2-40x-16\right)\mathrm{e}^{2\ln(x)-\frac{9\ln(1+2x)}{4}}}{35x^2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2(2x+1)\left(\frac{d^2}{dx^2}y(x)\right)-2x(4-x)\left(\frac{d}{dx}y(x)\right)-(7+5x)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=\frac{(7+5x)y(x)}{4x^2(2x+1)}-\frac{(-4+x)\left(\frac{d}{dx}y(x)\right)}{2(2x+1)x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)+\frac{(-4+x)\left(\frac{d}{dx}y(x)\right)}{2(2x+1)x}-\frac{(7+5x)y(x)}{4x^2(2x+1)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=\frac{-4+x}{2(2x+1)x},P_3(x)=-\frac{7+5x}{4x^2(2x+1)}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

$\left.(x\cdot P_2(x))\right|_{x=0}=-2$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left.(x^2\cdot P_3(x))\right|_{x=0}=-\frac{7}{4}$

- ◦ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$4x^2(2x+1)\left(\frac{d^2}{dx^2}y(x)\right)+2x(-4+x)\left(\frac{d}{dx}y(x)\right)+(-5x-7)\,y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

$x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ◦ Shift index using $k->k-m$

$x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ◦ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..3$

$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- ◦ Shift index using $k->k+2-m$

$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

Rewrite ODE with series expansions

$$a_0(1+2r)(-7+2r)x^r + \left(\sum_{k=1}^{\infty} (a_k(2k+2r+1)(2k+2r-7) + a_{k-1}(2k-1+2r)(4k-9+4r))x$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(1+2r)(-7+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{-\frac{1}{2}, \frac{7}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $8\left(k-\frac{9}{4}+r\right)\left(k+r-\frac{1}{2}\right)a_{k-1} + 4a_k\left(k+r+\frac{1}{2}\right)\left(k+r-\frac{7}{2}\right) = 0$
- Shift index using $k- >k+1$
  $8\left(k-\frac{5}{4}+r\right)\left(k+r+\frac{1}{2}\right)a_k + 4a_{k+1}\left(k+\frac{3}{2}+r\right)\left(k-\frac{5}{2}+r\right) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = -\frac{(4k+4r-5)(2k+2r+1)a_k}{(2k+3+2r)(2k-5+2r)}$
- Recursion relation for $r = -\frac{1}{2}$
  $a_{k+1} = -\frac{2(4k-7)ka_k}{(2k+2)(2k-6)}$
- Series not valid for $r = -\frac{1}{2}$, division by 0 in the recursion relation at $k = 3$
  $a_{k+1} = -\frac{2(4k-7)ka_k}{(2k+2)(2k-6)}$
- Recursion relation for $r = \frac{7}{2}$
  $a_{k+1} = -\frac{(4k+9)(2k+8)a_k}{(2k+10)(2k+2)}$
- Solution for $r = \frac{7}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{7}{2}}, a_{k+1} = -\frac{(4k+9)(2k+8)a_k}{(2k+10)(2k+2)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.025 (sec)
Leaf size : 34

```
dsolve(4*x^2*(2*x+1)*diff(diff(y(x),x),x)-2*x*(-x+4)*diff(y(x),x)-(7+5*x)*y(x) = 0,y(x),
```

$$y = \frac{c_1 + \frac{c_2(5x^3-10x^2-40x-16)}{(2x+1)^{5/4}}}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.275 (sec)
Leaf size : 111

```
DSolve[{4*x^2*(1+2*x)*D[y[x],{x,2}]-2*x*(4-x)*D[y[x],x]-(7+5*x)*y[x]==0,{}},y[x],x,IncludeSi
```

$$y(x) \to \exp\left(\int_1^x -\frac{3K[1]+6}{8K[1]^2+4K[1]}dK[1] - \frac{1}{2}\int_1^x \left(\frac{9}{4K[2]+2} - \frac{2}{K[2]}\right) dK[2]\right) \left(c_2 \int_1^x \exp\left(-2\int_1^{K[3]} -\frac{3K[1]+6}{8K[1]^2+4K[1]}dK[1]\right) dK[3] + c_1\right)$$

2.1.176 Problem 178

Internal problem ID [9348]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 178
Date solved : `Monday, January 27, 2025 at 06:01:45 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$3x^2(3+x)\,y''-x(15+x)\,y'-20y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.225 (sec)

Writing the ode as

$$\left(3x^3+9x^2\right)y''+\left(-x^2-15x\right)y'-20y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=3x^3+9x^2\\B&=-x^2-15x\\C&=-20\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{7x^2+450x+1215}{36\left(x^2+3x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=7x^2+450x+1215\\t&=36\left(x^2+3x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{7x^2+450x+1215}{36\left(x^2+3x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.336: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^2 + 3x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -3$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{9\,(3 + x)^2} + \frac{15}{4x^2} - \frac{10}{9x} + \frac{10}{9\,(3 + x)}$$

For the pole at $x = -3$ let b be the coefficient of $\frac{1}{(3+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{2}{9}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{2}{3} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{3} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{7x^2 + 450x + 1215}{36\left(x^2 + 3x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{7}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{7x^2 + 450x + 1215}{36\left(x^2 + 3x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-3	2	0	$\frac{2}{3}$	$\frac{1}{3}$
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{6}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= -\frac{1}{6} - \left(-\frac{7}{6}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{9+3x} - \frac{3}{2x} + (-)(0) \\
&= \frac{1}{9+3x} - \frac{3}{2x} \\
&= -\frac{7x+27}{6x(3+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{9+3x} - \frac{3}{2x}\right)(1) + \left(\left(-\frac{1}{3(3+x)^2} + \frac{3}{2x^2}\right) + \left(\frac{1}{9+3x} - \frac{3}{2x}\right)^2 - \left(\frac{7x^2 + 450x + 1215}{36(x^2+3x)^2}\right)\right) = \frac{-27 + 7a_0}{3x(3+x)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{27}{7}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{27}{7}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{27}{7}\right) e^{\int \left(\frac{1}{9+3x} - \frac{3}{2x}\right) dx} \\
&= \left(x + \frac{27}{7}\right) e^{\frac{\ln(3+x)}{3} - \frac{3\ln(x)}{2}} \\
&= \frac{\left(x + \frac{27}{7}\right)(3+x)^{1/3}}{x^{3/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2-15x}{3x^3+9x^2}\, dx} \\
&= z_1 e^{-\frac{2\ln(3+x)}{3} + \frac{5\ln(x)}{6}} \\
&= z_1 \left(\frac{x^{5/6}}{(3+x)^{2/3}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{7x + 27}{7(3+x)^{1/3} x^{2/3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-15x}{3x^3+9x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{4\ln(3+x)}{3} + \frac{5\ln(x)}{3}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{21(3+x)^{5/3}(x^2 - 36x - 243)\, e^{-\frac{4\ln(3+x)}{3} + \frac{5\ln(x)}{3}}}{4(7x+27)\, x^{5/3}}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{7x+27}{7(3+x)^{1/3} x^{2/3}} \right) \\
&\quad + c_2 \left(\frac{7x+27}{7(3+x)^{1/3} x^{2/3}} \left(\frac{21(3+x)^{5/3} (x^2-36x-243) \mathrm{e}^{-\frac{4\ln(3+x)}{3}+\frac{5\ln(x)}{3}}}{4(7x+27) x^{5/3}} \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$3x^2(x+3)\left(\frac{d^2}{dx^2}y(x)\right) - x(15+x)\left(\frac{d}{dx}y(x)\right) - 20y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{20y(x)}{3x^2(x+3)} + \frac{(15+x)\left(\frac{d}{dx}y(x)\right)}{3x(x+3)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) - \frac{(15+x)\left(\frac{d}{dx}y(x)\right)}{3x(x+3)} - \frac{20y(x)}{3x^2(x+3)} = 0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{15+x}{3x(x+3)}, P_3(x) = -\frac{20}{3x^2(x+3)}\right]$

  - $(x+3)\cdot P_2(x)$ is analytic at $x=-3$

    $\left((x+3)\cdot P_2(x)\right)\Big|_{x=-3} = \frac{4}{3}$

  - $(x+3)^2\cdot P_3(x)$ is analytic at $x=-3$

    $\left((x+3)^2\cdot P_3(x)\right)\Big|_{x=-3} = 0$

  - $x=-3$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -3$

- Multiply by denominators

  $3x^2(x+3)\left(\frac{d^2}{dx^2}y(x)\right) - x(15+x)\left(\frac{d}{dx}y(x)\right) - 20y(x) = 0$

- Change variables using $x = u-3$ so that the regular singular point is at $u=0$

  $(3u^3-18u^2+27u)\left(\frac{d^2}{du^2}y(u)\right) + (-u^2-9u+36)\left(\frac{d}{du}y(u)\right) - 20y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$9a_0r(1+3r)\, u^{-1+r} + (9a_1(1+r)(4+3r) - a_0(18r^2-9r+20))\, u^r + \left(\sum_{k=1}^{\infty} (9a_{k+1}(k+1+r)(3k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$9r(1+3r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{1}{3}\right\}$$

- Each term must be 0

$$9a_1(1+r)(4+3r) - a_0(18r^2-9r+20) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$3(-6a_k + a_{k-1} + 9a_{k+1})\, k^2 + (6(-6a_k + a_{k-1} + 9a_{k+1})\, r + 9a_k - 10a_{k-1} + 63a_{k+1})\, k + 3(-6a_k +$$

- Shift index using $k- >k+1$

$$3(-6a_{k+1} + a_k + 9a_{k+2})\,(k+1)^2 + (6(-6a_{k+1} + a_k + 9a_{k+2})\, r + 9a_{k+1} - 10a_k + 63a_{k+2})\,(k+1)$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{3k^2a_k - 18k^2a_{k+1} + 6kra_k - 36kra_{k+1} + 3r^2a_k - 18r^2a_{k+1} - 4ka_k - 27ka_{k+1} - 4ra_k - 27ra_{k+1} - 29a_{k+1}}{9(3k^2+6kr+3r^2+13k+13r+14)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{3k^2a_k - 18k^2a_{k+1} - 4ka_k - 27ka_{k+1} - 29a_{k+1}}{9(3k^2+13k+14)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{3k^2a_k - 18k^2a_{k+1} - 4ka_k - 27ka_{k+1} - 29a_{k+1}}{9(3k^2+13k+14)}, 36a_1 - 20a_0 = 0\right]$$

- Revert the change of variables $u = x+3$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k(x+3)^k, a_{k+2} = -\frac{3k^2a_k - 18k^2a_{k+1} - 4ka_k - 27ka_{k+1} - 29a_{k+1}}{9(3k^2+13k+14)}, 36a_1 - 20a_0 = 0\right]$$

- Recursion relation for $r = -\frac{1}{3}$

$$a_{k+2} = -\frac{3k^2a_k - 18k^2a_{k+1} - 6ka_k - 15ka_{k+1} + \frac{5}{3}a_k - 22a_{k+1}}{9(3k^2+11k+10)}$$

- Solution for $r = -\frac{1}{3}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{1}{3}}, a_{k+2} = -\frac{3k^2a_k - 18k^2a_{k+1} - 6ka_k - 15ka_{k+1} + \frac{5}{3}a_k - 22a_{k+1}}{9(3k^2+11k+10)}, 18a_1 - 25a_0 = 0\right]$$

- Revert the change of variables $u = x+3$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k(x+3)^{k-\frac{1}{3}}, a_{k+2} = -\frac{3k^2a_k - 18k^2a_{k+1} - 6ka_k - 15ka_{k+1} + \frac{5}{3}a_k - 22a_{k+1}}{9(3k^2+11k+10)}, 18a_1 - 25a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k(x+3)^k\right) + \left(\sum_{k=0}^{\infty} b_k(x+3)^{k-\frac{1}{3}}\right), a_{k+2} = -\frac{3k^2a_k - 18k^2a_{k+1} - 4ka_k - 27ka_{k+1} - 29a_{k+1}}{9(3k^2+13k+14)}, 36$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.036 (sec)
Leaf size : 31

```
dsolve(3*x^2*(x+3)*diff(diff(y(x),x),x)-x*(15+x)*diff(y(x),x)-20*y(x) = 0,y(x),singsol=a
```

$$y = \frac{c_1(x^2 - 36x - 243) + \frac{c_2(7x+27)}{(x+3)^{1/3}}}{x^{2/3}}$$

Mathematica DSolve solution

Solving time : 0.522 (sec)
Leaf size : 123

```
DSolve[{3*x^2*(3+x)*D[y[x],{x,2}]-x*(15+x)*D[y[x],x]-20*y[x]==0,{}},y[x],x,IncludeSingularSolu
```

$$y(x) \to \frac{1}{7}(7x+27)\exp\left(\int_1^x \left(\frac{1}{3K[1]+9} - \frac{3}{2K[1]}\right) dK[1] - \frac{1}{2}\int_1^x -\frac{K[2]+15}{3K[2]^2+9K[2]} dK[2]\right)\left(c_2\int_1^x \frac{49\exp\left(-2\int_1^{K[3]}\left(\frac{1}{3K[1]+9} - \frac{3}{2K[1]}\right) dK[1]\right)}{(7K[3]+27)^2} dK[3] + c_1\right)$$

2.1.177 Problem 179

Internal problem ID [9349]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 179
Date solved : `Monday, January 27, 2025 at 06:01:46 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+x)\,y''+x(1-10x)\,y'-(9-10x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.239 (sec)

Writing the ode as

$$x^2(1+x)\,y''+\left(-10x^2+x\right)y'+(10x-9)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2(1+x)\\ B&=-10x^2+x\\ C&=10x-9\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{80x^2-28x+35}{4\left(x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=80x^2-28x+35\\ t&=4\left(x^2+x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{80x^2-28x+35}{4\left(x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.338: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4-2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+x)^2$. There is a pole at $x=0$ of order 2. There is a pole at $x=-1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1,2,4,6,12]$$

Attempting to find a solution using case $n=1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{49}{2x} + \frac{49}{2\,(1+x)} + \frac{35}{4x^2} + \frac{143}{4\,(1+x)^2}$$

For the pole at $x=-1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{143}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{13}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{11}{2}\end{aligned}$$

For the pole at $x=0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{80x^2 - 28x + 35}{4\left(x^2 + x\right)^2}$$

Since the gcd$(s, t) = 1$. This gives $b = 20$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 5 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -4
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{80x^2 - 28x + 35}{4\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{13}{2}$	$-\frac{11}{2}$
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	5	-4

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 5$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= 5 - (4) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{13}{2(1 + x)} - \frac{5}{2x} + (0) \\
&= \frac{13}{2(1 + x)} - \frac{5}{2x} \\
&= \frac{8x - 5}{2x(1 + x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{13}{2(1+x)} - \frac{5}{2x}\right)(1) + \left(\left(-\frac{13}{2(1+x)^2} + \frac{5}{2x^2}\right) + \left(\frac{13}{2(1+x)} - \frac{5}{2x}\right)^2 - \left(\frac{80x^2-28x+35}{4(x^2+x)^2}\right)\right) = 0$$

$$\frac{-5-8a_0}{x(1+x)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{5}{8}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - \frac{5}{8}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \left(x - \frac{5}{8}\right) e^{\int \left(\frac{13}{2(1+x)} - \frac{5}{2x}\right) dx} \\
&= \left(x - \frac{5}{8}\right) e^{-\frac{5\ln(x)}{2} + \frac{13\ln(1+x)}{2}} \\
&= \frac{\left(x - \frac{5}{8}\right)(1+x)^{13/2}}{x^{5/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-10x^2+x}{x^2(1+x)}\, dx} \\
&= z_1 e^{-\frac{\ln(x)}{2} + \frac{11\ln(1+x)}{2}} \\
&= z_1 \left(\frac{(1+x)^{11/2}}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(1+x)^{12}\left(x - \frac{5}{8}\right)}{x^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-10x^2+x}{x^2(1+x)}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\ln(x)+11\ln(1+x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{8\,e^{-\ln(x)+11\ln(1+x)} x(715x^4 + 572x^3 + 234x^2 + 52x + 5)}{6435(8x-5)(1+x)^{23}}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(1+x)^{12} \left(x - \frac{5}{8}\right)}{x^3} \right) \\
&\quad + c_2 \left(\frac{(1+x)^{12} \left(x - \frac{5}{8}\right)}{x^3} \left(-\frac{8\, \mathrm{e}^{-\ln(x)+11\ln(1+x)} x(715x^4 + 572x^3 + 234x^2 + 52x + 5)}{6435\,(8x-5)\,(1+x)^{23}} \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right) + x(1-10x)\left(\frac{d}{dx}y(x)\right) - (9-10x)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{(-9+10x)y(x)}{(x+1)x^2} + \frac{(-1+10x)\left(\frac{d}{dx}y(x)\right)}{x(x+1)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x) - \frac{(-1+10x)\left(\frac{d}{dx}y(x)\right)}{x(x+1)} + \frac{(-9+10x)y(x)}{(x+1)x^2} = 0$

□ Check to see if x_0 is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{-1+10x}{x(x+1)}, P_3(x) = \frac{-9+10x}{(x+1)x^2}\right]$
  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$
    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1} = -11$
  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$
    $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$
  - $x=-1$is a regular singular point
    Check to see if x_0 is a regular singular point
    $x_0 = -1$
- Multiply by denominators
  $x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right) - x(-1+10x)\left(\frac{d}{dx}y(x)\right) + (-9+10x)\,y(x) = 0$
- Change variables using $x = u-1$ so that the regular singular point is at $u=0$
  $(u^3 - 2u^2 + u)\left(\frac{d^2}{du^2}y(u)\right) + (-10u^2 + 21u - 11)\left(\frac{d}{du}y(u)\right) + (-19+10u)\,y(u) = 0$
- Assume series solution for $y(u)$
  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions
  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$
    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$
  - Shift index using $k->k-m$
    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$
- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$
- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$
- Shift index using $k- >k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-12+r)\, u^{-1+r} + (a_1(1+r)(-11+r) - a_0(2r^2-23r+19))\, u^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)(k-1\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-12+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 12\}$
- Each term must be 0

  $a_1(1+r)(-11+r) - a_0(2r^2-23r+19) = 0$
- Each term in the series must be 0, giving the recursion relation

  $(-2a_k + a_{k-1} + a_{k+1})\, k^2 + ((-4a_k + 2a_{k-1} + 2a_{k+1})\, r + 23a_k - 13a_{k-1} - 10a_{k+1})\, k + (-2a_k + a_{k-1}$
- Shift index using $k- >k+1$

  $(-2a_{k+1} + a_k + a_{k+2})(k+1)^2 + ((-4a_{k+1} + 2a_k + 2a_{k+2})\, r + 23a_{k+1} - 13a_k - 10a_{k+2})(k+1) + ($
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + 2kra_k - 4kra_{k+1} + r^2a_k - 2r^2a_{k+1} - 11ka_k + 19ka_{k+1} - 11ra_k + 19ra_{k+1} + 10a_k + 2a_{k+1}}{k^2+2kr+r^2-8k-8r-20}$
- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} - 11ka_k + 19ka_{k+1} + 10a_k + 2a_{k+1}}{k^2-8k-20}$
- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 10$

  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} - 11ka_k + 19ka_{k+1} + 10a_k + 2a_{k+1}}{k^2-8k-20}$
- Recursion relation for $r = 12$

  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + 13ka_k - 29ka_{k+1} + 22a_k - 58a_{k+1}}{k^2+16k+28}$
- Solution for $r = 12$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+12}, a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + 13ka_k - 29ka_{k+1} + 22a_k - 58a_{k+1}}{k^2+16k+28}, 13a_1 - 31a_0 = 0\right]$
- Revert the change of variables $u = x+1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^{k+12}, a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + 13ka_k - 29ka_{k+1} + 22a_k - 58a_{k+1}}{k^2+16k+28}, 13a_1 - 31a_0 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.009 (sec)
Leaf size : 82

```
dsolve(x^2*(x+1)*diff(diff(y(x),x),x)+x*(1-10*x)*diff(y(x),x)-(9-10*x)*y(x) = 0,y(x),s
```

$$y = \frac{8c_2x^{13} + 91c_2x^{12} + 468c_2x^{11} + 1430c_2x^{10} + 2860c_2x^9 + 3861c_2x^8 + 3432c_2x^7 + 1716c_2x^6 + 715c_1x^4 + 5}{x^3}$$

Mathematica DSolve solution

Solving time : 0.504 (sec)
Leaf size : 123

```
DSolve[{x^2*(1+x)*D[y[x],{x,2}]+x*(1-10*x)*D[y[x],x]-(9-10*x)*y[x]==0,{}},y[x],x,IncludeSing
```

$$y(x) \to \frac{1}{8}(8x-5)\exp\left(\int_1^x \frac{8K[1]-5}{2K[1](K[1]+1)}dK[1] - \frac{1}{2}\int_1^x \left(\frac{1}{K[2]} - \frac{11}{K[2]+1}\right)dK[2]\right)\left(c_2\int_1^x \frac{64\exp\left(-2\int_1^{K[3]}\frac{8K[1]-5}{2K[1](K[1]+1)}dK[1]\right)}{(5-8K[3])^2}dK[3] + c_1\right)$$

2.1.178 Problem 180

Internal problem ID [9350]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 180
Date solved : `Monday, January 27, 2025 at 06:01:46 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+x)\,y''+3x^2y'-(6-x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.211 (sec)

Writing the ode as

$$x^2(1+x)\,y''+3x^2y'+(x-6)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2(1+x)\\B&=3x^2\\C&=x-6\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-x^2+20x+24}{4\left(x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=-x^2+20x+24\\t&=4\left(x^2+x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-x^2+20x+24}{4\left(x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.340: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{x} + \frac{6}{x^2} + \frac{3}{4\,(1 + x)^2} + \frac{7}{1 + x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2+20x+24}{4\left(x^2+x\right)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b} = \frac{1}{2}\\ \alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2+20x+24}{4\left(x^2+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right)\\ &= \frac{1}{2} - \left(-\frac{1}{2}\right)\\ &= 1\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (-)[\sqrt{r}]_\infty\\ &= \frac{3}{2\,(1+x)} - \frac{2}{x} + (-)\,(0)\\ &= \frac{3}{2\,(1+x)} - \frac{2}{x}\\ &= -\frac{x+4}{2x\,(1+x)}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=x+a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{3}{2\left(1+x\right)}-\frac{2}{x}\right)(1)+\left(\left(-\frac{3}{2\left(1+x\right)^2}+\frac{2}{x^2}\right)+\left(\frac{3}{2\left(1+x\right)}-\frac{2}{x}\right)^2-\left(\frac{-x^2+20x+24}{4\left(x^2+x\right)^2}\right)\right)=0$$

$$\frac{-4+a_0}{x\left(1+x\right)}=0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0=4\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x)=x+4$$

Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x)&=pe^{\int \omega\,dx}\\
&=(x+4)\,\mathrm{e}^{\int \left(\frac{3}{2(1+x)}-\frac{2}{x}\right)dx}\\
&=(x+4)\,\mathrm{e}^{\frac{3\ln(1+x)}{2}-2\ln(x)}\\
&=\frac{(x+4)\left(1+x\right)^{3/2}}{x^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1&=z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\
&=z_1e^{-\int \frac{1}{2}\frac{3x^2}{x^2(1+x)}\,dx}\\
&=z_1e^{-\frac{3\ln(1+x)}{2}}\\
&=z_1\left(\frac{1}{\left(1+x\right)^{3/2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\frac{x+4}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2&=y_1\int \frac{e^{\int -\frac{3x^2}{x^2(1+x)}\,dx}}{\left(y_1\right)^2}\,dx\\
&=y_1\int \frac{e^{-3\ln(1+x)}}{\left(y_1\right)^2}\,dx\\
&=y_1\left(\ln\left(1+x\right)-\frac{1}{18\left(1+x\right)^2}+\frac{14}{27\left(1+x\right)}+\frac{256}{27\left(x+4\right)}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x+4}{x^2}\right) + c_2 \left(\frac{x+4}{x^2}\left(\ln(1+x) - \frac{1}{18(1+x)^2} + \frac{14}{27(1+x)} + \frac{256}{27(x+4)}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right) + 3x^2\left(\frac{d}{dx}y(x)\right) - (-x+6)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(-6+x)y(x)}{(x+1)x^2} - \frac{3\left(\frac{d}{dx}y(x)\right)}{x+1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{3\left(\frac{d}{dx}y(x)\right)}{x+1} + \frac{(-6+x)y(x)}{(x+1)x^2} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{3}{x+1}, P_3(x) = \frac{-6+x}{(x+1)x^2}\right]$

  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1} = 3$

  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$

  - $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right) + 3x^2\left(\frac{d}{dx}y(x)\right) + (-6+x)\,y(x) = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(u^3-2u^2+u)\left(\frac{d^2}{du^2}y(u)\right) + (3u^2-6u+3)\left(\frac{d}{du}y(u)\right) + (-7+u)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

  - Shift index using $k->k-m$

    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(2+r)\, u^{-1+r} + (a_1(1+r)\,(3+r) - a_0(2r^2+4r+7))\, u^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+r+1)\,(k+3+r)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 0\}$

- Each term must be 0

  $a_1(1+r)\,(3+r) - a_0(2r^2+4r+7) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k-1}(k+r)^2 + a_{k+1}(k+r+1)\,(k+3+r) - 2\left(k^2 + (2r+2)\,k + r^2 + 2r + \frac{7}{2}\right) a_k = 0$

- Shift index using $k->k+1$

  $a_k(k+r+1)^2 + a_{k+2}(k+r+2)\,(k+4+r) - 2\left((k+1)^2 + (2r+2)\,(k+1) + r^2 + 2r + \frac{7}{2}\right) a_{k+1}$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + 2kra_k - 4kra_{k+1} + r^2a_k - 2r^2a_{k+1} + 2ka_k - 8ka_{k+1} + 2ra_k - 8ra_{k+1} + a_k - 13a_{k+1}}{(k+r+2)(k+4+r)}$

- Recursion relation for $r = -2$

  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} - 2ka_k + a_k - 5a_{k+1}}{k(k+2)}$

- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 0$

  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} - 2ka_k + a_k - 5a_{k+1}}{k(k+2)}$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + 2ka_k - 8ka_{k+1} + a_k - 13a_{k+1}}{(k+2)(k+4)}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + 2ka_k - 8ka_{k+1} + a_k - 13a_{k+1}}{(k+2)(k+4)}, 3a_1 - 7a_0 = 0\right]$

- Revert the change of variables $u = x+1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + 2ka_k - 8ka_{k+1} + a_k - 13a_{k+1}}{(k+2)(k+4)}, 3a_1 - 7a_0 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 45

```
dsolve(x^2*(x+1)*diff(diff(y(x),x),x)+3*diff(y(x),x)*x^2-(-x+6)*y(x) = 0,y(x),singsol=al
```

$$y = \frac{c_1(x+4) + \frac{c_2\left(6(x+4)(x+1)^2 \ln(x+1)+60x^2+129x+68\right)}{(x+1)^2}}{x^2}$$

Mathematica DSolve solution

Solving time : 0.497 (sec)
Leaf size : 91

```
DSolve[{x^2*(1+x)*D[y[x],{x,2}]+3*x^2*D[y[x],x]-(6-x)*y[x]==0,{}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to \frac{(x+4)\exp\left(\int_1^x \left(\frac{3}{2(K[1]+1)} - \frac{2}{K[1]}\right) dK[1]\right)\left(c_2 \int_1^x \frac{\exp\left(-2\int_1^{K[2]}\left(\frac{3}{2(K[1]+1)} - \frac{2}{K[1]}\right) dK[1]\right)}{(K[2]+4)^2} dK[2] + c_1\right)}{(x+1)^{3/2}}$$

2.1.179 Problem 181

Internal problem ID [9351]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 181
Date solved : `Monday, January 27, 2025 at 06:01:47 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+2x)\,y''-2x(3+14x)\,y'+(6+100x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.200 (sec)

Writing the ode as

$$\left(2x^3+x^2\right)y''+\left(-28x^2-6x\right)y'+(6+100x)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=2x^3+x^2\\ B&=-28x^2-6x \\ C&=6+100x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{24x^2-16x+6}{\left(2x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=24x^2-16x+6\\ t&=\left(2x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{24x^2-16x+6}{\left(2x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.342: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{20}{\left(x + \frac{1}{2}\right)^2} + \frac{40}{x + \frac{1}{2}} - \frac{40}{x} + \frac{6}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 20$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 5 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -4 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{24x^2 - 16x + 6}{(2x^2 + x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{24x^2 - 16x + 6}{(2x^2 + x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	3	-2
$-\frac{1}{2}$	2	0	5	-4

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	3	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 3$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 3 - (3) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{2}{x} + \frac{5}{x+\frac{1}{2}} + (0) \\
&= -\frac{2}{x} + \frac{5}{x+\frac{1}{2}} \\
&= \frac{-2+6x}{2x^2+x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{2}{x} + \frac{5}{x+\frac{1}{2}}\right)(0) + \left(\left(\frac{2}{x^2} - \frac{5}{\left(x+\frac{1}{2}\right)^2}\right) + \left(-\frac{2}{x} + \frac{5}{x+\frac{1}{2}}\right)^2 - \left(\frac{24x^2 - 16x + 6}{\left(2x^2+x\right)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(-\frac{2}{x} + \frac{5}{x+\frac{1}{2}}\right) dx} \\ &= \frac{(1+2x)^5}{x^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-28x^2-6x}{2x^3+x^2}\, dx} \\ &= z_1 e^{4\ln(1+2x)+3\ln(x)} \\ &= z_1\left((1+2x)^4 x^3\right) \end{aligned}$$

Which simplifies to

$$y_1 = (1+2x)^9 x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-28x^2-6x}{2x^3+x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{8\ln(1+2x)+6\ln(x)}}{(y_1)^2}\, dx \\ &= y_1\left(-\frac{\left(2016x^4 + 672x^3 + 144x^2 + 18x + 1\right)\mathrm{e}^{8\ln(1+2x)+6\ln(x)}}{20160\,(1+2x)^{17} x^6}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left((1+2x)^9 x\right) + c_2\left((1+2x)^9 x\left(-\frac{\left(2016x^4 + 672x^3 + 144x^2 + 18x + 1\right)\mathrm{e}^{8\ln(1+2x)+6\ln(x)}}{20160\,(1+2x)^{17} x^6}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(2x+1)\left(\frac{d^2}{dx^2}y(x)\right)-2x(3+14x)\left(\frac{d}{dx}y(x)\right)+(6+100x)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{2(3+50x)y(x)}{x^2(2x+1)}+\frac{2(3+14x)\left(\frac{d}{dx}y(x)\right)}{x(2x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)-\frac{2(3+14x)\left(\frac{d}{dx}y(x)\right)}{x(2x+1)}+\frac{2(3+50x)y(x)}{x^2(2x+1)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=-\frac{2(3+14x)}{x(2x+1)},P_3(x)=\frac{2(3+50x)}{x^2(2x+1)}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

$\left.(x\cdot P_2(x))\right|_{x=0}=-6$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left.(x^2\cdot P_3(x))\right|_{x=0}=6$

- ◦ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$x^2(2x+1)\left(\frac{d^2}{dx^2}y(x)\right)-2x(3+14x)\left(\frac{d}{dx}y(x)\right)+(6+100x)\,y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

$x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ◦ Shift index using $k->k-m$

$x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ◦ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..3$

$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- ◦ Shift index using $k->k+2-m$

$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

Rewrite ODE with series expansions

$$a_0(-1+r)(-6+r)x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)(k+r-6) + 2a_{k-1}(k+r-6)(k-11+r)\right) x^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+r)(-6+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{1, 6\}$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-6)\left((2k+2r-22)\,a_{k-1} + a_k(k+r-1)\right) = 0$
- Shift index using $k->k+1$
  $(k+r-5)\left((2k+2r-20)\,a_k + a_{k+1}(k+r)\right) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = -\frac{2(k+r-10)a_k}{k+r}$
- Recursion relation for $r = 1$; series terminates at $k = 9$
  $a_{k+1} = -\frac{2(k-9)a_k}{k+1}$
- Recursion relation that defines the terminating series solution of the ODE for $r = 1$
  $\left[y(x) = \sum_{k=0}^{8} a_k x^{k+1}, a_{k+1} = -\frac{2(k-9)a_k}{k+1}\right]$
- Recursion relation for $r = 6$; series terminates at $k = 4$
  $a_{k+1} = -\frac{2(k-4)a_k}{k+6}$
- Apply recursion relation for $k = 0$
  $a_1 = \frac{4a_0}{3}$
- Apply recursion relation for $k = 1$
  $a_2 = \frac{6a_1}{7}$
- Express in terms of a_0
  $a_2 = \frac{8a_0}{7}$
- Apply recursion relation for $k = 2$
  $a_3 = \frac{a_2}{2}$
- Express in terms of a_0
  $a_3 = \frac{4a_0}{7}$
- Apply recursion relation for $k = 3$
  $a_4 = \frac{2a_3}{9}$
- Express in terms of a_0
  $a_4 = \frac{8a_0}{63}$
- Terminating series solution of the ODE for $r = 6$. Use reduction of order to find the second linea
  $y(x) = a_0 \cdot \left(1 + \frac{4}{3}x + \frac{8}{7}x^2 + \frac{4}{7}x^3 + \frac{8}{63}x^4\right)$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{8} a_k x^{k+1}\right) + b_0 \cdot \left(1 + \frac{4}{3}x + \frac{8}{7}x^2 + \frac{4}{7}x^3 + \frac{8}{63}x^4\right), a_{k+1} = -\frac{2(k-9)a_k}{k+1}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 62

```
dsolve(x^2*(2*x+1)*diff(diff(y(x),x),x)-2*x*(3+14*x)*diff(y(x),x)+(6+100*x)*y(x) = 0,y
```

$$y = 8c_2x^{10}+36c_2x^9+72c_2x^8+84c_2x^7+63c_2x^6+2016c_1x^5+672c_1x^4+144c_1x^3+18c_1x^2+c_1x$$

Mathematica DSolve solution

Solving time : 0.282 (sec)
Leaf size : 105

```
DSolve[{x^2*(1+2*x)*D[y[x],{x,2}]-2*x*(3+14*x)*D[y[x],x]+(6+100*x)*y[x]==0,{}},y[x],x,Includ
```

$$y(x) \to \exp\left(\int_1^x \left(\frac{10}{2K[1]+1}-\frac{2}{K[1]}\right)dK[1]-\frac{1}{2}\int_1^x -\frac{28K[2]+6}{2K[2]^2+K[2]}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]}\left(\frac{10}{2K[1]+1}-\frac{2}{K[1]}\right)dK[1]\right)dK[3]+c_1\right)$$

2.1.180 Problem 182

Internal problem ID [9352]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 182
Date solved : `Monday, January 27, 2025 at 06:01:48 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+x)\,y''-x(6+11x)\,y'+(6+32x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.218 (sec)

Writing the ode as

$$x^2(1+x)\,y''+\left(-11x^2-6x\right)y'+(6+32x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(1+x)\\ B &= -11x^2-6x \\ C &= 6+32x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{15x^2+4x+24}{4\left(x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15x^2+4x+24 \\ t &= 4\left(x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{15x^2+4x+24}{4\left(x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.344: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{35}{4\,(1+x)^2} + \frac{11}{1+x} - \frac{11}{x} + \frac{6}{x^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15x^2 + 4x + 24}{4\left(x^2 + x\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15x^2 + 4x + 24}{4\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{7}{2}$	$-\frac{5}{2}$
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= \frac{5}{2} - \left(\frac{3}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{7}{2\left(1 + x\right)} - \frac{2}{x} + (0) \\
&= \frac{7}{2\left(1 + x\right)} - \frac{2}{x} \\
&= \frac{3x - 4}{2x\left(1 + x\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{7}{2(1+x)} - \frac{2}{x}\right)(1) + \left(\left(-\frac{7}{2(1+x)^2} + \frac{2}{x^2}\right) + \left(\frac{7}{2(1+x)} - \frac{2}{x}\right)^2 - \left(\frac{15x^2 + 4x + 24}{4(x^2+x)^2}\right)\right) = 0$$

$$\frac{-4 - 3a_0}{x(1+x)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{4}{3}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - \frac{4}{3}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \left(x - \frac{4}{3}\right) e^{\int \left(\frac{7}{2(1+x)} - \frac{2}{x}\right) dx} \\
&= \left(x - \frac{4}{3}\right) e^{-2\ln(x) + \frac{7\ln(1+x)}{2}} \\
&= \frac{\left(x - \frac{4}{3}\right)(1+x)^{7/2}}{x^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-11x^2 - 6x}{x^2(1+x)}\, dx} \\
&= z_1 e^{3\ln(x) + \frac{5\ln(1+x)}{2}} \\
&= z_1 \left(x^3 (1+x)^{5/2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x(1+x)^6 \left(x - \frac{4}{3}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-11x^2 - 6x}{x^2(1+x)}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{6\ln(x) + 5\ln(1+x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{3\, e^{6\ln(x) + 5\ln(1+x)}(35x^3 + 42x^2 + 21x + 4)}{140(3x-4)\, x^6 (1+x)^{11}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(x(1+x)^6\left(x-\frac{4}{3}\right)\right) \\
&\quad + c_2\left(x(1+x)^6\left(x-\frac{4}{3}\right)\left(-\frac{3\,\mathrm{e}^{6\ln(x)+5\ln(1+x)}(35x^3+42x^2+21x+4)}{140\,(3x-4)\,x^6\,(1+x)^{11}}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)-x(6+11x)\left(\frac{d}{dx}y(x)\right)+(6+32x)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{2(3+16x)y(x)}{(x+1)x^2}+\frac{(6+11x)\left(\frac{d}{dx}y(x)\right)}{x(x+1)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)-\frac{(6+11x)\left(\frac{d}{dx}y(x)\right)}{x(x+1)}+\frac{2(3+16x)y(x)}{(x+1)x^2}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=-\frac{6+11x}{x(x+1)},P_3(x)=\frac{2(3+16x)}{(x+1)x^2}\right]$
- ○ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=-5$
- ○ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$
- ○ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)-x(6+11x)\left(\frac{d}{dx}y(x)\right)+(6+32x)\,y(x)=0$
- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^3-2u^2+u)\left(\frac{d^2}{du^2}y(u)\right)+(-11u^2+16u-5)\left(\frac{d}{du}y(u)\right)+(-26+32u)\,y(u)=0$
- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_ku^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_ku^{k+r+m}$
- ○ Shift index using $k->k-m$

  $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$
- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$
- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$
- Shift index using $k- >k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-6+r)\, u^{-1+r} + (a_1(1+r)\,(-5+r) - 2a_0(r^2-9r+13))\, u^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(k-5\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-6+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 6\}$
- Each term must be 0

  $a_1(1+r)\,(-5+r) - 2a_0(r^2-9r+13) = 0$
- Each term in the series must be 0, giving the recursion relation

  $(-2a_k + a_{k-1} + a_{k+1})\, k^2 + 2((-2a_k + a_{k-1} + a_{k+1})\, r + 9a_k - 7a_{k-1} - 2a_{k+1})\, k + (-2a_k + a_{k-1}$
- Shift index using $k- >k+1$

  $(-2a_{k+1} + a_k + a_{k+2})\,(k+1)^2 + 2((-2a_{k+1} + a_k + a_{k+2})\, r + 9a_{k+1} - 7a_k - 2a_{k+2})\,(k+1) + ($
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + 2kra_k - 4kra_{k+1} + r^2a_k - 2r^2a_{k+1} - 12ka_k + 14ka_{k+1} - 12ra_k + 14ra_{k+1} + 32a_k - 10a_{k+1}}{k^2+2kr+r^2-2k-2r-8}$
- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} - 12ka_k + 14ka_{k+1} + 32a_k - 10a_{k+1}}{k^2-2k-8}$
- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 4$

  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} - 12ka_k + 14ka_{k+1} + 32a_k - 10a_{k+1}}{k^2-2k-8}$
- Recursion relation for $r = 6$

  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} - 10ka_{k+1} - 4a_k + 2a_{k+1}}{k^2+10k+16}$
- Solution for $r = 6$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+6}, a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} - 10ka_{k+1} - 4a_k + 2a_{k+1}}{k^2+10k+16}, 7a_1 + 10a_0 = 0\right]$
- Revert the change of variables $u = x+1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x+1)^{k+6}, a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} - 10ka_{k+1} - 4a_k + 2a_{k+1}}{k^2+10k+16}, 7a_1 + 10a_0 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 45

```
dsolve(x^2*(x+1)*diff(diff(y(x),x),x)-x*(6+11*x)*diff(y(x),x)+(6+32*x)*y(x) = 0,y(x),sin
```

$$y = 3c_1x^8 + 14c_1x^7 + 21c_1x^6 + 35c_2x^4 + 42c_2x^3 + 21c_2x^2 + 4c_2x$$

Mathematica DSolve solution

Solving time : 0.528 (sec)
Leaf size : 122

```
DSolve[{x^2*(1+x)*D[y[x],{x,2}]-x*(6+11*x)*D[y[x],x]+(6+32*x)*y[x]==0,{}},y[x],x,IncludeSingula
```

$$y(x) \to \frac{1}{3}(3x-4)\exp\left(\int_1^x \left(\frac{7}{2(K[1]+1)} - \frac{2}{K[1]}\right) dK[1] - \frac{1}{2}\int_1^x \left(-\frac{5}{K[2]+1} - \frac{6}{K[2]}\right) dK[2]\right) \left(c_2 \int_1^x \frac{9\exp\left(-2\int_1^{K[3]} \frac{3K[1]-4}{2K[1](K[1]+1)} dK[1]\right)}{(4-3K[3])^2} dK[3] + c_1\right)$$

2.1.181 Problem 183

Internal problem ID [9353]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 183
Date solved : Monday, January 27, 2025 at 06:01:48 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$4x^2(1+x)\,y''+4x(1+4x)\,y'-(49+27x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.212 (sec)

Writing the ode as

$$\left(4x^3+4x^2\right)y''+\left(16x^2+4x\right)y'+(-27x-49)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=4x^3+4x^2\\B&=16x^2+4x\\C&=-27x-49\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{35x^2+80x+48}{4\left(x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=35x^2+80x+48\\t&=4\left(x^2+x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{35x^2+80x+48}{4\left(x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.346: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{12}{x^2} - \frac{4}{x} + \frac{4}{1+x} + \frac{3}{4\,(1+x)^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 12$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 4 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -3 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{35x^2 + 80x + 48}{4\left(x^2 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{35}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{35x^2 + 80x + 48}{4\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
0	2	0	4	-3

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= \frac{7}{2} - \left(\frac{7}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(1+x)} + \frac{4}{x} + (0) \\
&= -\frac{1}{2(1+x)} + \frac{4}{x} \\
&= \frac{7x+8}{2x(1+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(1+x)} + \frac{4}{x}\right)(0) + \left(\left(\frac{1}{2(1+x)^2} - \frac{4}{x^2}\right) + \left(-\frac{1}{2(1+x)} + \frac{4}{x}\right)^2 - \left(\frac{35x^2 + 80x + 48}{4(x^2+x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2(1+x)} + \frac{4}{x}\right) dx} \\ &= \frac{x^4}{\sqrt{1+x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{16x^2+4x}{4x^3+4x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2} - \frac{3\ln(1+x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}\,(1+x)^{3/2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{7/2}}{(1+x)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{16x^2+4x}{4x^3+4x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\ln(x) - 3\ln(1+x)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{(7x+6)(1+x)^3\, \mathrm{e}^{-\ln(x) - 3\ln(1+x)}}{42x^6}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{x^{7/2}}{(1+x)^2}\right) + c_2 \left(\frac{x^{7/2}}{(1+x)^2}\left(-\frac{(7x+6)(1+x)^3\, \mathrm{e}^{-\ln(x) - 3\ln(1+x)}}{42x^6}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)+4x(4x+1)\left(\frac{d}{dx}y(x)\right)-(49+27x)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{(49+27x)y(x)}{4x^2(x+1)}-\frac{(4x+1)\left(\frac{d}{dx}y(x)\right)}{x(x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{(4x+1)\left(\frac{d}{dx}y(x)\right)}{x(x+1)}-\frac{(49+27x)y(x)}{4x^2(x+1)}=0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=\frac{4x+1}{x(x+1)},P_3(x)=-\frac{49+27x}{4x^2(x+1)}\right]$

- ○ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=3$

- ○ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

- ○ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $4x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)+4x(4x+1)\left(\frac{d}{dx}y(x)\right)+(-27x-49)\,y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(4u^3-8u^2+4u)\left(\frac{d^2}{du^2}y(u)\right)+(16u^2-28u+12)\left(\frac{d}{du}y(u)\right)+(-27u-22)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

- ○ Shift index using $k->k-m$

  $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

- ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ○ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

- ○ Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$4a_0r(2+r)\,u^{-1+r} + (4a_1(1+r)\,(3+r) - 2a_0(4r^2+10r+11))\,u^r + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+1+r)\,(k+3\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $4r(2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-2, 0\}$
- Each term must be 0
  $4a_1(1+r)\,(3+r) - 2a_0(4r^2+10r+11) = 0$
- Each term in the series must be 0, giving the recursion relation
  $4(-2a_k + a_{k-1} + a_{k+1})\,k^2 + 4(2(-2a_k + a_{k-1} + a_{k+1})\,r - 5a_k + a_{k-1} + 4a_{k+1})\,k + 4(-2a_k + a_{k-1} +$
- Shift index using $k- >k+1$
  $4(-2a_{k+1} + a_k + a_{k+2})\,(k+1)^2 + 4(2(-2a_{k+1} + a_k + a_{k+2})\,r - 5a_{k+1} + a_k + 4a_{k+2})\,(k+1) + 4(-2$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{4k^2a_k-8k^2a_{k+1}+8kra_k-16kra_{k+1}+4r^2a_k-8r^2a_{k+1}+12ka_k-36ka_{k+1}+12ra_k-36ra_{k+1}-27a_k-50a_{k+1}}{4(k^2+2kr+r^2+6k+6r+8)}$
- Recursion relation for $r = -2$
  $a_{k+2} = -\frac{4k^2a_k-8k^2a_{k+1}-4ka_k-4ka_{k+1}-35a_k-10a_{k+1}}{4(k^2+2k)}$
- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 0$
  $a_{k+2} = -\frac{4k^2a_k-8k^2a_{k+1}-4ka_k-4ka_{k+1}-35a_k-10a_{k+1}}{4(k^2+2k)}$
- Recursion relation for $r = 0$
  $a_{k+2} = -\frac{4k^2a_k-8k^2a_{k+1}+12ka_k-36ka_{k+1}-27a_k-50a_{k+1}}{4(k^2+6k+8)}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{4k^2a_k-8k^2a_{k+1}+12ka_k-36ka_{k+1}-27a_k-50a_{k+1}}{4(k^2+6k+8)}, 12a_1 - 22a_0 = 0\right]$
- Revert the change of variables $u = x+1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+2} = -\frac{4k^2a_k-8k^2a_{k+1}+12ka_k-36ka_{k+1}-27a_k-50a_{k+1}}{4(k^2+6k+8)}, 12a_1 - 22a_0 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.039 (sec)
Leaf size : 26

```
dsolve(4*x^2*(x+1)*diff(diff(y(x),x),x)+4*x*(4*x+1)*diff(y(x),x)-(49+27*x)*y(x) = 0,y(
```

$$y = \frac{c_1 x^7 + 7c_2 x + 6c_2}{(x+1)^2 x^{7/2}}$$

Mathematica DSolve solution

Solving time : 0.266 (sec)
Leaf size : 102

```
DSolve[{4*x^2*(1+x)*D[y[x],{x,2}]+4*x*(1+4*x)*D[y[x],x]-(49+27*x)*y[x]==0,{}},y[x],x,Include
```

$$y(x) \to \exp\left(\int_1^x \left(\frac{4}{K[1]} - \frac{1}{2(K[1]+1)}\right) dK[1] - \frac{1}{2}\int_1^x \left(\frac{3}{K[2]+1} + \frac{1}{K[2]}\right) dK[2]\right) \left(c_2 \int_1^x \exp\left(-2\int_1^{K[3]} \frac{7K[1]+8}{2K[1]^2+2K[1]} dK[1]\right) dK[3] + c_1\right)$$

2.1.182 Problem 184

Internal problem ID [9354]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 184
Date solved : `Monday, January 27, 2025 at 06:01:49 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(x^2+1\right)y''-x\left(-2x^2+7\right)y'+12y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.301 (sec)

Writing the ode as

$$\left(x^4+x^2\right)y''+\left(2x^3-7x\right)y'+12y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2 \\ B &= 2x^3-7x \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-30x^2+15}{4\left(x^3+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -30x^2+15 \\ t &= 4\left(x^3+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-30x^2+15}{4\left(x^3+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.348: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{45}{16\,(x-i)^2} + \frac{45}{16\,(x+i)^2} + \frac{75i}{16\,(x-i)} - \frac{75i}{16\,(x+i)} + \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{45}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{4} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{45}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{4}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-30x^2 + 15}{4\left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
i	2	0	$\frac{9}{4}$	$-\frac{5}{4}$
$-i$	2	0	$\frac{9}{4}$	$-\frac{5}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= 0 - (0) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{5}{2x} - \frac{5}{4\left(x-i\right)} - \frac{5}{4\left(x+i\right)} + (0) \\
&= \frac{5}{2x} - \frac{5}{4\left(x-i\right)} - \frac{5}{4\left(x+i\right)} \\
&= \frac{5}{2x\left(x^2+1\right)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{5}{2x} - \frac{5}{4(x-i)} - \frac{5}{4(x+i)}\right)(0) + \left(\left(-\frac{5}{2x^2} + \frac{5}{4(x-i)^2} + \frac{5}{4(x+i)^2}\right) + \left(\frac{5}{2x} - \frac{5}{4(x-i)} - \frac{}{4(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(\frac{5}{2x} - \frac{5}{4(x-i)} - \frac{5}{4(x+i)}\right) dx} \\ &= \frac{x^{5/2}}{(x^2+1)^{5/4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x^3-7x}{x^4+x^2}\, dx} \\ &= z_1 e^{\frac{7\ln(x)}{2} - \frac{9\ln(x^2+1)}{4}} \\ &= z_1 \left(\frac{x^{7/2}}{(x^2+1)^{9/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^6}{(x^2+1)^{7/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3-7x}{x^4+x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{7\ln(x) - \frac{9\ln(x^2+1)}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{(x^2+1)^{7/2}}{4x^4} - \frac{3(x^2+1)^{7/2}}{8x^2} + \frac{3(x^2+1)^{5/2}}{8} + \frac{5(x^2+1)^{3/2}}{8} + \frac{15\sqrt{x^2+1}}{8} \right. \\ &\qquad \left. - \frac{15 \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)}{8}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^6}{(x^2+1)^{7/2}}\right) \\
&\quad + c_2\left(\frac{x^6}{(x^2+1)^{7/2}}\left(-\frac{(x^2+1)^{7/2}}{4x^4} - \frac{3(x^2+1)^{7/2}}{8x^2} + \frac{3(x^2+1)^{5/2}}{8} + \frac{5(x^2+1)^{3/2}}{8} + \frac{15\sqrt{x^2+1}}{8} - \frac{15\operatorname{arctanh}\left(\right.}{8}\right.\right.
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) - x(-2x^2+7)\left(\frac{d}{dx}y(x)\right) + 12y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{12y(x)}{x^2(x^2+1)} - \frac{(2x^2-7)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(2x^2-7)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)} + \frac{12y(x)}{x^2(x^2+1)} = 0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{2x^2-7}{x(x^2+1)}, P_3(x) = \frac{12}{x^2(x^2+1)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -7$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 12$

  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + x(2x^2-7)\left(\frac{d}{dx}y(x)\right) + 12y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-2+r)(-6+r)\, x^r + a_1(-1+r)(-5+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-2)(k+r-6) + a_{k-2}(k+$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-2+r)(-6+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{2, 6\}$
- Each term must be 0
  $a_1(-1+r)(-5+r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-2)(a_k(k+r-6) + a_{k-2}(k+r-1)) = 0$
- Shift index using $k->k+2$
  $(k+r)(a_{k+2}(k-4+r) + a_k(k+r+1)) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{a_k(k+r+1)}{k-4+r}$
- Recursion relation for $r = 2$
  $a_{k+2} = -\frac{a_k(k+3)}{k-2}$
- Series not valid for $r = 2$, division by 0 in the recursion relation at $k = 2$
  $a_{k+2} = -\frac{a_k(k+3)}{k-2}$
- Recursion relation for $r = 6$
  $a_{k+2} = -\frac{a_k(k+7)}{k+2}$
- Solution for $r = 6$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+6}, a_{k+2} = -\frac{a_k(k+7)}{k+2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.036 (sec)
Leaf size : 56

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)-x*(-2*x^2+7)*diff(y(x),x)+12*y(x) = 0,y(x),sings
```

$$y = \frac{x^2 \left(-15 \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) c_2 x^4 + c_2 (8x^4 - 9x^2 - 2)\sqrt{x^2+1} + c_1 x^4\right)}{(x^2+1)^{7/2}}$$

Mathematica DSolve solution

Solving time : 0.178 (sec)
Leaf size : 96

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]-x*(7-2*x^2)*D[y[x],x]+12*y[x]==0,{}},y[x],x,IncludeSingularS
```

$$y(x) \to \exp\left(\int_1^x \frac{5}{2(K[1]^3+K[1])} dK[1] - \frac{1}{2}\int_1^x \frac{2K[2]^2-7}{K[2]^3+K[2]} dK[2]\right)\left(c_2 \int_1^x \exp\left(-2\int_1^{K[3]} \frac{5}{2(K[1]^3+K[1])} dK[1]\right) dK[3] + c_1\right)$$

2.1.183 Problem 185

Internal problem ID [9355]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 185
Date solved : `Monday, January 27, 2025 at 06:01:50 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x\left(-x^2+7\right)y' + 12y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.266 (sec)

Writing the ode as

$$x^2y'' + \left(x^3-7x\right)y' + 12y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^3-7x \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4-12x^2+15}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4-12x^2+15 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4-12x^2+15}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.350: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{4} - 3 + \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{x} - \frac{21}{4x^3} - \frac{63}{2x^5} - \frac{3465}{16x^7} - \frac{13041}{8x^9} - \frac{417501}{32x^{11}} - \frac{1744659}{16x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^4 - 12x^2 + 15}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{x^2}{4} - 3\right) + \left(\frac{15}{4x^2}\right) \\ &= \frac{x^2}{4} - 3 + \frac{15}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is -3. Now b can be found.

$$\begin{aligned} b &= (-3) - (0) \\ &= -3 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-3}{\frac{1}{2}} - 1\right) &= -\frac{7}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-3}{\frac{1}{2}} - 1\right) &= \frac{5}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 12x^2 + 15}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	$-\frac{7}{2}$	$\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{5}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{5}{2} - \left(\frac{5}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{5}{2x} + (-)\left(\frac{x}{2}\right) \\ &= \frac{5}{2x} - \frac{x}{2} \\ &= \frac{5}{2x} - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{5}{2x} - \frac{x}{2}\right)(0) + \left(\left(-\frac{5}{2x^2} - \frac{1}{2}\right) + \left(\frac{5}{2x} - \frac{x}{2}\right)^2 - \left(\frac{x^4 - 12x^2 + 15}{4x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{5}{2x} - \frac{x}{2}\right) dx} \\ &= x^{5/2} e^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^3-7x}{x^2}\,dx} \\
&= z_1 e^{-\frac{x^2}{4}+\frac{7\ln(x)}{2}} \\
&= z_1 \left(x^{7/2}\mathrm{e}^{-\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^6\mathrm{e}^{-\frac{x^2}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^3-7x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}+7\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{4x^4}-\frac{\mathrm{e}^{\frac{x^2}{2}}}{8x^2}-\frac{\mathrm{Ei}_1\left(-\frac{x^2}{2}\right)}{16}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^6\mathrm{e}^{-\frac{x^2}{2}}\right) + c_2\left(x^6\mathrm{e}^{-\frac{x^2}{2}}\left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{4x^4}-\frac{\mathrm{e}^{\frac{x^2}{2}}}{8x^2}-\frac{\mathrm{Ei}_1\left(-\frac{x^2}{2}\right)}{16}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right) - x(-x^2+7)\left(\frac{d}{dx}y(x)\right) + 12y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{12y(x)}{x^2} - \frac{(x^2-7)\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x) + \frac{(x^2-7)\left(\frac{d}{dx}y(x)\right)}{x} + \frac{12y(x)}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

$\left[P_2(x) = \frac{x^2-7}{x}, P_3(x) = \frac{12}{x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -7$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 12$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(x^2-7)\left(\frac{d}{dx}y(x)\right) + 12y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- ◦ Shift index using $k- >k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- ◦ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)\,(-6+r)\,x^r + a_1(-1+r)\,(-5+r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-2)\,(k+r-6) + a_{k-2}(k+r\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)\,(-6+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{2, 6\}$

- Each term must be 0

  $a_1(-1+r)\,(-5+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)\,(a_k(k+r-6) + a_{k-2}) = 0$

- Shift index using $k- >k+2$

  $(k+r)\,(a_{k+2}(k-4+r) + a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{k-4+r}$

- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{a_k}{k-2}$

- Series not valid for $r = 2$, division by 0 in the recursion relation at $k = 2$

  $a_{k+2} = -\frac{a_k}{k-2}$

- Recursion relation for $r = 6$

  $a_{k+2} = -\frac{a_k}{k+2}$

- Solution for $r = 6$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+6}, a_{k+2} = -\frac{a_k}{k+2}, a_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 47

```
dsolve(x^2*diff(diff(y(x),x),x)-x*(-x^2+7)*diff(y(x),x)+12*y(x) = 0,y(x),singsol=all)
```

$$y = x^2 \left(\mathrm{Ei}_1\left(-\frac{x^2}{2}\right) \mathrm{e}^{-\frac{x^2}{2}} c_2 x^4 + \mathrm{e}^{-\frac{x^2}{2}} c_1 x^4 + 2c_2 x^2 + 4c_2\right)$$

Mathematica DSolve solution

Solving time : 0.213 (sec)
Leaf size : 68

```
DSolve[{x^2*D[y[x],{x,2}]-x*(7-x^2)*D[y[x],x]+12*y[x]==0,{}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{1}{16} e^{\frac{1}{2}(-x^2-5)} \left(c_2 x^6 \operatorname{ExpIntegralEi}\left(\frac{x^2}{2}\right) + 16e^5 c_1 x^6 - 2c_2 e^{\frac{x^2}{2}} (x^2+2)\, x^2\right)$$

2.1.184 Problem 186

Internal problem ID [9356]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 186
Date solved : `Monday, January 27, 2025 at 06:01:50 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x(2x^2+1)\, y' - (-10x^2+1)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.329 (sec)

Writing the ode as

$$x^2y'' + (2x^3+x)\, y' + (10x^2-1)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 2x^3 + x \\ C &= 10x^2 - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^4 - 32x^2 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^4 - 32x^2 + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4 - 32x^2 + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.352: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = x^2 - 8 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x - \frac{4}{x} - \frac{61}{8x^3} - \frac{61}{2x^5} - \frac{19337}{128x^7} - \frac{26779}{32x^9} - \frac{5083557}{1024x^{11}} - \frac{7896633}{256x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^4 - 32x^2 + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (x^2 - 8) + \left(\frac{3}{4x^2}\right) \\ &= x^2 - 8 + \frac{3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is -8. Now b can be found.

$$\begin{aligned} b &= (-8) - (0) \\ &= -8 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= x \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-8}{1} - 1\right) &= -\frac{9}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-8}{1} - 1\right) &= \frac{7}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^4 - 32x^2 + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	x	$-\frac{9}{2}$	$\frac{7}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{7}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{7}{2} - \left(\frac{3}{2}\right) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{3}{2x} + (-)\,(x) \\ &= \frac{3}{2x} - x \\ &= \frac{3}{2x} - x \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{3}{2x} - x\right)(2x + a_1) + \left(\left(-\frac{3}{2x^2} - 1\right) + \left(\frac{3}{2x} - x\right)^2 - \left(\frac{4x^4 - 32x^2 + 3}{4x^2}\right) \right) = 0$$

$$\frac{2x^2 a_1 + (4a_0 + 8)\,x + 3a_1}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -2, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2-2\right) \mathrm{e}^{\int \left(\frac{3}{2x}-x\right)dx} \\
&= \left(x^2-2\right) \mathrm{e}^{-\frac{x^2}{2}+\frac{3\ln(x)}{2}} \\
&= \left(x^2-2\right) x^{3/2} \mathrm{e}^{-\frac{x^2}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^3+x}{x^2} \, dx} \\
&= z_1 e^{-\frac{x^2}{2}-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x^2}{2}}}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x \,\mathrm{e}^{-x^2} \left(x^2-2\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3+x}{x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-x^2-\ln(x)}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-x^2-\ln(x)} \mathrm{e}^{2x^2}}{x^2 \left(x^2-2\right)^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x \,\mathrm{e}^{-x^2} \left(x^2-2\right)\right) + c_2 \left(x \,\mathrm{e}^{-x^2} \left(x^2-2\right) \left(\int \frac{\mathrm{e}^{-x^2-\ln(x)} \mathrm{e}^{2x^2}}{x^2 \left(x^2-2\right)^2} dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right)+x(2x^2+1)\left(\frac{d}{dx}y(x)\right)-(-10x^2+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(10x^2-1)y(x)}{x^2}-\frac{(2x^2+1)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{(2x^2+1)\left(\frac{d}{dx}y(x)\right)}{x}+\frac{(10x^2-1)y(x)}{x^2}=0$

☐ Check to see if $x_0=0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x)=\frac{2x^2+1}{x}, P_3(x)=\frac{10x^2-1}{x^2}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=1$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=-1$

- ○ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right)+x(2x^2+1)\left(\frac{d}{dx}y(x)\right)+(10x^2-1)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

- ○ Shift index using $k->k-m$

  $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ○ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ○ Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)(-1+r)\,x^r+a_1(2+r)\,r\,x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_k(k+r+1)(k+r-1)+2a_{k-2}(k+3+r)\right)x^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(1+r)(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 1\}$
- Each term must be 0

  $a_1(2+r)\, r = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)(k+r-1) + 2a_{k-2}(k+3+r) = 0$
- Shift index using $k- >k+2$

  $a_{k+2}(k+3+r)(k+r+1) + 2a_k(k+r+5) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2a_k(k+r+5)}{(k+3+r)(k+r+1)}$
- Recursion relation for $r=-1$

  $a_{k+2} = -\frac{2a_k(k+4)}{(k+2)k}$
- Solution for $r=-1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{2a_k(k+4)}{(k+2)k}, a_1 = 0\right]$
- Recursion relation for $r=1$

  $a_{k+2} = -\frac{2a_k(k+6)}{(k+4)(k+2)}$
- Solution for $r=1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{2a_k(k+6)}{(k+4)(k+2)}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+2} = -\frac{2a_k(4+k)}{(k+2)k}, a_1 = 0, b_{k+2} = -\frac{2b_k(k+6)}{(4+k)(k+2)}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solut
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
```

```
    <- elementary form could result into a too large expression - returning special
  <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.032 (sec)
Leaf size : 23

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(2*x^2+1)*diff(y(x),x)-(-10*x^2+1)*y(x) = 0,y(x),sin
```

$$y = -\frac{x\,\mathrm{e}^{-x^2}(x^2-2)\,(c_1-2c_2)}{2}$$

Mathematica DSolve solution

Solving time : 0.373 (sec)
Leaf size : 51

```
DSolve[{x^2*D[y[x],{x,2}]+x*(1+2*x^2)*D[y[x],x]-(1-10*x^2)*y[x]==0,{}},y[x],x,IncludeSingula
```

$$y(x) \to e^{-x^2}x(x^2-2)\left(c_2\int_1^x \frac{e^{K[1]^2}}{K[1]^3\,(K[1]^2-2)^2}dK[1]+c_1\right)$$

2.1.185 Problem 187

Internal problem ID [9357]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 187
Date solved : `Monday, January 27, 2025 at 06:01:51 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x\left(-2x^2+1\right)y' - 4\left(2x^2+1\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.249 (sec)

Writing the ode as

$$x^2y'' + \left(-2x^3+x\right)y' + \left(-8x^2-4\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x^3+x \\ C &= -8x^2-4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^4+24x^2+15}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^4+24x^2+15 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4+24x^2+15}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.354: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = x^2 + 6 + \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x + \frac{3}{x} - \frac{21}{8x^3} + \frac{63}{8x^5} - \frac{3465}{128x^7} + \frac{13041}{128x^9} - \frac{417501}{1024x^{11}} + \frac{1744659}{1024x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^4 + 24x^2 + 15}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (x^2 + 6) + \left(\frac{15}{4x^2}\right) \\ &= x^2 + 6 + \frac{15}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is 6. Now b can be found.

$$\begin{aligned} b &= (6) - (0) \\ &= 6 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= x \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{6}{1} - 1\right) &&= \frac{5}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{6}{1} - 1\right) &&= -\frac{7}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^4 + 24x^2 + 15}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	x	$\frac{5}{2}$	$-\frac{7}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^+) \\ &= \frac{5}{2} - \left(\frac{5}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{5}{2x} + (x) \\ &= \frac{5}{2x} + x \\ &= \frac{5}{2x} + x \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{5}{2x} + x\right)(0) + \left(\left(-\frac{5}{2x^2} + 1\right) + \left(\frac{5}{2x} + x\right)^2 - \left(\frac{4x^4 + 24x^2 + 15}{4x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{5}{2x} + x\right) dx} \\ &= x^{5/2} e^{\frac{x^2}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x^3+x}{x^2}\,dx} \\ &= z_1 e^{\frac{x^2}{2}-\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{e^{\frac{x^2}{2}}}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^2 e^{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^3+x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{x^2-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(-\frac{e^{-x^2}}{4x^4}+\frac{e^{-x^2}}{4x^2}-\frac{\mathrm{Ei}_1(x^2)}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(x^2 e^{x^2}\right) + c_2\left(x^2 e^{x^2}\left(-\frac{e^{-x^2}}{4x^4}+\frac{e^{-x^2}}{4x^2}-\frac{\mathrm{Ei}_1(x^2)}{4}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(-2x^2+1)\left(\frac{d}{dx}y(x)\right) - 4(2x^2+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = \frac{4(2x^2+1)y(x)}{x^2} + \frac{(2x^2-1)\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) - \frac{(2x^2-1)\left(\frac{d}{dx}y(x)\right)}{x} - \frac{4(2x^2+1)y(x)}{x^2} = 0$
- □ Check to see if $x_0 = 0$ is a regular singular point
  - ○ Define functions
    $\left[P_2(x) = -\frac{2x^2-1}{x}, P_3(x) = -\frac{4(2x^2+1)}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x=0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$

- $x^2 \cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -4$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) - x(2x^2-1)\left(\frac{d}{dx}y(x)\right) + (-8x^2-4)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)(-2+r)x^r + a_1(3+r)(-1+r)x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+2)(k+r-2) - 2a_{k-2}(k+r\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 2\}$

- Each term must be 0

  $a_1(3+r)(-1+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+r+2)(a_k(k+r-2) - 2a_{k-2}) = 0$

- Shift index using $k->k+2$

  $(k+r+4)(a_{k+2}(k+r) - 2a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2a_k}{k+r}$

- Recursion relation for $r=-2$

  $a_{k+2} = \frac{2a_k}{k-2}$

- Series not valid for $r=-2$, division by 0 in the recursion relation at $k=2$
  $a_{k+2}=\frac{2a_k}{k-2}$
- Recursion relation for $r=2$
  $a_{k+2}=\frac{2a_k}{k+2}$
- Solution for $r=2$
  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2}=\frac{2a_k}{k+2}, a_1=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 41

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(-2*x^2+1)*diff(y(x),x)-4*(2*x^2+1)*y(x) = 0,y(x),sing
```

$$y=\frac{-\mathrm{e}^{x^2}\,\mathrm{Ei}_1\left(x^2\right) c_2 x^4+c_1 x^4 \mathrm{e}^{x^2}+c_2 x^2-c_2}{x^2}$$

Mathematica DSolve solution

Solving time : 0.114 (sec)
Leaf size : 46

```
DSolve[{x^2*D[y[x],{x,2}]+x*(1-2*x^2)*D[y[x],x]-4*(1+2*x^2)*y[x]==0,{}},y[x],x,IncludeSingular
```

$$y(x)\to \frac{c_2\left(e^{x^2} x^4 \,\mathrm{ExpIntegralEi}\left(-x^2\right)+x^2-1\right)}{4x^2}+c_1 e^{x^2} x^2$$

2.1.186 Problem 188

Internal problem ID [9358]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 188
Date solved : `Monday, January 27, 2025 at 06:01:52 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x\left(-3x^2+1\right)y' - 4\left(-3x^2+1\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.580 (sec)

Writing the ode as

$$x^2y'' + \left(-3x^3+x\right)y' + \left(12x^2-4\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -3x^3+x \\ C &= 12x^2-4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9x^4 - 60x^2 + 15}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9x^4 - 60x^2 + 15 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{9x^4 - 60x^2 + 15}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.356: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9x^2}{4} - 15 + \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3x}{2} - \frac{5}{x} - \frac{85}{12x^3} - \frac{425}{18x^5} - \frac{41225}{432x^7} - \frac{278375}{648x^9} - \frac{1787125}{864x^{11}} - \frac{40534375}{3888x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{3x}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{9x^4 - 60x^2 + 15}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{9x^2}{4} - 15\right) + \left(\frac{15}{4x^2}\right) \\ &= \frac{9x^2}{4} - 15 + \frac{15}{4x^2}\end{aligned}$$

We see that the coefficient of the term x in the quotient is -15. Now b can be found.

$$\begin{aligned}b &= (-15) - (0) \\ &= -15\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{3x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{-15}{\frac{3}{2}} - 1\right) = -\frac{11}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{-15}{\frac{3}{2}} - 1\right) = \frac{9}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{9x^4 - 60x^2 + 15}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{3x}{2}$	$-\frac{11}{2}$	$\frac{9}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{9}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^+) \\ &= \frac{9}{2} - \left(\frac{5}{2}\right) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{5}{2x} + (-)\left(\frac{3x}{2}\right) \\ &= \frac{5}{2x} - \frac{3x}{2} \\ &= \frac{5}{2x} - \frac{3x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{5}{2x} - \frac{3x}{2}\right)(2x + a_1) + \left(\left(-\frac{5}{2x^2} - \frac{3}{2}\right) + \left(\frac{5}{2x} - \frac{3x}{2}\right)^2 - \left(\frac{9x^4 - 60x^2 + 15}{4x^2}\right)\right) = 0$$

$$\frac{3x^2 a_1 + 6(2 + a_0)\, x + 5a_1}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -2, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega \, dx} \\ &= (x^2 - 2)\, e^{\int \left(\frac{5}{2x} - \frac{3x}{2}\right) dx} \\ &= (x^2 - 2)\, e^{-\frac{3x^2}{4} + \frac{5\ln(x)}{2}} \\ &= (x^2 - 2)\, x^{5/2} e^{-\frac{3x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-3x^3+x}{x^2}\, dx} \\ &= z_1 e^{\frac{3x^2}{4} - \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{e^{\frac{3x^2}{4}}}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (x^2 - 2)\, x^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3x^3+x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\frac{3x^2}{2} - \ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{e^{\frac{3x^2}{2} - \ln(x)}}{(x^2 - 2)^2\, x^4} dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left((x^2 - 2)\, x^2\right) + c_2\left((x^2 - 2)\, x^2 \left(\int \frac{e^{\frac{3x^2}{2} - \ln(x)}}{(x^2 - 2)^2\, x^4} dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right)+x(-3x^2+1)\left(\frac{d}{dx}y(x)\right)-4(-3x^2+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{4(3x^2-1)y(x)}{x^2}+\frac{(3x^2-1)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)-\frac{(3x^2-1)\left(\frac{d}{dx}y(x)\right)}{x}+\frac{4(3x^2-1)y(x)}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(x)=-\frac{3x^2-1}{x}, P_3(x)=\frac{4(3x^2-1)}{x^2}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=1$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=-4$

  - $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right)-(3x^2-1)\,x\left(\frac{d}{dx}y(x)\right)+(12x^2-4)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

    $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)\,(-2+r)\,x^r+a_1(3+r)\,(-1+r)\,x^{1+r}+\left(\sum_{k=2}^{\infty}(a_k(k+r+2)\,(k+r-2)-3a_{k-2}(k-6+\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(2+r)(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 2\}$
- Each term must be 0

  $a_1(3+r)(-1+r)=0$
- Solve for the dependent coefficient(s)

  $a_1=0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)(k+r-2)-3a_{k-2}(k-6+r)=0$
- Shift index using $k->k+2$

  $a_{k+2}(k+4+r)(k+r)-3a_k(k+r-4)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{3a_k(k+r-4)}{(k+4+r)(k+r)}$
- Recursion relation for $r=-2$; series terminates at $k=6$

  $a_{k+2}=\frac{3a_k(k-6)}{(k+2)(k-2)}$
- Series not valid for $r=-2$, division by 0 in the recursion relation at $k=2$

  $a_{k+2}=\frac{3a_k(k-6)}{(k+2)(k-2)}$
- Recursion relation for $r=2$; series terminates at $k=2$

  $a_{k+2}=\frac{3a_k(k-2)}{(k+6)(k+2)}$
- Solution for $r=2$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2}=\frac{3a_k(k-2)}{(k+6)(k+2)}, a_1=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric sol
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returr
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.032` (sec)
Leaf size : 19

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(-3*x^2+1)*diff(y(x),x)-4*(-3*x^2+1)*y(x) = 0,y(x),sin
```

$$y = -\frac{x^2(x^2-2)(c_1-c_2)}{2}$$

Mathematica DSolve solution

Solving time : `0.41` (sec)
Leaf size : 50

```
DSolve[{x^2*D[y[x],{x,2}]+x*(1-3*x^2)*D[y[x],x]-4*(1-3*x^2)*y[x]==0,{}},y[x],x,IncludeSingularS
```

$$y(x) \to x^2(x^2-2)\left(c_2\int_1^x \frac{e^{\frac{3K[1]^2}{2}}}{K[1]^5\left(K[1]^2-2\right)^2}dK[1]+c_1\right)$$

2.1.187 Problem 189

Internal problem ID [9359]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 189
Date solved : `Monday, January 27, 2025 at 06:01:53 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(x^2+1\right)y''+x\left(11x^2+5\right)y'+24x^2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.380 (sec)

Writing the ode as

$$\left(x^4+x^2\right)y''+\left(11x^3+5x\right)y'+24x^2y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^4+x^2\\B&=11x^3+5x\\C&=24x^2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{3x^4+6x^2+15}{4\left(x^3+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=3x^4+6x^2+15\\t&=4\left(x^3+x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{3x^4+6x^2+15}{4\left(x^3+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.358: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{9i}{4(x-i)} - \frac{9i}{4(x+i)} + \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{3x^4 + 6x^2 + 15}{4\left(x^3 + x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3x^4 + 6x^2 + 15}{4\left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\
&= \frac{3}{2} - \left(\frac{3}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^+}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{3}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)} + (0) \\ &= -\frac{3}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)} \\ &= -\frac{3}{2x} + \frac{3x}{x^2+1}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)}\right)(0) + \left(\left(\frac{3}{2x^2} - \frac{3}{2(x-i)^2} - \frac{3}{2(x+i)^2}\right) + \left(-\frac{3}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= pe^{\int \omega\, dx} \\ &= e^{\int \left(-\frac{3}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)}\right) dx} \\ &= \frac{(x^2+1)^{3/2}}{x^{3/2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{11x^3+5x}{x^4+x^2}\, dx} \\ &= z_1 e^{-\frac{5\ln(x)}{2} - \frac{3\ln(x^2+1)}{2}} \\ &= z_1 \left(\frac{1}{x^{5/2}(x^2+1)^{3/2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{11x^3+5x}{x^4+x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-5\ln(x) - 3\ln(x^2+1)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{(x^2+1)(2x^2+1)\, x^5 e^{-5\ln(x) - 3\ln(x^2+1)}}{4}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x^4}\right) + c_2 \left(\frac{1}{x^4}\left(-\frac{(x^2+1)(2x^2+1)\,x^5 \mathrm{e}^{-5\ln(x)-3\ln(x^2+1)}}{4}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + x(11x^2+5)\left(\frac{d}{dx}y(x)\right) + 24x^2 y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{24y(x)}{x^2+1} - \frac{(11x^2+5)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(11x^2+5)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)} + \frac{24y(x)}{x^2+1} = 0$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{11x^2+5}{x(x^2+1)}, P_3(x) = \frac{24}{x^2+1}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 5$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  - $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + (11x^2+5)\left(\frac{d}{dx}y(x)\right) + 24xy(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x \cdot y(x)$ to series expansion

    $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

  - Shift index using $k->k-1$

    $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..2$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 1..3$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(4+r)\, x^{-1+r} + a_1(1+r)\,(5+r)\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+r+1)\,(k+5+r) + a_{k-1}(k+5+r)\,(k+3\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(4+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-4, 0\}$

- Each term must be 0

  $a_1(1+r)\,(5+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+5+r)\,(a_{k+1}(k+r+1) + a_{k-1}(k+3+r)) = 0$

- Shift index using $k->k+1$

  $(k+r+6)\,(a_{k+2}(k+2+r) + a_k(k+r+4)) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r+4)}{k+2+r}$

- Recursion relation for $r = -4$

  $a_{k+2} = -\frac{a_k k}{k-2}$

- Series not valid for $r = -4$, division by 0 in the recursion relation at $k = 2$

  $a_{k+2} = -\frac{a_k k}{k-2}$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{a_k(k+4)}{k+2}$

- Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(k+4)}{k+2}, 5a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 28

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)+x*(11*x^2+5)*diff(y(x),x)+24*x^2*y(x) = 0,y(x)
```

$$y = \frac{c_1 x^4 + 2c_2 x^2 + c_2}{\left(x^2+1\right)^2 x^4}$$

Mathematica DSolve solution

Solving time : 0.223 (sec)
Leaf size : 112

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]+x*(5+11*x^2)*D[y[x],x]+24*x^2*y[x]==0,{}},y[x],x,IncludeSi
```

$$y(x) \to \exp\left(\int_1^x \frac{3K[1]^2+5}{2\left(K[1]^3+K[1]\right)}dK[1] - \frac{1}{2}\int_1^x \frac{11K[2]^2+5}{K[2]^3+K[2]}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{3K[1]^2+5}{2\left(K[1]^3+K[1]\right)}dK[1]\right)dK[3] + c_1\right)$$

2.1.188 Problem 190

Internal problem ID [9360]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 190
Date solved : `Monday, January 27, 2025 at 06:01:53 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2\left(x^2+1\right)y''+8xy'-\left(-x^2+35\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.334 (sec)

Writing the ode as

$$\left(4x^4+4x^2\right)y''+8xy'+\left(x^2-35\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=4x^4+4x^2\\ B&=8x \\ C&=x^2-35 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-x^4+22x^2+35}{4\left(x^3+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=-x^4+22x^2+35\\ t&=4\left(x^3+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-x^4+22x^2+35}{4\left(x^3+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.360: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{35}{4x^2} + \frac{3}{4\,(x-i)^2} + \frac{3}{4\,(x+i)^2} + \frac{21i}{4\,(x-i)} - \frac{21i}{4\,(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{-x^4 + 22x^2 + 35}{4\left(x^3+x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-x^4 + 22x^2 + 35}{4\left(x^3+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+ + \alpha_{c_3}^+\right)\\
&= \frac{1}{2} - \left(\frac{1}{2}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{+}}{x-c_3}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{5}{2x} + \frac{3}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)} + (-)\left(0\right) \\
&= -\frac{5}{2x} + \frac{3}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)} \\
&= -\frac{5}{2x} + \frac{3x}{x^2+1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{5}{2x} + \frac{3}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)}\right)(0) + \left(\left(\frac{5}{2x^2} - \frac{3}{2\left(x-i\right)^2} - \frac{3}{2\left(x+i\right)^2}\right) + \left(-\frac{5}{2x} + \frac{3}{2\left(x-i\right)} + \right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{5}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)}\right) dx} \\
&= \frac{\left(x^2+1\right)^{3/2}}{x^{5/2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{8x}{4x^4+4x^2}\, dx} \\
&= z_1 e^{\frac{\ln\left(x^2+1\right)}{2} - \ln(x)} \\
&= z_1 \left(\frac{\sqrt{x^2+1}}{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\left(x^2+1\right)^2}{x^{7/2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{8x}{4x^4+4x^2}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{\ln\left(x^2+1\right) - 2\ln(x)}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\frac{1}{x^2+1} - \frac{1}{4\left(x^2+1\right)^2} + \frac{\ln\left(x^2+1\right)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{(x^2+1)^2}{x^{7/2}} \right) + c_2 \left(\frac{(x^2+1)^2}{x^{7/2}} \left(\frac{1}{x^2+1} - \frac{1}{4(x^2+1)^2} + \frac{\ln(x^2+1)}{2} \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + 8x\left(\frac{d}{dx}y(x)\right) - (-x^2+35)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(x^2-35)y(x)}{4x^2(x^2+1)} - \frac{2\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{2\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)} + \frac{(x^2-35)y(x)}{4x^2(x^2+1)} = 0$

- □ Check to see if x_0 is a regular singular point
  - ◦ Define functions

    $\left[P_2(x) = \frac{2}{(x^2+1)x}, P_3(x) = \frac{x^2-35}{4x^2(x^2+1)}\right]$

  - ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$

  - ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{35}{4}$

  - ◦ $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + 8x\left(\frac{d}{dx}y(x)\right) + (x^2-35)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - ◦ Shift index using $k- >k-m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - ◦ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r}$

  - ◦ Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(7+2r)(-5+2r)\, x^r + a_1(9+2r)(-3+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+7)(2k+2r-5) + a_k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(7+2r)(-5+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{-\frac{7}{2}, \frac{5}{2}\right\}$
- Each term must be 0
  $a_1(9+2r)(-3+2r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $4\left(k-\frac{5}{2}+r\right)\left(\left(k-\frac{5}{2}+r\right)a_{k-2} + a_k\left(k+r+\frac{7}{2}\right)\right) = 0$
- Shift index using $k->k+2$
  $4\left(k-\frac{1}{2}+r\right)\left(\left(k-\frac{1}{2}+r\right)a_k + a_{k+2}\left(k+\frac{11}{2}+r\right)\right) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{(2k+2r-1)a_k}{2k+11+2r}$
- Recursion relation for $r = -\frac{7}{2}$; series terminates at $k = 4$
  $a_{k+2} = -\frac{(2k-8)a_k}{2k+4}$
- Solution for $r = -\frac{7}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{7}{2}}, a_{k+2} = -\frac{(2k-8)a_k}{2k+4}, a_1 = 0\right]$
- Recursion relation for $r = \frac{5}{2}$
  $a_{k+2} = -\frac{(2k+4)a_k}{2k+16}$
- Solution for $r = \frac{5}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{2}}, a_{k+2} = -\frac{(2k+4)a_k}{2k+16}, a_1 = 0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{7}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{5}{2}}\right), a_{k+2} = -\frac{(2k-8)a_k}{2k+4}, a_1 = 0, b_{k+2} = -\frac{(2k+4)b_k}{2k+16}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
```

```
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.039 (sec)
Leaf size : 42

```
dsolve(4*x^2*(x^2+1)*diff(diff(y(x),x),x)+8*diff(y(x),x)*x-(-x^2+35)*y(x) = 0,y(x),sings
```

$$y = \frac{(x^2+1)^2 c_2 \ln(x^2+1) + \left(2x^2+\frac{3}{2}\right) c_2 + c_1 (x^2+1)^2}{x^{7/2}}$$

Mathematica DSolve solution

Solving time : 0.174 (sec)
Leaf size : 101

```
DSolve[{4*x^2*(1+x^2)*D[y[x],{x,2}]+8*x*D[y[x],x]-(35-x^2)*y[x]==0,{}},y[x],x,IncludeSingularSo
```

$$y(x) \to \exp\left(\int_1^x \frac{K[1]^2-5}{2(K[1]^3+K[1])} dK[1] - \frac{1}{2}\int_1^x \frac{2}{K[2]^3+K[2]} dK[2]\right)\left(c_2 \int_1^x \exp\left(-2\int_1^{K[3]} \frac{K[1]^2-5}{2(K[1]^3+K[1])} dK[1]\right) dK[3] + c_1\right)$$

2.1.189 Problem 191

Internal problem ID [9361]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 191
Date solved : Monday, January 27, 2025 at 06:01:54 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$x^2\left(x^2+1\right)y''-x\left(-x^2+5\right)y'-\left(25x^2+7\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.354 (sec)

Writing the ode as

$$\left(x^4+x^2\right)y''+\left(x^3-5x\right)y'+\left(-25x^2-7\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2 \\ B &= x^3-5x \\ C &= -25x^2-7 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{99x^4+150x^2+63}{4\left(x^3+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 99x^4+150x^2+63 \\ t &= 4\left(x^3+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{99x^4+150x^2+63}{4\left(x^3+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.362: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4\,(x-i)^2} + \frac{3}{4\,(x+i)^2} - \frac{15i}{4\,(x-i)} + \frac{15i}{4\,(x+i)} + \frac{63}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{63}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{7}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{99x^4 + 150x^2 + 63}{4\left(x^3 + x\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{99}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{9}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{99x^4 + 150x^2 + 63}{4\left(x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{9}{2}$	$-\frac{7}{2}$
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{11}{2}$	$-\frac{9}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{9}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= -\frac{9}{2} - \left(-\frac{9}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{7}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)} + (-)(0) \\ &= -\frac{7}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)} \\ &= -\frac{7}{2x} - \frac{x}{x^2+1}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{7}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)}\right)(0) + \left(\left(\frac{7}{2x^2} + \frac{1}{2(x-i)^2} + \frac{1}{2(x+i)^2}\right) + \left(-\frac{7}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(-\frac{7}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)}\right) dx} \\ &= \frac{1}{\sqrt{x^2+1}\, x^{7/2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^3-5x}{x^4+x^2}\, dx} \\ &= z_1 e^{-\frac{3\ln(x^2+1)}{2} + \frac{5\ln(x)}{2}} \\ &= z_1 \left(\frac{x^{5/2}}{(x^2+1)^{3/2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{(x^2+1)^2\, x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{x^3-5x}{x^4+x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-3\ln(x^2+1)+5\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{x^3(4x^2+5)\,(x^2+1)^3\, e^{-3\ln(x^2+1)+5\ln(x)}}{40}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{\left(x^2+1\right)^2 x}\right) + c_2\left(\frac{1}{\left(x^2+1\right)^2 x}\left(\frac{x^3(4x^2+5)\left(x^2+1\right)^3 \mathrm{e}^{-3\ln(x^2+1)+5\ln(x)}}{40}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) - x(-x^2+5)\left(\frac{d}{dx}y(x)\right) - (25x^2+7)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = \frac{(25x^2+7)y(x)}{x^2(x^2+1)} - \frac{(x^2-5)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x) + \frac{(x^2-5)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)} - \frac{(25x^2+7)y(x)}{x^2(x^2+1)} = 0$
- □ Check to see if x_0 is a regular singular point
  - ○ Define functions
    $\left[P_2(x) = \frac{x^2-5}{x(x^2+1)}, P_3(x) = -\frac{25x^2+7}{x^2(x^2+1)}\right]$
  - ○ $x \cdot P_2(x)$ is analytic at $x=0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = -5$
  - ○ $x^2 \cdot P_3(x)$ is analytic at $x=0$
    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -7$
  - ○ $x = 0$is a regular singular point
    Check to see if x_0 is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + x(x^2-5)\left(\frac{d}{dx}y(x)\right) + (-25x^2-7)\,y(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- □ Rewrite ODE with series expansions
  - ○ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$
    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - ○ Shift index using $k->k-m$
    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - ○ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$
    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$
  - ○ Shift index using $k->k+1-m$

$$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- > k+2-m$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+r)\,(-7+r)\, x^r + a_1(2+r)\,(-6+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+1)\,(k+r-7) + a_{k-2}(k+3+r$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
$(1+r)\,(-7+r) = 0$
- Values of r that satisfy the indicial equation
$r \in \{-1, 7\}$
- Each term must be 0
$a_1(2+r)\,(-6+r) = 0$
- Solve for the dependent coefficient(s)
$a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
$(k+r-7)\,(a_k(k+r+1) + a_{k-2}(k+3+r)) = 0$
- Shift index using $k- > k+2$
$(k+r-5)\,(a_{k+2}(k+3+r) + a_k(k+r+5)) = 0$
- Recursion relation that defines series solution to ODE
$a_{k+2} = -\frac{a_k(k+r+5)}{k+3+r}$
- Recursion relation for $r = -1$
$a_{k+2} = -\frac{a_k(k+4)}{k+2}$
- Solution for $r = -1$
$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k(k+4)}{k+2}, a_1 = 0\right]$
- Recursion relation for $r = 7$
$a_{k+2} = -\frac{a_k(k+12)}{k+10}$
- Solution for $r = 7$
$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+7}, a_{k+2} = -\frac{a_k(k+12)}{k+10}, a_1 = 0\right]$
- Combine solutions and rename parameters
$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+7}\right), a_{k+2} = -\frac{a_k(4+k)}{k+2}, a_1 = 0, b_{k+2} = -\frac{b_k(k+12)}{k+10}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
```

```
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 29

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)-x*(-x^2+5)*diff(y(x),x)-(25*x^2+7)*y(x) = 0,y(
```

$$y = \frac{4c_2x^{10} + 5c_2x^8 + c_1}{x\left(x^2+1\right)^2}$$

Mathematica DSolve solution

Solving time : 0.209 (sec)
Leaf size : 110

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]-x*(5-x^2)*D[y[x],x]-(7+25*x^2)*y[x]==0,{}},y[x],x,IncludeS
```

$$y(x) \to \exp\left(\int_1^x -\frac{9K[1]^2+7}{2\left(K[1]^3+K[1]\right)}dK[1] - \frac{1}{2}\int_1^x \frac{K[2]^2-5}{K[2]^3+K[2]}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} -\frac{9K[1]^2+7}{2\left(K[1]^3+K[1]\right)}dK[1]\right)dK[3] + c_1\right)$$

2.1.190 Problem 192

Internal problem ID [9362]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 192
Date solved : `Monday, January 27, 2025 at 06:01:55 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(x^2+1)\, y''+x(2x^2+5)\, y'-21y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.324 (sec)

Writing the ode as

$$(x^4+x^2)\, y''+(2x^3+5x)\, y'-21y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2 \\ B &= 2x^3+5x \\ C &= -21 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{78x^2+99}{4\,(x^3+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 78x^2+99 \\ t &= 4(x^3+x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{78x^2+99}{4\,(x^3+x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.364: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{99}{4x^2} + \frac{21}{16\,(x-i)^2} + \frac{21}{16\,(x+i)^2} + \frac{219i}{16\,(x-i)} - \frac{219i}{16\,(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{99}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{9}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{78x^2 + 99}{4(x^3 + x)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{11}{2}$	$-\frac{9}{2}$
i	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
$-i$	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\
&= 1 - (-1) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^+}{x - c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{9}{2x} + \frac{7}{4(x-i)} + \frac{7}{4(x+i)} + (-)(0) \\
&= -\frac{9}{2x} + \frac{7}{4(x-i)} + \frac{7}{4(x+i)} \\
&= -\frac{9}{2x} + \frac{7x}{2x^2+2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(-\frac{9}{2x} + \frac{7}{4(x-i)} + \frac{7}{4(x+i)}\right)(2x + a_1) + \left(\left(\frac{9}{2x^2} - \frac{7}{4(x-i)^2} - \frac{7}{4(x+i)^2}\right) + \left(-\frac{9}{2x} + \frac{7}{4(x-\right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 8, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + 8$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x^2 + 8)\, e^{\int \left(-\frac{9}{2x} + \frac{7}{4(x-i)} + \frac{7}{4(x+i)}\right) dx} \\ &= (x^2 + 8)\, e^{\frac{7\ln(x^2+1)}{4} - \frac{9\ln(x)}{2}} \\ &= \frac{(x^2 + 8)(x^2 + 1)^{7/4}}{x^{9/2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x^3+5x}{x^4+x^2} \, dx} \\ &= z_1 e^{\frac{3\ln(x^2+1)}{4} - \frac{5\ln(x)}{2}} \\ &= z_1 \left(\frac{(x^2 + 1)^{3/4}}{x^{5/2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2 + 1)^{5/2}(x^2 + 8)}{x^7}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3+5x}{x^4+x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{3\ln(x^2+1)}{2} - 5\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{(35x^6 + 140x^4 + 168x^2 + 64)\, x^5 e^{\frac{3\ln(x^2+1)}{2} - 5\ln(x)}}{35 (x^2 + 1)^4 (x^2 + 8)}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(x^2+1)^{5/2}(x^2+8)}{x^7}\right) \\
&\quad + c_2\left(\frac{(x^2+1)^{5/2}(x^2+8)}{x^7}\left(-\frac{(35x^6+140x^4+168x^2+64)\,x^5 \mathrm{e}^{\frac{3\ln(x^2+1)}{2}-5\ln(x)}}{35(x^2+1)^4(x^2+8)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(2x^2+5)\left(\frac{d}{dx}y(x)\right)-21y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{21y(x)}{x^2(x^2+1)}-\frac{(2x^2+5)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{(2x^2+5)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}-\frac{21y(x)}{x^2(x^2+1)}=0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{2x^2+5}{x(x^2+1)}, P_3(x)=-\frac{21}{x^2(x^2+1)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=5$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=-21$

  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(2x^2+5)\left(\frac{d}{dx}y(x)\right)-21y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..4$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(7+r)(-3+r)\, x^r + a_1(8+r)(-2+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+7)(k+r-3) + a_{k-2}(k-2$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(7+r)(-3+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-7, 3\}$
- Each term must be 0
  $a_1(8+r)(-2+r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_k(k+r+7)(k+r-3) + a_{k-2}(k-2+r)(k+r-1) = 0$
- Shift index using $k->k+2$
  $a_{k+2}(k+9+r)(k+r-1) + a_k(k+r)(k+r+1) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{a_k(k+r)(k+r+1)}{(k+9+r)(k+r-1)}$
- Recursion relation for $r = -7$; series terminates at $k = 6$
  $a_{k+2} = -\frac{a_k(k-7)(k-6)}{(k+2)(k-8)}$
- Solution for $r = -7$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-7}, a_{k+2} = -\frac{a_k(k-7)(k-6)}{(k+2)(k-8)}, a_1 = 0\right]$
- Recursion relation for $r = 3$
  $a_{k+2} = -\frac{a_k(k+3)(k+4)}{(k+12)(k+2)}$
- Solution for $r = 3$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = -\frac{a_k(k+3)(k+4)}{(k+12)(k+2)}, a_1 = 0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-7}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), a_{k+2} = -\frac{a_k(k-7)(k-6)}{(k+2)(k-8)}, a_1 = 0, b_{k+2} = -\frac{b_k(k+3)(4+k)}{(k+12)(k+2)}, b_1 = 0\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.023 (sec)
Leaf size : 41

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)+x*(2*x^2+5)*diff(y(x),x)-21*y(x) = 0,y(x),singso
```

$$y = \frac{c_1(x^2+1)^{5/2}(x^2+8)+35c_2\left(x^6+4x^4+\frac{24}{5}x^2+\frac{64}{35}\right)}{x^7}$$

Mathematica DSolve solution

Solving time : 0.61 (sec)
Leaf size : 126

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]+x*(5+2*x^2)*D[y[x],x]-21*y[x]==0,{}},y[x],x,IncludeSingularSo
```

$$y(x) \to (x^2+8)\exp\left(\int_1^x -\frac{2K[1]^2+9}{2(K[1]^3+K[1])}dK[1] - \frac{1}{2}\int_1^x \frac{2K[2]^2+5}{K[2]^3+K[2]}dK[2]\right)\left(c_2\int_1^x \frac{\exp\left(-2\int_1^{K[3]} -\frac{2K[1]^2+9}{2(K[1]^3+K[1])}dK[1]\right)}{(K[3]^2+8)^2}dK[3] + c_1\right)$$

2.1.191 Problem 193

Internal problem ID [9363]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 193
Date solved : `Monday, January 27, 2025 at 06:01:56 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2\left(x^2+1\right)y''+4x\left(x^2+2\right)y'-\left(x^2+15\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.249 (sec)

Writing the ode as

$$\left(4x^4+4x^2\right)y''+\left(4x^3+8x\right)y'+\left(-x^2-15\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^4+4x^2 \\ B &= 4x^3+8x \\ C &= -x^2-15 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{10x^2+15}{4\left(x^3+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 10x^2+15 \\ t &= 4\left(x^3+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{10x^2+15}{4\left(x^3+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.366: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4x^2} + \frac{5}{16\,(x-i)^2} + \frac{5}{16\,(x+i)^2} + \frac{35i}{16\,(x-i)} - \frac{35i}{16\,(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{10x^2 + 15}{4(x^3 + x)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
i	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-i$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^+}{x - c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{2x} + \frac{5}{4(x-i)} + \frac{5}{4(x+i)} + (-)(0) \\
&= -\frac{3}{2x} + \frac{5}{4(x-i)} + \frac{5}{4(x+i)} \\
&= -\frac{3}{2x} + \frac{5x}{2x^2 + 2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{3}{2x}+\frac{5}{4(x-i)}+\frac{5}{4(x+i)}\right)(0)+\left(\left(\frac{3}{2x^2}-\frac{5}{4(x-i)^2}-\frac{5}{4(x+i)^2}\right)+\left(-\frac{3}{2x}+\frac{5}{4(x-i)}+\frac{5}{4(x\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x)&=pe^{\int \omega\,dx}\\
&=e^{\int\left(-\frac{3}{2x}+\frac{5}{4(x-i)}+\frac{5}{4(x+i)}\right)dx}\\
&=\frac{(x^2+1)^{5/4}}{x^{3/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1&=z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\
&=z_1e^{-\int \frac{1}{2}\frac{4x^3+8x}{4x^4+4x^2}\,dx}\\
&=z_1e^{-\ln(x)+\frac{\ln\left(x^2+1\right)}{4}}\\
&=z_1\left(\frac{(x^2+1)^{1/4}}{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\frac{(x^2+1)^{3/2}}{x^{5/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2&=y_1\int \frac{e^{\int -\frac{4x^3+8x}{4x^4+4x^2}\,dx}}{(y_1)^2}\,dx\\
&=y_1\int \frac{e^{-2\ln(x)+\frac{\ln\left(x^2+1\right)}{2}}}{(y_1)^2}\,dx\\
&=y_1\left(-\frac{(3x^2+2)\,x^2e^{-2\ln(x)+\frac{\ln\left(x^2+1\right)}{2}}}{3\,(x^2+1)^2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y&=c_1y_1+c_2y_2\\
&=c_1\left(\frac{(x^2+1)^{3/2}}{x^{5/2}}\right)+c_2\left(\frac{(x^2+1)^{3/2}}{x^{5/2}}\left(-\frac{(3x^2+2)\,x^2e^{-2\ln(x)+\frac{\ln\left(x^2+1\right)}{2}}}{3\,(x^2+1)^2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)+4x(x^2+2)\left(\frac{d}{dx}y(x)\right)-(x^2+15)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=\frac{(x^2+15)y(x)}{4x^2(x^2+1)}-\frac{(x^2+2)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)+\frac{(x^2+2)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}-\frac{(x^2+15)y(x)}{4x^2(x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=\frac{x^2+2}{x(x^2+1)}, P_3(x)=-\frac{x^2+15}{4x^2(x^2+1)}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

$\left(x\cdot P_2(x)\right)\bigg|_{x=0}=2$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=-\frac{15}{4}$

- ◦ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$4x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)+4x(x^2+2)\left(\frac{d}{dx}y(x)\right)+(-x^2-15)\,y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

$x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ◦ Shift index using $k->k-m$

$x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ◦ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..4$

$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- ◦ Shift index using $k->k+2-m$

$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

Rewrite ODE with series expansions

$$a_0(5+2r)(-3+2r)x^r + a_1(7+2r)(-1+2r)x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+5)(2k+2r-3) + a_{k-2}(\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(5+2r)(-3+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{-\frac{5}{2}, \frac{3}{2}\right\}$
- Each term must be 0
  $a_1(7+2r)(-1+2r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $4\left(\left(k-\frac{5}{2}+r\right)a_{k-2} + \left(k+r+\frac{5}{2}\right)a_k\right)\left(k+r-\frac{3}{2}\right) = 0$
- Shift index using $k- >k+2$
  $4\left(\left(k-\frac{1}{2}+r\right)a_k + \left(k+\frac{9}{2}+r\right)a_{k+2}\right)\left(k+\frac{1}{2}+r\right) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{(2k+2r-1)a_k}{2k+9+2r}$
- Recursion relation for $r = -\frac{5}{2}$
  $a_{k+2} = -\frac{(2k-6)a_k}{2k+4}$
- Solution for $r = -\frac{5}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}, a_{k+2} = -\frac{(2k-6)a_k}{2k+4}, a_1 = 0\right]$
- Recursion relation for $r = \frac{3}{2}$
  $a_{k+2} = -\frac{(2k+2)a_k}{2k+12}$
- Solution for $r = \frac{3}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2} = -\frac{(2k+2)a_k}{2k+12}, a_1 = 0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+2} = -\frac{(2k-6)a_k}{2k+4}, a_1 = 0, b_{k+2} = -\frac{(2k+2)b_k}{2k+12}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.045 (sec)
Leaf size : 27

```
dsolve(4*x^2*(x^2+1)*diff(diff(y(x),x),x)+4*x*(x^2+2)*diff(y(x),x)-(x^2+15)*y(x) = 0,y
```

$$y = \frac{c_2(x^2+1)^{3/2} + 3c_1x^2 + 2c_1}{x^{5/2}}$$

Mathematica DSolve solution

Solving time : 0.212 (sec)
Leaf size : 110

```
DSolve[{4*x^2*(1+x^2)*D[y[x],{x,2}]+4*x*(2+x^2)*D[y[x],x]-(15+x^2)*y[x]==0,{}},y[x],x,Includ
```

$$y(x) \to \exp\left(\int_1^x \frac{2K[1]^2-3}{2\left(K[1]^3+K[1]\right)}dK[1] - \frac{1}{2}\int_1^x \frac{K[2]^2+2}{K[2]^3+K[2]}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{2K[1]^2-3}{2\left(K[1]^3+K[1]\right)}dK[1]\right)dK[3] + c_1\right)$$

2.1.192 Problem 194

Internal problem ID [9364]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 194
Date solved : `Monday, January 27, 2025 at 06:01:56 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - \frac{2(t+1)\, y'}{t^2+2t-1} + \frac{2y}{t^2+2t-1} = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.270 (sec)

Writing the ode as

$$y'' + \frac{(-2t-2)\, y'}{t^2+2t-1} + \frac{2y}{t^2+2t-1} = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{-2t-2}{t^2+2t-1} \\ C &= \frac{2}{t^2+2t-1} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{6}{\left(t^2+2t-1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 6 \\ t &= \left(t^2+2t-1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{6}{\left(t^2+2t-1\right)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.368: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = \left(t^2+2t-1\right)^2$. There is a pole at $t = \sqrt{2}-1$ of order 2. There is a pole at $t = -1-\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4\left(t-\sqrt{2}+1\right)^2} + \frac{3}{4\left(t+1+\sqrt{2}\right)^2} - \frac{3\sqrt{2}}{8\left(t-\sqrt{2}+1\right)} + \frac{3\sqrt{2}}{8\left(t+1+\sqrt{2}\right)}$$

For the pole at $t = \sqrt{2}-1$ let b be the coefficient of $\frac{1}{\left(t-\sqrt{2}+1\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $t=-1-\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(t+1+\sqrt{2}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{2} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $4>2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{6}{\left(t^{2}+2t-1\right)^{2}}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$\sqrt{2}-1$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-1-\sqrt{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{+}\right) \\
&= 1-(1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{t-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^{-}}{t-c_1}\right)+\left((+)[\sqrt{r}]_{c_2}+\frac{\alpha_{c_2}^{+}}{t-c_2}\right)+(-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(t-\sqrt{2}+1\right)}+\frac{3}{2\left(t+1+\sqrt{2}\right)}+(-)\,(0) \\
&= -\frac{1}{2\left(t-\sqrt{2}+1\right)}+\frac{3}{2\left(t+1+\sqrt{2}\right)} \\
&= \frac{t+1-2\sqrt{2}}{t^{2}+2t-1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\left(t-\sqrt{2}+1\right)} + \frac{3}{2\left(t+1+\sqrt{2}\right)}\right)(0) + \left(\left(\frac{1}{2\left(t-\sqrt{2}+1\right)^2} - \frac{3}{2\left(t+1+\sqrt{2}\right)^2}\right) + \left(-\frac{}{2\,(t -}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2\left(t-\sqrt{2}+1\right)} + \frac{3}{2\left(t+1+\sqrt{2}\right)}\right) dt} \\
&= \frac{\left(t+1+\sqrt{2}\right)^{3/2}}{\sqrt{t-\sqrt{2}+1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{\frac{-2t-2}{t^2+2t-1}}{1}\, dt} \\
&= z_1 e^{\frac{\ln\left(t^2+2t-1\right)}{2}} \\
&= z_1\left(\sqrt{t^2+2t-1}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{t^2+2t-1}\,\left(t+1+\sqrt{2}\right)^{3/2}}{\sqrt{t-\sqrt{2}+1}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{\frac{-2t-2}{t^2+2t-1}}{1}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{\ln\left(t^2+2t-1\right)}}{(y_1)^2}\, dt \\
&= y_1\left(\frac{\sqrt{2}}{\left(t+1+\sqrt{2}\right)^2} - \frac{1}{t+1+\sqrt{2}}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{t^2+2t-1}\,\left(t+1+\sqrt{2}\right)^{3/2}}{\sqrt{t-\sqrt{2}+1}}\right) \\
&\quad + c_2 \left(\frac{\sqrt{t^2+2t-1}\,\left(t+1+\sqrt{2}\right)^{3/2}}{\sqrt{t-\sqrt{2}+1}} \left(\frac{\sqrt{2}}{\left(t+1+\sqrt{2}\right)^2} - \frac{1}{t+1+\sqrt{2}}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dt^2}y(t) - \frac{2(t+1)\left(\frac{d}{dt}y(t)\right)}{t^2+2t-1} + \frac{2y(t)}{t^2+2t-1} = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dt^2}y(t)$

- □ Check to see if t_0 is a regular singular point
  - ◦ Define functions

    $\left[P_2(t) = -\frac{2(t+1)}{t^2+2t-1}, P_3(t) = \frac{2}{t^2+2t-1}\right]$
  - ◦ $\left(t+\sqrt{2}+1\right) \cdot P_2(t)$ is analytic at $t = -\sqrt{2}-1$

    $\left(\left(t+\sqrt{2}+1\right) \cdot P_2(t)\right)\Big|_{t=-\sqrt{2}-1} = 0$
  - ◦ $\left(t+\sqrt{2}+1\right)^2 \cdot P_3(t)$ is analytic at $t = -\sqrt{2}-1$

    $\left(\left(t+\sqrt{2}+1\right)^2 \cdot P_3(t)\right)\Big|_{t=-\sqrt{2}-1} = 0$
  - ◦ $t = -\sqrt{2}-1$is a regular singular point

    Check to see if t_0 is a regular singular point

    $t_0 = -\sqrt{2}-1$
- Multiply by denominators

  $(t^2+2t-1)\left(\frac{d^2}{dt^2}y(t)\right) + (-2t-2)\left(\frac{d}{dt}y(t)\right) + 2y(t) = 0$
- Change variables using $t = u-\sqrt{2}-1$ so that the regular singular point is at $u = 0$

  $\left(u^2-2u\sqrt{2}\right)\left(\frac{d^2}{du^2}y(u)\right) + \left(-2u+2\sqrt{2}\right)\left(\frac{d}{du}y(u)\right) + 2y(u) = 0$
- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$
- □ Rewrite ODE with series expansions
  - ◦ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
  - ◦ Shift index using $k- >k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - ◦ Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$
  - ◦ Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$-2\sqrt{2}\,r(r-2)\,a_0 u^{r-1} + \left(\sum_{k=0}^{\infty} \left(-2\sqrt{2}\,(k+1+r)(k+r-1)\,a_{k+1} + a_k(k+r-1)(k+r-2)\right) u$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2\sqrt{2}\,r(r-2) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$
- Each term in the series must be 0, giving the recursion relation

  $\left(-2a_{k+1}(k+1+r)\sqrt{2} + a_k(k+r-2)\right)(k+r-1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r-2)\sqrt{2}}{4(k+1+r)}$
- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+1} = \frac{a_k(k-2)\sqrt{2}}{4(k+1)}$
- Apply recursion relation for $k = 0$

  $a_1 = -\frac{a_0\sqrt{2}}{2}$
- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1\sqrt{2}}{8}$
- Express in terms of a_0

  $a_2 = \frac{a_0}{8}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - \frac{u\sqrt{2}}{2} + \frac{u^2}{8}\right)$
- Revert the change of variables $u = t + \sqrt{2} + 1$

  $\left[y(t) = a_0\left(\frac{(-2t-2)\sqrt{2}}{8} + \frac{t^2}{8} + \frac{t}{4} + \frac{3}{8}\right)\right]$
- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k k\sqrt{2}}{4(k+3)}$
- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k k\sqrt{2}}{4(k+3)}\right]$
- Revert the change of variables $u = t + \sqrt{2} + 1$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k\left(t+\sqrt{2}+1\right)^{k+2}, a_{k+1} = \frac{a_k k\sqrt{2}}{4(k+3)}\right]$
- Combine solutions and rename parameters

  $\left[y(t) = a_0\left(\frac{(-2t-2)\sqrt{2}}{8} + \frac{t^2}{8} + \frac{t}{4} + \frac{3}{8}\right) + \left(\sum_{k=0}^{\infty} b_k\left(t+\sqrt{2}+1\right)^{k+2}\right), b_{k+1} = \frac{b_k k\sqrt{2}}{4(k+3)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
```

```
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 15

```
dsolve(diff(diff(y(t),t),t)-2*(t+1)/(t^2+2*t-1)*diff(y(t),t)+2/(t^2+2*t-1)*y(t) = 0,y(t)
```

$$y = c_2 t^2 + c_1 t + c_1 + c_2$$

Mathematica DSolve solution

Solving time : 0.449 (sec)
Leaf size : 98

```
DSolve[{D[y[t],{t,2}]-2*(t+1)/(t^2+2*t-1)*D[y[t],t]+2/(t^2+2*t-1)*y[t]==0,{}},y[t],t,IncludeSi
```

$$y(t) \to \sqrt{t^2+2t-1}\exp\left(\int_1^t \frac{K[1]+2\sqrt{2}+1}{K[1](K[1]+2)-1}dK[1]\right)\left(c_2\int_1^t \exp\left(-2\int_1^{K[2]} \frac{K[1]+2\sqrt{2}+1}{K[1](K[1]+2)-1}dK[1]\right)dK[\right. \\ \left. + c_1\right)$$

2.1.193 Problem 195

Internal problem ID [9365]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 195
Date solved : `Monday, January 27, 2025 at 06:01:57 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - 4ty' + (4t^2 - 2)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.056 (sec)

Writing the ode as

$$y'' - 4ty' + (4t^2 - 2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4t \\ C &= 4t^2 - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.370: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4t}{1}\, dt} \\ &= z_1 e^{t^2} \\ &= z_1 \left(\mathrm{e}^{t^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{t^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4t}{1}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{2t^2}}{(y_1)^2}\, dt \\
&= y_1(t)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(e^{t^2}\right) + c_2\left(e^{t^2}(t)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\frac{d^2}{dt^2}y(t) - 4t\left(\frac{d}{dt}y(t)\right) + (4t^2 - 2)\, y(t) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dt^2}y(t)$
- Assume series solution for $y(t)$
  $y(t) = \sum_{k=0}^{\infty} a_k t^k$
- □ Rewrite ODE with series expansions
  - Convert $t^m \cdot y(t)$ to series expansion for $m = 0..2$
    $t^m \cdot y(t) = \sum_{k=\max(0,-m)}^{\infty} a_k t^{k+m}$
  - Shift index using $k- >k - m$
    $t^m \cdot y(t) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} t^k$
  - Convert $t \cdot \left(\frac{d}{dt}y(t)\right)$ to series expansion
    $t \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=0}^{\infty} a_k k\, t^k$
  - Convert $\frac{d^2}{dt^2}y(t)$ to series expansion
    $\frac{d^2}{dt^2}y(t) = \sum_{k=2}^{\infty} a_k k(k-1)\, t^{k-2}$
  - Shift index using $k- >k + 2$
    $\frac{d^2}{dt^2}y(t) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, t^k$

  Rewrite ODE with series expansions
  $2a_2 - 2a_0 + (6a_3 - 6a_1)\, t + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)(k+1) - 2a_k(2k+1) + 4a_{k-2}\right) t^k\right) = 0$
- The coefficients of each power of t must be 0
  $[2a_2 - 2a_0 = 0, 6a_3 - 6a_1 = 0]$
- Solve for the dependent coefficient(s)
  $\{a_2 = a_0, a_3 = a_1\}$
- Each term in the series must be 0, giving the recursion relation
  $(k^2 + 3k + 2)\, a_{k+2} - 4a_k k - 2a_k + 4a_{k-2} = 0$

- Shift index using $k->k+2$
  $$\left((k+2)^2+3k+8\right)a_{k+4}-4a_{k+2}(k+2)-2a_{k+2}+4a_k=0$$
- Recursion relation that defines the series solution to the ODE
  $$\left[y(t)=\sum_{k=0}^{\infty}a_k t^k, a_{k+4}=\frac{2(2ka_{k+2}-2a_k+5a_{k+2})}{k^2+7k+12}, a_2=a_0, a_3=a_1\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 14

```
dsolve(diff(diff(y(t),t),t)-4*t*diff(y(t),t)+(4*t^2-2)*y(t) = 0,y(t),singsol=all)
```

$$y=\mathrm{e}^{t^2}(c_2t+c_1)$$

Mathematica DSolve solution

Solving time : 0.021 (sec)
Leaf size : 18

```
DSolve[{D[y[t],{t,2}]-4*t*D[y[t],t]+(4*t^2-2)*y[t]==0,{}},y[t],t,IncludeSingularSolutions->True
```

$$y(t)\to e^{t^2}(c_2t+c_1)$$

2.1.194 Problem 196

Internal problem ID [9366]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 196
Date solved : `Monday, January 27, 2025 at 06:01:57 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-t^2+1\right)y''-2ty'+2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.236 (sec)

Writing the ode as

$$\left(-t^2+1\right)y''-2ty'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-t^2+1\\B&=-2t\\C&=2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{2t^2-3}{\left(t^2-1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=2t^2-3\\t&=\left(t^2-1\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(t)=\left(\frac{2t^2-3}{\left(t^2-1\right)^2}\right)z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.372: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (t^2-1)^2$. There is a pole at $t = 1$ of order 2. There is a pole at $t = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{4(t-1)} - \frac{1}{4(t+1)^2} - \frac{5}{4(t+1)} - \frac{1}{4(t-1)^2}$$

For the pole at $t = 1$ let b be the coefficient of $\frac{1}{(t-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $t = -1$ let b be the coefficient of $\frac{1}{(t+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2t^2 - 3}{(t^2 - 1)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2t^2 - 3}{(t^2 - 1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{t - c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2t - 2} + \frac{1}{2t + 2} + (0) \\
&= \frac{1}{2t - 2} + \frac{1}{2t + 2} \\
&= \frac{t}{t^2 - 1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 1$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2t-2} + \frac{1}{2t+2}\right)(1) + \left(\left(-\frac{1}{2(t-1)^2} - \frac{1}{2(t+1)^2}\right) + \left(\frac{1}{2t-2} + \frac{1}{2t+2}\right)^2 - \left(\frac{2t^2-3}{(t^2-1)^2}\right)\right) = 0$$

$$-\frac{2a_0}{t^2-1} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = t$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= (t)\, e^{\int \left(\frac{1}{2t-2} + \frac{1}{2t+2}\right) dt} \\ &= (t)\sqrt{(t-1)(t+1)} \\ &= t\sqrt{t^2-1} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2t}{-t^2+1} \, dt} \\ &= z_1 e^{-\frac{\ln(t-1)}{2} - \frac{\ln(t+1)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{t-1}\,\sqrt{t+1}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{t\sqrt{t^2-1}}{\sqrt{t-1}\,\sqrt{t+1}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2t}{-t^2+1} \, dt}}{(y_1)^2} \, dt \\ &= y_1 \int \frac{e^{-\ln(t-1) - \ln(t+1)}}{(y_1)^2} \, dt \\ &= y_1 \left(-\frac{\ln(t+1)}{2} + \frac{\ln(t-1)}{2} + \frac{1}{t}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{t\sqrt{t^2-1}}{\sqrt{t-1}\,\sqrt{t+1}}\right) + c_2 \left(\frac{t\sqrt{t^2-1}}{\sqrt{t-1}\,\sqrt{t+1}} \left(-\frac{\ln(t+1)}{2} + \frac{\ln(t-1)}{2} + \frac{1}{t}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(-t^2+1)\left(\frac{d^2}{dt^2}y(t)\right)-2t\left(\frac{d}{dt}y(t)\right)+2y(t)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dt^2}y(t)$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t)=\frac{2y(t)}{t^2-1}-\frac{2\left(\frac{d}{dt}y(t)\right)t}{t^2-1}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dt^2}y(t)+\frac{2\left(\frac{d}{dt}y(t)\right)t}{t^2-1}-\frac{2y(t)}{t^2-1}=0$

□ Check to see if t_0 is a regular singular point

  - Define functions

    $\left[P_2(t)=\frac{2t}{t^2-1}, P_3(t)=-\frac{2}{t^2-1}\right]$

  - $(t+1)\cdot P_2(t)$ is analytic at $t=-1$

    $\left((t+1)\cdot P_2(t)\right)\bigg|_{t=-1}=1$

  - $(t+1)^2\cdot P_3(t)$ is analytic at $t=-1$

    $\left((t+1)^2\cdot P_3(t)\right)\bigg|_{t=-1}=0$

  - $t=-1$is a regular singular point

  Check to see if t_0 is a regular singular point

  $t_0=-1$

- Multiply by denominators

  $(t^2-1)\left(\frac{d^2}{dt^2}y(t)\right)+2t\left(\frac{d}{dt}y(t)\right)-2y(t)=0$

- Change variables using $t=u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(2u-2)\left(\frac{d}{du}y(u)\right)-2y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r^2u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)^2+a_k(k+r+2)(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-2r^2 = 0$

- Values of r that satisfy the indicial equation
  $r = 0$
- Each term in the series must be 0, giving the recursion relation
  $-2a_{k+1}(k+1)^2 + a_k(k+2)(k-1) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$
- Recursion relation for $r = 0$; series terminates at $k = 1$
  $a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$
- Apply recursion relation for $k = 0$
  $a_1 = -a_0$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second linea
  $y(u) = a_0 \cdot (-u+1)$
- Revert the change of variables $u = t+1$
  $[y(t) = -a_0 t]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 25

```
dsolve((-t^2+1)*diff(diff(y(t),t),t)-2*t*diff(y(t),t)+2*y(t) = 0,y(t),singsol=all)
```

$$y = -\frac{c_2 \ln(t+1)\, t}{2} + \frac{c_2 \ln(t-1)\, t}{2} + c_1 t + c_2$$

Mathematica DSolve solution

Solving time : `0.022` (sec)
Leaf size : 33

```
DSolve[{(1-t^2)*D[y[t],{t,2}]-2*t*D[y[t],t]+2*y[t]==0,{}},y[t],t,IncludeSingularSolutions->Tru
```

$$y(t) \to c_1 t - \frac{1}{2} c_2 (t \log(1-t) - t \log(t+1) + 2)$$

2.1.195 Problem 197

Internal problem ID [9367]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 197
Date solved : `Monday, January 27, 2025 at 06:01:58 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(t^{2}+1\right) y''-2 t y'+2 y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.258 (sec)

Writing the ode as

$$\left(t^{2}+1\right) y''-2 t y'+2 y=0 \tag{1}$$

$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^{2}+1 \\ B &= -2 t \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=y e^{\int \frac{B}{2 A} d t}$$

Then (2) becomes

$$z''(t)=r z(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2 A B'-2 B A'+B^{2}-4 A C}{4 A^{2}} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{-3}{\left(t^{2}+1\right)^{2}} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= \left(t^{2}+1\right)^{2} \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t)=\left(-\frac{3}{\left(t^{2}+1\right)^{2}}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.374: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (t^2+1)^2$. There is a pole at $t = i$ of order 2. There is a pole at $t = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4\,(t-i)^2} + \frac{3}{4\,(t+i)^2} + \frac{3i}{4\,(t-i)} - \frac{3i}{4\,(t+i)}$$

For the pole at $t = i$ let b be the coefficient of $\frac{1}{(t-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $t = -i$ let b be the coefficient of $\frac{1}{(t+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= 0\\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{(t^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right)\\ &= 1 - (1)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{t-c_2}\right) + (-)[\sqrt{r}]_\infty\\ &= -\frac{1}{2(t-i)} + \frac{3}{2(t+i)} + (-)(0)\\ &= -\frac{1}{2(t-i)} + \frac{3}{2(t+i)}\\ &= \frac{t-2i}{t^2+1}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2(t-i)}+\frac{3}{2(t+i)}\right)(0)+\left(\left(\frac{1}{2(t-i)^2}-\frac{3}{2(t+i)^2}\right)+\left(-\frac{1}{2(t-i)}+\frac{3}{2(t+i)}\right)^2-\left(-\frac{3}{(t^2+1}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned} z_1(t) &= pe^{\int \omega\,dt} \\ &= e^{\int \left(-\frac{1}{2(t-i)}+\frac{3}{2(t+i)}\right)dt} \\ &= \frac{(t^2+1)^{3/2}}{(it+1)^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2t}{t^2+1}\,dt} \\ &= z_1 e^{\frac{\ln(t^2+1)}{2}} \\ &= z_1\left(\sqrt{t^2+1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(t^2+1)^2}{(it+1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2t}{t^2+1}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{\ln(t^2+1)}}{(y_1)^2}\,dt \\ &= y_1\left(-\frac{t}{(t+i)^2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{(t^2+1)^2}{(it+1)^2}\right) + c_2\left(\frac{(t^2+1)^2}{(it+1)^2}\left(-\frac{t}{(t+i)^2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 16

```
dsolve((t^2+1)*diff(diff(y(t),t),t)-2*t*diff(y(t),t)+2*y(t) = 0,y(t),singsol=all)
```

$$y = c_2t^2 + c_1t - c_2$$

Mathematica DSolve solution

Solving time : 0.329 (sec)
Leaf size : 79

```
DSolve[{(1+t^2)*D[y[t],{t,2}]-2*t*D[y[t],t]+2*y[t]==0,{}},y[t],t,IncludeSingularSolutions->T
```

$$y(t) \to \sqrt{t^2+1}\exp\left(\int_1^t \frac{K[1]+2i}{K[1]^2+1}dK[1]\right)\left(c_2\int_1^t \exp\left(-2\int_1^{K[2]} \frac{K[1]+2i}{K[1]^2+1}dK[1]\right)dK[2] + c_1\right)$$

2.1.196 Problem 198

Internal problem ID [9368]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 198
Date solved : `Monday, January 27, 2025 at 06:01:59 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-t^2+1\right) y''-2ty'+6y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.267 (sec)

Writing the ode as

$$\left(-t^2+1\right) y''-2ty'+6y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -t^2+1 \\ B &= -2t \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{6t^2-7}{\left(t^2-1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 6t^2-7 \\ t &= \left(t^2-1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t)=\left(\frac{6t^2-7}{\left(t^2-1\right)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.375: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (t^2 - 1)^2$. There is a pole at $t = 1$ of order 2. There is a pole at $t = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{13}{4\,(t-1)} - \frac{13}{4\,(t+1)} - \frac{1}{4\,(t-1)^2} - \frac{1}{4\,(t+1)^2}$$

For the pole at $t = 1$ let b be the coefficient of $\frac{1}{(t-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $t = -1$ let b be the coefficient of $\frac{1}{(t+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{6t^2 - 7}{(t^2 - 1)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{6t^2 - 7}{(t^2 - 1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	3	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 3$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - (\alpha_{c_1}^- + \alpha_{c_2}^-) \\
&= 3 - (1) \\
&= 2
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{t - c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2t - 2} + \frac{1}{2t + 2} + (0) \\
&= \frac{1}{2t - 2} + \frac{1}{2t + 2} \\
&= \frac{t}{t^2 - 1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 2$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = t^2 + a_1 t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{1}{2t-2} + \frac{1}{2t+2}\right)(2t + a_1) + \left(\left(-\frac{1}{2(t-1)^2} - \frac{1}{2(t+1)^2}\right) + \left(\frac{1}{2t-2} + \frac{1}{2t+2}\right)^2 - \left(\frac{6t^2-7}{(t^2-1)^2}\right.\right.$$

$$\frac{-4a_1 t - 6a_0 - 2}{t^2 - 1}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{1}{3}, a_1 = 0\right\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = t^2 - \frac{1}{3}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= \left(t^2 - \frac{1}{3}\right) e^{\int \left(\frac{1}{2t-2} + \frac{1}{2t+2}\right) dt} \\ &= \left(t^2 - \frac{1}{3}\right) \sqrt{(t-1)(t+1)} \\ &= \frac{(3t^2 - 1)\sqrt{t^2 - 1}}{3} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2t}{-t^2+1} \, dt} \\ &= z_1 e^{-\frac{\ln(t-1)}{2} - \frac{\ln(t+1)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{t-1}\sqrt{t+1}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(3t^2 - 1)\sqrt{t^2 - 1}}{3\sqrt{t-1}\sqrt{t+1}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2t}{-t^2+1} \, dt}}{(y_1)^2} \, dt \\ &= y_1 \int \frac{e^{-\ln(t-1) - \ln(t+1)}}{(y_1)^2} \, dt \\ &= y_1 \left(\frac{9\ln(t-1)}{8} - \frac{9\ln(t+1)}{8} + \frac{9t}{4\left(t^2 - \frac{1}{3}\right)}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(3t^2-1)\sqrt{t^2-1}}{3\sqrt{t-1}\,\sqrt{t+1}}\right) + c_2\left(\frac{(3t^2-1)\sqrt{t^2-1}}{3\sqrt{t-1}\,\sqrt{t+1}}\left(\frac{9\ln(t-1)}{8} - \frac{9\ln(t+1)}{8} + \frac{9t}{4\left(t^2-\frac{1}{3}\right)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$(-t^2+1)\left(\frac{d^2}{dt^2}y(t)\right) - 2t\left(\frac{d}{dt}y(t)\right) + 6y(t) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dt^2}y(t)$
- Isolate 2nd derivative
  $\frac{d^2}{dt^2}y(t) = \frac{6y(t)}{t^2-1} - \frac{2\left(\frac{d}{dt}y(t)\right)t}{t^2-1}$
- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d^2}{dt^2}y(t) + \frac{2\left(\frac{d}{dt}y(t)\right)t}{t^2-1} - \frac{6y(t)}{t^2-1} = 0$

☐ Check to see if t_0 is a regular singular point
  - Define functions
    $\left[P_2(t) = \frac{2t}{t^2-1}, P_3(t) = -\frac{6}{t^2-1}\right]$
  - $(t+1)\cdot P_2(t)$ is analytic at $t=-1$
    $\left((t+1)\cdot P_2(t)\right)\Big|_{t=-1} = 1$
  - $(t+1)^2\cdot P_3(t)$ is analytic at $t=-1$
    $\left((t+1)^2\cdot P_3(t)\right)\Big|_{t=-1} = 0$
  - $t=-1$is a regular singular point
    Check to see if t_0 is a regular singular point
    $t_0 = -1$
- Multiply by denominators
  $(t^2-1)\left(\frac{d^2}{dt^2}y(t)\right) + 2t\left(\frac{d}{dt}y(t)\right) - 6y(t) = 0$
- Change variables using $t = u-1$ so that the regular singular point is at $u=0$
  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (2u-2)\left(\frac{d}{du}y(u)\right) - 6y(u) = 0$
- Assume series solution for $y(u)$
  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$
    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$
    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r^2u^{-1+r} + \left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)^2 + a_k(k+r+3)(k+r-2)\right)u^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-2r^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 0$
- Each term in the series must be 0, giving the recursion relation
  $-2a_{k+1}(k+1)^2 + a_k(k+3)(k-2) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_k(k+3)(k-2)}{2(k+1)^2}$
- Recursion relation for $r = 0$; series terminates at $k = 2$
  $a_{k+1} = \frac{a_k(k+3)(k-2)}{2(k+1)^2}$
- Apply recursion relation for $k = 0$
  $a_1 = -3a_0$
- Apply recursion relation for $k = 1$
  $a_2 = -\frac{a_1}{2}$
- Express in terms of a_0
  $a_2 = \frac{3a_0}{2}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li
  $y(u) = a_0 \cdot \left(1 - 3u + \frac{3}{2}u^2\right)$
- Revert the change of variables $u = t + 1$
  $\left[y(t) = a_0\left(\frac{3t^2}{2} - \frac{1}{2}\right)\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 44

```
dsolve((-t^2+1)*diff(diff(y(t),t),t)-2*t*diff(y(t),t)+6*y(t) = 0,y(t),singsol=all)
```

$$y = \frac{c_2(3t^2-1)\ln(t-1)}{2} + \frac{(-3t^2+1)\, c_2 \ln(t+1)}{2} - 3c_1t^2 + 3c_2t + c_1$$

Mathematica DSolve solution

Solving time : `0.023` (sec)
Leaf size : 55

```
DSolve[{(1-t^2)*D[y[t],{t,2}]-2*t*D[y[t],t]+6*y[t]==0,{}},y[t],t,IncludeSingularSolutions->Tru
```

$$y(t) \to \frac{1}{2}c_1\left(3t^2-1\right) - \frac{1}{4}c_2\left(\left(3t^2-1\right)\log(1-t) + \left(1-3t^2\right)\log(t+1) + 6t\right)$$

2.1.197 Problem 199

Internal problem ID [9369]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 199
Date solved : `Monday, January 27, 2025 at 06:01:59 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(2t+1)\,y''-4(t+1)\,y'+4y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.230 (sec)

Writing the ode as

$$(2t+1)\,y''+(-4t-4)\,y'+4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2t+1\\ B &= -4t-4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=y e^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{4t^2+2}{\left(2t+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4t^2+2 \\ t &= \left(2t+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t)=\left(\frac{4t^2+2}{\left(2t+1\right)^2}\right)z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.377: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2t+1)^2$. There is a pole at $t = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1,2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{3}{4\left(t+\frac{1}{2}\right)^2} - \frac{1}{t+\frac{1}{2}}$$

For the pole at $t = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(t+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 - \frac{1}{2t} + \frac{1}{2t^2} - \frac{1}{4t^3} + \frac{3}{32t^4} - \frac{3}{64t^5} + \frac{1}{32t^6} - \frac{1}{64t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4t^2+2}{4t^2+4t+1} \\ &= Q + \frac{R}{4t^2+4t+1} \\ &= (1) + \left(\frac{-4t+1}{4t^2+4t+1}\right) \\ &= 1 + \frac{-4t+1}{4t^2+4t+1} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{1} - 0\right) &&= -\frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{1} - 0\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4t^2+2}{(2t+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{2\left(t+\frac{1}{2}\right)} + (1) \\ &= -\frac{1}{2\left(t+\frac{1}{2}\right)} + 1 \\ &= \frac{2t}{2t+1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\left(t+\frac{1}{2}\right)} + 1\right)(0) + \left(\left(\frac{1}{2\left(t+\frac{1}{2}\right)^2}\right) + \left(-\frac{1}{2\left(t+\frac{1}{2}\right)} + 1\right)^2 - \left(\frac{4t^2+2}{(2t+1)^2}\right) \right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2\left(t+\frac{1}{2}\right)}+1\right) dt} \\
&= \frac{\mathrm{e}^{t}}{\sqrt{2t+1}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4t-4}{2t+1} \, dt} \\
&= z_1 e^{t+\frac{\ln(2t+1)}{2}} \\
&= z_1 \left(\sqrt{2t+1}\, \mathrm{e}^{t}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{2t}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4t-4}{2t+1} \, dt}}{(y_1)^2} \, dt \\
&= y_1 \int \frac{e^{2t+\ln(2t+1)}}{(y_1)^2} \, dt \\
&= y_1 \left(-\frac{(t+1)\,\mathrm{e}^{2t+\ln(2t+1)}\mathrm{e}^{-4t}}{2t+1}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{2t}\right) + c_2 \left(\mathrm{e}^{2t} \left(-\frac{(t+1)\,\mathrm{e}^{2t+\ln(2t+1)}\mathrm{e}^{-4t}}{2t+1}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(2t+1)\left(\frac{d^2}{dt^2}y(t)\right)-4(t+1)\left(\frac{d}{dt}y(t)\right)+4y(t)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dt^2}y(t)$

- Isolate 2nd derivative

$\frac{d^2}{dt^2}y(t)=-\frac{4y(t)}{2t+1}+\frac{4(t+1)\left(\frac{d}{dt}y(t)\right)}{2t+1}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d^2}{dt^2}y(t)-\frac{4(t+1)\left(\frac{d}{dt}y(t)\right)}{2t+1}+\frac{4y(t)}{2t+1}=0$

□ Check to see if $t_0=-\frac{1}{2}$ is a regular singular point

  ◦ Define functions

  $\left[P_2(t)=-\frac{4(t+1)}{2t+1}, P_3(t)=\frac{4}{2t+1}\right]$

  ◦ $\left(t+\frac{1}{2}\right)\cdot P_2(t)$ is analytic at $t=-\frac{1}{2}$

  $\left(\left(t+\frac{1}{2}\right)\cdot P_2(t)\right)\bigg|_{t=-\frac{1}{2}}=-1$

  ◦ $\left(t+\frac{1}{2}\right)^2\cdot P_3(t)$ is analytic at $t=-\frac{1}{2}$

  $\left(\left(t+\frac{1}{2}\right)^2\cdot P_3(t)\right)\bigg|_{t=-\frac{1}{2}}=0$

  ◦ $t=-\frac{1}{2}$is a regular singular point

Check to see if $t_0=-\frac{1}{2}$ is a regular singular point

$t_0=-\frac{1}{2}$

- Multiply by denominators

$(2t+1)\left(\frac{d^2}{dt^2}y(t)\right)+(-4t-4)\left(\frac{d}{dt}y(t)\right)+4y(t)=0$

- Change variables using $t=u-\frac{1}{2}$ so that the regular singular point is at $u=0$

$2u\left(\frac{d^2}{du^2}y(u)\right)+(-4u-2)\left(\frac{d}{du}y(u)\right)+4y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  ◦ Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-1}$

  ◦ Shift index using $k->k+1$

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)(k+r)\,u^{k+r}$

Rewrite ODE with series expansions

$2a_0r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}(2a_{k+1}(k+1+r)(k+r-1)-4a_k(k+r-1))\,u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $2r(-2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 2\}$
- Each term in the series must be 0, giving the recursion relation
  $2(a_{k+1}(k+1+r) - 2a_k)(k+r-1) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{2a_k}{k+1+r}$
- Recursion relation for $r = 0$
  $a_{k+1} = \frac{2a_k}{k+1}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{2a_k}{k+1}\right]$
- Revert the change of variables $u = t + \frac{1}{2}$
  $\left[y(t) = \sum_{k=0}^{\infty} a_k \left(t+\frac{1}{2}\right)^k, a_{k+1} = \frac{2a_k}{k+1}\right]$
- Recursion relation for $r = 2$
  $a_{k+1} = \frac{2a_k}{k+3}$
- Solution for $r = 2$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{2a_k}{k+3}\right]$
- Revert the change of variables $u = t + \frac{1}{2}$
  $\left[y(t) = \sum_{k=0}^{\infty} a_k \left(t+\frac{1}{2}\right)^{k+2}, a_{k+1} = \frac{2a_k}{k+3}\right]$
- Combine solutions and rename parameters
  $\left[y(t) = \left(\sum_{k=0}^{\infty} a_k \left(t+\frac{1}{2}\right)^k\right) + \left(\sum_{k=0}^{\infty} b_k \left(t+\frac{1}{2}\right)^{k+2}\right), a_{k+1} = \frac{2a_k}{k+1}, b_{k+1} = \frac{2b_k}{k+3}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 15

```
dsolve((2*t+1)*diff(diff(y(t),t),t)-4*(t+1)*diff(y(t),t)+4*y(t) = 0,y(t),singsol=all)
```

$$y = c_2 e^{2t} + c_1 t + c_1$$

Mathematica DSolve solution

Solving time : 0.182 (sec)
Leaf size : 88

```
DSolve[{(2*t+1)*D[y[t],{t,2}]-4*(t+1)*D[y[t],t]+4*y[t]==0,{}},y[t],t,IncludeSingularSolutions->
```

$$y(t) \to \exp\left(\int_1^t \frac{2K[1]}{2K[1]+1}dK[1] - \frac{1}{2}\int_1^t \left(-2-\frac{2}{2K[2]+1}\right)dK[2]\right)\left(c_2\int_1^t \exp\left(-2\int_1^{K[3]} \frac{2K[1]}{2K[1]+1}dK[1]\right)dK[3] + c_1\right)$$

2.1.198 Problem 200

Internal problem ID [9370]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 200
Date solved : `Monday, January 27, 2025 at 06:02:00 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$t^2y'' + ty' + \left(t^2 - \frac{1}{4}\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.136 (sec)

Writing the ode as

$$t^2y'' + ty' + \left(t^2 - \frac{1}{4}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= t \\ C &= t^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.379: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{t}{t^2}\, dt} \\ &= z_1 e^{-\frac{\ln(t)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{t}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(t)}{\sqrt{t}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{t}{t^2}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{-\ln(t)}}{(y_1)^2}\, dt \\ &= y_1(\tan(t)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\cos(t)}{\sqrt{t}}\right) + c_2\left(\frac{\cos(t)}{\sqrt{t}}(\tan(t))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dt^2}y(t)\right)t^2 + t\left(\frac{d}{dt}y(t)\right) + \left(t^2 - \frac{1}{4}\right)y(t) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dt^2}y(t)$
- Isolate 2nd derivative
  $\frac{d^2}{dt^2}y(t) = -\frac{(4t^2-1)y(t)}{4t^2} - \frac{\frac{d}{dt}y(t)}{t}$
- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dt^2}y(t) + \frac{\frac{d}{dt}y(t)}{t} + \frac{(4t^2-1)y(t)}{4t^2} = 0$
- ☐ Check to see if $t_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(t) = \frac{1}{t}, P_3(t) = \frac{4t^2-1}{4t^2}\right]$
  - $t \cdot P_2(t)$ is analytic at $t = 0$
    $\left.(t \cdot P_2(t))\right|_{t=0} = 1$
  - $t^2 \cdot P_3(t)$ is analytic at $t = 0$
    $\left.(t^2 \cdot P_3(t))\right|_{t=0} = -\frac{1}{4}$
  - $t = 0$is a regular singular point
    Check to see if $t_0 = 0$ is a regular singular point
    $t_0 = 0$
- Multiply by denominators
  $4\left(\frac{d^2}{dt^2}y(t)\right)t^2 + 4t\left(\frac{d}{dt}y(t)\right) + (4t^2 - 1)y(t) = 0$
- Assume series solution for $y(t)$
  $y(t) = \sum_{k=0}^{\infty} a_k t^{k+r}$
- ☐ Rewrite ODE with series expansions

- Convert $t^m \cdot y(t)$ to series expansion for $m = 0..2$

  $t^m \cdot y(t) = \sum_{k=0}^{\infty} a_k t^{k+r+m}$
- Shift index using $k- > k - m$

  $t^m \cdot y(t) = \sum_{k=m}^{\infty} a_{k-m} t^{k+r}$
- Convert $t \cdot \left(\frac{d}{dt} y(t)\right)$ to series expansion

  $t \cdot \left(\frac{d}{dt} y(t)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, t^{k+r}$
- Convert $t^2 \cdot \left(\frac{d^2}{dt^2} y(t)\right)$ to series expansion

  $t^2 \cdot \left(\frac{d^2}{dt^2} y(t)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)(-1+2r)\, t^r + a_1(3+2r)(1+2r)\, t^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)(2k+2r-1) + 4a_{k-2})\, t^k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0

  $a_1(3+2r)(1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$
- Shift index using $k- > k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$
- Solution for $r = -\frac{1}{2}$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$
- Solution for $r = \frac{1}{2}$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(t) = \left(\sum_{k=0}^{\infty} a_k t^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.037 (sec)
Leaf size : 17

```
dsolve(t^2*diff(diff(y(t),t),t)+t*diff(y(t),t)+(t^2-1/4)*y(t) = 0,y(t),singsol=all)
```

$$y = \frac{c_2 \cos(t) + c_1 \sin(t)}{\sqrt{t}}$$

Mathematica DSolve solution

Solving time : 0.034 (sec)
Leaf size : 39

```
DSolve[{t^2*D[y[t],{t,2}]+t*D[y[t],t]+(t^2-1/4)*y[t]==0,{}},y[t],t,IncludeSingularSolutions-
```

$$y(t) \to \frac{e^{-it}(2c_1 - ic_2e^{2it})}{2\sqrt{t}}$$

2.1.199 Problem 201

Internal problem ID [9371]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 201
Date solved : Monday, January 27, 2025 at 06:02:01 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$y'' - \frac{2ty'}{t^2+1} + \frac{2y}{t^2+1} = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.270 (sec)

Writing the ode as

$$y'' - \frac{2ty'}{t^2+1} + \frac{2y}{t^2+1} = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -\frac{2t}{t^2+1} \\ C &= \frac{2}{t^2+1} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{(t^2+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= (t^2+1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(-\frac{3}{(t^2+1)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.381: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (t^2+1)^2$. There is a pole at $t = i$ of order 2. There is a pole at $t = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(t-i)^2} + \frac{3}{4(t+i)^2} + \frac{3i}{4(t-i)} - \frac{3i}{4(t+i)}$$

For the pole at $t = i$ let b be the coefficient of $\frac{1}{(t-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $t = -i$ let b be the coefficient of $\frac{1}{(t+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{3}{(t^2+1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{t-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(t-i)} + \frac{3}{2(t+i)} + (-)(0) \\
&= -\frac{1}{2(t-i)} + \frac{3}{2(t+i)} \\
&= \frac{t-2i}{t^2+1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(t-i)} + \frac{3}{2(t+i)}\right)(0) + \left(\left(\frac{1}{2(t-i)^2} - \frac{3}{2(t+i)^2}\right) + \left(-\frac{1}{2(t-i)} + \frac{3}{2(t+i)}\right)^2 - \left(-\frac{}{(t^2}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= e^{\int \left(-\frac{1}{2(t-i)} + \frac{3}{2(t+i)}\right) dt} \\ &= \frac{(t^2+1)^{3/2}}{(it+1)^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-\frac{2t}{t^2+1}}{1} \, dt} \\ &= z_1 e^{\frac{\ln(t^2+1)}{2}} \\ &= z_1 \left(\sqrt{t^2+1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(t^2+1)^2}{(it+1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-\frac{2t}{t^2+1}}{1} \, dt}}{(y_1)^2} \, dt \\ &= y_1 \int \frac{e^{\ln(t^2+1)}}{(y_1)^2} \, dt \\ &= y_1 \left(-\frac{t}{(t+i)^2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{(t^2+1)^2}{(it+1)^2}\right) + c_2 \left(\frac{(t^2+1)^2}{(it+1)^2} \left(-\frac{t}{(t+i)^2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 16

```
dsolve(diff(diff(y(t),t),t)-2*t/(t^2+1)*diff(y(t),t)+2/(t^2+1)*y(t) = 0,y(t),singsol=all
```

$$y = c_2t^2 + c_1t - c_2$$

Mathematica DSolve solution

Solving time : 0.302 (sec)
Leaf size : 79

```
DSolve[{D[y[t],{t,2}]-2*t/(1+t^2)*D[y[t],t]+2/(1+t^2)*y[t]==0,{}},y[t],t,IncludeSingularSoluti
```

$$y(t) \to \sqrt{t^2+1} \exp\left(\int_1^t \frac{K[1]+2i}{K[1]^2+1}dK[1]\right) \left(c_2 \int_1^t \exp\left(-2\int_1^{K[2]} \frac{K[1]+2i}{K[1]^2+1}dK[1]\right) dK[2] + c_1\right)$$

2.1.200 Problem 202

Internal problem ID [9372]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 202
Date solved : `Monday, January 27, 2025 at 06:02:01 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + \left(t^2+2t+1\right) y' - (4+4t)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.264 (sec)

Writing the ode as

$$y'' + (1+t)^2 y' + (-4-4t)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= (1+t)^2 \\ C &= -4-4t \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^4+4t^3+6t^2+24t+21}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^4+4t^3+6t^2+24t+21 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{21}{4} + 6t + \frac{1}{4}t^4 + t^3 + \frac{3}{2}t^2\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.382: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 4 \\ &= -4 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{2} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{t^2}{2} + t + \frac{1}{2} + \frac{5}{t} - \frac{5}{t^2} + \frac{5}{t^3} - \frac{30}{t^4} + \frac{105}{t^5} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 2$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i t^i \\ &= \frac{1}{2}t^2 + t + \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^1 = t$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}t^4 + t^3 + \frac{3}{2}t^2 + t + \frac{1}{4}$$

This shows that the coefficient of t in the above is 1. Now we need to find the coefficient of t in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of t in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^4 + 4t^3 + 6t^2 + 24t + 21}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{21}{4} + 6t + \frac{1}{4}t^4 + t^3 + \frac{3}{2}t^2\right) + (0) \\ &= \frac{21}{4} + 6t + \frac{1}{4}t^4 + t^3 + \frac{3}{2}t^2 \end{aligned}$$

We see that the coefficient of the term $\frac{1}{t}$ in the quotient is 6. Now b can be found.

$$\begin{aligned} b &= (6) - (1) \\ &= 5 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2}t^2 + t + \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{5}{\frac{1}{2}} - 2\right) &= 4 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{5}{\frac{1}{2}} - 2\right) &= -6 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{21}{4} + 6t + \frac{1}{4}t^4 + t^3 + \frac{3}{2}t^2$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-4	$\frac{1}{2}t^2 + t + \frac{1}{2}$	4	-6

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 4$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 4 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{1}{2}t^2 + t + \frac{1}{2}\right) \\ &= \frac{1}{2}t^2 + t + \frac{1}{2} \\ &= \frac{(1+t)^2}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 4$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = t^4 + a_3t^3 + a_2t^2 + a_1t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(12t^2 + 6ta_3 + 2a_2) + 2\left(\frac{1}{2}t^2 + t + \frac{1}{2}\right)(4t^3 + 3t^2a_3 + 2ta_2 + a_1) + \left((1+t) + \left(\frac{1}{2}t^2 + t + \frac{1}{2}\right)^2 - \left(\frac{21}{4} + 6t\right.\right.$$

$$(-a_3 + 4)\, t^4 + 2(2 - a_2 + a_3)\, t^3 + 3(4 - a_1 + a_3)\, t^2 + 2(-2a_0 - a_1 + a_2 + 3a_3)\, t -$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 5, a_1 = 8, a_2 = 6, a_3 = 4\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = t^4 + 4t^3 + 6t^2 + 8t + 5$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= pe^{\int \omega\, dt} \\ &= \left(t^4 + 4t^3 + 6t^2 + 8t + 5\right) e^{\int \left(\frac{1}{2}t^2 + t + \frac{1}{2}\right) dt} \\ &= \left(t^4 + 4t^3 + 6t^2 + 8t + 5\right) e^{\frac{(1+t)^3}{6}} \\ &= (1+t)\left(t^3 + 3t^2 + 3t + 5\right) e^{\frac{(1+t)^3}{6}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{(1+t)^2}{1}\, dt} \\ &= z_1 e^{-\frac{(1+t)^3}{6}} \\ &= z_1 \left(e^{-\frac{(1+t)^3}{6}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (1+t)\left(t^3+3t^2+3t+5\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{(1+t)^2}{1}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{-\frac{(1+t)^3}{3}}}{(y_1)^2}\, dt \\ &= y_1 \left(\int \frac{e^{-\frac{(1+t)^3}{3}}}{(1+t)^2\,(t^3+3t^2+3t+5)^2} dt\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left((1+t)\left(t^3+3t^2+3t+5\right)\right) \\ &\quad + c_2\left((1+t)\left(t^3+3t^2+3t+5\right)\left(\int \frac{e^{-\frac{(1+t)^3}{3}}}{(1+t)^2\,(t^3+3t^2+3t+5)^2} dt\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\frac{d^2}{dt^2}y(t) + (t^2+2t+1)\left(\frac{d}{dt}y(t)\right) - (4t+4)\,y(t) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dt^2}y(t)$
- Isolate 2nd derivative
  $\frac{d^2}{dt^2}y(t) = -(t^2+2t+1)\left(\frac{d}{dt}y(t)\right) + (4t+4)\,y(t)$
- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dt^2}y(t) + (t^2+2t+1)\left(\frac{d}{dt}y(t)\right) + (-4t-4)\,y(t) = 0$
- Assume series solution for $y(t)$
  $y(t) = \sum_{k=0}^{\infty} a_k t^k$

□ Rewrite ODE with series expansions
  - Convert $t^m \cdot y(t)$ to series expansion for $m = 0..1$
    $t^m \cdot y(t) = \sum_{k=\max(0,-m)}^{\infty} a_k t^{k+m}$
  - Shift index using $k->k-m$
    $t^m \cdot y(t) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} t^k$
  - Convert $t^m \cdot \left(\frac{d}{dt}y(t)\right)$ to series expansion for $m = 0..2$

$$t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, t^{k-1+m}$$

- Shift index using $k- >k+1-m$

$$t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, t^k$$

- Convert $\frac{d^2}{dt^2}y(t)$ to series expansion

$$\frac{d^2}{dt^2}y(t) = \sum_{k=2}^{\infty} a_k k(k-1)\, t^{k-2}$$

- Shift index using $k- >k+2$

$$\frac{d^2}{dt^2}y(t) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, t^k$$

Rewrite ODE with series expansions

$$2a_2 + a_1 - 4a_0 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_{k+1}(k+1) + 2a_k(k-2) + a_{k-1}(k-5)\right) t^k\right) = 0$$

- Each term must be 0

$$2a_2 + a_1 - 4a_0 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$k^2 a_{k+2} + (2a_k + a_{k-1} + a_{k+1} + 3a_{k+2})\, k - 4a_k - 5a_{k-1} + a_{k+1} + 2a_{k+2} = 0$$

- Shift index using $k- >k+1$

$$(k+1)^2 a_{k+3} + (2a_{k+1} + a_k + a_{k+2} + 3a_{k+3})\,(k+1) - 4a_{k+1} - 5a_k + a_{k+2} + 2a_{k+3} = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y(t) = \sum_{k=0}^{\infty} a_k t^k, a_{k+3} = -\frac{a_k k + 2a_{k+1} k + k a_{k+2} - 4a_k - 2a_{k+1} + 2a_{k+2}}{k^2+5k+6}, 2a_2 + a_1 - 4a_0 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Mo
      <- Heun successful: received ODE is equivalent to the  HeunT  ODE, case  c = 0
   Special function solution also has integrals. Returning default Liouvillian solution.
```

```
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 1.993 (sec)
Leaf size : 60

```
dsolve(diff(diff(y(t),t),t)+(t^2+2*t+1)*diff(y(t),t)-(4*t+4)*y(t) = 0,y(t),singsol=all
```

$$y = (t+1)\left(t^3+3t^2+3t+5\right)\left(c_2\left(\int \frac{e^{-\frac{t\left(t^2+3t+3\right)}{3}}}{(t+1)^2\left(t^3+3t^2+3t+5\right)^2}dt\right)+c_1\right)$$

Mathematica DSolve solution

Solving time : 4.118 (sec)
Leaf size : 78

```
DSolve[{D[y[t],{t,2}]+(t^2+2*t+1)*D[y[t],t]-(4+4*t)*y[t]==0,{}},y[t],t,IncludeSingularSoluti
```

$$y(t) \to (t+1)\left(t^3+3t^2+3t+5\right)\left(c_2\int_1^t \frac{e^{-\frac{1}{3}K[1](K[1]^2+3K[1]+3)}}{(K[1]+1)^2\left(K[1]^3+3K[1]^2+3K[1]+5\right)^2}dK[1]+c_1\right)$$

2.1.201 Problem 204

Internal problem ID [9373]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 204
Date solved : `Monday, January 27, 2025 at 06:02:02 PM`
CAS classification : `[_Laguerre]`

Solve

$$2ty'' + (1 - 2t) y' - y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.223 (sec)

Writing the ode as

$$2ty'' + (1 - 2t) y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2t \\ B &= 1 - 2t \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4t^2 + 4t - 3}{16t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4t^2 + 4t - 3 \\ t &= 16t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{4t^2 + 4t - 3}{16t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.384: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{3}{16t^2} + \frac{1}{4t}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{4t} - \frac{1}{4t^2} + \frac{1}{8t^3} - \frac{1}{8t^4} + \frac{1}{8t^5} - \frac{9}{64t^6} + \frac{21}{128t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4t^2 + 4t - 3}{16t^2} \\ &= Q + \frac{R}{16t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{4t - 3}{16t^2}\right) \\ &= \frac{1}{4} + \frac{4t - 3}{16t^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is 4. Dividing this by leading coefficient in t which is 16 gives $\frac{1}{4}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{4}\right) - (0) \\ &= \frac{1}{4} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{4}}{\frac{1}{2}} - 0\right) &&= \frac{1}{4} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{4}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4t^2 + 4t - 3}{16t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= \frac{1}{4} - \left(\frac{1}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1}\right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{4t} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2} + \frac{1}{4t} \\ &= \frac{1}{2} + \frac{1}{4t} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2} + \frac{1}{4t}\right)(0) + \left(\left(-\frac{1}{4t^2}\right) + \left(\frac{1}{2} + \frac{1}{4t}\right)^2 - \left(\frac{4t^2+4t-3}{16t^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega\, dt} \\ &= e^{\int \left(\frac{1}{2} + \frac{1}{4t}\right) dt} \\ &= t^{1/4} e^{\frac{t}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1-2t}{2t}\,dt} \\ &= z_1 e^{\frac{t}{2}-\frac{\ln(t)}{4}} \\ &= z_1 \left(\frac{e^{\frac{t}{2}}}{t^{1/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1-2t}{2t}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{t-\frac{\ln(t)}{2}}}{(y_1)^2}\,dt \\ &= y_1\left(\sqrt{\pi}\,\mathrm{erf}\left(\sqrt{t}\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(e^t\right) + c_2\left(e^t\left(\sqrt{\pi}\,\mathrm{erf}\left(\sqrt{t}\right)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2\left(\frac{d^2}{dt^2}y(t)\right)t + (1-2t)\left(\frac{d}{dt}y(t)\right) - y(t) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dt^2}y(t)$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = \frac{y(t)}{2t} + \frac{(2t-1)\left(\frac{d}{dt}y(t)\right)}{2t}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d^2}{dt^2}y(t) - \frac{(2t-1)\left(\frac{d}{dt}y(t)\right)}{2t} - \frac{y(t)}{2t} = 0$

- □ Check to see if $t_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(t) = -\frac{2t-1}{2t}, P_3(t) = -\frac{1}{2t}\right]$

  - $t \cdot P_2(t)$ is analytic at $t = 0$

$(t \cdot P_2(t))\Big|_{t=0} = \frac{1}{2}$

- ◦ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $(t^2 \cdot P_3(t))\Big|_{t=0} = 0$

- ◦ $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $2\left(\frac{d^2}{dt^2}y(t)\right)t + (1-2t)\left(\frac{d}{dt}y(t)\right) - y(t) = 0$

- Assume series solution for $y(t)$

  $y(t) = \sum_{k=0}^{\infty} a_k t^{k+r}$

- □ Rewrite ODE with series expansions

  - ◦ Convert $t^m \cdot \left(\frac{d}{dt}y(t)\right)$ to series expansion for $m = 0..1$

    $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$

  - ◦ Shift index using $k- >k+1-m$

    $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$

  - ◦ Convert $t \cdot \left(\frac{d^2}{dt^2}y(t)\right)$ to series expansion

    $t \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, t^{k+r-1}$

  - ◦ Shift index using $k- >k+1$

    $t \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-1+2r)\, t^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(2k+2r+1) - a_k(2k+2r+1)\right) t^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2(a_{k+1}(k+1+r) - a_k)\left(k+r+\frac{1}{2}\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1+r}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 0$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^k, a_{k+1} = \frac{a_k}{k+1}\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = \frac{a_k}{k+\frac{3}{2}}$

- Solution for $r = \frac{1}{2}$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k}{k+\frac{3}{2}}\right]$

- Combine solutions and rename parameters

$$\left[y(t) = \left(\sum_{k=0}^{\infty} a_k t^k\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+\frac{1}{2}}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+\frac{3}{2}}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.031 (sec)
Leaf size : 15

```
dsolve(2*t*diff(diff(y(t),t),t)+(-2*t+1)*diff(y(t),t)-y(t) = 0,y(t),singsol=all)
```

$$y = e^t \left(\operatorname{erf}\left(\sqrt{t}\right) c_1 + c_2\right)$$

Mathematica DSolve solution

Solving time : 0.123 (sec)
Leaf size : 21

```
DSolve[{2*t*D[y[t],{t,2}]+(1-2*t)*D[y[t],t]-y[t]==0,{}},y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to e^t \left(c_1 - c_2 \Gamma\left(\frac{1}{2}, t\right)\right)$$

2.1.202 Problem 205

Internal problem ID [9374]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 205
Date solved : `Monday, January 27, 2025 at 06:02:03 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2ty'' + (1+t)\,y' - 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.394 (sec)

Writing the ode as

$$2ty'' + (1+t)\,y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2t \\ B &= 1+t \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 + 18t - 3}{16t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^2 + 18t - 3 \\ t &= 16t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 + 18t - 3}{16t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.386: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16} - \frac{3}{16t^2} + \frac{9}{8t}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{4} + \frac{9}{4t} - \frac{21}{2t^2} + \frac{189}{2t^3} - \frac{1071}{t^4} + \frac{13608}{t^5} - \frac{370629}{2t^6} + \frac{5288409}{2t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{4} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{16}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 + 18t - 3}{16t^2} \\ &= Q + \frac{R}{16t^2} \\ &= \left(\frac{1}{16}\right) + \left(\frac{18t - 3}{16t^2}\right) \\ &= \frac{1}{16} + \frac{18t - 3}{16t^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is 18. Dividing this by leading coefficient in t which is 16 gives $\frac{9}{8}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{9}{8}\right) - (0) \\ &= \frac{9}{8} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{4} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{9}{8}}{\frac{1}{4}} - 0\right) &&= \frac{9}{4} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{9}{8}}{\frac{1}{4}} - 0\right) &&= -\frac{9}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 + 18t - 3}{16t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{4}$	$\frac{9}{4}$	$-\frac{9}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{9}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\ &= \frac{9}{4} - \left(\frac{1}{4}\right) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{4t} + \left(\frac{1}{4}\right) \\ &= \frac{1}{4t} + \frac{1}{4} \\ &= \frac{1+t}{4t} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 2$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = t^2 + a_1 t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{1}{4t} + \frac{1}{4}\right)(2t + a_1) + \left(\left(-\frac{1}{4t^2}\right) + \left(\frac{1}{4t} + \frac{1}{4}\right)^2 - \left(\frac{t^2 + 18t - 3}{16t^2}\right)\right) = 0$$

$$\frac{(-a_1 + 6)\, t - 2a_0 + a_1}{2t} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 3, a_1 = 6\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = t^2 + 6t + 3$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= \left(t^2+6t+3\right) e^{\int \left(\frac{1}{4t}+\frac{1}{4}\right)dt} \\ &= \left(t^2+6t+3\right) e^{\frac{t}{4}+\frac{\ln(t)}{4}} \\ &= \left(t^2+6t+3\right) t^{1/4} e^{\frac{t}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1+t}{2t}\, dt} \\ &= z_1 e^{-\frac{t}{4}-\frac{\ln(t)}{4}} \\ &= z_1 \left(\frac{e^{-\frac{t}{4}}}{t^{1/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = t^2+6t+3$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1+t}{2t}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{-\frac{t}{2}-\frac{\ln(t)}{2}}}{(y_1)^2}\, dt \\ &= y_1 \left(\int \frac{e^{-\frac{t}{2}-\frac{\ln(t)}{2}}}{\left(t^2+6t+3\right)^2} dt\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(t^2+6t+3\right) + c_2\left(t^2+6t+3\left(\int \frac{e^{-\frac{t}{2}-\frac{\ln(t)}{2}}}{\left(t^2+6t+3\right)^2} dt\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$2\left(\frac{d^2}{dt^2}y(t)\right)t + (t+1)\left(\frac{d}{dt}y(t)\right) - 2y(t) = 0$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dt^2}y(t)$
- Isolate 2nd derivative
$\frac{d^2}{dt^2}y(t) = \frac{y(t)}{t} - \frac{(t+1)\left(\frac{d}{dt}y(t)\right)}{2t}$
- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
$\frac{d^2}{dt^2}y(t) + \frac{(t+1)\left(\frac{d}{dt}y(t)\right)}{2t} - \frac{y(t)}{t} = 0$
- □ Check to see if $t_0 = 0$ is a regular singular point
  - Define functions
  $\left[P_2(t) = \frac{t+1}{2t}, P_3(t) = -\frac{1}{t}\right]$
  - $t \cdot P_2(t)$ is analytic at $t = 0$
  $\left.(t \cdot P_2(t))\right|_{t=0} = \frac{1}{2}$
  - $t^2 \cdot P_3(t)$ is analytic at $t = 0$
  $\left.(t^2 \cdot P_3(t))\right|_{t=0} = 0$
  - $t = 0$is a regular singular point
  Check to see if $t_0 = 0$ is a regular singular point
  $t_0 = 0$
- Multiply by denominators
$2\left(\frac{d^2}{dt^2}y(t)\right)t + (t+1)\left(\frac{d}{dt}y(t)\right) - 2y(t) = 0$
- Assume series solution for $y(t)$
$y(t) = \sum_{k=0}^{\infty} a_k t^{k+r}$
- □ Rewrite ODE with series expansions
  - Convert $t^m \cdot \left(\frac{d}{dt}y(t)\right)$ to series expansion for $m = 0..1$
  $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$
  - Shift index using $k- >k+1-m$
  $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$
  - Convert $t \cdot \left(\frac{d^2}{dt^2}y(t)\right)$ to series expansion
  $t \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, t^{k+r-1}$
  - Shift index using $k- >k+1$
  $t \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, t^{k+r}$
  Rewrite ODE with series expansions
  $a_0 r(-1+2r)\, t^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(2k+1+2r) + a_k(k+r-2)\right) t^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation
$r(-1+2r) = 0$
- Values of r that satisfy the indicial equation
$r \in \left\{0, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation
  $2(k+1+r)\left(k+r+\frac{1}{2}\right)a_{k+1}+a_k(k+r-2)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=-\frac{a_k(k+r-2)}{(k+1+r)(2k+1+2r)}$
- Recursion relation for $r=0$; series terminates at $k=2$
  $a_{k+1}=-\frac{a_k(k-2)}{(k+1)(2k+1)}$
- Apply recursion relation for $k=0$
  $a_1=2a_0$
- Apply recursion relation for $k=1$
  $a_2=\frac{a_1}{6}$
- Express in terms of a_0
  $a_2=\frac{a_0}{3}$
- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li
  $y(t)=a_0\cdot\left(1+2t+\frac{1}{3}t^2\right)$
- Recursion relation for $r=\frac{1}{2}$
  $a_{k+1}=-\frac{a_k\left(k-\frac{3}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}$
- Solution for $r=\frac{1}{2}$
  $\left[y(t)=\sum_{k=0}^{\infty}a_k t^{k+\frac{1}{2}}, a_{k+1}=-\frac{a_k\left(k-\frac{3}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$
- Combine solutions and rename parameters
  $\left[y(t)=a_0\cdot\left(1+2t+\frac{1}{3}t^2\right)+\left(\sum_{k=0}^{\infty}b_k t^{k+\frac{1}{2}}\right), b_{k+1}=-\frac{b_k\left(k-\frac{3}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special functio
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.046 (sec)
Leaf size : 56

```
dsolve(2*t*diff(diff(y(t),t),t)+(t+1)*diff(y(t),t)-2*y(t) = 0,y(t),singsol=all)
```

$$y = c_1\sqrt{\pi}\,(t^2+6t+3)\,\mathrm{erf}\left(\frac{\sqrt{2}\,\sqrt{t}}{2}\right)+5c_1\sqrt{2}\left(\sqrt{t}+\frac{t^{3/2}}{5}\right)\mathrm{e}^{-\frac{t}{2}}+c_2\left(t^2+6t+3\right)$$

Mathematica DSolve solution

Solving time : 11.026 (sec)
Leaf size : 58

```
DSolve[{2*t*D[y[t],{t,2}]+(1+t)*D[y[t],t]-2*y[t]==0,{}},y[t],t,IncludeSingularSolutions->True]
```

$$y(t)\to\left(t^2+6t+3\right)\left(c_2\int_1^t\frac{e^{-\frac{K[1]}{2}-\frac{1}{2}}}{\sqrt{K[1]}\,(K[1]^2+6K[1]+3)^2}dK[1]+c_1\right)$$

2.1.203 Problem 206

Internal problem ID [9375]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 206
Date solved : `Monday, January 27, 2025 at 06:02:03 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2t^2y'' - ty' + (1+t)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.212 (sec)

Writing the ode as

$$2t^2y'' - ty' + (1+t)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2t^2 \\ B &= -t \\ C &= 1+t \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3-8t}{16t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3-8t \\ t &= 16t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{-3-8t}{16t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.388: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16t^2$. There is a pole at $t = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16t^2} - \frac{1}{2t}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 1, \; e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(t)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(1 - (1)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{t-c} \\ &= \frac{1}{2}\left(\frac{1}{(t-(0))}\right) \\ &= \frac{1}{2t} \end{aligned}$$

Now we search for a monic polynomial $p(t)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(t)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2t} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2t} + \frac{1+8t}{16t^2} = 0$$

Solving for ω gives

$$\omega = \frac{1 + 2\sqrt{2}\,\sqrt{-t}}{4t}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= e^{\int \omega\, dt} \\ &= \mathrm{e}^{\int \frac{1+2\sqrt{2}\,\sqrt{-t}}{4t} dt} \\ &= t^{1/4} \mathrm{e}^{\sqrt{2}\,\sqrt{-t}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-t}{2t^2}\,dt} \\ &= z_1 e^{\frac{\ln(t)}{4}} \\ &= z_1 \left(t^{1/4}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{t}\,\mathrm{e}^{\sqrt{2}\sqrt{-t}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-t}{2t^2}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{\frac{\ln(t)}{2}}}{(y_1)^2}\,dt \\ &= y_1 \left(-\frac{\sqrt{2}\sqrt{-t}\left(1-\mathrm{e}^{-2\sqrt{2}\sqrt{-t}}\right)}{2\sqrt{t}}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\sqrt{t}\,\mathrm{e}^{\sqrt{2}\sqrt{-t}}\right) + c_2\left(\sqrt{t}\,\mathrm{e}^{\sqrt{2}\sqrt{-t}}\left(-\frac{\sqrt{2}\sqrt{-t}\left(1-\mathrm{e}^{-2\sqrt{2}\sqrt{-t}}\right)}{2\sqrt{t}}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2\left(\frac{d^2}{dt^2}y(t)\right)t^2 - t\left(\frac{d}{dt}y(t)\right) + (t+1)\,y(t) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dt^2}y(t)$
- Isolate 2nd derivative
  $\frac{d^2}{dt^2}y(t) = -\frac{(t+1)y(t)}{2t^2} + \frac{\frac{d}{dt}y(t)}{2t}$
- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d^2}{dt^2}y(t) - \frac{\frac{d}{dt}y(t)}{2t} + \frac{(t+1)y(t)}{2t^2} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(t) = -\frac{1}{2t}, P_3(t) = \frac{t+1}{2t^2}\right]$
  - $t \cdot P_2(t)$ is analytic at $t = 0$

$\left(t \cdot P_2(t)\right)\bigg|_{t=0} = -\frac{1}{2}$

- $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $\left(t^2 \cdot P_3(t)\right)\bigg|_{t=0} = \frac{1}{2}$

- $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $2\left(\frac{d^2}{dt^2}y(t)\right) t^2 - t\left(\frac{d}{dt}y(t)\right) + (t+1)\, y(t) = 0$

- Assume series solution for $y(t)$

  $y(t) = \sum_{k=0}^{\infty} a_k t^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $t^m \cdot y(t)$ to series expansion for $m = 0..1$

    $t^m \cdot y(t) = \sum_{k=0}^{\infty} a_k t^{k+r+m}$

  - Shift index using $k- >k - m$

    $t^m \cdot y(t) = \sum_{k=m}^{\infty} a_{k-m} t^{k+r}$

  - Convert $t \cdot \left(\frac{d}{dt}y(t)\right)$ to series expansion

    $t \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, t^{k+r}$

  - Convert $t^2 \cdot \left(\frac{d^2}{dt^2}y(t)\right)$ to series expansion

    $t^2 \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)\,(-1+r)\, t^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)\,(k+r-1) + a_{k-1}\right) t^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)\,(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{1, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k+r-\frac{1}{2}\right)(k+r-1)\, a_k + a_{k-1} = 0$

- Shift index using $k- >k + 1$

  $2\left(k+\frac{1}{2}+r\right)(k+r)\, a_{k+1} + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{(2k+1+2r)(k+r)}$

- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{a_k}{(2k+3)(k+1)}$

- Solution for $r = 1$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = -\frac{a_k}{(2k+3)(k+1)}\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = -\frac{a_k}{(2k+2)\left(k+\frac{1}{2}\right)}$

- Solution for $r = \frac{1}{2}$

$$\left[y(t)=\sum_{k=0}^{\infty} a_k t^{k+\frac{1}{2}}, a_{k+1}=-\frac{a_k}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y(t)=\left(\sum_{k=0}^{\infty} a_k t^{k+1}\right)+\left(\sum_{k=0}^{\infty} b_k t^{k+\frac{1}{2}}\right), a_{k+1}=-\frac{a_k}{(2k+3)(k+1)}, b_{k+1}=-\frac{b_k}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 29

```
dsolve(2*t^2*diff(diff(y(t),t),t)-t*diff(y(t),t)+(t+1)*y(t) = 0,y(t),singsol=all)
```

$$y=\sqrt{t}\left(c_1 \sin\left(\sqrt{2}\,\sqrt{t}\right)+c_2 \cos\left(\sqrt{2}\,\sqrt{t}\right)\right)$$

Mathematica DSolve solution

Solving time : 0.079 (sec)
Leaf size : 62

```
DSolve[{2*t^2*D[y[t],{t,2}]-t*D[y[t],t]+(1+t)*y[t]==0,{}},y[t],t,IncludeSingularSolutions->True
```

$$y(t)\to \frac{1}{2}e^{-i\sqrt{2}\sqrt{t}}\sqrt{t}\left(2c_1 e^{2i\sqrt{2}\sqrt{t}}+i\sqrt{2}c_2\right)$$

2.1.204 Problem 207

Internal problem ID [9376]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 207
Date solved : Monday, January 27, 2025 at 06:02:04 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$2t^2y'' + (t^2 - t)\, y' + y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.230 (sec)

Writing the ode as

$$2t^2y'' + (t^2 - t)\, y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2t^2 \\ B &= t^2 - t \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 - 2t - 3}{16t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^2 - 2t - 3 \\ t &= 16t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 - 2t - 3}{16t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.390: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16} - \frac{3}{16t^2} - \frac{1}{8t}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{4} - \frac{1}{4t} - \frac{1}{2t^2} - \frac{1}{2t^3} - \frac{1}{t^4} - \frac{2}{t^5} - \frac{9}{2t^6} - \frac{21}{2t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{4}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{16}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{t^2 - 2t - 3}{16t^2} \\ &= Q + \frac{R}{16t^2} \\ &= \left(\frac{1}{16}\right) + \left(\frac{-2t - 3}{16t^2}\right) \\ &= \frac{1}{16} + \frac{-2t - 3}{16t^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 16 gives $-\frac{1}{8}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{1}{8}\right) - (0) \\ &= -\frac{1}{8}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{4} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{8}}{\frac{1}{4}} - 0\right) &&= -\frac{1}{4} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{8}}{\frac{1}{4}} - 0\right) &&= \frac{1}{4}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 - 2t - 3}{16t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{4} - \left(\frac{1}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{4t} + (-)\left(\frac{1}{4}\right) \\ &= \frac{1}{4t} - \frac{1}{4} \\ &= -\frac{t-1}{4t} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{4t} - \frac{1}{4}\right)(0) + \left(\left(-\frac{1}{4t^2}\right) + \left(\frac{1}{4t} - \frac{1}{4}\right)^2 - \left(\frac{t^2-2t-3}{16t^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= e^{\int \left(\frac{1}{4t} - \frac{1}{4}\right) dt} \\ &= t^{1/4} e^{-\frac{t}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{t^2-t}{2t^2}\,dt} \\
&= z_1 e^{-\frac{t}{4}+\frac{\ln(t)}{4}} \\
&= z_1\left(t^{1/4}\mathrm{e}^{-\frac{t}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \sqrt{t}\,\mathrm{e}^{-\frac{t}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{t^2-t}{2t^2}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-\frac{t}{2}+\frac{\ln(t)}{2}}}{(y_1)^2}\,dt \\
&= y_1\left(-i\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,\sqrt{t}}{2}\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{t}\,\mathrm{e}^{-\frac{t}{2}}\right) + c_2\left(\sqrt{t}\,\mathrm{e}^{-\frac{t}{2}}\left(-i\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,\sqrt{t}}{2}\right)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2\left(\frac{d^2}{dt^2}y(t)\right)t^2 + (t^2-t)\left(\frac{d}{dt}y(t)\right) + y(t) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dt^2}y(t)$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = -\frac{y(t)}{2t^2} - \frac{(t-1)\left(\frac{d}{dt}y(t)\right)}{2t}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dt^2}y(t) + \frac{(t-1)\left(\frac{d}{dt}y(t)\right)}{2t} + \frac{y(t)}{2t^2} = 0$

☐ Check to see if $t_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(t) = \frac{t-1}{2t}, P_3(t) = \frac{1}{2t^2}\right]$

- ○ $t \cdot P_2(t)$ is analytic at $t = 0$

$\left(t \cdot P_2(t)\right)\Big|_{t=0} = -\frac{1}{2}$

- ◦ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $\left(t^2 \cdot P_3(t)\right)\Big|_{t=0} = \frac{1}{2}$

- ◦ $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $2\left(\frac{d^2}{dt^2}y(t)\right)t^2 + t(t-1)\left(\frac{d}{dt}y(t)\right) + y(t) = 0$

- Assume series solution for $y(t)$

  $y(t) = \sum_{k=0}^{\infty} a_k t^{k+r}$

- □ Rewrite ODE with series expansions

  - ◦ Convert $t^m \cdot \left(\frac{d}{dt}y(t)\right)$ to series expansion for $m = 1..2$

    $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$

  - ◦ Shift index using $k- >k+1-m$

    $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$

  - ◦ Convert $t^2 \cdot \left(\frac{d^2}{dt^2}y(t)\right)$ to series expansion

    $t^2 \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)(-1+r)\, t^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)(k+r-1) + a_{k-1}(k+r-1)\right) t^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{1, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(\left(k+r-\frac{1}{2}\right)a_k + \frac{a_{k-1}}{2}\right)(k+r-1) = 0$

- Shift index using $k- >k+1$

  $2\left(\left(k+\frac{1}{2}+r\right)a_{k+1} + \frac{a_k}{2}\right)(k+r) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{2k+1+2r}$

- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{a_k}{2k+3}$

- Solution for $r = 1$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = -\frac{a_k}{2k+3}\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = -\frac{a_k}{2k+2}$

- Solution for $r = \frac{1}{2}$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k+\frac{1}{2}}, a_{k+1} = -\frac{a_k}{2k+2}\right]$

- Combine solutions and rename parameters

$$\left[y(t)=\left(\sum_{k=0}^{\infty}a_k t^{k+1}\right)+\left(\sum_{k=0}^{\infty}b_k t^{k+\frac{1}{2}}\right), a_{k+1}=-\frac{a_k}{2k+3}, b_{k+1}=-\frac{b_k}{2k+2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Whittaker successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - return
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.020 (sec)
Leaf size : 47

```
dsolve(2*t^2*diff(diff(y(t),t),t)+(t^2-t)*diff(y(t),t)+y(t) = 0,y(t),singsol=all)
```

$$y=\frac{e^{-\frac{t}{2}}\left(\operatorname{erf}\left(\frac{\sqrt{2}\sqrt{-t}}{2}\right)2^{3/4}\sqrt{\pi}\,c_1 t+4\sqrt{t}\,\sqrt{-t}\,c_2\right)}{4\sqrt{-t}}$$

Mathematica DSolve solution

Solving time : 0.125 (sec)
Leaf size : 46

```
DSolve[{2*t^2*D[y[t],{t,2}]+(t^2-t)*D[y[t],t]+y[t]==0,{}},y[t],t,IncludeSingularSolutions->T
```

$$y(t)\to e^{-t/2}\left(c_2\sqrt{t}+\sqrt{2}c_1\sqrt{-t}\Gamma\left(\frac{1}{2},-\frac{t}{2}\right)\right)$$

2.1.205 Problem 208

Internal problem ID [9377]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 208
Date solved : Monday, January 27, 2025 at 06:02:05 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$t^2y'' + \left(-t^2 + t\right) y' - y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.207 (sec)

Writing the ode as

$$t^2y'' + \left(-t^2 + t\right) y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= -t^2 + t \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 - 2t + 3}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^2 - 2t + 3 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 - 2t + 3}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.392: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4t^2} - \frac{1}{2t}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2t} + \frac{1}{2t^2} + \frac{1}{2t^3} + \frac{1}{4t^4} - \frac{1}{4t^5} - \frac{3}{4t^6} - \frac{3}{4t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 - 2t + 3}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2t + 3}{4t^2}\right) \\ &= \frac{1}{4} + \frac{-2t + 3}{4t^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 - 2t + 3}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2t} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} - \frac{1}{2t} \\
&= \frac{t-1}{2t}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2} - \frac{1}{2t}\right)(0) + \left(\left(\frac{1}{2t^2}\right) + \left(\frac{1}{2} - \frac{1}{2t}\right)^2 - \left(\frac{t^2-2t+3}{4t^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega\, dt} \\
&= e^{\int \left(\frac{1}{2} - \frac{1}{2t}\right) dt} \\
&= \frac{e^{\frac{t}{2}}}{\sqrt{t}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-t^2+t}{t^2}\,dt} \\ &= z_1 e^{\frac{t}{2}-\frac{\ln(t)}{2}} \\ &= z_1\left(\frac{e^{\frac{t}{2}}}{\sqrt{t}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^t}{t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-t^2+t}{t^2}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{t-\ln(t)}}{(y_1)^2}\,dt \\ &= y_1\left(-(1+t)\,t\,e^{t-\ln(t)}e^{-2t}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{e^t}{t}\right) + c_2\left(\frac{e^t}{t}\left(-(1+t)\,t\,e^{t-\ln(t)}e^{-2t}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dt^2}y(t)\right)t^2 + (-t^2+t)\left(\frac{d}{dt}y(t)\right) - y(t) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dt^2}y(t)$
- Isolate 2nd derivative
  $\frac{d^2}{dt^2}y(t) = \frac{y(t)}{t^2} + \frac{(t-1)\left(\frac{d}{dt}y(t)\right)}{t}$
- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d^2}{dt^2}y(t) - \frac{(t-1)\left(\frac{d}{dt}y(t)\right)}{t} - \frac{y(t)}{t^2} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(t) = -\frac{t-1}{t}, P_3(t) = -\frac{1}{t^2}\right]$
  - $t \cdot P_2(t)$ is analytic at $t = 0$

$(t \cdot P_2(t))\Big|_{t=0} = 1$

- ◦ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $(t^2 \cdot P_3(t))\Big|_{t=0} = -1$

- ◦ $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $\left(\frac{d^2}{dt^2}y(t)\right)t^2 - t(t-1)\left(\frac{d}{dt}y(t)\right) - y(t) = 0$

- Assume series solution for $y(t)$

  $y(t) = \sum_{k=0}^{\infty} a_k t^{k+r}$

- □ Rewrite ODE with series expansions

  - ◦ Convert $t^m \cdot \left(\frac{d}{dt}y(t)\right)$ to series expansion for $m = 1..2$

    $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$

  - ◦ Shift index using $k- >k+1-m$

    $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$

  - ◦ Convert $t^2 \cdot \left(\frac{d^2}{dt^2}y(t)\right)$ to series expansion

    $t^2 \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)(-1+r)\, t^r + \left(\sum_{k=1}^{\infty} (a_k(k+r+1)(k+r-1) - a_{k-1}(k+r-1))\, t^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 1\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)(a_k(k+r+1) - a_{k-1}) = 0$

- Shift index using $k- >k+1$

  $(k+r)(a_{k+1}(k+2+r) - a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+2+r}$

- Recursion relation for $r = -1$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = -1$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k-1}, a_{k+1} = \frac{a_k}{k+1}\right]$

- Recursion relation for $r = 1$

  $a_{k+1} = \frac{a_k}{k+3}$

- Solution for $r = 1$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = \frac{a_k}{k+3}\right]$

- Combine solutions and rename parameters

  $\left[y(t) = \left(\sum_{k=0}^{\infty} a_k t^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+1}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 17

```
dsolve(t^2*diff(diff(y(t),t),t)+(-t^2+t)*diff(y(t),t)-y(t) = 0,y(t),singsol=all)
```

$$y = \frac{c_2 e^t + c_1 t + c_1}{t}$$

Mathematica DSolve solution

Solving time : 0.272 (sec)
Leaf size : 80

```
DSolve[{t^2*D[y[t],{t,2}]+(t-t^2)*D[y[t],t]-y[t]==0,{}},y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \exp\left(\int_1^t \left(1-\frac{1}{K[1]}\right) dK[1]\right) \left(\int_1^t \exp\left(-\int_1^{K[2]} \left(1-\frac{1}{K[1]}\right) dK[1]\right) c_1 dK[2]+c_2\right)$$

$$y(t) \to c_2 \exp\left(\int_1^t \left(1-\frac{1}{K[1]}\right) dK[1]\right)$$

2.1.206 Problem 209

Internal problem ID [9378]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 209
Date solved : `Monday, January 27, 2025 at 06:02:05 PM`
CAS classification : `[_Lienard]`

Solve

$$ty'' - \left(t^2 + 2\right) y' + ty = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.262 (sec)

Writing the ode as

$$ty'' + \left(-t^2 - 2\right) y' + ty = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= -t^2 - 2 \\ C &= t \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A} dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^4 - 2t^2 + 8}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^4 - 2t^2 + 8 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^4 - 2t^2 + 8}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.394: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 4 \\
&= -2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{t^2}{4} - \frac{1}{2} + \frac{2}{t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\
&= \sum_{i=0}^{1} a_i t^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $t^v = t^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{t}{2} - \frac{1}{2t} + \frac{7}{4t^3} + \frac{7}{4t^5} - \frac{21}{16t^7} - \frac{119}{16t^9} - \frac{189}{32t^{11}} + \frac{791}{32t^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i t^i \\ &= \frac{t}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{t^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^4 - 2t^2 + 8}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{t^2}{4} - \frac{1}{2}\right) + \left(\frac{2}{t^2}\right) \\ &= \frac{t^2}{4} - \frac{1}{2} + \frac{2}{t^2} \end{aligned}$$

We see that the coefficient of the term t in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{t}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 1\right) &&= -1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 1\right) &&= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^4 - 2t^2 + 8}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{t}{2}$	-1	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -1$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{t} + \left(\frac{t}{2}\right) \\ &= -\frac{1}{t} + \frac{t}{2} \\ &= -\frac{1}{t} + \frac{t}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{t} + \frac{t}{2}\right)(0) + \left(\left(\frac{1}{t^2} + \frac{1}{2}\right) + \left(-\frac{1}{t} + \frac{t}{2}\right)^2 - \left(\frac{t^4 - 2t^2 + 8}{4t^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= e^{\int \left(-\frac{1}{t} + \frac{t}{2}\right) dt} \\ &= \frac{e^{\frac{t^2}{4}}}{t} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-t^2-2}{t}\,dt} \\
&= z_1 e^{\frac{t^2}{4}+\ln(t)} \\
&= z_1\left(t\,\mathrm{e}^{\frac{t^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\frac{t^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-t^2-2}{t}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{\frac{t^2}{2}+2\ln(t)}}{(y_1)^2}\,dt \\
&= y_1\left(-t\,\mathrm{e}^{-\frac{t^2}{2}} + \frac{\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,t}{2}\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{\frac{t^2}{2}}\right) + c_2\left(\mathrm{e}^{\frac{t^2}{2}}\left(-t\,\mathrm{e}^{-\frac{t^2}{2}} + \frac{\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,t}{2}\right)}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dt^2}y(t)\right)t - (t^2+2)\left(\frac{d}{dt}y(t)\right) + ty(t) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dt^2}y(t)$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = -y(t) + \frac{(t^2+2)\left(\frac{d}{dt}y(t)\right)}{t}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dt^2}y(t) - \frac{(t^2+2)\left(\frac{d}{dt}y(t)\right)}{t} + y(t) = 0$

☐ Check to see if $t_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(t) = -\frac{t^2+2}{t}, P_3(t) = 1\right]$

- $t \cdot P_2(t)$ is analytic at $t = 0$

  $(t \cdot P_2(t))\Big|_{t=0} = -2$

- $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $(t^2 \cdot P_3(t))\Big|_{t=0} = 0$

- $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $\left(\frac{d^2}{dt^2}y(t)\right)t + (-t^2 - 2)\left(\frac{d}{dt}y(t)\right) + ty(t) = 0$

- Assume series solution for $y(t)$

  $y(t) = \sum_{k=0}^{\infty} a_k t^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $t \cdot y(t)$ to series expansion

    $t \cdot y(t) = \sum_{k=0}^{\infty} a_k t^{k+r+1}$

  - Shift index using $k- > k - 1$

    $t \cdot y(t) = \sum_{k=1}^{\infty} a_{k-1} t^{k+r}$

  - Convert $t^m \cdot \left(\frac{d}{dt}y(t)\right)$ to series expansion for $m = 0..2$

    $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$

  - Shift index using $k- > k + 1 - m$

    $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$

  - Convert $t \cdot \left(\frac{d^2}{dt^2}y(t)\right)$ to series expansion

    $t \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, t^{k+r-1}$

  - Shift index using $k- > k + 1$

    $t \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)(k+r)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-3+r)\, t^{-1+r} + a_1(1+r)(-2+r)\, t^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+r+1)(k-2+r) - a_{k-1}(k-2+r))\, t^k$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 3\}$

- Each term must be 0

  $a_1(1+r)(-2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k-2+r)(a_{k+1}(k+r+1) - a_{k-1}) = 0$

- Shift index using $k- > k + 1$

  $(k+r-1)(a_{k+2}(k+2+r) - a_k) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+2} = \frac{a_k}{k+2+r}$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{a_k}{k+2}$

- Solution for $r = 0$

  $$\left[y(t) = \sum_{k=0}^{\infty} a_k t^k, a_{k+2} = \frac{a_k}{k+2}, -2a_1 = 0\right]$$

- Recursion relation for $r = 3$

  $a_{k+2} = \frac{a_k}{k+5}$

- Solution for $r = 3$

  $$\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k+3}, a_{k+2} = \frac{a_k}{k+5}, 4a_1 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[y(t) = \left(\sum_{k=0}^{\infty} a_k t^k\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+3}\right), a_{k+2} = \frac{a_k}{k+2}, -2a_1 = 0, b_{k+2} = \frac{b_k}{5+k}, 4b_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 33

```
dsolve(t*diff(diff(y(t),t),t)-(t^2+2)*diff(y(t),t)+y(t)*t = 0,y(t),singsol=all)
```

$$y = \left(c_2\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,t}{2}\right) + c_1\right) e^{\frac{t^2}{2}} - 2c_2 t$$

Mathematica DSolve solution

Solving time : 0.15 (sec)
Leaf size : 56

```
DSolve[{t*D[y[t],{t,2}]-(t^2+2)*D[y[t],t]+t*y[t]==0,{}},y[t],t,IncludeSingularSolutions->Tru
```

$$y(t) \to \frac{1}{2}e\left(\sqrt{2\pi}c_2 e^{\frac{t^2}{2}}\mathrm{erf}\left(\frac{t}{\sqrt{2}}\right) + 2c_1 e^{\frac{t^2}{2}} - 2c_2 t\right)$$

2.1.207 Problem 210

Internal problem ID [9379]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 210
Date solved : `Monday, January 27, 2025 at 06:02:06 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$t^2y'' + t(t+1)\,y' - y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.205 (sec)

Writing the ode as

$$t^2y'' + \left(t^2+t\right)y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= t^2+t \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2+2t+3}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^2+2t+3 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2+2t+3}{4t^2}\right)z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.396: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{1}{2t} + \frac{3}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{2t} + \frac{1}{2t^2} - \frac{1}{2t^3} + \frac{1}{4t^4} + \frac{1}{4t^5} - \frac{3}{4t^6} + \frac{3}{4t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 + 2t + 3}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{2t + 3}{4t^2}\right) \\ &= \frac{1}{4} + \frac{2t + 3}{4t^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is 2. Dividing this by leading coefficient in t which is 4 gives $\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{2}\right) - (0) \\ &= \frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 + 2t + 3}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2t} + (-)\left(\frac{1}{2}\right) \\ &= -\frac{1}{2t} - \frac{1}{2} \\ &= -\frac{t+1}{2t} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{2t} - \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2t^2}\right) + \left(-\frac{1}{2t} - \frac{1}{2}\right)^2 - \left(\frac{t^2+2t+3}{4t^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= e^{\int \left(-\frac{1}{2t} - \frac{1}{2}\right) dt} \\ &= \frac{e^{-\frac{t}{2}}}{\sqrt{t}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{t^2+t}{t^2}\,dt} \\ &= z_1 e^{-\frac{t}{2}-\frac{\ln(t)}{2}} \\ &= z_1\left(\frac{\mathrm{e}^{-\frac{t}{2}}}{\sqrt{t}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-t}}{t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{t^2+t}{t^2}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{-t-\ln(t)}}{(y_1)^2}\,dt \\ &= y_1\left((-1+t)\,t\,\mathrm{e}^{-t-\ln(t)}\mathrm{e}^{2t}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\mathrm{e}^{-t}}{t}\right) + c_2\left(\frac{\mathrm{e}^{-t}}{t}\left((-1+t)\,t\,\mathrm{e}^{-t-\ln(t)}\mathrm{e}^{2t}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dt^2}y(t)\right)t^2 + t(t+1)\left(\frac{d}{dt}y(t)\right) - y(t) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dt^2}y(t)$
- Isolate 2nd derivative
  $\frac{d^2}{dt^2}y(t) = \frac{y(t)}{t^2} - \frac{(t+1)\left(\frac{d}{dt}y(t)\right)}{t}$
- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d^2}{dt^2}y(t) + \frac{(t+1)\left(\frac{d}{dt}y(t)\right)}{t} - \frac{y(t)}{t^2} = 0$

☐ Check to see if $t_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(t) = \frac{t+1}{t}, P_3(t) = -\frac{1}{t^2}\right]$
  - $t \cdot P_2(t)$ is analytic at $t = 0$

$(t \cdot P_2(t))\bigg|_{t=0} = 1$

- ○ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $(t^2 \cdot P_3(t))\bigg|_{t=0} = -1$

- ○ $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $\left(\frac{d^2}{dt^2}y(t)\right)t^2 + t(t+1)\left(\frac{d}{dt}y(t)\right) - y(t) = 0$

- Assume series solution for $y(t)$

  $y(t) = \sum_{k=0}^{\infty} a_k t^{k+r}$

- □ Rewrite ODE with series expansions

  - ○ Convert $t^m \cdot \left(\frac{d}{dt}y(t)\right)$ to series expansion for $m = 1..2$

    $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$

  - ○ Shift index using $k- >k+1-m$

    $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$

  - ○ Convert $t^2 \cdot \left(\frac{d^2}{dt^2}y(t)\right)$ to series expansion

    $t^2 \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)(-1+r)\, t^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)(k+r-1) + a_{k-1}(k+r-1)\right) t^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 1\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)(a_k(k+r+1) + a_{k-1}) = 0$

- Shift index using $k- >k+1$

  $(k+r)(a_{k+1}(k+2+r) + a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{k+2+r}$

- Recursion relation for $r = -1$

  $a_{k+1} = -\frac{a_k}{k+1}$

- Solution for $r = -1$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k-1}, a_{k+1} = -\frac{a_k}{k+1}\right]$

- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{a_k}{k+3}$

- Solution for $r = 1$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = -\frac{a_k}{k+3}\right]$

- Combine solutions and rename parameters

  $\left[y(t) = \left(\sum_{k=0}^{\infty} a_k t^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+1}\right), a_{k+1} = -\frac{a_k}{k+1}, b_{k+1} = -\frac{b_k}{k+3}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 20

```
dsolve(t^2*diff(diff(y(t),t),t)+t*(t+1)*diff(y(t),t)-y(t) = 0,y(t),singsol=all)
```

$$y = \frac{c_2 \mathrm{e}^{-t} + c_1(t-1)}{t}$$

Mathematica DSolve solution

Solving time : 0.625 (sec)
Leaf size : 54

```
DSolve[{t^2*D[y[t],{t,2}]+t*(t+1)*D[y[t],t]-y[t]==0,{}},y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{e^{-t-1}\left(\int_1^t e^{K[1]+1} c_1 K[1] dK[1] + c_2\right)}{t}$$
$$y(t) \to \frac{c_2 e^{-t-1}}{t}$$

2.1.208 Problem 211

Internal problem ID [9380]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 211
Date solved : `Monday, January 27, 2025 at 06:02:07 PM`
CAS classification : `[_Laguerre]`

Solve

$$ty'' - (4+t)\,y' + 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.253 (sec)

Writing the ode as

$$ty'' + (-4-t)\,y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= -4-t \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2+24}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^2+24 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2+24}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.398: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{6}{t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{6}{t^2} - \frac{36}{t^4} + \frac{432}{t^6} - \frac{6480}{t^8} + \frac{108864}{t^{10}} - \frac{1959552}{t^{12}} + \frac{36951552}{t^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{t^2 + 24}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{6}{t^2}\right) \\ &= \frac{1}{4} + \frac{6}{t^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is 0. Dividing this by leading coefficient in t which is 4 gives 0. Now b can be found.

$$\begin{aligned}b &= (0) - (0) \\ &= 0\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{\frac{1}{2}} - 0\right) &&= 0 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{\frac{1}{2}} - 0\right) &&= 0\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 + 24}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= 0 - (-2) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{2}{t} + (-)\left(\frac{1}{2}\right) \\ &= -\frac{2}{t} - \frac{1}{2} \\ &= -\frac{4+t}{2t} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 2$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = t^2 + a_1 t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (2) + 2\left(-\frac{2}{t} - \frac{1}{2}\right)(2t + a_1) + \left(\left(\frac{2}{t^2}\right) + \left(-\frac{2}{t} - \frac{1}{2}\right)^2 - \left(\frac{t^2+24}{4t^2}\right)\right) &= 0 \\ \frac{(a_1 - 6)\,t + 2a_0 - 4a_1}{t} &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 12, a_1 = 6\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = t^2 + 6t + 12$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= \left(t^2+6t+12\right) e^{\int \left(-\frac{2}{t}-\frac{1}{2}\right)dt} \\ &= \left(t^2+6t+12\right) e^{-\frac{t}{2}-2\ln(t)} \\ &= \frac{\left(t^2+6t+12\right) e^{-\frac{t}{2}}}{t^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4-t}{t}\, dt} \\ &= z_1 e^{\frac{t}{2}+2\ln(t)} \\ &= z_1 \left(t^2 e^{\frac{t}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = t^2+6t+12$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4-t}{t}\, dt}}{\left(y_1\right)^2}\, dt \\ &= y_1 \int \frac{e^{t+4\ln(t)}}{\left(y_1\right)^2}\, dt \\ &= y_1 \left(\frac{\left(t^2-6t+12\right) e^{t+4\ln(t)}}{\left(t^2+6t+12\right) t^4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(t^2+6t+12\right) + c_2 \left(t^2+6t+12\left(\frac{\left(t^2-6t+12\right) e^{t+4\ln(t)}}{\left(t^2+6t+12\right) t^4}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dt^2}y(t)\right) t - (t+4)\left(\frac{d}{dt}y(t)\right) + 2y(t) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dt^2}y(t)$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = -\frac{2y(t)}{t} + \frac{(t+4)\left(\frac{d}{dt}y(t)\right)}{t}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dt^2}y(t) - \frac{(t+4)\left(\frac{d}{dt}y(t)\right)}{t} + \frac{2y(t)}{t} = 0$

- ☐ Check to see if $t_0 = 0$ is a regular singular point
  - ◦ Define functions
    $\left[P_2(t) = -\frac{t+4}{t}, P_3(t) = \frac{2}{t}\right]$
  - ◦ $t \cdot P_2(t)$ is analytic at $t = 0$
    $\left(t \cdot P_2(t)\right)\Big|_{t=0} = -4$
  - ◦ $t^2 \cdot P_3(t)$ is analytic at $t = 0$
    $\left(t^2 \cdot P_3(t)\right)\Big|_{t=0} = 0$
  - ◦ $t = 0$is a regular singular point
    Check to see if $t_0 = 0$ is a regular singular point
    $t_0 = 0$
- • Multiply by denominators
  $\left(\frac{d^2}{dt^2}y(t)\right)t + (-t-4)\left(\frac{d}{dt}y(t)\right) + 2y(t) = 0$
- • Assume series solution for $y(t)$
  $y(t) = \sum_{k=0}^{\infty} a_k t^{k+r}$
- ☐ Rewrite ODE with series expansions
  - ◦ Convert $t^m \cdot \left(\frac{d}{dt}y(t)\right)$ to series expansion for $m = 0..1$
    $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$
  - ◦ Shift index using $k- >k+1-m$
    $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$
  - ◦ Convert $t \cdot \left(\frac{d^2}{dt^2}y(t)\right)$ to series expansion
    $t \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, t^{k+r-1}$
  - ◦ Shift index using $k- >k+1$
    $t \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, t^{k+r}$
    Rewrite ODE with series expansions
    $a_0 r(-5+r)\, t^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k-4+r) - a_k(k+r-2)\right) t^{k+r}\right) = 0$
- • $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-5+r) = 0$
- • Values of r that satisfy the indicial equation
  $r \in \{0, 5\}$
- • Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)(k-4+r) - a_k(k+r-2) = 0$
- • Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_k(k+r-2)}{(k+1+r)(k-4+r)}$
- • Recursion relation for $r = 0$; series terminates at $k = 2$
  $a_{k+1} = \frac{a_k(k-2)}{(k+1)(k-4)}$
- • Apply recursion relation for $k = 0$
  $a_1 = \frac{a_0}{2}$
- • Apply recursion relation for $k = 1$

$a_2 = \frac{a_1}{6}$

- Express in terms of a_0

  $a_2 = \frac{a_0}{12}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(t) = a_0 \cdot \left(1 + \frac{1}{2}t + \frac{1}{12}t^2\right)$

- Recursion relation for $r = 5$

  $a_{k+1} = \frac{a_k(k+3)}{(k+6)(k+1)}$

- Solution for $r = 5$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k+5}, a_{k+1} = \frac{a_k(k+3)}{(k+6)(k+1)}\right]$

- Combine solutions and rename parameters

  $\left[y(t) = a_0 \cdot \left(1 + \frac{1}{2}t + \frac{1}{12}t^2\right) + \left(\sum_{k=0}^{\infty} b_k t^{5+k}\right), b_{k+1} = \frac{b_k(k+3)}{(k+6)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 27

```
dsolve(t*diff(diff(y(t),t),t)-(4+t)*diff(y(t),t)+2*y(t) = 0,y(t),singsol=all)
```

$$y = c_1 \left(t^2 + 6t + 12\right) + c_2 \mathrm{e}^t \left(t^2 - 6t + 12\right)$$

Mathematica DSolve solution

Solving time : 0.08 (sec)
Leaf size : 87

```
DSolve[{t*D[y[t],{t,2}]-(4+t)*D[y[t],t]+2*y[t]==0,{}},y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{2e^{\frac{t+5}{2}}\sqrt{t}\left(\left(c_2t^2 - 6ic_1t + 12c_2\right)\cosh\left(\frac{t}{2}\right) + i\left(c_1\left(t^2+12\right) + 6ic_2t\right)\sinh\left(\frac{t}{2}\right)\right)}{\sqrt{\pi}\sqrt{-it}}$$

2.1.209 Problem 212

Internal problem ID [9381]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 212
Date solved : `Monday, January 27, 2025 at 06:02:07 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$t^2y'' + \left(t^2 - 3t\right) y' + 3y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.247 (sec)

Writing the ode as

$$t^2y'' + \left(t^2 - 3t\right) y' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= t^2 - 3t \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 - 6t + 3}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^2 - 6t + 3 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 - 6t + 3}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.400: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4t^2} - \frac{3}{2t}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{3}{2t} - \frac{3}{2t^2} - \frac{9}{2t^3} - \frac{63}{4t^4} - \frac{243}{4t^5} - \frac{999}{4t^6} - \frac{4293}{4t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{t^2 - 6t + 3}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-6t + 3}{4t^2}\right) \\ &= \frac{1}{4} + \frac{-6t + 3}{4t^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -6. Dividing this by leading coefficient in t which is 4 gives $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{3}{2}\right) - (0) \\ &= -\frac{3}{2}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &&= \frac{3}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 - 6t + 3}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{3}{2} - \left(\frac{3}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{t-c_1}\right) + (-)[\sqrt{r}]_\infty \\ &= \frac{3}{2t} + (-)\left(\frac{1}{2}\right) \\ &= \frac{3}{2t} - \frac{1}{2} \\ &= -\frac{t-3}{2t} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{3}{2t} - \frac{1}{2}\right)(0) + \left(\left(-\frac{3}{2t^2}\right) + \left(\frac{3}{2t} - \frac{1}{2}\right)^2 - \left(\frac{t^2-6t+3}{4t^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega\, dt} \\ &= e^{\int \left(\frac{3}{2t} - \frac{1}{2}\right) dt} \\ &= t^{3/2} e^{-\frac{t}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{t^2-3t}{t^2}\,dt} \\
&= z_1 e^{-\frac{t}{2}+\frac{3\ln(t)}{2}} \\
&= z_1\left(t^{3/2}\mathrm{e}^{-\frac{t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = t^3\mathrm{e}^{-t}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{t^2-3t}{t^2}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-t+3\ln(t)}}{(y_1)^2}\,dt \\
&= y_1\left(-\frac{\mathrm{e}^t}{2t^2}-\frac{\mathrm{e}^t}{2t}-\frac{\mathrm{Ei}_1(-t)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(t^3\mathrm{e}^{-t}\right) + c_2\left(t^3\mathrm{e}^{-t}\left(-\frac{\mathrm{e}^t}{2t^2}-\frac{\mathrm{e}^t}{2t}-\frac{\mathrm{Ei}_1(-t)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dt^2}y(t)\right)t^2 + (t^2-3t)\left(\frac{d}{dt}y(t)\right) + 3y(t) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dt^2}y(t)$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = -\frac{3y(t)}{t^2} - \frac{(-3+t)\left(\frac{d}{dt}y(t)\right)}{t}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d^2}{dt^2}y(t) + \frac{(-3+t)\left(\frac{d}{dt}y(t)\right)}{t} + \frac{3y(t)}{t^2} = 0$

☐ Check to see if $t_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(t) = \frac{-3+t}{t}, P_3(t) = \frac{3}{t^2}\right]$

- ○ $t \cdot P_2(t)$ is analytic at $t = 0$

$$\left(t \cdot P_2(t)\right)\bigg|_{t=0} = -3$$

- ∘ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

$$\left(t^2 \cdot P_3(t)\right)\bigg|_{t=0} = 3$$

- ∘ $t = 0$is a regular singular point

Check to see if $t_0 = 0$ is a regular singular point

$t_0 = 0$

- Multiply by denominators

$$\left(\frac{d^2}{dt^2}y(t)\right)t^2 + (-3+t)\,t\left(\frac{d}{dt}y(t)\right) + 3y(t) = 0$$

- Assume series solution for $y(t)$

$$y(t) = \sum_{k=0}^{\infty} a_k t^{k+r}$$

□ Rewrite ODE with series expansions

- ∘ Convert $t^m \cdot \left(\frac{d}{dt}y(t)\right)$ to series expansion for $m = 1..2$

$$t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,t^{k+r-1+m}$$

- ∘ Shift index using $k- >k+1-m$

$$t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,t^{k+r}$$

- ∘ Convert $t^2 \cdot \left(\frac{d^2}{dt^2}y(t)\right)$ to series expansion

$$t^2 \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,t^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+r)\,(-3+r)\,t^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)\,(k+r-3) + a_{k-1}(k+r-1)\right)t^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(-1+r)\,(-3+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{1, 3\}$

- Each term in the series must be 0, giving the recursion relation

$(k+r-1)\,(a_k(k+r-3) + a_{k-1}) = 0$

- Shift index using $k- >k+1$

$(k+r)\,(a_{k+1}(k-2+r) + a_k) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = -\frac{a_k}{k-2+r}$

- Recursion relation for $r = 1$

$a_{k+1} = -\frac{a_k}{k-1}$

- Series not valid for $r = 1$, division by 0 in the recursion relation at $k = 1$

$a_{k+1} = -\frac{a_k}{k-1}$

- Recursion relation for $r = 3$

$a_{k+1} = -\frac{a_k}{k+1}$

- Solution for $r = 3$

$$\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k+3}, a_{k+1} = -\frac{a_k}{k+1}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 34

```
dsolve(t^2*diff(diff(y(t),t),t)+(t^2-3*t)*diff(y(t),t)+3*y(t) = 0,y(t),singsol=all)
```

$$y = t\left(\mathrm{e}^{-t} \operatorname{Ei}_1\left(-t\right) c_2 t^2 + \mathrm{e}^{-t} c_1 t^2 + c_2 t + c_2\right)$$

Mathematica DSolve solution

Solving time : 48.836 (sec)
Leaf size : 50

```
DSolve[{t^2*D[y[t],{t,2}]+(t^2-3*t)*D[y[t],t]+3*y[t]==0,{}},y[t],t,IncludeSingularSolutions->T
```

$$y(t) \to e^{-t} t^3 \left(\int_1^t \frac{e^{K[1]} c_1}{K[1]^3} dK[1] + c_2\right)$$
$$y(t) \to c_2 e^{-t} t^3$$

2.1.210 Problem 213

Internal problem ID [9382]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 213
Date solved : `Monday, January 27, 2025 at 06:02:08 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$ty'' + ty' + 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.226 (sec)

Writing the ode as

$$ty'' + ty' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= t \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t-8}{4t} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t - 8 \\ t &= 4t \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t-8}{4t}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.402: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t$. There is a pole at $t = 0$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $t = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{2}{t} - \frac{4}{t^2} - \frac{16}{t^3} - \frac{80}{t^4} - \frac{448}{t^5} - \frac{2688}{t^6} - \frac{16896}{t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t-8}{4t} \\ &= Q + \frac{R}{4t} \\ &= \left(\frac{1}{4}\right) + \left(-\frac{2}{t}\right) \\ &= \frac{1}{4} - \frac{2}{t} \end{aligned}$$

Since the degree of t is 1, then we see that the coefficient of the term 1 in the remainder R is -8. Dividing this by leading coefficient in t which is 4 gives -2. Now b can be found.

$$\begin{aligned} b &= (-2) - (0) \\ &= -2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{-2}{\frac{1}{2}} - 0\right) = -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{-2}{\frac{1}{2}} - 0\right) = 2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t-8}{4t}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	-2	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= 2 - (1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{t} + (-)\left(\frac{1}{2}\right) \\ &= \frac{1}{t} - \frac{1}{2} \\ &= \frac{1}{t} - \frac{1}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 1$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{t} - \frac{1}{2}\right)(1) + \left(\left(-\frac{1}{t^2}\right) + \left(\frac{1}{t} - \frac{1}{2}\right)^2 - \left(\frac{t-8}{4t}\right) \right) &= 0 \\ \frac{2+a_0}{t} &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -2\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = -2 + t$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= (-2+t)\, e^{\int \left(\frac{1}{t} - \frac{1}{2}\right) dt} \\ &= (-2+t)\, e^{-\frac{t}{2} + \ln(t)} \\ &= (-2+t)\, t\, e^{-\frac{t}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{t}{t}\,dt} \\ &= z_1 e^{-\frac{t}{2}} \\ &= z_1\left(\mathrm{e}^{-\frac{t}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-t}(-2+t)\,t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{t}{t}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{-t}}{(y_1)^2}\,dt \\ &= y_1\left(-\frac{\mathrm{e}^t(-t+1)}{2(2-t)\,t} - \frac{\mathrm{Ei}_1(-t)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-t}(-2+t)\,t\right) + c_2\left(\mathrm{e}^{-t}(-2+t)\,t\left(-\frac{\mathrm{e}^t(-t+1)}{2(2-t)\,t} - \frac{\mathrm{Ei}_1(-t)}{2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dt^2}y(t)\right)t + t\left(\frac{d}{dt}y(t)\right) + 2y(t) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dt^2}y(t)$
- Isolate 2nd derivative
  $\frac{d^2}{dt^2}y(t) = -\frac{2y(t)}{t} - \frac{d}{dt}y(t)$
- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dt^2}y(t) + \frac{d}{dt}y(t) + \frac{2y(t)}{t} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(t) = 1, P_3(t) = \frac{2}{t}\right]$
  - $t \cdot P_2(t)$ is analytic at $t = 0$
    $\left.(t \cdot P_2(t))\right|_{t=0} = 0$

- $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $\left(t^2 \cdot P_3(t)\right)\Big|_{t=0} = 0$

- $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $\left(\frac{d^2}{dt^2}y(t)\right)t + t\left(\frac{d}{dt}y(t)\right) + 2y(t) = 0$

- Assume series solution for $y(t)$

  $y(t) = \sum_{k=0}^{\infty} a_k t^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $t \cdot \left(\frac{d}{dt}y(t)\right)$ to series expansion

  $t \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r}$

- Convert $t \cdot \left(\frac{d^2}{dt^2}y(t)\right)$ to series expansion

  $t \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, t^{k+r-1}$

- Shift index using $k- >k+1$

  $t \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-1+r)\, t^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k+r) + a_k(k+r+2)\right) t^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 1\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k+r) + a_k(k+r+2) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r+2)}{(k+1+r)(k+r)}$

- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{a_k(k+2)}{(k+1)k}$

- Solution for $r = 0$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^k, a_{k+1} = -\frac{a_k(k+2)}{(k+1)k}\right]$

- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{a_k(k+3)}{(k+2)(k+1)}$

- Solution for $r = 1$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = -\frac{a_k(k+3)}{(k+2)(k+1)}\right]$

- Combine solutions and rename parameters

  $\left[y(t) = \left(\sum_{k=0}^{\infty} a_k t^k\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+1}\right), a_{k+1} = -\frac{a_k(k+2)}{(k+1)k}, b_{k+1} = -\frac{b_k(k+3)}{(k+2)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 35

```
dsolve(t*diff(diff(y(t),t),t)+t*diff(y(t),t)+2*y(t) = 0,y(t),singsol=all)
```

$$y = tc_2 \mathrm{e}^{-t}(t-2)\,\mathrm{Ei}_1(-t) + c_1 \mathrm{e}^{-t}(t-2)\,t + c_2(t-1)$$

Mathematica DSolve solution

Solving time : `0.298` (sec)
Leaf size : 43

```
DSolve[{t*D[y[t],{t,2}]+t*D[y[t],t]+2*y[t]==0,{}},y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to e^{-t}(t-2)t\left(c_2 \int_1^t \frac{e^{K[1]}}{(K[1]-2)^2 K[1]^2} dK[1] + c_1\right)$$

2.1.211 Problem 214

Internal problem ID [9383]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 214
Date solved : `Monday, January 27, 2025 at 06:02:09 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$ty'' + \left(-t^2 + 1\right) y' + 4ty = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.657 (sec)

Writing the ode as

$$ty'' + \left(-t^2 + 1\right) y' + 4ty = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= -t^2 + 1 \\ C &= 4t \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^4 - 20t^2 - 1}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^4 - 20t^2 - 1 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^4 - 20t^2 - 1}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.404: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{t^2}{4} - 5 - \frac{1}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{1} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{t}{2} - \frac{5}{t} - \frac{101}{4t^3} - \frac{505}{2t^5} - \frac{50601}{16t^7} - \frac{355015}{8t^9} - \frac{21351501}{32t^{11}} - \frac{168167525}{16t^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i t^i \\ &= \frac{t}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{t^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{t^4 - 20t^2 - 1}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{t^2}{4} - 5\right) + \left(-\frac{1}{4t^2}\right) \\ &= \frac{t^2}{4} - 5 - \frac{1}{4t^2}\end{aligned}$$

We see that the coefficient of the term t in the quotient is -5. Now b can be found.

$$\begin{aligned}b &= (-5) - (0) \\ &= -5\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{t}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-5}{\frac{1}{2}} - 1\right) &= -\frac{11}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-5}{\frac{1}{2}} - 1\right) &= \frac{9}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^4 - 20t^2 - 1}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{t}{2}$	$-\frac{11}{2}$	$\frac{9}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{9}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{9}{2} - \left(\frac{1}{2}\right) \\ &= 4 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2t} + (-)\left(\frac{t}{2}\right) \\ &= \frac{1}{2t} - \frac{t}{2} \\ &= \frac{1}{2t} - \frac{t}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 4$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(12t^2 + 6ta_3 + 2a_2\right) + 2\left(\frac{1}{2t} - \frac{t}{2}\right)\left(4t^3 + 3a_3t^2 + 2a_2t + a_1\right) + \left(\left(-\frac{1}{2t^2} - \frac{1}{2}\right) + \left(\frac{1}{2t} - \frac{t}{2}\right)^2 - \left(\frac{t^4 - 20t}{4t^2}\right.\right.$$

$$\frac{t^4 a_3 + 2(8+a_2)\,t^3 + 3(a_1 + 3a_3)\,t^2 + 4(a_0 + a_2)\,t}{t}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 8, a_1 = 0, a_2 = -8, a_3 = 0\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = t^4 - 8t^2 + 8$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \left(t^4 - 8t^2 + 8\right) \mathrm{e}^{\int \left(\frac{1}{2t} - \frac{t}{2}\right) dt} \\
&= \left(t^4 - 8t^2 + 8\right) \mathrm{e}^{-\frac{t^2}{4} + \frac{\ln(t)}{2}} \\
&= \left(t^4 - 8t^2 + 8\right) \sqrt{t}\, \mathrm{e}^{-\frac{t^2}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-t^2+1}{t} \, dt} \\
&= z_1 e^{\frac{t^2}{4} - \frac{\ln(t)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{t^2}{4}}}{\sqrt{t}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = t^4 - 8t^2 + 8$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-t^2+1}{t} \, dt}}{(y_1)^2} \, dt \\
&= y_1 \int \frac{e^{\frac{t^2}{2} - \ln(t)}}{(y_1)^2} \, dt \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{t^2}{2} - \ln(t)}}{\left(t^4 - 8t^2 + 8\right)^2} dt\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(t^4 - 8t^2 + 8\right) + c_2 \left(t^4 - 8t^2 + 8 \left(\int \frac{\mathrm{e}^{\frac{t^2}{2} - \ln(t)}}{\left(t^4 - 8t^2 + 8\right)^2} dt\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dt^2}y(t)\right)t+(-t^2+1)\left(\frac{d}{dt}y(t)\right)+4ty(t)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dt^2}y(t)$

- Isolate 2nd derivative

$\frac{d^2}{dt^2}y(t)=-4y(t)+\frac{(t^2-1)\left(\frac{d}{dt}y(t)\right)}{t}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dt^2}y(t)-\frac{(t^2-1)\left(\frac{d}{dt}y(t)\right)}{t}+4y(t)=0$

□ Check to see if $t_0=0$ is a regular singular point

- Define functions

$\left[P_2(t)=-\frac{t^2-1}{t},P_3(t)=4\right]$

- $t\cdot P_2(t)$ is analytic at $t=0$

$(t\cdot P_2(t))\bigg|_{t=0}=1$

- $t^2\cdot P_3(t)$ is analytic at $t=0$

$(t^2\cdot P_3(t))\bigg|_{t=0}=0$

- $t=0$is a regular singular point

Check to see if $t_0=0$ is a regular singular point

$t_0=0$

- Multiply by denominators

$\left(\frac{d^2}{dt^2}y(t)\right)t+(-t^2+1)\left(\frac{d}{dt}y(t)\right)+4ty(t)=0$

- Assume series solution for $y(t)$

$y(t)=\sum_{k=0}^{\infty}a_k t^{k+r}$

□ Rewrite ODE with series expansions

- Convert $t\cdot y(t)$ to series expansion

$t\cdot y(t)=\sum_{k=0}^{\infty}a_k t^{k+r+1}$

- Shift index using $k->k-1$

$t\cdot y(t)=\sum_{k=1}^{\infty}a_{k-1}t^{k+r}$

- Convert $t^m\cdot\left(\frac{d}{dt}y(t)\right)$ to series expansion for $m=0..2$

$t^m\cdot\left(\frac{d}{dt}y(t)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,t^{k+r-1+m}$

- Shift index using $k->k+1-m$

$t^m\cdot\left(\frac{d}{dt}y(t)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,t^{k+r}$

- Convert $t\cdot\left(\frac{d^2}{dt^2}y(t)\right)$ to series expansion

$t\cdot\left(\frac{d^2}{dt^2}y(t)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,t^{k+r-1}$

- Shift index using $k->k+1$

$t\cdot\left(\frac{d^2}{dt^2}y(t)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+r+1)(k+r)\,t^{k+r}$

Rewrite ODE with series expansions

$$a_0 r^2 t^{-1+r} + a_1 (1+r)^2 t^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)^2 - a_{k-1}(k-5+r)\right) t^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 0$
- Each term must be 0
  $a_1 (1+r)^2 = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1)^2 - a_{k-1}(k-5) = 0$
- Shift index using $k- >k+1$
  $a_{k+2}(k+2)^2 - a_k(k-4) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{a_k(k-4)}{(k+2)^2}$
- Recursion relation for $r = 0$; series terminates at $k = 4$
  $a_{k+2} = \frac{a_k(k-4)}{(k+2)^2}$
- Solution for $r = 0$
  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^k, a_{k+2} = \frac{a_k(k-4)}{(k+2)^2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solut
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returnin
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.030 (sec)
Leaf size : 21

```
dsolve(t*diff(diff(y(t),t),t)+(-t^2+1)*diff(y(t),t)+4*y(t)*t = 0,y(t),singsol=all)
```

$$y = \frac{(t^4 - 8t^2 + 8)(c_1 + 2c_2)}{8}$$

Mathematica DSolve solution

Solving time : 0.622 (sec)
Leaf size : 65

```
DSolve[{t*D[y[t],{t,2}]+(1-t^2)*D[y[t],t]+4*t*y[t]==0,{}},y[t],t,IncludeSingularSolutions->T
```

$$y(t) \to \sqrt{e}(t^4 - 8t^2 + 8) \left(c_2 \int_1^t \frac{e^{\frac{K[1]^2}{2} - 1}}{K[1] (K[1]^4 - 8K[1]^2 + 8)^2} dK[1] + c_1 \right)$$

2.1.212 Problem 215

Internal problem ID [9384]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 215
Date solved : `Monday, January 27, 2025 at 06:02:10 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$t^2y'' - t(1+t)\,y' + y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.225 (sec)

Writing the ode as

$$t^2y'' + \left(-t^2 - t\right) y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= -t^2 - t \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 + 2t - 1}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^2 + 2t - 1 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 + 2t - 1}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.406: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{4t^2} + \frac{1}{2t}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{2t} - \frac{1}{2t^2} + \frac{1}{2t^3} - \frac{3}{4t^4} + \frac{5}{4t^5} - \frac{9}{4t^6} + \frac{17}{4t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 + 2t - 1}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{2t - 1}{4t^2}\right) \\ &= \frac{1}{4} + \frac{2t - 1}{4t^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is 2. Dividing this by leading coefficient in t which is 4 gives $\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{2}\right) - (0) \\ &= \frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 + 2t - 1}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^+) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{t-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{2t} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2} + \frac{1}{2t} \\ &= \frac{1+t}{2t} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2} + \frac{1}{2t}\right)(0) + \left(\left(-\frac{1}{2t^2}\right) + \left(\frac{1}{2} + \frac{1}{2t}\right)^2 - \left(\frac{t^2+2t-1}{4t^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega\, dt} \\ &= \mathrm{e}^{\int \left(\frac{1}{2} + \frac{1}{2t}\right) dt} \\ &= \sqrt{t}\, \mathrm{e}^{\frac{t}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-t^2-t}{t^2}\,dt} \\
&= z_1 e^{\frac{t}{2}+\frac{\ln(t)}{2}} \\
&= z_1\left(\sqrt{t}\,\mathrm{e}^{\frac{t}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = t\,\mathrm{e}^t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-t^2-t}{t^2}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{t+\ln(t)}}{(y_1)^2}\,dt \\
&= y_1(-\operatorname{Ei}_1(t))
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(t\,\mathrm{e}^t\right) + c_2\left(t\,\mathrm{e}^t(-\operatorname{Ei}_1(t))\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dt^2}y(t)\right)t^2 - t(t+1)\left(\frac{d}{dt}y(t)\right) + y(t) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dt^2}y(t)$
- Isolate 2nd derivative
  $\frac{d^2}{dt^2}y(t) = \frac{(t+1)\left(\frac{d}{dt}y(t)\right)}{t} - \frac{y(t)}{t^2}$
- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d^2}{dt^2}y(t) + \frac{y(t)}{t^2} - \frac{(t+1)\left(\frac{d}{dt}y(t)\right)}{t} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(t) = -\frac{t+1}{t}, P_3(t) = \frac{1}{t^2}\right]$
  - $t \cdot P_2(t)$ is analytic at $t = 0$
    $\left(t \cdot P_2(t)\right)\Big|_{t=0} = -1$
  - $t^2 \cdot P_3(t)$ is analytic at $t = 0$

$$\left(t^2 \cdot P_3(t)\right)\Big|_{t=0} = 1$$

- ◦ $t = 0$is a regular singular point
  Check to see if $t_0 = 0$ is a regular singular point
  $t_0 = 0$
- Multiply by denominators
  $\left(\frac{d^2}{dt^2}y(t)\right)t^2 - t(t+1)\left(\frac{d}{dt}y(t)\right) + y(t) = 0$
- Assume series solution for $y(t)$
  $y(t) = \sum_{k=0}^{\infty} a_k t^{k+r}$
- □ Rewrite ODE with series expansions
  - ◦ Convert $t^m \cdot \left(\frac{d}{dt}y(t)\right)$ to series expansion for $m = 1..2$
    $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$
  - ◦ Shift index using $k- >k+1-m$
    $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$
  - ◦ Convert $t^2 \cdot \left(\frac{d^2}{dt^2}y(t)\right)$ to series expansion
    $t^2 \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, t^{k+r}$

  Rewrite ODE with series expansions
  $a_0(-1+r)^2 t^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)^2 - a_{k-1}(k+r-1)\right) t^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 1$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-1)\left(a_k(k+r-1) - a_{k-1}\right) = 0$
- Shift index using $k- >k+1$
  $(k+r)\left(a_{k+1}(k+r) - a_k\right) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_k}{k+r}$
- Recursion relation for $r = 1$
  $a_{k+1} = \frac{a_k}{k+1}$
- Solution for $r = 1$
  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = \frac{a_k}{k+1}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
```

```
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 15

```
dsolve(t^2*diff(diff(y(t),t),t)-t*(t+1)*diff(y(t),t)+y(t) = 0,y(t),singsol=all)
```

$$y = e^t t(c_1 + c_2 \operatorname{Ei}_1(t))$$

Mathematica DSolve solution

Solving time : 35.864 (sec)
Leaf size : 44

```
DSolve[{t^2*D[y[t],{t,2}]-t*(1+t)*D[y[t],t]+y[t]==0,{}},y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to e^t t \left(\int_1^t \frac{e^{-K[1]} c_1}{K[1]} dK[1] + c_2 \right)$$
$$y(t) \to c_2 e^t t$$

2.1.213 Problem 216

Internal problem ID [9385]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 216
Date solved : `Monday, January 27, 2025 at 06:02:10 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + 4xy' + \left(4x^2 + 6\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.162 (sec)

Writing the ode as

$$y'' + 4xy' + \left(4x^2 + 6\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4x \\ C &= 4x^2 + 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.408: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x}{1}\, dx} \\ &= z_1 e^{-x^2} \\ &= z_1 \left(\mathrm{e}^{-x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x^2} \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2x^2}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-x^2}\cos(2x)\right) + c_2 \left(\mathrm{e}^{-x^2}\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + 4x\left(\frac{d}{dx}y(x)\right) + (4x^2+6)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k->k-m$

  $x^m \cdot y(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

  $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k->k+2$

  $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 + 6a_0 + (6a_3 + 10a_1)\,x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 2a_k(2k+3) + 4a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 + 6a_0 = 0, 6a_3 + 10a_1 = 0]$

- Solve for the dependent coefficient(s)

  $\left\{a_2 = -3a_0, a_3 = -\frac{5a_1}{3}\right\}$

- Each term in the series must be 0, giving the recursion relation
  $$(k^2+3k+2)\, a_{k+2}+4a_k k+6a_k+4a_{k-2}=0$$
- Shift index using $k->k+2$
  $$\left((k+2)^2+3k+8\right) a_{k+4}+4a_{k+2}(k+2)+6a_{k+2}+4a_k=0$$
- Recursion relation that defines the series solution to the ODE
  $$\left[y(x)=\sum_{k=0}^{\infty} a_k x^k, a_{k+4}=-\frac{2(2ka_{k+2}+2a_k+7a_{k+2})}{k^2+7k+12}, a_2=-3a_0, a_3=-\frac{5a_1}{3}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.016` (sec)
Leaf size : 24

```
dsolve(diff(diff(y(x),x),x)+4*diff(y(x),x)*x+(4*x^2+6)*y(x) = 0,y(x),singsol=all)
```

$$y=\mathrm{e}^{-x^2}\left(\cos(2x)\, c_1+\sin(2x)\, c_2\right)$$

Mathematica DSolve solution

Solving time : `0.044` (sec)
Leaf size : 37

```
DSolve[{D[y[x],{x,2}]+4*x*D[y[x],x]+(4*x^2+6)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tru
```

$$y(x)\to \frac{1}{4}e^{-x(x+2i)}\left(4c_1-ic_2e^{4ix}\right)$$

2.1.214 Problem 217

Internal problem ID [9386]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 217
Date solved : `Monday, January 27, 2025 at 06:02:11 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-z^2+1\right) y''-3zy'+\lambda y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.474 (sec)

Writing the ode as

$$\left(-z^2+1\right) y''-3zy'+\lambda y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -z^2+1 \\ B &= -3z \\ C &= \lambda \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(z)=ye^{\int \frac{B}{2A}\,dz}$$

Then (2) becomes

$$z''(z)=rz(z) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{4\lambda\, z^2+3z^2-4\lambda-6}{4\left(z^2-1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4\lambda\, z^2+3z^2-4\lambda-6 \\ t &= 4\left(z^2-1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(z)=\left(\frac{4\lambda\, z^2+3z^2-4\lambda-6}{4\left(z^2-1\right)^2}\right) z(z) \tag{7}$$

Equation (7) is now solved. After finding $z(z)$ then y is found using the inverse transformation

$$y = z(z)\, e^{-\int \frac{B}{2A}\, dz}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.410: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(z^2-1)^2$. There is a pole at $z = 1$ of order 2. There is a pole at $z = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16\,(z-1)^2} + \frac{\frac{9}{16} + \frac{\lambda}{2}}{z-1} - \frac{3}{16\,(z+1)^2} + \frac{-\frac{\lambda}{2} - \frac{9}{16}}{z+1}$$

For the pole at $z = 1$ let b be the coefficient of $\frac{1}{(z-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

For the pole at $z = -1$ let b be the coefficient of $\frac{1}{(z+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{z^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{4\lambda\, z^2 + 3z^2 - 4\lambda - 6}{4\left(z^2 - 1\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 1$. Hence

$$\begin{aligned} E_\infty &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{2\} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
1	2	$\{1, 2, 3\}$
-1	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 1,\ e_2 = 1,\ e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(z)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(2 - (1 + (1))) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{z - c} \\ &= \frac{1}{2}\left(\frac{1}{(z - (1))} + \frac{1}{(z - (-1))}\right) \\ &= \frac{1}{2z - 2} + \frac{1}{2z + 2} \end{aligned}$$

Now we search for a monic polynomial $p(z)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(z)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2z - 2} + \frac{1}{2z + 2} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{2z-2} + \frac{1}{2z+2}\right) w + \frac{-4\lambda\, z^2 - 3z^2 + 4\lambda + 4}{4\left(z^2-1\right)^2} = 0$$

Solving for ω gives

$$\omega = \frac{z + 2\sqrt{\left(z^2-1\right)\left(\lambda+1\right)}}{2\left(z-1\right)\left(z+1\right)}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(z) &= e^{\int \omega \, dz} \\ &= e^{\int \frac{z+2\sqrt{\left(z^2-1\right)\left(\lambda+1\right)}}{2(z-1)(z+1)} dz} \\ &= \left(z^2-1\right)^{1/4} \left(\frac{\sqrt{\left(z^2-1\right)\left(\lambda+1\right)}\,\sqrt{\lambda+1} + \lambda z + z}{\sqrt{\lambda+1}}\right)^{\sqrt{\lambda+1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dz} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-3z}{-z^2+1}\, dz} \\ &= z_1 e^{-\frac{3\ln(z-1)}{4} - \frac{3\ln(z+1)}{4}} \\ &= z_1 \left(\frac{1}{\left(z-1\right)^{3/4}\left(z+1\right)^{3/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(z^2-1\right)^{1/4}\left(\sqrt{\lambda+1}\left(z+\sqrt{z^2-1}\right)\right)^{\sqrt{\lambda+1}}}{\left(z-1\right)^{3/4}\left(z+1\right)^{3/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dz}}{y_1^2}\, dz$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3z}{-z^2+1}\, dz}}{\left(y_1\right)^2}\, dz \\ &= y_1 \int \frac{e^{-\frac{3\ln(z-1)}{2} - \frac{3\ln(z+1)}{2}}}{\left(y_1\right)^2}\, dz \\ &= y_1 \left(-\frac{\left(\sqrt{\lambda+1}\left(z+\sqrt{z^2-1}\right)\right)^{-2\sqrt{\lambda+1}}}{2\sqrt{\lambda+1}}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\left(z^2-1\right)^{1/4}\left(\sqrt{\lambda+1}\left(z+\sqrt{z^2-1}\right)\right)^{\sqrt{\lambda+1}}}{\left(z-1\right)^{3/4}\left(z+1\right)^{3/4}}\right) + c_2 \left(\frac{\left(z^2-1\right)^{1/4}\left(\sqrt{\lambda+1}\left(z+\sqrt{z^2-1}\right)\right)^{\sqrt{\lambda+1}}}{\left(z-1\right)^{3/4}\left(z+1\right)^{3/4}}\left(-\frac{\left(\sqrt{\lambda+\right.}}{}\right.\right. \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$(-z^2+1)\left(\frac{d^2}{dz^2}y(z)\right)-3z\left(\frac{d}{dz}y(z)\right)+\lambda y(z)=0$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dz^2}y(z)$
- Isolate 2nd derivative
$\frac{d^2}{dz^2}y(z)=\frac{\lambda y(z)}{z^2-1}-\frac{3z\left(\frac{d}{dz}y(z)\right)}{z^2-1}$
- Group terms with $y(z)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
$\frac{d^2}{dz^2}y(z)+\frac{3z\left(\frac{d}{dz}y(z)\right)}{z^2-1}-\frac{\lambda y(z)}{z^2-1}=0$

☐ Check to see if z_0 is a regular singular point
  - Define functions
  $\left[P_2(z)=\frac{3z}{z^2-1}, P_3(z)=-\frac{\lambda}{z^2-1}\right]$
  - $(z+1)\cdot P_2(z)$ is analytic at $z=-1$
  $\left((z+1)\cdot P_2(z)\right)\Big|_{z=-1}=\frac{3}{2}$
  - $(z+1)^2\cdot P_3(z)$ is analytic at $z=-1$
  $\left((z+1)^2\cdot P_3(z)\right)\Big|_{z=-1}=0$
  - $z=-1$is a regular singular point
  Check to see if z_0 is a regular singular point
  $z_0=-1$
- Multiply by denominators
$(z^2-1)\left(\frac{d^2}{dz^2}y(z)\right)+3z\left(\frac{d}{dz}y(z)\right)-\lambda y(z)=0$
- Change variables using $z=u-1$ so that the regular singular point is at $u=0$
$(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(3u-3)\left(\frac{d}{du}y(u)\right)-\lambda y(u)=0$
- Assume series solution for $y(u)$
$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$
  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$
  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$
  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$
  - Shift index using $k->k+2-m$
  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$
  Rewrite ODE with series expansions
  $-a_0r(1+2r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-a_{k+1}(k+1+r)(2k+3+2r)+a_k(k^2+2kr+r^2+2k-\lambda+2r)\right)\right.$
- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-r(1+2r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,-\frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-2(k+1+r)\left(k+r+\frac{3}{2}\right)a_{k+1}+\left(k^2+(2r+2)\,k+r^2+2r-\lambda\right)a_k=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{(k^2+2kr+r^2+2k-\lambda+2r)\,a_k}{(k+1+r)\,(2k+3+2r)}$$

- Recursion relation for $r=0$

$$a_{k+1}=\frac{(k^2+2k-\lambda)\,a_k}{(k+1)\,(2k+3)}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty}a_k u^k,a_{k+1}=\frac{(k^2+2k-\lambda)\,a_k}{(k+1)\,(2k+3)}\right]$$

- Revert the change of variables $u=z+1$

$$\left[y(z)=\sum_{k=0}^{\infty}a_k(z+1)^k,a_{k+1}=\frac{(k^2+2k-\lambda)\,a_k}{(k+1)\,(2k+3)}\right]$$

- Recursion relation for $r=-\frac{1}{2}$

$$a_{k+1}=\frac{(k^2+k-\lambda-\frac{3}{4})\,a_k}{(k+\frac{1}{2})\,(2k+2)}$$

- Solution for $r=-\frac{1}{2}$

$$\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k-\frac{1}{2}},a_{k+1}=\frac{(k^2+k-\lambda-\frac{3}{4})\,a_k}{(k+\frac{1}{2})\,(2k+2)}\right]$$

- Revert the change of variables $u=z+1$

$$\left[y(z)=\sum_{k=0}^{\infty}a_k(z+1)^{k-\frac{1}{2}},a_{k+1}=\frac{(k^2+k-\lambda-\frac{3}{4})\,a_k}{(k+\frac{1}{2})\,(2k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y(z)=\left(\sum_{k=0}^{\infty}a_k(z+1)^k\right)+\left(\sum_{k=0}^{\infty}b_k(z+1)^{k-\frac{1}{2}}\right),a_{k+1}=\frac{(k^2+2k-\lambda)\,a_k}{(k+1)\,(2k+3)},b_{k+1}=\frac{(k^2+k-\lambda-\frac{3}{4})\,b_k}{(k+\frac{1}{2})\,(2k+2)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.073 (sec)
Leaf size : 49

```
dsolve((-z^2+1)*diff(diff(y(z),z),z)-3*z*diff(y(z),z)+lambda*y(z) = 0,y(z),singsol=all)
```

$$y(z)=\frac{c_1\left(z+\sqrt{z^2-1}\right)^{\sqrt{\lambda+1}}+c_2\left(z+\sqrt{z^2-1}\right)^{-\sqrt{\lambda+1}}}{\sqrt{z^2-1}}$$

Mathematica DSolve solution

Solving time : 0.04 (sec)
Leaf size : 54

```
DSolve[{(1-z^2)*D[y[z],{z,2}]-3*z*D[y[z],z]+\[Lambda]*y[z]==0,{}},y[z],z,IncludeSingularSolu
```

$$y(z) \to \frac{c_1 P_{\sqrt{\lambda+1}-\frac{1}{2}}^{\frac{1}{2}}(z) + c_2 Q_{\sqrt{\lambda+1}-\frac{1}{2}}^{\frac{1}{2}}(z)}{\sqrt[4]{z^2-1}}$$

2.1.215 Problem 218

Internal problem ID [9387]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 218
Date solved : `Monday, January 27, 2025 at 06:02:12 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4zy'' + 2(1 - z)\, y' - y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.237 (sec)

Writing the ode as

$$\begin{aligned} 4zy'' + (-2z + 2)\, y' - y &= 0 &(1)\\ Ay'' + By' + Cy &= 0 &(2) \end{aligned}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4z\\ B &= -2z + 2 &(3)\\ C &= -1 \end{aligned}$$

Applying the Liouville transformation on the dependent variable gives

$$z(z) = y e^{\int \frac{B}{2A}\, dz}$$

Then (2) becomes

$$z''(z) = rz(z) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} &(5)\\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{z^2 + 2z - 3}{16z^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= z^2 + 2z - 3\\ t &= 16z^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(z) = \left(\frac{z^2 + 2z - 3}{16z^2}\right) z(z) \tag{7}$$

Equation (7) is now solved. After finding $z(z)$ then y is found using the inverse transformation

$$y = z(z)\, e^{-\int \frac{B}{2A}\, dz}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.412: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16z^2$. There is a pole at $z = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16} + \frac{1}{8z} - \frac{3}{16z^2}$$

For the pole at $z = 0$ let b be the coefficient of $\frac{1}{z^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving z^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i z^i \\ &= \sum_{i=0}^{0} a_i z^i \end{aligned} \tag{8}$$

Let a be the coefficient of $z^v = z^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{4} + \frac{1}{4z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{z^4} + \frac{2}{z^5} - \frac{9}{2z^6} + \frac{21}{2z^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i z^i \\ &= \frac{1}{4}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $z^{v-1} = z^{-1} = \frac{1}{z}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{16}$$

This shows that the coefficient of $\frac{1}{z}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{z}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{z}$ in r will be the coefficient in R of the term in z of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{z^2 + 2z - 3}{16z^2} \\ &= Q + \frac{R}{16z^2} \\ &= \left(\frac{1}{16}\right) + \left(\frac{2z - 3}{16z^2}\right) \\ &= \frac{1}{16} + \frac{2z - 3}{16z^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term z in the remainder R is 2. Dividing this by leading coefficient in t which is 16 gives $\frac{1}{8}$. Now b can be found.

$$\begin{aligned}b &= \left(\frac{1}{8}\right) - (0) \\ &= \frac{1}{8}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{4} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{1}{8}}{\frac{1}{4}} - 0\right) &= \frac{1}{4} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{1}{8}}{\frac{1}{4}} - 0\right) &= -\frac{1}{4}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{z^2 + 2z - 3}{16z^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= \frac{1}{4} - \left(\frac{1}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{z-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{z-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{4z} + \left(\frac{1}{4}\right) \\ &= \frac{1}{4} + \frac{1}{4z} \\ &= \frac{z+1}{4z} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(z)$ of degree $d = 0$ to solve the ode. The polynomial $p(z)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(z) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{4} + \frac{1}{4z}\right)(0) + \left(\left(-\frac{1}{4z^2}\right) + \left(\frac{1}{4} + \frac{1}{4z}\right)^2 - \left(\frac{z^2+2z-3}{16z^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(z) &= p e^{\int \omega \, dz} \\ &= e^{\int \left(\frac{1}{4} + \frac{1}{4z}\right) dz} \\ &= z^{1/4} e^{\frac{z}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dz} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2z+2}{4z}\,dz} \\
&= z_1 e^{\frac{z}{4}-\frac{\ln(z)}{4}} \\
&= z_1\left(\frac{e^{\frac{z}{4}}}{z^{1/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = e^{\frac{z}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dz}}{y_1^2}\,dz$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2z+2}{4z}\,dz}}{(y_1)^2}\,dz \\
&= y_1 \int \frac{e^{\frac{z}{2}-\frac{\ln(z)}{2}}}{(y_1)^2}\,dz \\
&= y_1\left(\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,\sqrt{z}}{2}\right)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(e^{\frac{z}{2}}\right) + c_2\left(e^{\frac{z}{2}}\left(\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,\sqrt{z}}{2}\right)\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4z\left(\frac{d^2}{dz^2}y(z)\right) + 2(1-z)\left(\frac{d}{dz}y(z)\right) - y(z) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dz^2}y(z)$

- Isolate 2nd derivative

  $\frac{d^2}{dz^2}y(z) = \frac{y(z)}{4z} + \frac{(z-1)\left(\frac{d}{dz}y(z)\right)}{2z}$

- Group terms with $y(z)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d^2}{dz^2}y(z) - \frac{(z-1)\left(\frac{d}{dz}y(z)\right)}{2z} - \frac{y(z)}{4z} = 0$

☐ Check to see if $z_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(z) = -\frac{z-1}{2z}, P_3(z) = -\frac{1}{4z}\right]$

  ◦ $z \cdot P_2(z)$ is analytic at $z = 0$

$(z \cdot P_2(z)) \Big|_{z=0} = \frac{1}{2}$

- ◦ $z^2 \cdot P_3(z)$ is analytic at $z = 0$

  $(z^2 \cdot P_3(z)) \Big|_{z=0} = 0$

- ◦ $z = 0$is a regular singular point

  Check to see if $z_0 = 0$ is a regular singular point

  $z_0 = 0$

- Multiply by denominators

  $4z\left(\frac{d^2}{dz^2}y(z)\right) + (-2z+2)\left(\frac{d}{dz}y(z)\right) - y(z) = 0$

- Assume series solution for $y(z)$

  $y(z) = \sum_{k=0}^{\infty} a_k z^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $z^m \cdot \left(\frac{d}{dz}y(z)\right)$ to series expansion for $m = 0..1$

  $z^m \cdot \left(\frac{d}{dz}y(z)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, z^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

  $z^m \cdot \left(\frac{d}{dz}y(z)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, z^{k+r}$

- ◦ Convert $z \cdot \left(\frac{d^2}{dz^2}y(z)\right)$ to series expansion

  $z \cdot \left(\frac{d^2}{dz^2}y(z)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, z^{k+r-1}$

- ◦ Shift index using $k->k+1$

  $z \cdot \left(\frac{d^2}{dz^2}y(z)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, z^{k+r}$

  Rewrite ODE with series expansions

  $2a_0r(-1+2r)\, z^{-1+r} + \left(\sum_{k=0}^{\infty} (2a_{k+1}(k+1+r)(2k+2r+1) - a_k(2k+2r+1))\, z^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $2r(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r+\frac{1}{2}\right)\left(a_{k+1}(k+1+r) - \frac{a_k}{2}\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{2(k+1+r)}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{2(k+1)}$

- Solution for $r = 0$

  $\left[y(z) = \sum_{k=0}^{\infty} a_k z^k, a_{k+1} = \frac{a_k}{2(k+1)}\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = \frac{a_k}{2\left(k+\frac{3}{2}\right)}$

- Solution for $r = \frac{1}{2}$

  $\left[y(z) = \sum_{k=0}^{\infty} a_k z^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k}{2\left(k+\frac{3}{2}\right)}\right]$

- Combine solutions and rename parameters

$$\left[y(z)=\left(\sum_{k=0}^{\infty} a_k z^k\right)+\left(\sum_{k=0}^{\infty} b_k z^{k+\frac{1}{2}}\right), a_{k+1}=\frac{a_k}{2(k+1)}, b_{k+1}=\frac{b_k}{2(k+\frac{3}{2})}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.033 (sec)
Leaf size : 22

```
dsolve(4*z*diff(diff(y(z),z),z)+2*(1-z)*diff(y(z),z)-y(z) = 0,y(z),singsol=all)
```

$$y(z)=\mathrm{e}^{\frac{z}{2}}\left(\operatorname{erf}\left(\frac{\sqrt{2}\,\sqrt{z}}{2}\right)c_1+c_2\right)$$

Mathematica DSolve solution

Solving time : 0.223 (sec)
Leaf size : 44

```
DSolve[{4*z*D[y[z],{z,2}]+2*(1-z)*D[y[z],z]-y[z]==0,{}},y[z],z,IncludeSingularSolutions->True]
```

$$y(z)\to e^{\frac{z}{2}-\frac{1}{4}}\left(\sqrt{e}c_1-\sqrt{2}c_2\Gamma\left(\frac{1}{2},\frac{z}{2}\right)\right)$$

2.1.216 Problem 219

Internal problem ID [9388]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 219
Date solved : `Monday, January 27, 2025 at 06:02:12 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$f''+2(z-1)\,f'+4f=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.223 (sec)

Writing the ode as

$$f''+(2z-2)\,f'+4f=0 \tag{1}$$
$$Af''+Bf'+Cf=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=1\\ B&=2z-2\\ C&=4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(z)=fe^{\int \frac{B}{2A}\,dz}$$

Then (2) becomes

$$z''(z)=rz(z) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{z^2-2z-2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=z^2-2z-2\\ t&=1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(z)=\left(z^2-2z-2\right)z(z) \tag{7}$$

Equation (7) is now solved. After finding $z(z)$ then f is found using the inverse transformation

$$f = z(z)\, e^{-\int \frac{B}{2A}\, dz}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.414: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving z^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i z^i \\ &= \sum_{i=0}^{1} a_i z^i \end{aligned} \tag{8}$$

Let a be the coefficient of $z^v = z^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx z - 1 - \frac{3}{2z} - \frac{3}{2z^2} - \frac{21}{8z^3} - \frac{39}{8z^4} - \frac{159}{16z^5} - \frac{339}{16z^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i z^i \\ &= z - 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $z^{v-1} = z^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = z^2 - 2z + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{z^2 - 2z - 2}{1} \\ &= Q + \frac{R}{1} \\ &= \left(z^2 - 2z - 2\right) + (0) \\ &= z^2 - 2z - 2 \end{aligned}$$

We see that the coefficient of the term $\frac{1}{z}$ in the quotient is -2. Now b can be found.

$$\begin{aligned} b &= (-2) - (1) \\ &= -3 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= z - 1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-3}{1} - 1\right) &= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-3}{1} - 1\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = z^2 - 2z - 2$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$z - 1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{z-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\,(z-1) \\ &= 1-z \\ &= 1-z\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(z)$ of degree $d=1$ to solve the ode. The polynomial $p(z)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(z) = z + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2(1-z)\,(1) + \left((-1) + (1-z)^2 - (z^2-2z-2)\right) &= 0 \\ 2 + 2a_0 &= 0\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1\}$$

Substituting these coefficients in $p(z)$ in eq. (2A) results in

$$p(z) = z - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(z) &= pe^{\int \omega\,dz} \\ &= (z-1)\,\mathrm{e}^{\int (1-z)dz} \\ &= (z-1)\,\mathrm{e}^{z-\frac{1}{2}z^2} \\ &= (z-1)\,\mathrm{e}^{-\frac{z(-2+z)}{2}}\end{aligned}$$

The first solution to the original ode in f is found from

$$\begin{aligned}f_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dz} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2z-2}{1}\,dz} \\ &= z_1 e^{z-\frac{1}{2}z^2} \\ &= z_1\left(\mathrm{e}^{-\frac{z(-2+z)}{2}}\right)\end{aligned}$$

Which simplifies to

$$f_1 = \mathrm{e}^{-z(-2+z)}(z-1)$$

The second solution f_2 to the original ode is found using reduction of order

$$f_2 = f_1 \int \frac{e^{\int -\frac{B}{A}\,dz}}{f_1^2}\,dz$$

Substituting gives

$$\begin{aligned}
f_2 &= f_1 \int \frac{e^{\int -\frac{2z-2}{1}\, dz}}{(f_1)^2}\, dz \\
&= f_1 \int \frac{e^{-z^2+2z}}{(f_1)^2}\, dz \\
&= f_1 \left(-\frac{e^{(z-1)^2-1}}{z-1} - i\sqrt{\pi}\, e^{-1} \operatorname{erf}\left(i(z-1)\right) \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
f &= c_1 f_1 + c_2 f_2 \\
&= c_1 \left(e^{-z(-2+z)}(z-1)\right) + c_2 \left(e^{-z(-2+z)}(z-1) \left(-\frac{e^{(z-1)^2-1}}{z-1} - i\sqrt{\pi}\, e^{-1} \operatorname{erf}\left(i(z-1)\right) \right) \right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\frac{d^2}{dz^2} f(z) + 2(z-1)\left(\frac{d}{dz} f(z)\right) + 4f(z) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dz^2} f(z)$
- Isolate 2nd derivative
  $\frac{d^2}{dz^2} f(z) = -2(z-1)\left(\frac{d}{dz} f(z)\right) - 4f(z)$
- Group terms with $f(z)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dz^2} f(z) + (-2+2z)\left(\frac{d}{dz} f(z)\right) + 4f(z) = 0$
- Assume series solution for $f(z)$
  $f(z) = \sum_{k=0}^{\infty} a_k z^k$

☐ Rewrite DE with series expansions

- ◦ Convert $z^m \cdot \left(\frac{d}{dz} f(z)\right)$ to series expansion for $m = 0..1$
  $z^m \cdot \left(\frac{d}{dz} f(z)\right) = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, z^{k-1+m}$
- ◦ Shift index using $k- >k+1-m$
  $z^m \cdot \left(\frac{d}{dz} f(z)\right) = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, z^k$
- ◦ Convert $\frac{d^2}{dz^2} f(z)$ to series expansion
  $\frac{d^2}{dz^2} f(z) = \sum_{k=2}^{\infty} a_k k(k-1)\, z^{k-2}$
- ◦ Shift index using $k- >k+2$
  $\frac{d^2}{dz^2} f(z) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, z^k$

Rewrite DE with series expansions
$\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)(k+1) - 2a_{k+1}(k+1) + 2a_k(k+2)\right) z^k = 0$

- Each term in the series must be 0, giving the recursion relation

$$k^2 a_{k+2} + (2a_k - 2a_{k+1} + 3a_{k+2})\, k + 4a_k - 2a_{k+1} + 2a_{k+2} = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[f(z) = \sum_{k=0}^{\infty} a_k z^k, a_{k+2} = -\frac{2(a_k k - a_{k+1} k + 2a_k - a_{k+1})}{k^2 + 3k + 2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 43

```
dsolve(diff(diff(f(z),z),z)+2*(z-1)*diff(f(z),z)+4*f(z) = 0,f(z),singsol=all)
```

$$f(z) = -\sqrt{\pi}\, \operatorname{erf}(i(z-1))\, c_2 (z-1)\, e^{-(z-1)^2} + c_1 e^{-z(z-2)}(z-1) + i c_2$$

Mathematica DSolve solution

Solving time : 0.133 (sec)
Leaf size : 72

```
DSolve[{D[ f[z],{z,2}]+2*(z-a)*D[ f[z],z]+4*f[z]==0,{}},f[z],z,IncludeSingularSolutions->True]
```

$$f(z) \to e^{z(2a-z)}\left(-\sqrt{\pi} c_2 \sqrt{(a-z)^2} \operatorname{erfi}\left(\sqrt{(a-z)^2}\right) + c_2 e^{(a-z)^2} - 2ac_1 + 2c_1 z\right)$$

2.1.217 Problem 220

Internal problem ID [9389]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 220
Date solved : `Monday, January 27, 2025 at 06:02:13 PM`
CAS classification : `[_Lienard]`

Solve

$$zy'' - 2y' + zy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.267 (sec)

Writing the ode as

$$zy'' - 2y' + zy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= z \\ B &= -2 \\ C &= z \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(z) = ye^{\int \frac{B}{2A}\, dz}$$

Then (2) becomes

$$z''(z) = rz(z) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-z^2 + 2}{z^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -z^2 + 2 \\ t &= z^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(z) = \left(\frac{-z^2 + 2}{z^2}\right) z(z) \tag{7}$$

Equation (7) is now solved. After finding $z(z)$ then y is found using the inverse transformation

$$y = z(z)\, e^{-\int \frac{B}{2A}\, dz}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.416: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = z^2$. There is a pole at $z = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -1 + \frac{2}{z^2}$$

For the pole at $z = 0$ let b be the coefficient of $\frac{1}{z^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving z^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i z^i \\ &= \sum_{i=0}^{0} a_i z^i \end{aligned} \tag{8}$$

Let a be the coefficient of $z^v = z^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx i - \frac{i}{z^2} - \frac{i}{2z^4} - \frac{i}{2z^6} - \frac{5i}{8z^8} - \frac{7i}{8z^{10}} - \frac{21i}{16z^{12}} - \frac{33i}{16z^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i z^i \\ &= i \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $z^{v-1} = z^{-1} = \frac{1}{z}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -1$$

This shows that the coefficient of $\frac{1}{z}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{z}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{z}$ in r will be the coefficient in R of the term in z of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-z^2 + 2}{z^2} \\ &= Q + \frac{R}{z^2} \\ &= (-1) + \left(\frac{2}{z^2}\right) \\ &= -1 + \frac{2}{z^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term z in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= i \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{0}{i} - 0\right) = 0 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{0}{i} - 0\right) = 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-z^2 + 2}{z^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	i	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{z-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{z-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{z} + (-)(i) \\ &= -\frac{1}{z} - i \\ &= -\frac{1}{z} - i \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(z)$ of degree $d = 1$ to solve the ode. The polynomial $p(z)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(z) = z + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{z} - i\right)(1) + \left(\left(\frac{1}{z^2}\right) + \left(-\frac{1}{z} - i\right)^2 - \left(\frac{-z^2+2}{z^2}\right) \right) = 0$$

$$\frac{2ia_0 - 2}{z} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -i\}$$

Substituting these coefficients in $p(z)$ in eq. (2A) results in

$$p(z) = z - i$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(z) &= p e^{\int \omega \, dz} \\ &= (z-i)\, e^{\int \left(-\frac{1}{z} - i\right) dz} \\ &= (z-i)\, e^{-\ln(z) - iz} \\ &= \frac{(z-i)\, e^{-iz}}{z} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dz} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{z}\,dz} \\ &= z_1 e^{\ln(z)} \\ &= z_1(z) \end{aligned}$$

Which simplifies to

$$y_1 = (z-i)\,e^{-iz}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dz}}{y_1^2}\,dz$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{z}\,dz}}{(y_1)^2}\,dz \\ &= y_1 \int \frac{e^{2\ln(z)}}{(y_1)^2}\,dz \\ &= y_1 \left(\frac{(iz-1)\,e^{2iz}}{-2z+2i}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left((z-i)\,e^{-iz}\right) + c_2\left((z-i)\,e^{-iz}\left(\frac{(iz-1)\,e^{2iz}}{-2z+2i}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$z\left(\frac{d^2}{dz^2}y(z)\right) - 2\frac{d}{dz}y(z) + y(z)\,z = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dz^2}y(z)$

- Isolate 2nd derivative

  $\frac{d^2}{dz^2}y(z) = -y(z) + \frac{2\left(\frac{d}{dz}y(z)\right)}{z}$

- Group terms with $y(z)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dz^2}y(z) - \frac{2\left(\frac{d}{dz}y(z)\right)}{z} + y(z) = 0$

- □ Check to see if $z_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(z) = -\frac{2}{z}, P_3(z) = 1\right]$

  - $z \cdot P_2(z)$ is analytic at $z = 0$

    $(z \cdot P_2(z))\Big|_{z=0} = -2$

- ◦ $z^2 \cdot P_3(z)$ is analytic at $z = 0$
  $\left(z^2 \cdot P_3(z)\right)\Big|_{z=0} = 0$
- ◦ $z = 0$is a regular singular point
  Check to see if $z_0 = 0$ is a regular singular point
  $z_0 = 0$
- • Multiply by denominators
  $z\left(\frac{d^2}{dz^2}y(z)\right) - 2\frac{d}{dz}y(z) + y(z)\,z = 0$
- • Assume series solution for $y(z)$
  $y(z) = \sum_{k=0}^{\infty} a_k z^{k+r}$
- □ Rewrite ODE with series expansions
  - ◦ Convert $z \cdot y(z)$ to series expansion
    $z \cdot y(z) = \sum_{k=0}^{\infty} a_k z^{k+r+1}$
  - ◦ Shift index using $k- > k-1$
    $z \cdot y(z) = \sum_{k=1}^{\infty} a_{k-1} z^{k+r}$
  - ◦ Convert $\frac{d}{dz}y(z)$ to series expansion
    $\frac{d}{dz}y(z) = \sum_{k=0}^{\infty} a_k(k+r)\, z^{k+r-1}$
  - ◦ Shift index using $k- > k+1$
    $\frac{d}{dz}y(z) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, z^{k+r}$
  - ◦ Convert $z \cdot \left(\frac{d^2}{dz^2}y(z)\right)$ to series expansion
    $z \cdot \left(\frac{d^2}{dz^2}y(z)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, z^{k+r-1}$
  - ◦ Shift index using $k- > k+1$
    $z \cdot \left(\frac{d^2}{dz^2}y(z)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, z^{k+r}$

  Rewrite ODE with series expansions
  $a_0 r(-3+r)\, z^{-1+r} + a_1(1+r)\,(-2+r)\, z^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k-2+r) + a_{k-1}\right) z^{k+r}\right) = 0$
- • $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-3+r) = 0$
- • Values of r that satisfy the indicial equation
  $r \in \{0, 3\}$
- • Each term must be 0
  $a_1(1+r)\,(-2+r) = 0$
- • Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+r+1)\,(k-2+r) + a_{k-1} = 0$
- • Shift index using $k- > k+1$
  $a_{k+2}(k+2+r)\,(k+r-1) + a_k = 0$
- • Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{a_k}{(k+2+r)(k+r-1)}$
- • Recursion relation for $r = 0$
  $a_{k+2} = -\frac{a_k}{(k+2)(k-1)}$
- • Solution for $r = 0$

$$\left[y(z)=\sum_{k=0}^{\infty} a_k z^k, a_{k+2}=-\frac{a_k}{(k+2)(k-1)}, -2a_1=0\right]$$

- Recursion relation for $r=3$

  $a_{k+2}=-\frac{a_k}{(k+5)(k+2)}$

- Solution for $r=3$

$$\left[y(z)=\sum_{k=0}^{\infty} a_k z^{k+3}, a_{k+2}=-\frac{a_k}{(k+5)(k+2)}, 4a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y(z)=\left(\sum_{k=0}^{\infty} a_k z^k\right)+\left(\sum_{k=0}^{\infty} b_k z^{k+3}\right), a_{k+2}=-\frac{a_k}{(k+2)(k-1)}, -2a_1=0, b_{k+2}=-\frac{b_k}{(5+k)(k+2)}, 4b_1=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.021 (sec)
Leaf size : 23

```
dsolve(z*diff(diff(y(z),z),z)-2*diff(y(z),z)+z*y(z) = 0,y(z),singsol=all)
```

$$y(z)=(c_1 z+c_2)\cos(z)+\sin(z)(c_2 z-c_1)$$

Mathematica DSolve solution

Solving time : 0.049 (sec)
Leaf size : 39

```
DSolve[{z*D[y[z],{z,2}]-2*D[y[z],z]+z*y[z]==0,{}},y[z],z,IncludeSingularSolutions->True]
```

$$y(z)\to -\sqrt{\frac{2}{\pi}}((c_1 z+c_2)\cos(z)+(c_2 z-c_1)\sin(z))$$

2.1.218 Problem 221

Internal problem ID [9390]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 221
Date solved : `Monday, January 27, 2025 at 06:02:14 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$zy'' + (2z-3)\,y' + \frac{4y}{z} = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.276 (sec)

Writing the ode as

$$zy'' + (2z-3)\,y' + \frac{4y}{z} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= z \\ B &= 2z-3 \\ C &= \frac{4}{z} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(z) = ye^{\int \frac{B}{2A}\,dz}$$

Then (2) becomes

$$z''(z) = rz(z) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4z^2 - 12z - 1}{4z^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4z^2 - 12z - 1 \\ t &= 4z^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(z) = \left(\frac{4z^2 - 12z - 1}{4z^2}\right) z(z) \tag{7}$$

Equation (7) is now solved. After finding $z(z)$ then y is found using the inverse transformation

$$y = z(z)\, e^{-\int \frac{B}{2A}\, dz}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.418: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4z^2$. There is a pole at $z = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 - \frac{1}{4z^2} - \frac{3}{z}$$

For the pole at $z = 0$ let b be the coefficient of $\frac{1}{z^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving z^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i z^i \\
&= \sum_{i=0}^{0} a_i z^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $z^v = z^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 - \frac{3}{2z} - \frac{5}{4z^2} - \frac{15}{8z^3} - \frac{115}{32z^4} - \frac{495}{64z^5} - \frac{2285}{128z^6} - \frac{11055}{256z^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i z^i \\
&= 1
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $z^{v-1} = z^{-1} = \frac{1}{z}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{z}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{z}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{z}$ in r will be the coefficient in R of the term in z of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{4z^2 - 12z - 1}{4z^2} \\
&= Q + \frac{R}{4z^2} \\
&= (1) + \left(\frac{-12z - 1}{4z^2}\right) \\
&= 1 + \frac{-12z - 1}{4z^2}
\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term z in the remainder R is -12. Dividing this by leading coefficient in t which is 4 gives -3. Now b can be found.

$$\begin{aligned}
b &= (-3) - (0) \\
&= -3
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 1 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-3}{1} - 0\right) &&= -\frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-3}{1} - 0\right) &&= \frac{3}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4z^2 - 12z - 1}{4z^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$-\frac{3}{2}$	$\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^+) \\ &= \frac{3}{2} - \left(\frac{1}{2}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{z-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{z-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2z} + (-)(1) \\ &= \frac{1}{2z} - 1 \\ &= \frac{1}{2z} - 1 \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(z)$ of degree $d = 1$ to solve the ode. The polynomial $p(z)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(z) = z + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2z} - 1\right)(1) + \left(\left(-\frac{1}{2z^2}\right) + \left(\frac{1}{2z} - 1\right)^2 - \left(\frac{4z^2 - 12z - 1}{4z^2}\right) \right) = 0$$

$$\frac{1 + 2a_0}{z} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{ a_0 = -\frac{1}{2} \right\}$$

Substituting these coefficients in $p(z)$ in eq. (2A) results in

$$p(z) = z - \frac{1}{2}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(z) &= p e^{\int \omega \, dz} \\ &= \left(z - \frac{1}{2}\right) e^{\int \left(\frac{1}{2z} - 1\right) dz} \\ &= \left(z - \frac{1}{2}\right) e^{-z + \frac{\ln(z)}{2}} \\ &= \frac{(-1 + 2z)\sqrt{z}\, e^{-z}}{2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dz} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2z-3}{z} \, dz} \\ &= z_1 e^{-z + \frac{3\ln(z)}{2}} \\ &= z_1 \left(z^{3/2} e^{-z}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{z^2 e^{-2z}(-1 + 2z)}{2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dz}}{y_1^2} \, dz$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2z-3}{z} \, dz}}{(y_1)^2} \, dz \\ &= y_1 \int \frac{e^{-2z + 3\ln(z)}}{(y_1)^2} \, dz \\ &= y_1 \left(-4 \operatorname{Ei}_1(-2z) - \frac{4 e^{2z}}{-1 + 2z}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{z^2 e^{-2z}(-1 + 2z)}{2}\right) + c_2 \left(\frac{z^2 e^{-2z}(-1 + 2z)}{2} \left(-4 \operatorname{Ei}_1(-2z) - \frac{4 e^{2z}}{-1 + 2z}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$z\left(\frac{d^2}{dz^2}y(z)\right)+(-3+2z)\left(\frac{d}{dz}y(z)\right)+\frac{4y(z)}{z}=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dz^2}y(z)$

- Isolate 2nd derivative

  $\frac{d^2}{dz^2}y(z)=-\frac{4y(z)}{z^2}-\frac{(-3+2z)\left(\frac{d}{dz}y(z)\right)}{z}$

- Group terms with $y(z)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dz^2}y(z)+\frac{(-3+2z)\left(\frac{d}{dz}y(z)\right)}{z}+\frac{4y(z)}{z^2}=0$

□ Check to see if $z_0=0$ is a regular singular point

- ○ Define functions

  $\left[P_2(z)=\frac{-3+2z}{z},P_3(z)=\frac{4}{z^2}\right]$

- ○ $z\cdot P_2(z)$ is analytic at $z=0$

  $\left.(z\cdot P_2(z))\right|_{z=0}=-3$

- ○ $z^2\cdot P_3(z)$ is analytic at $z=0$

  $\left.(z^2\cdot P_3(z))\right|_{z=0}=4$

- ○ $z=0$is a regular singular point

  Check to see if $z_0=0$ is a regular singular point

  $z_0=0$

- Multiply by denominators

  $\left(\frac{d^2}{dz^2}y(z)\right)z^2+z(-3+2z)\left(\frac{d}{dz}y(z)\right)+4y(z)=0$

- Assume series solution for $y(z)$

  $y(z)=\sum_{k=0}^{\infty}a_k z^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $z^m\cdot\left(\frac{d}{dz}y(z)\right)$ to series expansion for $m=1..2$

  $z^m\cdot\left(\frac{d}{dz}y(z)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,z^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

  $z^m\cdot\left(\frac{d}{dz}y(z)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,z^{k+r}$

- ○ Convert $z^2\cdot\left(\frac{d^2}{dz^2}y(z)\right)$ to series expansion

  $z^2\cdot\left(\frac{d^2}{dz^2}y(z)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,z^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)^2\,z^r+\left(\sum_{k=1}^{\infty}\left(a_k(k+r-2)^2+2a_{k-1}(k+r-1)\right)z^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)^2=0$

- Values of r that satisfy the indicial equation

  $r=2$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-2)^2+2a_{k-1}(k+r-1)=0$

- Shift index using $k->k+1$

$$a_{k+1}(k+r-1)^2 + 2a_k(k+r) = 0$$

- Recursion relation that defines series solution to ODE
  $a_{k+1} = -\frac{2a_k(k+r)}{(k+r-1)^2}$
- Recursion relation for $r = 2$
  $a_{k+1} = -\frac{2a_k(k+2)}{(k+1)^2}$
- Solution for $r = 2$
  $\left[y(z) = \sum_{k=0}^{\infty} a_k z^{k+2}, a_{k+1} = -\frac{2a_k(k+2)}{(k+1)^2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 36

```
dsolve(z*diff(diff(y(z),z),z)+(2*z-3)*diff(y(z),z)+4/z*y(z) = 0,y(z),singsol=all)
```

$$y(z) = 2\left(c_2 e^{-2z}\left(-\frac{1}{2}+z\right)\mathrm{Ei}_1(-2z) + c_1\left(-\frac{1}{2}+z\right)e^{-2z} + \frac{c_2}{2}\right)z^2$$

Mathematica DSolve solution

Solving time : 0.939 (sec)
Leaf size : 55

```
DSolve[{z*D[y[z],{z,2}]+(2*z-3)*D[y[z],z]+4/z*y[z]==0,{}},y[z],z,IncludeSingularSolutions->True
```

$$y(z) \to \frac{1}{2}e^{-2z}z^2(2z-1)\left(c_2\int_1^z \frac{4e^{2K[1]}}{(1-2K[1])^2K[1]}dK[1] + c_1\right)$$

2.1.219 Problem 222

Internal problem ID [9391]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 222
Date solved : `Monday, January 27, 2025 at 06:02:14 PM`
CAS classification : `[_erf]`

Solve

$$y'' + 2xy' + 4y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.188 (sec)

Writing the ode as

$$y'' + 2xy' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 3}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 3 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(x^2 - 3\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.420: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x - \frac{3}{2x} - \frac{9}{8x^3} - \frac{27}{16x^5} - \frac{405}{128x^7} - \frac{1701}{256x^9} - \frac{15309}{1024x^{11}} - \frac{72171}{2048x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= x \qquad (10)
\end{aligned}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 3}{1} \\
&= Q + \frac{R}{1} \\
&= (x^2 - 3) + (0) \\
&= x^2 - 3
\end{aligned}
$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is -3. Now b can be found.

$$
\begin{aligned}
b &= (-3) - (0) \\
&= -3
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= x \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-3}{1} - 1\right) &= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-3}{1} - 1\right) &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = x^2 - 3$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	x	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$
\begin{aligned}
d &= \alpha_\infty^- \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\,(x) \\ &= -x \\ &= -x\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2(-x)\,(1) + \left((-1) + (-x)^2 - (x^2-3)\right) &= 0 \\ 2a_0 &= 0\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= pe^{\int \omega\, dx} \\ &= (x)\,\mathrm{e}^{\int -x dx} \\ &= (x)\,\mathrm{e}^{-\frac{x^2}{2}} \\ &= x\,\mathrm{e}^{-\frac{x^2}{2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x}{1}\, dx} \\ &= z_1 e^{-\frac{x^2}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x^2}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x^2} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x^2}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\mathrm{e}^{x^2}}{x} + \sqrt{\pi}\,\mathrm{erfi}\,(x)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x^2} x\right) + c_2\left(\mathrm{e}^{-x^2} x\left(-\frac{\mathrm{e}^{x^2}}{x} + \sqrt{\pi}\,\mathrm{erfi}\,(x)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + 2x\left(\frac{d}{dx}y(x)\right) + 4y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite DE with series expansions

  ◦ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

  $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

  $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 2a_k(k+2)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+2)\left(k a_{k+2} + 2a_k + a_{k+2}\right) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{2a_k}{k+1}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 25

```
dsolve(diff(diff(y(x),x),x)+2*diff(y(x),x)*x+4*y(x) = 0,y(x),singsol=all)
```

$$y = x\left(c_2\sqrt{\pi}\,\mathrm{erfi}\left(x\right)+c_1\right)\mathrm{e}^{-x^2}-c_2$$

Mathematica DSolve solution

Solving time : 0.037 (sec)
Leaf size : 51

```
DSolve[{D[y[x],{x,2}]+2*x*D[y[x],x]+4*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to e^{-x^2}\left(-\sqrt{\pi}c_2\sqrt{x^2}\mathrm{erfi}\left(\sqrt{x^2}\right)+c_2e^{x^2}+2c_1x\right)$$

2.1.220 Problem 223

Internal problem ID [9392]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 223
Date solved : `Monday, January 27, 2025 at 06:02:15 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + xy' + 3y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.246 (sec)

Writing the ode as

$$y'' + xy' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 10}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 10 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{5}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.422: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{5}{2x} - \frac{25}{4x^3} - \frac{125}{4x^5} - \frac{3125}{16x^7} - \frac{21875}{16x^9} - \frac{328125}{32x^{11}} - \frac{2578125}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 10}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} - \frac{5}{2}\right) + (0) \\ &= \frac{x^2}{4} - \frac{5}{2}\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{5}{2}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{5}{2}\right) - (0) \\ &= -\frac{5}{2}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= -3 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= 2\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{5}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 2 \end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0+(-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=x^2+a_1x+a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (2)+2\left(-\frac{x}{2}\right)(2x+a_1)+\left(\left(-\frac{1}{2}\right)+\left(-\frac{x}{2}\right)^2-\left(\frac{x^2}{4}-\frac{5}{2}\right)\right) &= 0 \\ a_1x+2a_0+2 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0=-1, a_1=0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x)=x^2-1$$

Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\,dx} \\ &= \left(x^2-1\right)e^{\int -\frac{x}{2}dx} \\ &= \left(x^2-1\right)e^{-\frac{x^2}{4}} \\ &= \left(x^2-1\right)e^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1e^{-\int \frac{1}{2}\frac{x}{1}\,dx} \\ &= z_1e^{-\frac{x^2}{4}} \\ &= z_1\left(e^{-\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^2}{2}}\left(x^2-1\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{(x^2-1)^2} dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-\frac{x^2}{2}}\left(x^2-1\right)\right) + c_2\left(\mathrm{e}^{-\frac{x^2}{2}}\left(x^2-1\right)\left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{(x^2-1)^2} dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + x\left(\frac{d}{dx}y(x)\right) + 3y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  - Convert $x\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x\cdot\left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k->k+2$

    $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+3)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $\left(k^2+3k+2\right) a_{k+2} + a_k(k+3) = 0$

- Recursion relation that defines the series solution to the ODE

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(k+3)}{k^2+3k+2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special fu
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.026 (sec)
Leaf size : 42

```
dsolve(diff(diff(y(x),x),x)+diff(y(x),x)*x+3*y(x) = 0,y(x),singsol=all)
```

$$y = -(x-1)(x+1)\left(c_1\sqrt{\pi}\,\mathrm{erfi}\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}-c_2\right)\mathrm{e}^{-\frac{x^2}{2}}+2c_1x$$

Mathematica DSolve solution

Solving time : 0.201 (sec)
Leaf size : 52

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]+3*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-\frac{x^2}{2}}(x^2-1)\left(c_2\int_1^x \frac{e^{\frac{K[1]^2}{2}}}{(K[1]^2-1)^2}dK[1]+c_1\right)$$

2.1.221 Problem 224

Internal problem ID [9393]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 224
Date solved : `Monday, January 27, 2025 at 06:02:16 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - x^2 y' - 3xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.250 (sec)

Writing the ode as

$$y'' - x^2 y' - 3xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x^2 \\ C &= -3x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x(x^3 + 8)}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x(x^3 + 8) \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x(x^3 + 8)}{4} \right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.424: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 4 \\ &= -4 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{2} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x^2}{2} + \frac{2}{x} - \frac{4}{x^4} + \frac{16}{x^7} - \frac{80}{x^{10}} + \frac{448}{x^{13}} - \frac{2688}{x^{16}} + \frac{16896}{x^{19}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 2$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i x^i \\ &= \frac{x^2}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^1 = x$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^4}{4}$$

This shows that the coefficient of x in the above is 0. Now we need to find the coefficient of x in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x(x^3+8)}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^4 + 2x\right) + (0) \\ &= \frac{1}{4}x^4 + 2x\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is 2. Now b can be found.

$$\begin{aligned}b &= (2) - (0) \\ &= 2\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{x^2}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{2}{\frac{1}{2}} - 2\right) &&= 1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{2}{\frac{1}{2}} - 2\right) &&= -3\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x(x^3+8)}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-4	$\frac{x^2}{2}$	1	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 1$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{x^2}{2}\right) \\ &= \frac{x^2}{2} \\ &= \frac{x^2}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{x^2}{2}\right)(1) + \left((x) + \left(\frac{x^2}{2}\right)^2 - \left(\frac{x(x^3+8)}{4}\right)\right) = 0$$
$$-xa_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= (x)\,\mathrm{e}^{\int \frac{x^2}{2} dx} \\ &= (x)\,\mathrm{e}^{\frac{x^3}{6}} \\ &= x\,\mathrm{e}^{\frac{x^3}{6}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x^2}{1}\, dx} \\ &= z_1 e^{\frac{x^3}{6}} \\ &= z_1\left(\mathrm{e}^{\frac{x^3}{6}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{\frac{x^3}{3}} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{x^3}{3}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{e^{-\frac{x^3}{3}}}{x^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{\frac{x^3}{3}} x\right) + c_2 \left(e^{\frac{x^3}{3}} x \left(\int \frac{e^{-\frac{x^3}{3}}}{x^2} dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2} y(x) - x^2 \left(\frac{d}{dx} y(x)\right) - 3 x y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2} y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x \cdot y(x)$ to series expansion

    $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k- >k-1$

    $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert $x^2 \cdot \left(\frac{d}{dx} y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^{k+1}$

  - Shift index using $k- >k-1$

    $x^2 \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=1}^{\infty} a_{k-1} (k-1)\, x^k$

  - Convert $\frac{d^2}{dx^2} y(x)$ to series expansion

$\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- Shift index using $k- >k+2$

  $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_{k-1}(k+2)\right) x^k\right) = 0$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+2)\,(ka_{k+2} - a_{k-1} + a_{k+2}) = 0$

- Shift index using $k- >k+1$

  $(k+3)\,((k+1)\,a_{k+3} - a_k + a_{k+3}) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{a_k}{k+2}, 2a_2 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 58

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x^2-3*x*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{9\,\mathrm{e}^{\frac{x^3}{6}}\,\mathrm{WhittakerM}\left(\frac{1}{3}, \frac{5}{6}, \frac{x^3}{3}\right) c_2 x^3 + 9c_1 x^2 \mathrm{e}^{\frac{x^3}{3}} + 5\,3^{2/3} c_2 (x^3)^{1/3}\,(x^3+2)}{9x}$$

Mathematica DSolve solution

Solving time : 0.074 (sec)
Leaf size : 51

```
DSolve[{D[y[x],{x,2}]-x^2*D[y[x],x]-3*x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{9} e^{\frac{x^3}{3}} \left(9c_1 x - 3^{2/3} c_2 \sqrt[3]{x^3}\,\Gamma\left(-\frac{1}{3}, \frac{x^3}{3}\right)\right)$$

2.1.222 Problem 225

Internal problem ID [9394]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 225
Date solved : `Monday, January 27, 2025 at 06:02:16 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-4x^2+1\right)y''-20xy'-16y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.239 (sec)

Writing the ode as

$$\left(-4x^2+1\right)y''-20xy'-16y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-4x^2+1\\B&=-20x\\C&=-16\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-4x^2+6}{\left(4x^2-1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=-4x^2+6\\t&=\left(4x^2-1\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-4x^2+6}{\left(4x^2-1\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.426: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (4x^2 - 1)^2$. There is a pole at $x = \frac{1}{2}$ of order 2. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16\left(x - \frac{1}{2}\right)^2} - \frac{7}{8\left(x - \frac{1}{2}\right)} + \frac{5}{16\left(x + \frac{1}{2}\right)^2} + \frac{7}{8\left(x + \frac{1}{2}\right)}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4} \end{aligned}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x + \frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-4x^2 + 6}{(4x^2 - 1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-4x^2 + 6}{(4x^2 - 1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{1}{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-\frac{1}{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^- + \alpha_{c_2}^-) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{4\left(x - \frac{1}{2}\right)} - \frac{1}{4\left(x + \frac{1}{2}\right)} + (-)(0) \\
&= -\frac{1}{4\left(x - \frac{1}{2}\right)} - \frac{1}{4\left(x + \frac{1}{2}\right)} \\
&= -\frac{2x}{4x^2 - 1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{4\left(x-\frac{1}{2}\right)} - \frac{1}{4\left(x+\frac{1}{2}\right)}\right)(1) + \left(\left(\frac{1}{4\left(x-\frac{1}{2}\right)^2} + \frac{1}{4\left(x+\frac{1}{2}\right)^2}\right) + \left(-\frac{1}{4\left(x-\frac{1}{2}\right)} - \frac{1}{4\left(x+\frac{1}{2}\right)}\right)^2 - \left(\right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, e^{\int \left(-\frac{1}{4\left(x-\frac{1}{2}\right)} - \frac{1}{4\left(x+\frac{1}{2}\right)}\right) dx} \\
&= (x) \frac{1}{\left((2x-1)(2x+1)\right)^{1/4}} \\
&= \frac{x}{\left(4x^2-1\right)^{1/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-20x}{-4x^2+1}\, dx} \\
&= z_1 e^{-\frac{5\ln\left(4x^2-1\right)}{4}} \\
&= z_1 \left(\frac{1}{\left(4x^2-1\right)^{5/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x}{\left(4x^2-1\right)^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-20x}{-4x^2+1}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{5\ln\left(4x^2-1\right)}{2}}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\frac{\left(4x^2-1\right)^{3/2}}{x} - 4x\sqrt{4x^2-1} + \ln\left(x\sqrt{4} + \sqrt{4x^2-1}\right)\sqrt{4}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x}{(4x^2-1)^{3/2}}\right) \\
&\quad + c_2 \left(\frac{x}{(4x^2-1)^{3/2}} \left(\frac{(4x^2-1)^{3/2}}{x} - 4x\sqrt{4x^2-1} + \ln\left(x\sqrt{4} + \sqrt{4x^2-1}\right)\sqrt{4}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(-4x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) - 20x\left(\frac{d}{dx}y(x)\right) - 16y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{16y(x)}{4x^2-1} - \frac{20x\left(\frac{d}{dx}y(x)\right)}{4x^2-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{20x\left(\frac{d}{dx}y(x)\right)}{4x^2-1} + \frac{16y(x)}{4x^2-1} = 0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{20x}{4x^2-1}, P_3(x) = \frac{16}{4x^2-1}\right]$

  - $\left(x+\frac{1}{2}\right)\cdot P_2(x)$ is analytic at $x = -\frac{1}{2}$

    $\left(\left(x+\frac{1}{2}\right)\cdot P_2(x)\right)\bigg|_{x=-\frac{1}{2}} = \frac{5}{2}$

  - $\left(x+\frac{1}{2}\right)^2\cdot P_3(x)$ is analytic at $x = -\frac{1}{2}$

    $\left(\left(x+\frac{1}{2}\right)^2\cdot P_3(x)\right)\bigg|_{x=-\frac{1}{2}} = 0$

  - $x = -\frac{1}{2}$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -\frac{1}{2}$

- Multiply by denominators

  $(4x^2-1)\left(\frac{d^2}{dx^2}y(x)\right) + 20x\left(\frac{d}{dx}y(x)\right) + 16y(x) = 0$

- Change variables using $x = u - \frac{1}{2}$ so that the regular singular point is at $u = 0$

  $(4u^2-4u)\left(\frac{d^2}{du^2}y(u)\right) + (20u-10)\left(\frac{d}{du}y(u)\right) + 16y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0 r(3+2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)(2k+5+2r) + 4a_k(k+r+2)^2\right) u^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, -\frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $4a_k(k+r+2)^2 - 4(k+1+r)\left(k+r+\frac{5}{2}\right) a_{k+1} = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k(k+r+2)^2}{(k+1+r)(2k+5+2r)}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{2a_k(k+2)^2}{(k+1)(2k+5)}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{2a_k(k+2)^2}{(k+1)(2k+5)}\right]$

- Revert the change of variables $u = x + \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k\left(x+\frac{1}{2}\right)^k, a_{k+1} = \frac{2a_k(k+2)^2}{(k+1)(2k+5)}\right]$

- Recursion relation for $r = -\frac{3}{2}$

  $a_{k+1} = \frac{2a_k\left(k+\frac{1}{2}\right)^2}{\left(k-\frac{1}{2}\right)(2k+2)}$

- Solution for $r = -\frac{3}{2}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{3}{2}}, a_{k+1} = \frac{2a_k\left(k+\frac{1}{2}\right)^2}{\left(k-\frac{1}{2}\right)(2k+2)}\right]$

- Revert the change of variables $u = x + \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k\left(x+\frac{1}{2}\right)^{k-\frac{3}{2}}, a_{k+1} = \frac{2a_k\left(k+\frac{1}{2}\right)^2}{\left(k-\frac{1}{2}\right)(2k+2)}\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k\left(x+\frac{1}{2}\right)^k\right) + \left(\sum_{k=0}^{\infty} b_k\left(x+\frac{1}{2}\right)^{k-\frac{3}{2}}\right), a_{k+1} = \frac{2a_k(k+2)^2}{(k+1)(2k+5)}, b_{k+1} = \frac{2b_k\left(k+\frac{1}{2}\right)^2}{\left(k-\frac{1}{2}\right)(2k+2)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.035 (sec)
Leaf size : 48

```
dsolve((-4*x^2+1)*diff(diff(y(x),x),x)-20*diff(y(x),x)*x-16*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{2 \ln \left(2x + \sqrt{4x^2 - 1}\right) c_2 x + c_1 x - \sqrt{4x^2 - 1}\, c_2}{\left(4x^2 - 1\right)^{3/2}}$$

Mathematica DSolve solution

Solving time : 0.169 (sec)
Leaf size : 57

```
DSolve[{(1-4*x^2)*D[y[x],{x,2}]-20*x*D[y[x],x]-16*y[x]==0,{}},y[x],x,IncludeSingularSolution
```

$$y(x) \to \frac{-2c_2 x \arcsin(2x) - c_2\sqrt{1 - 4x^2} + c_1 x}{\sqrt[4]{1 - 4x^2}\left(4x^2 - 1\right)^{5/4}}$$

2.1.223 Problem 226

Internal problem ID [9395]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 226
Date solved : `Monday, January 27, 2025 at 06:02:17 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(x^2-1\right) y''-6xy'+12y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.192 (sec)

Writing the ode as

$$\left(x^2-1\right) y''-6xy'+12y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=x^2-1\\ B&=-6x\\ C&=12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{15}{\left(x^2-1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=15\\ t&=\left(x^2-1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{15}{\left(x^2-1\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.428: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4(x+1)^2} + \frac{15}{4(x+1)} + \frac{15}{4(x-1)^2} - \frac{15}{4(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= 0\\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{15}{(x^2-1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
-1	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right)\\
&= 1 - (1)\\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (-)[\sqrt{r}]_\infty\\
&= -\frac{3}{2\,(x-1)} + \frac{5}{2\,(x+1)} + (-)\,(0)\\
&= -\frac{3}{2\,(x-1)} + \frac{5}{2\,(x+1)}\\
&= \frac{x-4}{x^2-1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{2(x-1)} + \frac{5}{2(x+1)}\right)(0) + \left(\left(\frac{3}{2(x-1)^2} - \frac{5}{2(x+1)^2}\right) + \left(-\frac{3}{2(x-1)} + \frac{5}{2(x+1)}\right)^2 - \left(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{3}{2(x-1)} + \frac{5}{2(x+1)}\right) dx} \\ &= \frac{(x+1)^{5/2}}{(x-1)^{3/2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-6x}{x^2-1} \, dx} \\ &= z_1 e^{\frac{3\ln(x-1)}{2} + \frac{3\ln(x+1)}{2}} \\ &= z_1 \left((x-1)^{3/2} (x+1)^{3/2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (x+1)^4$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-6x}{x^2-1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{3\ln(x-1)+3\ln(x+1)}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{x(x^2+1)\,\mathrm{e}^{3\ln(x-1)+3\ln(x+1)}}{(x+1)^7 (x-1)^3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left((x+1)^4\right) + c_2 \left((x+1)^4 \left(-\frac{x(x^2+1)\,\mathrm{e}^{3\ln(x-1)+3\ln(x+1)}}{(x+1)^7 (x-1)^3}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x^2-1)\left(\frac{d^2}{dx^2}y(x)\right)-6x\left(\frac{d}{dx}y(x)\right)+12y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{12y(x)}{x^2-1}+\frac{6\left(\frac{d}{dx}y(x)\right)x}{x^2-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x)-\frac{6\left(\frac{d}{dx}y(x)\right)x}{x^2-1}+\frac{12y(x)}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

 ◦ Define functions

 $\left[P_2(x)=-\frac{6x}{x^2-1},P_3(x)=\frac{12}{x^2-1}\right]$

 ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

 $\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1}=-3$

 ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

 $\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1}=0$

 ◦ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1$

- Multiply by denominators

$(x^2-1)\left(\frac{d^2}{dx^2}y(x)\right)-6x\left(\frac{d}{dx}y(x)\right)+12y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(-6u+6)\left(\frac{d}{du}y(u)\right)+12y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

 ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

 $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

 ◦ Shift index using $k->k+1-m$

 $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

 ◦ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

 $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

 ◦ Shift index using $k->k+2-m$

 $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

Rewrite ODE with series expansions

$-2a_0r(-4+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)(k+r-3)+a_k(k+r-3)(k+r-4)\right)u^{k+r}\right)=$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-2r(-4+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{0,4\}$
- Each term in the series must be 0, giving the recursion relation

  $((-2k-2r-2)\,a_{k+1}+a_k(k+r-4))\,(k+r-3)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k(k+r-4)}{2(k+1+r)}$
- Recursion relation for $r=0$; series terminates at $k=4$

  $a_{k+1}=\frac{a_k(k-4)}{2(k+1)}$
- Apply recursion relation for $k=0$

  $a_1=-2a_0$
- Apply recursion relation for $k=1$

  $a_2=-\frac{3a_1}{4}$
- Express in terms of a_0

  $a_2=\frac{3a_0}{2}$
- Apply recursion relation for $k=2$

  $a_3=-\frac{a_2}{3}$
- Express in terms of a_0

  $a_3=-\frac{a_0}{2}$
- Apply recursion relation for $k=3$

  $a_4=-\frac{a_3}{8}$
- Express in terms of a_0

  $a_4=\frac{a_0}{16}$
- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li

  $y(u)=a_0\cdot\left(1-2u+\frac{3}{2}u^2-\frac{1}{2}u^3+\frac{1}{16}u^4\right)$
- Revert the change of variables $u=x+1$

  $\left[y(x)=\frac{a_0(x-1)^4}{16}\right]$
- Recursion relation for $r=4$

  $a_{k+1}=\frac{a_k k}{2(k+5)}$
- Solution for $r=4$

  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+4},a_{k+1}=\frac{a_k k}{2(k+5)}\right]$
- Revert the change of variables $u=x+1$

  $\left[y(x)=\sum_{k=0}^{\infty}a_k(x+1)^{k+4},a_{k+1}=\frac{a_k k}{2(k+5)}\right]$
- Combine solutions and rename parameters

  $\left[y(x)=\frac{a_0(x-1)^4}{16}+\left(\sum_{k=0}^{\infty}b_k(x+1)^{4+k}\right),b_{k+1}=\frac{b_k k}{2(5+k)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 25

```
dsolve((x^2-1)*diff(diff(y(x),x),x)-6*diff(y(x),x)*x+12*y(x) = 0,y(x),singsol=all)
```

$$y = c_2 x^4 + c_1 x^3 + 6c_2 x^2 + c_1 x + c_2$$

Mathematica DSolve solution

Solving time : 0.37 (sec)
Leaf size : 75

```
DSolve[{(x^2-1)*D[y[x],{x,2}]-6*x*D[y[x],x]+12*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tru
```

$$y(x) \to (x^2 - 1)^{3/2} \exp\left(\int_1^x \frac{K[1]+4}{K[1]^2-1} dK[1]\right) \left(c_2 \int_1^x \exp\left(-2\int_1^{K[2]} \frac{K[1]+4}{K[1]^2-1} dK[1]\right) dK[2] + c_1\right)$$

2.1.224 Problem 227

Internal problem ID [9396]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 227
Date solved : `Monday, January 27, 2025 at 06:02:18 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + xy' + (2+x)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.277 (sec)

Writing the ode as

$$y'' + xy' + (2+x)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 2+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x - 6}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 4x - 6 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 - x - \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.430: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - 1 - \frac{5}{2x} - \frac{5}{x^2} - \frac{65}{4x^3} - \frac{115}{2x^4} - \frac{885}{4x^5} - \frac{1785}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v=1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} - 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 - x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v=0$ or not. Since $v=1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x - 6}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 - x - \frac{3}{2}\right) + (0) \\ &= \frac{1}{4}x^2 - x - \frac{3}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{2}\right) - (1) \\ &= -\frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} - 1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= -3 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= 2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 - x - \frac{3}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2} - 1$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2} - 1\right) \\ &= 1 - \frac{x}{2} \\ &= 1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (2) + 2\left(1 - \frac{x}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2}\right) + \left(1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 - x - \frac{3}{2}\right)\right) &= 0 \\ (2 + x)\,a_1 + 4x + 2a_0 + 2 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 3, a_1 = -4\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 4x + 3$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\,dx} \\ &= (x^2 - 4x + 3)\,e^{\int (1 - \frac{x}{2})dx} \\ &= (x^2 - 4x + 3)\,e^{x - \frac{1}{4}x^2} \\ &= (x^2 - 4x + 3)\,e^{-\frac{x(-4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{1}\,dx} \\ &= z_1 e^{-\frac{x^2}{4}} \\ &= z_1\left(e^{-\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \left(x^2 - 4x + 3\right) e^{-\frac{x(-2+x)}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{e^{-\frac{x^2}{2}} e^{x(-2+x)}}{(x^2 - 4x + 3)^2} dx \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\left(x^2 - 4x + 3\right) e^{-\frac{x(-2+x)}{2}} \right) + c_2 \left(\left(x^2 - 4x + 3\right) e^{-\frac{x(-2+x)}{2}} \left(\int \frac{e^{-\frac{x^2}{2}} e^{x(-2+x)}}{(x^2 - 4x + 3)^2} dx \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2}y(x) + x\left(\frac{d}{dx}y(x)\right) + (x+2)\, y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$
  $x^m \cdot y(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$
- ○ Shift index using $k- > k - m$
  $x^m \cdot y(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$
- ○ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion
  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$
- ○ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion
  $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$
- ○ Shift index using $k- > k + 2$

$$\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\,x^k$$

Rewrite ODE with series expansions

$$2a_2 + 2a_0 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)(k+1) + a_k(k+2) + a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $$2a_2 + 2a_0 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2 + 3k + 2)\,a_{k+2} + a_k k + 2a_k + a_{k-1} = 0$$

- Shift index using $k- >k+1$

  $$\left((k+1)^2 + 3k + 5\right) a_{k+3} + a_{k+1}(k+1) + 2a_{k+1} + a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = -\frac{k a_{k+1} + a_k + 3a_{k+1}}{k^2 + 5k + 6}, 2a_2 + 2a_0 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special fu
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.017 (sec)
Leaf size : 78

```
dsolve(diff(diff(y(x),x),x)+diff(y(x),x)*x+(x+2)*y(x) = 0,y(x),singsol=all)
```

$$y = \left(c_2(x-3)\,\mathrm{e}^{-\frac{(x-2)^2}{2}}(x-1)\left(\mathrm{erf}\left(\frac{\sqrt{2}\,\sqrt{-(x-2)^2}}{2}\right)-1\right)\sqrt{\pi}-\sqrt{2}\,\sqrt{-(x-2)^2}\,c_2 - c_1\mathrm{e}^{-\frac{(x-2)^2}{2}}(x-1)\,(x-3)\right)\mathrm{e}^{-x}$$

Mathematica DSolve solution

Solving time : 0.696 (sec)
Leaf size : 63

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]+(2+x)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{x-\frac{x^2}{2}}(x^2-4x+3)\left(c_2\int_1^x \frac{e^{\frac{1}{2}(K[1]-4)K[1]}}{(K[1]-3)^2(K[1]-1)^2}dK[1]+c_1\right)$$

2.1.225 Problem 228

Internal problem ID [9397]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 228
Date solved : `Monday, January 27, 2025 at 06:02:18 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(2x^2+1\right)y''+7xy'+2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.335 (sec)

Writing the ode as

$$\left(2x^2+1\right)y''+7xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=2x^2+1\\ B&=7x\\ C&=2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{5x^2+6}{4\left(2x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=5x^2+6\\ t&=4\left(2x^2+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{5x^2+6}{4\left(2x^2+1\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.432: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 + 1)^2$. There is a pole at $x = \frac{i\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{i\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{64\left(x - \frac{i\sqrt{2}}{2}\right)^2} - \frac{7}{64\left(x + \frac{i\sqrt{2}}{2}\right)^2} - \frac{17i\sqrt{2}}{64\left(x - \frac{i\sqrt{2}}{2}\right)} + \frac{17i\sqrt{2}}{64\left(x + \frac{i\sqrt{2}}{2}\right)}$$

For the pole at $x = \frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{8} \end{aligned}$$

For the pole at $x = -\frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8}\\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{5x^2+6}{4\left(2x^2+1\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4}\\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{5x^2+6}{4\left(2x^2+1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{i\sqrt{2}}{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$
$-\frac{i\sqrt{2}}{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right)\\
&= \frac{5}{4} - \left(\frac{1}{4}\right)\\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4i\sqrt{2}} + (0) \\
&= \frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4i\sqrt{2}} \\
&= \frac{x}{4x^2 + 2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4i\sqrt{2}}\right)(1) + \left(\left(-\frac{1}{8\left(x - \frac{i\sqrt{2}}{2}\right)^2} - \frac{1}{8\left(x + \frac{i\sqrt{2}}{2}\right)^2}\right) + \left(\frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, e^{\int \left(\frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4i\sqrt{2}}\right) dx} \\
&= (x) \left(\left(i\sqrt{2} - 2x\right)\left(2x + i\sqrt{2}\right)\right)^{1/8} \\
&= x\left(-4x^2 - 2\right)^{1/8}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{7x}{2x^2 + 1} \, dx} \\
&= z_1 e^{-\frac{7 \ln\left(2x^2 + 1\right)}{8}} \\
&= z_1 \left(\frac{1}{\left(2x^2 + 1\right)^{7/8}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{2^{7/8} x \left(-4x^2 - 2\right)^{1/8}}{\left(4x^2 + 2\right)^{7/8}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{7x}{2x^2+1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\frac{7\ln(2x^2+1)}{4}}}{(y_1)^2}\,dx \\ &= y_1 \left(\int \frac{2^{1/4}(4x^2+2)^{7/4}}{4\,(2x^2+1)^{7/4}\,x^2\,(-4x^2-2)^{1/4}}dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{2^{7/8}x(-4x^2-2)^{1/8}}{(4x^2+2)^{7/8}}\right) + c_2 \left(\frac{2^{7/8}x(-4x^2-2)^{1/8}}{(4x^2+2)^{7/8}}\left(\int \frac{2^{1/4}(4x^2+2)^{7/4}}{4\,(2x^2+1)^{7/4}\,x^2\,(-4x^2-2)^{1/4}}dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Legendre successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.039 (sec)
Leaf size : 37

```
dsolve((2*x^2+1)*diff(diff(y(x),x),x)+7*diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_1 \operatorname{LegendreP}\left(\frac{1}{4}, \frac{3}{4}, i\sqrt{2}\, x\right) + c_2 \operatorname{LegendreQ}\left(\frac{1}{4}, \frac{3}{4}, i\sqrt{2}\, x\right)}{(2x^2+1)^{3/8}}$$

Mathematica DSolve solution

Solving time : 0.062 (sec)
Leaf size : 66

```
DSolve[{(1+2*x^2)*D[y[x],{x,2}]+7*x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to \frac{c_2 Q_{\frac{1}{4}}^{\frac{3}{4}}(i\sqrt{2}x)}{(2x^2+1)^{3/8}} + \frac{2i\sqrt[4]{2}c_1 x}{(2x^2+1)^{3/4} \operatorname{Gamma}\left(\frac{1}{4}\right)}$$

2.1.226 Problem 229

Internal problem ID [9398]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 229
Date solved : `Monday, January 27, 2025 at 06:02:19 PM`
CAS classification : `[_Lienard]`

Solve

$$4y'' + xy' + 4y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.249 (sec)

Writing the ode as

$$4y'' + xy' + 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4 \\ B &= x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 56}{64} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 56$$
$$t = 64$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{64} - \frac{7}{8}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.433: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{8} - \frac{7}{2x} - \frac{49}{x^3} - \frac{1372}{x^5} - \frac{48020}{x^7} - \frac{1882384}{x^9} - \frac{79060128}{x^{11}} - \frac{3478645632}{x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{8}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{8} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{64}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 56}{64} \\ &= Q + \frac{R}{64} \\ &= \left(\frac{x^2}{64} - \frac{7}{8}\right) + (0) \\ &= \frac{x^2}{64} - \frac{7}{8} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{7}{8}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{7}{8}\right) - (0) \\ &= -\frac{7}{8} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{8} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{7}{8}}{\frac{1}{8}} - 1\right) &&= -4 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{7}{8}}{\frac{1}{8}} - 1\right) &&= 3 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{64} - \frac{7}{8}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{8}$	-4	3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 3$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 3 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{8}\right) \\ &= -\frac{x}{8} \\ &= -\frac{x}{8} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=3$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (6x+2a_2) + 2\left(-\frac{x}{8}\right)\left(3x^2 + 2xa_2 + a_1\right) + \left(\left(-\frac{1}{8}\right) + \left(-\frac{x}{8}\right)^2 - \left(\frac{x^2}{64} - \frac{7}{8}\right)\right) &= 0 \\ 6x + 2a_2 + \frac{1}{4}a_2x^2 + \frac{1}{2}a_1x + \frac{3}{4}a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0, a_1 = -12, a_2 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^3 - 12x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= \left(x^3 - 12x\right)e^{\int -\frac{x}{8}dx} \\ &= \left(x^3 - 12x\right)e^{-\frac{x^2}{16}} \\ &= x\left(x^2 - 12\right)e^{-\frac{x^2}{16}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{4}\, dx} \\ &= z_1 e^{-\frac{x^2}{16}} \\ &= z_1\left(e^{-\frac{x^2}{16}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^2}{8}} x(x^2 - 12)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{4}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{8}}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{\frac{x^2}{8}}}{x^2 (x^2 - 12)^2} dx \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-\frac{x^2}{8}} x(x^2 - 12) \right) + c_2 \left(\mathrm{e}^{-\frac{x^2}{8}} x(x^2 - 12) \left(\int \frac{\mathrm{e}^{\frac{x^2}{8}}}{x^2 (x^2 - 12)^2} dx \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4 \frac{d^2}{dx^2} y(x) + x \left(\frac{d}{dx} y(x) \right) + 4y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2} y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2} y(x) = -\frac{x \left(\frac{d}{dx} y(x) \right)}{4} - y(x)$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2} y(x) + \frac{x \left(\frac{d}{dx} y(x) \right)}{4} + y(x) = 0$

- Multiply by denominators

  $4 \frac{d^2}{dx^2} y(x) + x \left(\frac{d}{dx} y(x) \right) + 4y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  - Convert $x \cdot \left(\frac{d}{dx} y(x) \right)$ to series expansion

    $x \cdot \left(\frac{d}{dx} y(x) \right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d^2}{dx^2} y(x)$ to series expansion

    $\frac{d^2}{dx^2} y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k+2$

$\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\,x^k$

Rewrite DE with series expansions

$\sum_{k=0}^{\infty} \left(4a_{k+2}(k+2)(k+1) + a_k(k+4)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $4(k^2+3k+2)\,a_{k+2} + a_k(k+4) = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(k+4)}{4(k^2+3k+2)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric sol
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.021 (sec)
Leaf size : 34

```
dsolve(4*diff(diff(y(x),x),x)+diff(y(x),x)*x+4*y(x) = 0,y(x),singsol=all)
```

$$y = -\frac{\left(-12\,\mathrm{hypergeom}\left(\left[-\frac{3}{2}\right],\left[\frac{1}{2}\right],\frac{x^2}{8}\right)c_2 + xc_1(x^2-12)\right)\mathrm{e}^{-\frac{x^2}{8}}}{12}$$

Mathematica DSolve solution

Solving time : 0.091 (sec)
Leaf size : 122

```
DSolve[{4*D[y[x],{x,2}]+x*D[y[x],x]+4*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-\frac{x^2}{8}}\left(\sqrt{2\pi}c_2(x^2-12)\,x^2\operatorname{erfi}\left(\frac{\sqrt{x^2}}{2\sqrt{2}}\right)+4\sqrt{x^2}\left(2\sqrt{2}c_1x^3-c_2e^{\frac{x^2}{8}}x^2+8c_2e^{\frac{x^2}{8}}-24\sqrt{2}c_1x\right)\right)}{32\sqrt{x^2}}$$

2.1.227 Problem 230

Internal problem ID [9399]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 230
Date solved : `Monday, January 27, 2025 at 06:02:20 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + xy' - 4y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.217 (sec)

Writing the ode as

$$y'' + xy' - 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 18}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 18 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} + \frac{9}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.435: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + \frac{9}{2x} - \frac{81}{4x^3} + \frac{729}{4x^5} - \frac{32805}{16x^7} + \frac{413343}{16x^9} - \frac{11160261}{32x^{11}} + \frac{157837977}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 18}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} + \frac{9}{2}\right) + (0) \\ &= \frac{x^2}{4} + \frac{9}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{9}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{9}{2}\right) - (0) \\ &= \frac{9}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{9}{2}}{\frac{1}{2}} - 1\right) &= 4 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{9}{2}}{\frac{1}{2}} - 1\right) &= -5 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} + \frac{9}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	4	-5

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 4$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 4 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{x}{2}\right) \\ &= \frac{x}{2} \\ &= \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(12x^2 + 6xa_3 + 2a_2) + 2\left(\frac{x}{2}\right)(4x^3 + 3x^2a_3 + 2xa_2 + a_1) + \left(\left(\frac{1}{2}\right) + \left(\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} + \frac{9}{2}\right)\right) = 0$$

$$-a_3x^3 + (-2a_2 + 12)\,x^2 + (-3a_1 + 6a_3)\,x - 4a_0 + 2a_2 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 3, a_1 = 0, a_2 = 6, a_3 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 + 6x^2 + 3$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= (x^4 + 6x^2 + 3)\, e^{\int \frac{x}{2} dx} \\ &= (x^4 + 6x^2 + 3)\, e^{\frac{x^2}{4}} \\ &= (x^4 + 6x^2 + 3)\, e^{\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{1}\, dx} \\ &= z_1 e^{-\frac{x^2}{4}} \\ &= z_1 \left(e^{-\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^4 + 6x^2 + 3$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{e^{-\frac{x^2}{2}}}{(x^4 + 6x^2 + 3)^2} dx \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (x^4 + 6x^2 + 3) + c_2 \left(x^4 + 6x^2 + 3 \left(\int \frac{e^{-\frac{x^2}{2}}}{(x^4 + 6x^2 + 3)^2} dx \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special functio
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.039` (sec)
Leaf size : 47

```
dsolve(diff(diff(y(x),x),x)+diff(y(x),x)*x-4*y(x) = 0,y(x),singsol=all)
```

$$y = xc_1(x^2+5)\sqrt{2}\,\mathrm{e}^{-\frac{x^2}{2}} + (x^4+6x^2+3)\left(\sqrt{\pi}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right)c_1 + c_2\right)$$

Mathematica DSolve solution

Solving time : `0.019` (sec)
Leaf size : 43

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]-4*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{-\frac{x^2}{2}}\,\mathrm{HermiteH}\left(-5, \frac{x}{\sqrt{2}}\right) + \frac{1}{3}c_2(x^4+6x^2+3)$$

2.1.228 Problem 231

Internal problem ID [9400]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 231
Date solved : `Monday, January 27, 2025 at 06:02:20 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4xy'' - xy' + 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.226 (sec)

Writing the ode as

$$4xy'' - xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x \\ B &= -x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x - 32}{64x} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x - 32 \\ t &= 64x \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x - 32}{64x}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.436: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64x$. There is a pole at $x = 0$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{8} - \frac{2}{x} - \frac{16}{x^2} - \frac{256}{x^3} - \frac{5120}{x^4} - \frac{114688}{x^5} - \frac{2752512}{x^6} - \frac{69206016}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{8}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{8} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{64}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x-32}{64x} \\ &= Q + \frac{R}{64x} \\ &= \left(\frac{1}{64}\right) + \left(-\frac{1}{2x}\right) \\ &= \frac{1}{64} - \frac{1}{2x} \end{aligned}$$

Since the degree of t is 1, then we see that the coefficient of the term 1 in the remainder R is -32. Dividing this by leading coefficient in t which is 64 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{8} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{8}} - 0\right) &= -2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{8}} - 0\right) &= 2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x-32}{64x}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{8}$	-2	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= 2 - (1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{x} + (-)\left(\frac{1}{8}\right) \\ &= \frac{1}{x} - \frac{1}{8} \\ &= \frac{1}{x} - \frac{1}{8} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x} - \frac{1}{8}\right)(1) + \left(\left(-\frac{1}{x^2}\right) + \left(\frac{1}{x} - \frac{1}{8}\right)^2 - \left(\frac{x-32}{64x}\right)\right) = 0$$

$$\frac{8 + a_0}{4x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -8\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = -8 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (-8 + x)\, e^{\int \left(\frac{1}{x} - \frac{1}{8}\right) dx} \\ &= (-8 + x)\, e^{-\frac{x}{8} + \ln(x)} \\ &= (-8 + x)\, x\, e^{-\frac{x}{8}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{4x}\,dx} \\ &= z_1 e^{\frac{x}{8}} \\ &= z_1\left(\mathrm{e}^{\frac{x}{8}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (-8+x)\,x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{4x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\frac{x}{4}}}{(y_1)^2}\,dx \\ &= y_1\left(-\frac{\mathrm{e}^{\frac{x}{4}}}{64x} - \frac{\mathrm{Ei}_1\left(-\frac{x}{4}\right)}{128} - \frac{\mathrm{e}^{\frac{x}{4}}}{256\left(-2+\frac{x}{4}\right)}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1((-8+x)\,x) + c_2\left((-8+x)\,x\left(-\frac{\mathrm{e}^{\frac{x}{4}}}{64x} - \frac{\mathrm{Ei}_1\left(-\frac{x}{4}\right)}{128} - \frac{\mathrm{e}^{\frac{x}{4}}}{256\left(-2+\frac{x}{4}\right)}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 33

```
dsolve(4*x*diff(diff(y(x),x),x)-diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{x c_2 (x-8) \operatorname{Ei}_1\left(-\frac{x}{4}\right)}{16} + \frac{c_2 (x-4) \, \mathrm{e}^{\frac{x}{4}}}{4} + c_1 x (x-8)$$

Mathematica DSolve solution

Solving time : 0.296 (sec)
Leaf size : 42

```
DSolve[{4*x*D[y[x],{x,2}]-x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to (x-8) x \left(c_2 \int_1^x \frac{e^{\frac{K[1]}{4}}}{(K[1]-8)^2 K[1]^2} dK[1] + c_1\right)$$

2.1.229 Problem 232

Internal problem ID [9401]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 232
Date solved : `Monday, January 27, 2025 at 06:02:21 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$6x^2y'' + x(1+18x)\,y' + (1+12x)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.273 (sec)

Writing the ode as

$$6x^2y'' + \left(18x^2 + x\right)y' + (1+12x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 6x^2 \\ B &= 18x^2 + x \\ C &= 1 + 12x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{324x^2 - 252x - 35}{144x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 324x^2 - 252x - 35 \\ t &= 144x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{324x^2 - 252x - 35}{144x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.437: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 144x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{4} - \frac{7}{4x} - \frac{35}{144x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3}{2} - \frac{7}{12x} - \frac{7}{36x^2} - \frac{49}{648x^3} - \frac{245}{5832x^4} - \frac{343}{13122x^5} - \frac{66199}{3779136x^6} - \frac{837949}{68024448x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{3}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{324x^2 - 252x - 35}{144x^2} \\ &= Q + \frac{R}{144x^2} \\ &= \left(\frac{9}{4}\right) + \left(\frac{-252x - 35}{144x^2}\right) \\ &= \frac{9}{4} + \frac{-252x - 35}{144x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -252. Dividing this by leading coefficient in t which is 144 gives $-\frac{7}{4}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{7}{4}\right) - (0) \\ &= -\frac{7}{4} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{3}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{-\frac{7}{4}}{\frac{3}{2}} - 0\right) = -\frac{7}{12} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{-\frac{7}{4}}{\frac{3}{2}} - 0\right) = \frac{7}{12} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{324x^2 - 252x - 35}{144x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{12}$	$\frac{5}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{3}{2}$	$-\frac{7}{12}$	$\frac{7}{12}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{7}{12}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{7}{12} - \left(\frac{7}{12}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{7}{12x} + (-)\left(\frac{3}{2}\right) \\ &= \frac{7}{12x} - \frac{3}{2} \\ &= \frac{7}{12x} - \frac{3}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered} (0) + 2\left(\frac{7}{12x} - \frac{3}{2}\right)(0) + \left(\left(-\frac{7}{12x^2}\right) + \left(\frac{7}{12x} - \frac{3}{2}\right)^2 - \left(\frac{324x^2 - 252x - 35}{144x^2}\right)\right) = 0 \\ 0 = 0 \end{gathered}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{7}{12x} - \frac{3}{2}\right) dx} \\ &= x^{7/12} e^{-\frac{3x}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{18x^2+x}{6x^2}\,dx} \\ &= z_1 e^{-\frac{3x}{2}-\frac{\ln(x)}{12}} \\ &= z_1\left(\frac{\mathrm{e}^{-\frac{3x}{2}}}{x^{1/12}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}\,\mathrm{e}^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{18x^2+x}{6x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-3x-\frac{\ln(x)}{6}}}{(y_1)^2}\,dx \\ &= y_1\left(\int \frac{\mathrm{e}^{-3x-\frac{\ln(x)}{6}}\mathrm{e}^{6x}}{x}\,dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\sqrt{x}\,\mathrm{e}^{-3x}\right) + c_2\left(\sqrt{x}\,\mathrm{e}^{-3x}\left(\int \frac{\mathrm{e}^{-3x-\frac{\ln(x)}{6}}\mathrm{e}^{6x}}{x}\,dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$6x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(1+18x)\left(\frac{d}{dx}y(x)\right) + (1+12x)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(1+12x)y(x)}{6x^2} - \frac{(1+18x)\left(\frac{d}{dx}y(x)\right)}{6x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(1+18x)\left(\frac{d}{dx}y(x)\right)}{6x} + \frac{(1+12x)y(x)}{6x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1+18x}{6x}, P_3(x) = \frac{1+12x}{6x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = \frac{1}{6}$
- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{1}{6}$
- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $6x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(1+18x)\left(\frac{d}{dx}y(x)\right) + (1+12x)\,y(x) = 0$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- Shift index using $k- >k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
- Shift index using $k- >k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+3r)\,(-1+2r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(3k+3r-1)\,(2k+2r-1) + 6a_{k-1}(3k+3r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+3r)\,(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2}, \frac{1}{3}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $6\left(\left(k+r-\frac{1}{2}\right)a_k + 3a_{k-1}\right)\left(k-\frac{1}{3}+r\right) = 0$
- Shift index using $k- >k+1$

  $6\left(\left(k+\frac{1}{2}+r\right)a_{k+1} + 3a_k\right)\left(k+\frac{2}{3}+r\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{6a_k}{2k+1+2r}$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = -\frac{6a_k}{2k+2}$
- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{6a_k}{2k+2}\right]$

- Recursion relation for $r = \frac{1}{3}$

  $a_{k+1} = -\frac{6a_k}{2k+\frac{5}{3}}$
- Solution for $r = \frac{1}{3}$

  $$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+1} = -\frac{6a_k}{2k+\frac{5}{3}}\right]$$
- Combine solutions and rename parameters

  $$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+1} = -\frac{6a_k}{2k+2}, b_{k+1} = -\frac{6b_k}{2k+\frac{5}{3}}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special functio
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.024 (sec)
Leaf size : 40

```
dsolve(6*x^2*diff(diff(y(x),x),x)+x*(1+18*x)*diff(y(x),x)+(1+12*x)*y(x) = 0,y(x),sings
```

$$y = \frac{-\frac{c_2(-x)^{5/6}3^{5/6}}{3} + x\,\mathrm{e}^{-3x}\left(c_2\Gamma\left(\frac{5}{6}\right) - c_2\Gamma\left(\frac{5}{6}, -3x\right) + c_1\right)}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.32 (sec)
Leaf size : 47

```
DSolve[{6*x^2*D[y[x],{x,2}]+x*(1+18*x)*D[y[x],x]+(1+12*x)*y[x]==0,{}},y[x],x,IncludeSingular
```

$$y(x) \to e^{-3x}\left(\frac{\sqrt[6]{3}c_2x^{4/3}\Gamma\left(-\frac{1}{6}, -3x\right)}{(-x)^{5/6}} + c_1\sqrt{x}\right)$$

2.1.230 Problem 233

Internal problem ID [9402]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 233
Date solved : `Monday, January 27, 2025 at 06:02:22 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$3x^2y'' - x(x+8)\,y' + 6y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 3.613 (sec)

Writing the ode as

$$3x^2y'' + \left(-x^2-8x\right)y' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^2 \\ B &= -x^2-8x \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2+16x+40}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2+16x+40 \\ t &= 36x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2+16x+40}{36x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.439: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{36} + \frac{10}{9x^2} + \frac{4}{9x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{10}{9}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{3} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{2}{3} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{6} + \frac{4}{3x} - \frac{2}{x^2} + \frac{16}{x^3} - \frac{140}{x^4} + \frac{1312}{x^5} - \frac{12944}{x^6} + \frac{132736}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{6} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{36}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 16x + 40}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{1}{36}\right) + \left(\frac{16x + 40}{36x^2}\right) \\ &= \frac{1}{36} + \frac{16x + 40}{36x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 16. Dividing this by leading coefficient in t which is 36 gives $\frac{4}{9}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{4}{9}\right) - (0) \\ &= \frac{4}{9} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{6} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{4}{9}}{\frac{1}{6}} - 0\right) &&= \frac{4}{3} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{4}{9}}{\frac{1}{6}} - 0\right) &&= -\frac{4}{3} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 16x + 40}{36x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{3}$	$-\frac{2}{3}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{6}$	$\frac{4}{3}$	$-\frac{4}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{4}{3}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= \frac{4}{3} - \left(-\frac{2}{3}\right) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{2}{3x} + \left(\frac{1}{6}\right) \\ &= -\frac{2}{3x} + \frac{1}{6} \\ &= \frac{-4+x}{6x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(-\frac{2}{3x} + \frac{1}{6}\right)(2x + a_1) + \left(\left(\frac{2}{3x^2}\right) + \left(-\frac{2}{3x} + \frac{1}{6}\right)^2 - \left(\frac{x^2+16x+40}{36x^2}\right)\right) = 0$$

$$\frac{(-a_1-2)\,x - 2a_0 - 4a_1}{3x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 4, a_1 = -2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 2x + 4$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 2x + 4\right) e^{\int \left(-\frac{2}{3x} + \frac{1}{6}\right) dx} \\
&= \left(x^2 - 2x + 4\right) e^{\frac{x}{6} - \frac{2\ln(x)}{3}} \\
&= \frac{\left(x^2 - 2x + 4\right) e^{\frac{x}{6}}}{x^{2/3}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x^2 - 8x}{3x^2} \, dx} \\
&= z_1 e^{\frac{x}{6} + \frac{4\ln(x)}{3}} \\
&= z_1 \left(x^{4/3} e^{\frac{x}{6}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^{2/3} e^{\frac{x}{3}} \left(x^2 - 2x + 4\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2 - 8x}{3x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{x}{3} + \frac{8\ln(x)}{3}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{e^{\frac{x}{3} + \frac{8\ln(x)}{3}} e^{-\frac{2x}{3}}}{x^{4/3} \left(x^2 - 2x + 4\right)^2} dx \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{2/3} e^{\frac{x}{3}} \left(x^2 - 2x + 4\right)\right) + c_2 \left(x^{2/3} e^{\frac{x}{3}} \left(x^2 - 2x + 4\right) \left(\int \frac{e^{\frac{x}{3} + \frac{8\ln(x)}{3}} e^{-\frac{2x}{3}}}{x^{4/3} \left(x^2 - 2x + 4\right)^2} dx \right) \right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$3x^2\left(\frac{d^2}{dx^2}y(x)\right)-x(x+8)\left(\frac{d}{dx}y(x)\right)+6y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{2y(x)}{x^2}+\frac{(x+8)\left(\frac{d}{dx}y(x)\right)}{3x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)-\frac{(x+8)\left(\frac{d}{dx}y(x)\right)}{3x}+\frac{2y(x)}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{x+8}{3x},P_3(x)=\frac{2}{x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=-\frac{8}{3}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=2$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $3x^2\left(\frac{d^2}{dx^2}y(x)\right)-x(x+8)\left(\frac{d}{dx}y(x)\right)+6y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+3r)\,(-3+r)\,x^r+\left(\sum_{k=1}^{\infty}\left(a_k(3k+3r-2)\,(k+r-3)-a_{k-1}(k+r-1)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+3r)\,(-3+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{3,\frac{2}{3}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $3\left(k+r-\frac{2}{3}\right)(k+r-3)\,a_k-a_{k-1}(k+r-1)=0$

- Shift index using $k->k+1$

$3\left(k+\frac{1}{3}+r\right)(k-2+r)\,a_{k+1}-a_k(k+r)=0$

- Recursion relation that defines series solution to ODE

$a_{k+1}=\frac{a_k(k+r)}{(3k+1+3r)(k-2+r)}$

- Recursion relation for $r=3$

$a_{k+1}=\frac{a_k(k+3)}{(3k+10)(k+1)}$

- Solution for $r=3$

$\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1}=\frac{a_k(k+3)}{(3k+10)(k+1)}\right]$

- Recursion relation for $r=\frac{2}{3}$

$a_{k+1}=\frac{a_k\left(k+\frac{2}{3}\right)}{(3k+3)\left(k-\frac{4}{3}\right)}$

- Solution for $r=\frac{2}{3}$

$\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+\frac{2}{3}}, a_{k+1}=\frac{a_k\left(k+\frac{2}{3}\right)}{(3k+3)\left(k-\frac{4}{3}\right)}\right]$

- Combine solutions and rename parameters

$\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^{k+3}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{2}{3}}\right), a_{k+1}=\frac{a_k(k+3)}{(3k+10)(k+1)}, b_{k+1}=\frac{b_k\left(k+\frac{2}{3}\right)}{(3k+3)\left(k-\frac{4}{3}\right)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solut
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.052 (sec)
Leaf size : 38

```
dsolve(3*x^2*diff(diff(y(x),x),x)-x*(x+8)*diff(y(x),x)+6*y(x) = 0,y(x),singsol=all)
```

$$y = c_2\left(x^{2/3}-\frac{x^{5/3}}{2}+\frac{x^{8/3}}{4}\right)\mathrm{e}^{\frac{x}{3}}+c_1\,\mathrm{hypergeom}\left([3],\left[\frac{10}{3}\right],\frac{x}{3}\right)x^3$$

Mathematica DSolve solution

Solving time : 0.818 (sec)
Leaf size : 99

```
DSolve[{3*x^2*D[y[x],{x,2}]-x*(x+8)*D[y[x],x]+6*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x)\to x^{4/3}\left(x^2-2x+4\right)\exp\left(\frac{1}{6}\left(6\int_1^x\frac{K[1]-4}{6K[1]}dK[1]+x+8\right)\right)\left(c_2\int_1^x\frac{\exp\left(-2\int_1^{K[2]}\frac{K[1]-4}{6K[1]}dK[1]\right)}{\left(K[2]^2-2K[2]+4\right)^2}dK[2]+c_1\right)$$

2.1.231 Problem 234

Internal problem ID [9403]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 234
Date solved : `Monday, January 27, 2025 at 06:02:26 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2y'' - x(1+2x)\,y' + 2(4x-1)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.737 (sec)

Writing the ode as

$$2x^2y'' + \left(-2x^2 - x\right)y' + (8x-2)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= -2x^2 - x \\ C &= 8x - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 - 60x + 21}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2 - 60x + 21 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 - 60x + 21}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.441: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{21}{16x^2} - \frac{15}{4x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{4} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{15}{4x} - \frac{51}{4x^2} - \frac{765}{8x^3} - \frac{3519}{4x^4} - \frac{144585}{16x^5} - \frac{6358527}{64x^6} - \frac{146409525}{128x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 - 60x + 21}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-60x + 21}{16x^2}\right) \\ &= \frac{1}{4} + \frac{-60x + 21}{16x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -60. Dividing this by leading coefficient in t which is 16 gives $-\frac{15}{4}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{15}{4}\right) - (0) \\ &= -\frac{15}{4} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{15}{4}}{\frac{1}{2}} - 0\right) &&= -\frac{15}{4} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{15}{4}}{\frac{1}{2}} - 0\right) &&= \frac{15}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^2 - 60x + 21}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{15}{4}$	$\frac{15}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{15}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^+) \\ &= \frac{15}{4} - \left(\frac{7}{4}\right) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{7}{4x} + (-)\left(\frac{1}{2}\right) \\ &= \frac{7}{4x} - \frac{1}{2} \\ &= \frac{7}{4x} - \frac{1}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{7}{4x} - \frac{1}{2}\right)(2x + a_1) + \left(\left(-\frac{7}{4x^2}\right) + \left(\frac{7}{4x} - \frac{1}{2}\right)^2 - \left(\frac{4x^2 - 60x + 21}{16x^2}\right)\right) = 0$$

$$\frac{2(9 + a_1)\,x + 4a_0 + 7a_1}{2x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{63}{4}, a_1 = -9\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 9x + \frac{63}{4}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 9x + \frac{63}{4}\right) e^{\int \left(\frac{7}{4x} - \frac{1}{2}\right) dx} \\
&= \left(x^2 - 9x + \frac{63}{4}\right) e^{-\frac{x}{2} + \frac{7\ln(x)}{4}} \\
&= \frac{(4x^2 - 36x + 63)\, x^{7/4} e^{-\frac{x}{2}}}{4}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-2x^2 - x}{2x^2} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x)}{4}} \\
&= z_1 \left(x^{1/4} e^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^4 - 9x^3 + \frac{63}{4} x^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^2 - x}{2x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{x + \frac{\ln(x)}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{e^{x + \frac{\ln(x)}{2}}}{\left(x^4 - 9x^3 + \frac{63}{4}x^2\right)^2} dx \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^4 - 9x^3 + \frac{63}{4}x^2\right) + c_2 \left(x^4 - 9x^3 + \frac{63}{4}x^2 \left(\int \frac{e^{x + \frac{\ln(x)}{2}}}{\left(x^4 - 9x^3 + \frac{63}{4}x^2\right)^2} dx \right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2\left(\frac{d^2}{dx^2}y(x)\right) - x(2x+1)\left(\frac{d}{dx}y(x)\right) + 2(4x-1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(4x-1)y(x)}{x^2} + \frac{(2x+1)\left(\frac{d}{dx}y(x)\right)}{2x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) - \frac{(2x+1)\left(\frac{d}{dx}y(x)\right)}{2x} + \frac{(4x-1)y(x)}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2x+1}{2x}, P_3(x) = \frac{4x-1}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -\frac{1}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -1$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x^2\left(\frac{d^2}{dx^2}y(x)\right) - x(2x+1)\left(\frac{d}{dx}y(x)\right) + (8x-2)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)(-2+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)(k+r-2) - 2a_{k-1}(k-5+r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(1+2r)(-2+r)=0$

- Values of r that satisfy the indicial equation

$r\in\left\{2,-\frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

$2\left(k+r+\frac{1}{2}\right)(k+r-2)a_k-2a_{k-1}(k-5+r)=0$

- Shift index using $k->k+1$

$2\left(k+\frac{3}{2}+r\right)(k+r-1)a_{k+1}-2a_k(k+r-4)=0$

- Recursion relation that defines series solution to ODE

$a_{k+1}=\frac{2a_k(k+r-4)}{(2k+3+2r)(k+r-1)}$

- Recursion relation for $r=2$; series terminates at $k=2$

$a_{k+1}=\frac{2a_k(k-2)}{(2k+7)(k+1)}$

- Apply recursion relation for $k=0$

$a_1=-\frac{4a_0}{7}$

- Apply recursion relation for $k=1$

$a_2=-\frac{a_1}{9}$

- Express in terms of a_0

$a_2=\frac{4a_0}{63}$

- Terminating series solution of the ODE for $r=2$. Use reduction of order to find the second linea

$y(x)=a_0\cdot\left(1-\frac{4}{7}x+\frac{4}{63}x^2\right)$

- Recursion relation for $r=-\frac{1}{2}$

$a_{k+1}=\frac{2a_k\left(k-\frac{9}{2}\right)}{(2k+2)\left(k-\frac{3}{2}\right)}$

- Solution for $r=-\frac{1}{2}$

$\left[y(x)=\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{2}},a_{k+1}=\frac{2a_k\left(k-\frac{9}{2}\right)}{(2k+2)\left(k-\frac{3}{2}\right)}\right]$

- Combine solutions and rename parameters

$\left[y(x)=a_0\cdot\left(1-\frac{4}{7}x+\frac{4}{63}x^2\right)+\left(\sum_{k=0}^{\infty}b_k x^{k-\frac{1}{2}}\right),b_{k+1}=\frac{2b_k\left(k-\frac{9}{2}\right)}{(2k+2)\left(k-\frac{3}{2}\right)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solut
```

```
    -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
    <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
    -> Trying to convert hypergeometric functions to elementary form...
    <- elementary form for at least one hypergeometric solution is achieved - return
  <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.053 (sec)
Leaf size : 32

```
dsolve(2*x^2*diff(diff(y(x),x),x)-x*(2*x+1)*diff(y(x),x)+2*(-1+4*x)*y(x) = 0,y(x),sing
```

$$y = \frac{c_1 x^2(4x^2 - 36x + 63)}{63} + \frac{c_2 \operatorname{hypergeom}\left(\left[-\frac{9}{2}\right], \left[-\frac{3}{2}\right], x\right)}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 1.857 (sec)
Leaf size : 61

```
DSolve[{2*x^2*D[y[x],{x,2}]-x*(1+2*x)*D[y[x],x]+2*(4*x-1)*y[x]==0,{}},y[x],x,IncludeSingular
```

$$y(x) \to \frac{1}{4}x^2\left(4x^2 - 36x + 63\right)\left(c_2 \int_1^x \frac{16e^{K[1]}}{K[1]^{7/2}\left(4K[1]^2 - 36K[1] + 63\right)^2} dK[1] + c_1\right)$$

2.1.232 Problem 235

Internal problem ID [9404]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 235
Date solved : `Monday, January 27, 2025 at 06:02:27 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' - 4x^2y' + (1+2x)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.221 (sec)

Writing the ode as

$$4x^2y'' - 4x^2y' + (1+2x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x^2 \\ C &= 1+2x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 2x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 2x - 1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 2x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.443: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} - \frac{1}{2x^2} - \frac{1}{2x^3} - \frac{3}{4x^4} - \frac{5}{4x^5} - \frac{9}{4x^6} - \frac{17}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 2x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-2x - 1}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 2x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\left(\frac{1}{2}\right) \\ &= \frac{1}{2x} - \frac{1}{2} \\ &= -\frac{x-1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2x} - \frac{1}{2}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^2-2x-1}{4x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{1}{2x} - \frac{1}{2}\right) dx} \\ &= \sqrt{x}\, e^{-\frac{x}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x^2}{4x^2}\,dx} \\ &= z_1 e^{\frac{x}{2}} \\ &= z_1\left(\mathrm{e}^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^2}{4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{x}}{(y_1)^2}\,dx \\ &= y_1(-\operatorname{Ei}_1(-x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\sqrt{x}\right) + c_2\left(\sqrt{x}(-\operatorname{Ei}_1(-x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2\left(\frac{d^2}{dx^2}y(x)\right) - 4x^2\left(\frac{d}{dx}y(x)\right) + (2x+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(2x+1)y(x)}{4x^2} + \frac{d}{dx}y(x)$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) - \frac{d}{dx}y(x) + \frac{(2x+1)y(x)}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -1, P_3(x) = \frac{2x+1}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 0$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{1}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) - 4x^2\left(\frac{d}{dx}y(x)\right) + (2x+1)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k-m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^2 \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r+1}$

  - Shift index using $k- >k-1$

    $x^2 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\,x^{k+r}$

  - Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)^2\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)^2 - 2a_{k-1}(2k-3+2r)\right)x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = \frac{1}{2}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(2k+2r-1)^2 + (-4k-4r+6)\,a_{k-1} = 0$

- Shift index using $k- >k+1$

  $a_{k+1}(2k+1+2r)^2 + a_k(-4k-4r+2) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k(2k+2r-1)}{(2k+1+2r)^2}$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = \frac{4a_k k}{(2k+2)^2}$

- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{4a_k k}{(2k+2)^2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 17

```
dsolve(4*x^2*diff(diff(y(x),x),x)-4*diff(y(x),x)*x^2+(2*x+1)*y(x) = 0,y(x),singsol=all)
```

$$y = (\mathrm{Ei}_1(-x)\, c_2 + c_1)\sqrt{x}$$

Mathematica DSolve solution

Solving time : 0.205 (sec)
Leaf size : 33

```
DSolve[{4*x^2*D[y[x],{x,2}]-4*x^2*D[y[x],x]+(1+2*x)*y[x]==0,{}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \sqrt{x}\left(c_2 \int_1^x \frac{e^{K[1]}}{K[1]} dK[1] + c_1\right)$$

2.1.233 Problem 236

Internal problem ID [9405]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 236
Date solved : `Monday, January 27, 2025 at 06:02:28 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x(3-2x)\,y' + (1-2x)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.248 (sec)

Writing the ode as

$$x^2y'' + \left(-2x^2+3x\right)y' + (1-2x)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x^2+3x \\ C &= 1-2x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2-4x-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2-4x-1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2-4x-1}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.445: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 2\\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 - \frac{1}{x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{0} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 - \frac{1}{2x} - \frac{1}{4x^2} - \frac{1}{8x^3} - \frac{3}{32x^4} - \frac{5}{64x^5} - \frac{9}{128x^6} - \frac{17}{256x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 - 4x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (1) + \left(\frac{-4x - 1}{4x^2}\right) \\ &= 1 + \frac{-4x - 1}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-1}{1} - 0\right) &= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-1}{1} - 0\right) &= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^2 - 4x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^+) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)(1) \\ &= \frac{1}{2x} - 1 \\ &= \frac{1}{2x} - 1 \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2x} - 1\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} - 1\right)^2 - \left(\frac{4x^2 - 4x - 1}{4x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{1}{2x} - 1\right) dx} \\ &= \sqrt{x}\, e^{-x} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x^2+3x}{x^2}\,dx} \\
&= z_1 e^{x-\frac{3\ln(x)}{2}} \\
&= z_1\left(\frac{e^x}{x^{3/2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^2+3x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x-3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(-\operatorname{Ei}_1(-2x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x}(-\operatorname{Ei}_1(-2x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(3-2x)\left(\frac{d}{dx}y(x)\right) + (-2x+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = \frac{(2x-1)y(x)}{x^2} + \frac{(2x-3)\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x) - \frac{(2x-3)\left(\frac{d}{dx}y(x)\right)}{x} - \frac{(2x-1)y(x)}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{2x-3}{x}, P_3(x) = -\frac{2x-1}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

$\left(x \cdot P_2(x)\right)\Big|_{x=0} = 3$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 1$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) - (2x-3)\,x\left(\frac{d}{dx}y(x)\right) + (-2x+1)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)^2\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)^2 - 2a_{k-1}(k+r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = -1$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)^2 - 2a_{k-1}(k+r) = 0$

- Shift index using $k- >k+1$

  $a_{k+1}(k+2+r)^2 - 2a_k(k+r+1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k(k+r+1)}{(k+2+r)^2}$

- Recursion relation for $r = -1$

  $a_{k+1} = \frac{2a_k k}{(k+1)^2}$

- Solution for $r = -1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = \frac{2a_k k}{(k+1)^2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(-2*x+3)*diff(y(x),x)+(1-2*x)*y(x) = 0,y(x),singsol=
```

$$y = \frac{c_2 \operatorname{Ei}_1(-2x) + c_1}{x}$$

Mathematica DSolve solution

Solving time : 0.2 (sec)
Leaf size : 33

```
DSolve[{x^2*D[y[x],{x,2}]+x*(3-2*x)*D[y[x],x]+(1-2*x)*y[x]==0,{}},y[x],x,IncludeSingularSolu
```

$$y(x) \rightarrow \frac{c_2 \int_1^x \frac{e^{2K[1]}}{K[1]} dK[1] + c_1}{x}$$

2.1.234 Problem 237

Internal problem ID [9406]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 237
Date solved : `Monday, January 27, 2025 at 06:02:28 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x(3+x)\,y' + (4-x)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.280 (sec)

Writing the ode as

$$x^2y'' + \left(-x^2-3x\right)y' + (4-x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2-3x \\ C &= 4-x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2+10x-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2+10x-1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2+10x-1}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.447: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{5}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{5}{2x} - \frac{13}{2x^2} + \frac{65}{2x^3} - \frac{819}{4x^4} + \frac{5785}{4x^5} - \frac{43797}{4x^6} + \frac{347425}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 10x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{10x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{10x - 1}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 10. Dividing this by leading coefficient in t which is 4 gives $\frac{5}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{5}{2}\right) - (0) \\ &= \frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{5}{2}}{\frac{1}{2}} - 0\right) &= \frac{5}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{5}{2}}{\frac{1}{2}} - 0\right) &= -\frac{5}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 10x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{5}{2}$	$-\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^+) \\ &= \frac{5}{2} - \left(\frac{1}{2}\right) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2x} + \frac{1}{2} \\ &= \frac{1+x}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered} (2) + 2\left(\frac{1}{2x} + \frac{1}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} + \frac{1}{2}\right)^2 - \left(\frac{x^2 + 10x - 1}{4x^2}\right)\right) = 0 \\ \frac{(-a_1 + 4)\, x - 2a_0 + a_1}{x} = 0 \end{gathered}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2, a_1 = 4\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + 4x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x^2+4x+2\right) e^{\int \left(\frac{1}{2x}+\frac{1}{2}\right)dx} \\ &= \left(x^2+4x+2\right) e^{\frac{x}{2}+\frac{\ln(x)}{2}} \\ &= \left(x^2+4x+2\right) \sqrt{x}\, e^{\frac{x}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x^2-3x}{x^2} \, dx} \\ &= z_1 e^{\frac{x}{2}+\frac{3\ln(x)}{2}} \\ &= z_1 \left(x^{3/2} e^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^2 e^x \left(x^2+4x+2\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-3x}{x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{x+3\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{e^{-x}(-x-3)}{4\left(x^2+4x+2\right)} - \frac{\mathrm{Ei}_1(x)}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(x^2 e^x \left(x^2+4x+2\right)\right) + c_2 \left(x^2 e^x \left(x^2+4x+2\right) \left(-\frac{e^{-x}(-x-3)}{4\left(x^2+4x+2\right)} - \frac{\mathrm{Ei}_1(x)}{4}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2 \left(\frac{d^2}{dx^2}y(x)\right) - x(x+3)\left(\frac{d}{dx}y(x)\right) + (4-x)\, y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{(-4+x)y(x)}{x^2} + \frac{(x+3)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x) - \frac{(x+3)\left(\frac{d}{dx}y(x)\right)}{x} - \frac{(-4+x)y(x)}{x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - ◦ Define functions

    $\left[P_2(x) = -\frac{x+3}{x}, P_3(x) = -\frac{-4+x}{x^2}\right]$
  - ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -3$
  - ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 4$
  - ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) - x(x+3)\left(\frac{d}{dx}y(x)\right) + (4-x)\,y(x) = 0$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - ◦ Shift index using $k- >k-m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - ◦ Shift index using $k- >k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - ◦ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

    Rewrite ODE with series expansions

    $a_0(-2+r)^2\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-2)^2 - a_{k-1}(k+r)\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 2$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-2)^2 - a_{k-1}(k+r) = 0$
- Shift index using $k- >k+1$

  $a_{k+1}(k+r-1)^2 - a_k(k+r+1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r+1)}{(k+r-1)^2}$
- Recursion relation for $r = 2$

$a_{k+1} = \frac{a_k(k+3)}{(k+1)^2}$

- Solution for $r = 2$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k(k+3)}{(k+1)^2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 42

```
dsolve(x^2*diff(diff(y(x),x),x)-x*(x+3)*diff(y(x),x)+(-x+4)*y(x) = 0,y(x),singsol=all)
```

$$y = \left(c_2 \mathrm{e}^{x} \left(x^2+4x+2\right) \mathrm{Ei}_1\left(x\right) + c_1 \left(x^2+4x+2\right) \mathrm{e}^{x} - c_2(x+3)\right) x^2$$

Mathematica DSolve solution

Solving time : `0.427` (sec)
Leaf size : 60

```
DSolve[{x^2*D[y[x],{x,2}]-x*(3+x)*D[y[x],x]+(4-x)*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to e^{x+2} x^2 \left(x^2+4x+2\right) \left(c_2 \int_1^x \frac{e^{-K[1]-1}}{K[1] \left(K[1]^2+4K[1]+2\right)^2} dK[1] + c_1\right)$$

2.1.235 Problem 238

Internal problem ID [9407]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 238
Date solved : `Monday, January 27, 2025 at 06:02:29 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x(3-x)\,y' + y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.276 (sec)

Writing the ode as

$$x^2y'' + \left(-x^2+3x\right)y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2+3x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2-6x-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2-6x-1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2-6x-1}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.449: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{3}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{3}{2x} - \frac{5}{2x^2} - \frac{15}{2x^3} - \frac{115}{4x^4} - \frac{495}{4x^5} - \frac{2285}{4x^6} - \frac{11055}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 6x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-6x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-6x - 1}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -6. Dividing this by leading coefficient in t which is 4 gives $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{3}{2}\right) - (0) \\ &= -\frac{3}{2}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &&= \frac{3}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 6x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{3}{2} - \left(\frac{1}{2}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\left(\frac{1}{2}\right) \\ &= \frac{1}{2x} - \frac{1}{2} \\ &= -\frac{-1+x}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered} (0) + 2\left(\frac{1}{2x} - \frac{1}{2}\right)(1) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^2-6x-1}{4x^2}\right)\right) = 0 \\ \frac{1+a_0}{x} = 0 \end{gathered}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = -1 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (-1+x)\, e^{\int \left(\frac{1}{2x}-\frac{1}{2}\right)dx} \\
&= (-1+x)\, e^{-\frac{x}{2}+\frac{\ln(x)}{2}} \\
&= (-1+x)\sqrt{x}\, e^{-\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2+3x}{x^2}\, dx} \\
&= z_1 e^{\frac{x}{2}-\frac{3\ln(x)}{2}} \\
&= z_1 \left(\frac{e^{\frac{x}{2}}}{x^{3/2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{-1+x}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+3x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x-3\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{e^x}{-1+x} - \mathrm{Ei}_1(-x)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{-1+x}{x}\right) + c_2\left(\frac{-1+x}{x}\left(-\frac{e^x}{-1+x} - \mathrm{Ei}_1(-x)\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right)+x(-x+3)\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
$\frac{d^2}{dx^2}y(x)=-\frac{y(x)}{x^2}+\frac{(x-3)\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
$\frac{d^2}{dx^2}y(x)-\frac{(x-3)\left(\frac{d}{dx}y(x)\right)}{x}+\frac{y(x)}{x^2}=0$
- ☐ Check to see if $x_0=0$ is a regular singular point
  - Define functions
  $\left[P_2(x)=-\frac{x-3}{x},P_3(x)=\frac{1}{x^2}\right]$
  - $x\cdot P_2(x)$ is analytic at $x=0$
  $\left.(x\cdot P_2(x))\right|_{x=0}=3$
  - $x^2\cdot P_3(x)$ is analytic at $x=0$
  $\left.(x^2\cdot P_3(x))\right|_{x=0}=1$
  - $x=0$is a regular singular point
  Check to see if $x_0=0$ is a regular singular point
  $x_0=0$
- Multiply by denominators
$x^2\left(\frac{d^2}{dx^2}y(x)\right)-x(x-3)\left(\frac{d}{dx}y(x)\right)+y(x)=0$
- Assume series solution for $y(x)$
$y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$
  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k->k+1-m$
  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion
  $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$
  Rewrite ODE with series expansions
  $a_0(1+r)^2\,x^r+\left(\sum_{k=1}^{\infty}\left(a_k(k+r+1)^2-a_{k-1}(k+r-1)\right)x^{k+r}\right)=0$
- $a_0$cannot be 0 by assumption, giving the indicial equation
$(1+r)^2=0$
- Values of r that satisfy the indicial equation
$r=-1$
- Each term in the series must be 0, giving the recursion relation
$a_k(k+r+1)^2-a_{k-1}(k+r-1)=0$
- Shift index using $k->k+1$

$a_{k+1}(k+2+r)^2 - a_k(k+r) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r)}{(k+2+r)^2}$

- Recursion relation for $r = -1$; series terminates at $k = 1$

  $a_{k+1} = \frac{a_k(k-1)}{(k+1)^2}$

- Apply recursion relation for $k = 0$

  $a_1 = -a_0$

- Terminating series solution of the ODE for $r = -1$. Use reduction of order to find the second

  $y(x) = a_0 \cdot (1 - x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 28

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(-x+3)*diff(y(x),x)+y(x) = 0,y(x),singsol=all)
```

$$y = \frac{\mathrm{Ei}_1(-x)\, c_2(x-1) + \mathrm{e}^x c_2 + c_1(x-1)}{x}$$

Mathematica DSolve solution

Solving time : `0.415` (sec)
Leaf size : 40

```
DSolve[{x^2*D[y[x],{x,2}]+x*(3-x)*D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tru
```

$$y(x) \to \frac{(x-1)\left(c_2 \int_1^x \frac{e^{K[1]}}{(K[1]-1)^2 K[1]} dK[1] + c_1\right)}{x}$$

2.1.236 Problem 239

Internal problem ID [9408]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 239
Date solved : `Monday, January 27, 2025 at 06:02:30 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - \left(2\sqrt{5}-1\right)xy' + \left(\frac{19}{4} - 3x^2\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.146 (sec)

Writing the ode as

$$x^2y'' + \left(-2x\sqrt{5}+x\right)y' + \left(\frac{19}{4} - 3x^2\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x\sqrt{5}+x \\ C &= \frac{19}{4} - 3x^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 3z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.451: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 3$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\sqrt{3}\, x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x\sqrt{5}+x}{x^2}\, dx}\\ &= z_1 e^{\ln(x)\sqrt{5}-\frac{\ln(x)}{2}}\\ &= z_1\left(x^{\sqrt{5}-\frac{1}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = x^{\sqrt{5}-\frac{1}{2}}\mathrm{e}^{-\sqrt{3}\, x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x\sqrt{5}+x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\ln(x)\left(2\sqrt{5}-1\right)}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{\mathrm{e}^{\ln(x)\left(2\sqrt{5}-1\right)} x^{1-2\sqrt{5}} \mathrm{e}^{2\sqrt{3}\,x} \sqrt{3}}{6}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(x^{\sqrt{5}-\frac{1}{2}} \mathrm{e}^{-\sqrt{3}\,x}\right) + c_2 \left(x^{\sqrt{5}-\frac{1}{2}} \mathrm{e}^{-\sqrt{3}\,x} \left(\frac{\mathrm{e}^{\ln(x)\left(2\sqrt{5}-1\right)} x^{1-2\sqrt{5}} \mathrm{e}^{2\sqrt{3}\,x} \sqrt{3}}{6}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2 \left(\frac{d^2}{dx^2} y(x)\right) - \left(-1 + 2\sqrt{5}\right) x \left(\frac{d}{dx} y(x)\right) + \left(\frac{19}{4} - 3x^2\right) y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2} y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2} y(x) = \frac{(12x^2-19) y(x)}{4x^2} + \frac{\left(-1+2\sqrt{5}\right)\left(\frac{d}{dx} y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2} y(x) - \frac{\left(-1+2\sqrt{5}\right)\left(\frac{d}{dx} y(x)\right)}{x} - \frac{(12x^2-19) y(x)}{4x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{-1+2\sqrt{5}}{x}, P_3(x) = -\frac{12x^2-19}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1 - 2\sqrt{5}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{19}{4}$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4x^2 \left(\frac{d^2}{dx^2} y(x)\right) - 4\left(-1 + 2\sqrt{5}\right) x \left(\frac{d}{dx} y(x)\right) + \left(-12x^2 + 19\right) y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- > k - m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$
  - Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $\left(1+2\sqrt{5}-2r\right)\left(-1+2\sqrt{5}-2r\right) a_0 x^r + \left(-1+2\sqrt{5}-2r\right)\left(-3+2\sqrt{5}-2r\right) a_1 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(\left(-\right.\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $\left(1+2\sqrt{5}-2r\right)\left(-1+2\sqrt{5}-2r\right) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}+\sqrt{5}, \frac{1}{2}+\sqrt{5}\right\}$
- Each term must be 0

  $\left(-1+2\sqrt{5}-2r\right)\left(-3+2\sqrt{5}-2r\right) a_1 = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $-8a_k(k+r)\sqrt{5} + \left(4k^2+8kr+4r^2+19\right) a_k - 12a_{k-2} = 0$
- Shift index using $k- > k + 2$

  $-8a_{k+2}(k+2+r)\sqrt{5} + \left(4(k+2)^2 + 8(k+2)\,r + 4r^2 + 19\right) a_{k+2} - 12a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{12a_k}{-35+8k\sqrt{5}+8\sqrt{5}\,r-4k^2-8kr-4r^2+16\sqrt{5}-16k-16r}$
- Recursion relation for $r = -\frac{1}{2}+\sqrt{5}$

  $a_{k+2} = -\frac{12a_k}{-27+8k\sqrt{5}+8\sqrt{5}\left(-\frac{1}{2}+\sqrt{5}\right)-4k^2-8k\left(-\frac{1}{2}+\sqrt{5}\right)-4\left(-\frac{1}{2}+\sqrt{5}\right)^2-16k}$
- Solution for $r = -\frac{1}{2}+\sqrt{5}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}+\sqrt{5}}, a_{k+2} = -\frac{12a_k}{-27+8k\sqrt{5}+8\sqrt{5}\left(-\frac{1}{2}+\sqrt{5}\right)-4k^2-8k\left(-\frac{1}{2}+\sqrt{5}\right)-4\left(-\frac{1}{2}+\sqrt{5}\right)^2-16k}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{2}+\sqrt{5}$

  $a_{k+2} = -\frac{12a_k}{-43+8k\sqrt{5}+8\sqrt{5}\left(\frac{1}{2}+\sqrt{5}\right)-4k^2-8k\left(\frac{1}{2}+\sqrt{5}\right)-4\left(\frac{1}{2}+\sqrt{5}\right)^2-16k}$
- Solution for $r = \frac{1}{2}+\sqrt{5}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}+\sqrt{5}}, a_{k+2} = -\frac{12a_k}{-43+8k\sqrt{5}+8\sqrt{5}\left(\frac{1}{2}+\sqrt{5}\right)-4k^2-8k\left(\frac{1}{2}+\sqrt{5}\right)-4\left(\frac{1}{2}+\sqrt{5}\right)^2-16k}, a_1 = 0\right]$
- Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}+\sqrt{5}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}+\sqrt{5}}\right), a_{k+2}=-\frac{12a_k}{-27+8k\sqrt{5}+8\sqrt{5}\left(-\frac{1}{2}+\sqrt{5}\right)-4k^2-8k\left(-\frac{1}{2}+\sqrt{5}\right)-4\left(-\right.}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 29

```
dsolve(x^2*diff(diff(y(x),x),x)-(-1+2*5^(1/2))*x*diff(y(x),x)+(19/4-3*x^2)*y(x) = 0,y(x)
```

$$y=x^{-\frac{1}{2}+\sqrt{5}}\left(c_1\sinh\left(\sqrt{3}\,x\right)+c_2\cosh\left(\sqrt{3}\,x\right)\right)$$

Mathematica DSolve solution

Solving time : 0.094 (sec)
Leaf size : 53

```
DSolve[{x^2*D[y[x],{x,2}]-(2*Sqrt[5]-1)*x*D[y[x],x]+(19/4-3*x^2)*y[x]==0,{}},y[x],x,IncludeSin
```

$$y(x)\to\frac{1}{6}e^{-\sqrt{3}x}x^{\sqrt{5}-\frac{1}{2}}\left(\sqrt{3}c_2e^{2\sqrt{3}x}+6c_1\right)$$

2.1.237 Problem 240

Internal problem ID [9409]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 240
Date solved : `Monday, January 27, 2025 at 06:02:30 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x(x-3)\,y' + (4-x)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.232 (sec)

Writing the ode as

$$x^2y'' + \left(x^2-3x\right)y' + (4-x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2-3x \\ C &= 4-x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2-2x-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2-2x-1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2-2x-1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.453: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} - \frac{1}{2x^2} - \frac{1}{2x^3} - \frac{3}{4x^4} - \frac{5}{4x^5} - \frac{9}{4x^6} - \frac{17}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 2x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-2x - 1}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 2x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\left(\frac{1}{2}\right) \\ &= \frac{1}{2x} - \frac{1}{2} \\ &= -\frac{x-1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered} (0) + 2\left(\frac{1}{2x} - \frac{1}{2}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^2-2x-1}{4x^2}\right)\right) = 0 \\ 0 = 0 \end{gathered}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{1}{2x} - \frac{1}{2}\right) dx} \\ &= \sqrt{x}\, e^{-\frac{x}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2-3x}{x^2}\,dx} \\
&= z_1 e^{-\frac{x}{2}+\frac{3\ln(x)}{2}} \\
&= z_1\left(x^{3/2}\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^2\mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2-3x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x+3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(-\operatorname{Ei}_1(-x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(x^2\mathrm{e}^{-x}\right) + c_2\left(x^2\mathrm{e}^{-x}(-\operatorname{Ei}_1(-x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(x-3)\left(\frac{d}{dx}y(x)\right) + (4-x)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{(-4+x)y(x)}{x^2} - \frac{(x-3)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(x-3)\left(\frac{d}{dx}y(x)\right)}{x} - \frac{(-4+x)y(x)}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{x-3}{x}, P_3(x) = -\frac{-4+x}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -3$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 4$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(x-3)\left(\frac{d}{dx}y(x)\right) + (4-x)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k-m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)^2\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-2)^2 + a_{k-1}(k+r-2)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 2$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)\,(a_k(k+r-2) + a_{k-1}) = 0$

- Shift index using $k- >k+1$

  $(k+r-1)\,(a_{k+1}(k+r-1) + a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{k+r-1}$

- Recursion relation for $r = 2$

  $a_{k+1} = -\frac{a_k}{k+1}$

- Solution for $r = 2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = -\frac{a_k}{k+1}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 21

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(x-3)*diff(y(x),x)+(-x+4)*y(x) = 0,y(x),singsol=all)
```

$$y = e^{-x}x^2(\text{Ei}_1(-x)\, c_2 + c_1)$$

Mathematica DSolve solution

Solving time : 0.396 (sec)
Leaf size : 60

```
DSolve[{x^2*D[y[x],{x,2}]+x*(x-3)*D[y[x],x]+(4-x)*y[x]==0,{}},y[x],x,IncludeSingularSolution
```

$$y(x) \to \sqrt{x}\left(c_2 \int_1^x \frac{e^{K[2]}}{K[2]} dK[2] + c_1\right) \exp\left(\frac{1}{2}\left(-\int_1^x \left(1 - \frac{3}{K[1]}\right) dK[1] - x\right)\right)$$

2.1.238 Problem 241

Internal problem ID [9410]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 241
Date solved : `Monday, January 27, 2025 at 06:02:31 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x^2y' - (2 + x)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.208 (sec)

Writing the ode as

$$x^2y'' + x^2y' + (-x - 2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2 \\ C &= -x - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x + 8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 4x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.455: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{x^2} + \frac{1}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{x} + \frac{1}{x^2} - \frac{2}{x^3} + \frac{3}{x^4} - \frac{2}{x^5} - \frac{6}{x^6} + \frac{28}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{4x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{4x + 8}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$\begin{aligned} b &= (1) - (0) \\ &= 1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{1}{\frac{1}{2}} - 0\right) &&= 1 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{1}{\frac{1}{2}} - 0\right) &&= -1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 4x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	1	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + (-)\left(\frac{1}{2}\right) \\ &= -\frac{1}{x} - \frac{1}{2} \\ &= -\frac{2+x}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{x} - \frac{1}{2}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - \frac{1}{2}\right)^2 - \left(\frac{x^2+4x+8}{4x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{x} - \frac{1}{2}\right) dx} \\ &= \frac{e^{-\frac{x}{2}}}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^2}{x^2}\,dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-x}}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\ &= y_1\left(\left(x^2-2x+2\right)\mathrm{e}^{x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\mathrm{e}^{-x}}{x}\right) + c_2\left(\frac{\mathrm{e}^{-x}}{x}\left(\left(x^2-2x+2\right)\mathrm{e}^{x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x^2\left(\frac{d}{dx}y(x)\right) - (x+2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{(x+2)y(x)}{x^2} - \frac{d}{dx}y(x)$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{d}{dx}y(x) - \frac{(x+2)y(x)}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = 1, P_3(x) = -\frac{x+2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 0$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -2$$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) + x^2\left(\frac{d}{dx}y(x)\right) + (-x-2)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - ◦ Shift index using $k- >k-m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - ◦ Convert $x^2 \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r+1}$

  - ◦ Shift index using $k- >k-1$

    $x^2 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\,x^{k+r}$

  - ◦ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-2+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-2) + a_{k-1}(k+r-2)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)\left(a_k(k+r+1) + a_{k-1}\right) = 0$

- Shift index using $k- >k+1$

  $(k-1+r)\left(a_{k+1}(k+2+r) + a_k\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{k+2+r}$

- Recursion relation for $r = -1$

  $a_{k+1} = -\frac{a_k}{k+1}$

- Solution for $r = -1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = -\frac{a_k}{k+1}\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = -\frac{a_k}{k+4}$

- Solution for $r = 2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = -\frac{a_k}{k+4}\right]$

- Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty}a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+2}\right), a_{k+1}=-\frac{a_k}{k+1}, b_{k+1}=-\frac{b_k}{4+k}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 25

```
dsolve(x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x^2-(x+2)*y(x) = 0,y(x),singsol=all)
```

$$y=\frac{c_1 e^{-x}+c_2(x^2-2x+2)}{x}$$

Mathematica DSolve solution

Solving time : 0.262 (sec)
Leaf size : 40

```
DSolve[{x^2*D[y[x],{x,2}]+x^2*D[y[x],x]-(2+x)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True
```

$$y(x)\to\frac{e^{-x-1}\left(c_2\int_1^x e^{K[1]+2}K[1]^2dK[1]+c_1\right)}{x}$$

2.1.239 Problem 242

Internal problem ID [9411]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 242
Date solved : `Monday, January 27, 2025 at 06:02:31 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + 2x^2y' + \left(x - \frac{3}{4}\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.208 (sec)

Writing the ode as

$$x^2y'' + 2x^2y' + \left(x - \frac{3}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 2x^2 \\ C &= x - \frac{3}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 - 4x + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2 - 4x + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 - 4x + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.457: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 - \frac{1}{x} + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r} \approx 1 - \frac{1}{2x} + \frac{1}{4x^2} + \frac{1}{8x^3} + \frac{1}{32x^4} - \frac{1}{64x^5} - \frac{3}{128x^6} - \frac{3}{256x^7} + \dots \tag{9}
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a = 1
$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= 1
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = 1
$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{4x^2 - 4x + 3}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= (1) + \left(\frac{-4x + 3}{4x^2}\right) \\
&= 1 + \frac{-4x + 3}{4x^2}
\end{aligned}
$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$
\begin{aligned}
b &= (-1) - (0) \\
&= -1
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-1}{1} - 0\right) &= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-1}{1} - 0\right) &= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^2 - 4x + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (1) \\ &= 1 - \frac{1}{2x} \\ &= 1 - \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(1 - \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(1 - \frac{1}{2x}\right)^2 - \left(\frac{4x^2 - 4x + 3}{4x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \,dx} \\
&= \mathrm{e}^{\int (1-\frac{1}{2x})dx} \\
&= \frac{\mathrm{e}^{x}}{\sqrt{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^2}{x^2}\,dx} \\
&= z_1 e^{-x} \\
&= z_1 \left(\mathrm{e}^{-x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{\sqrt{x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{(1+2x)\,\mathrm{e}^{-2x}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{\sqrt{x}}\right) + c_2 \left(\frac{1}{\sqrt{x}}\left(-\frac{(1+2x)\,\mathrm{e}^{-2x}}{4}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right) + 2x^2\left(\frac{d}{dx}y(x)\right) + \left(x - \frac{3}{4}\right)y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{(4x-3)y(x)}{4x^2} - 2\frac{d}{dx}y(x)$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x) + 2\frac{d}{dx}y(x) + \frac{(4x-3)y(x)}{4x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - ◦ Define functions

    $\left[P_2(x) = 2, P_3(x) = \frac{4x-3}{4x^2}\right]$
  - ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 0$
  - ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{3}{4}$
  - ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- • Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) + 8x^2\left(\frac{d}{dx}y(x)\right) + (4x-3)\,y(x) = 0$
- • Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - ◦ Shift index using $k- >k-m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - ◦ Convert $x^2 \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r+1}$
  - ◦ Shift index using $k- >k-1$

    $x^2 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\,x^{k+r}$
  - ◦ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\,x^{k+r}$

    Rewrite ODE with series expansions

    $a_0(1+2r)\,(-3+2r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-3) + 4a_{k-1}(2k-1+2r)\right) x^{k+r}\right) = 0$
- • $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-3+2r) = 0$
- • Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{3}{2}\right\}$
- • Each term in the series must be 0, giving the recursion relation

  $4\left(k+r-\frac{3}{2}\right)\left(k+r+\frac{1}{2}\right)a_k + 8\left(k+r-\frac{1}{2}\right)a_{k-1} = 0$
- • Shift index using $k- >k+1$

  $4\left(k+r-\frac{1}{2}\right)\left(k+\frac{3}{2}+r\right)a_{k+1} + 8\left(k+r+\frac{1}{2}\right)a_k = 0$
- • Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{4(2k+2r+1)a_k}{(2k-1+2r)(2k+3+2r)}$
- • Recursion relation for $r = -\frac{1}{2}$

$a_{k+1} = -\frac{8ka_k}{(2k-2)(2k+2)}$

- Series not valid for $r = -\frac{1}{2}$, division by 0 in the recursion relation at $k = 1$

  $a_{k+1} = -\frac{8ka_k}{(2k-2)(2k+2)}$
- Recursion relation for $r = \frac{3}{2}$

  $a_{k+1} = -\frac{4(2k+4)a_k}{(2k+2)(2k+6)}$
- Solution for $r = \frac{3}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1} = -\frac{4(2k+4)a_k}{(2k+2)(2k+6)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.023 (sec)
Leaf size : 24

```
dsolve(x^2*diff(diff(y(x),x),x)+2*diff(y(x),x)*x^2+(x-3/4)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{2c_2 e^{-2x} x + c_2 e^{-2x} + c_1}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.227 (sec)
Leaf size : 33

```
DSolve[{x^2*D[y[x],{x,2}]+2*x^2*D[y[x],x]+(x-3/4)*y[x]==0,{}},y[x],x,IncludeSingularSolution
```

$$y(x) \to \frac{c_2 \int_1^x e^{-2K[1]} K[1] dK[1] + c_1}{\sqrt{x}}$$

2.1.240 Problem 243

Internal problem ID [9412]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 243
Date solved : `Monday, January 27, 2025 at 06:02:32 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+x)\,y''+x^2y'-2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.184 (sec)

Writing the ode as

$$x^2(1+x)\,y''+x^2y'-2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(1+x)\\ B &= x^2\\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-x^2+8x+8}{4\left(x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2+8x+8\\ t &= 4\left(x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-x^2+8x+8}{4\left(x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.459: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{x} + \frac{2}{x^2} - \frac{1}{4\,(1+x)^2} + \frac{2}{1+x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2+8x+8}{4\left(x^2+x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2+8x+8}{4\left(x^2+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2+2x} - \frac{1}{x} + (-)(0) \\
&= \frac{1}{2+2x} - \frac{1}{x} \\
&= -\frac{x+2}{2x(1+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2+2x} - \frac{1}{x}\right)(1) + \left(\left(-\frac{1}{2(1+x)^2} + \frac{1}{x^2}\right) + \left(\frac{1}{2+2x} - \frac{1}{x}\right)^2 - \left(\frac{-x^2+8x+8}{4(x^2+x)^2}\right)\right) = 0$$

$$\frac{-2+a_0}{x(1+x)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+2)\, e^{\int \left(\frac{1}{2+2x} - \frac{1}{x}\right) dx} \\
&= (x+2)\, e^{\frac{\ln(1+x)}{2} - \ln(x)} \\
&= \frac{(x+2)\sqrt{1+x}}{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2}{x^2(1+x)}\, dx} \\
&= z_1 e^{-\frac{\ln(1+x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{1+x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x+2}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2}{x^2(1+x)}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\ln(1+x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\ln(1+x) + \frac{4}{x+2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{x+2}{x}\right) + c_2\left(\frac{x+2}{x}\left(\ln(1+x) + \frac{4}{x+2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right) + x^2\left(\frac{d}{dx}y(x)\right) - 2y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{2y(x)}{(x+1)x^2} - \frac{\frac{d}{dx}y(x)}{x+1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{\frac{d}{dx}y(x)}{x+1} - \frac{2y(x)}{(x+1)x^2} = 0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x+1}, P_3(x) = -\frac{2}{(x+1)x^2}\right]$

  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1} = 1$

  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$

  - $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right) + x^2\left(\frac{d}{dx}y(x)\right) - 2y(x) = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(u^3 - 2u^2 + u)\left(\frac{d^2}{du^2}y(u)\right) + (u^2 - 2u + 1)\left(\frac{d}{du}y(u)\right) - 2y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 u^{-1+r} + \left(a_1(1+r)^2 - 2a_0(r^2+1)\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)^2 - 2a_k(k^2+2kr+r^2+1) + \right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 0$
- Each term must be 0
  $a_1(1+r)^2 - 2a_0(r^2+1) = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1)^2 - 2a_k(k^2+1) + a_{k-1}(k-1)^2 = 0$
- Shift index using $k- >k+1$
  $a_{k+2}(k+2)^2 - 2a_{k+1}\left((k+1)^2+1\right) + k^2 a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 4k a_{k+1} - 4a_{k+1}}{(k+2)^2}$
- Recursion relation for $r = 0$
  $a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 4k a_{k+1} - 4a_{k+1}}{(k+2)^2}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 4k a_{k+1} - 4a_{k+1}}{(k+2)^2}, a_1 - 2a_0 = 0\right]$
- Revert the change of variables $u = x+1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x+1)^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 4k a_{k+1} - 4a_{k+1}}{(k+2)^2}, a_1 - 2a_0 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 27

```
dsolve(x^2*(x+1)*diff(diff(y(x),x),x)+diff(y(x),x)*x^2-2*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_2(x+2)\ln(x+1) + c_1 x + 2c_1 + 4c_2}{x}$$

Mathematica DSolve solution

Solving time : `0.484` (sec)
Leaf size : 87

```
DSolve[{x^2*(1+x)*D[y[x],{x,2}]+x^2*D[y[x],x]-2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to \frac{(x+2)\exp\left(\int_1^x \left(\frac{1}{2K[1]+2} - \frac{1}{K[1]}\right) dK[1]\right)\left(c_2 \int_1^x \frac{\exp\left(-2\int_1^{K[2]}\left(\frac{1}{2K[1]+2} - \frac{1}{K[1]}\right) dK[1]\right)}{(K[2]+2)^2} dK[2] + c_1\right)}{\sqrt{x+1}}$$

2.1.241 Problem 244

Internal problem ID [9413]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 244
Date solved : `Monday, January 27, 2025 at 06:02:33 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x(x^2+6)\,y' + 6y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.389 (sec)

Writing the ode as

$$x^2y'' + (x^3+6x)\,y' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^3+6x \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2+14}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2+14 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} + \frac{7}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.461: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + \frac{7}{2x} - \frac{49}{4x^3} + \frac{343}{4x^5} - \frac{12005}{16x^7} + \frac{117649}{16x^9} - \frac{2470629}{32x^{11}} + \frac{27176919}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 14}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} + \frac{7}{2}\right) + (0) \\ &= \frac{x^2}{4} + \frac{7}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{7}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{7}{2}\right) - (0) \\ &= \frac{7}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{7}{2}}{\frac{1}{2}} - 1\right) &&= 3 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{7}{2}}{\frac{1}{2}} - 1\right) &&= -4 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} + \frac{7}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	3	-4

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 3$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 3 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{x}{2}\right) \\ &= \frac{x}{2} \\ &= \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 3$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (6x + 2a_2) + 2\left(\frac{x}{2}\right)(3x^2 + 2xa_2 + a_1) + \left(\left(\frac{1}{2}\right) + \left(\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} + \frac{7}{2}\right)\right) &= 0 \\ -a_2 x^2 + (-2a_1 + 6)\, x - 3a_0 + 2a_2 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0, a_1 = 3, a_2 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^3 + 3x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x^3 + 3x)\, e^{\int \frac{x}{2} dx} \\ &= (x^3 + 3x)\, e^{\frac{x^2}{4}} \\ &= x(x^2 + 3)\, e^{\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^3+6x}{x^2} \, dx} \\ &= z_1 e^{-\frac{x^2}{4} - 3\ln(x)} \\ &= z_1 \left(\frac{e^{-\frac{x^2}{4}}}{x^3}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2+3}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^3+6x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{2}-6\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{e^{-\frac{x^2}{2}-6\ln(x)} x^4}{(x^2+3)^2} dx \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{x^2+3}{x^2} \right) + c_2 \left(\frac{x^2+3}{x^2} \left(\int \frac{e^{-\frac{x^2}{2}-6\ln(x)} x^4}{(x^2+3)^2} dx \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2 \left(\frac{d^2}{dx^2} y(x) \right) + x(x^2+6) \left(\frac{d}{dx} y(x) \right) + 6y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2} y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2} y(x) = -\frac{6y(x)}{x^2} - \frac{(x^2+6)\left(\frac{d}{dx} y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2} y(x) + \frac{(x^2+6)\left(\frac{d}{dx} y(x)\right)}{x} + \frac{6y(x)}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{x^2+6}{x}, P_3(x) = \frac{6}{x^2} \right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left. (x \cdot P_2(x)) \right|_{x=0} = 6$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left. (x^2 \cdot P_3(x)) \right|_{x=0} = 6$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(x^2+6)\left(\frac{d}{dx}y(x)\right) + 6y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  ◦ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(3+r)\,(2+r)\, x^r + a_1(4+r)\,(3+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+3)\,(k+r+2) + a_{k-2}(k-2+r))\, x^{k}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(3+r)\,(2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-3, -2\}$

- Each term must be 0

  $a_1(4+r)\,(3+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+3)\,(k+r+2) + a_{k-2}(k-2+r) = 0$

- Shift index using $k- >k+2$

  $a_{k+2}(k+5+r)\,(k+4+r) + a_k(k+r) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r)}{(k+5+r)(k+4+r)}$

- Recursion relation for $r = -3$

  $a_{k+2} = -\frac{a_k(k-3)}{(k+2)(k+1)}$

- Solution for $r = -3$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2} = -\frac{a_k(k-3)}{(k+2)(k+1)}, a_1 = 0\right]$

- Recursion relation for $r = -2$; series terminates at $k = 2$

  $a_{k+2} = -\frac{a_k(k-2)}{(k+3)(k+2)}$

- Solution for $r = -2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{a_k(k-2)}{(k+3)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-3}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-2}\right), a_{k+2} = -\frac{a_k(k-3)}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{b_k(k-2)}{(k+3)(k+2)}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric sol
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returr
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.010 (sec)
Leaf size : 35

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(x^2+6)*diff(y(x),x)+6*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_2 \mathrm{e}^{-\frac{x^2}{2}} \operatorname{hypergeom}\left(\left[2\right], \left[\frac{1}{2}\right], \frac{x^2}{2}\right) + c_1 (x^2+3)\, x}{x^3}$$

Mathematica DSolve solution

Solving time : 0.596 (sec)
Leaf size : 50

```
DSolve[{x^2*D[y[x],{x,2}]+x*(6+x^2)*D[y[x],x]+6*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to \frac{(x^2+3)\left(c_2 \int_1^x \frac{e^{-\frac{1}{2}K[1]^2}}{K[1]^2 (K[1]^2+3)^2} dK[1] + c_1\right)}{x^2}$$

2.1.242 Problem 245

Internal problem ID [9414]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 245
Date solved : `Monday, January 27, 2025 at 06:02:33 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x(1-x)\,y' - y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.217 (sec)

Writing the ode as

$$x^2y'' + \left(-x^2+x\right)y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 + x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 2x + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 2x + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 2x + 3}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.463: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2x} + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{2x^2} + \frac{1}{2x^3} + \frac{1}{4x^4} - \frac{1}{4x^5} - \frac{3}{4x^6} - \frac{3}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 2x + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x+3}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-2x+3}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 2x + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} - \frac{1}{2x} \\
&= \frac{x-1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2} - \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(\frac{1}{2} - \frac{1}{2x}\right)^2 - \left(\frac{x^2-2x+3}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{2} - \frac{1}{2x}\right) dx} \\
&= \frac{e^{\frac{x}{2}}}{\sqrt{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x^2+x}{x^2}\,dx} \\ &= z_1 e^{\frac{x}{2}-\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{e^{\frac{x}{2}}}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^x}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{x-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(-(1+x)\,x\,e^{x-\ln(x)}e^{-2x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{e^x}{x}\right) + c_2\left(\frac{e^x}{x}\left(-(1+x)\,x\,e^{x-\ln(x)}e^{-2x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(1-x)\left(\frac{d}{dx}y(x)\right) - y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = \frac{y(x)}{x^2} + \frac{\left(\frac{d}{dx}y(x)\right)(x-1)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) - \frac{\left(\frac{d}{dx}y(x)\right)(x-1)}{x} - \frac{y(x)}{x^2} = 0$
- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{x-1}{x}, P_3(x) = -\frac{1}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

$(x \cdot P_2(x))\Big|_{x=0} = 1$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = -1$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) - x(x-1)\left(\frac{d}{dx}y(x)\right) - y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - ○ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  - ○ Shift index using $k->k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - ○ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)(k+r-1) - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 1\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)(a_k(k+r+1) - a_{k-1}) = 0$

- Shift index using $k->k+1$

  $(k+r)(a_{k+1}(k+2+r) - a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+2+r}$

- Recursion relation for $r = -1$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = -1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = \frac{a_k}{k+1}\right]$

- Recursion relation for $r = 1$

  $a_{k+1} = \frac{a_k}{k+3}$

- Solution for $r = 1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = \frac{a_k}{k+3}\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(1-x)*diff(y(x),x)-y(x) = 0,y(x),singsol=all)
```

$$y = \frac{e^{x} c_2 + c_1 x + c_1}{x}$$

Mathematica DSolve solution

Solving time : 0.282 (sec)
Leaf size : 80

```
DSolve[{x^2*D[y[x],{x,2}]+x*(1-x)*D[y[x],x]-y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \exp\left(\int_1^x \left(1 - \frac{1}{K[1]}\right) dK[1]\right) \left(\int_1^x \exp\left(-\int_1^{K[2]} \left(1 - \frac{1}{K[1]}\right) dK[1]\right) c_1 dK[2] + c_2\right)$$

$$y(x) \to c_2 \exp\left(\int_1^x \left(1 - \frac{1}{K[1]}\right) dK[1]\right)$$

2.1.243 Problem 246

Internal problem ID [9415]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 246
Date solved : `Monday, January 27, 2025 at 06:02:34 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x(x+3)\, y' + 4y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.254 (sec)

Writing the ode as

$$x^2y'' + \left(-x^2-3x\right) y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2-3x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2+6x-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2+6x-1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2+6x-1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.465: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{4x^2} + \frac{3}{2x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{3}{2x} - \frac{5}{2x^2} + \frac{15}{2x^3} - \frac{115}{4x^4} + \frac{495}{4x^5} - \frac{2285}{4x^6} + \frac{11055}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 6x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{6x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{6x - 1}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 6. Dividing this by leading coefficient in t which is 4 gives $\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{3}{2}\right) - (0) \\ &= \frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{3}{2}}{\frac{1}{2}} - 0\right) &&= \frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{3}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{3}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 6x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{3}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+\right) \\
&= \frac{3}{2} - \left(\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2x} + \frac{1}{2} \\
&= \frac{1+x}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered}
(0) + 2\left(\frac{1}{2x} + \frac{1}{2}\right)(1) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} + \frac{1}{2}\right)^2 - \left(\frac{x^2+6x-1}{4x^2}\right)\right) = 0 \\
\frac{1-a_0}{x} = 0
\end{gathered}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 1 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (1+x)\, e^{\int \left(\frac{1}{2x}+\frac{1}{2}\right) dx} \\
&= (1+x)\, e^{\frac{x}{2}+\frac{\ln(x)}{2}} \\
&= (1+x)\sqrt{x}\, e^{\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2-3x}{x^2}\, dx} \\
&= z_1 e^{\frac{x}{2}+\frac{3\ln(x)}{2}} \\
&= z_1 \left(x^{3/2} e^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^2 e^x (1+x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-3x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x+3\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{e^{-x}}{-1-x} - \mathrm{Ei}_1(x)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^2 e^x (1+x)\right) + c_2 \left(x^2 e^x (1+x) \left(-\frac{e^{-x}}{-1-x} - \mathrm{Ei}_1(x)\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2 \left(\frac{d^2}{dx^2} y(x)\right) - x(x+3)\left(\frac{d}{dx} y(x)\right) + 4y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2} y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2} y(x) = -\frac{4y(x)}{x^2} + \frac{(x+3)\left(\frac{d}{dx} y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x) - \frac{(x+3)\left(\frac{d}{dx}y(x)\right)}{x} + \frac{4y(x)}{x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - ◦ Define functions

    $\left[P_2(x) = -\frac{x+3}{x}, P_3(x) = \frac{4}{x^2}\right]$
  - ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -3$
  - ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 4$
  - ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- • Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) - x(x+3)\left(\frac{d}{dx}y(x)\right) + 4y(x) = 0$
- • Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
  - ◦ Shift index using $k- >k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - ◦ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r}$

    Rewrite ODE with series expansions

    $a_0(-2+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-2)^2 - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$
- • $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)^2 = 0$
- • Values of r that satisfy the indicial equation

  $r = 2$
- • Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-2)^2 - a_{k-1}(k+r-1) = 0$
- • Shift index using $k- >k+1$

  $a_{k+1}(k+r-1)^2 - a_k(k+r) = 0$
- • Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r)}{(k+r-1)^2}$
- • Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k(k+2)}{(k+1)^2}$
- • Solution for $r = 2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k(k+2)}{(k+1)^2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 29

```
dsolve(x^2*diff(diff(y(x),x),x)-x*(x+3)*diff(y(x),x)+4*y(x) = 0,y(x),singsol=all)
```

$$y = x^2(c_2 e^x (x+1) \operatorname{Ei}_1(x) + (x+1) c_1 e^x - c_2)$$

Mathematica DSolve solution

Solving time : 0.232 (sec)
Leaf size : 49

```
DSolve[{x^2*D[y[x],{x,2}]-x*(x+3)*D[y[x],x]+4*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to e^{x+2} x^2 (x+1) \left(c_2 \int_1^x \frac{e^{-K[1]-1}}{K[1](K[1]+1)^2} dK[1] + c_1 \right)$$

2.1.244 Problem 247

Internal problem ID [9416]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 247
Date solved : `Monday, January 27, 2025 at 06:02:35 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x^2y' - 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.228 (sec)

Writing the ode as

$$x^2y'' - x^2y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.467: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{2}{x^2} - \frac{4}{x^4} + \frac{16}{x^6} - \frac{80}{x^8} + \frac{448}{x^{10}} - \frac{2688}{x^{12}} + \frac{16896}{x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{2}{x^2}\right) \\ &= \frac{1}{4} + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 4 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{\frac{1}{2}} - 0\right) &= 0 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{\frac{1}{2}} - 0\right) &= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + (-)\left(\frac{1}{2}\right) \\ &= -\frac{1}{x} - \frac{1}{2} \\ &= -\frac{2+x}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{x} - \frac{1}{2}\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - \frac{1}{2}\right)^2 - \left(\frac{x^2+8}{4x^2}\right) \right) &= 0 \\ \frac{-2+a_0}{x} &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (2+x)\, e^{\int \left(-\frac{1}{x}-\frac{1}{2}\right) dx} \\
&= (2+x)\, e^{-\frac{x}{2}-\ln(x)} \\
&= \frac{(2+x)\, e^{-\frac{x}{2}}}{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2}{x^2} \, dx} \\
&= z_1 e^{\frac{x}{2}} \\
&= z_1 \left(e^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{2+x}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{x}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{(-2+x)\, e^{x}}{2+x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{2+x}{x}\right) + c_2 \left(\frac{2+x}{x} \left(\frac{(-2+x)\, e^{x}}{2+x}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2 \left(\frac{d^2}{dx^2} y(x)\right) - x^2 \left(\frac{d}{dx} y(x)\right) - 2y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2} y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2} y(x) = \frac{2y(x)}{x^2} + \frac{d}{dx} y(x)$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x) - \frac{d}{dx}y(x) - \frac{2y(x)}{x^2} = 0$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - ◦ Define functions

    $\left[P_2(x) = -1, P_3(x) = -\frac{2}{x^2}\right]$
  - ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 0$
  - ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -2$
  - ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) - x^2\left(\frac{d}{dx}y(x)\right) - 2y(x) = 0$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- □ Rewrite ODE with series expansions
  - ◦ Convert $x^2 \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r+1}$
  - ◦ Shift index using $k- >k-1$

    $x^2 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, x^{k+r}$
  - ◦ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\, x^{k+r}$

    Rewrite ODE with series expansions

    $a_0(1+r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-2) - a_{k-1}(k-1+r)\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-1, 2\}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)\,(k+r-2) - a_{k-1}(k-1+r) = 0$
- Shift index using $k- >k+1$

  $a_{k+1}(k+2+r)\,(k-1+r) - a_k(k+r) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r)}{(k+2+r)(k-1+r)}$
- Recursion relation for $r = -1$; series terminates at $k = 1$

  $a_{k+1} = \frac{a_k(k-1)}{(k+1)(k-2)}$
- Apply recursion relation for $k = 0$

  $a_1 = \frac{a_0}{2}$
- Terminating series solution of the ODE for $r = -1$. Use reduction of order to find the second

  $y(x) = a_0 \cdot \left(1 + \frac{x}{2}\right)$
- Recursion relation for $r = 2$

$a_{k+1} = \frac{a_k(k+2)}{(k+4)(k+1)}$

- Solution for $r = 2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k(k+2)}{(k+4)(k+1)}\right]$

- Combine solutions and rename parameters

  $\left[y(x) = a_0 \cdot \left(1 + \frac{x}{2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), b_{k+1} = \frac{b_k(k+2)}{(4+k)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 21

```
dsolve(x^2*diff(diff(y(x),x),x)-diff(y(x),x)*x^2-2*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_2 \mathrm{e}^{x}(x-2) + c_1(x+2)}{x}$$

Mathematica DSolve solution

Solving time : 0.068 (sec)
Leaf size : 74

```
DSolve[{x^2*D[y[x],{x,2}]-x^2*D[y[x],x]-2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \rightarrow -\frac{2e^{\frac{x+1}{2}}\left((c_1 x + 2ic_2)\cosh\left(\frac{x}{2}\right) - (ic_2 x + 2c_1)\sinh\left(\frac{x}{2}\right)\right)}{\sqrt{\pi}\sqrt{-ix}\sqrt{x}}$$

2.1.245 Problem 248

Internal problem ID [9417]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 248
Date solved : `Monday, January 27, 2025 at 06:02:35 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x^2y' - (3x+2)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.262 (sec)

Writing the ode as

$$x^2y'' - x^2y' + (-3x-2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 \\ C &= -3x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 12x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 12x + 8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 12x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.469: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{x^2} + \frac{3}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{3}{x} - \frac{7}{x^2} + \frac{42}{x^3} - \frac{301}{x^4} + \frac{2394}{x^5} - \frac{20342}{x^6} + \frac{180852}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 + 12x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{12x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{12x + 8}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 12. Dividing this by leading coefficient in t which is 4 gives 3. Now b can be found.

$$\begin{aligned}b &= (3) - (0) \\ &= 3\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{3}{\frac{1}{2}} - 0\right) &&= 3 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{3}{\frac{1}{2}} - 0\right) &&= -3\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 12x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	3	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 3$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+\right) \\ &= 3 - (2) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{2}{x} + \left(\frac{1}{2}\right) \\ &= \frac{2}{x} + \frac{1}{2} \\ &= \frac{4+x}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{2}{x} + \frac{1}{2}\right)(1) + \left(\left(-\frac{2}{x^2}\right) + \left(\frac{2}{x} + \frac{1}{2}\right)^2 - \left(\frac{x^2+12x+8}{4x^2}\right) \right) &= 0 \\ \frac{4-a_0}{x} &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 4\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 4 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (4+x)\, e^{\int \left(\frac{2}{x}+\frac{1}{2}\right) dx} \\ &= (4+x)\, e^{\frac{x}{2}+2\ln(x)} \\ &= (4+x)\, x^2 e^{\frac{x}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x^2}{x^2} \, dx} \\ &= z_1 e^{\frac{x}{2}} \\ &= z_1 \left(e^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^x (4+x)\, x^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^x}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{e^{-x}(x^3+3x^2-2x+2)}{24\,(4+x)\, x^3} + \frac{\mathrm{Ei}_1(x)}{24} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(e^x (4+x)\, x^2\right) + c_2 \left(e^x (4+x)\, x^2 \left(-\frac{e^{-x}(x^3+3x^2-2x+2)}{24\,(4+x)\, x^3} + \frac{\mathrm{Ei}_1(x)}{24} \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2 \left(\frac{d^2}{dx^2} y(x)\right) - x^2 \left(\frac{d}{dx} y(x)\right) - (3x+2)\, y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2} y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2} y(x) = \frac{(3x+2) y(x)}{x^2} + \frac{d}{dx} y(x)$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x) - \frac{d}{dx}y(x) - \frac{(3x+2)y(x)}{x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - ◦ Define functions

    $\left[P_2(x) = -1, P_3(x) = -\frac{3x+2}{x^2}\right]$
  - ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 0$
  - ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -2$
  - ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$
- • Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) - x^2\left(\frac{d}{dx}y(x)\right) + (-3x-2)\, y(x) = 0$
- • Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - ◦ Shift index using $k- >k - m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - ◦ Convert $x^2 \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r+1}$
  - ◦ Shift index using $k- >k - 1$

    $x^2 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, x^{k+r}$
  - ◦ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-2) - a_{k-1}(k+2+r)\right) x^{k+r}\right) = 0$
- • $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-2+r) = 0$
- • Values of r that satisfy the indicial equation

  $r \in \{-1, 2\}$
- • Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)\,(k+r-2) - a_{k-1}(k+2+r) = 0$
- • Shift index using $k- >k + 1$

  $a_{k+1}(k+2+r)\,(k-1+r) - a_k(k+r+3) = 0$
- • Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r+3)}{(k+2+r)(k-1+r)}$
- • Recursion relation for $r = -1$

$a_{k+1} = \frac{a_k(k+2)}{(k+1)(k-2)}$

- Series not valid for $r = -1$, division by 0 in the recursion relation at $k = 2$

  $a_{k+1} = \frac{a_k(k+2)}{(k+1)(k-2)}$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k(k+5)}{(k+4)(k+1)}$

- Solution for $r = 2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k(k+5)}{(k+4)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 48

```
dsolve(x^2*diff(diff(y(x),x),x)-diff(y(x),x)*x^2-(2+3*x)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_2 e^x x^3(x+4) \operatorname{Ei}_1(x) + c_1 x^3(x+4) e^x - c_2(x^3+3x^2-2x+2)}{x}$$

Mathematica DSolve solution

Solving time : 0.305 (sec)
Leaf size : 45

```
DSolve[{x^2*D[y[x],{x,2}]-x^2*D[y[x],x]-(3*x+2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to e^x x^2(x+4) \left(c_2 \int_1^x \frac{e^{-K[1]}}{K[1]^4(K[1]+4)^2} dK[1] + c_1\right)$$

2.1.246 Problem 249

Internal problem ID [9418]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 249
Date solved : `Monday, January 27, 2025 at 06:02:36 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x(5-x)\,y' + 4y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.336 (sec)

Writing the ode as

$$x^2y'' + \left(-x^2+5x\right)y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2+5x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2-10x-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2-10x-1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2-10x-1}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.471: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{4x^2} - \frac{5}{2x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{5}{2x} - \frac{13}{2x^2} - \frac{65}{2x^3} - \frac{819}{4x^4} - \frac{5785}{4x^5} - \frac{43797}{4x^6} - \frac{347425}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 10x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-10x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-10x - 1}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -10. Dividing this by leading coefficient in t which is 4 gives $-\frac{5}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{5}{2}\right) - (0) \\ &= -\frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 0\right) = -\frac{5}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 0\right) = \frac{5}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 10x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{5}{2}$	$\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{5}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{5}{2} - \left(\frac{1}{2}\right) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\left(\frac{1}{2}\right) \\ &= \frac{1}{2x} - \frac{1}{2} \\ &= -\frac{-1+x}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (2) + 2\left(\frac{1}{2x} - \frac{1}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^2 - 10x - 1}{4x^2}\right)\right) &= 0 \\ \frac{(a_1 + 4)\, x + 2a_0 + a_1}{x} &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2, a_1 = -4\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 4x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\,dx} \\
&= \left(x^2 - 4x + 2\right) \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{1}{2}\right)dx} \\
&= \left(x^2 - 4x + 2\right) \mathrm{e}^{-\frac{x}{2} + \frac{\ln(x)}{2}} \\
&= \left(x^2 - 4x + 2\right) \sqrt{x}\,\mathrm{e}^{-\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2+5x}{x^2}\,dx} \\
&= z_1 e^{\frac{x}{2} - \frac{5\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{x}{2}}}{x^{5/2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2 - 4x + 2}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+5x}{x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{x - 5\ln(x)}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(-\frac{\mathrm{e}^x (x-3)}{4\left(x^2 - 4x + 2\right)} - \frac{\mathrm{Ei}_1(-x)}{4}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^2 - 4x + 2}{x^2}\right) + c_2 \left(\frac{x^2 - 4x + 2}{x^2}\left(-\frac{\mathrm{e}^x (x-3)}{4\left(x^2 - 4x + 2\right)} - \frac{\mathrm{Ei}_1(-x)}{4}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right)+x(5-x)\left(\frac{d}{dx}y(x)\right)+4y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{4y(x)}{x^2}+\frac{(x-5)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)-\frac{(x-5)\left(\frac{d}{dx}y(x)\right)}{x}+\frac{4y(x)}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{x-5}{x},P_3(x)=\frac{4}{x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=5$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=4$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

$x^2\left(\frac{d^2}{dx^2}y(x)\right)-x(x-5)\left(\frac{d}{dx}y(x)\right)+4y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)^2\,x^r+\left(\sum_{k=1}^{\infty}\left(a_k(k+r+2)^2-a_{k-1}(k+r-1)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(2+r)^2=0$

- Values of r that satisfy the indicial equation

$r=-2$

- Each term in the series must be 0, giving the recursion relation

$a_k(k+r+2)^2-a_{k-1}(k+r-1)=0$

- Shift index using $k->k+1$

$a_{k+1}(k+3+r)^2 - a_k(k+r) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r)}{(k+3+r)^2}$

- Recursion relation for $r = -2$; series terminates at $k = 2$

  $a_{k+1} = \frac{a_k(k-2)}{(k+1)^2}$

- Apply recursion relation for $k = 0$

  $a_1 = -2a_0$

- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{4}$

- Express in terms of a_0

  $a_2 = \frac{a_0}{2}$

- Terminating series solution of the ODE for $r = -2$. Use reduction of order to find the second lin

  $y(x) = a_0 \cdot \left(1 - 2x + \frac{1}{2}x^2\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 41

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(-x+5)*diff(y(x),x)+4*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{(x^2 - 4x + 2)\, c_2 \operatorname{Ei}_1(-x) + c_2(x-3)\, e^x + c_1(x^2 - 4x + 2)}{x^2}$$

Mathematica DSolve solution

Solving time : 0.358 (sec)
Leaf size : 51

```
DSolve[{x^2*D[y[x],{x,2}]+x*(5-x)*D[y[x],x]+4*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tru
```

$$y(x) \to \frac{(x^2 - 4x + 2)\left(c_2 \int_1^x \frac{e^{K[1]}}{K[1](K[1]^2 - 4K[1] + 2)^2} dK[1] + c_1\right)}{x^2}$$

2.1.247 Problem 250

Internal problem ID [9419]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 250
Date solved : `Monday, January 27, 2025 at 06:02:37 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' + 4x(1-x)\,y' + (2x-9)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.224 (sec)

Writing the ode as

$$4x^2y'' + \left(-4x^2+4x\right)y' + (2x-9)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x^2+4x \\ C &= 2x-9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2-4x+8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2-4x+8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2-4x+8}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.473: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{x} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{1}{x^2} + \frac{2}{x^3} + \frac{3}{x^4} + \frac{2}{x^5} - \frac{6}{x^6} - \frac{28}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-4x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-4x + 8}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned}b &= (-1) - (0) \\ &= -1\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 0\right) &&= -1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 0\right) &&= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -1$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2} - \frac{1}{x} \\ &= \frac{x-2}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2} - \frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(\frac{1}{2} - \frac{1}{x}\right)^2 - \left(\frac{x^2 - 4x + 8}{4x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{1}{2} - \frac{1}{x}\right) dx} \\ &= \frac{e^{\frac{x}{2}}}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x^2+4x}{4x^2}\,dx} \\ &= z_1 e^{\frac{x}{2}-\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{e^{\frac{x}{2}}}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{x}}{x^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^2+4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{x-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(-\left(x^2+2x+2\right) x\, e^{x-\ln(x)} e^{-2x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{e^{x}}{x^{3/2}}\right) + c_2\left(\frac{e^{x}}{x^{3/2}}\left(-\left(x^2+2x+2\right) x\, e^{x-\ln(x)} e^{-2x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x(1-x)\left(\frac{d}{dx}y(x)\right) + (2x-9)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(2x-9)y(x)}{4x^2} + \frac{\left(\frac{d}{dx}y(x)\right)(x-1)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) - \frac{\left(\frac{d}{dx}y(x)\right)(x-1)}{x} + \frac{(2x-9)y(x)}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{x-1}{x}, P_3(x) = \frac{2x-9}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

$\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\frac{9}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) - 4x(x-1)\left(\frac{d}{dx}y(x)\right) + (2x-9)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- Rewrite ODE with series expansions

  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  - Shift index using $k- >k + 1 - m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(3+2r)\,(-3+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+3)\,(2k+2r-3) - 2a_{k-1}(2k+2r-3)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(3+2r)\,(-3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{3}{2}, \frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $4\left(\left(k+r+\frac{3}{2}\right) a_k - a_{k-1}\right)\left(k+r-\frac{3}{2}\right) = 0$

- Shift index using $k- >k + 1$

  $4\left(\left(k+\frac{5}{2}+r\right) a_{k+1} - a_k\right)\left(k-\frac{1}{2}+r\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k}{2k+5+2r}$

- Recursion relation for $r = -\frac{3}{2}$

  $a_{k+1} = \frac{2a_k}{2k+2}$

- Solution for $r = -\frac{3}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}, a_{k+1} = \frac{2a_k}{2k+2}\right]$

- Recursion relation for $r = \frac{3}{2}$

$a_{k+1} = \frac{2a_k}{2k+8}$

- Solution for $r = \frac{3}{2}$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1} = \frac{2a_k}{2k+8}\right]$$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+1} = \frac{2a_k}{2k+2}, b_{k+1} = \frac{2b_k}{2k+8}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.039 (sec)
Leaf size : 23

```
dsolve(4*x^2*diff(diff(y(x),x),x)+4*x*(1-x)*diff(y(x),x)+(2*x-9)*y(x) = 0,y(x),singsol
```

$$y = \frac{e^x c_1 + c_2(x^2 + 2x + 2)}{x^{3/2}}$$

Mathematica DSolve solution

Solving time : 0.236 (sec)
Leaf size : 38

```
DSolve[{4*x^2*D[y[x],{x,2}]+4*x*(1-x)*D[y[x],x]+(2*x-9)*y[x]==0,{}},y[x],x,IncludeSingularSo
```

$$y(x) \to \frac{e^x \left(c_2 \int_1^x e^{-K[1]} K[1]^2 dK[1] + c_1\right)}{x^{3/2}}$$

2.1.248 Problem 251

Internal problem ID [9420]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 251
Date solved : `Monday, January 27, 2025 at 06:02:37 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + 2x(2+x)\,y' + 2(1+x)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.250 (sec)

Writing the ode as

$$x^2y'' + \left(2x^2+4x\right)y' + (2x+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 2x^2+4x \\ C &= 2x+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2+x}{x} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2+x \\ t &= x \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2+x}{x}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.475: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x$. There is a pole at $x = 0$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{2x^3} - \frac{5}{8x^4} + \frac{7}{8x^5} - \frac{21}{16x^6} + \frac{33}{16x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2+x}{x} \\ &= Q + \frac{R}{x} \\ &= (1) + \left(\frac{2}{x}\right) \\ &= 1 + \frac{2}{x} \end{aligned}$$

Since the degree of t is 1, then we see that the coefficient of the term 1 in the remainder R is 2. Dividing this by leading coefficient in t which is 1 gives 2. Now b can be found.

$$\begin{aligned} b &= (2) - (0) \\ &= 2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{2}{1} - 0\right) &= 1 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{2}{1} - 0\right) &= -1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2+x}{x}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	1	1	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 1$ then

$$\begin{aligned} d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{x} + (1) \\ &= 1 + \frac{1}{x} \\ &= 1 + \frac{1}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(1 + \frac{1}{x}\right)(0) + \left(\left(-\frac{1}{x^2}\right) + \left(1 + \frac{1}{x}\right)^2 - \left(\frac{2+x}{x}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(1 + \frac{1}{x}\right) dx} \\ &= x\, e^{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{2x^2+4x}{x^2}\, dx} \\ &= z_1 e^{-x - 2\ln(x)} \\ &= z_1 \left(\frac{e^{-x}}{x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2+4x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2x-4\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{e^{-2x}}{3x} + \frac{e^{-2x}}{3} - \frac{2x\, e^{-2x}}{3} + \frac{4x^2\, \mathrm{Ei}_1(2x)}{3} \right. \\
&\quad \left. - \frac{4\, \mathrm{Ei}_1(2x)\, x^3 - 2\, e^{-2x} x^2 + x\, e^{-2x} - 6x\, \mathrm{Ei}_1(2x) + 2\, e^{-2x}}{3x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x}\left(-\frac{e^{-2x}}{3x} + \frac{e^{-2x}}{3} - \frac{2x\, e^{-2x}}{3} + \frac{4x^2\, \mathrm{Ei}_1(2x)}{3} \right.\right. \\
&\quad \left.\left. - \frac{4\, \mathrm{Ei}_1(2x)\, x^3 - 2\, e^{-2x} x^2 + x\, e^{-2x} - 6x\, \mathrm{Ei}_1(2x) + 2\, e^{-2x}}{3x}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 28

```
dsolve(x^2*diff(diff(y(x),x),x)+2*x*(x+2)*diff(y(x),x)+2*(x+1)*y(x) = 0,y(x),singsol=all
```

$$y = \frac{2\, \mathrm{Ei}_1(2x)\, c_2 x - c_2 e^{-2x} + c_1 x}{x^2}$$

Mathematica DSolve solution

Solving time : 0.239 (sec)
Leaf size : 33

```
DSolve[{x^2*D[y[x],{x,2}]+2*x*(2+x)*D[y[x],x]+2*(1+x)*y[x]==0,{}},y[x],x,IncludeSingularSolu
```

$$y(x) \to \frac{c_2 \int_1^x \frac{e^{-2K[1]}}{K[1]^2} dK[1] + c_1}{x}$$

2.1.249 Problem 252

Internal problem ID [9421]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 252
Date solved : `Monday, January 27, 2025 at 06:02:38 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x(1-x)\,y' + (1-x)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.243 (sec)

Writing the ode as

$$x^2y'' + \left(x^2-x\right)y' + (1-x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2-x \\ C &= 1-x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2+2x-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2+2x-1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2+2x-1}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.476: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{1}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{2x} - \frac{1}{2x^2} + \frac{1}{2x^3} - \frac{3}{4x^4} + \frac{5}{4x^5} - \frac{9}{4x^6} + \frac{17}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 + 2x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{2x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{2x - 1}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 2. Dividing this by leading coefficient in t which is 4 gives $\frac{1}{2}$. Now b can be found.

$$\begin{aligned}b &= \left(\frac{1}{2}\right) - (0) \\ &= \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) = -\frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 2x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2} + \frac{1}{2x} \\ &= \frac{x+1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2} + \frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2} + \frac{1}{2x}\right)^2 - \left(\frac{x^2+2x-1}{4x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{1}{2} + \frac{1}{2x}\right) dx} \\ &= \sqrt{x}\, e^{\frac{x}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^2-x}{x^2}\,dx} \\ &= z_1 e^{-\frac{x}{2}+\frac{\ln(x)}{2}} \\ &= z_1\left(\sqrt{x}\,\mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2-x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-x+\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(-\operatorname{Ei}_1(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(x) + c_2(x(-\operatorname{Ei}_1(x))) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) - x(1-x)\left(\frac{d}{dx}y(x)\right) + (1-x)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{(x-1)y(x)}{x^2} - \frac{\left(\frac{d}{dx}y(x)\right)(x-1)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{\left(\frac{d}{dx}y(x)\right)(x-1)}{x} - \frac{(x-1)y(x)}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

  $\left[P_2(x) = \frac{x-1}{x}, P_3(x) = -\frac{x-1}{x^2}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -1$

  ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 1$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(x-1)\left(\frac{d}{dx}y(x)\right) + (1-x)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)^2\,x^r + \left(\sum_{k=1}^{\infty}\left(a_k(k+r-1)^2 + a_{k-1}(k-2+r)\right)x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 1$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)^2 + a_{k-1}(k-2+r) = 0$

- Shift index using $k->k+1$

  $a_{k+1}(k+r)^2 + a_k(k+r-1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r-1)}{(k+r)^2}$

- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{a_k k}{(k+1)^2}$

- Solution for $r = 1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{a_k k}{(k+1)^2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 13

```
dsolve(x^2*diff(diff(y(x),x),x)-x*(1-x)*diff(y(x),x)+(1-x)*y(x) = 0,y(x),singsol=all)
```

$$y = x(\mathrm{Ei}_1(x)\,c_2 + c_1)$$

Mathematica DSolve solution

Solving time : 0.443 (sec)
Leaf size : 63

```
DSolve[{x^2*D[y[x],{x,2}]-x*(1-x)*D[y[x],x]+(1-x)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->
```

$$y(x) \to \sqrt{x}\left(c_2 \int_1^x \frac{e^{-K[2]-1}}{K[2]} dK[2] + c_1\right) \exp\left(\frac{1}{2}\left(-\int_1^x \left(1 - \frac{1}{K[1]}\right) dK[1] + x + 1\right)\right)$$

2.1.250 Problem 253

Internal problem ID [9422]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 253
Date solved : `Monday, January 27, 2025 at 06:02:39 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' + 4x(1+2x)\,y' + (4x-1)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.101 (sec)

Writing the ode as

$$4x^2y'' + \left(8x^2+4x\right)y' + (4x-1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= 8x^2+4x \\ C &= 4x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.478: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{8x^2+4x}{4x^2}\, dx} \\ &= z_1 e^{-x-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{\mathrm{e}^{-x}}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-2x}}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{8x^2+4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{e^{-2x-\ln(x)} x\, e^{4x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{-2x}}{\sqrt{x}}\right) + c_2 \left(\frac{e^{-2x}}{\sqrt{x}} \left(\frac{e^{-2x-\ln(x)} x\, e^{4x}}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x(2x+1)\left(\frac{d}{dx}y(x)\right) + (4x-1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(4x-1)y(x)}{4x^2} - \frac{(2x+1)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(2x+1)\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(4x-1)y(x)}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{2x+1}{x}, P_3(x) = \frac{4x-1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x(2x+1)\left(\frac{d}{dx}y(x)\right) + (4x-1)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y(x)=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^{m}\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

  $x^{m}\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m}\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^{2}\cdot\left(\frac{d^{2}}{dx^{2}}y(x)\right)$ to series expansion

  $x^{2}\cdot\left(\frac{d^{2}}{dx^{2}}y(x)\right)=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(1+2r)\,(-1+2r)\, x^{r}+\left(\sum_{k=1}^{\infty}\left(a_{k}(2k+2r+1)\,(2k+2r-1)+4a_{k-1}(2k+2r-1)\right) x^{k+r}\right)=0$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{-\frac{1}{2},\frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r-\frac{1}{2}\right)\left(a_{k}\left(k+r+\frac{1}{2}\right)+2a_{k-1}\right)=0$

- Shift index using $k->k+1$

  $4\left(k+r+\frac{1}{2}\right)\left(a_{k+1}\left(k+\frac{3}{2}+r\right)+2a_{k}\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{4a_{k}}{2k+3+2r}$

- Recursion relation for $r=-\frac{1}{2}$

  $a_{k+1}=-\frac{4a_{k}}{2k+2}$

- Solution for $r=-\frac{1}{2}$

  $\left[y(x)=\sum_{k=0}^{\infty} a_{k} x^{k-\frac{1}{2}}, a_{k+1}=-\frac{4a_{k}}{2k+2}\right]$

- Recursion relation for $r=\frac{1}{2}$

  $a_{k+1}=-\frac{4a_{k}}{2k+4}$

- Solution for $r=\frac{1}{2}$

  $\left[y(x)=\sum_{k=0}^{\infty} a_{k} x^{k+\frac{1}{2}}, a_{k+1}=-\frac{4a_{k}}{2k+4}\right]$

- Combine solutions and rename parameters

  $\left[y(x)=\left(\sum_{k=0}^{\infty} a_{k} x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_{k} x^{k+\frac{1}{2}}\right), a_{k+1}=-\frac{4a_{k}}{2k+2}, b_{k+1}=-\frac{4b_{k}}{2k+4}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.022 (sec)
Leaf size : 16

```
dsolve(4*x^2*diff(diff(y(x),x),x)+4*x*(2*x+1)*diff(y(x),x)+(-1+4*x)*y(x) = 0,y(x),sing
```

$$y = \frac{c_1 + c_2 e^{-2x}}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.035 (sec)
Leaf size : 26

```
DSolve[{4*x^2*D[y[x],{x,2}]+4*x*(1+2*x)*D[y[x],x]+(4*x-1)*y[x]==0,{}},y[x],x,IncludeSingular
```

$$y(x) \to \frac{2c_1 e^{-2x} + c_2}{2\sqrt{x}}$$

2.1.251 Problem 254

Internal problem ID [9423]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 254
Date solved : `Monday, January 27, 2025 at 06:02:39 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x(4+x)\,y' + (2+x)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.251 (sec)

Writing the ode as

$$x^2y'' + \left(x^2+4x\right)y' + (2+x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2+4x \\ C &= 2+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4+x}{4x} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4+x \\ t &= 4x \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4+x}{4x}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.480: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x$. There is a pole at $x = 0$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \frac{5}{x^4} + \frac{14}{x^5} - \frac{42}{x^6} + \frac{132}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4+x}{4x} \\ &= Q + \frac{R}{4x} \\ &= \left(\frac{1}{4}\right) + \left(\frac{1}{x}\right) \\ &= \frac{1}{4} + \frac{1}{x} \end{aligned}$$

Since the degree of t is 1, then we see that the coefficient of the term 1 in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$\begin{aligned} b &= (1) - (0) \\ &= 1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{1}{\frac{1}{2}} - 0\right) &= 1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{1}{\frac{1}{2}} - 0\right) &= -1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4+x}{4x}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	1	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 1$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{x} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2} + \frac{1}{x} \\ &= \frac{1}{2} + \frac{1}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2} + \frac{1}{x}\right)(0) + \left(\left(-\frac{1}{x^2}\right) + \left(\frac{1}{2} + \frac{1}{x}\right)^2 - \left(\frac{4+x}{4x}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{1}{2} + \frac{1}{x}\right) dx} \\ &= x\, e^{\frac{x}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^2+4x}{x^2}\, dx} \\ &= z_1 e^{-\frac{x}{2} - 2\ln(x)} \\ &= z_1 \left(\frac{e^{-\frac{x}{2}}}{x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+4x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-x-4\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{e^{-x}}{3x} + \frac{e^{-x}}{6} - \frac{x\,e^{-x}}{6} + \frac{x^2\,\mathrm{Ei}_1(x)}{6} - \frac{\mathrm{Ei}_1(x)\,x^3 - e^{-x}x^2 - 6x\,\mathrm{Ei}_1(x) + x\,e^{-x} + 4\,e^{-x}}{6x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x}\left(-\frac{e^{-x}}{3x} + \frac{e^{-x}}{6} - \frac{x\,e^{-x}}{6} + \frac{x^2\,\mathrm{Ei}_1(x)}{6}\right.\right. \\
&\qquad \left.\left. - \frac{\mathrm{Ei}_1(x)\,x^3 - e^{-x}x^2 - 6x\,\mathrm{Ei}_1(x) + x\,e^{-x} + 4\,e^{-x}}{6x}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 25

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(x+4)*diff(y(x),x)+(x+2)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{-c_2 e^{-x} + x(\mathrm{Ei}_1(x)\, c_2 + c_1)}{x^2}$$

Mathematica DSolve solution

Solving time : 0.218 (sec)
Leaf size : 36

```
DSolve[{x^2*D[y[x],{x,2}]+x*(4+x)*D[y[x],x]+(2+x)*y[x]==0,{}},y[x],x,IncludeSingularSolution
```

$$y(x) \to \frac{c_2 \int_1^x \frac{e^{-K[1]}}{K[1]^2} dK[1] + c_1}{e^2 x}$$

2.1.252 Problem 255

Internal problem ID [9424]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 255
Date solved : `Monday, January 27, 2025 at 06:02:40 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{9}{4}\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.279 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{9}{4}\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{9}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2 + 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2+2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.481: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -1 + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx i - \frac{i}{x^2} - \frac{i}{2x^4} - \frac{i}{2x^6} - \frac{5i}{8x^8} - \frac{7i}{8x^{10}} - \frac{21i}{16x^{12}} - \frac{33i}{16x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= i \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-x^2+2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-1) + \left(\frac{2}{x^2}\right) \\ &= -1 + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= i \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{i} - 0\right) &&= 0 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{i} - 0\right) &&= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2+2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	i	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + (-)(i) \\ &= -\frac{1}{x} - i \\ &= -\frac{1}{x} - i \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered} (0) + 2\left(-\frac{1}{x} - i\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - i\right)^2 - \left(\frac{-x^2+2}{x^2}\right) \right) = 0 \\ \frac{2ia_0 - 2}{x} = 0 \end{gathered}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -i\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - i$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x - i)\, \mathrm{e}^{\int \left(-\frac{1}{x} - i\right) dx} \\ &= (x - i)\, \mathrm{e}^{-\ln(x) - ix} \\ &= \frac{(x - i)\, \mathrm{e}^{-ix}}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x}{x^2} \, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x - i)\, \mathrm{e}^{-ix}}{x^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{(ix - 1)\, \mathrm{e}^{2ix}}{-2x + 2i}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{(x - i)\, \mathrm{e}^{-ix}}{x^{3/2}}\right) + c_2 \left(\frac{(x - i)\, \mathrm{e}^{-ix}}{x^{3/2}} \left(\frac{(ix - 1)\, \mathrm{e}^{2ix}}{-2x + 2i}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right)+x\left(\frac{d}{dx}y(x)\right)+\left(x^2-\frac{9}{4}\right)y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(4x^2-9)y(x)}{4x^2}-\frac{\frac{d}{dx}y(x)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{\frac{d}{dx}y(x)}{x}+\frac{(4x^2-9)y(x)}{4x^2}=0$

- ☐ Check to see if $x_0=0$ is a regular singular point
  - Define functions

    $\left[P_2(x)=\frac{1}{x}, P_3(x)=\frac{4x^2-9}{4x^2}\right]$
  - $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\bigg|_{x=0}=1$
  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\bigg|_{x=0}=-\frac{9}{4}$
  - $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right)+4x\left(\frac{d}{dx}y(x)\right)+(4x^2-9)y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

    $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_k x^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$
  - Convert $x\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r}$
  - Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(3+2r)(-3+2r)x^r+a_1(5+2r)(-1+2r)x^{1+r}+\left(\sum_{k=2}^{\infty}(a_k(2k+2r+3)(2k+2r-3)+4a_{\ldots}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(3+2r)(-3+2r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{-\frac{3}{2},\frac{3}{2}\right\}$

- Each term must be 0
  $a_1(5+2r)(-1+2r)=0$
- Solve for the dependent coefficient(s)
  $a_1=0$
- Each term in the series must be 0, giving the recursion relation
  $a_k(4k^2+8kr+4r^2-9)+4a_{k-2}=0$
- Shift index using $k->k+2$
  $a_{k+2}\left(4(k+2)^2+8(k+2)r+4r^2-9\right)+4a_k=0$
- Recursion relation that defines series solution to ODE
  $a_{k+2}=-\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+7}$
- Recursion relation for $r=-\frac{3}{2}$
  $a_{k+2}=-\frac{4a_k}{4k^2+4k-8}$
- Solution for $r=-\frac{3}{2}$
  $\left[y(x)=\sum_{k=0}^{\infty}a_k x^{k-\frac{3}{2}}, a_{k+2}=-\frac{4a_k}{4k^2+4k-8}, a_1=0\right]$
- Recursion relation for $r=\frac{3}{2}$
  $a_{k+2}=-\frac{4a_k}{4k^2+28k+40}$
- Solution for $r=\frac{3}{2}$
  $\left[y(x)=\sum_{k=0}^{\infty}a_k x^{k+\frac{3}{2}}, a_{k+2}=-\frac{4a_k}{4k^2+28k+40}, a_1=0\right]$
- Combine solutions and rename parameters
  $\left[y(x)=\left(\sum_{k=0}^{\infty}a_k x^{k-\frac{3}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{3}{2}}\right), a_{k+2}=-\frac{4a_k}{4k^2+4k-8}, a_1=0, b_{k+2}=-\frac{4b_k}{4k^2+28k+40}, b_1=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.048` (sec)
Leaf size : 30

```
dsolve(x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x+(x^2-9/4)*y(x) = 0,y(x),singsol=all)
```

$$y=\frac{-c_2(-x+i)\,\mathrm{e}^{-ix}+(x+i)\,c_1\mathrm{e}^{ix}}{x^{3/2}}$$

Mathematica DSolve solution

Solving time : 0.06 (sec)
Leaf size : 44

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-9/4)*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to -\frac{\sqrt{\frac{2}{\pi}}((c_1x + c_2)\cos(x) + (c_2x - c_1)\sin(x))}{x^{3/2}}$$

2.1.253 Problem 256

Internal problem ID [9425]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 256
Date solved : `Monday, January 27, 2025 at 06:02:41 PM`
CAS classification : `[_Lienard]`

Solve

$$xy'' + 2y' + xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.123 (sec)

Writing the ode as

$$xy'' + 2y' + xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.483: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\, dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1 \left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2} dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\cos(x)}{x}\right) + c_2 \left(\frac{\cos(x)}{x}(\tan(x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dx^2}y(x)\right)x + 2\frac{d}{dx}y(x) + xy(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -y(x) - \frac{2\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) + \frac{2\left(\frac{d}{dx}y(x)\right)}{x} + y(x) = 0$
- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{2}{x}, P_3(x) = 1\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $\left(\frac{d^2}{dx^2}y(x)\right)x + 2\frac{d}{dx}y(x) + xy(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x \cdot y(x)$ to series expansion
    $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+1}$
  - Shift index using $k- >k-1$

$x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

- Convert $\frac{d}{dx}y(x)$ to series expansion

  $\frac{d}{dx}y(x) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $\frac{d}{dx}y(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$

- Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $a_1(1+r)\,(2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1} = 0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)\,(k+3+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{(k+1)(k+2)}$

- Solution for $r = -1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+3)}$

- Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0, b_{k+2} = -\frac{b_k}{(k+2)(k+3)}, 2b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 17

```
dsolve(x*diff(diff(y(x),x),x)+2*diff(y(x),x)+x*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{\sin(x) c_1 + \cos(x) c_2}{x}$$

Mathematica DSolve solution

Solving time : 0.027 (sec)
Leaf size : 37

```
DSolve[{x*D[y[x],{x,2}]+2*D[y[x],x]+x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2c_1 e^{-ix} - ic_2 e^{ix}}{2x}$$

2.1.254 Problem 257

Internal problem ID [9426]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 257
Date solved : `Monday, January 27, 2025 at 06:02:41 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2xy'' + 5(1-2x)\,y' - 5y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.421 (sec)

Writing the ode as

$$2xy'' + (-10x+5)\,y' - 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x \\ B &= -10x+5 \\ C &= -5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{100x^2 - 60x + 5}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 100x^2 - 60x + 5 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{100x^2 - 60x + 5}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.485: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{25}{4} - \frac{15}{4x} + \frac{5}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{5}{2} - \frac{3}{4x} - \frac{1}{20x^2} - \frac{3}{200x^3} - \frac{1}{200x^4} - \frac{9}{5000x^5} - \frac{137}{200000x^6} - \frac{543}{2000000x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{5}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{5}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{25}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{100x^2 - 60x + 5}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{25}{4}\right) + \left(\frac{-60x + 5}{16x^2}\right) \\ &= \frac{25}{4} + \frac{-60x + 5}{16x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -60. Dividing this by leading coefficient in t which is 16 gives $-\frac{15}{4}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{15}{4}\right) - (0) \\ &= -\frac{15}{4} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{5}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{15}{4}}{\frac{5}{2}} - 0\right) &&= -\frac{3}{4} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{15}{4}}{\frac{5}{2}} - 0\right) &&= \frac{3}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{100x^2 - 60x + 5}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{5}{2}$	$-\frac{3}{4}$	$\frac{3}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{3}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{3}{4} - \left(-\frac{1}{4}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{4x} + (-)\left(\frac{5}{2}\right) \\ &= -\frac{1}{4x} - \frac{5}{2} \\ &= -\frac{1}{4x} - \frac{5}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{4x} - \frac{5}{2}\right)(1) + \left(\left(\frac{1}{4x^2}\right) + \left(-\frac{1}{4x} - \frac{5}{2}\right)^2 - \left(\frac{100x^2 - 60x + 5}{16x^2}\right) \right) = 0$$

$$\frac{-1 + 10a_0}{2x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{ a_0 = \frac{1}{10} \right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{1}{10}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x + \frac{1}{10}\right) e^{\int \left(-\frac{1}{4x} - \frac{5}{2}\right) dx} \\ &= \left(x + \frac{1}{10}\right) e^{-\frac{5x}{2} - \frac{\ln(x)}{4}} \\ &= \frac{(1 + 10x)\, e^{-\frac{5x}{2}}}{10 x^{1/4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-10x+5}{2x} \, dx} \\ &= z_1 e^{\frac{5x}{2} - \frac{5 \ln(x)}{4}} \\ &= z_1 \left(\frac{e^{\frac{5x}{2}}}{x^{5/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1 + 10x}{10 x^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-10x+5}{2x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{5x - \frac{5 \ln(x)}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(\int \frac{100\, e^{5x - \frac{5 \ln(x)}{2}} x^3}{(1 + 10x)^2} \, dx \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{1 + 10x}{10 x^{3/2}}\right) + c_2 \left(\frac{1 + 10x}{10 x^{3/2}} \left(\int \frac{100\, e^{5x - \frac{5 \ln(x)}{2}} x^3}{(1 + 10x)^2} \, dx \right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2\left(\frac{d^2}{dx^2}y(x)\right)x + 5(-2x+1)\left(\frac{d}{dx}y(x)\right) - 5y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{5y(x)}{2x} + \frac{5(2x-1)\left(\frac{d}{dx}y(x)\right)}{2x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) - \frac{5(2x-1)\left(\frac{d}{dx}y(x)\right)}{2x} - \frac{5y(x)}{2x} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{5(2x-1)}{2x}, P_3(x) = -\frac{5}{2x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{5}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2\left(\frac{d^2}{dx^2}y(x)\right)x + (-10x+5)\left(\frac{d}{dx}y(x)\right) - 5y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..1$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-1}$

  - Shift index using $k- >k+1$

    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(3+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(2k+5+2r) - 5a_k(2k+2r+1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, -\frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation
  $2(k+1+r)\left(k+r+\frac{5}{2}\right)a_{k+1}-10a_k\left(k+r+\frac{1}{2}\right)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{5a_k(2k+2r+1)}{(k+1+r)(2k+5+2r)}$
- Recursion relation for $r=0$
  $a_{k+1}=\frac{5a_k(2k+1)}{(k+1)(2k+5)}$
- Solution for $r=0$
  $\left[y(x)=\sum_{k=0}^{\infty}a_k x^k, a_{k+1}=\frac{5a_k(2k+1)}{(k+1)(2k+5)}\right]$
- Recursion relation for $r=-\frac{3}{2}$; series terminates at $k=1$
  $a_{k+1}=\frac{5a_k(2k-2)}{\left(k-\frac{1}{2}\right)(2k+2)}$
- Apply recursion relation for $k=0$
  $a_1=10a_0$
- Terminating series solution of the ODE for $r=-\frac{3}{2}$. Use reduction of order to find the second
  $y(x)=a_0\cdot(1+10x)$
- Combine solutions and rename parameters
  $\left[y(x)=\left(\sum_{k=0}^{\infty}a_k x^k\right)+b_0\cdot(1+10x), a_{k+1}=\frac{5a_k(2k+1)}{(k+1)(2k+5)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special functio
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.057 (sec)
Leaf size : 44

```
dsolve(2*x*diff(diff(y(x),x),x)+5*(1-2*x)*diff(y(x),x)-5*y(x) = 0,y(x),singsol=all)
```

$$y=\frac{10\sqrt{5}\,c_1\sqrt{\pi}\left(\frac{1}{10}+x\right)\operatorname{erfi}\left(\sqrt{5}\,\sqrt{x}\right)-10\,\mathrm{e}^{5x}c_1\sqrt{x}+10c_2\left(\frac{1}{10}+x\right)}{x^{3/2}}$$

Mathematica DSolve solution

Solving time : 0.039 (sec)
Leaf size : 40

```
DSolve[{2*x*D[y[x],{x,2}]+5*(1-2*x)*D[y[x],x]-5*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tr
```

$$y(x) \to c_2 L_{-\frac{1}{2}}^{\frac{3}{2}}(5x) + \frac{c_1(10x+1)}{10\sqrt{5}x^{3/2}}$$

2.1.255 Problem 258

Internal problem ID [9427]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 258
Date solved : `Monday, January 27, 2025 at 06:02:42 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.145 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.487: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\, dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos(x)}{\sqrt{x}}(\tan(x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x\left(\frac{d}{dx}y(x)\right) + \left(x^2 - \frac{1}{4}\right)y(x) = 0$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
$\frac{d^2}{dx^2}y(x) = -\frac{(4x^2-1)y(x)}{4x^2} - \frac{\frac{d}{dx}y(x)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
$\frac{d^2}{dx^2}y(x) + \frac{\frac{d}{dx}y(x)}{x} + \frac{(4x^2-1)y(x)}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point
- ◦ Define functions
$\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$
- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$
$\left.(x \cdot P_2(x))\right|_{x=0} = 1$
- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$
$\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$
- ◦ $x = 0$is a regular singular point
Check to see if $x_0 = 0$ is a regular singular point
$x_0 = 0$

- Multiply by denominators
$4x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x\left(\frac{d}{dx}y(x)\right) + (4x^2 - 1)\,y(x) = 0$
- Assume series solution for $y(x)$
$y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- Shift index using $k- > k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x \cdot \left(\frac{d}{dx} y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$
- Convert $x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-2})\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$
- Shift index using $k- > k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$
- Solution for $r = -\frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$
- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.040 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x+(x^2-1/4)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{\sin (x) c_1 + \cos (x) c_2}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.032 (sec)
Leaf size : 39

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-1/4)*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to \frac{e^{-ix}(2c_1 - ic_2 e^{2ix})}{2\sqrt{x}}$$

2.1.256 Problem 259

Internal problem ID [9428]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 259
Date solved : `Monday, January 27, 2025 at 06:02:42 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + (x+n)\,y' + (n+1)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.419 (sec)

Writing the ode as

$$xy'' + (x+n)\,y' + (n+1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= x+n \\ C &= n+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{n^2 - 2xn + x^2 - 2n - 4x}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= n^2 - 2xn + x^2 - 2n - 4x \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{n^2 - 2xn + x^2 - 2n - 4x}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.489: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{\frac{1}{4}n^2 - \frac{1}{2}n}{x^2} + \frac{-\frac{n}{2} - 1}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{1}{4}n^2 - \frac{1}{2}n$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{n}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = 1 - \frac{n}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}
$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} - \frac{3n^5}{2x^6} - \frac{3n^4}{2x^5} - \frac{3n^3}{2x^4} - \frac{3n^2}{2x^3} - \frac{3n}{2x^2} - \frac{77n^5}{2x^7} - \frac{53n^4}{2x^6} - \frac{67n^3}{4x^5} - \frac{37n^2}{4x^4} - \frac{4n}{x^3} - \frac{1075n^4}{4x^7} - \frac{491n^3}{4x^6} - \frac{93n^2}{2x^5} - \frac{13n}{x^4} - 7 \qquad (9)
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a = \frac{1}{2}
$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2} \qquad (10)
\end{aligned}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}
$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{n^2 - 2xn + x^2 - 2n - 4x}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{(-2n-4)\,x + n^2 - 2n}{4x^2}\right) \\
&= \frac{1}{4} + \frac{(-2n-4)\,x + n^2 - 2n}{4x^2}
\end{aligned}
$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is $-2n - 4$. Dividing this by leading coefficient in t which is 4 gives $-\frac{n}{2} - 1$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{n}{2} - 1\right) - (0) \\
&= -\frac{n}{2} - 1
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{n}{2}-1}{\frac{1}{2}} - 0\right) &&= -\frac{n}{2} - 1 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{n}{2}-1}{\frac{1}{2}} - 0\right) &&= \frac{n}{2} + 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{n^2 - 2xn + x^2 - 2n - 4x}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{n}{2}$	$1 - \frac{n}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{n}{2} - 1$	$\frac{n}{2} + 1$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{n}{2} + 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{n}{2} + 1 - \left(\frac{n}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{n}{2x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{n}{2x} - \frac{1}{2} \\
&= \frac{n - x}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{n}{2x}-\frac{1}{2}\right)(1)+\left(\left(-\frac{n}{2x^2}\right)+\left(\frac{n}{2x}-\frac{1}{2}\right)^2-\left(\frac{n^2-2xn+x^2-2n-4x}{4x^2}\right)\right)=0$$

$$\frac{n+a_0}{x}=0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0=-n\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x)=x-n$$

Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\, dx} \\
&= (x-n)\,\mathrm{e}^{\int \left(\frac{n}{2x}-\frac{1}{2}\right)dx} \\
&= (x-n)\,\mathrm{e}^{-\frac{x}{2}+\frac{n\ln(x)}{2}} \\
&= -(n-x)\,x^{\frac{n}{2}}\mathrm{e}^{-\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x+n}{x}\, dx} \\
&= z_1 e^{-\frac{x}{2}-\frac{n\ln(x)}{2}} \\
&= z_1\left(x^{-\frac{n}{2}}\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1=(x-n)\,\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1\int \frac{e^{\int -\frac{x+n}{x}\, dx}}{(y_1)^2}\, dx \\
&= y_1\int \frac{\mathrm{e}^{-n\ln(x)-x}}{(y_1)^2}\, dx \\
&= y_1\left(\int \frac{\mathrm{e}^{-n\ln(x)-x}\mathrm{e}^{2x}}{(x-n)^2}dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1y_1+c_2y_2 \\
&= c_1\left((x-n)\,\mathrm{e}^{-x}\right)+c_2\left((x-n)\,\mathrm{e}^{-x}\left(\int \frac{\mathrm{e}^{-n\ln(x)-x}\mathrm{e}^{2x}}{(x-n)^2}dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dx^2}y(x)\right)x+(x+n)\left(\frac{d}{dx}y(x)\right)+(n+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(n+1)y(x)}{x}-\frac{(x+n)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{(x+n)\left(\frac{d}{dx}y(x)\right)}{x}+\frac{(n+1)y(x)}{x}=0$

□ Check to see if $x_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{x+n}{x},P_3(x)=\frac{n+1}{x}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\Big|_{x=0}=n$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\Big|_{x=0}=0$

  - $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $\left(\frac{d^2}{dx^2}y(x)\right)x+(x+n)\left(\frac{d}{dx}y(x)\right)+(n+1)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..1$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-1}$

  - Shift index using $k->k+1$

    $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-1+r+n)\,x^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(k+r+n)+a_k(n+k+r+1)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+r+n)=0$

- Values of r that satisfy the indicial equation

  $r\in\{0,1-n\}$

- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)(k+r+n)+a_k(n+k+r+1)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=-\frac{a_k(n+k+r+1)}{(k+1+r)(k+r+n)}$
- Recursion relation for $r=0$
  $a_{k+1}=-\frac{a_k(n+k+1)}{(k+1)(k+n)}$
- Solution for $r=0$
  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^k, a_{k+1}=-\frac{a_k(n+k+1)}{(k+1)(k+n)}\right]$
- Recursion relation for $r=1-n$
  $a_{k+1}=-\frac{a_k(k+2)}{(k+2-n)(k+1)}$
- Solution for $r=1-n$
  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+1-n}, a_{k+1}=-\frac{a_k(k+2)}{(k+2-n)(k+1)}\right]$
- Combine solutions and rename parameters
  $\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+1-n}\right), a_{k+1}=-\frac{a_k(n+k+1)}{(k+1)(k+n)}, b_{k+1}=-\frac{b_k(k+2)}{(k+2-n)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solut
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special fu
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.022 (sec)
Leaf size : 42

```
dsolve(x*diff(diff(y(x),x),x)+(x+n)*diff(y(x),x)+(n+1)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{\left(c_2 x^{-n+1} \operatorname{hypergeom}\left(\left[-n\right], \left[-n+2\right], x\right) n + \left(-x+n\right) c_1\right) \mathrm{e}^{-x}}{n}$$

Mathematica DSolve solution

Solving time : 0.355 (sec)
Leaf size : 77

```
DSolve[{x*D[y[x],{x,2}]+(x+n)*D[y[x],x]+(n+1)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to e^{\int \frac{-n+x-1}{n-x} dx} \left(c_2 \int_1^x \exp\left(\int_1^{K[2]} -\frac{n^2 - 2K[1]n + (K[1]-2)K[1]}{(n-K[1])K[1]} dK[1] \right) dK[2] + c_1 \right)$$

2.1.257 Problem 260

Internal problem ID [9429]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 260
Date solved : `Monday, January 27, 2025 at 06:02:43 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^4y'' + xy' + y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.336 (sec)

Writing the ode as

$$x^4y'' + xy' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4 \\ B &= x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-10x^2 + 1}{4x^6} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -10x^2 + 1 \\ t &= 4x^6 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-10x^2 + 1}{4x^6}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.491: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^6$. There is a pole at $x = 0$ of order 6. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned} \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) \end{aligned}$$

The partial fraction decomposition of r is

$$r = -\frac{5}{2x^4} + \frac{1}{4x^6}$$

There is pole in r at $x = 0$ of order 6, hence $v = 3$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{1}{2x^3} - \frac{5}{2x} - \frac{25x}{4} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 3$ the above becomes

$$[\sqrt{r}]_c = \frac{1}{2x^3} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^3}$ is

$$a = \frac{1}{2}$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^4}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be $-\frac{5}{2}$. Therefore

$$\begin{aligned} b &= \left(-\frac{5}{2}\right) - (0) \\ &= -\frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{1}{2x^3} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} + 3\right) &&= -1 \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} + 3\right) &&= 4 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-10x^2 + 1}{4x^6}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	6	$\frac{1}{2x^3}$	-1	4

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^+) \\ &= 1 - (-1) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x^3} - \frac{1}{x} + (-)\,(0) \\
&= \frac{1}{2x^3} - \frac{1}{x} \\
&= \frac{1}{2x^3} - \frac{1}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{1}{2x^3} - \frac{1}{x}\right)(2x + a_1) + \left(\left(-\frac{3}{2x^4} + \frac{1}{x^2}\right) + \left(\frac{1}{2x^3} - \frac{1}{x}\right)^2 - \left(\frac{-10x^2 + 1}{4x^6}\right)\right) = 0$$

$$\frac{(2a_0 + 2)\,x + a_1}{x^3} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \left(x^2 - 1\right) e^{\int \left(\frac{1}{2x^3} - \frac{1}{x}\right) dx} \\
&= \left(x^2 - 1\right) e^{-\ln(x) - \frac{1}{4x^2}} \\
&= \frac{\left(x^2 - 1\right) e^{-\frac{1}{4x^2}}}{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{x^4}\, dx} \\
&= z_1 e^{\frac{1}{4x^2}} \\
&= z_1 \left(e^{\frac{1}{4x^2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2 - 1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^4}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\frac{1}{2x^2}}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{e^{\frac{1}{2x^2}} x^2}{(x^2-1)^2} dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{x^2-1}{x}\right) + c_2 \left(\frac{x^2-1}{x}\left(\int \frac{e^{\frac{1}{2x^2}} x^2}{(x^2-1)^2} dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.027 (sec)
Leaf size : 50

```
dsolve(diff(diff(y(x),x),x)*x^4+diff(y(x),x)*x+y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_1\sqrt{2}\,\sqrt{\pi}\,(x-1)\,(x+1)\,\mathrm{erfi}\left(\frac{\sqrt{2}}{2x}\right)+c_2x^2+2\,\mathrm{e}^{\frac{1}{2x^2}}c_1x-c_2}{x}$$

Mathematica DSolve solution

Solving time : 0.43 (sec)
Leaf size : 57

```
DSolve[{x^4*D[y[x],{x,2}]+x*D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to \frac{e^{3/2}(x^2-1)\left(c_2\int_1^x\frac{e^{\frac{1}{2K[1]^2}-3}K[1]^2}{(K[1]^2-1)^2}dK[1]+c_1\right)}{x}$$

2.1.258 Problem 261

Internal problem ID [9430]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 261
Date solved : `Monday, January 27, 2025 at 06:02:44 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + (2x^2 + x)\, y' - 4y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.248 (sec)

Writing the ode as

$$x^2y'' + (2x^2 + x)\, y' - 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 2x^2 + x \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 + 4x + 15}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2 + 4x + 15 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 + 4x + 15}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.492: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{15}{4x^2} + \frac{1}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{1}{2x} + \frac{7}{4x^2} - \frac{7}{8x^3} - \frac{35}{32x^4} + \frac{133}{64x^5} + \frac{63}{128x^6} - \frac{1239}{256x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 + 4x + 15}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (1) + \left(\frac{4x + 15}{4x^2}\right) \\ &= 1 + \frac{4x + 15}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$\begin{aligned} b &= (1) - (0) \\ &= 1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{1}{1} - 0\right) &= \frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{1}{1} - 0\right) &= -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^2 + 4x + 15}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= -\frac{1}{2} - \left(-\frac{3}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{2x} + (-)(1) \\
&= -\frac{3}{2x} - 1 \\
&= -\frac{3}{2x} - 1
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{3}{2x} - 1\right)(1) + \left(\left(\frac{3}{2x^2}\right) + \left(-\frac{3}{2x} - 1\right)^2 - \left(\frac{4x^2 + 4x + 15}{4x^2}\right) \right) &= 0 \\
\frac{-3 + 2a_0}{x} &= 0
\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{ a_0 = \frac{3}{2} \right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{3}{2}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x + \frac{3}{2}\right) e^{\int \left(-\frac{3}{2x} - 1\right) dx} \\ &= \left(x + \frac{3}{2}\right) e^{-x - \frac{3\ln(x)}{2}} \\ &= \frac{(3 + 2x)\, e^{-x}}{2x^{3/2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x^2 + x}{x^2}\, dx} \\ &= z_1 e^{-x - \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{e^{-x}}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{-2x}(3 + 2x)}{2x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2 + x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-2x - \ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{(2x^2 - 4x + 3)\, x\, e^{-2x - \ln(x)} e^{4x}}{6 + 4x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{e^{-2x}(3 + 2x)}{2x^2}\right) + c_2 \left(\frac{e^{-2x}(3 + 2x)}{2x^2} \left(\frac{(2x^2 - 4x + 3)\, x\, e^{-2x - \ln(x)} e^{4x}}{6 + 4x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right)+(2x^2+x)\left(\frac{d}{dx}y(x)\right)-4y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{4y(x)}{x^2}-\frac{(2x+1)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{(2x+1)\left(\frac{d}{dx}y(x)\right)}{x}-\frac{4y(x)}{x^2}=0$

- □ Check to see if $x_0=0$ is a regular singular point
  - ∘ Define functions

    $\left[P_2(x)=\frac{2x+1}{x},P_3(x)=-\frac{4}{x^2}\right]$

  - ∘ $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\Big|_{x=0}=1$

  - ∘ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\Big|_{x=0}=-4$

  - ∘ $x=0$is a regular singular point

    Check to see if $x_0=0$ is a regular singular point

    $x_0=0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right)+x(2x+1)\left(\frac{d}{dx}y(x)\right)-4y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

- □ Rewrite ODE with series expansions
  - ∘ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  - ∘ Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - ∘ Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

    Rewrite ODE with series expansions

    $a_0(2+r)\,(-2+r)\,x^r+\left(\sum_{k=1}^{\infty}\left(a_k(k+r+2)\,(k+r-2)+2a_{k-1}(k+r-1)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{-2,2\}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)\,(k+r-2)+2a_{k-1}(k+r-1)=0$

- Shift index using $k->k+1$

$a_{k+1}(k+3+r)(k+r-1)+2a_k(k+r)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{2a_k(k+r)}{(k+3+r)(k+r-1)}$
- Recursion relation for $r=-2$; series terminates at $k=2$

  $a_{k+1}=-\frac{2a_k(k-2)}{(k+1)(k-3)}$
- Apply recursion relation for $k=0$

  $a_1=-\frac{4a_0}{3}$
- Apply recursion relation for $k=1$

  $a_2=-\frac{a_1}{2}$
- Express in terms of a_0

  $a_2=\frac{2a_0}{3}$
- Terminating series solution of the ODE for $r=-2$. Use reduction of order to find the second lin

  $y(x)=a_0\cdot\left(1-\frac{4}{3}x+\frac{2}{3}x^2\right)$
- Recursion relation for $r=2$

  $a_{k+1}=-\frac{2a_k(k+2)}{(k+5)(k+1)}$
- Solution for $r=2$

  $\left[y(x)=\sum_{k=0}^{\infty}a_kx^{k+2}, a_{k+1}=-\frac{2a_k(k+2)}{(k+5)(k+1)}\right]$
- Combine solutions and rename parameters

  $\left[y(x)=a_0\cdot\left(1-\frac{4}{3}x+\frac{2}{3}x^2\right)+\left(\sum_{k=0}^{\infty}b_kx^{k+2}\right), b_{k+1}=-\frac{2b_k(k+2)}{(5+k)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 31

```
dsolve(x^2*diff(diff(y(x),x),x)+(2*x^2+x)*diff(y(x),x)-4*y(x) = 0,y(x),singsol=all)
```

$$y=\frac{c_2\mathrm{e}^{-2x}(2x+3)+2c_1\left(x^2-2x+\frac{3}{2}\right)}{x^2}$$

Mathematica DSolve solution

Solving time : 0.721 (sec)
Leaf size : 71

```
DSolve[{x^2*D[y[x],{x,2}]+(x+2*x^2)*D[y[x],x]-4*y[x]==2,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to \frac{e^{-2x}\left(c_2(2x+3)\int_1^x \frac{4e^{2K[1]}K[1]^3}{(2K[1]+3)^2}dK[1] - e^{2x}x^2 + c_1(2x+3)\right)}{2x^2}$$

2.1.259 Problem 262

Internal problem ID [9431]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 262
Date solved : `Monday, January 27, 2025 at 06:02:45 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(4x^3-14x^2-2x\right)y''-\left(6x^2-7x+1\right)y'+\left(6x-1\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.446 (sec)

Writing the ode as

$$\left(4x^3-14x^2-2x\right)y''+\left(-6x^2+7x-1\right)y'+\left(6x-1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=4x^3-14x^2-2x\\ B&=-6x^2+7x-1\\ C&=6x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-12x^4+156x^3+297x^2-78x-3}{16\left(2x^3-7x^2-x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=-12x^4+156x^3+297x^2-78x-3\\ t&=16\left(2x^3-7x^2-x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-12x^4+156x^3+297x^2-78x-3}{16\left(2x^3-7x^2-x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.494: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(2x^3 - 7x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{7}{4} + \frac{\sqrt{57}}{4}$ of order 2. There is a pole at $x = \frac{7}{4} - \frac{\sqrt{57}}{4}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4\left(x - \frac{7}{4} - \frac{\sqrt{57}}{4}\right)^2} + \frac{3}{4\left(x - \frac{7}{4} + \frac{\sqrt{57}}{4}\right)^2} + \frac{\frac{9}{8} - \frac{29\sqrt{57}}{152}}{x - \frac{7}{4} - \frac{\sqrt{57}}{4}} + \frac{\frac{9}{8} + \frac{29\sqrt{57}}{152}}{x - \frac{7}{4} + \frac{\sqrt{57}}{4}} - \frac{9}{4x} - \frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = \frac{7}{4}+\frac{\sqrt{57}}{4}$ let b be the coefficient of $\frac{1}{\left(x-\frac{7}{4}-\frac{\sqrt{57}}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

For the pole at $x = \frac{7}{4}-\frac{\sqrt{57}}{4}$ let b be the coefficient of $\frac{1}{\left(x-\frac{7}{4}+\frac{\sqrt{57}}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-12x^4+156x^3+297x^2-78x-3}{16\left(2x^3-7x^2-x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{3}{4}\\ \alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{4}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-12x^4+156x^3+297x^2-78x-3}{16\left(2x^3-7x^2-x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
$\frac{7}{4}+\frac{\sqrt{57}}{4}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$\frac{7}{4}-\frac{\sqrt{57}}{4}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{4}$ then

$$\begin{aligned}d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}+\alpha_{c_3}^{-}\right)\\ &= \frac{1}{4} - \left(-\frac{3}{4}\right)\\ &= 1\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{4x} - \frac{1}{2\left(x-\frac{7}{4}-\frac{\sqrt{57}}{4}\right)} - \frac{1}{2\left(x-\frac{7}{4}+\frac{\sqrt{57}}{4}\right)} + (-)(0) \\
&= \frac{1}{4x} - \frac{1}{2\left(x-\frac{7}{4}-\frac{\sqrt{57}}{4}\right)} - \frac{1}{2\left(x-\frac{7}{4}+\frac{\sqrt{57}}{4}\right)} \\
&= \frac{-6x^2+7x-1}{8x^3-28x^2-4x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{4x} - \frac{1}{2\left(x-\frac{7}{4}-\frac{\sqrt{57}}{4}\right)} - \frac{1}{2\left(x-\frac{7}{4}+\frac{\sqrt{57}}{4}\right)} \right)(1) + \left(\left(-\frac{1}{4x^2} + \frac{1}{2\left(x-\frac{7}{4}-\frac{\sqrt{57}}{4}\right)^2} + \frac{1}{2\left(x-\frac{7}{4}+\right.}\right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\, dx} \\
&= (x-1)\,\mathrm{e}^{\int \left(\frac{1}{4x} - \frac{1}{2\left(x-\frac{7}{4}-\frac{\sqrt{57}}{4}\right)} - \frac{1}{2\left(x-\frac{7}{4}+\frac{\sqrt{57}}{4}\right)} \right) dx} \\
&= (x-1)\,\mathrm{e}^{\frac{\left(-57+7\sqrt{57}\right)\sqrt{57}\,\ln\left(4x-7+\sqrt{57}\right)}{-798+114\sqrt{57}} + \frac{2\ln(x)}{\left(7+\sqrt{57}\right)\left(-7+\sqrt{57}\right)} - \frac{\left(57+7\sqrt{57}\right)\sqrt{57}\,\ln\left(4x-7-\sqrt{57}\right)}{2\left(399+57\sqrt{57}\right)}} \\
&= \frac{(x-1)\,x^{1/4}}{\sqrt{4x-7+\sqrt{57}}\,\sqrt{4x-7-\sqrt{57}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-6x^2+7x-1}{4x^3-14x^2-2x}\, dx} \\
&= z_1 e^{-\frac{\ln(x)}{4} + \frac{\ln\left(2x^2-7x-1\right)}{2}} \\
&= z_1 \left(\frac{\sqrt{2x^2-7x-1}}{x^{1/4}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x-1)\sqrt{2}}{4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-6x^2+7x-1}{4x^3-14x^2-2x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{\ln(x)}{2}+\ln(2x^2-7x-1)}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{16x(2x+1)\,\mathrm{e}^{-\frac{\ln(x)}{2}+\ln(2x^2-7x-1)}}{(x-1)(2x^2-7x-1)}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{(x-1)\sqrt{2}}{4}\right) + c_2\left(\frac{(x-1)\sqrt{2}}{4}\left(\frac{16x(2x+1)\,\mathrm{e}^{-\frac{\ln(x)}{2}+\ln(2x^2-7x-1)}}{(x-1)(2x^2-7x-1)}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$(4x^3-14x^2-2x)\left(\frac{d^2}{dx^2}y(x)\right) - (6x^2-7x+1)\left(\frac{d}{dx}y(x)\right) + (6x-1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{(6x-1)y(x)}{2x(2x^2-7x-1)} + \frac{(6x^2-7x+1)\left(\frac{d}{dx}y(x)\right)}{2x(2x^2-7x-1)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) - \frac{(6x^2-7x+1)\left(\frac{d}{dx}y(x)\right)}{2x(2x^2-7x-1)} + \frac{(6x-1)y(x)}{2x(2x^2-7x-1)} = 0$
- ☐ Check to see if x_0 is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{6x^2-7x+1}{2x(2x^2-7x-1)}, P_3(x) = \frac{6x-1}{2x(2x^2-7x-1)}\right]$
  - $x \cdot P_2(x)$ is analytic at $x=0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{1}{2}$
  - $x^2 \cdot P_3(x)$ is analytic at $x=0$
    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$
  - $x = 0$is a regular singular point
    Check to see if x_0 is a regular singular point
    $x_0 = 0$

- Multiply by denominators
  $2x(2x^2-7x-1)\left(\frac{d^2}{dx^2}y(x)\right)+(-6x^2+7x-1)\left(\frac{d}{dx}y(x)\right)+(6x-1)\,y(x)=0$
- Assume series solution for $y(x)$
  $y(x)=\sum_{k=0}^{\infty} a_k x^{k+r}$
- □ Rewrite ODE with series expansions
  - Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$
    $x^m\cdot y(x)=\sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k->k-m$
    $x^m\cdot y(x)=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..2$
    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k->k+1-m$
    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=1..3$
    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,x^{k+r-2+m}$
  - Shift index using $k->k+2-m$
    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions
  $-a_0r(-1+2r)\,x^{-1+r}+\left(-a_1(1+r)(1+2r)-a_0(14r^2-21r+1)\right)x^r+\left(\sum_{k=1}^{\infty}\left(-a_{k+1}(k+1+r\right.\right.$
- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-r(-1+2r)=0$
- Values of r that satisfy the indicial equation
  $r\in\left\{0,\frac{1}{2}\right\}$
- Each term must be 0
  $-a_1(1+r)(1+2r)-a_0(14r^2-21r+1)=0$
- Each term in the series must be 0, giving the recursion relation
  $(-14a_k+4a_{k-1}-2a_{k+1})\,k^2+((-28a_k+8a_{k-1}-4a_{k+1})\,r+21a_k-18a_{k-1}-3a_{k+1})\,k+(-14a$
- Shift index using $k->k+1$
  $(-14a_{k+1}+4a_k-2a_{k+2})\,(k+1)^2+((-28a_{k+1}+8a_k-4a_{k+2})\,r+21a_{k+1}-18a_k-3a_{k+2})\,(k+$
- Recursion relation that defines series solution to ODE
  $a_{k+2}=\frac{4k^2a_k-14k^2a_{k+1}+8kra_k-28kra_{k+1}+4r^2a_k-14r^2a_{k+1}-10ka_k-7ka_{k+1}-10ra_k-7ra_{k+1}+6a_k+6a_{k+1}}{2k^2+4kr+2r^2+7k+7r+6}$
- Recursion relation for $r=0$
  $a_{k+2}=\frac{4k^2a_k-14k^2a_{k+1}-10ka_k-7ka_{k+1}+6a_k+6a_{k+1}}{2k^2+7k+6}$
- Solution for $r=0$
  $\left[y(x)=\sum_{k=0}^{\infty}a_kx^k, a_{k+2}=\frac{4k^2a_k-14k^2a_{k+1}-10ka_k-7ka_{k+1}+6a_k+6a_{k+1}}{2k^2+7k+6}, -a_1-a_0=0\right]$
- Recursion relation for $r=\frac{1}{2}$
  $a_{k+2}=\frac{4k^2a_k-14k^2a_{k+1}-6ka_k-21ka_{k+1}+2a_k-a_{k+1}}{2k^2+9k+10}$
- Solution for $r=\frac{1}{2}$

$$\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2}=\frac{4k^2 a_k-14k^2 a_{k+1}-6k a_k-21k a_{k+1}+2a_k-a_{k+1}}{2k^2+9k+10}, -3a_1+6a_0=0\right]$$

- Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2}=\frac{4k^2 a_k-14k^2 a_{k+1}-10k a_k-7k a_{k+1}+6a_k+6a_{k+1}}{2k^2+7k+6}, -a_1-a_0=0, b_k$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 21

```
dsolve((4*x^3-14*x^2-2*x)*diff(diff(y(x),x),x)-(6*x^2-7*x+1)*diff(y(x),x)+(6*x-1)*y(x) =
```

$$y=c_2\sqrt{x}+c_1(x-1)+2c_2x^{3/2}$$

Mathematica DSolve solution

Solving time : 1.932 (sec)
Leaf size : 155

```
DSolve[{(4*x^3-14*x^2-2*x)*D[y[x],{x,2}]-(6*x^2-7*x+1)*D[y[x],x]+(6*x-1)*y[x]==0,{}},y[x],x,In
```

$$y(x)\to (x-1)\exp\left(\int_1^x \frac{6K[1]^2-7K[1]+1}{-8K[1]^3+28K[1]^2+4K[1]}dK[1]-\frac{1}{2}\int_1^x\left(\frac{7-4K[2]}{K[2](2K[2]-7)-1}+\frac{1}{2K[2]}\right)dK[2]\right)\left(c_2\int_1^x\frac{\exp\left(-2\int_1^{K[3]}\frac{6K[1]^2-7K[1]+1}{-8K[1]^3+28K[1]^2+4K[1]}dK[1]\right)}{(K[3]-1)^2}dK[3]+c_1\right)$$

2.1.260 Problem 263

Internal problem ID [9432]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 263
Date solved : `Monday, January 27, 2025 at 06:02:46 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x^2y' + (x-2)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.204 (sec)

Writing the ode as

$$x^2y'' + x^2y' + (x-2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2 \\ C &= x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 4x + 8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.496: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{x^2} - \frac{1}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{1}{x^2} + \frac{2}{x^3} + \frac{3}{x^4} + \frac{2}{x^5} - \frac{6}{x^6} - \frac{28}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-4x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-4x + 8}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned}b &= (-1) - (0) \\ &= -1\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 0\right) &= -1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 0\right) &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -1$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2} - \frac{1}{x} \\ &= \frac{x-2}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2} - \frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(\frac{1}{2} - \frac{1}{x}\right)^2 - \left(\frac{x^2-4x+8}{4x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{2} - \frac{1}{x}\right) dx} \\ &= \frac{\mathrm{e}^{\frac{x}{2}}}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^2}{x^2}\,dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\ &= y_1\left(-\left(x^2+2x+2\right)\mathrm{e}^{-x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x}\left(-\left(x^2+2x+2\right)\mathrm{e}^{-x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x^2\left(\frac{d}{dx}y(x)\right) + (x-2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{(x-2)y(x)}{x^2} - \frac{d}{dx}y(x)$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x) + \frac{d}{dx}y(x) + \frac{(x-2)y(x)}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = 1, P_3(x) = \frac{x-2}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = 0$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -2$$

- $x = 0$is a regular singular point
  Check to see if $x_0 = 0$ is a regular singular point
  $x_0 = 0$

- Multiply by denominators
  $x^2\left(\frac{d^2}{dx^2}y(x)\right) + x^2\left(\frac{d}{dx}y(x)\right) + (x-2)\, y(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$
  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- Shift index using $k- > k - m$
  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x^2 \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion
  $x^2 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r+1}$
- Shift index using $k- > k - 1$
  $x^2 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, x^{k+r}$
- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion
  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\, x^{k+r}$

  Rewrite ODE with series expansions
  $a_0(1+r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-2) + a_{k-1}(k+r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(1+r)\,(-2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-1, 2\}$
- Each term in the series must be 0, giving the recursion relation
  $a_k(k+r+1)\,(k+r-2) + a_{k-1}(k+r) = 0$
- Shift index using $k- > k + 1$
  $a_{k+1}(k+2+r)\,(k-1+r) + a_k(k+r+1) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = -\frac{a_k(k+r+1)}{(k+2+r)(k-1+r)}$
- Recursion relation for $r = -1$
  $a_{k+1} = -\frac{a_k k}{(k+1)(k-2)}$
- Series not valid for $r = -1$, division by 0 in the recursion relation at $k = 2$
  $a_{k+1} = -\frac{a_k k}{(k+1)(k-2)}$
- Recursion relation for $r = 2$
  $a_{k+1} = -\frac{a_k(k+3)}{(k+4)(k+1)}$
- Solution for $r = 2$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = -\frac{a_k(k+3)}{(k+4)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 24

```
dsolve(x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x^2+(x-2)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_2 \mathrm{e}^{-x} (x^2 + 2x + 2) + c_1}{x}$$

Mathematica DSolve solution

Solving time : 0.196 (sec)
Leaf size : 33

```
DSolve[{x^2*D[y[x],{x,2}]+x^2*D[y[x],x]+(x-2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to \frac{c_2 \int_1^x e^{-K[1]} K[1]^2 dK[1] + c_1}{x}$$

2.1.261 Problem 264

Internal problem ID [9433]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 264
Date solved : `Monday, January 27, 2025 at 06:02:46 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x^2y' + (x-2)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.204 (sec)

Writing the ode as

$$x^2y'' - x^2y' + (x-2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 \\ C &= x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 4x + 8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.498: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{x^2} - \frac{1}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{1}{x^2} + \frac{2}{x^3} + \frac{3}{x^4} + \frac{2}{x^5} - \frac{6}{x^6} - \frac{28}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-4x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-4x + 8}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned}b &= (-1) - (0) \\ &= -1\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 0\right) &= -1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 0\right) &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -1$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2} - \frac{1}{x} \\ &= \frac{x-2}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2} - \frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(\frac{1}{2} - \frac{1}{x}\right)^2 - \left(\frac{x^2-4x+8}{4x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{1}{2} - \frac{1}{x}\right) dx} \\ &= \frac{e^{\frac{x}{2}}}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x^2}{x^2}\,dx} \\ &= z_1 e^{\frac{x}{2}} \\ &= z_1 \left(\mathrm{e}^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^x}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^x}{(y_1)^2}\,dx \\ &= y_1\left(-\left(x^2+2x+2\right)\mathrm{e}^{-x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\mathrm{e}^x}{x}\right) + c_2\left(\frac{\mathrm{e}^x}{x}\left(-\left(x^2+2x+2\right)\mathrm{e}^{-x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) - x^2\left(\frac{d}{dx}y(x)\right) + (x-2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(x-2)y(x)}{x^2} + \frac{d}{dx}y(x)$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) - \frac{d}{dx}y(x) + \frac{(x-2)y(x)}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -1, P_3(x) = \frac{x-2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$(x^2 \cdot P_3(x))\bigg|_{x=0} = -2$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) - x^2\left(\frac{d}{dx}y(x)\right) + (x-2)\, y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum\limits_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

  $x^m \cdot y(x) = \sum\limits_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y(x) = \sum\limits_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y(x)\right) = \sum\limits_{k=0}^{\infty} a_k(k+r)\, x^{k+r+1}$

- Shift index using $k- >k-1$

  $x^2 \cdot \left(\frac{d}{dx}y(x)\right) = \sum\limits_{k=1}^{\infty} a_{k-1}(k-1+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum\limits_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-2+r)\, x^r + \left(\sum\limits_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-2) - a_{k-1}(k+r-2)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)\,(a_k(k+r+1) - a_{k-1}) = 0$

- Shift index using $k- >k+1$

  $(k-1+r)\,(a_{k+1}(k+2+r) - a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+2+r}$

- Recursion relation for $r = -1$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = -1$

  $\left[y(x) = \sum\limits_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = \frac{a_k}{k+1}\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+4}$

- Solution for $r = 2$

  $\left[y(x) = \sum\limits_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k}{k+4}\right]$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{4+k}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 23

```
dsolve(x^2*diff(diff(y(x),x),x)-diff(y(x),x)*x^2+(x-2)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{e^x c_1 + c_2(x^2 + 2x + 2)}{x}$$

Mathematica DSolve solution

Solving time : 0.207 (sec)
Leaf size : 36

```
DSolve[{x^2*D[y[x],{x,2}]-x^2*D[y[x],x]+(x-2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True
```

$$y(x) \to \frac{e^x \left(c_2 \int_1^x e^{-K[1]} K[1]^2 dK[1] + c_1\right)}{x}$$

2.1.262 Problem 265

Internal problem ID [9434]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 265
Date solved : `Monday, January 27, 2025 at 06:02:47 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1-4x)\,y''-\frac{xy'}{2}-\frac{3xy}{4}=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.297 (sec)

Writing the ode as

$$\left(-4x^3+x^2\right)y''-\frac{xy'}{2}-\frac{3xy}{4}=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=-4x^3+x^2\\ B&=-\frac{x}{2}\\ C&=-\frac{3x}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-48x^2-20x+5}{16\left(4x^2-x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=-48x^2-20x+5\\ t&=16\left(4x^2-x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-48x^2 - 20x + 5}{16 (4x^2 - x)^2} \right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.500: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(4x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{1}{4}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{4x} + \frac{5}{16x^2} - \frac{3}{16 \left(x - \frac{1}{4}\right)^2} - \frac{5}{4 \left(x - \frac{1}{4}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

For the pole at $x = \frac{1}{4}$ let b be the coefficient of $\frac{1}{(x-\frac{1}{4})^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-48x^2 - 20x + 5}{16\,(4x^2 - x)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_\infty &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{-1, 2, 5\}$
$\frac{1}{4}$	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
2	$\{1, 2, 3\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 2,\; e_2 = 1,\; e_\infty = 3$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(3 - (2 + (1))) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x - c} \\ &= \frac{1}{2}\left(\frac{2}{(x - (0))} + \frac{1}{(x - (\frac{1}{4}))}\right) \\ &= \frac{1}{x} + \frac{1}{2x - \frac{1}{2}} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + (3\theta^2 + 3\theta' - 4r)\, p' + (\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r')\, p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p}\\ &= \frac{1}{x} + \frac{1}{2x - \frac{1}{2}}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{x} + \frac{1}{2x - \frac{1}{2}}\right) w + \frac{144x^2 - 12x - 5}{16x^2 (-1 + 4x)^2} = 0$$

Solving for ω gives

$$\omega = \frac{12x - 2 + 3\sqrt{1 - 4x}}{4x (-1 + 4x)}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega \, dx}\\ &= e^{\int \frac{12x - 2 + 3\sqrt{1-4x}}{4x(-1+4x)} dx}\\ &= \frac{(-1 + 4x)^{1/4} \sqrt{x}\,\sqrt{2} \left(\frac{\sqrt{1-4x}+1}{\sqrt{x}}\right)^{3/2}}{4}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}\\ &= z_1 e^{-\int \frac{1}{2} \frac{-\frac{x}{2}}{-4x^3+x^2} \, dx}\\ &= z_1 e^{-\frac{\ln(-1+4x)}{4} + \frac{\ln(x)}{4}}\\ &= z_1 \left(\frac{x^{1/4}}{(-1 + 4x)^{1/4}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/4}\sqrt{2}\left(\sqrt{1 - 4x} + 1\right)\sqrt{\frac{\sqrt{1-4x}+1}{\sqrt{x}}}}{4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-\frac{x}{2}}{-4x^3+x^2} \, dx}}{(y_1)^2} \, dx\\ &= y_1 \int \frac{e^{-\frac{\ln(-1+4x)}{2} + \frac{\ln(x)}{2}}}{(y_1)^2} \, dx\\ &= y_1 \left(\frac{2\,\mathrm{e}^{-\frac{\ln(-1+4x)}{2} + \frac{\ln(1-4x)}{2}} \left(-\left(\sqrt{1 - 4x} + 1\right)^2 + 2\sqrt{1 - 4x} + 2\right)^{3/2}}{3\left(\sqrt{1 - 4x} + 1\right)^3}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{1/4}\sqrt{2}\left(\sqrt{1-4x}+1\right)\sqrt{\frac{\sqrt{1-4x}+1}{\sqrt{x}}}}{4}\right) \\
&\quad + c_2 \left(\frac{x^{1/4}\sqrt{2}\left(\sqrt{1-4x}+1\right)\sqrt{\frac{\sqrt{1-4x}+1}{\sqrt{x}}}}{4}\right)\left(\frac{2\,\mathrm{e}^{-\frac{\ln(-1+4x)}{2}+\frac{\ln(1-4x)}{2}}\left(-\left(\sqrt{1-4x}+1\right)^2+2\sqrt{1-4x}+2\right)^{3/2}}{3\left(\sqrt{1-4x}+1\right)^3}\right.
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(1-4x)\left(\frac{d^2}{dx^2}y(x)\right)-\frac{x\left(\frac{d}{dx}y(x)\right)}{2}-\frac{3xy(x)}{4}=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{3y(x)}{4x(4x-1)}-\frac{\frac{d}{dx}y(x)}{2x(4x-1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{\frac{d}{dx}y(x)}{2x(4x-1)}+\frac{3y(x)}{4x(4x-1)}=0$

□ Check to see if x_0 is a regular singular point

  ∘ Define functions

    $\left[P_2(x)=\frac{1}{2x(4x-1)}, P_3(x)=\frac{3}{4x(4x-1)}\right]$

  ∘ $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\Big|_{x=0}=-\frac{1}{2}$

  ∘ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\Big|_{x=0}=0$

  ∘ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4x(4x-1)\left(\frac{d^2}{dx^2}y(x)\right)+2\frac{d}{dx}y(x)+3y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ∘ Convert $\frac{d}{dx}y(x)$ to series expansion

    $\frac{d}{dx}y(x)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1}$

  ∘ Shift index using $k->k+1$

    $\frac{d}{dx}y(x)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,x^{k+r}$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 1..2$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k- > k+2-m$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r(-3+2r)\,x^{-1+r} + \left(\sum_{k=0}^{\infty} (-2a_{k+1}(k+1+r)(2k-1+2r) + a_k(4k+4r-1)(4k+4r-3))\,x$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(-3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $-4(k+1+r)\left(k+r-\frac{1}{2}\right)a_{k+1} + 16\left(k+r-\frac{3}{4}\right)a_k\left(k+r-\frac{1}{4}\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{(4k+4r-3)a_k(4k+4r-1)}{2(k+1+r)(2k-1+2r)}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{(4k-3)a_k(4k-1)}{2(k+1)(2k-1)}$

- Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{(4k-3)a_k(4k-1)}{2(k+1)(2k-1)}\right]$

- Recursion relation for $r = \frac{3}{2}$

  $a_{k+1} = \frac{(4k+3)a_k(4k+5)}{2\left(k+\frac{5}{2}\right)(2k+2)}$

- Solution for $r = \frac{3}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1} = \frac{(4k+3)a_k(4k+5)}{2\left(k+\frac{5}{2}\right)(2k+2)}\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+1} = \frac{(4k-3)a_k(4k-1)}{2(k+1)(2k-1)}, b_{k+1} = \frac{(4k+3)b_k(4k+5)}{2\left(k+\frac{5}{2}\right)(2k+2)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.017 (sec)
Leaf size : 46

```
dsolve(x^2*(1-4*x)*diff(diff(y(x),x),x)-1/2*diff(y(x),x)*x-3/4*x*y(x) = 0,y(x),singsol
```

$$y = -\frac{\sqrt{2}\left(c_1(x-1)\sqrt{1-4x} - 2c_2 x^{3/2} + c_1(3x-1)\right)}{\left(1+\sqrt{1-4x}\right)^{3/2}}$$

Mathematica DSolve solution

Solving time : 4.075 (sec)
Leaf size : 125

```
DSolve[{x^2*(1-4*x)*D[y[x],{x,2}]+((1-(3/2))*x-(6-4*(3/2))*x^2)*D[y[x],x]+(3/2)*(1-(3/2))*x*
```

$$y(x) \to \frac{\sqrt[4]{4x-1}\left(6c_1\left(\sqrt{4x-1}-i\right)^{3/2} + ic_2\left(\sqrt{4x-1}+i\right)^{3/2}\right)\exp\left(-\frac{1}{2}\int_1^x -\frac{1}{2K[1]-8K[1]^2}dK[1]\right)}{6\sqrt[4]{\sqrt{4x-1}-i}\sqrt[4]{\sqrt{4x-1}+i}}$$

2.1.263 Problem 266

Internal problem ID [9435]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 266
Date solved : `Monday, January 27, 2025 at 06:02:47 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + (x^2+x)\,y' + (x-9)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.248 (sec)

Writing the ode as

$$x^2y'' + (x^2+x)\,y' + (x-9)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2+x \\ C &= x-9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2-2x+35}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2-2x+35 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2-2x+35}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.502: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{35}{4x^2} - \frac{1}{2x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{17}{2x^2} + \frac{17}{2x^3} - \frac{255}{4x^4} - \frac{833}{4x^5} + \frac{3213}{4x^6} + \frac{21709}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 2x + 35}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 35}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-2x + 35}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 2x + 35}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{5}{2}\right) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{5}{2x} + \left(\frac{1}{2}\right) \\ &= -\frac{5}{2x} + \frac{1}{2} \\ &= \frac{-5+x}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(-\frac{5}{2x} + \frac{1}{2}\right)(2x + a_1) + \left(\left(\frac{5}{2x^2}\right) + \left(-\frac{5}{2x} + \frac{1}{2}\right)^2 - \left(\frac{x^2 - 2x + 35}{4x^2}\right) \right) = 0$$

$$\frac{(-a_1 - 8)\, x - 2a_0 - 5a_1}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 20, a_1 = -8\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 8x + 20$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x^2 - 8x + 20\right) e^{\int \left(-\frac{5}{2x} + \frac{1}{2}\right) dx} \\ &= \left(x^2 - 8x + 20\right) e^{\frac{x}{2} - \frac{5 \ln(x)}{2}} \\ &= \frac{\left(x^2 - 8x + 20\right) e^{\frac{x}{2}}}{x^{5/2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x^2 + x}{x^2} \, dx} \\ &= z_1 e^{-\frac{x}{2} - \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{e^{-\frac{x}{2}}}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2 - 8x + 20}{x^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2 + x}{x^2} \, dx}}{\left(y_1\right)^2} \, dx \\ &= y_1 \int \frac{e^{-x - \ln(x)}}{\left(y_1\right)^2} \, dx \\ &= y_1 \left(-\frac{\left(x^3 + 9x^2 + 36x + 60\right) x \, e^{-x - \ln(x)}}{x^2 - 8x + 20}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{x^2 - 8x + 20}{x^3}\right) + c_2 \left(\frac{x^2 - 8x + 20}{x^3} \left(-\frac{\left(x^3 + 9x^2 + 36x + 60\right) x \, e^{-x - \ln(x)}}{x^2 - 8x + 20}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right)+(x^2+x)\left(\frac{d}{dx}y(x)\right)+(x-9)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(x-9)y(x)}{x^2}-\frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{x}+\frac{(x-9)y(x)}{x^2}=0$

☐ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{x+1}{x},P_3(x)=\frac{x-9}{x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=-9$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right)+x(x+1)\left(\frac{d}{dx}y(x)\right)+(x-9)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  ◦ Shift index using $k->k-m$

  $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(3+r)\,(-3+r)\,x^r+\left(\sum_{k=1}^{\infty}\left(a_k(k+r+3)\,(k+r-3)+a_{k-1}(k+r)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(3+r)(-3+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{-3, 3\}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+3)(k+r-3)+a_{k-1}(k+r)=0$
- Shift index using $k->k+1$

  $a_{k+1}(k+4+r)(k-2+r)+a_k(k+r+1)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{a_k(k+r+1)}{(k+4+r)(k-2+r)}$
- Recursion relation for $r=-3$; series terminates at $k=2$

  $a_{k+1}=-\frac{a_k(k-2)}{(k+1)(k-5)}$
- Apply recursion relation for $k=0$

  $a_1=-\frac{2a_0}{5}$
- Apply recursion relation for $k=1$

  $a_2=-\frac{a_1}{8}$
- Express in terms of a_0

  $a_2=\frac{a_0}{20}$
- Terminating series solution of the ODE for $r=-3$. Use reduction of order to find the second lin

  $y(x)=a_0\cdot\left(1-\frac{2}{5}x+\frac{1}{20}x^2\right)$
- Recursion relation for $r=3$

  $a_{k+1}=-\frac{a_k(k+4)}{(k+7)(k+1)}$
- Solution for $r=3$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1}=-\frac{a_k(k+4)}{(k+7)(k+1)}\right]$
- Combine solutions and rename parameters

  $\left[y(x)=a_0\cdot\left(1-\frac{2}{5}x+\frac{1}{20}x^2\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), b_{k+1}=-\frac{b_k(4+k)}{(k+7)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 38

```
dsolve(x^2*diff(diff(y(x),x),x)+(x^2+x)*diff(y(x),x)+(x-9)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_2 \mathrm{e}^{-x}(x^3 + 9x^2 + 36x + 60) + c_1(x^2 - 8x + 20)}{x^3}$$

Mathematica DSolve solution

Solving time : 0.626 (sec)
Leaf size : 96

```
DSolve[{x^2*D[y[x],{x,2}]+(x+x^2)*D[y[x],x]+(x-9)*y[x]==0,{}},y[x],x,IncludeSingularSolution
```

$$y(x) \to \frac{(x^2 - 8x + 20)\exp\left(\int_1^x \frac{K[1]-5}{2K[1]}dK[1] - \frac{x}{2}\right)\left(c_2 \int_1^x \frac{\exp\left(-2\int_1^{K[2]} \frac{K[1]-5}{2K[1]}dK[1]\right)}{(K[2]^2-8K[2]+20)^2}dK[2] + c_1\right)}{\sqrt{x}}$$

2.1.264 Problem 267

Internal problem ID [9436]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 267
Date solved : `Monday, January 27, 2025 at 06:02:48 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x(x+1)\,y' + (3x-1)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.267 (sec)

Writing the ode as

$$x^2y'' + \left(x^2+x\right)y' + (3x-1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2 + x \\ C &= 3x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 10x + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 10x + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 10x + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.504: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4x^2} - \frac{5}{2x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{5}{2x} - \frac{11}{2x^2} - \frac{55}{2x^3} - \frac{671}{4x^4} - \frac{4565}{4x^5} - \frac{33231}{4x^6} - \frac{253275}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 10x + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-10x + 3}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-10x + 3}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -10. Dividing this by leading coefficient in t which is 4 gives $-\frac{5}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{5}{2}\right) - (0) \\ &= -\frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 0\right) &= -\frac{5}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 0\right) &= \frac{5}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 10x + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{5}{2}$	$\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{5}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{5}{2} - \left(\frac{3}{2}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{3}{2x} + (-)\left(\frac{1}{2}\right) \\ &= \frac{3}{2x} - \frac{1}{2} \\ &= -\frac{-3+x}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{2x} - \frac{1}{2}\right)(1) + \left(\left(-\frac{3}{2x^2}\right) + \left(\frac{3}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^2 - 10x + 3}{4x^2}\right)\right) = 0$$

$$\frac{3 + a_0}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -3\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = -3 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= pe^{\int \omega\, dx} \\
&= (-3+x)\,\mathrm{e}^{\int \left(\frac{3}{2x}-\frac{1}{2}\right)dx} \\
&= (-3+x)\,\mathrm{e}^{-\frac{x}{2}+\frac{3\ln(x)}{2}} \\
&= (-3+x)\,x^{3/2}\mathrm{e}^{-\frac{x}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2+x}{x^2}\, dx} \\
&= z_1 e^{-\frac{x}{2}-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{2}}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x\,\mathrm{e}^{-x}(-3+x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-x-\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{\mathrm{e}^{x}}{27\,(-3+x)} - \frac{\mathrm{Ei}_1(-x)}{6} - \frac{7\,\mathrm{e}^{x}}{54x} - \frac{\mathrm{e}^{x}}{18x^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x\,\mathrm{e}^{-x}(-3+x)\right) + c_2\left(x\,\mathrm{e}^{-x}(-3+x)\left(-\frac{\mathrm{e}^{x}}{27\,(-3+x)} - \frac{\mathrm{Ei}_1(-x)}{6} - \frac{7\,\mathrm{e}^{x}}{54x} - \frac{\mathrm{e}^{x}}{18x^2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right)+x(x+1)\left(\frac{d}{dx}y(x)\right)+(3x-1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(3x-1)y(x)}{x^2}-\frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{x}+\frac{(3x-1)y(x)}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=\frac{x+1}{x},P_3(x)=\frac{3x-1}{x^2}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=1$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=-1$

- ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right)+x(x+1)\left(\frac{d}{dx}y(x)\right)+(3x-1)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ◦ Shift index using $k->k-m$

  $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ◦ Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-1+r)\,x^r+\left(\sum_{k=1}^{\infty}\left(a_k(k+r+1)\,(k+r-1)+a_{k-1}(k+2+r)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(1+r)(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 1\}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)(k+r-1) + a_{k-1}(k+2+r) = 0$
- Shift index using $k- >k+1$

  $a_{k+1}(k+2+r)(k+r) + a_k(k+r+3) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r+3)}{(k+2+r)(k+r)}$
- Recursion relation for $r = -1$

  $a_{k+1} = -\frac{a_k(k+2)}{(k+1)(k-1)}$
- Series not valid for $r = -1$, division by 0 in the recursion relation at $k = 1$

  $a_{k+1} = -\frac{a_k(k+2)}{(k+1)(k-1)}$
- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{a_k(k+4)}{(k+3)(k+1)}$
- Solution for $r = 1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{a_k(k+4)}{(k+3)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.007` (sec)
Leaf size : 48

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(x+1)*diff(y(x),x)+(3*x-1)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{x^2 c_2 e^{-x}(x-3)\operatorname{Ei}_1(-x) + x^2 c_1 (x-3) e^{-x} + c_2(x^2-2x-1)}{x}$$

Mathematica DSolve solution

Solving time : 0.319 (sec)
Leaf size : 43

```
DSolve[{x^2*D[y[x],{x,2}]+x*(x+1)*D[y[x],x]+(3*x-1)*y[x]==0,{}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to e^{-x}(x-3)x\left(c_2\int_1^x \frac{e^{K[1]}}{(K[1]-3)^2K[1]^3}dK[1]+c_1\right)$$

2.1.265 Problem 268

Internal problem ID [9437]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 268
Date solved : `Monday, January 27, 2025 at 06:02:49 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - (x^2 + 4x)\, y' + 4y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.243 (sec)

Writing the ode as

$$x^2y'' + (-x^2 - 4x)\, y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 - 4x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 8x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 8x + 8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 8x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.506: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{x^2} + \frac{2}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{2}{x} - \frac{2}{x^2} + \frac{8}{x^3} - \frac{36}{x^4} + \frac{176}{x^5} - \frac{912}{x^6} + \frac{4928}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 + 8x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{8x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{8x + 8}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 8. Dividing this by leading coefficient in t which is 4 gives 2. Now b can be found.

$$\begin{aligned}b &= (2) - (0) \\ &= 2\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{2}{\frac{1}{2}} - 0\right) &&= 2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{2}{\frac{1}{2}} - 0\right) &&= -2\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 8x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	2	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - (\alpha_{c_1}^+) \\
&= 2 - (2) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{2}{x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} + \frac{2}{x} \\
&= \frac{x+4}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2} + \frac{2}{x}\right)(0) + \left(\left(-\frac{2}{x^2}\right) + \left(\frac{1}{2} + \frac{2}{x}\right)^2 - \left(\frac{x^2+8x+8}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{2} + \frac{2}{x}\right) dx} \\
&= x^2 e^{\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x^2-4x}{x^2}\,dx} \\ &= z_1 e^{\frac{x}{2}+2\ln(x)} \\ &= z_1\left(x^2 \mathrm{e}^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^4 \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-4x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{x+4\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(-\frac{\mathrm{e}^{-x}}{3x^3}+\frac{\mathrm{e}^{-x}}{6x^2}-\frac{\mathrm{e}^{-x}}{6x}+\frac{\mathrm{Ei}_1(x)}{6}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(x^4\mathrm{e}^x\right) + c_2\left(x^4\mathrm{e}^x\left(-\frac{\mathrm{e}^{-x}}{3x^3}+\frac{\mathrm{e}^{-x}}{6x^2}-\frac{\mathrm{e}^{-x}}{6x}+\frac{\mathrm{Ei}_1(x)}{6}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right) - (x^2+4x)\left(\frac{d}{dx}y(x)\right) + 4y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{4y(x)}{x^2} + \frac{(x+4)\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) - \frac{(x+4)\left(\frac{d}{dx}y(x)\right)}{x} + \frac{4y(x)}{x^2} = 0$
- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{x+4}{x}, P_3(x) = \frac{4}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -4$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 4$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) - x(x+4)\left(\frac{d}{dx}y(x)\right) + 4y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  - ◦ Shift index using $k- >k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - ◦ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)(-4+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)(k+r-4) - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)(-4+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{1, 4\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_k(k+r-4) - a_{k-1}\right) = 0$

- Shift index using $k- >k+1$

  $(k+r)\left(a_{k+1}(k-3+r) - a_k\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k-3+r}$

- Recursion relation for $r = 1$

  $a_{k+1} = \frac{a_k}{k-2}$

- Series not valid for $r = 1$, division by 0 in the recursion relation at $k = 2$

  $a_{k+1} = \frac{a_k}{k-2}$

- Recursion relation for $r = 4$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 4$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+4}, a_{k+1} = \frac{a_k}{k+1}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 35

```
dsolve(x^2*diff(diff(y(x),x),x)-(x^2+4*x)*diff(y(x),x)+4*y(x) = 0,y(x),singsol=all)
```

$$y = x\left(\mathrm{Ei}_1\left(x\right) \mathrm{e}^{x} c_2 x^3 + \mathrm{e}^{x} x^3 c_1 - c_2\left(x^2 - x + 2\right)\right)$$

Mathematica DSolve solution

Solving time : 60.067 (sec)
Leaf size : 39

```
DSolve[{x^2*D[y[x],{x,2}]-(x^2+4*x)*D[y[x],x]+4*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to e^{x+4} x^4 \left(\int_1^x \frac{e^{-K[1]-4} c_1}{K[1]^4} dK[1] + c_2\right)$$

2.1.266 Problem 269

Internal problem ID [9438]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 269
Date solved : Monday, January 27, 2025 at 06:02:49 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$2x^2y'' - (3x+2)\,y' + \frac{(2x-1)\,y}{x} = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.489 (sec)

Writing the ode as

$$2x^2y'' + (-3x-2)\,y' + \left(2-\frac{1}{x}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= -3x-2 \\ C &= 2-\frac{1}{x} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5x^2+36x+4}{16x^4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 5x^2+36x+4 \\ t &= 16x^4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5x^2 + 36x + 4}{16x^4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.508: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^4$. There is a pole at $x = 0$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned} \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) \end{aligned}$$

The partial fraction decomposition of r is

$$r = \frac{9}{4x^3} + \frac{1}{4x^4} + \frac{5}{16x^2}$$

There is pole in r at $x = 0$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{1}{2x^2} + \frac{9}{4x} - \frac{19}{4} + \frac{171x}{8} - \frac{475x^2}{4} + \frac{11799x^3}{16} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{1}{2x^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^2}$ is

$$a = \frac{1}{2}$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be $\frac{9}{4}$. Therefore

$$\begin{aligned} b &= \left(\frac{9}{4}\right) - (0) \\ &= \frac{9}{4} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{1}{2x^2} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) = \frac{1}{2}\left(\frac{\frac{9}{4}}{\frac{1}{2}} + 2\right) = \frac{13}{4} \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) = \frac{1}{2}\left(-\frac{\frac{9}{4}}{\frac{1}{2}} + 2\right) = -\frac{5}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{5x^2 + 36x + 4}{16x^4}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{5x^2 + 36x + 4}{16x^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	4	$\frac{1}{2x^2}$	$\frac{13}{4}$	$-\frac{5}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{4} - \left(-\frac{5}{4}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x^2} - \frac{5}{4x} + (-)(0) \\ &= -\frac{1}{2x^2} - \frac{5}{4x} \\ &= \frac{-2-5x}{4x^2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x^2} - \frac{5}{4x}\right)(1) + \left(\left(\frac{1}{x^3} + \frac{5}{4x^2}\right) + \left(-\frac{1}{2x^2} - \frac{5}{4x}\right)^2 - \left(\frac{5x^2+36x+4}{16x^4}\right)\right) = 0$$

$$\frac{-2+5a_0}{2x^2} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{2}{5}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{2}{5}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x + \frac{2}{5}\right) \mathrm{e}^{\int \left(-\frac{1}{2x^2} - \frac{5}{4x}\right) dx} \\ &= \left(x + \frac{2}{5}\right) \mathrm{e}^{\frac{1}{2x} - \frac{5 \ln(x)}{4}} \\ &= \frac{(2 + 5x)\, \mathrm{e}^{\frac{1}{2x}}}{5x^{5/4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-3x-2}{2x^2} \, dx} \\ &= z_1 e^{-\frac{1}{2x} + \frac{3 \ln(x)}{4}} \\ &= z_1 \left(x^{3/4} \mathrm{e}^{-\frac{1}{2x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{2 + 5x}{5\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3x-2}{2x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\frac{1}{x} + \frac{3 \ln(x)}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(\int \frac{25\, \mathrm{e}^{-\frac{1}{x} + \frac{3 \ln(x)}{2}} x}{(2 + 5x)^2} dx \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{2 + 5x}{5\sqrt{x}}\right) + c_2 \left(\frac{2 + 5x}{5\sqrt{x}} \left(\int \frac{25\, \mathrm{e}^{-\frac{1}{x} + \frac{3 \ln(x)}{2}} x}{(2 + 5x)^2} dx \right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solut
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returnin
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.100 (sec)
Leaf size : 35

```
dsolve(2*x^2*diff(diff(y(x),x),x)-(2+3*x)*diff(y(x),x)+(-1+2*x)/x*y(x) = 0,y(x),singsol=
```

$$y = \frac{c_2 \mathrm{e}^{-\frac{1}{x}} \operatorname{hypergeom}\left([2], \left[-\frac{1}{2}\right], \frac{1}{x}\right) x^{5/2} + 5c_1 x + 2c_1}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.509 (sec)
Leaf size : 65

```
DSolve[{2*x^2*D[y[x],{x,2}]-(3*x+2)*D[y[x],x]+(2*x-1)/x*y[x]==0,{}},y[x],x,IncludeSingularSolu
```

$$y(x) \to \frac{\sqrt{e}(5x+2)\left(c_2 \int_1^x \frac{25e^{-\frac{5}{2}-\frac{1}{K[1]}} K[1]^{5/2}}{(5K[1]+2)^2} dK[1] + c_1\right)}{5\sqrt{x}}$$

2.1.267 Problem 270

Internal problem ID [9439]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 270
Date solved : `Monday, January 27, 2025 at 06:02:50 PM`
CAS classification : `[_Jacobi]`

Solve

$$x(1-x)\,y''+\left(\frac{3}{2}-2x\right)y'-\frac{y}{4}=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.229 (sec)

Writing the ode as

$$\left(-x^2+x\right)y''+\left(\frac{3}{2}-2x\right)y'-\frac{y}{4}=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=-x^2+x\\ B&=\frac{3}{2}-2x\\ C&=-\frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-4x^2+4x-3}{16\left(x^2-x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=-4x^2+4x-3\\ t&=16\left(x^2-x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-4x^2+4x-3}{16\left(x^2-x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.509: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2-x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{8x} - \frac{3}{16x^2} + \frac{1}{-8+8x} - \frac{3}{16\left(-1+x\right)^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{-4x^2 + 4x - 3}{16\left(x^2 - x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-4x^2 + 4x - 3}{16\left(x^2 - x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
1	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{4x} + \frac{1}{-4+4x} + (-)(0) \\
&= \frac{1}{4x} + \frac{1}{-4+4x} \\
&= \frac{2x-1}{4x(-1+x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{4x} + \frac{1}{-4+4x}\right)(0) + \left(\left(-\frac{1}{4x^2} - \frac{1}{4(-1+x)^2}\right) + \left(\frac{1}{4x} + \frac{1}{-4+4x}\right)^2 - \left(\frac{-4x^2+4x-3}{16(x^2-x)^2}\right)\right) = 0
$$

$$
0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(\frac{1}{4x} + \frac{1}{-4+4x}\right) dx} \\
&= (x(-1+x))^{1/4}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{\frac{3}{2}-2x}{-x^2+x}\, dx} \\
&= z_1 e^{-\frac{\ln(-1+x)}{4} - \frac{3\ln(x)}{4}} \\
&= z_1 \left(\frac{1}{(-1+x)^{1/4}\, x^{3/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(x(-1+x))^{1/4}}{(-1+x)^{1/4}\, x^{3/4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{\frac{3}{2}-2x}{-x^2+x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{\ln(-1+x)}{2} - \frac{3\ln(x)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\ln\left(-\frac{1}{2} + x + \sqrt{x^2-x}\right)\right)
\end{aligned}
$$

Therefore the solution is

$$\begin{aligned}y &= c_1y_1 + c_2y_2\\ &= c_1\left(\frac{(x(-1+x))^{1/4}}{(-1+x)^{1/4}\,x^{3/4}}\right) + c_2\left(\frac{(x(-1+x))^{1/4}}{(-1+x)^{1/4}\,x^{3/4}}\left(\ln\left(-\frac{1}{2}+x+\sqrt{x^2-x}\right)\right)\right)\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x(1-x)\left(\frac{d^2}{dx^2}y(x)\right) + \left(\frac{3}{2}-2x\right)\left(\frac{d}{dx}y(x)\right) - \frac{y(x)}{4} = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{y(x)}{4x(x-1)} - \frac{(4x-3)\left(\frac{d}{dx}y(x)\right)}{2x(x-1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(4x-3)\left(\frac{d}{dx}y(x)\right)}{2x(x-1)} + \frac{y(x)}{4x(x-1)} = 0$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{4x-3}{2x(x-1)}, P_3(x) = \frac{1}{4x(x-1)}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = \frac{3}{2}$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$
  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4x(x-1)\left(\frac{d^2}{dx^2}y(x)\right) + (8x-6)\left(\frac{d}{dx}y(x)\right) + y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..1$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 1..2$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,x^{k+r-2+m}$
  - Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0r(1+2r)\,x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)\,(2k+3+2r) + a_k(2k+2r+1)^2\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-2r(1+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{0, -\frac{1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $a_k(2k+2r+1)^2 - 4(k+1+r)\left(k+r+\frac{3}{2}\right)a_{k+1} = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_k(2k+2r+1)^2}{2(k+1+r)(2k+3+2r)}$
- Recursion relation for $r = 0$
  $a_{k+1} = \frac{a_k(2k+1)^2}{2(k+1)(2k+3)}$
- Solution for $r = 0$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k(2k+1)^2}{2(k+1)(2k+3)}\right]$
- Recursion relation for $r = -\frac{1}{2}$
  $a_{k+1} = \frac{2a_k k^2}{\left(k+\frac{1}{2}\right)(2k+2)}$
- Solution for $r = -\frac{1}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = \frac{2a_k k^2}{\left(k+\frac{1}{2}\right)(2k+2)}\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{2}}\right), a_{k+1} = \frac{a_k(2k+1)^2}{2(k+1)(2k+3)}, b_{k+1} = \frac{2b_k k^2}{\left(k+\frac{1}{2}\right)(2k+2)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.023 (sec)
Leaf size : 32

```
dsolve(x*(1-x)*diff(diff(y(x),x),x)+(3/2-2*x)*diff(y(x),x)-1/4*y(x) = 0,y(x),singsol=a
```

$$y = \frac{-c_2 \ln(2) + c_2 \ln\left(2x - 1 + 2\sqrt{(x-1)\,x}\right) + c_1}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.283 (sec)
Leaf size : 104

```
DSolve[{x*(1-x)*D[y[x],{x,2}]+(3/2-2*x)*D[y[x],x]-1/4*y[x]==0,{}},y[x],x,IncludeSingularSolu
```

$$y(x) \to \exp\left(\int_1^x \frac{1}{4}\left(\frac{1}{K[1]} + \frac{1}{K[1]-1}\right) dK[1] - \frac{1}{2}\int_1^x \frac{1}{2}\left(\frac{3}{K[2]} + \frac{1}{K[2]-1}\right) dK[2]\right)\left(c_2 \int_1^x \exp\left(-2\int_1^{K[3]} \frac{1-2K[1]}{4K[1]-4K[1]^2} dK[1]\right) dK[3] + c_1\right)$$

2.1.268 Problem 271

Internal problem ID [9440]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 271
Date solved : `Monday, January 27, 2025 at 06:02:51 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x(1-x)\,y''+xy'-y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.229 (sec)

Writing the ode as

$$\left(-2x^2+2x\right)y''+xy'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=-2x^2+2x\\ B&=x\\ C&=-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-3x+8}{16x\left(-1+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=-3x+8\\ t&=16x(-1+x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-3x+8}{16x\left(-1+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.511: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x(-1+x)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{2x} - \frac{1}{2(-1+x)} + \frac{5}{16(-1+x)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x + 8}{16x(-1+x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x + 8}{16x(-1+x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1
1	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - (\alpha_{c_1}^- + \alpha_{c_2}^-) \\
&= \frac{3}{4} - \left(\frac{3}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{x} - \frac{1}{4(-1+x)} + (0) \\
&= \frac{1}{x} - \frac{1}{4(-1+x)} \\
&= \frac{1}{x} - \frac{1}{-4+4x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x} - \frac{1}{4(-1+x)}\right)(0) + \left(\left(-\frac{1}{x^2} + \frac{1}{4(-1+x)^2}\right) + \left(\frac{1}{x} - \frac{1}{4(-1+x)}\right)^2 - \left(\frac{-3x+8}{16x(-1+x)^2}\right)\right) =$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{1}{x} - \frac{1}{4(-1+x)}\right) dx} \\ &= \frac{x}{(-1+x)^{1/4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{-2x^2+2x}\, dx} \\ &= z_1 e^{\frac{\ln(-1+x)}{4}} \\ &= z_1\left((-1+x)^{1/4}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{-2x^2+2x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\frac{\ln(-1+x)}{2}}}{(y_1)^2}\, dx \\ &= y_1\left(-\frac{\sqrt{-1+x}}{x} + \arctan\left(\sqrt{-1+x}\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(x) + c_2\left(x\left(-\frac{\sqrt{-1+x}}{x} + \arctan\left(\sqrt{-1+x}\right)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 25

```
dsolve(2*x*(1-x)*diff(diff(y(x),x),x)+diff(y(x),x)*x-y(x) = 0,y(x),singsol=all)
```

$$y = c_1 x + \arctan\left(\sqrt{x-1}\right) x c_2 - \sqrt{x-1}\, c_2$$

Mathematica DSolve solution

Solving time : 0.422 (sec)
Leaf size : 75

```
DSolve[{2*x*(1-x)*D[y[x],{x,2}]+x*D[y[x],x]-y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \sqrt[4]{2-2x}\exp\left(\int_1^x \left(\frac{1}{K[1]} + \frac{1}{4-4K[1]}\right) dK[1]\right)\left(c_2 \int_1^x \exp\left(-2\int_1^{K[2]}\left(\frac{1}{K[1]} + \frac{1}{4-4K[1]}\right) dK[1]\right) dK[2] + c_1\right)$$

2.1.269 Problem 272

Internal problem ID [9441]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 272
Date solved : `Monday, January 27, 2025 at 06:02:52 PM`
CAS classification : `[_Jacobi]`

Solve

$$2x(1-x)\,y''+(1-11x)\,y'-10y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.212 (sec)

Writing the ode as

$$\left(-2x^2+2x\right)y''+(1-11x)\,y'-10y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=-2x^2+2x\\ B&=1-11x\\ C&=-10 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-3x^2+66x-3}{16\left(x^2-x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=-3x^2+66x-3\\ t&=16\left(x^2-x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-3x^2+66x-3}{16\left(x^2-x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.512: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4\left(-1+x\right)^{2}} + \frac{15}{4x} - \frac{3}{16x^{2}} - \frac{15}{4\left(-1+x\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 + 66x - 3}{16\left(x^2 - x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x^2 + 66x - 3}{16\left(x^2 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
1	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= \frac{1}{4} - \left(-\frac{3}{4}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{3}{4x} - \frac{3}{2(-1+x)} + (-)(0) \\
&= \frac{3}{4x} - \frac{3}{2(-1+x)} \\
&= -\frac{3(x+1)}{4x(-1+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=x+a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{3}{4x}-\frac{3}{2\left(-1+x\right)}\right)(1)+\left(\left(-\frac{3}{4x^2}+\frac{3}{2\left(-1+x\right)^2}\right)+\left(\frac{3}{4x}-\frac{3}{2\left(-1+x\right)}\right)^2-\left(\frac{-3x^2+66x-3}{16\left(x^2-x\right)^2}\right)\right.$$
$$\frac{-3+3a_0}{2x\left(-1+x\right)}=$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0=1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x)=x+1$$

Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x)&=pe^{\int\omega\,dx}\\
&=(x+1)\,e^{\int\left(\frac{3}{4x}-\frac{3}{2(-1+x)}\right)dx}\\
&=(x+1)\,e^{-\frac{3\ln(-1+x)}{2}+\frac{3\ln(x)}{4}}\\
&=\frac{(x+1)\,x^{3/4}}{\left(-1+x\right)^{3/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1&=z_1e^{\int-\frac{1}{2}\frac{B}{A}\,dx}\\
&=z_1e^{-\int\frac{1}{2}\frac{1-11x}{-2x^2+2x}\,dx}\\
&=z_1e^{-\frac{5\ln(-1+x)}{2}-\frac{\ln(x)}{4}}\\
&=z_1\left(\frac{1}{\left(-1+x\right)^{5/2}x^{1/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\frac{\sqrt{x}\,(x+1)}{\left(-1+x\right)^4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int\frac{e^{\int-\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2&=y_1\int\frac{e^{\int-\frac{1-11x}{-2x^2+2x}\,dx}}{\left(y_1\right)^2}\,dx\\
&=y_1\int\frac{e^{-5\ln(-1+x)-\frac{\ln(x)}{2}}}{\left(y_1\right)^2}\,dx\\
&=y_1\left(\frac{2(x^2+6x+1)\left(-1+x\right)^5e^{-5\ln(-1+x)-\frac{\ln(x)}{2}}}{x+1}\right)
\end{aligned}$$

Therefore the solution is

$$y = c_1y_1 + c_2y_2$$
$$= c_1\left(\frac{\sqrt{x}\,(x+1)}{(-1+x)^4}\right) + c_2\left(\frac{\sqrt{x}\,(x+1)}{(-1+x)^4}\left(\frac{2(x^2+6x+1)\,(-1+x)^5\,\mathrm{e}^{-5\ln(-1+x)-\frac{\ln(x)}{2}}}{x+1}\right)\right)$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$2x(1-x)\left(\frac{d^2}{dx^2}y(x)\right) + (1-11x)\left(\frac{d}{dx}y(x)\right) - 10y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{5y(x)}{x(x-1)} - \frac{(11x-1)\left(\frac{d}{dx}y(x)\right)}{2x(x-1)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x) + \frac{(11x-1)\left(\frac{d}{dx}y(x)\right)}{2x(x-1)} + \frac{5y(x)}{x(x-1)} = 0$
- □ Check to see if x_0 is a regular singular point
  - ◦ Define functions
    $\left[P_2(x) = \frac{11x-1}{2x(x-1)}, P_3(x) = \frac{5}{x(x-1)}\right]$
  - ◦ $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{1}{2}$
  - ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$
  - ◦ $x = 0$is a regular singular point
    Check to see if x_0 is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $2x(x-1)\left(\frac{d^2}{dx^2}y(x)\right) + (11x-1)\left(\frac{d}{dx}y(x)\right) + 10y(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- □ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..1$
    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - ◦ Shift index using $k->k+1-m$
    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - ◦ Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 1..2$
    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,x^{k+r-2+m}$
  - ◦ Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0 r(-1+2r)\,x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(2k+1+2r) + a_k(2k+2r+5)\,(k+r+2)\right) x^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-2(k+1+r)\left(k+r+\frac{1}{2}\right)a_{k+1} + 2(k+r+2)\left(k+r+\frac{5}{2}\right)a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{(k+r+2)(2k+2r+5)a_k}{(k+1+r)(2k+1+2r)}$$

- Recursion relation for $r=0$

$$a_{k+1} = \frac{(k+2)(2k+5)a_k}{(k+1)(2k+1)}$$

- Solution for $r=0$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{(k+2)(2k+5)a_k}{(k+1)(2k+1)}\right]$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+1} = \frac{\left(k+\frac{5}{2}\right)(2k+6)a_k}{\left(k+\frac{3}{2}\right)(2k+2)}$$

- Solution for $r=\frac{1}{2}$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{\left(k+\frac{5}{2}\right)(2k+6)a_k}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = \frac{(k+2)(2k+5)a_k}{(k+1)(2k+1)}, b_{k+1} = \frac{\left(k+\frac{5}{2}\right)(2k+6)b_k}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.024 (sec)
Leaf size : 29

```
dsolve(2*x*(1-x)*diff(diff(y(x),x),x)+(1-11*x)*diff(y(x),x)-10*y(x) = 0,y(x),singsol=all
```

$$y = \frac{c_1(x^2+6x+1) + c_2\sqrt{x}\,(x+1)}{(x-1)^4}$$

Mathematica DSolve solution

Solving time : `0.548` (sec)
Leaf size : 119

```
DSolve[{2*x*(1-x)*D[y[x],{x,2}]+(1-11*x)*D[y[x],x]-10*y[x]==0,{}},y[x],x,IncludeSingularSolu
```

$$y(x) \to (x+1)\exp\left(\int_1^x \frac{3K[1]+3}{4K[1]-4K[1]^2}dK[1] - \frac{1}{2}\int_1^x \left(\frac{1}{2K[2]} + \frac{5}{K[2]-1}\right)dK[2]\right)\left(c_2\int_1^x \frac{\exp\left(-2\int_1^{K[3]}\frac{3K[1]+3}{4K[1]-4K[1]^2}dK[1]\right)}{(K[3]+1)^2}dK[3] + c_1\right)$$

2.1.270 Problem 273

Internal problem ID [9442]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 273
Date solved : Monday, January 27, 2025 at 06:02:52 PM
CAS classification : [_Jacobi]

Solve

$$x(1-x)\,y''+\frac{(1-2x)\,y'}{3}+\frac{20y}{9}=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.234 (sec)

Writing the ode as

$$\left(-x^2+x\right)y''+\left(-\frac{2x}{3}+\frac{1}{3}\right)y'+\frac{20y}{9}=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-x^2+x\\ B&=-\frac{2x}{3}+\frac{1}{3}\\ C&=\frac{20}{9}\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{72x^2-72x-5}{36\left(x^2-x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=72x^2-72x-5\\ t&=36\left(x^2-x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{72x^2 - 72x - 5}{36\left(x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.514: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{41}{18x} - \frac{5}{36\left(-1+x\right)^2} + \frac{41}{18\left(-1+x\right)} - \frac{5}{36x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6} \end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{72x^2 - 72x - 5}{36\left(x^2 - x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{72x^2 - 72x - 5}{36\left(x^2 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{6}$	$\frac{1}{6}$
1	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{6x} + \frac{5}{6\left(-1+x\right)} + (0) \\
&= \frac{1}{6x} + \frac{5}{6\left(-1+x\right)} \\
&= \frac{-1+6x}{6x\left(-1+x\right)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{6x} + \frac{5}{6\left(-1+x\right)}\right)(1) + \left(\left(-\frac{1}{6x^2} - \frac{5}{6\left(-1+x\right)^2}\right) + \left(\frac{1}{6x} + \frac{5}{6\left(-1+x\right)}\right)^2 - \left(\frac{72x^2 - 72x - 5}{36\left(x^2-x\right)^2}\right.\right.
$$

$$
\frac{-1-6a_0}{3x\left(-1+x\right)}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = -\frac{1}{6}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = -\frac{1}{6} + x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(-\frac{1}{6} + x\right) \mathrm{e}^{\int \left(\frac{1}{6x} + \frac{5}{6(-1+x)}\right) dx} \\
&= \left(-\frac{1}{6} + x\right) \mathrm{e}^{\frac{\ln(x)}{6} + \frac{5\ln(-1+x)}{6}} \\
&= \left(-\frac{1}{6} + x\right) x^{1/6}(-1+x)^{5/6}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-\frac{2x}{3}+\frac{1}{3}}{-x^2+x}\, dx} \\
&= z_1 e^{-\frac{\ln(x(-1+x))}{6}} \\
&= z_1 \left(\frac{1}{\left(x\left(-1+x\right)\right)^{1/6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\left(-1+6x\right) x^{1/6}(-1+x)^{5/6}}{6\left(x\left(-1+x\right)\right)^{1/6}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-\frac{2x}{3}+\frac{1}{3}}{-x^2+x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{\ln(x(-1+x))}{3}}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{54x^{2/3}(-5+6x)}{5\,(-1+6x)\,(-1+x)^{2/3}}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{(-1+6x)\, x^{1/6}(-1+x)^{5/6}}{6\,(x\,(-1+x))^{1/6}}\right) + c_2 \left(\frac{(-1+6x)\, x^{1/6}(-1+x)^{5/6}}{6\,(x\,(-1+x))^{1/6}}\left(-\frac{54x^{2/3}(-5+6x)}{5\,(-1+6x)\,(-1+x)^{2/3}}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x(1-x)\left(\frac{d^2}{dx^2}y(x)\right) + \frac{(-2x+1)\left(\frac{d}{dx}y(x)\right)}{3} + \frac{20y(x)}{9} = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{20y(x)}{9x(x-1)} - \frac{(2x-1)\left(\frac{d}{dx}y(x)\right)}{3x(x-1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(2x-1)\left(\frac{d}{dx}y(x)\right)}{3x(x-1)} - \frac{20y(x)}{9x(x-1)} = 0$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{2x-1}{3x(x-1)}, P_3(x) = -\frac{20}{9x(x-1)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{1}{3}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $9x(x-1)\left(\frac{d^2}{dx^2}y(x)\right) + (6x-3)\left(\frac{d}{dx}y(x)\right) - 20y(x) = 0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..1$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=1..2$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$
  - Shift index using $k->k+2-m$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $-3a_0r(-2+3r)\,x^{-1+r}+\left(\sum_{k=0}^{\infty}(-3a_{k+1}(k+1+r)\,(3k+1+3r)+a_k(3k+3r+4)\,(3k+3r-5\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-3r(-2+3r)=0$
- Values of r that satisfy the indicial equation

  $r\in\left\{0,\frac{2}{3}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $-9\left(k+r+\frac{1}{3}\right)(k+1+r)\,a_{k+1}+9\left(k+\frac{4}{3}+r\right)\left(k+r-\frac{5}{3}\right)a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{(3k+3r+4)(3k+3r-5)a_k}{3(3k+1+3r)(k+1+r)}$
- Recursion relation for $r=0$

  $a_{k+1}=\frac{(3k+4)(3k-5)a_k}{3(3k+1)(k+1)}$
- Solution for $r=0$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^k, a_{k+1}=\frac{(3k+4)(3k-5)a_k}{3(3k+1)(k+1)}\right]$
- Recursion relation for $r=\frac{2}{3}$; series terminates at $k=1$

  $a_{k+1}=\frac{(3k+6)(3k-3)a_k}{3(3k+3)\left(k+\frac{5}{3}\right)}$
- Apply recursion relation for $k=0$

  $a_1=-\frac{6a_0}{5}$
- Terminating series solution of the ODE for $r=\frac{2}{3}$. Use reduction of order to find the second li

  $y(x)=a_0\cdot\left(-\frac{6x}{5}+1\right)$
- Combine solutions and rename parameters

  $\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^k\right)+b_0\cdot\left(-\frac{6x}{5}+1\right), a_{k+1}=\frac{(3k+4)(3k-5)a_k}{3(3k+1)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 27

```
dsolve(x*(1-x)*diff(diff(y(x),x),x)+1/3*(1-2*x)*diff(y(x),x)+20/9*y(x) = 0,y(x),singsol=
```

$$y = c_1(6x-5)\,x^{2/3}+c_2(6x-1)\,(x-1)^{2/3}$$

Mathematica DSolve solution

Solving time : 0.499 (sec)
Leaf size : 93

```
DSolve[{x*(1-x)*D[y[x],{x,2}]+1/3*(1-2*x)*D[y[x],x]+20/9*y[x]==0,{}},y[x],x,IncludeSingularSolu
```

$$y(x) \to \frac{\left(3c_2(x-1)\,\mathrm{Gamma}\left(\frac{4}{3}\right)Q_1^{\frac{2}{3}}(2x-1)+c_1(5-6x)(1-x)^{2/3}\sqrt[3]{x}\right)\exp\left(\int_1^x \frac{1-2K[1]}{3K[1]-3K[1]^2}dK[1]\right)}{3(x-1)\,\mathrm{Gamma}\left(\frac{4}{3}\right)}$$

2.1.271 Problem 274

Internal problem ID [9443]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 274
Date solved : `Monday, January 27, 2025 at 06:02:53 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4y''+\frac{3(-x^2+2)\,y}{(-x^2+1)^2}=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.181 (sec)

Writing the ode as

$$4y''+\frac{(-3x^2+6)\,y}{(x^2-1)^2}=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=4\\ B&=0\\ C&=\frac{-3x^2+6}{(x^2-1)^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{3x^2-6}{4(x^2-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=3x^2-6\\ t&=4\left(x^2-1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^2 - 6}{4\left(x^2 - 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.516: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{16\,(x-1)} - \frac{9}{16\,(x+1)} - \frac{3}{16\,(x+1)^2} - \frac{3}{16\,(x-1)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{3x^2 - 6}{4(x^2-1)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3x^2 - 6}{4(x^2-1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{4}$	$\frac{1}{4}$
-1	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\
&= \frac{3}{2} - \left(\frac{3}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{3}{4(x-1)} + \frac{3}{4(x+1)} + (0) \\
&= \frac{3}{4(x-1)} + \frac{3}{4(x+1)} \\
&= \frac{3x}{2x^2-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{4(x-1)} + \frac{3}{4(x+1)}\right)(0) + \left(\left(-\frac{3}{4(x-1)^2} - \frac{3}{4(x+1)^2}\right) + \left(\frac{3}{4(x-1)} + \frac{3}{4(x+1)}\right)^2 - \left(\frac{3x^2}{4(x^2}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(\frac{3}{4(x-1)} + \frac{3}{4(x+1)}\right) dx} \\
&= (x^2-1)^{3/4}
\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= (x^2-1)^{3/4}
\end{aligned}$$

Which simplifies to

$$y_1 = (x^2-1)^{3/4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= (x^2-1)^{3/4} \int \frac{1}{(x^2-1)^{3/2}}\, dx \\
&= (x^2-1)^{3/4} \left(-\frac{(x-1)(x+1)x}{(x^2-1)^{3/2}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\left(x^2-1\right)^{3/4}\right) + c_2\left(\left(x^2-1\right)^{3/4}\left(-\frac{(x-1)\,(x+1)\,x}{\left(x^2-1\right)^{3/2}}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4\frac{d^2}{dx^2}y(x) + \frac{3(-x^2+2)y(x)}{(-x^2+1)^2} = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{3(x^2-2)y(x)}{4(x^2-1)^2}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) - \frac{3(x^2-2)y(x)}{4(x^2-1)^2} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = 0, P_3(x) = -\frac{3(x^2-2)}{4(x^2-1)^2}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1} = 0$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1} = \frac{3}{16}$

  ◦ $x=-1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -1$

- Multiply by denominators

  $4(x^2-1)^2\left(\frac{d^2}{dx^2}y(x)\right) + (-3x^2+6)\,y(x) = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(4u^4-16u^3+16u^2)\left(\frac{d^2}{du^2}y(u)\right) + (-3u^2+6u+3)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot y(u)$ to series expansion for $m=0..2$

    $u^m\cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

  ◦ Shift index using $k->k-m$

    $u^m\cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  ◦ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=2..4$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+4r)(-3+4r)\, u^r + (a_1(3+4r)(1+4r) - 2a_0(8r^2-8r-3))\, u^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(4k+4r-1)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
$(-1+4r)(-3+4r) = 0$
- Values of r that satisfy the indicial equation
$r \in \left\{\frac{1}{4}, \frac{3}{4}\right\}$
- Each term must be 0
$a_1(3+4r)(1+4r) - 2a_0(8r^2-8r-3) = 0$
- Solve for the dependent coefficient(s)
$a_1 = \frac{2a_0(8r^2-8r-3)}{16r^2+16r+3}$
- Each term in the series must be 0, giving the recursion relation
$4(4a_k + a_{k-2} - 4a_{k-1})\, k^2 + 4(2(4a_k + a_{k-2} - 4a_{k-1})\, r - 4a_k - 5a_{k-2} + 12a_{k-1})\, k + 4(4a_k + a_{k-2} -$
- Shift index using $k->k+2$
$4(4a_{k+2} + a_k - 4a_{k+1})(k+2)^2 + 4(2(4a_{k+2} + a_k - 4a_{k+1})\, r - 4a_{k+2} - 5a_k + 12a_{k+1})(k+2) + 4(4$
- Recursion relation that defines series solution to ODE
$a_{k+2} = -\frac{4k^2a_k - 16k^2a_{k+1} + 8kra_k - 32kra_{k+1} + 4r^2a_k - 16r^2a_{k+1} - 4ka_k - 16ka_{k+1} - 4ra_k - 16ra_{k+1} - 3a_k + 6a_{k+1}}{16k^2+32kr+16r^2+48k+48r+35}$
- Recursion relation for $r = \frac{1}{4}$
$a_{k+2} = -\frac{4k^2a_k - 16k^2a_{k+1} - 2ka_k - 24ka_{k+1} - \frac{15}{4}a_k + a_{k+1}}{16k^2+56k+48}$
- Solution for $r = \frac{1}{4}$
$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{4}}, a_{k+2} = -\frac{4k^2a_k - 16k^2a_{k+1} - 2ka_k - 24ka_{k+1} - \frac{15}{4}a_k + a_{k+1}}{16k^2+56k+48}, a_1 = -\frac{9a_0}{8}\right]$
- Revert the change of variables $u = x+1$
$\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^{k+\frac{1}{4}}, a_{k+2} = -\frac{4k^2a_k - 16k^2a_{k+1} - 2ka_k - 24ka_{k+1} - \frac{15}{4}a_k + a_{k+1}}{16k^2+56k+48}, a_1 = -\frac{9a_0}{8}\right]$
- Recursion relation for $r = \frac{3}{4}$
$a_{k+2} = -\frac{4k^2a_k - 16k^2a_{k+1} + 2ka_k - 40ka_{k+1} - \frac{15}{4}a_k - 15a_{k+1}}{16k^2+72k+80}$
- Solution for $r = \frac{3}{4}$
$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{3}{4}}, a_{k+2} = -\frac{4k^2a_k - 16k^2a_{k+1} + 2ka_k - 40ka_{k+1} - \frac{15}{4}a_k - 15a_{k+1}}{16k^2+72k+80}, a_1 = -\frac{3a_0}{8}\right]$
- Revert the change of variables $u = x+1$
$\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^{k+\frac{3}{4}}, a_{k+2} = -\frac{4k^2a_k - 16k^2a_{k+1} + 2ka_k - 40ka_{k+1} - \frac{15}{4}a_k - 15a_{k+1}}{16k^2+72k+80}, a_1 = -\frac{3a_0}{8}\right]$
- Combine solutions and rename parameters
$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k(x+1)^{k+\frac{1}{4}}\right) + \left(\sum_{k=0}^{\infty} b_k(x+1)^{k+\frac{3}{4}}\right), a_{k+2} = -\frac{4k^2a_k - 16k^2a_{k+1} - 2ka_k - 24ka_{k+1} - \frac{15}{4}a_k + a_{k+1}}{16k^2+56k+48},\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 24

```
dsolve(4*diff(diff(y(x),x),x)+3*(-x^2+2)/(-x^2+1)^2*y(x) = 0,y(x),singsol=all)
```

$$y = c_1 \left(x^2-1\right)^{3/4}+c_2 \left(x^2-1\right)^{1/4} x$$

Mathematica DSolve solution

Solving time : 0.041 (sec)
Leaf size : 51

```
DSolve[{4*D[y[x],{x,2}]+3*(2-x^2)/(1-x^2)^2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tru
```

$$y(x) \to \sqrt{x^2-1}\left(c_2 Q_{\frac{1}{2}}^{\frac{1}{2}}(x)+\frac{\sqrt{\frac{2}{\pi}} c_1 x}{\sqrt[4]{1-x^2}}\right)$$

2.1.272 Problem 275

Internal problem ID [9444]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 275
Date solved : `Monday, January 27, 2025 at 06:02:54 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$u'' - \frac{2u'}{x} - a^2 u = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.274 (sec)

Writing the ode as

$$u'' - \frac{2u'}{x} - a^2 u = 0 \tag{1}$$
$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -\frac{2}{x} \\ C &= -a^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = u e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2x^2 + 2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= a^2x^2 + 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{a^2x^2+2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.518: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2} + a^2$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx a + \frac{1}{a\,x^2} - \frac{1}{2a^3x^4} + \frac{1}{2a^5x^6} - \frac{5}{8a^7x^8} + \frac{7}{8a^9x^{10}} - \frac{21}{16a^{11}x^{12}} + \frac{33}{16a^{13}x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = a$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= a \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = a^2$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{a^2x^2 + 2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (a^2) + \left(\frac{2}{x^2}\right) \\ &= \frac{2}{x^2} + a^2 \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= a \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{a} - 0\right) &= 0 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{a} - 0\right) &= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{a^2x^2 + 2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	a	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + (-)(a) \\ &= -\frac{1}{x} - a \\ &= \frac{-ax-1}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{x} - a\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - a\right)^2 - \left(\frac{a^2x^2+2}{x^2}\right) \right) &= 0 \\ \frac{2aa_0 - 2}{x} &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{ a_0 = \frac{1}{a} \right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{1}{a}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x + \frac{1}{a}\right) e^{\int (-\frac{1}{x} - a) dx} \\ &= \left(x + \frac{1}{a}\right) e^{-ax - \ln(x)} \\ &= \frac{(ax+1)\, e^{-ax}}{ax} \end{aligned}$$

The first solution to the original ode in u is found from

$$\begin{aligned} u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-\frac{2}{x}}{1} \, dx} \\ &= z_1 e^{\ln(x)} \\ &= z_1(x) \end{aligned}$$

Which simplifies to

$$u_1 = \frac{(ax+1)\, e^{-ax}}{a}$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{u_1^2} \, dx$$

Substituting gives

$$\begin{aligned} u_2 &= u_1 \int \frac{e^{\int -\frac{-\frac{2}{x}}{1} \, dx}}{(u_1)^2} \, dx \\ &= u_1 \int \frac{e^{2\ln(x)}}{(u_1)^2} \, dx \\ &= u_1 \left(\frac{(ax-1)\, e^{2ax}}{2a(ax+1)}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} u &= c_1 u_1 + c_2 u_2 \\ &= c_1 \left(\frac{(ax+1)\, e^{-ax}}{a}\right) + c_2 \left(\frac{(ax+1)\, e^{-ax}}{a} \left(\frac{(ax-1)\, e^{2ax}}{2a(ax+1)}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}u(x) - \frac{2\left(\frac{d}{dx}u(x)\right)}{x} - a^2u(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}u(x)$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{2}{x}, P_3(x) = -a^2\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$
  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $-a^2u(x)\,x + \left(\frac{d^2}{dx^2}u(x)\right)x - 2\frac{d}{dx}u(x) = 0$

- Assume series solution for $u(x)$

  $u(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x \cdot u(x)$ to series expansion

    $x \cdot u(x) = \sum_{k=0}^{\infty} a_k x^{k+r+1}$
  - Shift index using $k- >k-1$

    $x \cdot u(x) = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$
  - Convert $\frac{d}{dx}u(x)$ to series expansion

    $\frac{d}{dx}u(x) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1}$
  - Shift index using $k- >k+1$

    $\frac{d}{dx}u(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,x^{k+r}$
  - Convert $x \cdot \left(\frac{d^2}{dx^2}u(x)\right)$ to series expansion

    $x \cdot \left(\frac{d^2}{dx^2}u(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$
  - Shift index using $k- >k+1$

    $x \cdot \left(\frac{d^2}{dx^2}u(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-3+r)\,x^{-1+r} + a_1(1+r)\,(-2+r)\,x^r + \left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+r+1)\,(k-2+r) - a^2a_{k-1}\right)x^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{0, 3\}$

- Each term must be 0

  $a_1(1+r)(-2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)(k-2+r) - a^2 a_{k-1} = 0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)(k+r-1) - a^2 a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a^2 a_k}{(k+2+r)(k+r-1)}$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{a^2 a_k}{(k+2)(k-1)}$

- Solution for $r = 0$

  $\left[u(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a^2 a_k}{(k+2)(k-1)}, -2a_1 = 0\right]$

- Recursion relation for $r = 3$

  $a_{k+2} = \frac{a^2 a_k}{(k+5)(k+2)}$

- Solution for $r = 3$

  $\left[u(x) = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = \frac{a^2 a_k}{(k+5)(k+2)}, 4a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[u(x) = \left(\sum_{k=0}^{\infty} b_k x^k\right) + \left(\sum_{k=0}^{\infty} c_k x^{k+3}\right), b_{k+2} = \frac{a^2 b_k}{(k+2)(k-1)}, -2b_1 = 0, c_{k+2} = \frac{a^2 c_k}{(5+k)(k+2)}, 4c_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 28

```
dsolve(diff(diff(u(x),x),x)-2/x*diff(u(x),x)-a^2*u(x) = 0,u(x),singsol=all)
```

$$u = c_1 e^{ax}(ax-1) + c_2 e^{-ax}(ax+1)$$

Mathematica DSolve solution

Solving time : `0.104` (sec)
Leaf size : 68

```
DSolve[{D[u[x],{x,2}]-2/x*D[u[x],x]-a^2*u[x]==0,{}},u[x],x,IncludeSingularSolutions->True]
```

$$u(x) \to \frac{\sqrt{\frac{2}{\pi}}\sqrt{x}((iac_2x + c_1)\sinh(ax) - (ac_1x + ic_2)\cosh(ax))}{a\sqrt{-iax}}$$

2.1.273 Problem 276

Internal problem ID [9445]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 276
Date solved : Monday, January 27, 2025 at 06:02:54 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$u'' + \frac{2u'}{x} - a^2 u = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.103 (sec)

Writing the ode as

$$u'' + \frac{2u'}{x} - a^2 u = 0 \tag{1}$$

$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{2}{x} \\ C &= -a^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = u e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= a^2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(a^2\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.520: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = a^2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{\sqrt{a^2}\, x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in u is found from

$$\begin{aligned} u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{\frac{2}{x}}{1}\, dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1\left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$u_1 = \frac{e^{\operatorname{csgn}(a) a x}}{x}$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{u_1^2}\, dx$$

Substituting gives

$$\begin{aligned} u_2 &= u_1 \int \frac{e^{\int -\frac{\frac{2}{x}}{1}\, dx}}{(u_1)^2}\, dx \\ &= u_1 \int \frac{e^{-2\ln(x)}}{(u_1)^2}\, dx \\ &= u_1 \left(-\frac{\mathrm{e}^{-2\,\mathrm{csgn}(a)ax}}{2\,\mathrm{csgn}\,(a)\,a}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} u &= c_1 u_1 + c_2 u_2 \\ &= c_1 \left(\frac{\mathrm{e}^{\mathrm{csgn}(a)ax}}{x}\right) + c_2 \left(\frac{\mathrm{e}^{\mathrm{csgn}(a)ax}}{x}\left(-\frac{\mathrm{e}^{-2\,\mathrm{csgn}(a)ax}}{2\,\mathrm{csgn}\,(a)\,a}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}u(x) + \frac{2\left(\frac{d}{dx}u(x)\right)}{x} - a^2 u(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}u(x)$

□ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = -a^2\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$

  ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  ○ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $-a^2 u(x)\, x + \left(\frac{d^2}{dx^2}u(x)\right) x + 2\frac{d}{dx}u(x) = 0$

- Assume series solution for $u(x)$

  $u(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ○ Convert $x \cdot u(x)$ to series expansion

    $x \cdot u(x) = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

  ○ Shift index using $k- >k-1$

$x \cdot u(x) = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

- Convert $\frac{d}{dx}u(x)$ to series expansion

  $\frac{d}{dx}u(x) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$
- Shift index using $k->k+1$

  $\frac{d}{dx}u(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$
- Convert $x \cdot \left(\frac{d^2}{dx^2}u(x)\right)$ to series expansion

  $x \cdot \left(\frac{d^2}{dx^2}u(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$
- Shift index using $k->k+1$

  $x \cdot \left(\frac{d^2}{dx^2}u(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) - a^2 a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$
- Each term must be 0

  $a_1(1+r)\,(2+r) = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)\,(k+2+r) - a^2 a_{k-1} = 0$
- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)\,(k+3+r) - a^2 a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a^2 a_k}{(k+2+r)(k+3+r)}$
- Recursion relation for $r = -1$

  $a_{k+2} = \frac{a^2 a_k}{(k+1)(k+2)}$
- Solution for $r = -1$

  $\left[u(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = \frac{a^2 a_k}{(k+1)(k+2)}, 0 = 0\right]$
- Recursion relation for $r = 0$

  $a_{k+2} = \frac{a^2 a_k}{(k+2)(k+3)}$
- Solution for $r = 0$

  $\left[u(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a^2 a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[u(x) = \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} c_k x^k\right), b_{k+2} = \frac{a^2 b_k}{(k+1)(k+2)}, 0 = 0, c_{k+2} = \frac{a^2 c_k}{(k+2)(k+3)}, 2c_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 21

```
dsolve(diff(diff(u(x),x),x)+2/x*diff(u(x),x)-a^2*u(x) = 0,u(x),singsol=all)
```

$$u = \frac{c_1 \sinh(ax) + c_2 \cosh(ax)}{x}$$

Mathematica DSolve solution

Solving time : 0.03 (sec)
Leaf size : 35

```
DSolve[{D[u[x],{x,2}]+2/x*D[u[x],x]-a^2*u[x]==0,{}},u[x],x,IncludeSingularSolutions->True]
```

$$u(x) \to \frac{2ac_1 e^{-ax} + c_2 e^{ax}}{2ax}$$

2.1.274 Problem 277

Internal problem ID [9446]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 277
Date solved : `Monday, January 27, 2025 at 06:02:55 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$u'' + \frac{2u'}{x} + a^2 u = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.118 (sec)

Writing the ode as

$$u'' + \frac{2u'}{x} + a^2 u = 0 \tag{1}$$
$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{2}{x} \\ C &= a^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = u e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-a^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -a^2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-a^2\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.522: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -a^2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{\sqrt{-a^2}\, x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in u is found from

$$\begin{aligned} u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{\frac{2}{x}}{1}\, dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1 \left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$u_1 = \frac{\mathrm{e}^{\sqrt{-a^2}\, x}}{x}$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{u_1^2}\, dx$$

Substituting gives

$$\begin{aligned} u_2 &= u_1 \int \frac{e^{\int -\frac{\frac{2}{x}}{1}\, dx}}{(u_1)^2}\, dx \\ &= u_1 \int \frac{e^{-2\ln(x)}}{(u_1)^2}\, dx \\ &= u_1 \left(\frac{\sqrt{-a^2}\, \mathrm{e}^{-2\sqrt{-a^2}\, x}}{2a^2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} u &= c_1 u_1 + c_2 u_2 \\ &= c_1 \left(\frac{\mathrm{e}^{\sqrt{-a^2}\, x}}{x}\right) + c_2 \left(\frac{\mathrm{e}^{\sqrt{-a^2}\, x}}{x} \left(\frac{\sqrt{-a^2}\, \mathrm{e}^{-2\sqrt{-a^2}\, x}}{2a^2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}u(x) + \frac{2\left(\frac{d}{dx}u(x)\right)}{x} + a^2 u(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}u(x)$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = a^2\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$
  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $a^2 u(x)\, x + \left(\frac{d^2}{dx^2}u(x)\right) x + 2\frac{d}{dx}u(x) = 0$

- Assume series solution for $u(x)$

  $u(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x \cdot u(x)$ to series expansion

    $x \cdot u(x) = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

- Shift index using $k->k-1$

  $x \cdot u(x)=\sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

- Convert $\frac{d}{dx}u(x)$ to series expansion

  $\frac{d}{dx}u(x)=\sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $\frac{d}{dx}u(x)=\sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$

- Convert $x \cdot \left(\frac{d^2}{dx^2}u(x)\right)$ to series expansion

  $x \cdot \left(\frac{d^2}{dx^2}u(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $x \cdot \left(\frac{d^2}{dx^2}u(x)\right)=\sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(1+r)\, x^{-1+r}+a_1(1+r)\,(2+r)\, x^r+\left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+r+1)\,(k+2+r)+a^2 a_{k-1}\right) x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{-1,0\}$

- Each term must be 0

  $a_1(1+r)\,(2+r)=0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)\,(k+2+r)+a^2 a_{k-1}=0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)\,(k+3+r)+a^2 a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a^2 a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r=-1$

  $a_{k+2}=-\frac{a^2 a_k}{(k+1)(k+2)}$

- Solution for $r=-1$

  $\left[u(x)=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2}=-\frac{a^2 a_k}{(k+1)(k+2)}, 0=0\right]$

- Recursion relation for $r=0$

  $a_{k+2}=-\frac{a^2 a_k}{(k+2)(k+3)}$

- Solution for $r=0$

  $\left[u(x)=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=-\frac{a^2 a_k}{(k+2)(k+3)}, 2a_1=0\right]$

- Combine solutions and rename parameters

  $\left[u(x)=\left(\sum_{k=0}^{\infty} b_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} c_k x^k\right), b_{k+2}=-\frac{a^2 b_k}{(k+1)(k+2)}, 0=0, c_{k+2}=-\frac{a^2 c_k}{(k+2)(k+3)}, 2c_1=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 21

```
dsolve(diff(diff(u(x),x),x)+2/x*diff(u(x),x)+a^2*u(x) = 0,u(x),singsol=all)
```

$$u = \frac{c_1 \sin(ax) + c_2 \cos(ax)}{x}$$

Mathematica DSolve solution

Solving time : 0.032 (sec)
Leaf size : 42

```
DSolve[{D[u[x],{x,2}]+2/x*D[u[x],x]+a^2*u[x]==0,{}},u[x],x,IncludeSingularSolutions->True]
```

$$u(x) \to \frac{e^{-iax}\left(2c_1 - \frac{ic_2e^{2iax}}{a}\right)}{2x}$$

2.1.275 Problem 278

Internal problem ID [9447]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 278
Date solved : `Monday, January 27, 2025 at 06:02:55 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$u'' + \frac{4u'}{x} - a^2 u = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.269 (sec)

Writing the ode as

$$u'' + \frac{4u'}{x} - a^2 u = 0 \tag{1}$$

$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{4}{x} \\ C &= -a^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = u e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2 x^2 + 2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= a^2 x^2 + 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{a^2x^2+2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.524: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2-2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1,2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2} + a^2$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx a + \frac{1}{a\,x^2} - \frac{1}{2a^3x^4} + \frac{1}{2a^5x^6} - \frac{5}{8a^7x^8} + \frac{7}{8a^9x^{10}} - \frac{21}{16a^{11}x^{12}} + \frac{33}{16a^{13}x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = a$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= a
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = a^2$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{a^2x^2 + 2}{x^2} \\
&= Q + \frac{R}{x^2} \\
&= (a^2) + \left(\frac{2}{x^2}\right) \\
&= \frac{2}{x^2} + a^2
\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned}
b &= (0) - (0) \\
&= 0
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= a \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{a} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{a} - 0\right) &&= 0
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{a^2x^2 + 2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	a	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + (-)(a) \\ &= -\frac{1}{x} - a \\ &= \frac{-ax - 1}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x} - a\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - a\right)^2 - \left(\frac{a^2x^2 + 2}{x^2}\right)\right) = 0$$

$$\frac{2aa_0 - 2}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{1}{a}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{1}{a}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{1}{a}\right) e^{\int \left(-\frac{1}{x} - a\right) dx} \\
&= \left(x + \frac{1}{a}\right) e^{-ax - \ln(x)} \\
&= \frac{(ax + 1) e^{-ax}}{ax}
\end{aligned}$$

The first solution to the original ode in u is found from

$$\begin{aligned}
u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{\frac{4}{x}}{1} \, dx} \\
&= z_1 e^{-2\ln(x)} \\
&= z_1 \left(\frac{1}{x^2}\right)
\end{aligned}$$

Which simplifies to

$$u_1 = \frac{(ax + 1) e^{-ax}}{x^3 a}$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{u_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
u_2 &= u_1 \int \frac{e^{\int -\frac{\frac{4}{x}}{1} \, dx}}{(u_1)^2} \, dx \\
&= u_1 \int \frac{e^{-4\ln(x)}}{(u_1)^2} \, dx \\
&= u_1 \left(\frac{(ax - 1) e^{2ax}}{2a(ax + 1)}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1 \left(\frac{(ax + 1) e^{-ax}}{x^3 a}\right) + c_2 \left(\frac{(ax + 1) e^{-ax}}{x^3 a} \left(\frac{(ax - 1) e^{2ax}}{2a(ax + 1)}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}u(x) + \frac{4\left(\frac{d}{dx}u(x)\right)}{x} - a^2u(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}u(x)$

- ☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{4}{x}, P_3(x) = -a^2\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 4$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $-a^2u(x)\,x + \left(\frac{d^2}{dx^2}u(x)\right)x + 4\frac{d}{dx}u(x) = 0$

- Assume series solution for $u(x)$

  $u(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $x \cdot u(x)$ to series expansion

    $x \cdot u(x) = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

  - Shift index using $k- >k-1$

    $x \cdot u(x) = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

  - Convert $\frac{d}{dx}u(x)$ to series expansion

    $\frac{d}{dx}u(x) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1}$

  - Shift index using $k- >k+1$

    $\frac{d}{dx}u(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,x^{k+r}$

  - Convert $x \cdot \left(\frac{d^2}{dx^2}u(x)\right)$ to series expansion

    $x \cdot \left(\frac{d^2}{dx^2}u(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,x^{k+r-1}$

  - Shift index using $k- >k+1$

    $x \cdot \left(\frac{d^2}{dx^2}u(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)(k+r)\,x^{k+r}$

    Rewrite ODE with series expansions

    $a_0 r(3+r)\,x^{-1+r} + a_1(1+r)(4+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)(k+4+r) - a^2 a_{k-1}\right)x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(3+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{-3, 0\}$

- Each term must be 0

  $a_1(1+r)(4+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)(k+4+r) - a^2 a_{k-1} = 0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)(k+5+r) - a^2 a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a^2 a_k}{(k+2+r)(k+5+r)}$

- Recursion relation for $r = -3$

  $a_{k+2} = \frac{a^2 a_k}{(k-1)(k+2)}$

- Solution for $r = -3$

  $\left[u(x) = \sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2} = \frac{a^2 a_k}{(k-1)(k+2)}, -2a_1 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{a^2 a_k}{(k+2)(k+5)}$

- Solution for $r = 0$

  $\left[u(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a^2 a_k}{(k+2)(k+5)}, 4a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[u(x) = \left(\sum_{k=0}^{\infty} b_k x^{k-3}\right) + \left(\sum_{k=0}^{\infty} c_k x^k\right), b_{k+2} = \frac{a^2 b_k}{(k+2)(k-1)}, -2b_1 = 0, c_{k+2} = \frac{a^2 c_k}{(5+k)(k+2)}, 4c_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 32

```
dsolve(diff(diff(u(x),x),x)+4/x*diff(u(x),x)-a^2*u(x) = 0,u(x),singsol=all)
```

$$u = \frac{c_1 e^{ax}(ax-1) + c_2 e^{-ax}(ax+1)}{x^3}$$

Mathematica DSolve solution

Solving time : 0.073 (sec)
Leaf size : 68

```
DSolve[{D[u[x],{x,2}]+4/x*D[u[x],x]-a^2*u[x]==0,{}},u[x],x,IncludeSingularSolutions->True]
```

$$u(x) \to \frac{\sqrt{\frac{2}{\pi}}((iac_2x + c_1)\sinh(ax) - (ac_1x + ic_2)\cosh(ax))}{ax^{5/2}\sqrt{-iax}}$$

2.1.276 Problem 279

Internal problem ID [9448]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 279
Date solved : `Monday, January 27, 2025 at 06:02:56 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$u'' + \frac{4u'}{x} + a^2 u = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.316 (sec)

Writing the ode as

$$u'' + \frac{4u'}{x} + a^2 u = 0 \tag{1}$$

$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{4}{x} \\ C &= a^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = u e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-a^2x^2 + 2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -a^2x^2 + 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-a^2x^2+2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.526: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2} - a^2$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx ia - \frac{i}{a\,x^2} - \frac{i}{2a^3x^4} - \frac{i}{2a^5x^6} - \frac{5i}{8a^7x^8} - \frac{7i}{8a^9x^{10}} - \frac{21i}{16a^{11}x^{12}} - \frac{33i}{16a^{13}x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = ia$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= ia \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -a^2$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-a^2x^2 + 2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-a^2) + \left(\frac{2}{x^2}\right) \\ &= \frac{2}{x^2} - a^2 \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= ia \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{ia} - 0\right) &= 0 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{ia} - 0\right) &= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-a^2x^2 + 2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	ia	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + (-)\,(ia) \\ &= -\frac{1}{x} - ia \\ &= -\frac{1}{x} - ia \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x} - ia\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - ia\right)^2 - \left(\frac{-a^2x^2+2}{x^2}\right)\right) = 0$$

$$\frac{2iaa_0 - 2}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{i}{a}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - \frac{i}{a}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{i}{a}\right) e^{\int \left(-\frac{1}{x} - ia\right) dx} \\
&= \left(x - \frac{i}{a}\right) e^{-\ln(x) - iax} \\
&= \frac{(ax - i)\, e^{-iax}}{xa}
\end{aligned}$$

The first solution to the original ode in u is found from

$$\begin{aligned}
u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{\frac{4}{x}}{1} \, dx} \\
&= z_1 e^{-2\ln(x)} \\
&= z_1 \left(\frac{1}{x^2}\right)
\end{aligned}$$

Which simplifies to

$$u_1 = \frac{(ax - i)\, e^{-iax}}{x^3 a}$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{u_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
u_2 &= u_1 \int \frac{e^{\int -\frac{\frac{4}{x}}{1} \, dx}}{(u_1)^2} \, dx \\
&= u_1 \int \frac{e^{-4\ln(x)}}{(u_1)^2} \, dx \\
&= u_1 \left(\frac{(iax - 1)\, e^{2iax}}{2a\,(-ax + i)}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1 \left(\frac{(ax - i)\, e^{-iax}}{x^3 a}\right) + c_2 \left(\frac{(ax - i)\, e^{-iax}}{x^3 a} \left(\frac{(iax - 1)\, e^{2iax}}{2a\,(-ax + i)}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}u(x) + \frac{4\left(\frac{d}{dx}u(x)\right)}{x} + a^2u(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}u(x)$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{4}{x}, P_3(x) = a^2\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 4$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$
  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $a^2u(x)\,x + \left(\frac{d^2}{dx^2}u(x)\right)x + 4\frac{d}{dx}u(x) = 0$

- Assume series solution for $u(x)$

  $u(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x \cdot u(x)$ to series expansion

    $x \cdot u(x) = \sum_{k=0}^{\infty} a_k x^{k+r+1}$
  - Shift index using $k- >k-1$

    $x \cdot u(x) = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$
  - Convert $\frac{d}{dx}u(x)$ to series expansion

    $\frac{d}{dx}u(x) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1}$
  - Shift index using $k- >k+1$

    $\frac{d}{dx}u(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,x^{k+r}$
  - Convert $x \cdot \left(\frac{d^2}{dx^2}u(x)\right)$ to series expansion

    $x \cdot \left(\frac{d^2}{dx^2}u(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$
  - Shift index using $k- >k+1$

    $x \cdot \left(\frac{d^2}{dx^2}u(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(3+r)\,x^{-1+r} + a_1(1+r)\,(4+r)\,x^r + \left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+r+1)\,(k+4+r) + a^2a_{k-1}\right)x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(3+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{-3, 0\}$

- Each term must be 0

  $a_1(1+r)(4+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)(k+4+r) + a^2 a_{k-1} = 0$

- Shift index using $k- >k+1$

  $a_{k+2}(k+2+r)(k+5+r) + a^2 a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a^2 a_k}{(k+2+r)(k+5+r)}$

- Recursion relation for $r = -3$

  $a_{k+2} = -\frac{a^2 a_k}{(k-1)(k+2)}$

- Solution for $r = -3$

  $\left[u(x) = \sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2} = -\frac{a^2 a_k}{(k-1)(k+2)}, -2a_1 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{a^2 a_k}{(k+2)(k+5)}$

- Solution for $r = 0$

  $\left[u(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a^2 a_k}{(k+2)(k+5)}, 4a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[u(x) = \left(\sum_{k=0}^{\infty} b_k x^{k-3}\right) + \left(\sum_{k=0}^{\infty} c_k x^k\right), b_{k+2} = -\frac{a^2 b_k}{(k+2)(k-1)}, -2b_1 = 0, c_{k+2} = -\frac{a^2 c_k}{(5+k)(k+2)}, 4c_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.030 (sec)
Leaf size : 33

```
dsolve(diff(diff(u(x),x),x)+4/x*diff(u(x),x)+a^2*u(x) = 0,u(x),singsol=all)
```

$$u = \frac{(ac_1x + c_2)\cos(ax) + \sin(ax)(ac_2x - c_1)}{x^3}$$

Mathematica DSolve solution

Solving time : 0.072 (sec)
Leaf size : 57

```
DSolve[{D[u[x],{x,2}]+4/x*D[u[x],x]+a^2*u[x]==0,{}},u[x],x,IncludeSingularSolutions->True]
```

$$u(x) \to -\frac{\sqrt{\frac{2}{\pi}}((ac_1x + c_2)\cos(ax) + (ac_2x - c_1)\sin(ax))}{x^{3/2}(ax)^{3/2}}$$

2.1.277 Problem 280

Internal problem ID [9449]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 280
Date solved : Monday, January 27, 2025 at 06:02:57 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$y'' - a^2 y = \frac{6y}{x^2}$$

Solved as second order ode using Kovacic algorithm

Time used: 0.284 (sec)

Writing the ode as

$$y'' + \left(-a^2 - \frac{6}{x^2}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -a^2 - \frac{6}{x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2 x^2 + 6}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= a^2 x^2 + 6 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{a^2x^2+6}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.528: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = a^2 + \frac{6}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx a + \frac{3}{a\,x^2} - \frac{9}{2a^3x^4} + \frac{27}{2a^5x^6} - \frac{405}{8a^7x^8} + \frac{1701}{8a^9x^{10}} - \frac{15309}{16a^{11}x^{12}} + \frac{72171}{16a^{13}x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = a$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= a \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = a^2$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{a^2x^2 + 6}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (a^2) + \left(\frac{6}{x^2}\right) \\ &= a^2 + \frac{6}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= a \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{0}{a} - 0\right) = 0 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{0}{a} - 0\right) = 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{a^2x^2+6}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	a	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= 0 - (-2) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{2}{x} + (-)(a) \\ &= -\frac{2}{x} - a \\ &= \frac{-ax-2}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(-\frac{2}{x} - a\right)(2x + a_1) + \left(\left(\frac{2}{x^2}\right) + \left(-\frac{2}{x} - a\right)^2 - \left(\frac{a^2x^2+6}{x^2}\right) \right) = 0$$

$$\frac{2axa_1 + 4aa_0 - 6x - 4a_1}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{ a_0 = \frac{3}{a^2}, a_1 = \frac{3}{a} \right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + \frac{3x}{a} + \frac{3}{a^2}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \left(x^2 + \frac{3x}{a} + \frac{3}{a^2}\right) e^{\int \left(-\frac{2}{x} - a\right) dx} \\ &= \left(x^2 + \frac{3x}{a} + \frac{3}{a^2}\right) e^{-ax - 2\ln(x)} \\ &= \frac{(a^2x^2 + 3ax + 3)\, e^{-ax}}{a^2x^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{(a^2x^2 + 3ax + 3)\, e^{-ax}}{a^2x^2} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(a^2x^2 + 3ax + 3)\, e^{-ax}}{a^2x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \frac{(a^2x^2 + 3ax + 3)\, e^{-ax}}{a^2x^2} \int \frac{1}{\frac{(a^2x^2+3ax+3)^2 e^{-2ax}}{a^4x^4}}\, dx \\ &= \frac{(a^2x^2 + 3ax + 3)\, e^{-ax}}{a^2x^2} \left(\frac{(a^2x^2 - 3ax + 3)\, e^{2ax}}{2a\,(a^2x^2 + 3ax + 3)}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{(a^2x^2 + 3ax + 3)\, e^{-ax}}{a^2x^2}\right) + c_2 \left(\frac{(a^2x^2 + 3ax + 3)\, e^{-ax}}{a^2x^2} \left(\frac{(a^2x^2 - 3ax + 3)\, e^{2ax}}{2a\,(a^2x^2 + 3ax + 3)}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) - a^2y(x) = \frac{6y(x)}{x^2}$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{(a^2x^2+6)y(x)}{x^2}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lir

  $\frac{d^2}{dx^2}y(x) - \frac{(a^2x^2+6)y(x)}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = 0, P_3(x) = -\frac{a^2x^2+6}{x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 0$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = -6$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) + (-a^2x^2 - 6)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ◦ Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ◦ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)\,(-3+r)\,x^r + a_1(3+r)\,(-2+r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+2)\,(k+r-3) - a_{k-2}a^2)\,x^{k+r}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 3\}$

- Each term must be 0

  $a_1(3+r)\,(-2+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)(k+r-3) - a_{k-2}a^2 = 0$
- Shift index using $k- >k+2$

  $a_{k+2}(k+4+r)(k+r-1) - a_k a^2 = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k a^2}{(k+4+r)(k+r-1)}$
- Recursion relation for $r = -2$

  $a_{k+2} = \frac{a_k a^2}{(k+2)(k-3)}$
- Solution for $r = -2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = \frac{a_k a^2}{(k+2)(k-3)}, a_1 = 0\right]$
- Recursion relation for $r = 3$

  $a_{k+2} = \frac{a_k a^2}{(k+7)(k+2)}$
- Solution for $r = 3$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = \frac{a_k a^2}{(k+7)(k+2)}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} b_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} c_k x^{k+3}\right), b_{k+2} = \frac{b_k a^2}{(k+2)(k-3)}, b_1 = 0, c_{k+2} = \frac{c_k a^2}{(k+7)(k+2)}, c_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.008` (sec)
Leaf size : 48

```
dsolve(diff(diff(y(x),x),x)-a^2*y(x) = 6/x^2*y(x),y(x),singsol=all)
```

$$y = \frac{c_2 e^{-ax}(a^2x^2 + 3ax + 3) + c_1 e^{ax}(a^2x^2 - 3ax + 3)}{x^2}$$

Mathematica DSolve solution

Solving time : 0.123 (sec)
Leaf size : 90

```
DSolve[{D[y[x],{x,2}]-a^2*y[x]==6*y[x]/x^2,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt{\frac{2}{\pi}}((a^2c_2x^2 - 3iac_1x + 3c_2)\cosh(ax) + i(c_1(a^2x^2 + 3) + 3iac_2x)\sinh(ax))}{a^2x^{3/2}\sqrt{-iax}}$$

2.1.278 Problem 281

Internal problem ID [9450]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 281
Date solved : `Monday, January 27, 2025 at 06:02:57 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + n^2 y = \frac{6y}{x^2}$$

Solved as second order ode using Kovacic algorithm

Time used: 0.336 (sec)

Writing the ode as

$$y'' + \left(n^2 - \frac{6}{x^2}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= n^2 - \frac{6}{x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-n^2 x^2 + 6}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -n^2 x^2 + 6 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-n^2x^2+6}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.530: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2-2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1,2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -n^2 + \frac{6}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx in - \frac{3i}{n\,x^2} - \frac{9i}{2n^3x^4} - \frac{27i}{2n^5x^6} - \frac{405i}{8n^7x^8} - \frac{1701i}{8n^9x^{10}} - \frac{15309i}{16n^{11}x^{12}} - \frac{72171i}{16n^{13}x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = in$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= in \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -n^2$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-n^2x^2 + 6}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-n^2) + \left(\frac{6}{x^2}\right) \\ &= -n^2 + \frac{6}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= in \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{0}{in} - 0\right) = 0 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{0}{in} - 0\right) = 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{-n^2x^2+6}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	in	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=0$ then

$$\begin{aligned}d&=\alpha_\infty^--\left(\alpha_{c_1}^-\right)\\&=0-(-2)\\&=2\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega&=\left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^-}{x-c_1}\right)+(-)[\sqrt{r}]_\infty\\&=-\frac{2}{x}+(-)\,(in)\\&=-\frac{2}{x}-in\\&=-\frac{2}{x}-in\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=x^2+a_1x+a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2)+2\left(-\frac{2}{x}-in\right)(2x+a_1)+\left(\left(\frac{2}{x^2}\right)+\left(-\frac{2}{x}-in\right)^2-\left(\frac{-n^2x^2+6}{x^2}\right)\right)=0$$

$$\frac{(2ina_1-6)\,x+4ina_0-4a_1}{x}=0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0=-\frac{3}{n^2},a_1=-\frac{3i}{n}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - \frac{3ix}{n} - \frac{3}{n^2}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - \frac{3ix}{n} - \frac{3}{n^2}\right) e^{\int \left(-\frac{2}{x} - in\right) dx} \\
&= \left(x^2 - \frac{3ix}{n} - \frac{3}{n^2}\right) e^{-2\ln(x) - inx} \\
&= \frac{(n^2x^2 - 3inx - 3)\, e^{-inx}}{x^2n^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= \frac{(n^2x^2 - 3inx - 3)\, e^{-inx}}{x^2n^2}
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(n^2x^2 - 3inx - 3)\, e^{-inx}}{x^2n^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\
&= \frac{(n^2x^2 - 3inx - 3)\, e^{-inx}}{x^2n^2} \int \frac{1}{\frac{(n^2x^2 - 3inx - 3)^2 e^{-2inx}}{x^4n^4}} \, dx \\
&= \frac{(n^2x^2 - 3inx - 3)\, e^{-inx}}{x^2n^2} \left(\frac{(in^2x^2 - 3nx - 3i)\, e^{2inx}}{2n\,(-n^2x^2 + 3inx + 3)}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{(n^2x^2 - 3inx - 3)\, e^{-inx}}{x^2n^2}\right) + c_2\left(\frac{(n^2x^2 - 3inx - 3)\, e^{-inx}}{x^2n^2}\left(\frac{(in^2x^2 - 3nx - 3i)\, e^{2inx}}{2n\,(-n^2x^2 + 3inx + 3)}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + n^2 y(x) = \frac{6y(x)}{x^2}$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(n^2x^2-6)y(x)}{x^2}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(n^2x^2-6)y(x)}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = 0, P_3(x) = \frac{n^2x^2-6}{x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 0$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -6$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) + (n^2x^2 - 6)\, y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ○ Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ○ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)(-3+r)\, x^r + a_1(3+r)(-2+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+2)(k+r-3) + a_{k-2}n^2\right) x^{k+r}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 3\}$

- Each term must be 0

  $a_1(3+r)(-2+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)(k+r-3)+a_{k-2}n^2=0$
- Shift index using $k->k+2$

  $a_{k+2}(k+4+r)(k+r-1)+a_k n^2=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_k n^2}{(k+4+r)(k+r-1)}$
- Recursion relation for $r=-2$

  $a_{k+2}=-\frac{a_k n^2}{(k+2)(k-3)}$
- Solution for $r=-2$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2}=-\frac{a_k n^2}{(k+2)(k-3)}, a_1=0\right]$
- Recursion relation for $r=3$

  $a_{k+2}=-\frac{a_k n^2}{(k+7)(k+2)}$
- Solution for $r=3$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2}=-\frac{a_k n^2}{(k+7)(k+2)}, a_1=0\right]$
- Combine solutions and rename parameters

  $\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^{k-2}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), a_{k+2}=-\frac{a_k n^2}{(k+2)(k-3)}, a_1=0, b_{k+2}=-\frac{b_k n^2}{(k+7)(k+2)}, b_1=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.033` (sec)
Leaf size : 53

```
dsolve(diff(diff(y(x),x),x)+n^2*y(x) = 6/x^2*y(x),y(x),singsol=all)
```

$$y=\frac{(c_1 n^2 x^2+3c_2 n x-3c_1)\cos(nx)+\sin(nx)(c_2 n^2 x^2-3c_1 n x-3c_2)}{x^2}$$

Mathematica DSolve solution

Solving time : 0.123 (sec)
Leaf size : 79

```
DSolve[{D[y[x],{x,2}]+n^2*y[x]==6*y[x]/x^2,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{\sqrt{\frac{2}{\pi}}\sqrt{x}\left(\left(c_2\left(-n^2\right)x^2+3c_1nx+3c_2\right)\cos(nx)+\left(c_1\left(n^2x^2-3\right)+3c_2nx\right)\sin(nx)\right)}{(nx)^{5/2}}$$

2.1.279 Problem 282

Internal problem ID [9451]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 282
Date solved : `Monday, January 27, 2025 at 06:02:58 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' - \left(x^2 + \frac{1}{4}\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.079 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(-x^2 - \frac{1}{4}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= -x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.532: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-x}}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\, dx}}{(y_1)^2}\, dx\\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\, dx\\ &= y_1 \left(\frac{e^{2x}}{2}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2\\ &= c_1 \left(\frac{e^{-x}}{\sqrt{x}}\right) + c_2 \left(\frac{e^{-x}}{\sqrt{x}}\left(\frac{e^{2x}}{2}\right)\right)\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x\left(\frac{d}{dx}y(x)\right) - \left(x^2 + \frac{1}{4}\right) y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{(4x^2+1)y(x)}{4x^2} - \frac{\frac{d}{dx}y(x)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{\frac{d}{dx}y(x)}{x} - \frac{(4x^2+1)y(x)}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = -\frac{4x^2+1}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x\left(\frac{d}{dx}y(x)\right) + (-4x^2 - 1)\, y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) - 4a_{k-}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) - 4a_{k-2} = 0$

- Shift index using $k- >k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\,r + 4r^2 - 1\right) - 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = \frac{4a_k}{4k^2+12k+8}$

- Solution for $r = -\frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = \frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = \frac{4a_k}{4k^2+20k+24}$

- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = \frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = \frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.040 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x-(x^2+1/4)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_1 \sinh (x) + c_2 \cosh (x)}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.031 (sec)
Leaf size : 32

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]-(x^2+1/4)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to \frac{e^{-x}(c_2 e^{2x} + 2c_1)}{2\sqrt{x}}$$

2.1.280 Problem 283

Internal problem ID [9452]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 283
Date solved : `Monday, January 27, 2025 at 06:02:59 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \frac{(-9a^2 + 4x^2)\, y}{4a^2} = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.303 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(-\frac{9}{4} + \frac{x^2}{a^2}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= -\frac{9}{4} + \frac{x^2}{a^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2a^2 - x^2}{x^2a^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2a^2 - x^2 \\ t &= x^2a^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2a^2 - x^2}{x^2 a^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.534: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2 a^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{a^2} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx -\frac{33ia^{13}}{16x^{14}} - \frac{21ia^{11}}{16x^{12}} - \frac{7ia^9}{8x^{10}} - \frac{5ia^7}{8x^8} - \frac{ia^5}{2x^6} - \frac{ia^3}{2x^4} - \frac{ia}{x^2} + \frac{i}{a} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{i}{a}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{i}{a} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -\frac{1}{a^2}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2a^2 - x^2}{x^2 a^2} \\ &= Q + \frac{R}{x^2 a^2} \\ &= \left(-\frac{1}{a^2}\right) + \left(\frac{2}{x^2}\right) \\ &= -\frac{1}{a^2} + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{i}{a} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{\frac{i}{a}} - 0\right) &&= 0 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{\frac{i}{a}} - 0\right) &&= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2a^2 - x^2}{x^2 a^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{i}{a}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + (-)\left(\frac{i}{a}\right) \\ &= -\frac{1}{x} - \frac{i}{a} \\ &= -\frac{ix+a}{xa} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x} - \frac{i}{a}\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - \frac{i}{a}\right)^2 - \left(\frac{2a^2 - x^2}{x^2 a^2}\right)\right) = 0$$

$$\frac{2ia_0 - 2a}{xa} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -ia\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = -ia + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (-ia + x) \, \mathrm{e}^{\int \left(-\frac{1}{x} - \frac{i}{a}\right) dx} \\ &= (-ia + x) \, \mathrm{e}^{-\ln(x) - \frac{ix}{a}} \\ &= \frac{(-ia + x) \, \mathrm{e}^{-\frac{ix}{a}}}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x}{x^2} \, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(-ia + x) \, \mathrm{e}^{-\frac{ix}{a}}}{x^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{(ix + a) \, a (ia + x) \, \mathrm{e}^{\frac{2ix}{a}}}{2 \, (ia - x)^2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{(-ia + x) \, \mathrm{e}^{-\frac{ix}{a}}}{x^{3/2}}\right) + c_2 \left(\frac{(-ia + x) \, \mathrm{e}^{-\frac{ix}{a}}}{x^{3/2}} \left(-\frac{(ix + a) \, a (ia + x) \, \mathrm{e}^{\frac{2ix}{a}}}{2 \, (ia - x)^2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right)+x\left(\frac{d}{dx}y(x)\right)+\frac{(-9a^2+4x^2)y(x)}{4a^2}=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{(9a^2-4x^2)y(x)}{4a^2x^2}-\frac{\frac{d}{dx}y(x)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{\frac{d}{dx}y(x)}{x}-\frac{(9a^2-4x^2)y(x)}{4a^2x^2}=0$

- Multiply by denominators of the ODE

  $4\left(\frac{d^2}{dx^2}y(x)\right)x^2a^2+4\left(\frac{d}{dx}y(x)\right)x\,a^2-(9a^2-4x^2)\,y(x)=0$

- Make a change of variables

  $t=\ln(x)$

- ☐ Substitute the change of variables back into the ODE
  - ○ Calculate the 1st derivative of y with respect to x , using the chain rule

    $\frac{d}{dx}y(x)=\left(\frac{d}{dt}y(t)\right)\left(\frac{d}{dx}t(x)\right)$

  - ○ Compute derivative

    $\frac{d}{dx}y(x)=\frac{\frac{d}{dt}y(t)}{x}$

  - ○ Calculate the 2nd derivative of y with respect to x , using the chain rule

    $\frac{d^2}{dx^2}y(x)=\left(\frac{d^2}{dt^2}y(t)\right)\left(\frac{d}{dx}t(x)\right)^2+\left(\frac{d^2}{dx^2}t(x)\right)\left(\frac{d}{dt}y(t)\right)$

  - ○ Compute derivative

    $\frac{d^2}{dx^2}y(x)=\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $4\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}\right)x^2a^2+4\left(\frac{d}{dt}y(t)\right)a^2-(9a^2-4x^2)\,y(t)=0$

- Simplify

  $-9y(t)\,a^2+4y(t)\,x^2+4a^2\left(\frac{d^2}{dt^2}y(t)\right)=0$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t)=\frac{(9a^2-4x^2)y(t)}{4a^2}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d^2}{dt^2}y(t)-\frac{(9a^2-4x^2)y(t)}{4a^2}=0$

- Characteristic polynomial of ODE

  $r^2-\frac{9a^2-4x^2}{4a^2}=0$

- Factor the characteristic polynomial

  $\frac{4r^2a^2-9a^2+4x^2}{4a^2}=0$

- Roots of the characteristic polynomial

  $r=\left(\frac{\sqrt{9a^2-4x^2}}{2a},-\frac{\sqrt{9a^2-4x^2}}{2a}\right)$

- 1st solution of the ODE

  $y_1(t)=\mathrm{e}^{\frac{\sqrt{9a^2-4x^2}\,t}{2a}}$

- 2nd solution of the ODE

  $y_2(t)=\mathrm{e}^{-\frac{\sqrt{9a^2-4x^2}\,t}{2a}}$

- General solution of the ODE

  $y(t)=C1y_1(t)+C2y_2(t)$

- Substitute in solutions

  $$y(t) = C1\,\mathrm{e}^{\frac{\sqrt{9a^2-4x^2}\,t}{2a}} + C2\,\mathrm{e}^{-\frac{\sqrt{9a^2-4x^2}\,t}{2a}}$$

- Change variables back using $t = \ln(x)$

  $$y(x) = C1\,\mathrm{e}^{\frac{\sqrt{9a^2-4x^2}\,\ln(x)}{2a}} + C2\,\mathrm{e}^{-\frac{\sqrt{9a^2-4x^2}\,\ln(x)}{2a}}$$

- Simplify

  $$y(x) = C1\,x^{\frac{\sqrt{9a^2-4x^2}}{2a}} + C2\,x^{-\frac{\sqrt{9a^2-4x^2}}{2a}}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.049` (sec)
Leaf size : 37

```
dsolve(x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x+1/4*(-9*a^2+4*x^2)/a^2*y(x) = 0,y(x),si
```

$$y = \frac{c_2(ix+a)\,\mathrm{e}^{-\frac{ix}{a}} + (-ix+a)\,c_1\mathrm{e}^{\frac{ix}{a}}}{x^{3/2}}$$

Mathematica DSolve solution

Solving time : `0.077` (sec)
Leaf size : 62

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(4*x^2-9*a^2)/(4*a^2)*y[x]==0,{}},y[x],x,IncludeSingul
```

$$y(x) \to -\frac{\sqrt{\frac{2}{\pi}}\left((ac_2+c_1x)\cos\left(\frac{x}{a}\right)+(c_2x-ac_1)\sin\left(\frac{x}{a}\right)\right)}{x\sqrt{\frac{x}{a}}}$$

2.1.281 Problem 284

Internal problem ID [9453]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 284
Date solved : `Monday, January 27, 2025 at 06:02:59 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{25}{4}\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.303 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{25}{4}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{25}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 6}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2 + 6 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2+6}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.536: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -1 + \frac{6}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx i - \frac{3i}{x^2} - \frac{9i}{2x^4} - \frac{27i}{2x^6} - \frac{405i}{8x^8} - \frac{1701i}{8x^{10}} - \frac{15309i}{16x^{12}} - \frac{72171i}{16x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= i \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-x^2 + 6}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-1) + \left(\frac{6}{x^2}\right) \\ &= -1 + \frac{6}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= i \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{0}{i} - 0\right) = 0 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{0}{i} - 0\right) = 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2+6}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	i	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= 0 - (-2) \\
&= 2
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{2}{x} + (-)(i) \\
&= -\frac{2}{x} - i \\
&= -\frac{2}{x} - i
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(2) + 2\left(-\frac{2}{x} - i\right)(2x + a_1) + \left(\left(\frac{2}{x^2}\right) + \left(-\frac{2}{x} - i\right)^2 - \left(\frac{-x^2+6}{x^2}\right)\right) &= 0 \\
\frac{2ixa_1 + 4ia_0 - 6x - 4a_1}{x} &= 0
\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -3, a_1 = -3i\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 3ix - 3$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= \left(x^2 - 3ix - 3\right) \mathrm{e}^{\int \left(-\frac{2}{x} - i\right) dx} \\ &= \left(x^2 - 3ix - 3\right) \mathrm{e}^{-2\ln(x) - ix} \\ &= \frac{\left(x^2 - 3ix - 3\right) \mathrm{e}^{-ix}}{x^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x}{x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(x^2 - 3ix - 3\right) \mathrm{e}^{-ix}}{x^{5/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\, dx}}{\left(y_1\right)^2}\, dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{\left(y_1\right)^2}\, dx \\ &= y_1 \left(\frac{\left(i x^2 - 3x - 3i\right) \mathrm{e}^{2ix}}{-2x^2 + 6ix + 6}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\left(x^2 - 3ix - 3\right) \mathrm{e}^{-ix}}{x^{5/2}}\right) + c_2 \left(\frac{\left(x^2 - 3ix - 3\right) \mathrm{e}^{-ix}}{x^{5/2}} \left(\frac{\left(i x^2 - 3x - 3i\right) \mathrm{e}^{2ix}}{-2x^2 + 6ix + 6}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right)+x\left(\frac{d}{dx}y(x)\right)+\left(x^2-\frac{25}{4}\right)y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(4x^2-25)y(x)}{4x^2}-\frac{\frac{d}{dx}y(x)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{\frac{d}{dx}y(x)}{x}+\frac{(4x^2-25)y(x)}{4x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{1}{x},P_3(x)=\frac{4x^2-25}{4x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=-\frac{25}{4}$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right)+4x\left(\frac{d}{dx}y(x)\right)+(4x^2-25)y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

  ◦ Shift index using $k->k-m$

  $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r}$

  ◦ Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(5+2r)\,(-5+2r)\,x^r+a_1(7+2r)\,(-3+2r)\,x^{1+r}+\left(\sum_{k=2}^{\infty}(a_k(2k+2r+5)\,(2k+2r-5)+4a\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(5+2r)\,(-5+2r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{-\frac{5}{2},\frac{5}{2}\right\}$

- Each term must be 0

  $a_1(7+2r)(-3+2r)=0$
- Solve for the dependent coefficient(s)

  $a_1=0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(2k+2r+5)(2k+2r-5)+4a_{k-2}=0$
- Shift index using $k->k+2$

  $a_{k+2}(2k+9+2r)(2k-1+2r)+4a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{4a_k}{(2k+9+2r)(2k-1+2r)}$
- Recursion relation for $r=-\frac{5}{2}$

  $a_{k+2}=-\frac{4a_k}{(2k+4)(2k-6)}$
- Solution for $r=-\frac{5}{2}$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}, a_{k+2}=-\frac{4a_k}{(2k+4)(2k-6)}, a_1=0\right]$
- Recursion relation for $r=\frac{5}{2}$

  $a_{k+2}=-\frac{4a_k}{(2k+14)(2k+4)}$
- Solution for $r=\frac{5}{2}$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+\frac{5}{2}}, a_{k+2}=-\frac{4a_k}{(2k+14)(2k+4)}, a_1=0\right]$
- Combine solutions and rename parameters

  $\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{5}{2}}\right), a_{k+2}=-\frac{4a_k}{(2k+4)(2k-6)}, a_1=0, b_{k+2}=-\frac{4b_k}{(2k+14)(2k+4)}, b_1=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.051` (sec)
Leaf size : 43

```
dsolve(x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x+(x^2-25/4)*y(x) = 0,y(x),singsol=all)
```

$$y=\frac{-3c_2\left(ix-\frac{1}{3}x^2+1\right)\mathrm{e}^{-ix}+3\left(ix+\frac{1}{3}x^2-1\right)c_1\mathrm{e}^{ix}}{x^{5/2}}$$

Mathematica DSolve solution

Solving time : 0.075 (sec)
Leaf size : 59

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-25/4)*y[x]==0,{}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to -\frac{\sqrt{\frac{2}{\pi}}((-c_2x^2 + 3c_1x + 3c_2)\cos(x) + (c_1(x^2 - 3) + 3c_2x)\sin(x))}{x^{5/2}}$$

2.1.282 Problem 285

Internal problem ID [9454]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 285
Date solved : `Monday, January 27, 2025 at 06:03:00 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + qy' = \frac{2y}{x^2}$$

Solved as second order ode using Kovacic algorithm

Time used: 0.276 (sec)

Writing the ode as

$$y'' + qy' - \frac{2y}{x^2} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= q \\ C &= -\frac{2}{x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{q^2x^2 + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= q^2x^2 + 8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{q^2x^2+8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.538: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{q^2}{4} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{q}{2} + \frac{2}{q\,x^2} - \frac{4}{q^3 x^4} + \frac{16}{q^5 x^6} - \frac{80}{q^7 x^8} + \frac{448}{q^9 x^{10}} - \frac{2688}{q^{11} x^{12}} + \frac{16896}{q^{13} x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{q}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{q}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{q^2}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{q^2 x^2 + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{q^2}{4}\right) + \left(\frac{2}{x^2}\right) \\ &= \frac{q^2}{4} + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 4 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{q}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{\frac{q}{2}} - 0\right) &&= 0 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{\frac{q}{2}} - 0\right) &&= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{q^2x^2 + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{q}{2}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= 0 - (-1) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x} + (-)\left(\frac{q}{2}\right) \\
&= -\frac{1}{x} - \frac{q}{2} \\
&= -\frac{qx + 2}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x} - \frac{q}{2}\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - \frac{q}{2}\right)^2 - \left(\frac{q^2x^2 + 8}{4x^2}\right)\right) = 0$$

$$\frac{qa_0 - 2}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{2}{q}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{2}{q}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x + \frac{2}{q}\right) e^{\int \left(-\frac{1}{x} - \frac{q}{2}\right) dx} \\ &= \left(x + \frac{2}{q}\right) e^{-\frac{qx}{2} - \ln(x)} \\ &= \frac{(qx+2)\, e^{-\frac{qx}{2}}}{qx} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{q}{1} \, dx} \\ &= z_1 e^{-\frac{qx}{2}} \\ &= z_1 \left(e^{-\frac{qx}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{-qx}(qx+2)}{qx}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{q}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-qx}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{(qx-2)\, e^{qx}}{q\,(qx+2)}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{e^{-qx}(qx+2)}{qx}\right) + c_2 \left(\frac{e^{-qx}(qx+2)}{qx} \left(\frac{(qx-2)\, e^{qx}}{q\,(qx+2)}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + q\left(\frac{d}{dx}y(x)\right) = \frac{2y(x)}{x^2}$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -q\left(\frac{d}{dx}y(x)\right) + \frac{2y(x)}{x^2}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lir

  $\frac{d^2}{dx^2}y(x) + q\left(\frac{d}{dx}y(x)\right) - \frac{2y(x)}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = q, P_3(x) = -\frac{2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -2$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $q\left(\frac{d}{dx}y(x)\right)x^2 + x^2\left(\frac{d^2}{dx^2}y(x)\right) - 2y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^2 \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r+1}$

  ◦ Shift index using $k->k-1$

  $x^2 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\,x^{k+r}$

  ◦ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-2+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-2) + q a_{k-1}(k-1+r)\right)x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)\,(k+r-2) + q a_{k-1}(k-1+r) = 0$

- Shift index using $k->k+1$

  $a_{k+1}(k+2+r)\,(k-1+r) + q a_k(k+r) = 0$

- Recursion relation that defines series solution to ODE
  $a_{k+1} = -\frac{q a_k (k+r)}{(k+2+r)(k-1+r)}$
- Recursion relation for $r = -1$; series terminates at $k = 1$
  $a_{k+1} = -\frac{q a_k (k-1)}{(k+1)(k-2)}$
- Apply recursion relation for $k = 0$
  $a_1 = -\frac{q a_0}{2}$
- Terminating series solution of the ODE for $r = -1$. Use reduction of order to find the second lin
  $y(x) = a_0 \cdot \left(-\frac{qx}{2} + 1\right)$
- Recursion relation for $r = 2$
  $a_{k+1} = -\frac{q a_k (k+2)}{(k+4)(k+1)}$
- Solution for $r = 2$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = -\frac{q a_k (k+2)}{(k+4)(k+1)}\right]$
- Combine solutions and rename parameters
  $\left[y(x) = a_0 \cdot \left(-\frac{qx}{2} + 1\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), b_{k+1} = -\frac{q b_k (k+2)}{(4+k)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 28

```
dsolve(diff(diff(y(x),x),x)+q*diff(y(x),x) = 2/x^2*y(x),y(x),singsol=all)
```

$$y = \frac{c_2 \mathrm{e}^{-qx}(qx+2) + c_1(qx-2)}{x}$$

Mathematica DSolve solution

Solving time : 0.081 (sec)
Leaf size : 83

```
DSolve[{D[y[x],{x,2}]+q*D[y[x],x]==2*y[x]/x^2,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2qx^{3/2} e^{\frac{1}{2}-\frac{qx}{2}} \left((c_1 qx + 2ic_2)\cosh\left(\frac{qx}{2}\right) - (ic_2 qx + 2c_1)\sinh\left(\frac{qx}{2}\right)\right)}{\sqrt{\pi}(-iqx)^{5/2}}$$

2.1.283 Problem 286

Internal problem ID [9455]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 286
Date solved : `Monday, January 27, 2025 at 06:03:01 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$xy'' + 3y' + 4x^3y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.261 (sec)

Writing the ode as

$$xy'' + 3y' + 4x^3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 3 \\ C &= 4x^3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16x^4 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -16x^4 + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-16x^4 + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.540: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -4x^2 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 2ix - \frac{3i}{16x^3} - \frac{9i}{1024x^7} - \frac{27i}{32768x^{11}} - \frac{405i}{4194304x^{15}} - \frac{1701i}{134217728x^{19}} - \frac{15309i}{8589934592x^{23}} - \frac{72171i}{274877906944x^{27}} + \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 2i$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= 2ix \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -4x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-16x^4 + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (-4x^2) + \left(\frac{3}{4x^2}\right) \\ &= -4x^2 + \frac{3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 2ix \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{2i} - 1\right) &&= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{2i} - 1\right) &&= -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-16x^4 + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$2ix$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)\left(2ix\right) \\ &= -\frac{1}{2x} - 2ix \\ &= -\frac{1}{2x} - 2ix \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{2x} - 2ix\right)(0) + \left(\left(\frac{1}{2x^2} - 2i\right) + \left(-\frac{1}{2x} - 2ix\right)^2 - \left(\frac{-16x^4+3}{4x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(-\frac{1}{2x} - 2ix\right) dx} \\ &= \frac{e^{-ix^2}}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3}{x}\,dx} \\ &= z_1 e^{-\frac{3\ln(x)}{2}} \\ &= z_1\left(\frac{1}{x^{3/2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-ix^2}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3}{x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-3\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(-\frac{i\mathrm{e}^{2ix^2}}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\mathrm{e}^{-ix^2}}{x^2}\right) + c_2\left(\frac{\mathrm{e}^{-ix^2}}{x^2}\left(-\frac{i\mathrm{e}^{2ix^2}}{4}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dx^2}y(x)\right)x + 3\frac{d}{dx}y(x) + 4x^3y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -4x^2y(x) - \frac{3\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x) + \frac{3\left(\frac{d}{dx}y(x)\right)}{x} + 4x^2y(x) = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{3}{x}, P_3(x) = 4x^2\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

$\left(x \cdot P_2(x)\right)\Big|_{x=0} = 3$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $\left(\frac{d^2}{dx^2}y(x)\right)x + 3\frac{d}{dx}y(x) + 4x^3y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $x^3 \cdot y(x)$ to series expansion

    $x^3 \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+3}$

  - Shift index using $k- >k-3$

    $x^3 \cdot y(x) = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$

  - Convert $\frac{d}{dx}y(x)$ to series expansion

    $\frac{d}{dx}y(x) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1}$

  - Shift index using $k- >k+1$

    $\frac{d}{dx}y(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,x^{k+r}$

  - Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$

  - Shift index using $k- >k+1$

    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(2+r)\,x^{-1+r} + a_1(1+r)\,(3+r)\,x^r + a_2(2+r)\,(4+r)\,x^{1+r} + a_3(3+r)\,(5+r)\,x^{2+r} + \left(\sum_{k=3}^{\infty} (a_{k+\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 0\}$

- The coefficients of each power of x must be 0

  $[a_1(1+r)\,(3+r) = 0, a_2(2+r)\,(4+r) = 0, a_3(3+r)\,(5+r) = 0]$

- Solve for the dependent coefficient(s)

  $\{a_1 = 0, a_2 = 0, a_3 = 0\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k+r+3) + 4a_{k-3} = 0$

- Shift index using $k- >k+3$

  $a_{k+4}(k+4+r)\,(k+6+r) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+4} = -\frac{4a_k}{(k+4+r)(k+6+r)}$

- Recursion relation for $r = -2$

  $a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}$
- Solution for $r = -2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$
- Recursion relation for $r = 0$

  $a_{k+4} = -\frac{4a_k}{(k+4)(k+6)}$
- Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k}, a_{k+4} = -\frac{4a_k}{(k+4)(k+6)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k}\right), a_{4+k} = -\frac{4a_k}{(k+2)(4+k)}, a_1 = 0, a_2 = 0, a_3 = 0, b_{4+k} = -\frac{4b_k}{(4+k)(k}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 21

```
dsolve(x*diff(diff(y(x),x),x)+3*diff(y(x),x)+4*x^3*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_1 \sin\left(x^2\right) + c_2 \cos\left(x^2\right)}{x^2}$$

Mathematica DSolve solution

Solving time : 0.062 (sec)
Leaf size : 41

```
DSolve[{x*D[y[x],{x,2}]+3*D[y[x],x]+4*x^3*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{4c_1 e^{-ix^2} - ic_2 e^{ix^2}}{4x^2}$$

2.1.284 Problem 287

Internal problem ID [9456]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 287
Date solved : `Monday, January 27, 2025 at 06:03:01 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+2x\right)y''-2(x+1)\,y'+2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.209 (sec)

Writing the ode as

$$\left(x^2+2x\right)y''+(-2x-2)\,y'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2+2x\\B&=-2x-2\\C&=2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{3}{\left(x^2+2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=3\\t&=\left(x^2+2x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{3}{\left(x^2+2x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.542: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2} + \frac{3}{4(x+2)} - \frac{3}{4x} + \frac{3}{4(x+2)^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4>2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= 0\\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{(x^2+2x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right)\\ &= 1-(1)\\ &= 0\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (-)[\sqrt{r}]_\infty\\ &= -\frac{1}{2\,(x+2)} + \frac{3}{2x} + (-)\,(0)\\ &= -\frac{1}{2\,(x+2)} + \frac{3}{2x}\\ &= \frac{x+3}{x\,(x+2)}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2(x+2)}+\frac{3}{2x}\right)(0)+\left(\left(\frac{1}{2(x+2)^2}-\frac{3}{2x^2}\right)+\left(-\frac{1}{2(x+2)}+\frac{3}{2x}\right)^2-\left(\frac{3}{(x^2+2x)^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\,dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2(x+2)}+\frac{3}{2x}\right)dx} \\ &= \frac{x^{3/2}}{\sqrt{x+2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x-2}{x^2+2x}\,dx} \\ &= z_1 e^{\frac{\ln(x(x+2))}{2}} \\ &= z_1\left(\sqrt{x(x+2)}\right) \end{aligned}$$

Which simplifies to

$$y_1=\frac{\sqrt{x(x+2)}\,x^{3/2}}{\sqrt{x+2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1\int \frac{e^{\int -\frac{-2x-2}{x^2+2x}\,dx}}{(y_1)^2}\,dx \\ &= y_1\int \frac{e^{\ln(x(x+2))}}{(y_1)^2}\,dx \\ &= y_1\left(-\frac{1}{x}-\frac{1}{x^2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1y_1+c_2y_2 \\ &= c_1\left(\frac{\sqrt{x(x+2)}\,x^{3/2}}{\sqrt{x+2}}\right)+c_2\left(\frac{\sqrt{x(x+2)}\,x^{3/2}}{\sqrt{x+2}}\left(-\frac{1}{x}-\frac{1}{x^2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x^2+2x)\left(\frac{d^2}{dx^2}y(x)\right)-2(x+1)\left(\frac{d}{dx}y(x)\right)+2y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{2y(x)}{x(x+2)}+\frac{2(x+1)\left(\frac{d}{dx}y(x)\right)}{x(x+2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x)-\frac{2(x+1)\left(\frac{d}{dx}y(x)\right)}{x(x+2)}+\frac{2y(x)}{x(x+2)}=0$

□ Check to see if x_0 is a regular singular point

◦ Define functions

$\left[P_2(x)=-\frac{2(x+1)}{x(x+2)}, P_3(x)=\frac{2}{x(x+2)}\right]$

◦ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

$\left((x+2)\cdot P_2(x)\right)\bigg|_{x=-2}=-1$

◦ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

$\left((x+2)^2\cdot P_3(x)\right)\bigg|_{x=-2}=0$

◦ $x=-2$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-2$

- Multiply by denominators

$x(x+2)\left(\frac{d^2}{dx^2}y(x)\right)+(-2-2x)\left(\frac{d}{dx}y(x)\right)+2y(x)=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

$(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(2-2u)\left(\frac{d}{du}y(u)\right)+2y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

◦ Shift index using $k->k+1-m$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

◦ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

◦ Shift index using $k->k+2-m$

$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

Rewrite ODE with series expansions

$-2a_0r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)(k+r-1)+a_k(k+r-1)(k+r-2)\right)u^{k+r}\right)=$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-2r(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{0,2\}$

- Each term in the series must be 0, giving the recursion relation

  $((-2k-2r-2)\,a_{k+1}+a_k(k+r-2))\,(k+r-1)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k(k+r-2)}{2(k+1+r)}$

- Recursion relation for $r=0$; series terminates at $k=2$

  $a_{k+1}=\frac{a_k(k-2)}{2(k+1)}$

- Apply recursion relation for $k=0$

  $a_1=-a_0$

- Apply recursion relation for $k=1$

  $a_2=-\frac{a_1}{4}$

- Express in terms of a_0

  $a_2=\frac{a_0}{4}$

- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li

  $y(u)=a_0\cdot\left(1-u+\frac{1}{4}u^2\right)$

- Revert the change of variables $u=x+2$

  $\left[y(x)=\frac{a_0x^2}{4}\right]$

- Recursion relation for $r=2$

  $a_{k+1}=\frac{a_kk}{2(k+3)}$

- Solution for $r=2$

  $\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k+2}, a_{k+1}=\frac{a_kk}{2(k+3)}\right]$

- Revert the change of variables $u=x+2$

  $\left[y(x)=\sum_{k=0}^{\infty}a_k(x+2)^{k+2}, a_{k+1}=\frac{a_kk}{2(k+3)}\right]$

- Combine solutions and rename parameters

  $\left[y(x)=\frac{a_0x^2}{4}+\left(\sum_{k=0}^{\infty}b_k(x+2)^{k+2}\right), b_{k+1}=\frac{b_kk}{2(k+3)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 14

```
dsolve((x^2+2*x)*diff(diff(y(x),x),x)-2*(x+1)*diff(y(x),x)+2*y(x) = 0,y(x),singsol=all)
```

$$y = c_1 x^2 + c_2 x + c_2$$

Mathematica DSolve solution

Solving time : `0.184` (sec)
Leaf size : 100

```
DSolve[{(x^2+2*x)*D[y[x],{x,2}]-2*(x+1)*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \exp\left(\int_1^x \frac{K[1]+3}{K[1]^2+2K[1]}dK[1] - \frac{1}{2}\int_1^x -\frac{2(K[2]+1)}{K[2](K[2]+2)}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{K[1]+3}{K[1]^2+2K[1]}dK[1]\right)dK[3] + c_1\right)$$

2.1.285 Problem 288

Internal problem ID [9457]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 288
Date solved : `Monday, January 27, 2025 at 06:03:02 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^{2}+2 x \right) y''-2(x+1) y'+2 y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.209 (sec)

Writing the ode as

$$\left(x^{2}+2 x \right) y''+\left(-2 x -2\right) y'+2 y = 0 \tag{1}$$
$$A y'' + B y' + C y = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^{2}+2 x \\ B &= -2 x -2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2 A} \,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2 A B' - 2 B A' + B^2 - 4 A C}{4 A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r = \frac{3}{\left(x^{2}+2 x \right)^{2}} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= \left(x^{2}+2 x \right)^{2} \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{\left(x^{2}+2 x \right)^{2}}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.544: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 0\\ &= 4\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2} - \frac{3}{4x} + \frac{3}{4(x+2)} + \frac{3}{4(x+2)^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= 0\\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{(x^2+2x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right)\\ &= 1 - (1)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (-)[\sqrt{r}]_\infty\\ &= -\frac{1}{2(x+2)} + \frac{3}{2x} + (-)(0)\\ &= -\frac{1}{2(x+2)} + \frac{3}{2x}\\ &= \frac{x+3}{x(x+2)}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2\left(x+2\right)}+\frac{3}{2x}\right)(0)+\left(\left(\frac{1}{2\left(x+2\right)^{2}}-\frac{3}{2x^{2}}\right)+\left(-\frac{1}{2\left(x+2\right)}+\frac{3}{2x}\right)^{2}-\left(\frac{3}{\left(x^{2}+2x\right)^{2}}\right)\right)=0$$

$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2(x+2)}+\frac{3}{2x}\right)dx} \\ &= \frac{x^{3/2}}{\sqrt{x+2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x-2}{x^2+2x}\, dx} \\ &= z_1 e^{\frac{\ln(x(x+2))}{2}} \\ &= z_1\left(\sqrt{x\left(x+2\right)}\right) \end{aligned}$$

Which simplifies to

$$y_1=\frac{\sqrt{x\left(x+2\right)}\, x^{3/2}}{\sqrt{x+2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1\int \frac{e^{\int -\frac{-2x-2}{x^2+2x}\, dx}}{\left(y_1\right)^2}\, dx \\ &= y_1\int \frac{e^{\ln(x(x+2))}}{\left(y_1\right)^2}\, dx \\ &= y_1\left(-\frac{1}{x^2}-\frac{1}{x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\sqrt{x\left(x+2\right)}\, x^{3/2}}{\sqrt{x+2}}\right)+c_2\left(\frac{\sqrt{x\left(x+2\right)}\, x^{3/2}}{\sqrt{x+2}}\left(-\frac{1}{x^2}-\frac{1}{x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x^2+2x)\left(\frac{d^2}{dx^2}y(x)\right)-2(x+1)\left(\frac{d}{dx}y(x)\right)+2y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{2y(x)}{x(x+2)}+\frac{2(x+1)\left(\frac{d}{dx}y(x)\right)}{x(x+2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)-\frac{2(x+1)\left(\frac{d}{dx}y(x)\right)}{x(x+2)}+\frac{2y(x)}{x(x+2)}=0$

□ Check to see if x_0 is a regular singular point

- ∘ Define functions

$\left[P_2(x)=-\frac{2(x+1)}{x(x+2)}, P_3(x)=\frac{2}{x(x+2)}\right]$

- ∘ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

$\left((x+2)\cdot P_2(x)\right)\Big|_{x=-2}=-1$

- ∘ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

$\left((x+2)^2\cdot P_3(x)\right)\Big|_{x=-2}=0$

- ∘ $x=-2$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-2$

- Multiply by denominators

$x(x+2)\left(\frac{d^2}{dx^2}y(x)\right)+(-2-2x)\left(\frac{d}{dx}y(x)\right)+2y(x)=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

$(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(2-2u)\left(\frac{d}{du}y(u)\right)+2y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ∘ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ∘ Shift index using $k->k+1-m$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ∘ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

- ∘ Shift index using $k->k+2-m$

$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

Rewrite ODE with series expansions

$-2a_0r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)(k+r-1)+a_k(k+r-1)(k+r-2)\right)u^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-2r(-2+r)=0$

- Values of r that satisfy the indicial equation
  $r \in \{0, 2\}$
- Each term in the series must be 0, giving the recursion relation
  $((-2k-2r-2)\, a_{k+1} + a_k(k+r-2))\,(k+r-1)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_k(k+r-2)}{2(k+1+r)}$
- Recursion relation for $r=0$; series terminates at $k=2$
  $a_{k+1} = \frac{a_k(k-2)}{2(k+1)}$
- Apply recursion relation for $k=0$
  $a_1 = -a_0$
- Apply recursion relation for $k=1$
  $a_2 = -\frac{a_1}{4}$
- Express in terms of a_0
  $a_2 = \frac{a_0}{4}$
- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second linea
  $y(u) = a_0 \cdot \left(1 - u + \frac{1}{4}u^2\right)$
- Revert the change of variables $u = x+2$
  $\left[y(x) = \frac{a_0 x^2}{4}\right]$
- Recursion relation for $r=2$
  $a_{k+1} = \frac{a_k k}{2(k+3)}$
- Solution for $r=2$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k k}{2(k+3)}\right]$
- Revert the change of variables $u = x+2$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x+2)^{k+2}, a_{k+1} = \frac{a_k k}{2(k+3)}\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \frac{a_0 x^2}{4} + \left(\sum_{k=0}^{\infty} b_k (x+2)^{k+2}\right), b_{k+1} = \frac{b_k k}{2(k+3)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 14

```
dsolve((x^2+2*x)*diff(diff(y(x),x),x)-2*(x+1)*diff(y(x),x)+2*y(x) = 0,y(x),singsol=all
```

$$y = c_1 x^2 + c_2 x + c_2$$

Mathematica DSolve solution

Solving time : 0.159 (sec)
Leaf size : 100

```
DSolve[{(x^2+2*x)*D[y[x],{x,2}]-2*(x+1)*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to \exp\left(\int_1^x \frac{K[1]+3}{K[1]^2+2K[1]}dK[1] - \frac{1}{2}\int_1^x -\frac{2(K[2]+1)}{K[2](K[2]+2)}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{K[1]+3}{K[1]^2+2K[1]}dK[1]\right)dK[3] + c_1\right)$$

2.1.286 Problem 289

Internal problem ID [9458]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 289
Date solved : `Monday, January 27, 2025 at 06:03:03 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+1\right) y''-2 x y'+2 y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.258 (sec)

Writing the ode as

$$\left(x^2+1\right) y''-2 x y'+2 y=0 \tag{1}$$
$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= -2x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{-3}{\left(x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= \left(x^2+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{3}{\left(x^2+1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.546: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{3i}{4(x-i)} - \frac{3i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= 0\\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right)\\ &= 1 - (1)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (-)[\sqrt{r}]_\infty\\ &= -\frac{1}{2(x-i)} + \frac{3}{2(x+i)} + (-)(0)\\ &= -\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\\ &= \frac{x-2i}{x^2+1}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)}\right)(0) + \left(\left(\frac{1}{2\left(x-i\right)^{2}} - \frac{3}{2\left(x+i\right)^{2}}\right) + \left(-\frac{1}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)}\right)^{2} - \left(-\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right) dx} \\ &= \frac{\left(x^{2}+1\right)^{3/2}}{\left(ix+1\right)^{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2x}{x^{2}+1} \, dx} \\ &= z_1 e^{\frac{\ln\left(x^{2}+1\right)}{2}} \\ &= z_1 \left(\sqrt{x^{2}+1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(x^{2}+1\right)^{2}}{\left(ix+1\right)^{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^{2}} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^{2}+1} \, dx}}{\left(y_1\right)^{2}} \, dx \\ &= y_1 \int \frac{e^{\ln\left(x^{2}+1\right)}}{\left(y_1\right)^{2}} \, dx \\ &= y_1 \left(-\frac{x}{\left(x+i\right)^{2}}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\left(x^{2}+1\right)^{2}}{\left(ix+1\right)^{2}}\right) + c_2 \left(\frac{\left(x^{2}+1\right)^{2}}{\left(ix+1\right)^{2}} \left(-\frac{x}{\left(x+i\right)^{2}}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 16

```
dsolve((x^2+1)*diff(diff(y(x),x),x)-2*diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y = c_2 x^2 + c_1 x - c_2$$

Mathematica DSolve solution

Solving time : 0.313 (sec)
Leaf size : 79

```
DSolve[{(x^2+1)*D[y[x],{x,2}]-2*x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True
```

$$y(x) \to \sqrt{x^2+1}\exp\left(\int_1^x \frac{K[1]+2i}{K[1]^2+1}dK[1]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[2]}\frac{K[1]+2i}{K[1]^2+1}dK[1]\right)dK[2] + c_1\right)$$

2.1.287 Problem 290

Internal problem ID [9459]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 290
Date solved : `Monday, January 27, 2025 at 06:03:03 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+1\right) y''-2xy'+2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.270 (sec)

Writing the ode as

$$\left(x^2+1\right) y''-2xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= -2x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{-3}{\left(x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= \left(x^2+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{3}{\left(x^2+1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.547: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{3i}{4(x-i)} - \frac{3i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{3}{(x^2+1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\,(x-i)} + \frac{3}{2\,(x+i)} + (-)\,(0) \\
&= -\frac{1}{2\,(x-i)} + \frac{3}{2\,(x+i)} \\
&= \frac{x-2i}{x^2+1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2\left(x-i\right)}+\frac{3}{2\left(x+i\right)}\right)(0)+\left(\left(\frac{1}{2\left(x-i\right)^{2}}-\frac{3}{2\left(x+i\right)^{2}}\right)+\left(-\frac{1}{2\left(x-i\right)}+\frac{3}{2\left(x+i\right)}\right)^{2}-\left(-\frac{}{(x^{2}}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x-i)}+\frac{3}{2(x+i)}\right)dx} \\
&= \frac{\left(x^{2}+1\right)^{3/2}}{\left(ix+1\right)^{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2+1}\, dx} \\
&= z_1 e^{\frac{\ln\left(x^2+1\right)}{2}} \\
&= z_1\left(\sqrt{x^{2}+1}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(x^{2}+1\right)^{2}}{\left(ix+1\right)^{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2+1}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{\ln\left(x^2+1\right)}}{\left(y_1\right)^2}\, dx \\
&= y_1\left(-\frac{x}{\left(x+i\right)^{2}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\left(x^{2}+1\right)^{2}}{\left(ix+1\right)^{2}}\right)+c_2\left(\frac{\left(x^{2}+1\right)^{2}}{\left(ix+1\right)^{2}}\left(-\frac{x}{\left(x+i\right)^{2}}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 16

```
dsolve((x^2+1)*diff(diff(y(x),x),x)-2*diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y = c_2 x^2 + c_1 x - c_2$$

Mathematica DSolve solution

Solving time : 0.306 (sec)
Leaf size : 79

```
DSolve[{(x^2+1)*D[y[x],{x,2}]-2*x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to \sqrt{x^2+1}\exp\left(\int_1^x \frac{K[1]+2i}{K[1]^2+1}dK[1]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[2]}\frac{K[1]+2i}{K[1]^2+1}dK[1]\right)dK[2] + c_1\right)$$

2.1.288 Problem 291

Internal problem ID [9460]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 291
Date solved : `Monday, January 27, 2025 at 06:03:04 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - 4xy' + (4x^2 - 2)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.058 (sec)

Writing the ode as

$$y'' - 4xy' + (4x^2 - 2)\, y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4x \\ C &= 4x^2 - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.548: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{1}\, dx} \\ &= z_1 e^{x^2} \\ &= z_1\left(\mathrm{e}^{x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x^2}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{x^2}\right) + c_2\left(\mathrm{e}^{x^2}(x)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2}y(x) - 4x\left(\frac{d}{dx}y(x)\right) + (4x^2 - 2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dx^2}y(x)$
- Assume series solution for $y(x)$
$y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$
  $x^m \cdot y(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$
  - Shift index using $k- >k - m$
  $x^m \cdot y(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$
  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion
  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$
  - Convert $\frac{d^2}{dx^2}y(x)$ to series expansion
  $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$
  - Shift index using $k- >k + 2$
  $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions
  $2a_2 - 2a_0 + (6a_3 - 6a_1)\,x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) - 2a_k(2k+1) + 4a_{k-2})\, x^k\right) = 0$

- The coefficients of each power of x must be 0
$[2a_2 - 2a_0 = 0, 6a_3 - 6a_1 = 0]$
- Solve for the dependent coefficient(s)
$\{a_2 = a_0, a_3 = a_1\}$
- Each term in the series must be 0, giving the recursion relation
$(k^2 + 3k + 2)\,a_{k+2} - 4a_k k - 2a_k + 4a_{k-2} = 0$

- Shift index using $k->k+2$

  $$\left((k+2)^2+3k+8\right)a_{k+4}-4a_{k+2}(k+2)-2a_{k+2}+4a_k=0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y(x)=\sum_{k=0}^{\infty}a_k x^k, a_{k+4}=\frac{2(2ka_{k+2}-2a_k+5a_{k+2})}{k^2+7k+12}, a_2=a_0, a_3=a_1\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 14

```
dsolve(diff(diff(y(x),x),x)-4*diff(y(x),x)*x+(4*x^2-2)*y(x) = 0,y(x),singsol=all)
```

$$y=\mathrm{e}^{x^2}(c_2x+c_1)$$

Mathematica DSolve solution

Solving time : 0.021 (sec)
Leaf size : 18

```
DSolve[{D[y[x],{x,2}]-4*x*D[y[x],x]+(4*x^2-2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x)\to e^{x^2}(c_2x+c_1)$$

2.1.289 Problem 292

Internal problem ID [9461]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 292
Date solved : `Monday, January 27, 2025 at 06:03:04 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - 4xy' + (4x^2 - 2)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.053 (sec)

Writing the ode as

$$y'' - 4xy' + (4x^2 - 2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4x \\ C &= 4x^2 - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.550: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{1}\, dx} \\ &= z_1 e^{x^2} \\ &= z_1 \left(e^{x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x^2}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{x^2}\right) + c_2\left(\mathrm{e}^{x^2}(x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) - 4x\left(\frac{d}{dx}y(x)\right) + (4x^2 - 2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k + 2$

    $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 - 2a_0 + (6a_3 - 6a_1)\,x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - 2a_k(2k+1) + 4a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 - 2a_0 = 0, 6a_3 - 6a_1 = 0]$

- Solve for the dependent coefficient(s)

  $\{a_2 = a_0, a_3 = a_1\}$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\,a_{k+2} - 4a_k k - 2a_k + 4a_{k-2} = 0$

- Shift index using $k->k+2$
  $$\left((k+2)^2+3k+8\right)a_{k+4}-4a_{k+2}(k+2)-2a_{k+2}+4a_k=0$$
- Recursion relation that defines the series solution to the ODE
  $$\left[y(x)=\sum_{k=0}^{\infty}a_kx^k, a_{k+4}=\frac{2(2ka_{k+2}-2a_k+5a_{k+2})}{k^2+7k+12}, a_2=a_0, a_3=a_1\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 14

```
dsolve(diff(diff(y(x),x),x)-4*diff(y(x),x)*x+(4*x^2-2)*y(x) = 0,y(x),singsol=all)
```

$$y=\mathrm{e}^{x^2}(c_2x+c_1)$$

Mathematica DSolve solution

Solving time : 0.02 (sec)
Leaf size : 18

```
DSolve[{D[y[x],{x,2}]-4*x*D[y[x],x]+(4*x^2-2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x)\to e^{x^2}(c_2x+c_1)$$

2.1.290 Problem 293

Internal problem ID [9462]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 293
Date solved : Monday, January 27, 2025 at 06:03:05 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$(2x-3)\,y''-xy'+y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.353 (sec)

Writing the ode as

$$(2x-3)\,y''-xy'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x-3\\B&=-x\\C&=1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{x^2-8x+18}{4\,(2x-3)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=x^2-8x+18\\t&=4(2x-3)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x^2-8x+18}{4\,(2x-3)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.552: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x-3)^2$. There is a pole at $x = \frac{3}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16} + \frac{33}{64\left(x - \frac{3}{2}\right)^2} - \frac{5}{16\left(x - \frac{3}{2}\right)}$$

For the pole at $x = \frac{3}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{3}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{33}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{8} \\ \alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{8} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{4} - \frac{5}{8x} - \frac{11}{16x^2} - \frac{1}{32x^3} + \frac{245}{64x^4} + \frac{2591}{128x^5} + \frac{21117}{256x^6} + \frac{154743}{512x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{4} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{16}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 8x + 18}{16x^2 - 48x + 36} \\ &= Q + \frac{R}{16x^2 - 48x + 36} \\ &= \left(\frac{1}{16}\right) + \left(\frac{-5x + \frac{63}{4}}{16x^2 - 48x + 36}\right) \\ &= \frac{1}{16} + \frac{-5x + \frac{63}{4}}{16x^2 - 48x + 36} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -5. Dividing this by leading coefficient in t which is 16 gives $-\frac{5}{16}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{5}{16}\right) - (0) \\ &= -\frac{5}{16} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{4} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{16}}{\frac{1}{4}} - 0\right) &&= -\frac{5}{8} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{16}}{\frac{1}{4}} - 0\right) &&= \frac{5}{8}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 8x + 18}{4\left(2x - 3\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$\frac{3}{2}$	2	0	$\frac{11}{8}$	$-\frac{3}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{4}$	$-\frac{5}{8}$	$\frac{5}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{5}{8}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{5}{8} - \left(-\frac{3}{8}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{8\left(x - \frac{3}{2}\right)} + (-)\left(\frac{1}{4}\right) \\
&= -\frac{3}{8\left(x - \frac{3}{2}\right)} - \frac{1}{4} \\
&= -\frac{x}{4x - 6}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{3}{8\left(x-\frac{3}{2}\right)}-\frac{1}{4}\right)(1)+\left(\left(\frac{3}{8\left(x-\frac{3}{2}\right)^{2}}\right)+\left(-\frac{3}{8\left(x-\frac{3}{2}\right)}-\frac{1}{4}\right)^{2}-\left(\frac{x^{2}-8x+18}{4\left(2x-3\right)^{2}}\right)\right)=0$$

$$\frac{a_0}{2x-3}=0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0=0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x)=x$$

Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\,dx}\\
&= (x)\,\mathrm{e}^{\int\left(-\frac{3}{8\left(x-\frac{3}{2}\right)}-\frac{1}{4}\right)dx}\\
&= (x)\,\mathrm{e}^{-\frac{x}{4}-\frac{3\ln(2x-3)}{8}}\\
&= \frac{x\,\mathrm{e}^{-\frac{x}{4}}}{\left(2x-3\right)^{3/8}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{2x-3}\,dx}\\
&= z_1 e^{\frac{x}{4}+\frac{3\ln(2x-3)}{8}}\\
&= z_1\left(\left(2x-3\right)^{3/8}\mathrm{e}^{\frac{x}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1=x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int\frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1\int\frac{e^{\int -\frac{-x}{2x-3}\,dx}}{\left(y_1\right)^2}\,dx\\
&= y_1\int\frac{\mathrm{e}^{\frac{x}{2}+\frac{3\ln(2x-3)}{4}}}{\left(y_1\right)^2}\,dx\\
&= y_1\left(\int\frac{\mathrm{e}^{\frac{x}{2}+\frac{3\ln(2x-3)}{4}}}{x^2}dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1y_1+c_2y_2\\
&= c_1(x)+c_2\left(x\left(\int\frac{\mathrm{e}^{\frac{x}{2}+\frac{3\ln(2x-3)}{4}}}{x^2}dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(2x-3)\left(\frac{d^2}{dx^2}y(x)\right)-x\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{y(x)}{2x-3}+\frac{\left(\frac{d}{dx}y(x)\right)x}{2x-3}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)-\frac{\left(\frac{d}{dx}y(x)\right)x}{2x-3}+\frac{y(x)}{2x-3}=0$

☐ Check to see if $x_0=\frac{3}{2}$ is a regular singular point

◦ Define functions

$\left[P_2(x)=-\frac{x}{2x-3},P_3(x)=\frac{1}{2x-3}\right]$

◦ $\left(x-\frac{3}{2}\right)\cdot P_2(x)$ is analytic at $x=\frac{3}{2}$

$\left(\left(x-\frac{3}{2}\right)\cdot P_2(x)\right)\bigg|_{x=\frac{3}{2}}=-\frac{3}{4}$

◦ $\left(x-\frac{3}{2}\right)^2\cdot P_3(x)$ is analytic at $x=\frac{3}{2}$

$\left(\left(x-\frac{3}{2}\right)^2\cdot P_3(x)\right)\bigg|_{x=\frac{3}{2}}=0$

◦ $x=\frac{3}{2}$is a regular singular point

Check to see if $x_0=\frac{3}{2}$ is a regular singular point

$x_0=\frac{3}{2}$

- Multiply by denominators

$(2x-3)\left(\frac{d^2}{dx^2}y(x)\right)-x\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Change variables using $x=u+\frac{3}{2}$ so that the regular singular point is at $u=0$

$2u\left(\frac{d^2}{du^2}y(u)\right)+\left(-u-\frac{3}{2}\right)\left(\frac{d}{du}y(u)\right)+y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

◦ Shift index using $k->k+1-m$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

◦ Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-1}$

◦ Shift index using $k->k+1$

$u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$

Rewrite ODE with series expansions

$\frac{a_0r(-7+4r)u^{-1+r}}{2}+\left(\sum_{k=0}^{\infty}\left(\frac{a_{k+1}(k+1+r)(4k-3+4r)}{2}-a_k(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $\frac{r(-7+4r)}{2} = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{0, \frac{7}{4}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $2\left(k+r-\frac{3}{4}\right)(k+1+r)\, a_{k+1} - a_k(k+r-1) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{2a_k(k+r-1)}{(4k-3+4r)(k+1+r)}$
- Recursion relation for $r = 0$; series terminates at $k = 1$
  $a_{k+1} = \frac{2a_k(k-1)}{(4k-3)(k+1)}$
- Apply recursion relation for $k = 0$
  $a_1 = \frac{2a_0}{3}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second linea
  $y(u) = a_0 \cdot \left(1 + \frac{2u}{3}\right)$
- Revert the change of variables $u = x - \frac{3}{2}$
  $\left[y(x) = \frac{2a_0x}{3}\right]$
- Recursion relation for $r = \frac{7}{4}$
  $a_{k+1} = \frac{2a_k\left(k+\frac{3}{4}\right)}{(4k+4)\left(k+\frac{11}{4}\right)}$
- Solution for $r = \frac{7}{4}$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{7}{4}}, a_{k+1} = \frac{2a_k\left(k+\frac{3}{4}\right)}{(4k+4)\left(k+\frac{11}{4}\right)}\right]$
- Revert the change of variables $u = x - \frac{3}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k \left(x-\frac{3}{2}\right)^{k+\frac{7}{4}}, a_{k+1} = \frac{2a_k\left(k+\frac{3}{4}\right)}{(4k+4)\left(k+\frac{11}{4}\right)}\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \frac{2a_0x}{3} + \left(\sum_{k=0}^{\infty} b_k \left(x-\frac{3}{2}\right)^{k+\frac{7}{4}}\right), b_{k+1} = \frac{2b_k\left(k+\frac{3}{4}\right)}{(4k+4)\left(k+\frac{11}{4}\right)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
```

```
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returr
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.029 (sec)
Leaf size : 29

```
dsolve((2*x-3)*diff(diff(y(x),x),x)-diff(y(x),x)*x+y(x) = 0,y(x),singsol=all)
```

$$y = 2(2x-3)^{3/4}\left(x-\frac{3}{2}\right)c_1\,\mathrm{KummerM}\left(\frac{3}{4},\frac{11}{4},\frac{x}{2}-\frac{3}{4}\right)+c_2x$$

Mathematica DSolve solution

Solving time : 0.09 (sec)
Leaf size : 63

```
DSolve[{(2*x-3)*D[y[x],{x,2}]-x*D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to 2\ 2^{3/4}(2x-3)\left(c_2(2x-3)^{3/4}L_{-\frac{3}{4}}^{\frac{7}{4}}\left(\frac{x}{2}-\frac{3}{4}\right)+\frac{4\sqrt{2}c_1x}{2x-3}\right)$$

2.1.291 Problem 294

Internal problem ID [9463]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 294
Date solved : Monday, January 27, 2025 at 06:03:06 PM
CAS classification : [_Hermite]

Solve

$$y'' - xy' - 3y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.223 (sec)

Writing the ode as

$$y'' - xy' - 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 10}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 10 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} + \frac{5}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.554: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + \frac{5}{2x} - \frac{25}{4x^3} + \frac{125}{4x^5} - \frac{3125}{16x^7} + \frac{21875}{16x^9} - \frac{328125}{32x^{11}} + \frac{2578125}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 10}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} + \frac{5}{2}\right) + (0) \\ &= \frac{x^2}{4} + \frac{5}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{5}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{5}{2}\right) - (0) \\ &= \frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{5}{2}}{\frac{1}{2}} - 1\right) &= 2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{5}{2}}{\frac{1}{2}} - 1\right) &= -3 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} + \frac{5}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	2	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$, and since there are no poles, then

$$
\begin{aligned}
d &= \alpha_\infty^+ \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= (+)[\sqrt{r}]_\infty \\
&= 0 + \left(\frac{x}{2}\right) \\
&= \frac{x}{2} \\
&= \frac{x}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(2) + 2\left(\frac{x}{2}\right)(2x + a_1) + \left(\left(\frac{1}{2}\right) + \left(\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} + \frac{5}{2}\right)\right) &= 0 \\
-a_1 x - 2a_0 + 2 &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 1, a_1 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 + 1
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 + 1\right) e^{\int \frac{x}{2} dx} \\
&= \left(x^2 + 1\right) e^{\frac{x^2}{4}} \\
&= \left(x^2 + 1\right) e^{\frac{x^2}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{1} \, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(e^{\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{\frac{x^2}{2}} \left(x^2+1\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{\left(x^2+1\right)^2} dx \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\frac{x^2}{2}} \left(x^2+1\right)\right) + c_2 \left(\mathrm{e}^{\frac{x^2}{2}} \left(x^2+1\right) \left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{\left(x^2+1\right)^2} dx \right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2}y(x) - x\left(\frac{d}{dx}y(x)\right) - 3y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^k$
- □ Rewrite DE with series expansions
  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion
    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$
  - Convert $\frac{d^2}{dx^2}y(x)$ to series expansion
    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$
  - Shift index using $k- >k+2$
    $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions
  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k(k+3)\right) x^k = 0$
- Each term in the series must be 0, giving the recursion relation
  $\left(k^2+3k+2\right) a_{k+2} - a_k(k+3) = 0$

- Recursion relation that defines the series solution to the ODE
$$\left[y(x)=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=\frac{a_k(k+3)}{k^2+3k+2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.035 (sec)
Leaf size : 37

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x-3*y(x) = 0,y(x),singsol=all)
```

$$y=\left(x^2+1\right)\left(\sqrt{\pi}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right)c_1+c_2\right)\mathrm{e}^{\frac{x^2}{2}}+\sqrt{2}\,c_1 x$$

Mathematica DSolve solution

Solving time : 0.02 (sec)
Leaf size : 35

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-3*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to c_1\,\mathrm{HermiteH}\left(-3,\frac{x}{\sqrt{2}}\right)+c_2 e^{\frac{x^2}{2}}\left(x^2+1\right)$$

2.1.292 Problem 295

Internal problem ID [9464]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 295
Date solved : `Monday, January 27, 2025 at 06:03:06 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+1\right)y''-xy'+y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.235 (sec)

Writing the ode as

$$\left(x^2+1\right)y''-xy'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2+1\\B&=-x\\C&=1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{-x^2-6}{4\left(x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=-x^2-6\\t&=4\left(x^2+1\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-x^2-6}{4\left(x^2+1\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.556: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + 1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16 (x - i)^2} + \frac{5}{16 (x + i)^2} + \frac{7i}{16 (x - i)} - \frac{7i}{16 (x + i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 6}{4\left(x^2+1\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 - 6}{4\left(x^2+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-i$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{4\left(x-i\right)} - \frac{1}{4\left(x+i\right)} + (-)\left(0\right) \\
&= -\frac{1}{4\left(x-i\right)} - \frac{1}{4\left(x+i\right)} \\
&= -\frac{x}{2x^2+2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{4(x-i)} - \frac{1}{4(x+i)}\right)(1) + \left(\left(\frac{1}{4(x-i)^2} + \frac{1}{4(x+i)^2}\right) + \left(-\frac{1}{4(x-i)} - \frac{1}{4(x+i)}\right)^2 - \left(\frac{-}{4(\ldots}\right.\right.$$

$$\frac{(x^2+1\ldots}{(-x+i)^2(\ldots}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \,dx} \\
&= (x)\, e^{\int \left(-\frac{1}{4(x-i)} - \frac{1}{4(x+i)}\right) dx} \\
&= (x) \frac{1}{((-x+i)(x+i))^{1/4}} \\
&= \frac{x}{(-x^2-1)^{1/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{x^2+1} \,dx} \\
&= z_1 e^{\frac{\ln(x^2+1)}{4}} \\
&= z_1\left((x^2+1)^{1/4}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \left(\frac{1}{2} - \frac{i}{2}\right) x\sqrt{2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \,dx}}{y_1^2} \,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x^2+1} \,dx}}{(y_1)^2} \,dx \\
&= y_1 \int \frac{e^{\frac{\ln(x^2+1)}{2}}}{(y_1)^2} \,dx \\
&= y_1 \left(i\left(-\frac{(x^2+1)^{3/2}}{x} + x\sqrt{x^2+1} + \operatorname{arcsinh}(x)\right)\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\left(\frac{1}{2} - \frac{i}{2} \right) x\sqrt{2} \right) + c_2 \left(\left(\frac{1}{2} - \frac{i}{2} \right) x\sqrt{2} \left(i \left(-\frac{(x^2+1)^{3/2}}{x} + x\sqrt{x^2+1} + \operatorname{arcsinh}(x) \right) \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 23

```
dsolve((x^2+1)*diff(diff(y(x),x),x)-diff(y(x),x)*x+y(x) = 0,y(x),singsol=all)
```

$$
y = -\sqrt{x^2+1}\, c_2 + x(c_2 \operatorname{arcsinh}(x) + c_1)
$$

Mathematica DSolve solution

Solving time : 0.043 (sec)
Leaf size : 29

```
DSolve[{(1+x^2)*D[y[x],{x,2}]-x*D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$
y(x) \to c_2 x \operatorname{arcsinh}(x) - c_2\sqrt{x^2+1} + c_1 x
$$

2.1.293 Problem 296

Internal problem ID [9465]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 296
Date solved : Monday, January 27, 2025 at 06:03:07 PM
CAS classification : [_Hermite]

Solve

$$y'' - xy' + 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.241 (sec)

Writing the ode as

$$y'' - xy' + 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 10}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 10 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{5}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.557: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{5}{2x} - \frac{25}{4x^3} - \frac{125}{4x^5} - \frac{3125}{16x^7} - \frac{21875}{16x^9} - \frac{328125}{32x^{11}} - \frac{2578125}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$[\sqrt{r}]_\infty = \sum_{i=0}^{1} a_i x^i = \frac{x}{2} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 10}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} - \frac{5}{2}\right) + (0) \\ &= \frac{x^2}{4} - \frac{5}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{5}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{5}{2}\right) - (0) \\ &= -\frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) = -3 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) = 2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{5}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (2) + 2\left(-\frac{x}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{5}{2}\right)\right) &= 0 \\ a_1 x + 2a_0 + 2 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (x^2 - 1)\, e^{\int -\frac{x}{2} dx} \\ &= (x^2 - 1)\, e^{-\frac{x^2}{4}} \\ &= (x^2 - 1)\, e^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1\left(e^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^2 - 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{e^{\frac{x^2}{2}}}{(x^2-1)^2} dx \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (x^2 - 1) + c_2 \left(x^2 - 1 \left(\int \frac{e^{\frac{x^2}{2}}}{(x^2-1)^2} dx \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2}y(x) - x\left(\frac{d}{dx}y(x)\right) + 2y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions
  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion
    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$
  - Convert $\frac{d^2}{dx^2}y(x)$ to series expansion
    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$
  - Shift index using $k- >k+2$
    $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite DE with series expansions
  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_k(k-2)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation
  $(k^2 + 3k + 2)\, a_{k+2} - a_k(k-2) = 0$
- Recursion relation; series terminates at $k = 2$

$a_{k+2} = \frac{a_k(k-2)}{k^2+3k+2}$

- Apply recursion relation for $k = 0$

  $a_2 = -a_0$

- Terminating series solution of the ODE. Use reduction of order to find the second linearly indepe

  $y(x) = A_2x^2 + A_1x - a_0$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.027 (sec)
Leaf size : 42

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y = 2c_1 e^{\frac{x^2}{2}} x - (x-1)(x+1)\left(c_1\sqrt{\pi}\,\mathrm{erfi}\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2} - c_2\right)$$

Mathematica DSolve solution

Solving time : 0.203 (sec)
Leaf size : 43

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to (x^2-1)\left(c_2\int_1^x \frac{e^{\frac{K[1]^2}{2}}}{(K[1]^2-1)^2}dK[1] + c_1\right)$$

2.1.294 Problem 297

Internal problem ID [9466]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 297
Date solved : `Monday, January 27, 2025 at 06:03:07 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(-x^2+1\right) y''-y'+y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.667 (sec)

Writing the ode as

$$\left(-x^2+1\right) y''-y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=-x^2+1\\ B&=-1\\ C&=1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{4x^2-4x-3}{4\left(x^2-1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=4x^2-4x-3\\ t&=4\left(x^2-1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{4x^2-4x-3}{4\left(x^2-1\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.559: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16\,(x-1)^2} + \frac{5}{16\,(x+1)^2} - \frac{7}{16\,(x+1)} + \frac{7}{16\,(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{4x^2 - 4x - 3}{4\left(x^2 - 1\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 1$. Hence

$$\begin{aligned} E_\infty &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{2\} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
1	2	$\{1, 2, 3\}$
-1	2	$\{-1, 2, 5\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1,\ e_2 = -1,\ e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(2 - (1 + (-1))) \\ &= 1 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x - c} \\ &= \frac{1}{2}\left(\frac{1}{(x - (1))} + \frac{-1}{(x - (-1))}\right) \\ &= \frac{1}{2x - 2} - \frac{1}{2(x+1)} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 1$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 1$, then letting

$$p = x + a_0 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$\frac{4a_0 - 6}{(x+1)^2 (x-1)} = 0$$

And solving for p gives

$$p = x + \frac{3}{2}$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p}\\ &= \frac{1}{x+\frac{3}{2}} + \frac{1}{2x-2} - \frac{1}{2(x+1)}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{x+\frac{3}{2}} + \frac{1}{2x-2} - \frac{1}{2(x+1)}\right)w + \frac{-8x^3-4x^2+10x+7}{4(x^2-1)^2(2x+3)} = 0$$

Solving for ω gives

$$\omega = \frac{2\sqrt{5}\,\sqrt{(x-1)(x+1)}\,x + 2\sqrt{5}\,\sqrt{(x-1)(x+1)} + 2x^2+2x+1}{2(2x+3)(x-1)(x+1)}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega\,dx}\\ &= e^{\int \frac{2\sqrt{5}\,\sqrt{(x-1)(x+1)}\,x+2\sqrt{5}\,\sqrt{(x-1)(x+1)}+2x^2+2x+1}{2(2x+3)(x-1)(x+1)}dx}\\ &= \frac{\sqrt{2x+3}\,(x-1)^{1/4}\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}} 5^{1/4}}{(x+1)^{1/4}\sqrt{\frac{5\sqrt{x^2-1}+(2+3x)\sqrt{5}}{\sqrt{x^2-1}\sqrt{-\frac{(2x+3)^2}{x^2-1}}}}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-1}{-x^2+1}\,dx}\\ &= z_1 e^{\frac{\operatorname{arctanh}(x)}{2}}\\ &= z_1\left(\sqrt{\frac{x+1}{\sqrt{-x^2+1}}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{\frac{x+1}{\sqrt{-x^2+1}}}\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}}\sqrt{2x+3}\,(5x-5)^{1/4}}{\sqrt{\frac{i\left(3\sqrt{5}\,x+5\sqrt{x^2-1}+2\sqrt{5}\right)}{2x+3}}\,(x+1)^{1/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{-x^2+1}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{\operatorname{arctanh}(x)}}{(y_1)^2}\,dx\\ &= y_1\left(\int \frac{i\sqrt{x+1}\left(x+\sqrt{x^2-1}\right)^{-\sqrt{5}}\left(3\sqrt{5}\,x+5\sqrt{x^2-1}+2\sqrt{5}\right)}{(2x+3)^2\sqrt{5x-5}}dx\right)\end{aligned}$$

Therefore the solution is

$$y = c_1y_1 + c_2y_2$$

$$= c_1\left(\frac{\sqrt{\frac{x+1}{\sqrt{-x^2+1}}}\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}}\sqrt{2x+3}\,(5x-5)^{1/4}}{\sqrt{\frac{i\left(3\sqrt{5}\,x+5\sqrt{x^2-1}+2\sqrt{5}\right)}{2x+3}}\,(x+1)^{1/4}}\right)$$

$$+c_2\left(\frac{\sqrt{\frac{x+1}{\sqrt{-x^2+1}}}\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}}\sqrt{2x+3}\,(5x-5)^{1/4}}{\sqrt{\frac{i\left(3\sqrt{5}\,x+5\sqrt{x^2-1}+2\sqrt{5}\right)}{2x+3}}\,(x+1)^{1/4}}\left(\int \frac{i\sqrt{x+1}\left(x+\sqrt{x^2-1}\right)^{-\sqrt{5}}\left(3\sqrt{5}\,x+5\sqrt{x^2}\right.}{(2x+3)^2\sqrt{5x-5}}\right.\right.$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(-x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) - \frac{d}{dx}y(x) + y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{y(x)}{x^2-1} - \frac{\frac{d}{dx}y(x)}{x^2-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{\frac{d}{dx}y(x)}{x^2-1} - \frac{y(x)}{x^2-1} = 0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x^2-1}, P_3(x) = -\frac{1}{x^2-1}\right]$

  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1} = -\frac{1}{2}$

  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1} = 0$

  - $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(x^2-1)\left(\frac{d^2}{dx^2}y(x)\right) + \frac{d}{dx}y(x) - y(x) = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + \frac{d}{du}y(u) - y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $\frac{d}{du}y(u)$ to series expansion

    $\frac{d}{du}y(u) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1}$

  - Shift index using $k->k+1$

$\frac{d}{du}y(u) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k- > k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-a_0 r(-3+2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(2k-1+2r) + a_k(k^2+2kr+r^2-k-r-1)\right) u^{k+}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $-2(k+1+r)\left(k+r-\frac{1}{2}\right) a_{k+1} + \left(k^2+(2r-1)\, k+r^2-r-1\right) a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{(k^2+2kr+r^2-k-r-1)a_k}{(k+1+r)(2k-1+2r)}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{(k^2-k-1)a_k}{(k+1)(2k-1)}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{(k^2-k-1)a_k}{(k+1)(2k-1)}\right]$

- Revert the change of variables $u = x+1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+1} = \frac{(k^2-k-1)a_k}{(k+1)(2k-1)}\right]$

- Recursion relation for $r = \frac{3}{2}$

  $a_{k+1} = \frac{(k^2+2k-\frac{1}{4})a_k}{(k+\frac{5}{2})(2k+2)}$

- Solution for $r = \frac{3}{2}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{3}{2}}, a_{k+1} = \frac{(k^2+2k-\frac{1}{4})a_k}{(k+\frac{5}{2})(2k+2)}\right]$

- Revert the change of variables $u = x+1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^{k+\frac{3}{2}}, a_{k+1} = \frac{(k^2+2k-\frac{1}{4})a_k}{(k+\frac{5}{2})(2k+2)}\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k(x+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k(x+1)^{k+\frac{3}{2}}\right), a_{k+1} = \frac{(k^2-k-1)a_k}{(k+1)(2k-1)}, b_{k+1} = \frac{(k^2+2k-\frac{1}{4})b_k}{(k+\frac{5}{2})(2k+2)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.027 (sec)
Leaf size : 66

```
dsolve((-x^2+1)*diff(diff(y(x),x),x)-diff(y(x),x)+y(x) = 0,y(x),singsol=all)
```

$$y = c_1 \operatorname{hypergeom}\left(\left[\frac{\sqrt{5}}{2}-\frac{1}{2}, -\frac{1}{2}-\frac{\sqrt{5}}{2}\right], \left[-\frac{1}{2}\right], \frac{1}{2}+\frac{x}{2}\right) + 2c_2\sqrt{2+2x}\operatorname{hypergeom}\left(\left[1-\frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{2}+1\right], \left[\frac{5}{2}\right], \frac{1}{2}+\frac{x}{2}\right)(x+1)$$

Mathematica DSolve solution

Solving time : 6.1 (sec)
Leaf size : 210

```
DSolve[{(1-x^2)*D[y[x],{x,2}]-D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \left(\sqrt{x-1}-\sqrt{x+1}\right)^{-\frac{1}{2}-\frac{\sqrt{5}}{2}}\left(\sqrt{x-1}+\sqrt{x+1}\right)^{\frac{1}{2}\left(\sqrt{5}-1\right)}\left(\sqrt{x-1}-\sqrt{5}\sqrt{x+1}\right)\left(c_2\int_1^x -\frac{2e^{\operatorname{arctanh}(K[2])}\left(\sqrt{K[2]-1}-\sqrt{K[2]+1}\right)^{\sqrt{5}}\left(\sqrt{K[2]-1}+\sqrt{K[2]+1}\right)^{-\sqrt{5}}}{\left(\sqrt{K[2]-1}-\sqrt{5}\sqrt{K[2]+1}\right)^2}dK[2] + c_1\right)\exp\left(-\frac{1}{2}\int_1^x \frac{1}{K[1]^2-1}dK[1]-\frac{\operatorname{arctanh}(x)}{2}\right)$$

2.1.295 Problem 298

Internal problem ID [9467]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 298
Date solved : `Monday, January 27, 2025 at 06:03:09 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x(x+1)^2 y'' + \left(-x^2+1\right) y' + (x-1) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.161 (sec)

Writing the ode as

$$x(x+1)^2 y'' + \left(-x^2+1\right) y' + (x-1) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x(x+1)^2 \\ B &= -x^2+1 \\ C &= x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.561: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=-\frac{1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{2}-\left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^{-}}{x-c_1}\right)+(-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x}+(-)(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}
$$

Let

$$
p(x)=1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega \, dx} \\ &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x^2+1}{x(x+1)^2} \, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}+\ln(x+1)} \\ &= z_1 \left(\frac{x+1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x + 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+1}{x(x+1)^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\ln(x)+2\ln(x+1)}}{(y_1)^2} \, dx \\ &= y_1(\ln(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(x+1) + c_2(x + 1(\ln(x))) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x(x+1)^2 \left(\frac{d^2}{dx^2}y(x)\right) + (-x^2+1)\left(\frac{d}{dx}y(x)\right) + (x-1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(x-1)y(x)}{x(x+1)^2} + \frac{(x-1)\left(\frac{d}{dx}y(x)\right)}{x(x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) - \frac{(x-1)\left(\frac{d}{dx}y(x)\right)}{x(x+1)} + \frac{(x-1)y(x)}{x(x+1)^2} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{x-1}{x(x+1)}, P_3(x) = \frac{x-1}{x(x+1)^2}\right]$

  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1} = -2$

  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1} = 2$

  - $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $x(x+1)^2 \left(\frac{d^2}{dx^2}y(x)\right) - (x-1)(x+1)\left(\frac{d}{dx}y(x)\right) + (x-1)\,y(x) = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(u^3-u^2)\left(\frac{d^2}{du^2}y(u)\right) + (-u^2+2u)\left(\frac{d}{du}y(u)\right) + (u-2)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m\cdot y(u)$ to series expansion for $m = 0..1$

    $u^m\cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

  - Shift index using $k->k-m$

    $u^m\cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 2..3$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

$$u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$-a_{0}(-1+r)\,(-2+r)\,u^{r}+\left(\sum_{k=1}^{\infty}\left(-a_{k}(k+r-1)\,(k+r-2)+a_{k-1}(k+r-2)^{2}\right)u^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-(-1+r)\,(-2+r)=0$
- Values of r that satisfy the indicial equation

  $r\in\{1,2\}$
- Each term in the series must be 0, giving the recursion relation

  $-a_{k}(k+r-1)\,(k+r-2)+a_{k-1}(k+r-2)^{2}=0$
- Shift index using $k->k+1$

  $-a_{k+1}(k+r)\,(k+r-1)+a_{k}(k+r-1)^{2}=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_{k}(k+r-1)}{k+r}$
- Recursion relation for $r=1$

  $a_{k+1}=\frac{a_{k}k}{k+1}$
- Solution for $r=1$

  $\left[y(u)=\sum_{k=0}^{\infty}a_{k}u^{k+1},a_{k+1}=\frac{a_{k}k}{k+1}\right]$
- Revert the change of variables $u=x+1$

  $\left[y(x)=\sum_{k=0}^{\infty}a_{k}(x+1)^{k+1},a_{k+1}=\frac{a_{k}k}{k+1}\right]$
- Recursion relation for $r=2$

  $a_{k+1}=\frac{a_{k}(k+1)}{k+2}$
- Solution for $r=2$

  $\left[y(u)=\sum_{k=0}^{\infty}a_{k}u^{k+2},a_{k+1}=\frac{a_{k}(k+1)}{k+2}\right]$
- Revert the change of variables $u=x+1$

  $\left[y(x)=\sum_{k=0}^{\infty}a_{k}(x+1)^{k+2},a_{k+1}=\frac{a_{k}(k+1)}{k+2}\right]$
- Combine solutions and rename parameters

  $\left[y(x)=\left(\sum_{k=0}^{\infty}a_{k}(x+1)^{k+1}\right)+\left(\sum_{k=0}^{\infty}b_{k}(x+1)^{k+2}\right),a_{k+1}=\frac{a_{k}k}{k+1},b_{k+1}=\frac{b_{k}(k+1)}{k+2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 14

```
dsolve(x*(x+1)^2*diff(diff(y(x),x),x)+(-x^2+1)*diff(y(x),x)+(x-1)*y(x) = 0,y(x),singsol=
```

$$y = (x + 1) (c_2 \ln (x) + c_1)$$

Mathematica DSolve solution

Solving time : 0.25 (sec)
Leaf size : 45

```
DSolve[{x*(x+1)^2*D[y[x],{x,2}]+(1-x^2)*D[y[x],x]+(x-1)*y[x]==0,{}},y[x],x,IncludeSingularSolu
```

$$y(x) \to \sqrt{x}(c_2 \log(x) + c_1) \exp \left(-\frac{1}{2} \int_1^x \left(\frac{1}{K[1]} - \frac{2}{K[1] + 1}\right) dK[1]\right)$$

2.1.296 Problem 299

Internal problem ID [9468]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 299
Date solved : `Monday, January 27, 2025 at 06:03:09 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$2xy'' - y' + 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.201 (sec)

Writing the ode as

$$2xy'' - y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x \\ B &= -1 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5 - 16x}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 5 - 16x \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5 - 16x}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.563: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{x} + \frac{5}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{-1,2,5\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = -1, \; e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(1-(-1)) \\ &= 1 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{-1}{(x-(0))}\right) \\ &= -\frac{1}{2x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 1$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 1$, then letting

$$p = x + a_0 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$\frac{1-4a_0}{x^2} = 0$$

And solving for p gives

$$p = x + \frac{1}{4}$$

Now that $p(x)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{x+\frac{1}{4}} - \frac{1}{2x} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{x+\frac{1}{4}} - \frac{1}{2x}\right)w + \frac{64x^2 - 12x + 1}{64x^3 + 16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{16x\sqrt{-x} + 4x - 1}{4\,(4x+1)\,x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{16x\sqrt{-x}+4x-1}{4(4x+1)x} dx} \\
&= \frac{\left(2\sqrt{-x}-1\right) \mathrm{e}^{2\sqrt{-x}}}{\left(-x\right)^{1/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{2x} \, dx} \\
&= z_1 e^{\frac{\ln(x)}{4}} \\
&= z_1 \left(x^{1/4}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^{1/4}\left(2\sqrt{-x}-1\right) \mathrm{e}^{2\sqrt{-x}}}{\left(-x\right)^{1/4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{2x} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{\ln(x)}{2}}}{\left(y_1\right)^2} \, dx \\
&= y_1 \left(\frac{\mathrm{e}^{-4\sqrt{-x}}}{8} + \frac{\mathrm{e}^{-4\sqrt{-x}}}{8\sqrt{-x}-4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{1/4}\left(2\sqrt{-x}-1\right) \mathrm{e}^{2\sqrt{-x}}}{\left(-x\right)^{1/4}}\right) + c_2 \left(\frac{x^{1/4}\left(2\sqrt{-x}-1\right) \mathrm{e}^{2\sqrt{-x}}}{\left(-x\right)^{1/4}} \left(\frac{\mathrm{e}^{-4\sqrt{-x}}}{8} + \frac{\mathrm{e}^{-4\sqrt{-x}}}{8\sqrt{-x}-4}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2\left(\frac{d^2}{dx^2}y(x)\right)x-\frac{d}{dx}y(x)+2y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{y(x)}{x}+\frac{\frac{d}{dx}y(x)}{2x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)-\frac{\frac{d}{dx}y(x)}{2x}+\frac{y(x)}{x}=0$

□ Check to see if $x_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(x)=-\frac{1}{2x},P_3(x)=\frac{1}{x}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=-\frac{1}{2}$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=0$

  - $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2\left(\frac{d^2}{dx^2}y(x)\right)x-\frac{d}{dx}y(x)+2y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $\frac{d}{dx}y(x)$ to series expansion

    $\frac{d}{dx}y(x)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1}$

  - Shift index using $k->k+1$

    $\frac{d}{dx}y(x)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,x^{k+r}$

  - Convert $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-1}$

  - Shift index using $k->k+1$

    $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-3+2r)\,x^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(2k-1+2r)+2a_k\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+2r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{0,\frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation
  $2(k+1+r)\left(k+r-\frac{1}{2}\right)a_{k+1}+2a_k=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=-\frac{2a_k}{(k+1+r)(2k-1+2r)}$
- Recursion relation for $r=0$
  $a_{k+1}=-\frac{2a_k}{(k+1)(2k-1)}$
- Solution for $r=0$
  $\left[y(x)=\sum_{k=0}^{\infty}a_k x^k, a_{k+1}=-\frac{2a_k}{(k+1)(2k-1)}\right]$
- Recursion relation for $r=\frac{3}{2}$
  $a_{k+1}=-\frac{2a_k}{\left(k+\frac{5}{2}\right)(2k+2)}$
- Solution for $r=\frac{3}{2}$
  $\left[y(x)=\sum_{k=0}^{\infty}a_k x^{k+\frac{3}{2}}, a_{k+1}=-\frac{2a_k}{\left(k+\frac{5}{2}\right)(2k+2)}\right]$
- Combine solutions and rename parameters
  $\left[y(x)=\left(\sum_{k=0}^{\infty}a_k x^k\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{3}{2}}\right), a_{k+1}=-\frac{2a_k}{(k+1)(2k-1)}, b_{k+1}=-\frac{2b_k}{\left(k+\frac{5}{2}\right)(2k+2)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.017 (sec)
Leaf size : 36

```
dsolve(2*x*diff(diff(y(x),x),x)-diff(y(x),x)+2*y(x) = 0,y(x),singsol=all)
```

$$y=\left(2\sqrt{x}\,c_1+c_2\right)\cos\left(2\sqrt{x}\right)-\sin\left(2\sqrt{x}\right)\left(-2c_2\sqrt{x}+c_1\right)$$

Mathematica DSolve solution

Solving time : 0.209 (sec)
Leaf size : 74

```
DSolve[{2*x*D[y[x],{x,2}]-D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to e^{2i\sqrt{x}}\left(2\sqrt{x}+i\right)\left(c_2\int_1^x\frac{e^{-4i\sqrt{K[1]}}\sqrt{K[1]}}{\left(2\sqrt{K[1]}+i\right)^2}dK[1]+c_1\right)$$

2.1.297 Problem 300

Internal problem ID [9469]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 300
Date solved : `Monday, January 27, 2025 at 06:03:10 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + xy' - 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.220 (sec)

Writing the ode as

$$xy'' + xy' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x+8}{4x} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x + 8 \\ t &= 4x \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x+8}{4x}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.565: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x$. There is a pole at $x = 0$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{2}{x} - \frac{4}{x^2} + \frac{16}{x^3} - \frac{80}{x^4} + \frac{448}{x^5} - \frac{2688}{x^6} + \frac{16896}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x+8}{4x} \\ &= Q + \frac{R}{4x} \\ &= \left(\frac{1}{4}\right) + \left(\frac{2}{x}\right) \\ &= \frac{1}{4} + \frac{2}{x} \end{aligned}$$

Since the degree of t is 1, then we see that the coefficient of the term 1 in the remainder R is 8. Dividing this by leading coefficient in t which is 4 gives 2. Now b can be found.

$$\begin{aligned} b &= (2) - (0) \\ &= 2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{2}{\frac{1}{2}} - 0\right) &&= 2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{2}{\frac{1}{2}} - 0\right) &&= -2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x+8}{4x}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	2	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= 2 - (1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{x} + \left(\frac{1}{2}\right) \\ &= \frac{1}{x} + \frac{1}{2} \\ &= \frac{1}{x} + \frac{1}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x} + \frac{1}{2}\right)(1) + \left(\left(-\frac{1}{x^2}\right) + \left(\frac{1}{x} + \frac{1}{2}\right)^2 - \left(\frac{x+8}{4x}\right) \right) = 0$$

$$\frac{2-a_0}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (2+x)\, e^{\int \left(\frac{1}{x}+\frac{1}{2}\right) dx} \\ &= (2+x)\, e^{\frac{x}{2}+\ln(x)} \\ &= (2+x)\, x\, e^{\frac{x}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (2+x)\,x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{\mathrm{e}^{-x}}{4x} + \frac{\mathrm{Ei}_1(x)}{2} + \frac{\mathrm{e}^{-x}}{-8-4x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1((2+x)\,x) + c_2\left((2+x)\,x\left(-\frac{\mathrm{e}^{-x}}{4x} + \frac{\mathrm{Ei}_1(x)}{2} + \frac{\mathrm{e}^{-x}}{-8-4x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 28

```
dsolve(x*diff(diff(y(x),x),x)+diff(y(x),x)*x-2*y(x) = 0,y(x),singsol=all)
```

$$y = -\frac{(x+1)\, c_2 \mathrm{e}^{-x}}{2} + x(x+2)\left(c_1 + \frac{\mathrm{Ei}_1(x)\, c_2}{2}\right)$$

Mathematica DSolve solution

Solving time : 0.223 (sec)
Leaf size : 40

```
DSolve[{x*D[y[x],{x,2}]+x*D[y[x],x]-2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to x(x+2)\left(c_2 \int_1^x \frac{e^{-K[1]}}{K[1]^2(K[1]+2)^2} dK[1] + c_1\right)$$

2.1.298 Problem 301

Internal problem ID [9470]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 301
Date solved : `Monday, January 27, 2025 at 06:03:10 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x(x-1)^2 y''-2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.151 (sec)

Writing the ode as

$$x(x-1)^2 y''-2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=x(x-1)^2\\ B&=0\\ C&=-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{2}{x\left(x-1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=2\\ t&=x(x-1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{2}{x\left(x-1\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.566: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 0 \\ &= 3 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x(x-1)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 3 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 3 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{x-1} + \frac{2}{(x-1)^2} + \frac{2}{x}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $3>2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= 0\\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x(x-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1
1	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
3	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=0$ then

$$\begin{aligned}d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right)\\ &= 0-(0)\\ &= 0\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (+)[\sqrt{r}]_\infty\\ &= \frac{1}{x} - \frac{1}{x-1} + (0)\\ &= \frac{1}{x} - \frac{1}{x-1}\\ &= -\frac{1}{x(x-1)}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x} - \frac{1}{x-1}\right)(0) + \left(\left(-\frac{1}{x^2} + \frac{1}{(x-1)^2}\right) + \left(\frac{1}{x} - \frac{1}{x-1}\right)^2 - \left(\frac{2}{x(x-1)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{1}{x} - \frac{1}{x-1}\right) dx} \\ &= \frac{x}{x-1} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{x}{x-1} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x}{x-1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= \frac{x}{x-1} \int \frac{1}{\frac{x^2}{(x-1)^2}} \, dx \\ &= \frac{x}{x-1} \left(x - \frac{1}{x} - 2\ln(x) \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{x}{x-1} \right) + c_2 \left(\frac{x}{x-1} \left(x - \frac{1}{x} - 2\ln(x) \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x(x-1)^2\left(\frac{d^2}{dx^2}y(x)\right)-2y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{2y(x)}{x(x-1)^2}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)-\frac{2y(x)}{x(x-1)^2}=0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=0, P_3(x)=-\frac{2}{(x-1)^2 x}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=0$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x(x-1)^2\left(\frac{d^2}{dx^2}y(x)\right)-2y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=1..3$

  $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)x^{k+r-2+m}$

  ◦ Shift index using $k->k+2-m$

  $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)x^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-1+r)x^{-1+r}+(a_1(1+r)r-2a_0(r^2-r+1))x^r+\left(\sum_{k=1}^{\infty}(a_{k+1}(k+1+r)(k+r)-2a_k(k^2\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{0,1\}$

- Each term must be 0

  $a_1(1+r)r-2a_0(r^2-r+1)=0$

- Each term in the series must be 0, giving the recursion relation

  $(-2a_k+a_{k-1}+a_{k+1})k^2+((-4a_k+2a_{k-1}+2a_{k+1})r+2a_k-3a_{k-1}+a_{k+1})k+(-2a_k+a_{k-1}-$

- Shift index using $k->k+1$

  $(-2a_{k+1}+a_k+a_{k+2})(k+1)^2+((-4a_{k+1}+2a_k+2a_{k+2})r+2a_{k+1}-3a_k+a_{k+2})(k+1)+(-$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2kr a_k - 4kr a_{k+1} + r^2 a_k - 2r^2 a_{k+1} - k a_k - 2k a_{k+1} - r a_k - 2r a_{k+1} - 2a_{k+1}}{k^2 + 2kr + r^2 + 3k + 3r + 2}$
- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - k a_k - 2k a_{k+1} - 2a_{k+1}}{k^2 + 3k + 2}$
- Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - k a_k - 2k a_{k+1} - 2a_{k+1}}{k^2 + 3k + 2}, -2a_0 = 0\right]$
- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + k a_k - 6k a_{k+1} - 6a_{k+1}}{k^2 + 5k + 6}$
- Solution for $r = 1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + k a_k - 6k a_{k+1} - 6a_{k+1}}{k^2 + 5k + 6}, 2a_1 - 2a_0 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - k a_k - 2k a_{k+1} - 2a_{k+1}}{k^2 + 3k + 2}, -2a_0 = 0, b_{k+2} = -\frac{k^2 \ldots}{\ldots}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 27

```
dsolve(x*(x-1)^2*diff(diff(y(x),x),x)-2*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{2\ln(x)\, c_2 x - c_2 x^2 + c_1 x + c_2}{x - 1}$$

Mathematica DSolve solution

Solving time : `0.068` (sec)
Leaf size : 62

```
DSolve[{x*(x-1)^2*D[y[x],{x,2}]-2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \exp\left(\int_1^x \frac{1}{K[1] - K[1]^2} dK[1]\right)\left(c_2 \int_1^x \exp\left(-2\int_1^{K[2]} \frac{1}{K[1] - K[1]^2} dK[1]\right) dK[2] + c_1\right)$$

2.1.299 Problem 302

Internal problem ID [9471]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 302
Date solved : `Monday, January 27, 2025 at 06:03:11 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - 2xy' + x^2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.154 (sec)

Writing the ode as

$$y'' - 2xy' + x^2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2x \\ C &= x^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.568: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{1}\, dx} \\ &= z_1 e^{\frac{x^2}{2}} \\ &= z_1 \left(e^{\frac{x^2}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{\frac{x^2}{2}} \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x^2}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{\frac{x^2}{2}}\cos(x)\right) + c_2\left(\mathrm{e}^{\frac{x^2}{2}}\cos(x)(\tan(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) - 2x\left(\frac{d}{dx}y(x)\right) + x^2 y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^2 \cdot y(x)$ to series expansion

    $x^2 \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}$

  ◦ Shift index using $k->k-2$

    $x^2 \cdot y(x) = \sum_{k=2}^{\infty} a_{k-2} x^k$

  ◦ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k->k+2$

    $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 + (6a_3 - 2a_1)x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)(k+1) - 2a_k k + a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 = 0, 6a_3 - 2a_1 = 0]$

- Solve for the dependent coefficient(s)

  $\left\{a_2 = 0, a_3 = \frac{a_1}{3}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} - 2a_k k + a_{k-2} = 0$

- Shift index using $k->k+2$

  $$\left((k+2)^2+3k+8\right)a_{k+4}-2a_{k+2}(k+2)+a_k=0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y(x)=\sum_{k=0}^{\infty}a_kx^k, a_{k+4}=\frac{2ka_{k+2}-a_k+4a_{k+2}}{k^2+7k+12}, a_2=0, a_3=\frac{a_1}{3}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.014` (sec)
Leaf size : 20

```
dsolve(diff(diff(y(x),x),x)-2*diff(y(x),x)*x+x^2*y(x) = 0,y(x),singsol=all)
```

$$y=e^{\frac{x^2}{2}}(\cos(x)c_1+\sin(x)c_2)$$

Mathematica DSolve solution

Solving time : `0.033` (sec)
Leaf size : 39

```
DSolve[{D[y[x],{x,2}]-2*x*D[y[x],x]+x^2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to\frac{1}{2}e^{\frac{1}{2}x(x-2i)}\left(2c_1-ic_2e^{2ix}\right)$$

2.1.300 Problem 303

Internal problem ID [9472]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 303
Date solved : `Monday, January 27, 2025 at 06:03:11 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x\left(-x^2+2\right)y''-\left(x^2+4x+2\right)\left((1-x)\,y'+y\right)=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.509 (sec)

Writing the ode as

$$\left(-x^3+2x\right)y''+\left(x^3+3x^2-2x-2\right)y'+\left(-x^2-4x-2\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^3+2x \\ B &= x^3+3x^2-2x-2 \\ C &= -x^2-4x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{x^6+2x^5-5x^4-16x^3+24x^2+24x+12}{4\left(x^3-2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^6+2x^5-5x^4-16x^3+24x^2+24x+12 \\ t &= 4\left(x^3-2x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x^6+2x^5-5x^4-16x^3+24x^2+24x+12}{4\left(x^3-2x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.570: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 6 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 - 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \sqrt{2}$ of order 2. There is a pole at $x = -\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{2x} + \frac{3}{4x^2} + \frac{3}{4\left(x - \sqrt{2}\right)^2} + \frac{3}{4\left(x + \sqrt{2}\right)^2} + \frac{-\frac{5\sqrt{2}}{8} - \frac{1}{2}}{x - \sqrt{2}} + \frac{\frac{5\sqrt{2}}{8} - \frac{1}{2}}{x + \sqrt{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = \sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x - \sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x=-\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x+\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{2}\\
\alpha_c^- &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty)=0$ then

$$
v=\frac{-O_r(\infty)}{2}=\frac{0}{2}=0
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0\leq i\leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $x^v=x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r}\approx \frac{1}{2}+\frac{1}{2x}-\frac{1}{2x^2}-\frac{3}{2x^3}+\frac{21}{4x^4}-\frac{43}{4x^5}+\frac{135}{4x^6}-\frac{147}{4x^7}+\dots \tag{9}
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a=\frac{1}{2}
$$

From Eq. (9) the sum up to $v=0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i\\
&= \frac{1}{2}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1}=x^{-1}=\frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2=\frac{1}{4}
$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v=0$ or not. Since $v=0$ then starting from $r=\frac{s}{t}$ and doing long division in the form

$$
r=Q+\frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t}\\
&= \frac{x^6+2x^5-5x^4-16x^3+24x^2+24x+12}{4x^6-16x^4+16x^2}\\
&= Q+\frac{R}{4x^6-16x^4+16x^2}\\
&= \left(\frac{1}{4}\right)+\left(\frac{2x^5-x^4-16x^3+20x^2+24x+12}{4x^6-16x^4+16x^2}\right)\\
&= \frac{1}{4}+\frac{2x^5-x^4-16x^3+20x^2+24x+12}{4x^6-16x^4+16x^2}
\end{aligned}
$$

Since the degree of t is 6, then we see that the coefficient of the term x^5 in the remainder R is 2. Dividing this by leading coefficient in t which is 4 gives $\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{2}\right) - (0) \\ &= \frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^6 + 2x^5 - 5x^4 - 16x^3 + 24x^2 + 24x + 12}{4\left(x^3 - 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$\sqrt{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-\sqrt{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x - c_3}\right) + (+)[\sqrt{r}]_\infty \\ &= \frac{3}{2x} - \frac{1}{2\left(x - \sqrt{2}\right)} - \frac{1}{2\left(x + \sqrt{2}\right)} + \left(\frac{1}{2}\right) \\ &= \frac{3}{2x} - \frac{1}{2\left(x - \sqrt{2}\right)} - \frac{1}{2\left(x + \sqrt{2}\right)} + \frac{1}{2} \\ &= \frac{x^3 + x^2 - 2x - 6}{2x^3 - 4x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{2x} - \frac{1}{2\left(x-\sqrt{2}\right)} - \frac{1}{2\left(x+\sqrt{2}\right)} + \frac{1}{2}\right)(0) + \left(\left(-\frac{3}{2x^2} + \frac{1}{2\left(x-\sqrt{2}\right)^2} + \frac{1}{2\left(x+\sqrt{2}\right)^2}\right) + \left(\frac{3}{2x}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(\frac{3}{2x} - \frac{1}{2\left(x-\sqrt{2}\right)} - \frac{1}{2\left(x+\sqrt{2}\right)} + \frac{1}{2}\right) dx} \\
&= \frac{x^{3/2} \mathrm{e}^{\frac{x}{2}}}{\sqrt{x-\sqrt{2}}\, \sqrt{x+\sqrt{2}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^3+3x^2-2x-2}{-x^3+2x}\, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x)}{2} + \frac{\ln\left(x^2-2\right)}{2}} \\
&= z_1 \left(\sqrt{x}\, \sqrt{x^2-2}\, \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^2 \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^3+3x^2-2x-2}{-x^3+2x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x+\ln(x)+\ln\left(x^2-2\right)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{(x-1)\, \mathrm{e}^{x+\ln(x)+\ln\left(x^2-2\right)} \mathrm{e}^{-2x}}{x^3\left(x^2-2\right)}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^2 \mathrm{e}^x\right) + c_2\left(x^2 \mathrm{e}^x \left(-\frac{(x-1)\, \mathrm{e}^{x+\ln(x)+\ln\left(x^2-2\right)} \mathrm{e}^{-2x}}{x^3\left(x^2-2\right)}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x(-x^2+2)\left(\frac{d^2}{dx^2}y(x)\right)-(x^2+4x+2)\left((1-x)\left(\frac{d}{dx}y(x)\right)+y(x)\right)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{(x^2+4x+2)y(x)}{x(x^2-2)}+\frac{(x^2+4x+2)(x-1)\left(\frac{d}{dx}y(x)\right)}{x(x^2-2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x)-\frac{(x^2+4x+2)(x-1)\left(\frac{d}{dx}y(x)\right)}{x(x^2-2)}+\frac{(x^2+4x+2)y(x)}{x(x^2-2)}=0$

- □ Check to see if x_0 is a regular singular point
  - ○ Define functions

    $\left[P_2(x)=-\frac{(x-1)(x^2+4x+2)}{x(x^2-2)}, P_3(x)=\frac{x^2+4x+2}{x(x^2-2)}\right]$
  - ○ $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\bigg|_{x=0}=-1$
  - ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\bigg|_{x=0}=0$
  - ○ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$
- Multiply by denominators

$x(x^2-2)\left(\frac{d^2}{dx^2}y(x)\right)-(x-1)(x^2+4x+2)\left(\frac{d}{dx}y(x)\right)+(x^2+4x+2)y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - ○ Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

    $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_k x^{k+r+m}$
  - ○ Shift index using $k->k-m$

    $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$
  - ○ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..3$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$
  - ○ Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - ○ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=1..3$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,x^{k+r-2+m}$
  - ○ Shift index using $k->k+2-m$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$-2a_0r(-2+r)\,x^{-1+r}+(-2a_1(1+r)\,(-1+r)+2a_0(1+r))\,x^r+\big(-2a_2(2+r)\,r+2a_1(2+r)+$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $$-2r(-2+r)=0$$
- Values of r that satisfy the indicial equation
  $$r\in\{0,2\}$$
- The coefficients of each power of x must be 0
  $$\big[-2a_1(1+r)\,(-1+r)+2a_0(1+r)=0,-2a_2(2+r)\,r+2a_1(2+r)+a_0(-2+r)^2=0\big]$$
- Solve for the dependent coefficient(s)
  $$\left\{a_1=\frac{a_0}{-1+r},a_2=\frac{a_0(r^2-5r+10)}{2(r^2+r-2)}\right\}$$
- Each term in the series must be 0, giving the recursion relation
  $$a_{k-1}(k-3+r)^2-2a_{k+1}(k+r+1)\,(k+r-1)+(2a_k-a_{k-2})\,k+(2a_k-a_{k-2})\,r+2a_k+3a_{k-}$$
- Shift index using $k->k+2$
  $$a_{k+1}(k+r-1)^2-2a_{k+3}(k+3+r)\,(k+r+1)+(2a_{k+2}-a_k)\,(k+2)+(2a_{k+2}-a_k)\,r+2a_{k+2}$$
- Recursion relation that defines series solution to ODE
  $$a_{k+3}=\frac{k^2a_{k+1}+2kra_{k+1}+r^2a_{k+1}-a_kk-2ka_{k+1}+2ka_{k+2}-a_kr-2ra_{k+1}+2ra_{k+2}+a_k+a_{k+1}+6a_{k+2}}{2(k+3+r)(k+r+1)}$$
- Recursion relation for $r=0$
  $$a_{k+3}=\frac{k^2a_{k+1}-a_kk-2ka_{k+1}+2ka_{k+2}+a_k+a_{k+1}+6a_{k+2}}{2(k+3)(k+1)}$$
- Solution for $r=0$
  $$\left[y(x)=\sum_{k=0}^{\infty}a_kx^k,a_{k+3}=\frac{k^2a_{k+1}-a_kk-2ka_{k+1}+2ka_{k+2}+a_k+a_{k+1}+6a_{k+2}}{2(k+3)(k+1)},a_1=-a_0,a_2=-\frac{5a_0}{2}\right]$$
- Recursion relation for $r=2$
  $$a_{k+3}=\frac{k^2a_{k+1}-a_kk+2ka_{k+1}+2ka_{k+2}-a_k+a_{k+1}+10a_{k+2}}{2(k+5)(k+3)}$$
- Solution for $r=2$
  $$\left[y(x)=\sum_{k=0}^{\infty}a_kx^{k+2},a_{k+3}=\frac{k^2a_{k+1}-a_kk+2ka_{k+1}+2ka_{k+2}-a_k+a_{k+1}+10a_{k+2}}{2(k+5)(k+3)},a_1=a_0,a_2=\frac{a_0}{2}\right]$$
- Combine solutions and rename parameters
  $$\left[y(x)=\left(\sum_{k=0}^{\infty}a_kx^k\right)+\left(\sum_{k=0}^{\infty}b_kx^{k+2}\right),a_{k+3}=\frac{k^2a_{k+1}-ka_k-2ka_{k+1}+2ka_{k+2}+a_k+a_{k+1}+6a_{k+2}}{2(k+3)(k+1)},a_1=-a_0,a\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 17

```
dsolve(x*(-x^2+2)*diff(diff(y(x),x),x)-(x^2+4*x+2)*((1-x)*diff(y(x),x)+y(x)) = 0,y(x),si
```

$$y = c_1(x-1) + c_2 \mathrm{e}^x x^2$$

Mathematica DSolve solution

Solving time : 0.338 (sec)
Leaf size : 126

```
DSolve[{x*(2-x^2)*D[y[x],{x,2}]-(x^2+4*x+2)*((1-x)*D[y[x],x]+y[x])==0,{}},y[x],x,IncludeSingula
```

$$y(x) \to \exp\left(\int_1^x \left(-\frac{K[1]}{K[1]^2-2} + \frac{1}{2} + \frac{3}{2K[1]}\right) dK[1] - \frac{1}{2}\int_1^x \left(-\frac{2K[2]}{K[2]^2-2} - 1 - \frac{1}{K[2]}\right) dK[2]\right) \left(c_2 \int_1^x \exp\left(-2\int_1^{K[3]} \frac{K[1]^3+K[1]^2-2K[1]-6}{2K[1]\left(K[1]^2-2\right)} dK[1]\right) dK[3] + c_1\right)$$

2.1.301 Problem 304

Internal problem ID [9473]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 304
Date solved : `Monday, January 27, 2025 at 06:03:12 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+x)\,y''-(1+2x)\,(xy'-y)=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.202 (sec)

Writing the ode as

$$x^2(1+x)\,y''+\left(-2x^2-x\right)y'+(1+2x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=x^2(1+x)\\ B&=-2x^2-x\\ C&=1+2x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-4x-1}{4\left(x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=-4x-1\\ t&=4\left(x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-4x-1}{4\left(x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.572: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 3 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 3 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{2 + 2x} - \frac{1}{2x} + \frac{3}{4\,(1 + x)^2} - \frac{1}{4x^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $3 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-4x-1}{4\left(x^2+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
3	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 0 - (0) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(1+x\right)} + \frac{1}{2x} + (0) \\
&= -\frac{1}{2\left(1+x\right)} + \frac{1}{2x} \\
&= \frac{1}{2x\left(1+x\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0)+2\left(-\frac{1}{2(1+x)}+\frac{1}{2x}\right)(0)+\left(\left(\frac{1}{2(1+x)^2}-\frac{1}{2x^2}\right)+\left(-\frac{1}{2(1+x)}+\frac{1}{2x}\right)^2-\left(\frac{-4x-1}{4(x^2+x)^2}\right)\right)&=0\\
0&=0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx}\\
&= \mathrm{e}^{\int \left(-\frac{1}{2(1+x)}+\frac{1}{2x}\right)dx}\\
&= \frac{\sqrt{x}}{\sqrt{1+x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}\\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x^2-x}{x^2(1+x)}\, dx}\\
&= z_1 e^{\frac{\ln(x(1+x))}{2}}\\
&= z_1\left(\sqrt{x(1+x)}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\sqrt{x(1+x)}\,\sqrt{x}}{\sqrt{1+x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^2-x}{x^2(1+x)}\, dx}}{(y_1)^2}\, dx\\
&= y_1 \int \frac{e^{\ln(x(1+x))}}{(y_1)^2}\, dx\\
&= y_1(x+\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2\\
&= c_1\left(\frac{\sqrt{x(1+x)}\,\sqrt{x}}{\sqrt{1+x}}\right)+c_2\left(\frac{\sqrt{x(1+x)}\,\sqrt{x}}{\sqrt{1+x}}(x+\ln(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)-(2x+1)\left(x\left(\frac{d}{dx}y(x)\right)-y(x)\right)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{(2x+1)y(x)}{x^2(x+1)}+\frac{(2x+1)\left(\frac{d}{dx}y(x)\right)}{x(x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)-\frac{(2x+1)\left(\frac{d}{dx}y(x)\right)}{x(x+1)}+\frac{(2x+1)y(x)}{x^2(x+1)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=-\frac{2x+1}{x(x+1)},P_3(x)=\frac{2x+1}{x^2(x+1)}\right]$

- ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

$\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=-1$

- ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

$\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

- ◦ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1$

- Multiply by denominators

$x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)-x(2x+1)\left(\frac{d}{dx}y(x)\right)+(2x+1)y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(u^3-2u^2+u)\left(\frac{d^2}{du^2}y(u)\right)+(-2u^2+3u-1)\left(\frac{d}{du}y(u)\right)+(2u-1)y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

$u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

- ◦ Shift index using $k->k-m$

$u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ◦ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

- ◦ Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-2+r)\,u^{-1+r} + \left(a_1(1+r)(-1+r) - a_0(2r^2-5r+1)\right)u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 2\}$
- Each term must be 0
  $a_1(1+r)(-1+r) - a_0(2r^2-5r+1) = 0$
- Each term in the series must be 0, giving the recursion relation
  $(-2a_k + a_{k-1} + a_{k+1})\,k^2 + ((-4a_k + 2a_{k-1} + 2a_{k+1})\,r + 5a_k - 5a_{k-1})\,k + (-2a_k + a_{k-1} + a_{k+1})\,r^2$
- Shift index using $k- >k+1$
  $(-2a_{k+1} + a_k + a_{k+2})\,(k+1)^2 + ((-4a_{k+1} + 2a_k + 2a_{k+2})\,r + 5a_{k+1} - 5a_k)\,(k+1) + (-2a_{k+1} + a_k$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + 2kra_k - 4kra_{k+1} + r^2a_k - 2r^2a_{k+1} - 3ka_k + ka_{k+1} - 3ra_k + ra_{k+1} + 2a_k + 2a_{k+1}}{k^2+2kr+r^2+2k+2r}$
- Recursion relation for $r = 0$
  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} - 3ka_k + ka_{k+1} + 2a_k + 2a_{k+1}}{k^2+2k}$
- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$
  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} - 3ka_k + ka_{k+1} + 2a_k + 2a_{k+1}}{k^2+2k}$
- Recursion relation for $r = 2$
  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + ka_k - 7ka_{k+1} - 4a_{k+1}}{k^2+6k+8}$
- Solution for $r = 2$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + ka_k - 7ka_{k+1} - 4a_{k+1}}{k^2+6k+8}, 3a_1 + a_0 = 0\right]$
- Revert the change of variables $u = x+1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x+1)^{k+2}, a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + ka_k - 7ka_{k+1} - 4a_{k+1}}{k^2+6k+8}, 3a_1 + a_0 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 15

```
dsolve(x^2*(x+1)*diff(diff(y(x),x),x)-(2*x+1)*(diff(y(x),x)*x-y(x)) = 0,y(x),singsol=a
```

$$y = x(c_2 \ln(x) + c_2 x + c_1)$$

Mathematica DSolve solution

Solving time : 0.183 (sec)
Leaf size : 132

```
DSolve[{x^2*(1+x)*D[y[x],{x,2}]-(1+2*x)*(x*D[y[x],x]+y[x])==0,{}},y[x],x,IncludeSingularSolu
```

$$y(x) \to c_2 x^{1+\sqrt{2}}\,\text{Hypergeometric2F1}\left(-\frac{1}{2}+\sqrt{2}-\frac{\sqrt{17}}{2}, -\frac{1}{2}+\sqrt{2}+\frac{\sqrt{17}}{2}, 1+2\sqrt{2}, -x\right) + c_1 x^{1-\sqrt{2}}\,\text{Hypergeometric2F1}\left(\frac{1}{2}\left(-1-2\sqrt{2}-\sqrt{17}\right), \frac{1}{2}\left(-1-2\sqrt{2}+\sqrt{17}\right), 1 - 2\sqrt{2}, -x\right)$$

2.1.302 Problem 305

Internal problem ID [9474]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 305
Date solved : `Monday, January 27, 2025 at 06:03:13 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2(2-x)\,x^2y''-(4-x)\,xy'+(3-x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.161 (sec)

Writing the ode as

$$\left(-2x^3+4x^2\right)y''+\left(x^2-4x\right)y'+(3-x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-2x^3+4x^2\\B&=x^2-4x\\C&=3-x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-3}{16\left(-2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=-3\\t&=16(-2+x)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{3}{16\left(-2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.574: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(-2+x)^2$. There is a pole at $x = 2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16\,(-2+x)^2}$$

For the pole at $x = 2$ let b be the coefficient of $\frac{1}{(-2+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{3}{16\,(-2+x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{4}\\ \alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{4}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=-\frac{3}{16\,(-2+x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
2	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=\frac{1}{4}$ then

$$\begin{aligned}d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right)\\ &= \frac{1}{4}-\left(\frac{1}{4}\right)\\ &= 0\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^-}{x-c_1}\right)+(-)[\sqrt{r}]_\infty\\ &= \frac{1}{-8+4x}+(-)\,(0)\\ &= \frac{1}{-8+4x}\\ &= \frac{1}{-8+4x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{1}{-8+4x}\right)(0)+\left(\left(-\frac{1}{4\left(-2+x\right)^{2}}\right)+\left(\frac{1}{-8+4x}\right)^{2}-\left(-\frac{3}{16\left(-2+x\right)^{2}}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}z_1(x)&=pe^{\int \omega\, dx}\\&=\mathrm{e}^{\int \frac{1}{-8+4x}dx}\\&=\left(-2+x\right)^{1/4}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1&=z_1e^{\int -\frac{1}{2}\frac{B}{A}\, dx}\\&=z_1e^{-\int \frac{1}{2}\frac{x^2-4x}{-2x^3+4x^2}\, dx}\\&=z_1e^{\frac{\ln(x)}{2}-\frac{\ln(-2+x)}{4}}\\&=z_1\left(\frac{\sqrt{x}}{\left(-2+x\right)^{1/4}}\right)\end{aligned}$$

Which simplifies to

$$y_1=\sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}y_2&=y_1\int \frac{\mathrm{e}^{\int -\frac{x^2-4x}{-2x^3+4x^2}\, dx}}{\left(y_1\right)^2}\, dx\\&=y_1\int \frac{\mathrm{e}^{\ln(x)-\frac{\ln(-2+x)}{2}}}{\left(y_1\right)^2}\, dx\\&=y_1\left(\frac{2\,\mathrm{e}^{\ln(x)-\frac{\ln(-2+x)}{2}}\left(-2+x\right)}{x}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y&=c_1y_1+c_2y_2\\&=c_1\left(\sqrt{x}\right)+c_2\left(\sqrt{x}\left(\frac{2\,\mathrm{e}^{\ln(x)-\frac{\ln(-2+x)}{2}}\left(-2+x\right)}{x}\right)\right)\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$2(-x+2)\,x^2\left(\frac{d^2}{dx^2}y(x)\right)-x(4-x)\left(\frac{d}{dx}y(x)\right)+(-x+3)\,y(x)=0$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
$\frac{d^2}{dx^2}y(x)=-\frac{(x-3)y(x)}{2x^2(x-2)}+\frac{(-4+x)\left(\frac{d}{dx}y(x)\right)}{2x(x-2)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
$\frac{d^2}{dx^2}y(x)-\frac{(-4+x)\left(\frac{d}{dx}y(x)\right)}{2x(x-2)}+\frac{(x-3)y(x)}{2x^2(x-2)}=0$

☐ Check to see if x_0 is a regular singular point
  - Define functions
  $\left[P_2(x)=-\frac{-4+x}{2x(x-2)},P_3(x)=\frac{x-3}{2x^2(x-2)}\right]$
  - $x\cdot P_2(x)$ is analytic at $x=0$
  $\left.(x\cdot P_2(x))\right|_{x=0}=-1$
  - $x^2\cdot P_3(x)$ is analytic at $x=0$
  $\left.(x^2\cdot P_3(x))\right|_{x=0}=\frac{3}{4}$
  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point
  $x_0=0$
- Multiply by denominators
$2x^2(x-2)\left(\frac{d^2}{dx^2}y(x)\right)-x(-4+x)\left(\frac{d}{dx}y(x)\right)+(x-3)\,y(x)=0$
- Assume series solution for $y(x)$
$y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

☐ Rewrite ODE with series expansions
  - Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$
  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$
  - Shift index using $k->k-m$
  $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$
  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$
  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k->k+1-m$
  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..3$
  $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$
  - Shift index using $k->k+2-m$
  $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$-a_0(-1+2r)(-3+2r)x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(2k+2r-1)(2k+2r-3) + a_{k-1}(2k+2r-3)(k-2\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-(-1+2r)(-3+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $-4\left(\left(-\frac{k}{2}-\frac{r}{2}+1\right)a_{k-1} + \left(k+r-\frac{1}{2}\right)a_k\right)\left(k+r-\frac{3}{2}\right) = 0$
- Shift index using $k->k+1$
  $-4\left(\left(-\frac{k}{2}+\frac{1}{2}-\frac{r}{2}\right)a_k + \left(k+\frac{1}{2}+r\right)a_{k+1}\right)\left(k+r-\frac{1}{2}\right) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{(k+r-1)a_k}{2k+1+2r}$
- Recursion relation for $r = \frac{1}{2}$
  $a_{k+1} = \frac{\left(k-\frac{1}{2}\right)a_k}{2k+2}$
- Solution for $r = \frac{1}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{\left(k-\frac{1}{2}\right)a_k}{2k+2}\right]$
- Recursion relation for $r = \frac{3}{2}$
  $a_{k+1} = \frac{\left(k+\frac{1}{2}\right)a_k}{2k+4}$
- Solution for $r = \frac{3}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1} = \frac{\left(k+\frac{1}{2}\right)a_k}{2k+4}\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+1} = \frac{\left(k-\frac{1}{2}\right)a_k}{2k+2}, b_{k+1} = \frac{\left(k+\frac{1}{2}\right)b_k}{2k+4}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 19

```
dsolve(2*(-x+2)*x^2*diff(diff(y(x),x),x)-x*(-x+4)*diff(y(x),x)+(-x+3)*y(x) = 0,y(x),si
```

$$y = \sqrt{x}\, c_1 + c_2 \sqrt{x\,(x-2)}$$

Mathematica DSolve solution

Solving time : 0.299 (sec)
Leaf size : 57

```
DSolve[{2*(2-x)*x^2*D[y[x],{x,2}]-(4-x)*x*D[y[x],x]+(3-x)*y[x]==0,{}},y[x],x,IncludeSingularSol
```

$$y(x) \to \sqrt[4]{x-2}\left(2c_2\sqrt{x-2}+c_1\right)\exp\left(-\frac{1}{2}\int_1^x\left(\frac{1}{2(K[1]-2)}-\frac{1}{K[1]}\right)dK[1]\right)$$

2.1.303 Problem 306

Internal problem ID [9475]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 306
Date solved : `Monday, January 27, 2025 at 06:03:14 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(1-x)\,x^2y''+(5x-4)\,xy'+(6-9x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.187 (sec)

Writing the ode as

$$\left(-x^3+x^2\right)y''+\left(5x^2-4x\right)y'+(6-9x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-x^3+x^2\\ B&=5x^2-4x\\ C&=6-9x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-x+4}{4x\left(-1+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=-x+4\\ t&=4x(-1+x)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-x+4}{4x\left(-1+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.576: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 3 - 1 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x(-1+x)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= 1 \\
\alpha_c^- &= 1
\end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{-1+x} + \frac{1}{x} + \frac{3}{4\,(-1+x)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x+4}{4x\left(-1+x\right)^2}$$

Since the gcd$(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x+4}{4x\left(-1+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{x} - \frac{1}{2\left(-1+x\right)} + (-)\left(0\right) \\
&= \frac{1}{x} - \frac{1}{2\left(-1+x\right)} \\
&= \frac{-2+x}{2\left(-1+x\right)x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x} - \frac{1}{2(-1+x)}\right)(0) + \left(\left(-\frac{1}{x^2} + \frac{1}{2(-1+x)^2}\right) + \left(\frac{1}{x} - \frac{1}{2(-1+x)}\right)^2 - \left(\frac{-x+4}{4x(-1+x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{x} - \frac{1}{2(-1+x)}\right) dx} \\
&= \frac{x}{\sqrt{-1+x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{5x^2-4x}{-x^3+x^2} \, dx} \\
&= z_1 e^{2\ln(x) + \frac{\ln(-1+x)}{2}} \\
&= z_1 \left(x^2 \sqrt{-1+x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^3$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5x^2-4x}{-x^3+x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{4\ln(x) + \ln(-1+x)}}{(y_1)^2} \, dx \\
&= y_1 \left(\ln(x) + \frac{1}{x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^3\right) + c_2 \left(x^3 \left(\ln(x) + \frac{1}{x}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 18

```
dsolve((1-x)*x^2*diff(diff(y(x),x),x)+(5*x-4)*x*diff(y(x),x)+(6-9*x)*y(x) = 0,y(x),sin
```

$$y = x^2(\ln(x)\, c_2 x + c_1 x + c_2)$$

Mathematica DSolve solution

Solving time : 0.247 (sec)
Leaf size : 98

```
DSolve[{(1-x)*x^2*D[y[x],{x,2}]+(5*x-4)*x*D[y[x],x]+(6-9*x)*y[x]==0,{}},y[x],x,IncludeSingul
```

$$y(x) \to \exp\left(\int_1^x \left(\frac{1}{K[1]} + \frac{1}{2-2K[1]}\right) dK[1] - \frac{1}{2}\int_1^x \left(\frac{1}{1-K[2]} - \frac{4}{K[2]}\right) dK[2]\right)\left(c_2 \int_1^x \exp\left(-2\int_1^{K[3]} \frac{K[1]-2}{2(K[1]-1)K[1]} dK[1]\right) dK[3] + c_1\right)$$

2.1.304 Problem 307

Internal problem ID [9476]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 307
Date solved : `Monday, January 27, 2025 at 06:03:14 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + (4x^2 + 1)\,y' + 4x(x^2 + 1)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.161 (sec)

Writing the ode as

$$xy'' + (4x^2 + 1)\,y' + (4x^3 + 4x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 4x^2 + 1 \\ C &= 4x^3 + 4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.577: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=-\frac{1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2}-\left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^-}{x-c_1}\right)+(-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x}+(-)(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}
$$

Let

$$
p(x)=1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x^2+1}{x}\, dx} \\ &= z_1 e^{-x^2 - \frac{\ln(x)}{2}} \\ &= z_1\left(\frac{e^{-x^2}}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{-x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x^2+1}{x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-2x^2-\ln(x)}}{(y_1)^2}\, dx \\ &= y_1(\ln(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(e^{-x^2}\right) + c_2\left(e^{-x^2}(\ln(x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dx^2}y(x)\right)x+(4x^2+1)\left(\frac{d}{dx}y(x)\right)+4x(x^2+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=(-4x^2-4)\,y(x)-\frac{(4x^2+1)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{(4x^2+1)\left(\frac{d}{dx}y(x)\right)}{x}+(4x^2+4)\,y(x)=0$

- □ Check to see if $x_0=0$ is a regular singular point
  - Define functions

    $\left[P_2(x)=\frac{4x^2+1}{x}, P_3(x)=4x^2+4\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=1$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

  - $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $\left(\frac{d^2}{dx^2}y(x)\right)x+(4x^2+1)\left(\frac{d}{dx}y(x)\right)+4x(x^2+1)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x^m\cdot y(x)$ to series expansion for $m=1..3$

    $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..2$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-1}$

  - Shift index using $k->k+1$

    $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$a_0 r^2 x^{-1+r} + a_1(1+r)^2 x^r + \left(a_2(2+r)^2 + 4a_0(1+r)\right) x^{1+r} + \left(a_3(3+r)^2 + 4a_1(2+r)\right) x^{2+r} + \Big($$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 0$
- The coefficients of each power of x must be 0
  $\left[a_1(1+r)^2 = 0, a_2(2+r)^2 + 4a_0(1+r) = 0, a_3(3+r)^2 + 4a_1(2+r) = 0\right]$
- Solve for the dependent coefficient(s)
  $\left\{a_1 = 0, a_2 = -\frac{4a_0(1+r)}{r^2+4r+4}, a_3 = 0\right\}$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1)^2 + 4a_{k-1}k + 4a_{k-3} = 0$
- Shift index using $k- > k+3$
  $a_{k+4}(k+4)^2 + 4a_{k+2}(k+3) + 4a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+4} = -\frac{4(ka_{k+2}+a_k+3a_{k+2})}{(k+4)^2}$
- Recursion relation for $r = 0$
  $a_{k+4} = -\frac{4(ka_{k+2}+a_k+3a_{k+2})}{(k+4)^2}$
- Solution for $r = 0$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{4(ka_{k+2}+a_k+3a_{k+2})}{(k+4)^2}, a_1 = 0, a_2 = -a_0, a_3 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 17

```
dsolve(x*diff(diff(y(x),x),x)+(4*x^2+1)*diff(y(x),x)+4*x*(x^2+1)*y(x) = 0,y(x),singsol
```

$$y = e^{-x^2}(c_2 \ln(x) + c_1)$$

Mathematica DSolve solution

Solving time : 0.033 (sec)
Leaf size : 21

```
DSolve[{x*D[y[x],{x,2}]+(4*x^2+1)*D[y[x],x]+4*x*(x^2+1)*y[x]==0,{}},y[x],x,IncludeSingularSolu
```

$$y(x) \to e^{-x^2}(c_2 \log(x) + c_1)$$

2.1.305 Problem 309

Internal problem ID [9477]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 309
Date solved : `Monday, January 27, 2025 at 06:03:15 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - 2xy' + 8y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.227 (sec)

Writing the ode as

$$y'' - 2xy' + 8y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2x \\ C &= 8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = (x^2 - 9)\, z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.579: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x - \frac{9}{2x} - \frac{81}{8x^3} - \frac{729}{16x^5} - \frac{32805}{128x^7} - \frac{413343}{256x^9} - \frac{11160261}{1024x^{11}} - \frac{157837977}{2048x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 9}{1} \\ &= Q + \frac{R}{1} \\ &= (x^2 - 9) + (0) \\ &= x^2 - 9 \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is -9. Now b can be found.

$$\begin{aligned} b &= (-9) - (0) \\ &= -9 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= x \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-9}{1} - 1\right) &&= -5 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-9}{1} - 1\right) &&= 4 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = x^2 - 9$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	x	-5	4

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 4$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^{-} \\ &= 4 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)(x) \\ &= -x \\ &= -x\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(12x^2 + 6xa_3 + 2a_2) + 2(-x)\left(4x^3 + 3x^2a_3 + 2xa_2 + a_1\right) + \left((-1) + (-x)^2 - (x^2 - 9)\right) &= 0 \\ 2a_3x^3 + 4(3 + a_2)\, x^2 + 6(a_1 + a_3)\, x + 8a_0 + 2a_2 &= 0\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{3}{4}, a_1 = 0, a_2 = -3, a_3 = 0\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 - 3x^2 + \frac{3}{4}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= pe^{\int \omega\, dx} \\ &= \left(x^4 - 3x^2 + \frac{3}{4}\right) \mathrm{e}^{\int -x dx} \\ &= \left(x^4 - 3x^2 + \frac{3}{4}\right) \mathrm{e}^{-\frac{x^2}{2}} \\ &= \frac{(4x^4 - 12x^2 + 3)\, \mathrm{e}^{-\frac{x^2}{2}}}{4}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{1}\, dx} \\ &= z_1 e^{\frac{x^2}{2}} \\ &= z_1 \left(\mathrm{e}^{\frac{x^2}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = x^4 - 3x^2 + \frac{3}{4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{x^2}}{(y_1)^2}\,dx \\ &= y_1 \left(\int \frac{e^{x^2}}{\left(x^4 - 3x^2 + \frac{3}{4}\right)^2} dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(x^4 - 3x^2 + \frac{3}{4}\right) + c_2 \left(x^4 - 3x^2 + \frac{3}{4}\left(\int \frac{e^{x^2}}{\left(x^4 - 3x^2 + \frac{3}{4}\right)^2} dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special functio
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.028 (sec)
Leaf size : 44

```
dsolve(diff(diff(y(x),x),x)-2*diff(y(x),x)*x+8*y(x) = 0,y(x),singsol=all)
```

$$y = 2c_1 \left(2x^3 - 5x\right) e^{x^2} - 4\left(\operatorname{erfi}(x)\sqrt{\pi}\, c_1 - c_2\right)\left(x^4 - 3x^2 + \frac{3}{4}\right)$$

Mathematica DSolve solution

Solving time : `1.176` (sec)
Leaf size : 58

```
DSolve[{D[y[x],{x,2}]-2*x*D[y[x],x]+8*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{4}(4x^4 - 12x^2 + 3)\left(c_2 \int_1^x \frac{16e^{K[1]^2}}{(4K[1]^4 - 12K[1]^2 + 3)^2} dK[1] + c_1\right)$$

2.1.306 Problem 310

Internal problem ID [9478]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 310
Date solved : `Monday, January 27, 2025 at 06:03:15 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-x^2+1\right) y''-2xy'+12y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.276 (sec)

Writing the ode as

$$\left(-x^2+1\right) y''-2xy'+12y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+1 \\ B &= -2x \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{12x^2-13}{\left(x^2-1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 12x^2-13 \\ t &= \left(x^2-1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{12x^2-13}{\left(x^2-1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.580: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{25}{4(x-1)} - \frac{1}{4(x-1)^2} - \frac{1}{4(x+1)^2} - \frac{25}{4(x+1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{12x^2 - 13}{(x^2 - 1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 12$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 4 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -3
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{12x^2 - 13}{(x^2 - 1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	4	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 4$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= 4 - (1) \\
&= 3
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x-2} + \frac{1}{2x+2} + (0) \\
&= \frac{1}{2x-2} + \frac{1}{2x+2} \\
&= \frac{x}{x^2 - 1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 3$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(6x + 2a_2) + 2\left(\frac{1}{2x-2} + \frac{1}{2x+2}\right)(3x^2 + 2xa_2 + a_1) + \left(\left(-\frac{1}{2(x-1)^2} - \frac{1}{2(x+1)^2}\right) + \left(\frac{1}{2x-2} + \frac{1}{2x+}\right.\right.$$

$$\frac{-6a_2x^2 + (-10a_1 -}{x^2 -}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = 0, a_1 = -\frac{3}{5}, a_2 = 0\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^3 - \frac{3}{5}x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\,dx} \\
&= \left(x^3 - \frac{3}{5}x\right) e^{\int \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right)dx} \\
&= \left(x^3 - \frac{3}{5}x\right) \sqrt{(x-1)(x+1)} \\
&= \frac{(5x^3 - 3x)\sqrt{x^2-1}}{5}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{-x^2+1}\,dx} \\
&= z_1 e^{-\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x-1}\,\sqrt{x+1}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(5x^3 - 3x)\sqrt{x^2-1}}{5\sqrt{x-1}\,\sqrt{x+1}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{-x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x-1) - \ln(x+1)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{25\ln(x+1)}{8} + \frac{125x}{36\left(x^2 - \frac{3}{5}\right)} + \frac{25\ln(x-1)}{8} + \frac{25}{9x}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{(5x^3-3x)\sqrt{x^2-1}}{5\sqrt{x-1}\,\sqrt{x+1}}\right) \\
&\quad + c_2\left(\frac{(5x^3-3x)\sqrt{x^2-1}}{5\sqrt{x-1}\,\sqrt{x+1}}\left(-\frac{25\ln(x+1)}{8} + \frac{125x}{36\left(x^2-\frac{3}{5}\right)} + \frac{25\ln(x-1)}{8} + \frac{25}{9x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(-x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) - 2x\left(\frac{d}{dx}y(x)\right) + 12y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{12y(x)}{x^2-1} - \frac{2\left(\frac{d}{dx}y(x)\right)x}{x^2-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{2\left(\frac{d}{dx}y(x)\right)x}{x^2-1} - \frac{12y(x)}{x^2-1} = 0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{2x}{x^2-1}, P_3(x) = -\frac{12}{x^2-1}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1} = 1$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(x^2-1)\left(\frac{d^2}{dx^2}y(x)\right) + 2x\left(\frac{d}{dx}y(x)\right) - 12y(x) = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (2u-2)\left(\frac{d}{du}y(u)\right) - 12y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ∘ Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$
- ∘ Shift index using $k- > k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r^2u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)^2 + a_k(k+r+4)(k+r-3)\right) u^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 0$
- Each term in the series must be 0, giving the recursion relation

  $-2a_{k+1}(k+1)^2 + a_k(k+4)(k-3) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+4)(k-3)}{2(k+1)^2}$
- Recursion relation for $r = 0$; series terminates at $k = 3$

  $a_{k+1} = \frac{a_k(k+4)(k-3)}{2(k+1)^2}$
- Apply recursion relation for $k = 0$

  $a_1 = -6a_0$
- Apply recursion relation for $k = 1$

  $a_2 = -\frac{5a_1}{4}$
- Express in terms of a_0

  $a_2 = \frac{15a_0}{2}$
- Apply recursion relation for $k = 2$

  $a_3 = -\frac{a_2}{3}$
- Express in terms of a_0

  $a_3 = -\frac{5a_0}{2}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second linea

  $y(u) = a_0 \cdot \left(1 - 6u + \frac{15}{2}u^2 - \frac{5}{2}u^3\right)$
- Revert the change of variables $u = x + 1$

  $\left[y(x) = a_0\left(\frac{3}{2}x - \frac{5}{2}x^3\right)\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 55

```
dsolve((-x^2+1)*diff(diff(y(x),x),x)-2*diff(y(x),x)*x+12*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_2(5x^3-3x)\ln(x-1)}{24} + \frac{(-5x^3+3x)\, c_2 \ln(x+1)}{24} - \frac{5c_1 x^3}{3} + \frac{5c_2 x^2}{12} + c_1 x - \frac{c_2}{9}$$

Mathematica DSolve solution

Solving time : 0.03 (sec)
Leaf size : 59

```
DSolve[{(1-x^2)*D[y[x],{x,2}]-2*x*D[y[x],x]+12*y[x]==0,{}},y[x],x,IncludeSingularSolutions->
```

$$y(x) \to \frac{1}{2} c_1 x \left(5x^2-3\right) + c_2 \left(-\frac{5x^2}{2} - \frac{1}{4}\left(5x^2-3\right) x(\log(1-x) - \log(x+1)) + \frac{2}{3}\right)$$

2.1.307 Problem 311

Internal problem ID [9479]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 311
Date solved : Monday, January 27, 2025 at 06:03:16 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$x(x+2)\,y''+2(x+1)\,y'-2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.220 (sec)

Writing the ode as

$$\left(x^2+2x\right)y''+(2x+2)\,y'-2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2+2x\\B&=2x+2\\C&=-2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{2x^2+4x-1}{\left(x^2+2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=2x^2+4x-1\\t&=\left(x^2+2x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{2x^2+4x-1}{\left(x^2+2x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.582: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(x+2)^2} - \frac{1}{4x^2} + \frac{5}{4x} - \frac{5}{4(x+2)}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^2+4x-1}{(x^2+2x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2x^2+4x-1}{(x^2+2x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{1}{2}$	$\frac{1}{2}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x+4} + \frac{1}{2x} + (0) \\
&= \frac{1}{2x+4} + \frac{1}{2x} \\
&= \frac{x+1}{x(x+2)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x+4} + \frac{1}{2x}\right)(1) + \left(\left(-\frac{1}{2(x+2)^2} - \frac{1}{2x^2}\right) + \left(\frac{1}{2x+4} + \frac{1}{2x}\right)^2 - \left(\frac{2x^2+4x-1}{(x^2+2x)^2}\right)\right) = 0$$

$$\frac{2-2a_0}{x(x+2)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x+1)\, e^{\int \left(\frac{1}{2x+4} + \frac{1}{2x}\right) dx} \\ &= (x+1)\sqrt{x(x+2)} \\ &= (x+1)\sqrt{x(x+2)} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x+2}{x^2+2x} \, dx} \\ &= z_1 e^{-\frac{\ln(x(x+2))}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x(x+2)}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x + 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x+2}{x^2+2x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\ln(x(x+2))}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{1}{x+1} + \frac{\ln(x)}{2} - \frac{\ln(x+2)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(x+1) + c_2\left(x + 1\left(\frac{1}{x+1} + \frac{\ln(x)}{2} - \frac{\ln(x+2)}{2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x(x+2)\left(\frac{d^2}{dx^2}y(x)\right)+2(x+1)\left(\frac{d}{dx}y(x)\right)-2y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{2(x+1)\left(\frac{d}{dx}y(x)\right)}{x(x+2)}+\frac{2y(x)}{x(x+2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x)-\frac{2y(x)}{x(x+2)}+\frac{2(x+1)\left(\frac{d}{dx}y(x)\right)}{x(x+2)}=0$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x)=\frac{2(x+1)}{x(x+2)},P_3(x)=-\frac{2}{x(x+2)}\right]$
  - $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

    $\left.((x+2)\cdot P_2(x))\right|_{x=-2}=1$
  - $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

    $\left.((x+2)^2\cdot P_3(x))\right|_{x=-2}=0$
  - $x=-2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-2$
- Multiply by denominators

$x(x+2)\left(\frac{d^2}{dx^2}y(x)\right)+(2x+2)\left(\frac{d}{dx}y(x)\right)-2y(x)=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

$(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(2u-2)\left(\frac{d}{du}y(u)\right)-2y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$
  - Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r^2u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)^2+a_k(k+r+2)(k+r-1)\right)u^{k+r}\right)=0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

$-2r^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 0$

- Each term in the series must be 0, giving the recursion relation

  $-2a_{k+1}(k+1)^2 + a_k(k+2)(k-1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$

- Recursion relation for $r = 0$; series terminates at $k = 1$

  $a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$

- Apply recursion relation for $k = 0$

  $a_1 = -a_0$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot (-u+1)$

- Revert the change of variables $u = x + 2$

  $[y(x) = a_0(-x-1)]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 28

```
dsolve(x*(x+2)*diff(diff(y(x),x),x)+2*(x+1)*diff(y(x),x)-2*y(x) = 0,y(x),singsol=all)
```

$$y = -\frac{(x+1)\, c_2 \ln(x+2)}{2} + \frac{c_2(x+1)\ln(x)}{2} + c_1 x + c_1 + c_2$$

Mathematica DSolve solution

Solving time : 0.024 (sec)
Leaf size : 37

```
DSolve[{x*(x+2)*D[y[x],{x,2}]+2*(x+1)*D[y[x],x]-2*y[x]==0,{}},y[x],x,IncludeSingularSolution
```

$$y(x) \to c_1(x+1) - \frac{1}{2}c_2((x+1)\log(-x) - (x+1)\log(x+2) + 2)$$

2.1.308 Problem 313

Internal problem ID [9480]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 313
Date solved : Monday, January 27, 2025 at 06:03:17 PM
CAS classification :
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, '_with_symmetry_[0,F(x)]']]

Solve

$$x(x+2)\,y''+(x+1)\,y'-4y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.208 (sec)

Writing the ode as

$$\left(x^2+2x\right)y''+(x+1)\,y'-4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2+2x\\B&=x+1\\C&=-4\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{15x^2+30x-3}{4\left(x^2+2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=15x^2+30x-3\\t&=4\left(x^2+2x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15x^2+30x-3}{4\left(x^2+2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.584: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4-2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+2x)^2$. There is a pole at $x=0$ of order 2. There is a pole at $x=-2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1,2,4,6,12]$$

Attempting to find a solution using case $n=1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{33}{16\,(x+2)} - \frac{3}{16x^2} - \frac{3}{16\,(x+2)^2} + \frac{33}{16x}$$

For the pole at $x=-2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{15x^2 + 30x - 3}{4\left(x^2 + 2x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{15x^2 + 30x - 3}{4\left(x^2 + 2x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{3}{4}$	$\frac{1}{4}$
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\
&= \frac{5}{2} - \left(\frac{3}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{3}{4(x+2)} + \frac{3}{4x} + (0) \\
&= \frac{3}{4(x+2)} + \frac{3}{4x} \\
&= \frac{\frac{3x}{2}+\frac{3}{2}}{x(x+2)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{3}{4(x+2)} + \frac{3}{4x}\right)(1) + \left(\left(-\frac{3}{4(x+2)^2} - \frac{3}{4x^2}\right) + \left(\frac{3}{4(x+2)} + \frac{3}{4x}\right)^2 - \left(\frac{15x^2+30x-3}{4(x^2+2x)^2}\right)\right) = \frac{3-3a_0}{x(x+2)} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 1\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + 1
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+1)\, e^{\int \left(\frac{3}{4(x+2)} + \frac{3}{4x}\right) dx} \\
&= (x+1)\,(x(x+2))^{3/4} \\
&= (x+1)\,(x(x+2))^{3/4}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x+1}{x^2+2x} \, dx} \\
&= z_1 e^{-\frac{\ln(x(x+2))}{4}} \\
&= z_1 \left(\frac{1}{(x(x+2))^{1/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \sqrt{x(x+2)}\,(x+1)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x+1}{x^2+2x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x(x+2))}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{2x^2+4x+1}{\sqrt{x\,(x+2)}\,(x+1)}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\sqrt{x\,(x+2)}\,(x+1)\right) + c_2 \left(\sqrt{x\,(x+2)}\,(x+1)\left(-\frac{2x^2+4x+1}{\sqrt{x\,(x+2)}\,(x+1)}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x(x+2)\left(\frac{d^2}{dx^2}y(x)\right) + (x+1)\left(\frac{d}{dx}y(x)\right) - 4y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{4y(x)}{x(x+2)} - \frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{x(x+2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{x(x+2)} - \frac{4y(x)}{x(x+2)} = 0$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{x+1}{x(x+2)}, P_3(x) = -\frac{4}{x(x+2)}\right]$

  - $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

    $\left.\left((x+2)\cdot P_2(x)\right)\right|_{x=-2} = \frac{1}{2}$

  - $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

    $\left.\left((x+2)^2\cdot P_3(x)\right)\right|_{x=-2} = 0$

  - $x=-2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -2$

- Multiply by denominators

  $x(x+2)\left(\frac{d^2}{dx^2}y(x)\right) + (x+1)\left(\frac{d}{dx}y(x)\right) - 4y(x) = 0$

- Change variables using $x = u-2$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (u-1)\left(\frac{d}{du}y(u)\right) - 4y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

- □ Rewrite ODE with series expansions
  - ○ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$
  - ○ Shift index using $k->k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$
  - ○ Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$
  - ○ Shift index using $k->k+2-m$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-a_0r(-1+2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(2k+1+2r) + a_k(k+r+2)\,(k+r-2)\right) u^{k+r}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $-2(k+1+r)\left(k+r+\frac{1}{2}\right) a_{k+1} + a_k(k+r+2)\,(k+r-2) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r+2)(k+r-2)}{(k+1+r)(2k+1+2r)}$
- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+1} = \frac{a_k(k+2)(k-2)}{(k+1)(2k+1)}$
- Apply recursion relation for $k = 0$

  $a_1 = -4a_0$
- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{2}$
- Express in terms of a_0

  $a_2 = 2a_0$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot (2u^2 - 4u + 1)$
- Revert the change of variables $u = x + 2$

  $[y(x) = a_0(2x^2 + 4x + 1)]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}$
- Solution for $r = \frac{1}{2}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$
- Revert the change of variables $u = x + 2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+2)^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$
- Combine solutions and rename parameters

$$\left[y(x)=a_0(2x^2+4x+1)+\left(\sum_{k=0}^{\infty} b_k(x+2)^{k+\frac{1}{2}}\right), b_{k+1}=\frac{b_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
   Solution is available but has compositions of trig with ln functions of radicals. Att
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- linear_1 successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 28

```
dsolve(x*(x+2)*diff(diff(y(x),x),x)+(x+1)*diff(y(x),x)-4*y(x) = 0,y(x),singsol=all)
```

$$y=c_2(x+1)\sqrt{x(x+2)}+2\left(x^2+2x+\frac{1}{2}\right)c_1$$

Mathematica DSolve solution

Solving time : 1.805 (sec)
Leaf size : 45

```
DSolve[{x*(x+2)*D[y[x],{x,2}]+(x+1)*D[y[x],x]-4*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x)\to c_1\cosh\left(4\operatorname{arctanh}\left(\frac{1}{\sqrt{\frac{x}{x+2}}}\right)\right)+ic_2\sinh\left(4\operatorname{arctanh}\left(\frac{1}{\sqrt{\frac{x}{x+2}}}\right)\right)$$

2.1.309 Problem 314

Internal problem ID [9481]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 314
Date solved : `Monday, January 27, 2025 at 06:03:17 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(x-1)\,y''-xy'+y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.227 (sec)

Writing the ode as

$$(x-1)\,y''-xy'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x-1 \\ B &= -x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{x^2-4x+6}{4\,(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2-4x+6 \\ t &= 4(x-1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x^2-4x+6}{4\,(x-1)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.586: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4\,(x-1)^2} - \frac{1}{2\,(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 4x + 6}{4\left(x-1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(x-1\right)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2\left(x-1\right)} + \frac{1}{2} \\
&= \frac{x-2}{2x-2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2(x-1)}+\frac{1}{2}\right)(0)+\left(\left(\frac{1}{2(x-1)^2}\right)+\left(-\frac{1}{2(x-1)}+\frac{1}{2}\right)^2-\left(\frac{x^2-4x+6}{4(x-1)^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\,dx}\\
&= \mathrm{e}^{\int\left(-\frac{1}{2(x-1)}+\frac{1}{2}\right)dx}\\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{x-1}\,dx}\\
&= z_1 e^{\frac{x}{2}+\frac{\ln(x-1)}{2}}\\
&= z_1\left(\sqrt{x-1}\,\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1\int \frac{e^{\int -\frac{-x}{x-1}\,dx}}{(y_1)^2}\,dx\\
&= y_1\int \frac{e^{x+\ln(x-1)}}{(y_1)^2}\,dx\\
&= y_1\left(-\frac{x\,\mathrm{e}^{x+\ln(x-1)}\mathrm{e}^{-2x}}{x-1}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1y_1+c_2y_2\\
&= c_1(\mathrm{e}^{x})+c_2\left(\mathrm{e}^{x}\left(-\frac{x\,\mathrm{e}^{x+\ln(x-1)}\mathrm{e}^{-2x}}{x-1}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x-1)\left(\frac{d^2}{dx^2}y(x)\right)-x\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{y(x)}{x-1}+\frac{\left(\frac{d}{dx}y(x)\right)x}{x-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x)-\frac{\left(\frac{d}{dx}y(x)\right)x}{x-1}+\frac{y(x)}{x-1}=0$

□ Check to see if $x_0=1$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{x}{x-1},P_3(x)=\frac{1}{x-1}\right]$

  ◦ $(x-1)\cdot P_2(x)$ is analytic at $x=1$

  $\left((x-1)\cdot P_2(x)\right)\Big|_{x=1}=-1$

  ◦ $(x-1)^2\cdot P_3(x)$ is analytic at $x=1$

  $\left((x-1)^2\cdot P_3(x)\right)\Big|_{x=1}=0$

  ◦ $x=1$is a regular singular point

Check to see if $x_0=1$ is a regular singular point

$x_0=1$

- Multiply by denominators

$(x-1)\left(\frac{d^2}{dx^2}y(x)\right)-x\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Change variables using $x=u+1$ so that the regular singular point is at $u=0$

$u\left(\frac{d^2}{du^2}y(u)\right)+(-u-1)\left(\frac{d}{du}y(u)\right)+y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  ◦ Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-1}$

  ◦ Shift index using $k->k+1$

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)(k+r)\,u^{k+r}$

Rewrite ODE with series expansions

$a_0r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)(k+r-1)-a_k(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r)=0$

- Values of r that satisfy the indicial equation
$r \in \{0, 2\}$
- Each term in the series must be 0, giving the recursion relation
$(k+r-1)\left(a_{k+1}(k+1+r)-a_k\right)=0$
- Recursion relation that defines series solution to ODE
$a_{k+1}=\frac{a_k}{k+1+r}$
- Recursion relation for $r=0$
$a_{k+1}=\frac{a_k}{k+1}$
- Solution for $r=0$
$\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+1}=\frac{a_k}{k+1}\right]$
- Revert the change of variables $u=x-1$
$\left[y(x)=\sum_{k=0}^{\infty} a_k (x-1)^k, a_{k+1}=\frac{a_k}{k+1}\right]$
- Recursion relation for $r=2$
$a_{k+1}=\frac{a_k}{k+3}$
- Solution for $r=2$
$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$
- Revert the change of variables $u=x-1$
$\left[y(x)=\sum_{k=0}^{\infty} a_k (x-1)^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$
- Combine solutions and rename parameters
$\left[y(x)=\left(\sum_{k=0}^{\infty} a_k (x-1)^k\right)+\left(\sum_{k=0}^{\infty} b_k (x-1)^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+3}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 12

```
dsolve((x-1)*diff(diff(y(x),x),x)-diff(y(x),x)*x+y(x) = 0,y(x),singsol=all)
```

$$y = c_1 x + e^x c_2$$

Mathematica DSolve solution

Solving time : 0.16 (sec)
Leaf size : 90

```
DSolve[{(x-1)*D[y[x],{x,2}]-x*D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \exp\left(\int_1^x \frac{K[1]-2}{2(K[1]-1)}dK[1] - \frac{1}{2}\int_1^x -\frac{K[2]}{K[2]-1}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{K[1]-2}{2(K[1]-1)}dK[1]\right)dK[3] + c_1\right)$$

2.1.310 Problem 315

Internal problem ID [9482]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 315
Date solved : `Monday, January 27, 2025 at 06:03:18 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+1\right) y''-2xy'+2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.267 (sec)

Writing the ode as

$$\left(x^2+1\right) y''-2xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= -2x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-3}{\left(x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= \left(x^2+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{3}{\left(x^2+1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.588: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 + 1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{3i}{4(x-i)} - \frac{3i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= 0\\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right)\\ &= 1 - (1)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (-)[\sqrt{r}]_\infty\\ &= -\frac{1}{2\,(x-i)} + \frac{3}{2\,(x+i)} + (-)\,(0)\\ &= -\frac{1}{2\,(x-i)} + \frac{3}{2\,(x+i)}\\ &= \frac{x-2i}{x^2+1}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2\left(x-i\right)}+\frac{3}{2\left(x+i\right)}\right)(0)+\left(\left(\frac{1}{2\left(x-i\right)^{2}}-\frac{3}{2\left(x+i\right)^{2}}\right)+\left(-\frac{1}{2\left(x-i\right)}+\frac{3}{2\left(x+i\right)}\right)^{2}-\left(-\frac{}{(x^2}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\,dx} \\
&= e^{\int \left(-\frac{1}{2(x-i)}+\frac{3}{2(x+i)}\right)dx} \\
&= \frac{\left(x^2+1\right)^{3/2}}{\left(ix+1\right)^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2+1}\,dx} \\
&= z_1 e^{\frac{\ln\left(x^2+1\right)}{2}} \\
&= z_1\left(\sqrt{x^2+1}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(x^2+1\right)^2}{\left(ix+1\right)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2+1}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{\ln\left(x^2+1\right)}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(-\frac{x}{\left(x+i\right)^2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\left(x^2+1\right)^2}{\left(ix+1\right)^2}\right) + c_2\left(\frac{\left(x^2+1\right)^2}{\left(ix+1\right)^2}\left(-\frac{x}{\left(x+i\right)^2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 16

```
dsolve((x^2+1)*diff(diff(y(x),x),x)-2*diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y = c_2 x^2 + c_1 x - c_2$$

Mathematica DSolve solution

Solving time : 0.319 (sec)
Leaf size : 79

```
DSolve[{(1+x^2)*D[y[x],{x,2}]-2*x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to \sqrt{x^2+1}\exp\left(\int_1^x \frac{K[1]+2i}{K[1]^2+1}dK[1]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[2]} \frac{K[1]+2i}{K[1]^2+1}dK[1]\right)dK[2] + c_1\right)$$

2.1.311 Problem 316

Internal problem ID [9483]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 316
Date solved : `Monday, January 27, 2025 at 06:03:19 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2-2x+10\right)y''+xy'-4y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.354 (sec)

Writing the ode as

$$\left(x^2-2x+10\right)y''+xy'-4y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2-2x+10 \\ B &= x \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{15x^2-32x+180}{4\left(x^2-2x+10\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15x^2-32x+180 \\ t &= 4\left(x^2-2x+10\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{15x^2-32x+180}{4\left(x^2-2x+10\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.589: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 2x + 10)^2$. There is a pole at $x = 1 + 3i$ of order 2. There is a pole at $x = 1 - 3i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{-\frac{7}{36} + \frac{i}{24}}{(x - 1 - 3i)^2} + \frac{-\frac{7}{36} - \frac{i}{24}}{(x - 1 + 3i)^2} - \frac{149i}{216\,(x - 1 - 3i)} + \frac{149i}{216\,(x - 1 + 3i)}$$

For the pole at $x = 1 + 3i$ let b be the coefficient of $\frac{1}{(x-1-3i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{36} + \frac{i}{24}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} + \frac{i}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} - \frac{i}{12} \end{aligned}$$

For the pole at $x = 1 - 3i$ let b be the coefficient of $\frac{1}{(x-1+3i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{36} - \frac{i}{24}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} - \frac{i}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} + \frac{i}{12} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15x^2 - 32x + 180}{4\left(x^2 - 2x + 10\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15x^2 - 32x + 180}{4\left(x^2 - 2x + 10\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$1+3i$	2	0	$\frac{3}{4} + \frac{i}{12}$	$\frac{1}{4} - \frac{i}{12}$
$1-3i$	2	0	$\frac{3}{4} - \frac{i}{12}$	$\frac{1}{4} + \frac{i}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\
&= \frac{5}{2} - \left(\frac{3}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{\frac{3}{4} + \frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4} - \frac{i}{12}}{x-1+3i} + (0) \\
&= \frac{\frac{3}{4} + \frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4} - \frac{i}{12}}{x-1+3i} \\
&= \frac{3x-4}{2x^2-4x+20}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{3}{4} + \frac{i}{12}}{x - 1 - 3i} + \frac{\frac{3}{4} - \frac{i}{12}}{x - 1 + 3i}\right)(1) + \left(\left(\frac{-\frac{3}{4} - \frac{i}{12}}{(x - 1 - 3i)^2} + \frac{-\frac{3}{4} + \frac{i}{12}}{(x - 1 + 3i)^2}\right) + \left(\frac{\frac{3}{4} + \frac{i}{12}}{x - 1 - 3i} + \frac{\frac{3}{4} - \frac{i}{12}}{x - 1 + 3i \ldots}\right.\right.$$

$$- \frac{3}{(- \ldots}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{4}{3}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - \frac{4}{3}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{4}{3}\right) e^{\int \left(\frac{\frac{3}{4} + \frac{i}{12}}{x - 1 - 3i} + \frac{\frac{3}{4} - \frac{i}{12}}{x - 1 + 3i}\right) dx} \\
&= \left(x - \frac{4}{3}\right) e^{\frac{3 \ln\left(x^2 - 2x + 10\right)}{4} - \frac{\arctan\left(\frac{x}{3} - \frac{1}{3}\right)}{6}} \\
&= \frac{(3x - 4)\left(x^2 - 2x + 10\right)^{3/4} e^{-\frac{\arctan\left(\frac{x}{3} - \frac{1}{3}\right)}{6}}}{3}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{x^2 - 2x + 10} \, dx} \\
&= z_1 e^{-\frac{\ln\left(x^2 - 2x + 10\right)}{4} - \frac{\arctan\left(\frac{x}{3} - \frac{1}{3}\right)}{6}} \\
&= z_1 \left(\frac{e^{-\frac{\arctan\left(\frac{x}{3} - \frac{1}{3}\right)}{6}}}{\left(x^2 - 2x + 10\right)^{1/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x^2 - 2x + 10}\, e^{-\frac{\arctan\left(\frac{x}{3} - \frac{1}{3}\right)}{3}} (3x - 4)}{3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2-2x+10} dx}}{(y_1)^2} dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x^2-2x+10)}{2}-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}}{(y_1)^2} dx \\
&= y_1 \left(-\frac{9(3x^2-4x+15)\,\mathrm{e}^{-\frac{\ln(x^2-2x+10)}{2}-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}\,\mathrm{e}^{\frac{2\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}}{410\,(3x-4)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}(3x-4)}{3}\right) \\
&\quad + c_2 \left(\frac{\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}(3x-4)}{3}\left(-\frac{9(3x^2-4x+15)\,\mathrm{e}^{-\frac{\ln(x^2-2x+10)}{2}-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}\,\mathrm{e}^{\frac{2\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}}{410\,(3x-4)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.010` (sec)
Leaf size : 31

```
dsolve((x^2-2*x+10)*diff(diff(y(x),x),x)+diff(y(x),x)*x-4*y(x) = 0,y(x),singsol=all)
```

$$
y = 3c_2\left(x-\frac{4}{3}\right)(x-1+3i)^{\frac{1}{2}-\frac{i}{6}}(x-1-3i)^{\frac{1}{2}+\frac{i}{6}} + c_1\left(x^2-\frac{4}{3}x+5\right)
$$

Mathematica DSolve solution

Solving time : `0.754` (sec)
Leaf size : 125

```
DSolve[{(x^2-2*x+10)*D[y[x],{x,2}]+x*D[y[x],x]-4*y[x]==0,{}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{1}{3}(3x-4)\exp\left(\int_1^x \frac{3K[1]-4}{2(K[1]-2)K[1]+20}dK[1] - \frac{1}{2}\int_1^x \frac{K[2]}{(K[2]-2)K[2]+10}dK[2]\right)\left(c_2\int_1^x \frac{9\exp\left(-2\int_1^{K[3]} \frac{3K[1]-4}{2(K[1]^2-2K[1]+10)}dK[1]\right)}{(4-3K[3])^2}dK[3] + c_1\right)$$

2.1.312 Problem 317

Internal problem ID [9484]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 317
Date solved : `Monday, January 27, 2025 at 06:03:19 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2-2x+10\right)y''+xy'-4y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.364 (sec)

Writing the ode as

$$\left(x^2-2x+10\right)y''+xy'-4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2-2x+10 \\ B &= x \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{15x^2-32x+180}{4\left(x^2-2x+10\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15x^2-32x+180 \\ t &= 4\left(x^2-2x+10\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{15x^2-32x+180}{4\left(x^2-2x+10\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.590: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 2x + 10)^2$. There is a pole at $x = 1 + 3i$ of order 2. There is a pole at $x = 1 - 3i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{-\frac{7}{36} + \frac{i}{24}}{(x - 1 - 3i)^2} + \frac{-\frac{7}{36} - \frac{i}{24}}{(x - 1 + 3i)^2} - \frac{149i}{216\,(x - 1 - 3i)} + \frac{149i}{216\,(x - 1 + 3i)}$$

For the pole at $x = 1 + 3i$ let b be the coefficient of $\frac{1}{(x-1-3i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{36} + \frac{i}{24}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} + \frac{i}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} - \frac{i}{12} \end{aligned}$$

For the pole at $x = 1 - 3i$ let b be the coefficient of $\frac{1}{(x-1+3i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{36} - \frac{i}{24}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} - \frac{i}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} + \frac{i}{12} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15x^2 - 32x + 180}{4\left(x^2 - 2x + 10\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15x^2 - 32x + 180}{4\left(x^2 - 2x + 10\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$1+3i$	2	0	$\frac{3}{4} + \frac{i}{12}$	$\frac{1}{4} - \frac{i}{12}$
$1-3i$	2	0	$\frac{3}{4} - \frac{i}{12}$	$\frac{1}{4} + \frac{i}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\
&= \frac{5}{2} - \left(\frac{3}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{\frac{3}{4} + \frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4} - \frac{i}{12}}{x-1+3i} + (0) \\
&= \frac{\frac{3}{4} + \frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4} - \frac{i}{12}}{x-1+3i} \\
&= \frac{3x-4}{2x^2-4x+20}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{3}{4}+\frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4}-\frac{i}{12}}{x-1+3i}\right)(1) + \left(\left(\frac{-\frac{3}{4}-\frac{i}{12}}{\left(x-1-3i\right)^2} + \frac{-\frac{3}{4}+\frac{i}{12}}{\left(x-1+3i\right)^2}\right) + \left(\frac{\frac{3}{4}+\frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4}-\frac{i}{12}}{x-1+3i} \right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{4}{3}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - \frac{4}{3}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{4}{3}\right) e^{\int \left(\frac{\frac{3}{4}+\frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4}-\frac{i}{12}}{x-1+3i}\right) dx} \\
&= \left(x - \frac{4}{3}\right) e^{\frac{3\ln\left(x^2-2x+10\right)}{4} - \frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{6}} \\
&= \frac{(3x-4)\left(x^2-2x+10\right)^{3/4} e^{-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{6}}}{3}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2-2x+10} \, dx} \\
&= z_1 e^{-\frac{\ln\left(x^2-2x+10\right)}{4} - \frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{6}} \\
&= z_1 \left(\frac{e^{-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{6}}}{\left(x^2-2x+10\right)^{1/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x^2-2x+10}\, e^{-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}} (3x-4)}{3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2-2x+10}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln\left(x^2-2x+10\right)}{2}-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{9(3x^2-4x+15)\,\mathrm{e}^{-\frac{\ln\left(x^2-2x+10\right)}{2}-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}\,\mathrm{e}^{\frac{2\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}}{410\,(3x-4)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}(3x-4)}{3}\right) \\
&\quad + c_2\left(\frac{\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}(3x-4)}{3}\left(-\frac{9(3x^2-4x+15)\,\mathrm{e}^{-\frac{\ln\left(x^2-2x+10\right)}{2}-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}\,\mathrm{e}^{\frac{2\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}}{410\,(3x-4)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.010 (sec)
Leaf size : 31

```
dsolve((x^2-2*x+10)*diff(diff(y(x),x),x)+diff(y(x),x)*x-4*y(x) = 0,y(x),singsol=all)
```

$$
y = 3c_2\left(x-\frac{4}{3}\right)(x-1+3i)^{\frac{1}{2}-\frac{i}{6}}(x-1-3i)^{\frac{1}{2}+\frac{i}{6}} + c_1\left(x^2-\frac{4}{3}x+5\right)
$$

Mathematica DSolve solution

Solving time : 0.591 (sec)
Leaf size : 125

```
DSolve[{(x^2-2*x+10)*D[y[x],{x,2}]+x*D[y[x],x]-4*y[x]==0,{}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{1}{3}(3x-4)\exp\left(\int_1^x \frac{3K[1]-4}{2(K[1]-2)K[1]+20}dK[1] - \frac{1}{2}\int_1^x \frac{K[2]}{(K[2]-2)K[2]+10}dK[2]\right)\left(c_2\int_1^x \frac{9\exp\left(-2\int_1^{K[3]} \frac{3K[1]-4}{2(K[1]^2-2K[1]+10)}dK[1]\right)}{(4-3K[3])^2}dK[3] + c_1\right)$$

2.1.313 Problem 318

Internal problem ID [9485]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 318
Date solved : `Monday, January 27, 2025 at 06:03:20 PM`
CAS classification : `[_Hermite]`

Solve

$$y'' - xy' + 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.231 (sec)

Writing the ode as

$$y'' - xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 10}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 10 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{5}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.591: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{5}{2x} - \frac{25}{4x^3} - \frac{125}{4x^5} - \frac{3125}{16x^7} - \frac{21875}{16x^9} - \frac{328125}{32x^{11}} - \frac{2578125}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 10}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} - \frac{5}{2}\right) + (0) \\ &= \frac{x^2}{4} - \frac{5}{2}\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{5}{2}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{5}{2}\right) - (0) \\ &= -\frac{5}{2}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &= -3 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &= 2\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{5}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (2) + 2\left(-\frac{x}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{5}{2}\right)\right) &= 0 \\ a_1 x + 2a_0 + 2 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (x^2 - 1)\, e^{\int -\frac{x}{2} dx} \\ &= (x^2 - 1)\, e^{-\frac{x^2}{4}} \\ &= (x^2 - 1)\, e^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1\left(e^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^2 - 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{e^{\frac{x^2}{2}}}{(x^2-1)^2} dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (x^2 - 1) + c_2 \left(x^2 - 1 \left(\int \frac{e^{\frac{x^2}{2}}}{(x^2-1)^2} dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) - x\left(\frac{d}{dx}y(x)\right) + 2y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite DE with series expansions

- ∘ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

- ∘ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

  $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ∘ Shift index using $k- >k+2$

  $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_k(k-2)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} - a_k(k-2) = 0$

- Recursion relation; series terminates at $k = 2$

$a_{k+2} = \frac{a_k(k-2)}{k^2+3k+2}$

- Apply recursion relation for $k = 0$

  $a_2 = -a_0$

- Terminating series solution of the ODE. Use reduction of order to find the second linearly ind

  $y(x) = A_2 x^2 + A_1 x - a_0$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special functio
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.026 (sec)
Leaf size : 42

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y = 2c_1 e^{\frac{x^2}{2}} x - (x-1)(x+1)\left(c_1\sqrt{\pi}\, \operatorname{erfi}\left(\frac{\sqrt{2}\, x}{2}\right)\sqrt{2} - c_2\right)$$

Mathematica DSolve solution

Solving time : 0.203 (sec)
Leaf size : 43

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to (x^2-1)\left(c_2 \int_1^x \frac{e^{\frac{K[1]^2}{2}}}{(K[1]^2-1)^2} dK[1] + c_1\right)$$

2.1.314 Problem 319

Internal problem ID [9486]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 319
Date solved : `Monday, January 27, 2025 at 06:03:21 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(x+2)\,y''+xy'-y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.253 (sec)

Writing the ode as

$$(x+2)\,y''+xy'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=x+2\\ B&=x\\ C&=-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{x^2+4x+12}{4\,(x+2)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=x^2+4x+12\\ t&=4(x+2)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x^2+4x+12}{4\,(x+2)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.593: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x+2)^2$. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{(x+2)^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{2}{x^2} - \frac{8}{x^3} + \frac{20}{x^4} - \frac{32}{x^5} + \frac{16}{x^6} + \frac{64}{x^7} - \frac{80}{x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x + 12}{4x^2 + 16x + 16} \\ &= Q + \frac{R}{4x^2 + 16x + 16} \\ &= \left(\frac{1}{4}\right) + \left(\frac{8}{4x^2 + 16x + 16}\right) \\ &= \frac{1}{4} + \frac{8}{4x^2 + 16x + 16} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 4 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{\frac{1}{2}} - 0\right) &= 0 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{\frac{1}{2}} - 0\right) &= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 4x + 12}{4(x+2)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x+2} + (-)\left(\frac{1}{2}\right) \\ &= -\frac{1}{x+2} - \frac{1}{2} \\ &= -\frac{4+x}{2(x+2)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x+2} - \frac{1}{2}\right)(1) + \left(\left(\frac{1}{(x+2)^2} \right) + \left(-\frac{1}{x+2} - \frac{1}{2} \right)^2 - \left(\frac{x^2+4x+12}{4(x+2)^2} \right) \right) = 0$$

$$\frac{a_0 - 4}{x+2} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 4\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 4 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (4+x)\, \mathrm{e}^{\int \left(-\frac{1}{x+2}-\frac{1}{2}\right) dx} \\ &= (4+x)\, \mathrm{e}^{-\frac{x}{2}-\ln(x+2)} \\ &= \frac{(4+x)\, \mathrm{e}^{-\frac{x}{2}}}{x+2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x+2} \, dx} \\ &= z_1 e^{-\frac{x}{2}+\ln(x+2)} \\ &= z_1 \left((x+2)\, \mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}(4+x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x+2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-x+2\ln(x+2)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{x\, \mathrm{e}^{-x+2\ln(x+2)} \mathrm{e}^{2x}}{(4+x)\,(x+2)^2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-x}(4+x)\right) + c_2 \left(\mathrm{e}^{-x}(4+x) \left(\frac{x\, \mathrm{e}^{-x+2\ln(x+2)} \mathrm{e}^{2x}}{(4+x)\,(x+2)^2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x+2)\left(\frac{d^2}{dx^2}y(x)\right)+x\left(\frac{d}{dx}y(x)\right)-y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=\frac{y(x)}{x+2}-\frac{\left(\frac{d}{dx}y(x)\right)x}{x+2}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)+\frac{\left(\frac{d}{dx}y(x)\right)x}{x+2}-\frac{y(x)}{x+2}=0$

☐ Check to see if $x_0=-2$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=\frac{x}{x+2}, P_3(x)=-\frac{1}{x+2}\right]$

- ◦ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

  $\left((x+2)\cdot P_2(x)\right)\bigg|_{x=-2}=-2$

- ◦ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

  $\left((x+2)^2\cdot P_3(x)\right)\bigg|_{x=-2}=0$

- ◦ $x=-2$is a regular singular point

Check to see if $x_0=-2$ is a regular singular point

$x_0=-2$

- Multiply by denominators

$(x+2)\left(\frac{d^2}{dx^2}y(x)\right)+x\left(\frac{d}{dx}y(x)\right)-y(x)=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

$u\left(\frac{d^2}{du^2}y(u)\right)+(u-2)\left(\frac{d}{du}y(u)\right)-y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ◦ Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-1}$

- ◦ Shift index using $k->k+1$

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$

Rewrite ODE with series expansions

$a_0r(-3+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(k-2+r)+a_k(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 3\}$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k-2+r) + a_k(k+r-1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r-1)}{(k+1+r)(k-2+r)}$
- Recursion relation for $r = 0$; series terminates at $k = 1$

  $a_{k+1} = -\frac{a_k(k-1)}{(k+1)(k-2)}$
- Apply recursion relation for $k = 0$

  $a_1 = -\frac{a_0}{2}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second linea

  $y(u) = a_0 \cdot \left(1 - \frac{u}{2}\right)$
- Revert the change of variables $u = x + 2$

  $\left[y(x) = -\frac{a_0 x}{2}\right]$
- Recursion relation for $r = 3$

  $a_{k+1} = -\frac{a_k(k+2)}{(k+4)(k+1)}$
- Solution for $r = 3$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+3}, a_{k+1} = -\frac{a_k(k+2)}{(k+4)(k+1)}\right]$
- Revert the change of variables $u = x + 2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x+2)^{k+3}, a_{k+1} = -\frac{a_k(k+2)}{(k+4)(k+1)}\right]$
- Combine solutions and rename parameters

  $\left[y(x) = -\frac{a_0 x}{2} + \left(\sum_{k=0}^{\infty} b_k (x+2)^{k+3}\right), b_{k+1} = -\frac{b_k(k+2)}{(4+k)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 17

```
dsolve((x+2)*diff(diff(y(x),x),x)+diff(y(x),x)*x-y(x) = 0,y(x),singsol=all)
```

$$y = c_1 x + c_2 e^{-x}(x+4)$$

Mathematica DSolve solution

Solving time : 0.341 (sec)
Leaf size : 96

```
DSolve[{(x+2)*D[y[x],{x,2}]+x*D[y[x],x]-y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2i\left((c_1(x+2)+2ic_2)\cosh\left(\frac{x+2}{2}\right)-(ic_2(x+2)+2c_1)\sinh\left(\frac{x+2}{2}\right)\right)\exp\left(\int_1^x \frac{1-K[1]}{2K[1]+4}dK[1]\right)}{\sqrt{\pi}(-i(x+2))^{3/2}}$$

2.1.315 Problem 320

Internal problem ID [9487]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 320
Date solved : `Monday, January 27, 2025 at 06:03:21 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$\left(x^2+1\right) y''-6y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.217 (sec)

Writing the ode as

$$\left(x^2+1\right) y''-6y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= 0 \\ C &= -6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{6}{x^2+1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 6 \\ t &= x^2+1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{6}{x^2+1}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.595: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2 + 1$. There is a pole at $x = i$ of order 1. There is a pole at $x = -i$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = i$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{6}{x^2 + 1}$$

Since the $\gcd(s,t) = 1$. This gives $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{6}{x^2 + 1}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	3	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 3$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\ &= 3 - (1) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{x-i} + (0) \\ &= \frac{1}{x-i} \\ &= \frac{1}{x-i} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{1}{x-i}\right)(2x + a_1) + \left(\left(-\frac{1}{(x-i)^2}\right) + \left(\frac{1}{x-i}\right)^2 - \left(\frac{6}{x^2+1}\right)\right) = 0$$

$$2 + \frac{-4x - 2a_1}{-x+i} + \frac{-6x^2 - 6a_1 x - 6a_0}{x^2+1} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0, a_1 = i\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + ix$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 + ix\right) \mathrm{e}^{\int \frac{1}{x-i} dx} \\
&= \left(x^2 + ix\right) \mathrm{e}^{\frac{\ln\left(x^2+1\right)}{2} + i \arctan(x)} \\
&= x(x+i)\,(ix+1)
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= x(x+i)\,(ix+1)
\end{aligned}
$$

Which simplifies to

$$
y_1 = i x^3 + ix
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\
&= i x^3 + ix \int \frac{1}{\left(i x^3 + ix\right)^2} \, dx \\
&= i x^3 + ix \left(\frac{1}{x} + \frac{x}{2x^2+2} + \frac{3 \arctan\left(x\right)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(i x^3 + ix\right) + c_2 \left(i x^3 + ix \left(\frac{1}{x} + \frac{x}{2x^2+2} + \frac{3 \arctan\left(x\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
```

```
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 31

```
dsolve((x^2+1)*diff(diff(y(x),x),x)-6*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{3xc_2(x^2+1)\arctan(x)}{2} + c_1x^3 + \frac{3c_2x^2}{2} + c_1x + c_2$$

Mathematica DSolve solution

Solving time : 0.147 (sec)
Leaf size : 33

```
DSolve[{(x^2+1)*D[y[x],{x,2}]-6*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to (x^3+x)\left(c_2\int_1^x \frac{1}{(K[1]^3+K[1])^2}dK[1] + c_1\right)$$

2.1.316 Problem 321

Internal problem ID [9488]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 321
Date solved : `Monday, January 27, 2025 at 06:03:22 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+2\right) y''+3 x y'-y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.424 (sec)

Writing the ode as

$$\left(x^2+2\right) y''+3 x y'-y=0 \tag{1}$$
$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=x^2+2\\ B&=3 x\\ C&=-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2 A}\, d x}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2 A B'-2 B A'+B^2-4 A C}{4 A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{7 x^2+20}{4\left(x^2+2\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=7 x^2+20\\ t&=4\left(x^2+2\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{7 x^2+20}{4\left(x^2+2\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.596: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+2)^2$. There is a pole at $x = i\sqrt{2}$ of order 2. There is a pole at $x = -i\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16\left(x - i\sqrt{2}\right)^2} - \frac{3}{16\left(x + i\sqrt{2}\right)^2} - \frac{17i\sqrt{2}}{32\left(x - i\sqrt{2}\right)} + \frac{17i\sqrt{2}}{32\left(x + i\sqrt{2}\right)}$$

For the pole at $x = i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x - i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

For the pole at $x = -i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x + i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{7x^2 + 20}{4\left(x^2+2\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{7}{4}$. Hence

$$\begin{aligned} E_\infty &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{2\} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
$i\sqrt{2}$	2	$\{1,2,3\}$
$-i\sqrt{2}$	2	$\{1,2,3\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1,\ e_2 = 1,\ e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(2-(1+(1))) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{1}{\left(x-(i\sqrt{2})\right)} + \frac{1}{\left(x-(-i\sqrt{2})\right)}\right) \\ &= \frac{1}{2x-2i\sqrt{2}} + \frac{1}{2x+2i\sqrt{2}} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p}\\ &= \frac{1}{2x-2i\sqrt{2}} + \frac{1}{2x+2i\sqrt{2}}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{2x-2i\sqrt{2}} + \frac{1}{2x+2i\sqrt{2}}\right)w + \frac{7x^2+16}{4\left(\sqrt{2}+ix\right)^2\left(x+i\sqrt{2}\right)^2} = 0$$

Solving for ω gives

$$\omega = \frac{x+2\sqrt{2x^2+4}}{2x^2+4}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega\, dx}\\ &= \mathrm{e}^{\int \frac{x+2\sqrt{2x^2+4}}{2x^2+4}dx}\\ &= \left(x^2+2\right)^{1/4} \mathrm{e}^{\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{3x}{x^2+2}\, dx}\\ &= z_1 e^{-\frac{3\ln\left(x^2+2\right)}{4}}\\ &= z_1\left(\frac{1}{\left(x^2+2\right)^{3/4}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}}{\sqrt{x^2+2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{x^2+2}\, dx}}{\left(y_1\right)^2}\, dx\\ &= y_1 \int \frac{e^{-\frac{3\ln\left(x^2+2\right)}{2}}}{\left(y_1\right)^2}\, dx\\ &= y_1\left(\int \frac{\mathrm{e}^{-2\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}}{\sqrt{x^2+2}} dx\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{\sqrt{2}\,\mathrm{arcsinh}\left(\frac{\sqrt{2}\,x}{2}\right)}}{\sqrt{x^2+2}} \right) + c_2 \left(\frac{\mathrm{e}^{\sqrt{2}\,\mathrm{arcsinh}\left(\frac{\sqrt{2}\,x}{2}\right)}}{\sqrt{x^2+2}} \left(\int \frac{\mathrm{e}^{-2\sqrt{2}\,\mathrm{arcsinh}\left(\frac{\sqrt{2}\,x}{2}\right)}}{\sqrt{x^2+2}} dx \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.056` (sec)
Leaf size : 45

```
dsolve((x^2+2)*diff(diff(y(x),x),x)+3*diff(y(x),x)*x-y(x) = 0,y(x),singsol=all)
```

$$
y = \frac{c_2 \left(\sqrt{x^2+2}+x\right)^{-\sqrt{2}} + c_1 \left(\sqrt{x^2+2}+x\right)^{\sqrt{2}}}{\sqrt{x^2+2}}
$$

Mathematica DSolve solution

Solving time : `0.081` (sec)
Leaf size : 92

```
DSolve[{(x^2+2)*D[y[x],{x,2}]+3*x*D[y[x],x]-y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tru
```

$$
y(x) \to \frac{2^{3/4} c_1 \cos\left(2\sqrt{2}\arcsin\left(\frac{1}{2}\sqrt{2-i\sqrt{2}x}\right)\right)}{\sqrt{\pi}\sqrt{x^2+2}} + \frac{c_2 Q^{\frac{1}{2}}_{-\frac{1}{2}+\sqrt{2}}\left(\frac{ix}{\sqrt{2}}\right)}{\sqrt[4]{x^2+2}}
$$

2.1.317 Problem 322

Internal problem ID [9489]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 322
Date solved : `Monday, January 27, 2025 at 06:03:23 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(x-1)\,y''-xy'+y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.224 (sec)

Writing the ode as

$$(x-1)\,y''-xy'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x-1 \\ B &= -x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{x^2-4x+6}{4\,(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2-4x+6 \\ t &= 4(x-1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x^2-4x+6}{4\,(x-1)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.597: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4\,(x-1)^2} - \frac{1}{2\,(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 4x + 6}{4(x-1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(x-1)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2(x-1)} + \frac{1}{2} \\
&= \frac{x-2}{2x-2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0)+2\left(-\frac{1}{2(x-1)}+\frac{1}{2}\right)(0)+\left(\left(\frac{1}{2(x-1)^2}\right)+\left(-\frac{1}{2(x-1)}+\frac{1}{2}\right)^2-\left(\frac{x^2-4x+6}{4(x-1)^2}\right)\right)=0
$$

$$
0=0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\,dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x-1)}+\frac{1}{2}\right)dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x-1}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{x-1}\,dx} \\
&= z_1 e^{\frac{x}{2}+\frac{\ln(x-1)}{2}} \\
&= z_1\left(\sqrt{x-1}\,\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1=\mathrm{e}^x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1\int \frac{e^{\int -\frac{-x}{x-1}\,dx}}{(y_1)^2}\,dx \\
&= y_1\int \frac{e^{x+\ln(x-1)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{x\,\mathrm{e}^{x+\ln(x-1)}\mathrm{e}^{-2x}}{x-1}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1+c_2y_2 \\
&= c_1(\mathrm{e}^x)+c_2\left(\mathrm{e}^x\left(-\frac{x\,\mathrm{e}^{x+\ln(x-1)}\mathrm{e}^{-2x}}{x-1}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x-1)\left(\frac{d^2}{dx^2}y(x)\right)-x\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{y(x)}{x-1}+\frac{\left(\frac{d}{dx}y(x)\right)x}{x-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)-\frac{\left(\frac{d}{dx}y(x)\right)x}{x-1}+\frac{y(x)}{x-1}=0$

□ Check to see if $x_0=1$ is a regular singular point

- ◦ Define functions

$\left[P_2(x)=-\frac{x}{x-1}, P_3(x)=\frac{1}{x-1}\right]$

- ◦ $(x-1)\cdot P_2(x)$ is analytic at $x=1$

$\left((x-1)\cdot P_2(x)\right)\bigg|_{x=1}=-1$

- ◦ $(x-1)^2\cdot P_3(x)$ is analytic at $x=1$

$\left((x-1)^2\cdot P_3(x)\right)\bigg|_{x=1}=0$

- ◦ $x=1$is a regular singular point

Check to see if $x_0=1$ is a regular singular point

$x_0=1$

- Multiply by denominators

$(x-1)\left(\frac{d^2}{dx^2}y(x)\right)-x\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Change variables using $x=u+1$ so that the regular singular point is at $u=0$

$u\left(\frac{d^2}{du^2}y(u)\right)+(-u-1)\left(\frac{d}{du}y(u)\right)+y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ◦ Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-1}$

- ◦ Shift index using $k->k+1$

$u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$

Rewrite ODE with series expansions

$a_0r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(k+r-1)-a_k(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r)=0$

- Values of r that satisfy the indicial equation
  $r \in \{0,2\}$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-1)\,(a_{k+1}(k+1+r)-a_k)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{a_k}{k+1+r}$
- Recursion relation for $r=0$
  $a_{k+1}=\frac{a_k}{k+1}$
- Solution for $r=0$
  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+1}=\frac{a_k}{k+1}\right]$
- Revert the change of variables $u=x-1$
  $\left[y(x)=\sum_{k=0}^{\infty} a_k(x-1)^k, a_{k+1}=\frac{a_k}{k+1}\right]$
- Recursion relation for $r=2$
  $a_{k+1}=\frac{a_k}{k+3}$
- Solution for $r=2$
  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$
- Revert the change of variables $u=x-1$
  $\left[y(x)=\sum_{k=0}^{\infty} a_k(x-1)^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$
- Combine solutions and rename parameters
  $\left[y(x)=\left(\sum_{k=0}^{\infty} a_k(x-1)^k\right)+\left(\sum_{k=0}^{\infty} b_k(x-1)^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+3}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 12

```
dsolve((x-1)*diff(diff(y(x),x),x)-diff(y(x),x)*x+y(x) = 0,y(x),singsol=all)
```

$$y = c_1 x + e^{x} c_2$$

Mathematica DSolve solution

Solving time : `0.154` (sec)
Leaf size : 90

```
DSolve[{(x-1)*D[y[x],{x,2}]-x*D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to \exp\left(\int_1^x \frac{K[1]-2}{2(K[1]-1)}dK[1]-\frac{1}{2}\int_1^x -\frac{K[2]}{K[2]-1}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]}\frac{K[1]-2}{2(K[1]-1)}dK[1]\right)dK[3]+c_1\right)$$

2.1.318 Problem 325

Internal problem ID [9490]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 325
Date solved : `Monday, January 27, 2025 at 06:03:24 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + \left(\frac{5}{3}x + x^2\right)y' - \frac{y}{3} = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.412 (sec)

Writing the ode as

$$x^2y'' + \left(\frac{5}{3}x + x^2\right)y' - \frac{y}{3} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= \frac{5}{3}x + x^2 \\ C &= -\frac{1}{3} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9x^2 + 30x + 7}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9x^2 + 30x + 7 \\ t &= 36x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{9x^2 + 30x + 7}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.599: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{7}{36x^2} + \frac{5}{6x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{6} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{5}{6x} - \frac{1}{2x^2} + \frac{5}{6x^3} - \frac{59}{36x^4} + \frac{385}{108x^5} - \frac{2681}{324x^6} + \frac{19525}{972x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{9x^2 + 30x + 7}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{30x + 7}{36x^2}\right) \\ &= \frac{1}{4} + \frac{30x + 7}{36x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 30. Dividing this by leading coefficient in t which is 36 gives $\frac{5}{6}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{5}{6}\right) - (0) \\ &= \frac{5}{6} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{5}{6}}{\frac{1}{2}} - 0\right) &= \frac{5}{6} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{5}{6}}{\frac{1}{2}} - 0\right) &= -\frac{5}{6}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{9x^2 + 30x + 7}{36x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{5}{6}$	$-\frac{5}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{6}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\
&= \frac{5}{6} - \left(-\frac{1}{6}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{6x} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{6x} + \frac{1}{2} \\
&= -\frac{1}{6x} + \frac{1}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{6x}+\frac{1}{2}\right)(1)+\left(\left(\frac{1}{6x^2}\right)+\left(-\frac{1}{6x}+\frac{1}{2}\right)^2-\left(\frac{9x^2+30x+7}{36x^2}\right)\right)=0$$
$$\frac{-1-3a_0}{3x}=0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0=-\frac{1}{3}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x)=x-\frac{1}{3}$$

Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x)&=pe^{\int \omega\,dx}\\
&=\left(x-\frac{1}{3}\right)e^{\int\left(-\frac{1}{6x}+\frac{1}{2}\right)dx}\\
&=\left(x-\frac{1}{3}\right)e^{\frac{x}{2}-\frac{\ln(x)}{6}}\\
&=\frac{(-1+3x)\,e^{\frac{x}{2}}}{3x^{1/6}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1&=z_1e^{\int-\frac{1}{2}\frac{B}{A}\,dx}\\
&=z_1e^{-\int\frac{1}{2}\frac{\frac{5}{3}x+x^2}{x^2}\,dx}\\
&=z_1e^{-\frac{x}{2}-\frac{5\ln(x)}{6}}\\
&=z_1\left(\frac{e^{-\frac{x}{2}}}{x^{5/6}}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\frac{-1+3x}{3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int\frac{e^{\int-\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2&=y_1\int\frac{e^{\int-\frac{\frac{5}{3}x+x^2}{x^2}\,dx}}{(y_1)^2}\,dx\\
&=y_1\int\frac{e^{-x-\frac{5\ln(x)}{3}}}{(y_1)^2}\,dx\\
&=y_1\left(\int\frac{9\,e^{-x-\frac{5\ln(x)}{3}}x^2}{(-1+3x)^2}dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y&=c_1y_1+c_2y_2\\
&=c_1\left(\frac{-1+3x}{3x}\right)+c_2\left(\frac{-1+3x}{3x}\left(\int\frac{9\,e^{-x-\frac{5\ln(x)}{3}}x^2}{(-1+3x)^2}dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right)+\left(\frac{5}{3}x+x^2\right)\left(\frac{d}{dx}y(x)\right)-\frac{y(x)}{3}=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{y(x)}{3x^2}-\frac{(5+3x)\left(\frac{d}{dx}y(x)\right)}{3x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{(5+3x)\left(\frac{d}{dx}y(x)\right)}{3x}-\frac{y(x)}{3x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{5+3x}{3x},P_3(x)=-\frac{1}{3x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=\frac{5}{3}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=-\frac{1}{3}$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $3x^2\left(\frac{d^2}{dx^2}y(x)\right)+x(5+3x)\left(\frac{d}{dx}y(x)\right)-y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-1+3r)\,x^r+\left(\sum_{k=1}^{\infty}\left(a_k(k+r+1)\,(3k+3r-1)+3a_{k-1}(k+r-1)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-1+3r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{-1,\frac{1}{3}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $3(k+r+1)\left(k-\frac{1}{3}+r\right)a_k+3a_{k-1}(k+r-1)=0$

- Shift index using $k->k+1$

$3(k+2+r)\left(k+\frac{2}{3}+r\right)a_{k+1}+3a_k(k+r)=0$

- Recursion relation that defines series solution to ODE
  $a_{k+1}=-\frac{3a_k(k+r)}{(k+2+r)(3k+2+3r)}$
- Recursion relation for $r=-1$; series terminates at $k=1$
  $a_{k+1}=-\frac{3a_k(k-1)}{(k+1)(3k-1)}$
- Apply recursion relation for $k=0$
  $a_1=-3a_0$
- Terminating series solution of the ODE for $r=-1$. Use reduction of order to find the second lin
  $y(x)=a_0\cdot(1-3x)$
- Recursion relation for $r=\frac{1}{3}$
  $a_{k+1}=-\frac{3a_k\left(k+\frac{1}{3}\right)}{\left(k+\frac{7}{3}\right)(3k+3)}$
- Solution for $r=\frac{1}{3}$
  $\left[y(x)=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{3}}, a_{k+1}=-\frac{3a_k\left(k+\frac{1}{3}\right)}{\left(k+\frac{7}{3}\right)(3k+3)}\right]$
- Combine solutions and rename parameters
  $\left[y(x)=a_0\cdot(1-3x)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{3}}\right), b_{k+1}=-\frac{3b_k\left(k+\frac{1}{3}\right)}{\left(k+\frac{7}{3}\right)(3k+3)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solut
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.049 (sec)
Leaf size : 29

```
dsolve(x^2*diff(diff(y(x),x),x)+(5/3*x+x^2)*diff(y(x),x)-1/3*y(x) = 0,y(x),singsol=all
```

$$y = \frac{c_1 x^{4/3} \operatorname{hypergeom}\left([2], \left[\frac{7}{3}\right], x\right) \mathrm{e}^{-x} - 3c_2 x + c_2}{x}$$

Mathematica DSolve solution

Solving time : 0.763 (sec)
Leaf size : 52

```
DSolve[{x^2*D[y[x],{x,2}]+(5/3*x+x^2)*D[y[x],x]-1/3*y[x]==0,{}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to \frac{(3x-1)\left(c_2 \int_1^x \frac{9e^{-K[1]}\sqrt[3]{K[1]}}{(1-3K[1])^2} dK[1] + c_1\right)}{3x}$$

2.1.319 Problem 326

Internal problem ID [9491]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 326
Date solved : `Monday, January 27, 2025 at 06:03:24 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$2xy'' - y' + 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.201 (sec)

Writing the ode as

$$2xy'' - y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x \\ B &= -1 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5 - 16x}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 5 - 16x \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5 - 16x}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.601: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s)\\
&= 2 - 1\\
&= 1
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16x^2} - \frac{1}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\}\\
&= \{-1, 2, 5\}
\end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{-1,2,5\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = -1, \; e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(1 - (-1)) \\ &= 1 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{-1}{(x-(0))}\right) \\ &= -\frac{1}{2x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 1$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right) p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right) p = 0 \tag{1A}$$

Since $d = 1$, then letting

$$p = x + a_0 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$\frac{1 - 4a_0}{x^2} = 0$$

And solving for p gives

$$p = x + \frac{1}{4}$$

Now that $p(x)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{x+\frac{1}{4}} - \frac{1}{2x} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{x+\frac{1}{4}} - \frac{1}{2x}\right) w + \frac{64x^2 - 12x + 1}{64x^3 + 16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{16x\sqrt{-x} + 4x - 1}{4\,(4x+1)\,x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{16x\sqrt{-x}+4x-1}{4(4x+1)x} dx} \\
&= \frac{\left(2\sqrt{-x}-1\right) \mathrm{e}^{2\sqrt{-x}}}{(-x)^{1/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{2x} \, dx} \\
&= z_1 e^{\frac{\ln(x)}{4}} \\
&= z_1 \left(x^{1/4}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^{1/4}\left(2\sqrt{-x}-1\right) \mathrm{e}^{2\sqrt{-x}}}{(-x)^{1/4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{2x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{\ln(x)}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{\mathrm{e}^{-4\sqrt{-x}}}{8} + \frac{\mathrm{e}^{-4\sqrt{-x}}}{8\sqrt{-x}-4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{1/4}\left(2\sqrt{-x}-1\right) \mathrm{e}^{2\sqrt{-x}}}{(-x)^{1/4}}\right) + c_2 \left(\frac{x^{1/4}\left(2\sqrt{-x}-1\right) \mathrm{e}^{2\sqrt{-x}}}{(-x)^{1/4}} \left(\frac{\mathrm{e}^{-4\sqrt{-x}}}{8} + \frac{\mathrm{e}^{-4\sqrt{-x}}}{8\sqrt{-x}-4}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2\left(\frac{d^2}{dx^2}y(x)\right)x - \frac{d}{dx}y(x) + 2y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{y(x)}{x} + \frac{\frac{d}{dx}y(x)}{2x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) - \frac{\frac{d}{dx}y(x)}{2x} + \frac{y(x)}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{1}{2x}, P_3(x) = \frac{1}{x}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -\frac{1}{2}$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$
  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2\left(\frac{d^2}{dx^2}y(x)\right)x - \frac{d}{dx}y(x) + 2y(x) = 0$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $\frac{d}{dx}y(x)$ to series expansion

    $\frac{d}{dx}y(x) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$
  - Shift index using $k- >k+1$

    $\frac{d}{dx}y(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$
  - Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-1}$
  - Shift index using $k- >k+1$

    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-3+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(2k-1+2r) + 2a_k\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation
$2(k+1+r)\left(k+r-\frac{1}{2}\right)a_{k+1}+2a_k=0$
- Recursion relation that defines series solution to ODE
$a_{k+1}=-\frac{2a_k}{(k+1+r)(2k-1+2r)}$
- Recursion relation for $r=0$
$a_{k+1}=-\frac{2a_k}{(k+1)(2k-1)}$
- Solution for $r=0$
$\left[y(x)=\sum_{k=0}^{\infty}a_k x^k, a_{k+1}=-\frac{2a_k}{(k+1)(2k-1)}\right]$
- Recursion relation for $r=\frac{3}{2}$
$a_{k+1}=-\frac{2a_k}{\left(k+\frac{5}{2}\right)(2k+2)}$
- Solution for $r=\frac{3}{2}$
$\left[y(x)=\sum_{k=0}^{\infty}a_k x^{k+\frac{3}{2}}, a_{k+1}=-\frac{2a_k}{\left(k+\frac{5}{2}\right)(2k+2)}\right]$
- Combine solutions and rename parameters
$\left[y(x)=\left(\sum_{k=0}^{\infty}a_k x^k\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{3}{2}}\right), a_{k+1}=-\frac{2a_k}{(k+1)(2k-1)}, b_{k+1}=-\frac{2b_k}{\left(k+\frac{5}{2}\right)(2k+2)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.017` (sec)
Leaf size : 36

```
dsolve(2*x*diff(diff(y(x),x),x)-diff(y(x),x)+2*y(x) = 0,y(x),singsol=all)
```

$$y=\left(2\sqrt{x}\,c_1+c_2\right)\cos\left(2\sqrt{x}\right)-\sin\left(2\sqrt{x}\right)\left(-2c_2\sqrt{x}+c_1\right)$$

Mathematica DSolve solution

Solving time : `0.205` (sec)
Leaf size : 74

```
DSolve[{2*x*D[y[x],{x,2}]-D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to e^{2i\sqrt{x}}\left(2\sqrt{x}+i\right)\left(c_2\int_1^x\frac{e^{-4i\sqrt{K[1]}}\sqrt{K[1]}}{\left(2\sqrt{K[1]}+i\right)^2}dK[1]+c_1\right)$$

2.1.320 Problem 327

Internal problem ID [9492]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 327
Date solved : Monday, January 27, 2025 at 06:03:25 PM
CAS classification : [_Laguerre]

Solve

$$2xy'' - (3 + 2x) y' + y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.382 (sec)

Writing the ode as

$$2xy'' + (-3 - 2x) y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x \\ B &= -3 - 2x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 + 4x + 21}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2 + 4x + 21 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 + 4x + 21}{16x^2} \right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.603: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{1}{4x} + \frac{21}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{4x} + \frac{5}{4x^2} - \frac{5}{8x^3} - \frac{5}{4x^4} + \frac{35}{16x^5} + \frac{105}{64x^6} - \frac{1005}{128x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $([\sqrt{r}]_\infty)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$([\sqrt{r}]_\infty)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 + 4x + 21}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{4x + 21}{16x^2}\right) \\ &= \frac{1}{4} + \frac{4x + 21}{16x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 16 gives $\frac{1}{4}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{4}\right) - (0) \\ &= \frac{1}{4} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{4}}{\frac{1}{2}} - 0\right) &&= \frac{1}{4} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{4}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^2 + 4x + 21}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= \frac{1}{4} - \left(-\frac{3}{4}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{3}{4x} + \left(\frac{1}{2}\right) \\ &= -\frac{3}{4x} + \frac{1}{2} \\ &= -\frac{3}{4x} + \frac{1}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{4x} + \frac{1}{2}\right)(1) + \left(\left(\frac{3}{4x^2}\right) + \left(-\frac{3}{4x} + \frac{1}{2}\right)^2 - \left(\frac{4x^2+4x+21}{16x^2}\right)\right) = 0$$

$$\frac{-3-2a_0}{2x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{3}{2}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - \frac{3}{2}$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{3}{2}\right) e^{\int \left(-\frac{3}{4x} + \frac{1}{2}\right) dx} \\
&= \left(x - \frac{3}{2}\right) e^{\frac{x}{2} - \frac{3 \ln(x)}{4}} \\
&= \frac{(-3 + 2x) e^{\frac{x}{2}}}{2x^{3/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-3-2x}{2x} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{3 \ln(x)}{4}} \\
&= z_1 \left(x^{3/4} e^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{e^{x}(-3 + 2x)}{2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3-2x}{2x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{x + \frac{3 \ln(x)}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{4 e^{x + \frac{3 \ln(x)}{2}} e^{-2x}}{(-3 + 2x)^2} \, dx \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{x}(-3 + 2x)}{2} \right) + c_2 \left(\frac{e^{x}(-3 + 2x)}{2} \left(\int \frac{4 e^{x + \frac{3 \ln(x)}{2}} e^{-2x}}{(-3 + 2x)^2} \, dx \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2\left(\frac{d^2}{dx^2}y(x)\right)x-(2x+3)\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{y(x)}{2x}+\frac{(2x+3)\left(\frac{d}{dx}y(x)\right)}{2x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)-\frac{(2x+3)\left(\frac{d}{dx}y(x)\right)}{2x}+\frac{y(x)}{2x}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{2x+3}{2x},P_3(x)=\frac{1}{2x}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\Big|_{x=0}=-\frac{3}{2}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=0$

  ◦ $x=0$is a regular singular point

Check to see if $x_0=0$ is a regular singular point

$x_0=0$

- Multiply by denominators

$2\left(\frac{d^2}{dx^2}y(x)\right)x+(-2x-3)\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..1$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-1}$

  ◦ Shift index using $k->k+1$

  $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

Rewrite ODE with series expansions

$a_0r(-5+2r)\,x^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(2k-3+2r)-a_k(2k+2r-1)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-5+2r)=0$

- Values of r that satisfy the indicial equation

$r\in\left\{0,\frac{5}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation
  $2(k+1+r)\left(k+r-\frac{3}{2}\right)a_{k+1}-2\left(k+r-\frac{1}{2}\right)a_k=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{(2k+2r-1)a_k}{(k+1+r)(2k-3+2r)}$
- Recursion relation for $r=0$
  $a_{k+1}=\frac{(2k-1)a_k}{(k+1)(2k-3)}$
- Solution for $r=0$
  $\left[y(x)=\sum_{k=0}^{\infty}a_k x^k, a_{k+1}=\frac{(2k-1)a_k}{(k+1)(2k-3)}\right]$
- Recursion relation for $r=\frac{5}{2}$
  $a_{k+1}=\frac{(2k+4)a_k}{\left(k+\frac{7}{2}\right)(2k+2)}$
- Solution for $r=\frac{5}{2}$
  $\left[y(x)=\sum_{k=0}^{\infty}a_k x^{k+\frac{5}{2}}, a_{k+1}=\frac{(2k+4)a_k}{\left(k+\frac{7}{2}\right)(2k+2)}\right]$
- Combine solutions and rename parameters
  $\left[y(x)=\left(\sum_{k=0}^{\infty}a_k x^k\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{5}{2}}\right), a_{k+1}=\frac{(2k-1)a_k}{(k+1)(2k-3)}, b_{k+1}=\frac{(2k+4)b_k}{\left(k+\frac{7}{2}\right)(2k+2)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solut
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returnin
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.040 (sec)
Leaf size : 24

```
dsolve(2*x*diff(diff(y(x),x),x)-(2*x+3)*diff(y(x),x)+y(x) = 0,y(x),singsol=all)
```

$$y = c_1 \operatorname{hypergeom}\left([2], \left[\frac{7}{2}\right], x\right) x^{5/2} - \frac{2c_2\left(x - \frac{3}{2}\right) e^{x}}{3}$$

Mathematica DSolve solution

Solving time : 0.9 (sec)
Leaf size : 52

```
DSolve[{2*x*D[y[x],{x,2}]-(3+2*x)*D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tru
```

$$y(x) \to \frac{1}{2}e^{x}(2x-3)\left(c_2 \int_1^x \frac{4e^{-K[1]}K[1]^{3/2}}{(3-2K[1])^2}dK[1] + c_1\right)$$

2.1.321 Problem 328

Internal problem ID [9493]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 328
Date solved : `Monday, January 27, 2025 at 06:03:26 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2y''+3xy'+(2x-1)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.207 (sec)

Writing the ode as

$$2x^2y''+3xy'+(2x-1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=2x^2\\ B&=3x\\ C&=2x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{5-16x}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=5-16x\\ t&=16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{5-16x}{16x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.605: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{x} + \frac{5}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{-1, 2, 5\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = -1,\ e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(1-(-1)) \\ &= 1 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{-1}{(x-(0))}\right) \\ &= -\frac{1}{2x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 1$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 1$, then letting

$$p = x + a_0 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$\frac{1-4a_0}{x^2} = 0$$

And solving for p gives

$$p = x + \frac{1}{4}$$

Now that $p(x)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{x+\frac{1}{4}} - \frac{1}{2x} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{x+\frac{1}{4}} - \frac{1}{2x}\right)w + \frac{64x^2 - 12x + 1}{64x^3 + 16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{16x\sqrt{-x} + 4x - 1}{4\left(4x+1\right)x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{16x\sqrt{-x}+4x-1}{4(4x+1)x} dx} \\
&= \frac{\left(2\sqrt{-x}-1\right) \mathrm{e}^{2\sqrt{-x}}}{\left(-x\right)^{1/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x}{2x^2} \, dx} \\
&= z_1 e^{-\frac{3\ln(x)}{4}} \\
&= z_1 \left(\frac{1}{x^{3/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\left(2\sqrt{-x}-1\right) \mathrm{e}^{2\sqrt{-x}}}{x^{3/4} \left(-x\right)^{1/4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{2x^2} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{3\ln(x)}{2}}}{\left(y_1\right)^2} \, dx \\
&= y_1 \left(\frac{\mathrm{e}^{-4\sqrt{-x}}}{8} + \frac{\mathrm{e}^{-4\sqrt{-x}}}{8\sqrt{-x}-4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\left(2\sqrt{-x}-1\right) \mathrm{e}^{2\sqrt{-x}}}{x^{3/4} \left(-x\right)^{1/4}}\right) + c_2 \left(\frac{\left(2\sqrt{-x}-1\right) \mathrm{e}^{2\sqrt{-x}}}{x^{3/4} \left(-x\right)^{1/4}} \left(\frac{\mathrm{e}^{-4\sqrt{-x}}}{8} + \frac{\mathrm{e}^{-4\sqrt{-x}}}{8\sqrt{-x}-4}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2\left(\frac{d^2}{dx^2}y(x)\right)+3x\left(\frac{d}{dx}y(x)\right)+(2x-1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(2x-1)y(x)}{2x^2}-\frac{3\left(\frac{d}{dx}y(x)\right)}{2x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d^2}{dx^2}y(x)+\frac{3\left(\frac{d}{dx}y(x)\right)}{2x}+\frac{(2x-1)y(x)}{2x^2}=0$

☐ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{3}{2x},P_3(x)=\frac{2x-1}{2x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{3}{2}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=-\frac{1}{2}$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2x^2\left(\frac{d^2}{dx^2}y(x)\right)+3x\left(\frac{d}{dx}y(x)\right)+(2x-1)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  ◦ Shift index using $k->k-m$

  $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r}$

  ◦ Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-1+2r)\,x^r+\left(\sum_{k=1}^{\infty}\left(a_k(k+r+1)\,(2k+2r-1)+2a_{k-1}\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-1+2r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{-1,\frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation
  $2(k+r+1)\left(k+r-\frac{1}{2}\right)a_k+2a_{k-1}=0$
- Shift index using $k->k+1$
  $2(k+2+r)\left(k+\frac{1}{2}+r\right)a_{k+1}+2a_k=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=-\frac{2a_k}{(k+2+r)(2k+1+2r)}$
- Recursion relation for $r=-1$
  $a_{k+1}=-\frac{2a_k}{(k+1)(2k-1)}$
- Solution for $r=-1$
  $\left[y(x)=\sum_{k=0}^{\infty}a_k x^{k-1}, a_{k+1}=-\frac{2a_k}{(k+1)(2k-1)}\right]$
- Recursion relation for $r=\frac{1}{2}$
  $a_{k+1}=-\frac{2a_k}{\left(k+\frac{5}{2}\right)(2k+2)}$
- Solution for $r=\frac{1}{2}$
  $\left[y(x)=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}, a_{k+1}=-\frac{2a_k}{\left(k+\frac{5}{2}\right)(2k+2)}\right]$
- Combine solutions and rename parameters
  $\left[y(x)=\left(\sum_{k=0}^{\infty}a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{2}}\right), a_{k+1}=-\frac{2a_k}{(k+1)(2k-1)}, b_{k+1}=-\frac{2b_k}{\left(k+\frac{5}{2}\right)(2k+2)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 2.034 (sec)
Leaf size : 73

```
dsolve(2*x^2*diff(diff(y(x),x),x)+3*diff(y(x),x)*x+(-1+2*x)*y(x) = 0,y(x),singsol=all)
```

$$y=\frac{c_2\sqrt{\frac{(-2\sqrt{x}+i)(4x+1)}{2\sqrt{x}+i}}\,\mathrm{e}^{-2i\sqrt{x}}+c_1\sqrt{\frac{(2\sqrt{x}+i)(4x+1)}{-2\sqrt{x}+i}}\,\mathrm{e}^{2i\sqrt{x}}}{x}$$

Mathematica DSolve solution

Solving time : 0.211 (sec)
Leaf size : 77

```
DSolve[{2*x^2*D[y[x],{x,2}]+3*x*D[y[x],x]+(2*x-1)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->
```

$$y(x) \to \frac{e^{2i\sqrt{x}}(2\sqrt{x}+i)\left(c_2\int_1^x \frac{e^{-4i\sqrt{K[1]}}\sqrt{K[1]}}{\left(2\sqrt{K[1]}+i\right)^2}dK[1]+c_1\right)}{x}$$

2.1.322 Problem 329

Internal problem ID [9494]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 329
Date solved : `Monday, January 27, 2025 at 06:03:26 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + 2y' - xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.058 (sec)

Writing the ode as

$$xy'' + 2y' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.607: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\, dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1 \left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{-x}}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{e^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{e^{-x}}{x}\right) + c_2 \left(\frac{e^{-x}}{x}\left(\frac{e^{2x}}{2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dx^2}y(x)\right)x + 2\frac{d}{dx}y(x) - xy(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = y(x) - \frac{2\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x) + \frac{2\left(\frac{d}{dx}y(x)\right)}{x} - y(x) = 0$
- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{2}{x}, P_3(x) = -1\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $\left(\frac{d^2}{dx^2}y(x)\right)x + 2\frac{d}{dx}y(x) - xy(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- □ Rewrite ODE with series expansions
  - Convert $x \cdot y(x)$ to series expansion
    $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

- Shift index using $k->k-1$
  $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$
- Convert $\frac{d}{dx}y(x)$ to series expansion
  $\frac{d}{dx}y(x) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$
- Shift index using $k->k+1$
  $\frac{d}{dx}y(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$
- Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion
  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$
- Shift index using $k->k+1$
  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions
  $a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) - a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(1+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-1, 0\}$
- Each term must be 0
  $a_1(1+r)\,(2+r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+r+1)\,(k+2+r) - a_{k-1} = 0$
- Shift index using $k->k+1$
  $a_{k+2}(k+2+r)\,(k+3+r) - a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{a_k}{(k+2+r)(k+3+r)}$
- Recursion relation for $r=-1$
  $a_{k+2} = \frac{a_k}{(k+1)(k+2)}$
- Solution for $r=-1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = \frac{a_k}{(k+1)(k+2)}, 0 = 0\right]$
- Recursion relation for $r=0$
  $a_{k+2} = \frac{a_k}{(k+2)(k+3)}$
- Solution for $r=0$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = \frac{a_k}{(k+1)(k+2)}, 0 = 0, b_{k+2} = \frac{b_k}{(k+2)(k+3)}, 2b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 17

```
dsolve(x*diff(diff(y(x),x),x)+2*diff(y(x),x)-x*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_1 \sinh(x) + c_2 \cosh(x)}{x}$$

Mathematica DSolve solution

Solving time : 0.025 (sec)
Leaf size : 28

```
DSolve[{x*D[y[x],{x,2}]+2*D[y[x],x]-x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2c_1 e^{-x} + c_2 e^{x}}{2x}$$

2.1.323 Problem 330

Internal problem ID [9495]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 330
Date solved : `Monday, January 27, 2025 at 06:03:27 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.141 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.609: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\, dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2\left(\frac{\cos(x)}{\sqrt{x}}(\tan(x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x\left(\frac{d}{dx}y(x)\right) + \left(x^2 - \frac{1}{4}\right)y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(4x^2-1)y(x)}{4x^2} - \frac{\frac{d}{dx}y(x)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{\frac{d}{dx}y(x)}{x} + \frac{(4x^2-1)y(x)}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x\left(\frac{d}{dx}y(x)\right) + (4x^2 - 1)y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- Shift index using $k->k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$
- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$
- Shift index using $k->k+2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$
- Solution for $r = -\frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$
- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.039 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x+(x^2-1/4)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{\sin(x) c_1 + \cos(x) c_2}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.035 (sec)
Leaf size : 39

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-1/4)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to \frac{e^{-ix}(2c_1 - ic_2 e^{2ix})}{2\sqrt{x}}$$

2.1.324 Problem 331

Internal problem ID [9496]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 331
Date solved : `Monday, January 27, 2025 at 06:03:27 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + (x-6)\,y' - 3y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.264 (sec)

Writing the ode as

$$xy'' + (x-6)\,y' - 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= x-6 \\ C &= -3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2+48}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 48 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2+48}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.611: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1,2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{12}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 12$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 4 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -3 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{12}{x^2} - \frac{144}{x^4} + \frac{3456}{x^6} - \frac{103680}{x^8} + \frac{3483648}{x^{10}} - \frac{125411328}{x^{12}} + \frac{4729798656}{x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 48}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{12}{x^2}\right) \\ &= \frac{1}{4} + \frac{12}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 4 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{\frac{1}{2}} - 0\right) &= 0 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{\frac{1}{2}} - 0\right) &= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 48}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	4	-3

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= 0 - (-3) \\ &= 3 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{3}{x} + (-)\left(\frac{1}{2}\right) \\ &= -\frac{3}{x} - \frac{1}{2} \\ &= -\frac{6+x}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 3$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(6x + 2a_2) + 2\left(-\frac{3}{x} - \frac{1}{2}\right)(3x^2 + 2a_2x + a_1) + \left(\left(\frac{3}{x^2}\right) + \left(-\frac{3}{x} - \frac{1}{2}\right)^2 - \left(\frac{x^2+48}{4x^2}\right)\right) = 0$$

$$\frac{(a_2 - 12)\, x^2 + 2(a_1 - 5a_2)\, x + 3a_0 - 6a_1}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 120, a_1 = 60, a_2 = 12\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^3 + 12x^2 + 60x + 120$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x^3 + 12x^2 + 60x + 120\right) \mathrm{e}^{\int \left(-\frac{3}{x} - \frac{1}{2}\right) dx} \\ &= \left(x^3 + 12x^2 + 60x + 120\right) \mathrm{e}^{-\frac{x}{2} - 3\ln(x)} \\ &= \frac{\left(x^3 + 12x^2 + 60x + 120\right) \mathrm{e}^{-\frac{x}{2}}}{x^3} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x-6}{x} \, dx} \\ &= z_1 e^{-\frac{x}{2} + 3\ln(x)} \\ &= z_1 \left(x^3 \mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x} \left(x^3 + 12x^2 + 60x + 120\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x-6}{x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-x + 6\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{\left(x^3 - 12x^2 + 60x - 120\right) \mathrm{e}^{-x + 6\ln(x)} \mathrm{e}^{2x}}{\left(x^3 + 12x^2 + 60x + 120\right) x^6} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-x} \left(x^3 + 12x^2 + 60x + 120\right)\right) \\ &\quad + c_2 \left(\mathrm{e}^{-x} \left(x^3 + 12x^2 + 60x + 120\right) \left(\frac{\left(x^3 - 12x^2 + 60x - 120\right) \mathrm{e}^{-x + 6\ln(x)} \mathrm{e}^{2x}}{\left(x^3 + 12x^2 + 60x + 120\right) x^6} \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dx^2}y(x)\right)x+(-6+x)\left(\frac{d}{dx}y(x)\right)-3y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{3y(x)}{x}-\frac{(-6+x)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{(-6+x)\left(\frac{d}{dx}y(x)\right)}{x}-\frac{3y(x)}{x}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{-6+x}{x}, P_3(x)=-\frac{3}{x}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=-6$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $\left(\frac{d^2}{dx^2}y(x)\right)x+(-6+x)\left(\frac{d}{dx}y(x)\right)-3y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..1$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-1}$

  ◦ Shift index using $k->k+1$

  $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-7+r)\,x^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(k-6+r)+a_k(k+r-3)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-7+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{0,7\}$

- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)(k-6+r)+a_k(k+r-3)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=-\frac{a_k(k+r-3)}{(k+1+r)(k-6+r)}$
- Recursion relation for $r=0$; series terminates at $k=3$
  $a_{k+1}=-\frac{a_k(k-3)}{(k+1)(k-6)}$
- Apply recursion relation for $k=0$
  $a_1=-\frac{a_0}{2}$
- Apply recursion relation for $k=1$
  $a_2=-\frac{a_1}{5}$
- Express in terms of a_0
  $a_2=\frac{a_0}{10}$
- Apply recursion relation for $k=2$
  $a_3=-\frac{a_2}{12}$
- Express in terms of a_0
  $a_3=-\frac{a_0}{120}$
- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li
  $y(x)=a_0\cdot\left(1-\frac{1}{2}x+\frac{1}{10}x^2-\frac{1}{120}x^3\right)$
- Recursion relation for $r=7$
  $a_{k+1}=-\frac{a_k(k+4)}{(k+8)(k+1)}$
- Solution for $r=7$
  $\left[y(x)=\sum_{k=0}^{\infty}a_k x^{k+7}, a_{k+1}=-\frac{a_k(k+4)}{(k+8)(k+1)}\right]$
- Combine solutions and rename parameters
  $\left[y(x)=a_0\cdot\left(1-\frac{1}{2}x+\frac{1}{10}x^2-\frac{1}{120}x^3\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+7}\right), b_{k+1}=-\frac{b_k(4+k)}{(k+8)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 39

```
dsolve(x*diff(diff(y(x),x),x)+(x-6)*diff(y(x),x)-3*y(x) = 0,y(x),singsol=all)
```

$$y=c_1\left(x^3-12x^2+60x-120\right)+c_2\mathrm{e}^{-x}\left(x^3+12x^2+60x+120\right)$$

Mathematica DSolve solution

Solving time : 0.078 (sec)
Leaf size : 98

```
DSolve[{x*D[y[x],{x,2}]+(x-6)*D[y[x],x]-3*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2e^{-x/2}\sqrt{x}\left((c_1x^3+12ic_2x^2+60c_1x+120ic_2)\cosh\left(\frac{x}{2}\right)-(12c_1(x^2+10)+ic_2x(x^2+60))\sinh\left(\frac{x}{2}\right)\right)}{\sqrt{\pi}\sqrt{-ix}}$$

2.1.325 Problem 332

Internal problem ID [9497]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 332
Date solved : `Monday, January 27, 2025 at 06:03:28 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$x^4y'' + \lambda y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.237 (sec)

Writing the ode as

$$x^4y'' + \lambda y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4 \\ B &= 0 \\ C &= \lambda \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-\lambda}{x^4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -\lambda \\ t &= x^4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{\lambda}{x^4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.613: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^4$. There is a pole at $x = 0$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned} \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) \end{aligned}$$

The partial fraction decomposition of r is

$$r = -\frac{\lambda}{x^4}$$

There is pole in r at $x = 0$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{i\sqrt{\lambda}}{x^2} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{i\sqrt{\lambda}}{x^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^2}$ is

$$a = i\sqrt{\lambda}$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be 0. Therefore

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{i\sqrt{\lambda}}{x^2} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &&= \frac{1}{2}\left(\frac{0}{i\sqrt{\lambda}} + 2\right) &&= 1 \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &&= \frac{1}{2}\left(-\frac{0}{i\sqrt{\lambda}} + 2\right) &&= 1 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{\lambda}{x^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	4	$\frac{i\sqrt{\lambda}}{x^2}$	1	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{i\sqrt{\lambda}}{x^2} + \frac{1}{x} + (-)\,(0) \\
&= -\frac{i\sqrt{\lambda}}{x^2} + \frac{1}{x} \\
&= \frac{-i\sqrt{\lambda} + x}{x^2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{i\sqrt{\lambda}}{x^2} + \frac{1}{x}\right)(0) + \left(\left(\frac{2i\sqrt{\lambda}}{x^3} - \frac{1}{x^2}\right) + \left(-\frac{i\sqrt{\lambda}}{x^2} + \frac{1}{x}\right)^2 - \left(-\frac{\lambda}{x^4}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(-\frac{i\sqrt{\lambda}}{x^2} + \frac{1}{x}\right) dx} \\
&= x\, \mathrm{e}^{\frac{i\sqrt{\lambda}}{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= x\, \mathrm{e}^{\frac{i\sqrt{\lambda}}{x}}
\end{aligned}$$

Which simplifies to

$$y_1 = x\, \mathrm{e}^{\frac{i\sqrt{\lambda}}{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= x\, \mathrm{e}^{\frac{i\sqrt{\lambda}}{x}} \int \frac{1}{x^2 \mathrm{e}^{\frac{2i\sqrt{\lambda}}{x}}}\, dx \\
&= x\, \mathrm{e}^{\frac{i\sqrt{\lambda}}{x}} \left(-\frac{i\mathrm{e}^{-\frac{2i\sqrt{\lambda}}{x}}}{2\sqrt{\lambda}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x\, \mathrm{e}^{\frac{i\sqrt{\lambda}}{x}}\right) + c_2 \left(x\, \mathrm{e}^{\frac{i\sqrt{\lambda}}{x}} \left(-\frac{i\mathrm{e}^{-\frac{2i\sqrt{\lambda}}{x}}}{2\sqrt{\lambda}}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 31

```
dsolve(diff(diff(y(x),x),x)*x^4+lambda*y(x) = 0,y(x),singsol=all)
```

$$
y = x \left(c_1 \sinh\left(\frac{\sqrt{-\lambda}}{x}\right) + c_2 \cosh\left(\frac{\sqrt{-\lambda}}{x}\right)\right)
$$

Mathematica DSolve solution

Solving time : 0.142 (sec)
Leaf size : 56

```
DSolve[{x^4*D[y[x],{x,2}]+\[Lambda]*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$
y(x) \to c_1 x e^{-1+\frac{i\sqrt{\lambda}}{x}} - \frac{i c_2 x e^{1-\frac{i\sqrt{\lambda}}{x}}}{2\sqrt{\lambda}}
$$

2.1.326 Problem 333

Internal problem ID [9498]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 333
Date solved : `Monday, January 27, 2025 at 06:03:29 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' + 4xy' + \left(4x^2 - 25\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.287 (sec)

Writing the ode as

$$4x^2y'' + 4xy' + \left(4x^2 - 25\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= 4x \\ C &= 4x^2 - 25 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 6}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2 + 6 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 6}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.614: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s)\\
&= 2 - 2\\
&= 0
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1,2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -1 + \frac{6}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3\\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx i - \frac{3i}{x^2} - \frac{9i}{2x^4} - \frac{27i}{2x^6} - \frac{405i}{8x^8} - \frac{1701i}{8x^{10}} - \frac{15309i}{16x^{12}} - \frac{72171i}{16x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= i \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-x^2 + 6}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-1) + \left(\frac{6}{x^2}\right) \\ &= -1 + \frac{6}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= i \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{i} - 0\right) &&= 0 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{i} - 0\right) &&= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 + 6}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	i	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= 0 - (-2) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{2}{x} + (-)(i) \\ &= -\frac{2}{x} - i \\ &= -\frac{2}{x} - i \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(-\frac{2}{x} - i\right)(2x + a_1) + \left(\left(\frac{2}{x^2}\right) + \left(-\frac{2}{x} - i\right)^2 - \left(\frac{-x^2+6}{x^2}\right) \right) = 0$$

$$\frac{2ixa_1 + 4ia_0 - 6x - 4a_1}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -3, a_1 = -3i\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 3ix - 3$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \left(x^2 - 3ix - 3\right) e^{\int \left(-\frac{2}{x} - i\right) dx} \\ &= \left(x^2 - 3ix - 3\right) e^{-2\ln(x) - ix} \\ &= \frac{\left(x^2 - 3ix - 3\right) e^{-ix}}{x^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x}{4x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{1}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{5/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{(ix^2 - 3x - 3i)\,\mathrm{e}^{2ix}}{-2x^2 + 6ix + 6}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{5/2}}\right) + c_2\left(\frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{5/2}}\left(\frac{(ix^2 - 3x - 3i)\,\mathrm{e}^{2ix}}{-2x^2 + 6ix + 6}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x\left(\frac{d}{dx}y(x)\right) + (4x^2 - 25)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(4x^2-25)y(x)}{4x^2} - \frac{\frac{d}{dx}y(x)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{\frac{d}{dx}y(x)}{x} + \frac{(4x^2-25)y(x)}{4x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - ○ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-25}{4x^2}\right]$

  - ○ $x \cdot P_2(x)$ is analytic at $x = 0$

$$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{25}{4}$$

- $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$$4x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x\left(\frac{d}{dx}y(x)\right) + (4x^2 - 25)\, y(x) = 0$$

- Assume series solution for $y(x)$

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

$$x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

$$x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

$$x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(5+2r)\,(-5+2r)\, x^r + a_1(7+2r)\,(-3+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+5)\,(2k+2r-5) + 4a\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(5+2r)\,(-5+2r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{-\frac{5}{2}, \frac{5}{2}\right\}$

- Each term must be 0

$a_1(7+2r)\,(-3+2r) = 0$

- Solve for the dependent coefficient(s)

$a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

$a_k(2k+2r+5)\,(2k+2r-5) + 4a_{k-2} = 0$

- Shift index using $k- >k + 2$

$a_{k+2}(2k+9+2r)\,(2k-1+2r) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

$a_{k+2} = -\frac{4a_k}{(2k+9+2r)(2k-1+2r)}$

- Recursion relation for $r = -\frac{5}{2}$

$a_{k+2} = -\frac{4a_k}{(2k+4)(2k-6)}$

- Solution for $r = -\frac{5}{2}$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}, a_{k+2} = -\frac{4a_k}{(2k+4)(2k-6)}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{5}{2}$

  $a_{k+2} = -\frac{4a_k}{(2k+14)(2k+4)}$
- Solution for $r = \frac{5}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{2}}, a_{k+2} = -\frac{4a_k}{(2k+14)(2k+4)}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{5}{2}}\right), a_{k+2} = -\frac{4a_k}{(2k+4)(2k-6)}, a_1 = 0, b_{k+2} = -\frac{4b_k}{(2k+14)(2k+4)}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.049 (sec)
Leaf size : 43

```
dsolve(4*x^2*diff(diff(y(x),x),x)+4*diff(y(x),x)*x+(4*x^2-25)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{-3c_2\left(ix - \frac{1}{3}x^2 + 1\right) e^{-ix} + 3\left(ix + \frac{1}{3}x^2 - 1\right) c_1 e^{ix}}{x^{5/2}}$$

Mathematica DSolve solution

Solving time : 0.066 (sec)
Leaf size : 59

```
DSolve[{4*x^2*D[y[x],{x,2}]+4*x*D[y[x],x]+(4*x^2-25)*y[x]==0,{}},y[x],x,IncludeSingularSolution
```

$$y(x) \to -\frac{\sqrt{\frac{2}{\pi}}\left(\left(-c_2x^2 + 3c_1x + 3c_2\right)\cos(x) + \left(c_1\left(x^2 - 3\right) + 3c_2x\right)\sin(x)\right)}{x^{5/2}}$$

2.1.327 Problem 334

Internal problem ID [9499]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 334
Date solved : `Monday, January 27, 2025 at 06:03:29 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(36x^2 - \frac{1}{4}\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.155 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(36x^2 - \frac{1}{4}\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= 36x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-36}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -36 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -36z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.616: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -36$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(6x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(6x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{\tan(6x)}{6}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\cos(6x)}{\sqrt{x}}\right) + c_2\left(\frac{\cos(6x)}{\sqrt{x}}\left(\frac{\tan(6x)}{6}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x\left(\frac{d}{dx}y(x)\right) + \left(36x^2 - \frac{1}{4}\right)y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(144x^2-1)y(x)}{4x^2} - \frac{\frac{d}{dx}y(x)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{\frac{d}{dx}y(x)}{x} + \frac{(144x^2-1)y(x)}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{144x^2-1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x\left(\frac{d}{dx}y(x)\right) + (144x^2 - 1)y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - ◦ Shift index using $k- >k - m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - ◦ Convert $x \cdot \left(\frac{d}{dx} y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$
  - ◦ Convert $x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-1) + 144a_{k-2}\right.\right.$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 144a_{k-2} = 0$
- Shift index using $k- >k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 144a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{144a_k}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{144a_k}{4k^2+12k+8}$
- Solution for $r = -\frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{144a_k}{4k^2+12k+8}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{144a_k}{4k^2+20k+24}$
- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{144a_k}{4k^2+20k+24}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{144a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{144b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.039 (sec)
Leaf size : 21

```
dsolve(x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x+(36*x^2-1/4)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_1 \sin(6x) + c_2 \cos(6x)}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.041 (sec)
Leaf size : 39

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(36*x^2-1/4)*y[x]==0,{}},y[x],x,IncludeSingularSolutio
```

$$y(x) \to \frac{e^{-6ix}(12c_1 - ic_2 e^{12ix})}{12\sqrt{x}}$$

2.1.328 Problem 335

Internal problem ID [9500]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 335
Date solved : `Monday, January 27, 2025 at 06:03:30 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + (x^2-2)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.261 (sec)

Writing the ode as

$$x^2y'' + (x^2-2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 0 \\ C &= x^2-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2+2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2+2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2+2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.618: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -1 + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx i - \frac{i}{x^2} - \frac{i}{2x^4} - \frac{i}{2x^6} - \frac{5i}{8x^8} - \frac{7i}{8x^{10}} - \frac{21i}{16x^{12}} - \frac{33i}{16x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= i \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-x^2 + 2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-1) + \left(\frac{2}{x^2}\right) \\ &= -1 + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= i \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{i} - 0\right) &&= 0 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{i} - 0\right) &&= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 + 2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	i	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + (-)\,(i) \\ &= -\frac{1}{x} - i \\ &= -\frac{1}{x} - i \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{x} - i\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - i\right)^2 - \left(\frac{-x^2+2}{x^2}\right) \right) &= 0 \\ \frac{2ia_0 - 2}{x} &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -i\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - i$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (x-i)\, e^{\int \left(-\frac{1}{x} - i\right) dx} \\ &= (x-i)\, e^{-\ln(x) - ix} \\ &= \frac{(x-i)\, e^{-ix}}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{(x-i)\,\mathrm{e}^{-ix}}{x} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x-i)\,\mathrm{e}^{-ix}}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \frac{(x-i)\,\mathrm{e}^{-ix}}{x} \int \frac{1}{\frac{(x-i)^2\mathrm{e}^{-2ix}}{x^2}}\,dx \\ &= \frac{(x-i)\,\mathrm{e}^{-ix}}{x}\left(\frac{(ix-1)\,\mathrm{e}^{2ix}}{-2x+2i}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{(x-i)\,\mathrm{e}^{-ix}}{x}\right) + c_2\left(\frac{(x-i)\,\mathrm{e}^{-ix}}{x}\left(\frac{(ix-1)\,\mathrm{e}^{2ix}}{-2x+2i}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right) + (x^2-2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{(x^2-2)y(x)}{x^2}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) + \frac{(x^2-2)y(x)}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = 0, P_3(x) = \frac{x^2-2}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 0$
- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -2$
- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) + (x^2 - 2)\, y(x) = 0$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-2+r)\, x^r + a_1(2+r)\,(-1+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+1)\,(k+r-2) + a_{k-2}\right) x^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-1, 2\}$
- Each term must be 0

  $a_1(2+r)\,(-1+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)\,(k+r-2) + a_{k-2} = 0$
- Shift index using $k- >k + 2$

  $a_{k+2}(k+3+r)\,(k+r) + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+3+r)(k+r)}$
- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{(k+2)(k-1)}$
- Solution for $r = -1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+2)(k-1)}, a_1 = 0\right]$
- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{a_k}{(k+5)(k+2)}$
- Solution for $r = 2$

$$\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2}=-\frac{a_k}{(k+5)(k+2)}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2}=-\frac{a_k}{(k+2)(k-1)}, a_1=0, b_{k+2}=-\frac{b_k}{(5+k)(k+2)}, b_1=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.025 (sec)
Leaf size : 27

```
dsolve(x^2*diff(diff(y(x),x),x)+(x^2-2)*y(x) = 0,y(x),singsol=all)
```

$$y=\frac{(c_1 x+c_2)\cos(x)+\sin(x)(c_2 x-c_1)}{x}$$

Mathematica DSolve solution

Solving time : 0.02 (sec)
Leaf size : 21

```
DSolve[{x^2*D[y[x],{x,2}]+(x^2-2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to x(c_1 j_1(x)-c_2 y_1(x))$$

2.1.329 Problem 336

Internal problem ID [9501]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 336
Date solved : `Monday, January 27, 2025 at 06:03:31 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$xy'' + 3y' + x^3y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.272 (sec)

Writing the ode as

$$xy'' + 3y' + x^3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 3 \\ C &= x^3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4x^4 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4x^4 + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-4x^4 + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.620: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -x^2 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx ix - \frac{3i}{8x^3} - \frac{9i}{128x^7} - \frac{27i}{1024x^{11}} - \frac{405i}{32768x^{15}} - \frac{1701i}{262144x^{19}} - \frac{15309i}{4194304x^{23}} - \frac{72171i}{33554432x^{27}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= ix \end{aligned}\tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-4x^4 + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (-x^2) + \left(\frac{3}{4x^2}\right) \\ &= -x^2 + \frac{3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= ix \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{i} - 1\right) &&= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{i} - 1\right) &&= -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-4x^4 + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	ix	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)\,(ix) \\
&= -\frac{1}{2x} - ix \\
&= -\frac{1}{2x} - ix
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{2x} - ix\right)(0) + \left(\left(\frac{1}{2x^2} - i\right) + \left(-\frac{1}{2x} - ix\right)^2 - \left(\frac{-4x^4+3}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(-\frac{1}{2x} - ix\right) dx} \\
&= \frac{e^{-\frac{ix^2}{2}}}{\sqrt{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3}{x}\,dx} \\ &= z_1 e^{-\frac{3\ln(x)}{2}} \\ &= z_1\left(\frac{1}{x^{3/2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{-\frac{i x^2}{2}}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3}{x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-3\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(-\frac{i e^{i x^2}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{e^{-\frac{i x^2}{2}}}{x^2}\right) + c_2\left(\frac{e^{-\frac{i x^2}{2}}}{x^2}\left(-\frac{i e^{i x^2}}{2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dx^2}y(x)\right)x + 3\frac{d}{dx}y(x) + x^3 y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -x^2 y(x) - \frac{3\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{3\left(\frac{d}{dx}y(x)\right)}{x} + x^2 y(x) = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - ◦ Define functions

    $\left[P_2(x) = \frac{3}{x}, P_3(x) = x^2\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 3$
- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$
- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $\left(\frac{d^2}{dx^2}y(x)\right)x + 3\frac{d}{dx}y(x) + x^3y(x) = 0$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^3 \cdot y(x)$ to series expansion

  $x^3 \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+3}$
- Shift index using $k- >k-3$

  $x^3 \cdot y(x) = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$
- Convert $\frac{d}{dx}y(x)$ to series expansion

  $\frac{d}{dx}y(x) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$
- Shift index using $k- >k+1$

  $\frac{d}{dx}y(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$
- Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$
- Shift index using $k- >k+1$

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0 r(2+r)\, x^{-1+r} + a_1(1+r)\,(3+r)\, x^r + a_2(2+r)\,(4+r)\, x^{1+r} + a_3(3+r)\,(5+r)\, x^{2+r} + \left(\sum_{k=3}^{\infty} (a_{k+}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-2, 0\}$
- The coefficients of each power of x must be 0

  $[a_1(1+r)\,(3+r) = 0, a_2(2+r)\,(4+r) = 0, a_3(3+r)\,(5+r) = 0]$
- Solve for the dependent coefficient(s)

  $\{a_1 = 0, a_2 = 0, a_3 = 0\}$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k+r+3) + a_{k-3} = 0$
- Shift index using $k- >k+3$

  $a_{k+4}(k+4+r)\,(k+6+r) + a_k = 0$
- Recursion relation that defines series solution to ODE

$a_{k+4} = -\frac{a_k}{(k+4+r)(k+6+r)}$

- Recursion relation for $r = -2$

  $a_{k+4} = -\frac{a_k}{(k+2)(k+4)}$

- Solution for $r = -2$

  $$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+4} = -\frac{a_k}{(k+2)(k+4)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$$

- Recursion relation for $r = 0$

  $a_{k+4} = -\frac{a_k}{(k+4)(k+6)}$

- Solution for $r = 0$

  $$\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{a_k}{(k+4)(k+6)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{4+k} = -\frac{a_k}{(k+2)(4+k)}, a_1 = 0, a_2 = 0, a_3 = 0, b_{4+k} = -\frac{b_k}{(4+k)(k\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 25

```
dsolve(x*diff(diff(y(x),x),x)+3*diff(y(x),x)+x^3*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_1 \sin\left(\frac{x^2}{2}\right) + c_2 \cos\left(\frac{x^2}{2}\right)}{x^2}$$

Mathematica DSolve solution

Solving time : 0.067 (sec)
Leaf size : 43

```
DSolve[{x*D[y[x],{x,2}]+3*D[y[x],x]+x^3*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-\frac{ix^2}{2}}\left(2c_1 - ic_2 e^{ix^2}\right)}{2x^2}$$

2.1.330 Problem 337

Internal problem ID [9502]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 337
Date solved : `Monday, January 27, 2025 at 06:03:31 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + 4xy' + (x^2 + 2)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.127 (sec)

Writing the ode as

$$x^2y'' + 4xy' + (x^2 + 2)\, y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 4x \\ C &= x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.622: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{4x}{x^2}\, dx} \\ &= z_1 e^{-2\ln(x)} \\ &= z_1 \left(\frac{1}{x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-4\ln(x)}}{(y_1)^2}\, dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\cos(x)}{x^2}\right) + c_2\left(\frac{\cos(x)}{x^2}(\tan(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x\left(\frac{d}{dx}y(x)\right) + (x^2+2)\, y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{(x^2+2)y(x)}{x^2} - \frac{4\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) + \frac{4\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(x^2+2)y(x)}{x^2} = 0$
- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{4}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = 4$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x\left(\frac{d}{dx}y(x)\right) + (x^2+2)\, y(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$
    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx} y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)\,(1+r)\,x^r + a_1(3+r)\,(2+r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+2)\,(k+r+1) + a_{k-2}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-2, -1\}$
- Each term must be 0

  $a_1(3+r)\,(2+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)\,(k+r+1) + a_{k-2} = 0$
- Shift index using $k->k+2$

  $a_{k+2}(k+4+r)\,(k+3+r) + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+4+r)(k+3+r)}$
- Recursion relation for $r=-2$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$
- Solution for $r=-2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$
- Recursion relation for $r=-1$

  $a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$
- Solution for $r=-1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+2)(k+3)}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+4*diff(y(x),x)*x+(x^2+2)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{\sin(x) c_1 + \cos(x) c_2}{x^2}$$

Mathematica DSolve solution

Solving time : 0.029 (sec)
Leaf size : 37

```
DSolve[{x^2*D[y[x],{x,2}]+4*x*D[y[x],x]+(x^2+2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tr
```

$$y(x) \to \frac{2c_1 e^{-ix} - ic_2 e^{ix}}{2x^2}$$

2.1.331 Problem 338

Internal problem ID [9503]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 338
Date solved : `Monday, January 27, 2025 at 06:03:32 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$16x^2y'' + 32xy' + \left(x^4 - 12\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.276 (sec)

Writing the ode as

$$16x^2y'' + 32xy' + \left(x^4 - 12\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 16x^2 \\ B &= 32x \\ C &= x^4 - 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^4 + 12}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^4 + 12 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^4 + 12}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.624: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{x^2}{16} + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{ix}{4} - \frac{3i}{2x^3} - \frac{9i}{2x^7} - \frac{27i}{x^{11}} - \frac{405i}{2x^{15}} - \frac{1701i}{x^{19}} - \frac{15309i}{x^{23}} - \frac{144342i}{x^{27}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{i}{4}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{ix}{4}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -\frac{x^2}{16}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{-x^4 + 12}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(-\frac{x^2}{16}\right) + \left(\frac{3}{4x^2}\right) \\ &= -\frac{x^2}{16} + \frac{3}{4x^2}\end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned}b &= (0) - (0) \\ &= 0\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{ix}{4} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{\frac{i}{4}} - 1\right) &&= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{\frac{i}{4}} - 1\right) &&= -\frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^4 + 12}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{ix}{4}$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)\left(\frac{ix}{4}\right) \\
&= -\frac{1}{2x} - \frac{ix}{4} \\
&= -\frac{1}{2x} - \frac{ix}{4}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{2x} - \frac{ix}{4}\right)(0) + \left(\left(\frac{1}{2x^2} - \frac{i}{4}\right) + \left(-\frac{1}{2x} - \frac{ix}{4}\right)^2 - \left(\frac{-x^4+12}{16x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{1}{2x} - \frac{ix}{4}\right) dx} \\
&= \frac{e^{-\frac{ix^2}{8}}}{\sqrt{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{32x}{16x^2}\,dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1 \left(\frac{1}{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{-\frac{ix^2}{8}}}{x^{3/2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{32x}{16x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-2i\mathrm{e}^{\frac{ix^2}{4}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{-\frac{ix^2}{8}}}{x^{3/2}}\right) + c_2 \left(\frac{\mathrm{e}^{-\frac{ix^2}{8}}}{x^{3/2}} \left(-2i\mathrm{e}^{\frac{ix^2}{4}}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$16x^2\left(\frac{d^2}{dx^2}y(x)\right) + 32x\left(\frac{d}{dx}y(x)\right) + (x^4 - 12)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{(x^4-12)y(x)}{16x^2} - \frac{2\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x) + \frac{2\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(x^4-12)y(x)}{16x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{2}{x}, P_3(x) = \frac{x^4-12}{16x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

$(x \cdot P_2(x))\Big|_{x=0} = 2$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{3}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $16x^2\left(\frac{d^2}{dx^2}y(x)\right) + 32x\left(\frac{d}{dx}y(x)\right) + (x^4 - 12)\, y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..4$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $4a_0(3+2r)\,(-1+2r)\, x^r + 4a_1(5+2r)\,(1+2r)\, x^{1+r} + 4a_2(7+2r)\,(3+2r)\, x^{2+r} + 4a_3(9+2r)\,(5+$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $4(3+2r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{3}{2}, \frac{1}{2}\right\}$

- The coefficients of each power of x must be 0

  $[4a_1(5+2r)\,(1+2r) = 0, 4a_2(7+2r)\,(3+2r) = 0, 4a_3(9+2r)\,(5+2r) = 0]$

- Solve for the dependent coefficient(s)

  $\{a_1 = 0, a_2 = 0, a_3 = 0\}$

- Each term in the series must be 0, giving the recursion relation

  $16\left(k+r+\frac{3}{2}\right)\left(k+r-\frac{1}{2}\right) a_k + a_{k-4} = 0$

- Shift index using $k- >k + 4$

  $16\left(k+\frac{11}{2}+r\right)\left(k+\frac{7}{2}+r\right) a_{k+4} + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+4} = -\frac{a_k}{4(2k+11+2r)(2k+7+2r)}$

- Recursion relation for $r = -\frac{3}{2}$

  $a_{k+4} = -\frac{a_k}{4(2k+8)(2k+4)}$

- Solution for $r = -\frac{3}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}, a_{k+4} = -\frac{a_k}{4(2k+8)(2k+4)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+4} = -\frac{a_k}{4(2k+12)(2k+8)}$
- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+4} = -\frac{a_k}{4(2k+12)(2k+8)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{4+k} = -\frac{a_k}{4(2k+8)(2k+4)}, a_1 = 0, a_2 = 0, a_3 = 0, b_{4+k} = -\frac{}{4}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.037 (sec)
Leaf size : 25

```
dsolve(16*x^2*diff(diff(y(x),x),x)+32*diff(y(x),x)*x+(x^4-12)*y(x) = 0,y(x),singsol=al
```

$$y = \frac{c_1 \sin\left(\frac{x^2}{8}\right) + c_2 \cos\left(\frac{x^2}{8}\right)}{x^{3/2}}$$

Mathematica DSolve solution

Solving time : 0.112 (sec)
Leaf size : 48

```
DSolve[{16*x^2*D[y[x],{x,2}]+32*x*D[y[x],x]+(x^4-12)*y[x]==0,{}},y[x],x,IncludeSingularSolut
```

$$y(x) \to \frac{e^{-\frac{1}{2}-\frac{ix^2}{8}}\left(c_1 - 2ic_2 e^{1+\frac{ix^2}{4}}\right)}{x^{3/2}}$$

2.1.332 Problem 339

Internal problem ID [9504]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 339
Date solved : `Monday, January 27, 2025 at 06:03:32 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - x^2y' + xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.369 (sec)

Writing the ode as

$$y'' - x^2y' + xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x^2 \\ C &= x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x(x^3 - 8)}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x(x^3 - 8) \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x(x^3 - 8)}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.626: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 4 \\ &= -4 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{2} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x^2}{2} - \frac{2}{x} - \frac{4}{x^4} - \frac{16}{x^7} - \frac{80}{x^{10}} - \frac{448}{x^{13}} - \frac{2688}{x^{16}} - \frac{16896}{x^{19}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 2$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i x^i \\
&= \frac{x^2}{2}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^1 = x$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{x^4}{4}
$$

This shows that the coefficient of x in the above is 0. Now we need to find the coefficient of x in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of x in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x(x^3 - 8)}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^4 - 2x\right) + (0) \\
&= \frac{1}{4}x^4 - 2x
\end{aligned}
$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is -2. Now b can be found.

$$
\begin{aligned}
b &= (-2) - (0) \\
&= -2
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x^2}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-2}{\frac{1}{2}} - 2\right) &&= -3 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-2}{\frac{1}{2}} - 2\right) &&= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x(x^3 - 8)}{4}
$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-4	$\frac{x^2}{2}$	-3	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x^2}{2}\right) \\ &= -\frac{x^2}{2} \\ &= -\frac{x^2}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{x^2}{2}\right)(1) + \left((-x) + \left(-\frac{x^2}{2}\right)^2 - \left(\frac{x(x^3-8)}{4}\right)\right) &= 0 \\ x a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (x)\, e^{\int -\frac{x^2}{2} dx} \\ &= (x)\, e^{-\frac{x^3}{6}} \\ &= x\, e^{-\frac{x^3}{6}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x^2}{1}\, dx} \\ &= z_1 e^{\frac{x^3}{6}} \\ &= z_1 \left(e^{\frac{x^3}{6}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\frac{x^3}{3}}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{3^{2/3}(-1)^{1/3}\left(-\frac{3x^2(-1)^{2/3}\Gamma\left(\frac{2}{3}\right)}{(-x^3)^{2/3}} + \frac{3\,3^{1/3}(-1)^{2/3}e^{\frac{x^3}{3}}}{x} + \frac{3x^2(-1)^{2/3}\Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{(-x^3)^{2/3}}\right)}{9}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(x) + c_2\left(x\left(\frac{3^{2/3}(-1)^{1/3}\left(-\frac{3x^2(-1)^{2/3}\Gamma\left(\frac{2}{3}\right)}{(-x^3)^{2/3}} + \frac{3\,3^{1/3}(-1)^{2/3}e^{\frac{x^3}{3}}}{x} + \frac{3x^2(-1)^{2/3}\Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{(-x^3)^{2/3}}\right)}{9}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) - x^2\left(\frac{d}{dx}y(x)\right) + xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y(x)$ to series expansion

  $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ◦ Shift index using $k- >k-1$

  $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ◦ Convert $x^2 \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^{k+1}$

  ◦ Shift index using $k- >k-1$

$$x^2 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=1}^{\infty} a_{k-1}(k-1)\, x^k$$

- Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

$$\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- > k+2$

$$\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_{k-1}(k-2)\right) x^k\right) = 0$$

- Each term must be 0

$2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

$(k^2 + 3k + 2)\, a_{k+2} - a_{k-1}(k-2) = 0$

- Shift index using $k- > k+1$

$\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_k(k-1) = 0$

- Recursion relation that defines the series solution to the ODE

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{a_k(k-1)}{k^2+5k+6}, 2a_2 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 56

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x^2+x*y(x) = 0,y(x),singsol=all)
```

$$y = -\frac{\left(-c_2 3^{1/3} e^{\frac{x^3}{3}} - c_1 x\right) \left(-x^3\right)^{2/3} + c_2 x^3 \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)}{\left(-x^3\right)^{2/3}}$$

Mathematica DSolve solution

Solving time : 0.055 (sec)
Leaf size : 41

```
DSolve[{D[y[x],{x,2}]-x^2*D[y[x],x]+x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 x - \frac{c_2 \sqrt[3]{-x^3}\,\Gamma\left(-\frac{1}{3}, -\frac{x^3}{3}\right)}{3\sqrt[3]{3}}$$

2.1.333 Problem 340

Internal problem ID [9505]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 340
Date solved : `Monday, January 27, 2025 at 06:03:33 PM`
CAS classification : `[_Laguerre]`

Solve

$$xy'' - (x+2)\, y' + 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.205 (sec)

Writing the ode as

$$xy'' + (-x-2)\, y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -x-2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 4x + 8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.628: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{x^2} - \frac{1}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{1}{x^2} + \frac{2}{x^3} + \frac{3}{x^4} + \frac{2}{x^5} - \frac{6}{x^6} - \frac{28}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-4x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-4x + 8}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned}b &= (-1) - (0) \\ &= -1\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 0\right) &&= -1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 0\right) &&= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -1$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2} - \frac{1}{x} \\ &= \frac{x-2}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2} - \frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(\frac{1}{2} - \frac{1}{x}\right)^2 - \left(\frac{x^2-4x+8}{4x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{1}{2} - \frac{1}{x}\right) dx} \\ &= \frac{e^{\frac{x}{2}}}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x-2}{x}\,dx} \\ &= z_1 e^{\frac{x}{2}+\ln(x)} \\ &= z_1\left(x\,\mathrm{e}^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x-2}{x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{x+2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(-\frac{(x^2+2x+2)\,\mathrm{e}^{x+2\ln(x)}\mathrm{e}^{-2x}}{x^2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(-\frac{(x^2+2x+2)\,\mathrm{e}^{x+2\ln(x)}\mathrm{e}^{-2x}}{x^2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dx^2}y(x)\right)x - (x+2)\left(\frac{d}{dx}y(x)\right) + 2y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{2y(x)}{x} + \frac{(x+2)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) - \frac{(x+2)\left(\frac{d}{dx}y(x)\right)}{x} + \frac{2y(x)}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{x+2}{x}, P_3(x) = \frac{2}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -2$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $\left(\frac{d^2}{dx^2}y(x)\right)x + (-x-2)\left(\frac{d}{dx}y(x)\right) + 2y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- Rewrite ODE with series expansions

  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..1$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-1}$

  - Shift index using $k- >k+1$

    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-3+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-2) - a_k(k+r-2)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 3\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)\left(a_{k+1}(k+1+r) - a_k\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1+r}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k}{k+1}\right]$

- Recursion relation for $r = 3$

  $a_{k+1} = \frac{a_k}{k+4}$

- Solution for $r = 3$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = \frac{a_k}{k+4}\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{4+k}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 19

```
dsolve(x*diff(diff(y(x),x),x)-(x+2)*diff(y(x),x)+2*y(x) = 0,y(x),singsol=all)
```

$$y = e^x c_1 + c_2 (x^2 + 2x + 2)$$

Mathematica DSolve solution

Solving time : 0.216 (sec)
Leaf size : 35

```
DSolve[{x*D[y[x],{x,2}]-(x+2)*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{x+1} \left(c_2 \int_1^x e^{-K[1]} K[1]^2 dK[1] + c_1 \right)$$

2.1.334 Problem 341

Internal problem ID [9506]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 341
Date solved : `Monday, January 27, 2025 at 06:03:34 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + xy' + 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.222 (sec)

Writing the ode as

$$y'' + xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 6 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.630: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{2x} - \frac{9}{4x^3} - \frac{27}{4x^5} - \frac{405}{16x^7} - \frac{1701}{16x^9} - \frac{15309}{32x^{11}} - \frac{72171}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$[\sqrt{r}]_\infty = \sum_{i=0}^{1} a_i x^i$$
$$= \frac{x}{2} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 6}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} - \frac{3}{2}\right) + (0) \\ &= \frac{x^2}{4} - \frac{3}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{2}\right) - (0) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{3}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{3}{2}\right)\right) &= 0 \\ a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= (x)\,\mathrm{e}^{\int -\frac{x}{2} dx} \\ &= (x)\,\mathrm{e}^{-\frac{x^2}{4}} \\ &= x\,\mathrm{e}^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{1}\, dx} \\ &= z_1 e^{-\frac{x^2}{4}} \\ &= z_1\left(\mathrm{e}^{-\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^2}{2}} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-\frac{x^2}{2}} x \right) + c_2 \left(\mathrm{e}^{-\frac{x^2}{2}} x \left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2}y(x) + x\left(\frac{d}{dx}y(x)\right) + 2y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^k$
- □ Rewrite DE with series expansions
  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion
    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$
  - Convert $\frac{d^2}{dx^2}y(x)$ to series expansion
    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$
  - Shift index using $k- >k+2$
    $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions
  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+2)\right) x^k = 0$
- Each term in the series must be 0, giving the recursion relation

$$(k+2)(ka_{k+2}+a_k+a_{k+2})=0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y(x)=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=-\frac{a_k}{k+1}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 34

```
dsolve(diff(diff(y(x),x),x)+diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y=x\left(ic_2\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)+c_1\right)\mathrm{e}^{-\frac{x^2}{2}}+2c_2$$

Mathematica DSolve solution

Solving time : 0.051 (sec)
Leaf size : 69

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to -\sqrt{\frac{\pi}{2}}c_2e^{-\frac{x^2}{2}}\sqrt{x^2}\mathrm{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{2}}\right)+\sqrt{2}c_1e^{-\frac{x^2}{2}}x+c_2$$

2.1.335 Problem 342

Internal problem ID [9507]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 342
Date solved : Monday, January 27, 2025 at 06:03:34 PM
CAS classification : [_Gegenbauer]

Solve

$$\left(-x^2+1\right)y''-2xy'+2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.248 (sec)

Writing the ode as

$$\left(-x^2+1\right)y''-2xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-x^2+1\\B&=-2x\\C&=2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{2x^2-3}{\left(x^2-1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=2x^2-3\\t&=\left(x^2-1\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{2x^2-3}{\left(x^2-1\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.632: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{4(x+1)} + \frac{5}{4(x-1)} - \frac{1}{4(x-1)^2} - \frac{1}{4(x+1)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^2 - 3}{(x^2 - 1)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2x^2 - 3}{(x^2 - 1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x - 2} + \frac{1}{2x + 2} + (0) \\
&= \frac{1}{2x - 2} + \frac{1}{2x + 2} \\
&= \frac{x}{x^2 - 1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x-2} + \frac{1}{2x+2}\right)(1) + \left(\left(-\frac{1}{2(x-1)^2} - \frac{1}{2(x+1)^2}\right) + \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right)^2 - \left(\frac{2x^2-3}{(x^2-1)^2}\right)\right. \\ -\frac{2a_0}{x^2-1} =$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x)\, e^{\int \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right) dx} \\ &= (x)\, \sqrt{(x-1)(x+1)} \\ &= x\sqrt{x^2-1} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{-x^2+1} \, dx} \\ &= z_1 e^{-\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x-1}\,\sqrt{x+1}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{x+1}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{-x^2+1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\ln(x-1) - \ln(x+1)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + \frac{1}{x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{x\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{x+1}}\right) + c_2 \left(\frac{x\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{x+1}} \left(\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + \frac{1}{x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(-x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)-2x\left(\frac{d}{dx}y(x)\right)+2y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=\frac{2y(x)}{x^2-1}-\frac{2\left(\frac{d}{dx}y(x)\right)x}{x^2-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x)+\frac{2\left(\frac{d}{dx}y(x)\right)x}{x^2-1}-\frac{2y(x)}{x^2-1}=0$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x)=\frac{2x}{x^2-1},P_3(x)=-\frac{2}{x^2-1}\right]$
  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=1$
  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$
  - $x=-1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=-1$
- Multiply by denominators

$(x^2-1)\left(\frac{d^2}{dx^2}y(x)\right)+2x\left(\frac{d}{dx}y(x)\right)-2y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(2u-2)\left(\frac{d}{du}y(u)\right)-2y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$
  - Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r^2u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)^2+a_k(k+r+2)(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-2r^2 = 0$

- Values of r that satisfy the indicial equation
  $r = 0$
- Each term in the series must be 0, giving the recursion relation
  $-2a_{k+1}(k+1)^2 + a_k(k+2)(k-1) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$
- Recursion relation for $r = 0$; series terminates at $k = 1$
  $a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$
- Apply recursion relation for $k = 0$
  $a_1 = -a_0$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li
  $y(u) = a_0 \cdot (-u + 1)$
- Revert the change of variables $u = x + 1$
  $[y(x) = -a_0 x]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 25

```
dsolve((-x^2+1)*diff(diff(y(x),x),x)-2*diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y = -\frac{\ln(x+1)\, c_2 x}{2} + \frac{c_2 \ln(x-1)\, x}{2} + c_1 x + c_2$$

Mathematica DSolve solution

Solving time : 0.02 (sec)
Leaf size : 33

```
DSolve[{(1-x^2)*D[y[x],{x,2}]-2*x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to c_1 x - \frac{1}{2} c_2 (x \log(1-x) - x \log(x+1) + 2)$$

2.1.336 Problem 343

Internal problem ID [9508]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 343
Date solved : Monday, January 27, 2025 at 06:03:35 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$y'' - 4xy' + \left(4x^2 - 2\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.053 (sec)

Writing the ode as

$$y'' - 4xy' + \left(4x^2 - 2\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4x \\ C &= 4x^2 - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.634: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{1}\, dx} \\ &= z_1 e^{x^2} \\ &= z_1 \left(e^{x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x^2}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{x^2}\right) + c_2\left(\mathrm{e}^{x^2}(x)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) - 4x\left(\frac{d}{dx}y(x)\right) + (4x^2-2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  - ◦ Shift index using $k- > k-m$

    $x^m \cdot y(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  - ◦ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  - ◦ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - ◦ Shift index using $k- > k+2$

    $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 - 2a_0 + (6a_3 - 6a_1)\,x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)(k+1) - 2a_k(2k+1) + 4a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 - 2a_0 = 0, 6a_3 - 6a_1 = 0]$

- Solve for the dependent coefficient(s)

  $\{a_2 = a_0, a_3 = a_1\}$

- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)\,a_{k+2} - 4a_k k - 2a_k + 4a_{k-2} = 0$

- Shift index using $k- > k+2$

  $$\left((k+2)^2+3k+8\right)a_{k+4}-4a_{k+2}(k+2)-2a_{k+2}+4a_k=0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y(x)=\sum_{k=0}^{\infty}a_k x^k, a_{k+4}=\frac{2(2ka_{k+2}-2a_k+5a_{k+2})}{k^2+7k+12}, a_2=a_0, a_3=a_1\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 14

```
dsolve(diff(diff(y(x),x),x)-4*diff(y(x),x)*x+(4*x^2-2)*y(x) = 0,y(x),singsol=all)
```

$$y=\mathrm{e}^{x^2}(c_2x+c_1)$$

Mathematica DSolve solution

Solving time : 0.02 (sec)
Leaf size : 18

```
DSolve[{D[y[x],{x,2}]-4*x*D[y[x],x]+(4*x^2-2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x)\to e^{x^2}(c_2x+c_1)$$

2.1.337 Problem 344

Internal problem ID [9509]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 344
Date solved : `Monday, January 27, 2025 at 06:03:36 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-x^2+1\right)y''-2xy'+30y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.287 (sec)

Writing the ode as

$$\left(-x^2+1\right)y''-2xy'+30y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=-x^2+1\\ B&=-2x \\ C&=30 \end{aligned}\tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned}\tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{30x^2-31}{\left(x^2-1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=30x^2-31\\ t&=\left(x^2-1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{30x^2-31}{\left(x^2-1\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.636: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{61}{4(x+1)} - \frac{1}{4(x+1)^2} + \frac{61}{4(x-1)} - \frac{1}{4(x-1)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{30x^2 - 31}{(x^2 - 1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 30$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 6 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -5
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{30x^2 - 31}{(x^2 - 1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	6	-5

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 6$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= 6 - (1) \\
&= 5
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x-2} + \frac{1}{2x+2} + (0) \\
&= \frac{1}{2x-2} + \frac{1}{2x+2} \\
&= \frac{x}{x^2-1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 5$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(20x^3 + 12x^2a_4 + 6xa_3 + 2a_2) + 2\left(\frac{1}{2x-2} + \frac{1}{2x+2}\right)(5x^4 + 4x^3a_4 + 3x^2a_3 + 2xa_2 + a_1) + \left(\left(-\frac{1}{2(x-\right.\right.$$

$$\frac{-10a_4x^4 + (-18a_3 - 20}{}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = 0, a_1 = \frac{5}{21}, a_2 = 0, a_3 = -\frac{10}{9}, a_4 = 0\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^5 - \frac{10}{9}x^3 + \frac{5}{21}x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega \, dx} \\
&= \left(x^5 - \frac{10}{9}x^3 + \frac{5}{21}x\right) e^{\int \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right) dx} \\
&= \left(x^5 - \frac{10}{9}x^3 + \frac{5}{21}x\right) \sqrt{(x-1)(x+1)} \\
&= \frac{(63x^5 - 70x^3 + 15x)\sqrt{x^2-1}}{63}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{-x^2+1} \, dx} \\
&= z_1 e^{-\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x-1}\,\sqrt{x+1}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(63x^5 - 70x^3 + 15x)\sqrt{x^2-1}}{63\sqrt{x-1}\,\sqrt{x+1}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{-x^2+1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\ln(x-1) - \ln(x+1)}}{(y_1)^2} \, dx \\
&= y_1 \left(-\frac{3087\left(-23x^3 + \frac{935}{63}x\right)}{1600\left(x^4 - \frac{10}{9}x^2 + \frac{5}{21}\right)} + \frac{441}{25x} + \frac{3969\ln(x-1)}{128} - \frac{3969\ln(x+1)}{128}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(63x^5 - 70x^3 + 15x)\sqrt{x^2-1}}{63\sqrt{x-1}\,\sqrt{x+1}} \right) \\
&\quad + c_2 \left(\frac{(63x^5 - 70x^3 + 15x)\sqrt{x^2-1}}{63\sqrt{x-1}\,\sqrt{x+1}} \left(-\frac{3087\left(-23x^3 + \frac{935}{63}x\right)}{1600\left(x^4 - \frac{10}{9}x^2 + \frac{5}{21}\right)} + \frac{441}{25x} \right.\right. \\
&\quad \left.\left. + \frac{3969\ln(x-1)}{128} - \frac{3969\ln(x+1)}{128} \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(-x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) - 2x\left(\frac{d}{dx}y(x)\right) + 30y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{30y(x)}{x^2-1} - \frac{2\left(\frac{d}{dx}y(x)\right)x}{x^2-1}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{2\left(\frac{d}{dx}y(x)\right)x}{x^2-1} - \frac{30y(x)}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{2x}{x^2-1}, P_3(x) = -\frac{30}{x^2-1}\right]$
- ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1} = 1$
- ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$
- ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(x^2-1)\left(\frac{d^2}{dx^2}y(x)\right) + 2x\left(\frac{d}{dx}y(x)\right) - 30y(x) = 0$
- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (2u-2)\left(\frac{d}{du}y(u)\right) - 30y(u) = 0$
- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
- ◦ Shift index using $k->k+1-m$

$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k- > k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0 r^2 u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)^2 + a_k(k+r+6)\,(k+r-5)\right) u^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 0$
- Each term in the series must be 0, giving the recursion relation

  $-2a_{k+1}(k+1)^2 + a_k(k+6)\,(k-5) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+6)(k-5)}{2(k+1)^2}$
- Recursion relation for $r = 0$; series terminates at $k = 5$

  $a_{k+1} = \frac{a_k(k+6)(k-5)}{2(k+1)^2}$
- Apply recursion relation for $k = 0$

  $a_1 = -15a_0$
- Apply recursion relation for $k = 1$

  $a_2 = -\frac{7a_1}{2}$
- Express in terms of a_0

  $a_2 = \frac{105a_0}{2}$
- Apply recursion relation for $k = 2$

  $a_3 = -\frac{4a_2}{3}$
- Express in terms of a_0

  $a_3 = -70a_0$
- Apply recursion relation for $k = 3$

  $a_4 = -\frac{9a_3}{16}$
- Express in terms of a_0

  $a_4 = \frac{315a_0}{8}$
- Apply recursion relation for $k = 4$

  $a_5 = -\frac{a_4}{5}$
- Express in terms of a_0

  $a_5 = -\frac{63a_0}{8}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - 15u + \frac{105}{2}u^2 - 70u^3 + \frac{315}{8}u^4 - \frac{63}{8}u^5\right)$
- Revert the change of variables $u = x + 1$

  $\left[y(x) = a_0\left(-\frac{15}{8}x + \frac{35}{4}x^3 - \frac{63}{8}x^5\right)\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 71

```
dsolve((-x^2+1)*diff(diff(y(x),x),x)-2*diff(y(x),x)*x+30*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{21c_2x\left(x^4 - \frac{10}{9}x^2 + \frac{5}{21}\right)\ln(x-1)}{640} - \frac{21c_2x\left(x^4 - \frac{10}{9}x^2 + \frac{5}{21}\right)\ln(x+1)}{640} + \frac{21c_1x^5}{5} + \frac{21c_2x^4}{320} - \frac{14c_1x^3}{3} - \frac{49c_2x^2}{960} + c_1x + \frac{c_2}{225}$$

Mathematica DSolve solution

Solving time : 0.025 (sec)
Leaf size : 76

```
DSolve[{(1-x^2)*D[y[x],{x,2}]-2*x*D[y[x],x]+30*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tru
```

$$y(x) \to \frac{1}{8}c_1x\left(63x^4 - 70x^2 + 15\right) + c_2\left(-\frac{63x^4}{8} + \frac{49x^2}{8} - \frac{1}{16}\left(63x^4 - 70x^2 + 15\right)x(\log(1-x) - \log(x+1)) - \frac{8}{15}\right)$$

2.1.338 Problem 345

Internal problem ID [9510]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 345
Date solved : Monday, January 27, 2025 at 06:03:36 PM
CAS classification : [_Lienard]

Solve

$$xy'' + 2y' + xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.124 (sec)

Writing the ode as

$$xy'' + 2y' + xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.638: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\, dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1 \left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2} dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\cos(x)}{x}\right) + c_2 \left(\frac{\cos(x)}{x}(\tan(x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dx^2}y(x)\right)x + 2\frac{d}{dx}y(x) + xy(x) = 0$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
$\frac{d^2}{dx^2}y(x) = -y(x) - \frac{2\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
$\frac{d^2}{dx^2}y(x) + \frac{2\left(\frac{d}{dx}y(x)\right)}{x} + y(x) = 0$

□ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
  $\left[P_2(x) = \frac{2}{x}, P_3(x) = 1\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
  $\left.(x \cdot P_2(x))\right|_{x=0} = 2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$
  - $x = 0$is a regular singular point
  Check to see if $x_0 = 0$ is a regular singular point
  $x_0 = 0$

- Multiply by denominators
$\left(\frac{d^2}{dx^2}y(x)\right)x + 2\frac{d}{dx}y(x) + xy(x) = 0$
- Assume series solution for $y(x)$
$y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions
  - Convert $x \cdot y(x)$ to series expansion
  $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+1}$
  - Shift index using $k- >k-1$

$x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

- Convert $\frac{d}{dx}y(x)$ to series expansion

  $\frac{d}{dx}y(x) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $\frac{d}{dx}y(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$

- Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1})\, x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $a_1(1+r)\,(2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1} = 0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)\,(k+3+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{(k+1)(k+2)}$

- Solution for $r = -1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+3)}$

- Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0, b_{k+2} = -\frac{b_k}{(k+2)(k+3)}, 2b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 17

```
dsolve(x*diff(diff(y(x),x),x)+2*diff(y(x),x)+x*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{\sin(x) c_1 + \cos(x) c_2}{x}$$

Mathematica DSolve solution

Solving time : 0.027 (sec)
Leaf size : 37

```
DSolve[{x*D[y[x],{x,2}]+2*D[y[x],x]+x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2c_1 e^{-ix} - ic_2 e^{ix}}{2x}$$

2.1.339 Problem 346

Internal problem ID [9511]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 346
Date solved : `Monday, January 27, 2025 at 06:03:37 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + (2x+1)\,y' + (x+1)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.154 (sec)

Writing the ode as

$$xy'' + (2x+1)\,y' + (x+1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2x+1 \\ C &= x+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.640: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=-\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{2}-\left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^{-}}{x-c_1}\right)+(-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x}+(-)\,(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{1}{2x}\right)(0)+\left(\left(-\frac{1}{2x^2}\right)+\left(\frac{1}{2x}\right)^2-\left(-\frac{1}{4x^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}z_1(x)&=pe^{\int \omega\,dx}\\&=\mathrm{e}^{\int \frac{1}{2x}dx}\\&=\sqrt{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1&=z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\&=z_1e^{-\int \frac{1}{2}\frac{2x+1}{x}\,dx}\\&=z_1e^{-x-\frac{\ln(x)}{2}}\\&=z_1\left(\frac{\mathrm{e}^{-x}}{\sqrt{x}}\right)\end{aligned}$$

Which simplifies to

$$y_1=\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2&=y_1\int \frac{e^{\int -\frac{2x+1}{x}\,dx}}{(y_1)^2}\,dx\\&=y_1\int \frac{e^{-2x-\ln(x)}}{(y_1)^2}\,dx\\&=y_1(\ln(x))\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y&=c_1y_1+c_2y_2\\&=c_1\left(\mathrm{e}^{-x}\right)+c_2\left(\mathrm{e}^{-x}(\ln(x))\right)\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dx^2}y(x)\right)x+(2x+1)\left(\frac{d}{dx}y(x)\right)+(x+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{(x+1)y(x)}{x}-\frac{(2x+1)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x)+\frac{(2x+1)\left(\frac{d}{dx}y(x)\right)}{x}+\frac{(x+1)y(x)}{x}=0$

□ Check to see if $x_0=0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=\frac{2x+1}{x},P_3(x)=\frac{x+1}{x}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=1$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

- ◦ $x=0$is a regular singular point

Check to see if $x_0=0$ is a regular singular point

$x_0=0$

- Multiply by denominators

$\left(\frac{d^2}{dx^2}y(x)\right)x+(2x+1)\left(\frac{d}{dx}y(x)\right)+(x+1)\,y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

- ◦ Shift index using $k->k-m$

  $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..1$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ◦ Convert $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-1}$

- ◦ Shift index using $k->k+1$

  $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

Rewrite ODE with series expansions

$$a_0r^2x^{-1+r} + \left(a_1(1+r)^2 + a_0(1+2r)\right)x^r + \left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+1+r)^2 + a_k(2k+2r+1) + a_{k-1}\right)x^k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 0$
- Each term must be 0
  $a_1(1+r)^2 + a_0(1+2r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1)^2 + 2a_kk + a_k + a_{k-1} = 0$
- Shift index using $k->k+1$
  $a_{k+2}(k+2)^2 + 2a_{k+1}(k+1) + a_{k+1} + a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{2ka_{k+1}+a_k+3a_{k+1}}{(k+2)^2}$
- Recursion relation for $r = 0$
  $a_{k+2} = -\frac{2ka_{k+1}+a_k+3a_{k+1}}{(k+2)^2}$
- Solution for $r = 0$
  $\left[y(x) = \sum_{k=0}^{\infty} a_kx^k, a_{k+2} = -\frac{2ka_{k+1}+a_k+3a_{k+1}}{(k+2)^2}, a_1 + a_0 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 15

```
dsolve(x*diff(diff(y(x),x),x)+(2*x+1)*diff(y(x),x)+(x+1)*y(x) = 0,y(x),singsol=all)
```

$$y = e^{-x}(c_2 \ln(x) + c_1)$$

Mathematica DSolve solution

Solving time : 0.029 (sec)
Leaf size : 19

```
DSolve[{x*D[y[x],{x,2}]+(2*x+1)*D[y[x],x]+(x+1)*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to e^{-x}(c_2 \log(x) + c_1)$$

2.1.340 Problem 347

Internal problem ID [9512]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 347
Date solved : `Monday, January 27, 2025 at 06:03:37 PM`
CAS classification : `[_Jacobi]`

Solve

$$2x(x-1)\,y''-(x+1)\,y'+y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.187 (sec)

Writing the ode as

$$\left(2x^2-2x\right)y''+(-x-1)\,y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2-2x \\ B &= -x-1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-3x^2+18x-3}{16\left(x^2-x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3x^2+18x-3 \\ t &= 16\left(x^2-x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-3x^2+18x-3}{16\left(x^2-x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.642: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2} + \frac{3}{4x} - \frac{3}{4(x-1)} + \frac{3}{4(x-1)^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 + 18x - 3}{16\left(x^2 - x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x^2 + 18x - 3}{16\left(x^2 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= \frac{1}{4} - \left(\frac{1}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{3}{4x} - \frac{1}{2(x-1)} + (-)(0) \\
&= \frac{3}{4x} - \frac{1}{2(x-1)} \\
&= \frac{x-3}{4x(x-1)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{4x} - \frac{1}{2(x-1)}\right)(0) + \left(\left(-\frac{3}{4x^2} + \frac{1}{2(x-1)^2}\right) + \left(\frac{3}{4x} - \frac{1}{2(x-1)}\right)^2 - \left(\frac{-3x^2+18x-3}{16(x^2-x)^2}\right)\right) = 0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{3}{4x} - \frac{1}{2(x-1)}\right) dx} \\ &= \frac{x^{3/4}}{\sqrt{x-1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x-1}{2x^2-2x}\, dx} \\ &= z_1 e^{\frac{\ln(x-1)}{2} - \frac{\ln(x)}{4}} \\ &= z_1 \left(\frac{\sqrt{x-1}}{x^{1/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x-1}{2x^2-2x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\ln(x-1) - \frac{\ln(x)}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{2(x+1)\, e^{\ln(x-1) - \frac{\ln(x)}{2}}}{x-1}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\sqrt{x}\right) + c_2\left(\sqrt{x}\left(\frac{2(x+1)\, e^{\ln(x-1) - \frac{\ln(x)}{2}}}{x-1}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$2x(x-1)\left(\frac{d^2}{dx^2}y(x)\right)-(x+1)\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
$\frac{d^2}{dx^2}y(x)=-\frac{y(x)}{2x(x-1)}+\frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{2x(x-1)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
$\frac{d^2}{dx^2}y(x)-\frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{2x(x-1)}+\frac{y(x)}{2x(x-1)}=0$
- □ Check to see if x_0 is a regular singular point
  - ◦ Define functions
$\left[P_2(x)=-\frac{x+1}{2x(x-1)}, P_3(x)=\frac{1}{2x(x-1)}\right]$
  - ◦ $x\cdot P_2(x)$ is analytic at $x=0$
$\left.(x\cdot P_2(x))\right|_{x=0}=\frac{1}{2}$
  - ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$
$\left.(x^2\cdot P_3(x))\right|_{x=0}=0$
  - ◦ $x=0$is a regular singular point
Check to see if x_0 is a regular singular point
$x_0=0$
- Multiply by denominators
$2x(x-1)\left(\frac{d^2}{dx^2}y(x)\right)+(-x-1)\left(\frac{d}{dx}y(x)\right)+y(x)=0$
- Assume series solution for $y(x)$
$y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$
- □ Rewrite ODE with series expansions
  - ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..1$
$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$
  - ◦ Shift index using $k->k+1-m$
$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - ◦ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=1..2$
$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,x^{k+r-2+m}$
  - ◦ Shift index using $k->k+2-m$
$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)\,x^{k+r}$

Rewrite ODE with series expansions
$-a_0r(-1+2r)\,x^{-1+r}+\left(\sum_{k=0}^{\infty}(-a_{k+1}(k+1+r)(2k+1+2r)+a_k(2k+2r-1)(k+r-1))\,x^{k+r}\right)$
- $a_0$cannot be 0 by assumption, giving the indicial equation
$-r(-1+2r)=0$
- Values of r that satisfy the indicial equation

$r \in \left\{0, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $-2(k+1+r)\left(k+r+\frac{1}{2}\right)a_{k+1}+2\left(k+r-\frac{1}{2}\right)a_k(k+r-1)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{(2k+2r-1)a_k(k+r-1)}{(k+1+r)(2k+1+2r)}$

- Recursion relation for $r=0$; series terminates at $k=1$

  $a_{k+1}=\frac{(2k-1)a_k(k-1)}{(k+1)(2k+1)}$

- Apply recursion relation for $k=0$

  $a_1=a_0$

- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li

  $y(x)=a_0\cdot(x+1)$

- Recursion relation for $r=\frac{1}{2}$

  $a_{k+1}=\frac{2ka_k\left(k-\frac{1}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}$

- Solution for $r=\frac{1}{2}$

  $\left[y(x)=\sum_{k=0}^{\infty}a_kx^{k+\frac{1}{2}}, a_{k+1}=\frac{2ka_k\left(k-\frac{1}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$

- Combine solutions and rename parameters

  $\left[y(x)=a_0\cdot(x+1)+\left(\sum_{k=0}^{\infty}b_kx^{k+\frac{1}{2}}\right), b_{k+1}=\frac{2kb_k\left(k-\frac{1}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 14

```
dsolve(2*x*(x-1)*diff(diff(y(x),x),x)-(x+1)*diff(y(x),x)+y(x) = 0,y(x),singsol=all)
```

$$y = c_2\sqrt{x}+c_1x+c_1$$

Mathematica DSolve solution

Solving time : 0.266 (sec)
Leaf size : 106

```
DSolve[{2*x*(x-1)*D[y[x],{x,2}]-(x+1)*D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tr
```

$$y(x) \to \exp\left(\int_1^x \frac{K[1]-3}{4(K[1]-1)K[1]}dK[1] - \frac{1}{2}\int_1^x \frac{K[2]+1}{2K[2]-2K[2]^2}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{K[1]-3}{4(K[1]-1)K[1]}dK[1]\right)dK[3] + c_1\right)$$

2.1.341 Problem 348

Internal problem ID [9513]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 348
Date solved : `Monday, January 27, 2025 at 06:03:38 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + 2y' + 4xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.134 (sec)

Writing the ode as

$$xy'' + 2y' + 4xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= 4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.644: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\,dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1\left(\frac{1}{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(2x)}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\tan(2x)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(2x)}{x}\right) + c_2 \left(\frac{\cos(2x)}{x} \left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dx^2}y(x)\right)x + 2\frac{d}{dx}y(x) + 4xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -4y(x) - \frac{2\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{2\left(\frac{d}{dx}y(x)\right)}{x} + 4y(x) = 0$
- □ Check to see if $x_0 = 0$ is a regular singular point
  - ◦ Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = 4\right]$
  - ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$
  - ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$
  - ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$
- Multiply by denominators

  $\left(\frac{d^2}{dx^2}y(x)\right)x + 2\frac{d}{dx}y(x) + 4xy(x) = 0$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- □ Rewrite ODE with series expansions
  - ◦ Convert $x \cdot y(x)$ to series expansion

    $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

- ○ Shift index using $k- >k-1$

  $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$
- ○ Convert $\frac{d}{dx}y(x)$ to series expansion

  $\frac{d}{dx}y(x) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$
- ○ Shift index using $k- >k+1$

  $\frac{d}{dx}y(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$
- ○ Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$
- ○ Shift index using $k- >k+1$

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + 4a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$
- Each term must be 0

  $a_1(1+r)\,(2+r) = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)\,(k+2+r) + 4a_{k-1} = 0$
- Shift index using $k- >k+1$

  $a_{k+2}(k+2+r)\,(k+3+r) + 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{(k+2+r)(k+3+r)}$
- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{4a_k}{(k+1)(k+2)}$
- Solution for $r = -1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{4a_k}{(k+1)(k+2)}, 0 = 0\right]$
- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{4a_k}{(k+2)(k+3)}$
- Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{4a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = -\frac{4a_k}{(k+1)(k+2)}, 0 = 0, b_{k+2} = -\frac{4b_k}{(k+2)(k+3)}, 2b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 21

```
dsolve(x*diff(diff(y(x),x),x)+2*diff(y(x),x)+4*x*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_1 \sin(2x) + c_2 \cos(2x)}{x}$$

Mathematica DSolve solution

Solving time : 0.031 (sec)
Leaf size : 37

```
DSolve[{x*D[y[x],{x,2}]+2*D[y[x],x]+4*x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{4c_1 e^{-2ix} - ic_2 e^{2ix}}{4x}$$

2.1.342 Problem 349

Internal problem ID [9514]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 349
Date solved : `Monday, January 27, 2025 at 06:03:38 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + (2 - 2x)\, y' + (x - 2)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.066 (sec)

Writing the ode as

$$xy'' + (2 - 2x)\, y' + (x - 2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 - 2x \\ C &= x - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.646: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2-2x}{x}\, dx} \\ &= z_1 e^{x-\ln(x)} \\ &= z_1 \left(\frac{e^x}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^x}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2-2x}{x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x-2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{e^x}{x}\right) + c_2\left(\frac{e^x}{x}(x)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dx^2}y(x)\right)x + (-2x+2)\left(\frac{d}{dx}y(x)\right) + (x-2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{(x-2)y(x)}{x} + \frac{2\left(\frac{d}{dx}y(x)\right)(x-1)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) - \frac{2\left(\frac{d}{dx}y(x)\right)(x-1)}{x} + \frac{(x-2)y(x)}{x} = 0$
- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{2(x-1)}{x}, P_3(x) = \frac{x-2}{x}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $\left(\frac{d^2}{dx^2}y(x)\right)x + (-2x+2)\left(\frac{d}{dx}y(x)\right) + (x-2)\,y(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$
    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..1$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(1+r)\, x^{-1+r} + (a_1(1+r)\,(2+r) - 2a_0(1+r))\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(k+2+r) - 2a_k$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $a_1(1+r)\,(2+r) - 2a_0(1+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k+2+r) - 2a_k k - 2a_k r - 2a_k + a_{k-1} = 0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)\,(k+3+r) - 2a_{k+1}(k+1) - 2ra_{k+1} - 2a_{k+1} + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2ka_{k+1}+2ra_{k+1}-a_k+4a_{k+1}}{(k+2+r)(k+3+r)}$

- Recursion relation for $r = -1$

  $a_{k+2} = \frac{2ka_{k+1}-a_k+2a_{k+1}}{(k+1)(k+2)}$

- Solution for $r = -1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = \frac{2ka_{k+1}-a_k+2a_{k+1}}{(k+1)(k+2)}, 0 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{2ka_{k+1}-a_k+4a_{k+1}}{(k+2)(k+3)}$

- Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2ka_{k+1}-a_k+4a_{k+1}}{(k+2)(k+3)}, 2a_1 - 2a_0 = 0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = \frac{2ka_{k+1}-a_k+2a_{k+1}}{(k+1)(k+2)}, 0 = 0, b_{k+2} = \frac{2kb_{k+1}-b_k+4b_{k+1}}{(k+2)(k+3)}, 2b_1 -\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 15

```
dsolve(x*diff(diff(y(x),x),x)+(-2*x+2)*diff(y(x),x)+(x-2)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{e^x (c_1 x + c_2)}{x}$$

Mathematica DSolve solution

Solving time : 0.031 (sec)
Leaf size : 19

```
DSolve[{x*D[y[x],{x,2}]+(2-2*x)*D[y[x],x]+(x-2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tr
```

$$y(x) \to \frac{e^x (c_2 x + c_1)}{x}$$

2.1.343 Problem 350

Internal problem ID [9515]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 350
Date solved : `Monday, January 27, 2025 at 06:03:39 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + 6xy' + \left(4x^2+6\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.141 (sec)

Writing the ode as

$$x^2y'' + 6xy' + \left(4x^2+6\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 6x \\ C &= 4x^2+6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.648: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{6x}{x^2}\, dx} \\ &= z_1 e^{-3\ln(x)} \\ &= z_1 \left(\frac{1}{x^3}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(2x)}{x^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{6x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-6\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\tan(2x)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(2x)}{x^3}\right) + c_2 \left(\frac{\cos(2x)}{x^3} \left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2 \left(\frac{d^2}{dx^2} y(x)\right) + 6x \left(\frac{d}{dx} y(x)\right) + (4x^2 + 6)\, y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2} y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2} y(x) = -\frac{2(2x^2+3)y(x)}{x^2} - \frac{6\left(\frac{d}{dx} y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2} y(x) + \frac{6\left(\frac{d}{dx} y(x)\right)}{x} + \frac{2(2x^2+3)y(x)}{x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{6}{x}, P_3(x) = \frac{2(2x^2+3)}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 6$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 6$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2 \left(\frac{d^2}{dx^2} y(x)\right) + 6x \left(\frac{d}{dx} y(x)\right) + (4x^2 + 6)\, y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

$x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(3+r)\,(2+r)\, x^r + a_1(4+r)\,(3+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+3)\,(k+r+2) + 4a_{k-2}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(3+r)\,(2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-3, -2\}$
- Each term must be 0

  $a_1(4+r)\,(3+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+3)\,(k+r+2) + 4a_{k-2} = 0$
- Shift index using $k->k+2$

  $a_{k+2}(k+5+r)\,(k+4+r) + 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{(k+5+r)(k+4+r)}$
- Recursion relation for $r = -3$

  $a_{k+2} = -\frac{4a_k}{(k+2)(k+1)}$
- Solution for $r = -3$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2} = -\frac{4a_k}{(k+2)(k+1)}, a_1 = 0\right]$
- Recursion relation for $r = -2$

  $a_{k+2} = -\frac{4a_k}{(k+3)(k+2)}$
- Solution for $r = -2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{4a_k}{(k+3)(k+2)}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-3}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-2}\right), a_{k+2} = -\frac{4a_k}{(k+1)(k+2)}, a_1 = 0, b_{k+2} = -\frac{4b_k}{(k+2)(k+3)}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 21

```
dsolve(x^2*diff(diff(y(x),x),x)+6*diff(y(x),x)*x+(4*x^2+6)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_1 \sin(2x) + c_2 \cos(2x)}{x^3}$$

Mathematica DSolve solution

Solving time : 0.032 (sec)
Leaf size : 37

```
DSolve[{x^2*D[y[x],{x,2}]+6*x*D[y[x],x]+(4*x^2+6)*y[x]==0,{}},y[x],x,IncludeSingularSolution
```

$$y(x) \to \frac{4c_1 e^{-2ix} - ic_2 e^{2ix}}{4x^3}$$

2.1.344 Problem 351

Internal problem ID [9516]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 351
Date solved : `Monday, January 27, 2025 at 06:03:39 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + (1 - 2x)\, y' + (x - 1)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.156 (sec)

Writing the ode as

$$xy'' + (1 - 2x)\, y' + (x - 1)\, y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 1 - 2x \\ C &= x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.650: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=-\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2}-\left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^-}{x-c_1}\right)+(-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x}+(-)(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$
\begin{gathered}
(0)+2\left(\frac{1}{2x}\right)(0)+\left(\left(-\frac{1}{2x^2}\right)+\left(\frac{1}{2x}\right)^2-\left(-\frac{1}{4x^2}\right)\right)=0 \\
0=0
\end{gathered}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \frac{1}{2x} dx} \\
&= \sqrt{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1-2x}{x}\, dx} \\
&= z_1 e^{x-\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{\mathrm{e}^x}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1-2x}{x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2x-\ln(x)}}{(y_1)^2}\, dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(\ln(x)))
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dx^2}y(x)\right)x+(-2x+1)\left(\frac{d}{dx}y(x)\right)+(x-1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
$\frac{d^2}{dx^2}y(x)=-\frac{(x-1)y(x)}{x}+\frac{(2x-1)\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
$\frac{d^2}{dx^2}y(x)-\frac{(2x-1)\left(\frac{d}{dx}y(x)\right)}{x}+\frac{(x-1)y(x)}{x}=0$
- ☐ Check to see if $x_0=0$ is a regular singular point
  - Define functions
  $\left[P_2(x)=-\frac{2x-1}{x},P_3(x)=\frac{x-1}{x}\right]$
  - $x\cdot P_2(x)$ is analytic at $x=0$
  $\left.(x\cdot P_2(x))\right|_{x=0}=1$
  - $x^2\cdot P_3(x)$ is analytic at $x=0$
  $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$
  - $x=0$is a regular singular point
  Check to see if $x_0=0$ is a regular singular point
  $x_0=0$
- Multiply by denominators
$\left(\frac{d^2}{dx^2}y(x)\right)x+(-2x+1)\left(\frac{d}{dx}y(x)\right)+(x-1)\,y(x)=0$
- Assume series solution for $y(x)$
$y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$
  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$
  - Shift index using $k->k-m$
  $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$
  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..1$
  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k->k+1-m$
  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion
  $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-1}$
  - Shift index using $k->k+1$
  $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$
  Rewrite ODE with series expansions

$$a_0 r^2 x^{-1+r} + \left(a_1(1+r)^2 - a_0(1+2r)\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)^2 - a_k(2k+2r+1) + a_{k-1}\right) x^k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 0$
- Each term must be 0
  $a_1(1+r)^2 - a_0(1+2r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1)^2 + (-2k-1)\, a_k + a_{k-1} = 0$
- Shift index using $k- >k+1$
  $a_{k+2}(k+2)^2 + (-2k-3)\, a_{k+1} + a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{2ka_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}$
- Recursion relation for $r = 0$
  $a_{k+2} = \frac{2ka_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}$
- Solution for $r = 0$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2ka_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}, a_1 - a_0 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 13

```
dsolve(x*diff(diff(y(x),x),x)+(1-2*x)*diff(y(x),x)+(x-1)*y(x) = 0,y(x),singsol=all)
```

$$y = e^x (c_2 \ln(x) + c_1)$$

Mathematica DSolve solution

Solving time : 0.025 (sec)
Leaf size : 17

```
DSolve[{x*D[y[x],{x,2}]+(1-2*x)*D[y[x],x]+(x-1)*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to e^x (c_2 \log(x) + c_1)$$

2.1.345 Problem 352

Internal problem ID [9517]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 352
Date solved : Monday, January 27, 2025 at 06:03:40 PM
CAS classification : [_Jacobi]

Solve

$$x(1-x)\,y''+\left(\frac{1}{2}+2x\right)y'-2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.254 (sec)

Writing the ode as

$$\left(-x^2+x\right)y''+\left(\frac{1}{2}+2x\right)y'-2y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-x^2+x\\B&=\frac{1}{2}+2x\\C&=-2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{48x-3}{16\left(x^2-x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=48x-3\\t&=16\left(x^2-x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{48x - 3}{16\left(x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.652: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 3 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 3 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{21}{8x} - \frac{3}{16x^2} + \frac{45}{16\left(-1 + x\right)^2} - \frac{21}{8\left(-1 + x\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x=1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{45}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{9}{4}\\ \alpha_c^- &= \frac{1}{2}-\sqrt{1+4b}=-\frac{5}{4}\end{aligned}$$

Since the order of r at ∞ is $3>2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= 0\\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{48x-3}{16\left(x^2-x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
1	2	0	$\frac{9}{4}$	$-\frac{5}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
3	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=0$ then

$$\begin{aligned}d &= \alpha_\infty^+-\left(\alpha_{c_1}^-+\alpha_{c_2}^-\right)\\ &= 0-(-1)\\ &= 1\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^-}{x-c_1}\right)+\left((-)[\sqrt{r}]_{c_2}+\frac{\alpha_{c_2}^-}{x-c_2}\right)+(+)[\sqrt{r}]_\infty\\ &= \frac{1}{4x}-\frac{5}{4(-1+x)}+(0)\\ &= \frac{1}{4x}-\frac{5}{4(-1+x)}\\ &= -\frac{4x+1}{4x(-1+x)}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{4x} - \frac{5}{4(-1+x)}\right)(1) + \left(\left(-\frac{1}{4x^2} + \frac{5}{4(-1+x)^2}\right) + \left(\frac{1}{4x} - \frac{5}{4(-1+x)}\right)^2 - \left(\frac{48x-3}{16(x^2-x)^2}\right)\right) \frac{-1+4a_0}{2x(-1+x)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{1}{4}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{1}{4}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{1}{4}\right) e^{\int \left(\frac{1}{4x} - \frac{5}{4(-1+x)}\right) dx} \\
&= \left(x + \frac{1}{4}\right) e^{-\frac{5\ln(-1+x)}{4} + \frac{\ln(x)}{4}} \\
&= \frac{\left(x + \frac{1}{4}\right) x^{1/4}}{(-1+x)^{5/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{\frac{1}{2}+2x}{-x^2+x} \, dx} \\
&= z_1 e^{\frac{5\ln(-1+x)}{4} - \frac{\ln(x)}{4}} \\
&= z_1 \left(\frac{(-1+x)^{5/4}}{x^{1/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x + \frac{1}{4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{\frac{1}{2}+2x}{-x^2+x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{5\ln(-1+x)}{2} - \frac{\ln(x)}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(-\frac{\sqrt{-1+x}\,\sqrt{x}\left(12\ln\left(-\frac{1}{2} + x + \sqrt{x(-1+x)}\right)x - 4\sqrt{x(-1+x)}\,x + 3\ln\left(-\frac{1}{2} + x + \sqrt{x(-1\right.\right.}{\sqrt{x(-1+x)}\,(4x+1)}\right.
\end{aligned}$$

Therefore the solution is

$$y = c_1y_1 + c_2y_2$$
$$= c_1\left(x+\frac{1}{4}\right) + c_2\left(x + \frac{1}{4}\left(-\frac{\sqrt{-1+x}\,\sqrt{x}\left(12\ln\left(-\frac{1}{2}+x+\sqrt{x\left(-1+x\right)}\right)x - 4\sqrt{x\left(-1+x\right)}\,x + 3\ln\left(-\frac{1}{2}+x+\sqrt{x\left(-1+\right.}\right.\right.}{\sqrt{x\left(-1+x\right)}\,\left(4x+1\right)}\right.\right.$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x(1-x)\left(\frac{d^2}{dx^2}y(x)\right) + \left(\frac{1}{2}+2x\right)\left(\frac{d}{dx}y(x)\right) - 2y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{2y(x)}{x(x-1)} + \frac{(4x+1)\left(\frac{d}{dx}y(x)\right)}{2x(x-1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) - \frac{(4x+1)\left(\frac{d}{dx}y(x)\right)}{2x(x-1)} + \frac{2y(x)}{x(x-1)} = 0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x) = -\frac{4x+1}{2x(x-1)}, P_3(x) = \frac{2}{x(x-1)}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{1}{2}$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

- ○ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x(x-1)\left(\frac{d^2}{dx^2}y(x)\right) + (-4x-1)\left(\frac{d}{dx}y(x)\right) + 4y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..1$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ○ Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 1..2$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

- ∘ Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0r(-1+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)(2k+1+2r) + 2a_k(k+r-1)(k+r-2)\right) x^{k+r}$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $$-r(-1+2r) = 0$$
- Values of r that satisfy the indicial equation
  $$r \in \left\{0, \frac{1}{2}\right\}$$
- Each term in the series must be 0, giving the recursion relation
  $$-2(k+1+r)\left(k+r+\frac{1}{2}\right)a_{k+1} + 2a_k(k+r-1)(k+r-2) = 0$$
- Recursion relation that defines series solution to ODE
  $$a_{k+1} = \frac{2a_k(k+r-1)(k+r-2)}{(k+1+r)(2k+1+2r)}$$
- Recursion relation for $r=0$; series terminates at $k=1$
  $$a_{k+1} = \frac{2a_k(k-1)(k-2)}{(k+1)(2k+1)}$$
- Apply recursion relation for $k=0$
  $$a_1 = 4a_0$$
- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li
  $$y(x) = a_0 \cdot (4x+1)$$
- Recursion relation for $r=\frac{1}{2}$
  $$a_{k+1} = \frac{2a_k\left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}$$
- Solution for $r=\frac{1}{2}$
  $$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{2a_k\left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$
- Combine solutions and rename parameters
  $$\left[y(x) = a_0 \cdot (4x+1) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), b_{k+1} = \frac{2b_k\left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 53

```
dsolve(x*(1-x)*diff(diff(y(x),x),x)+(1/2+2*x)*diff(y(x),x)-2*y(x) = 0,y(x),singsol=all)
```

$$y = (-12x-3)\,c_2 \ln\left(2x-1+2\sqrt{(x-1)\,x}\right) + (4x+26)\,c_2\sqrt{(x-1)\,x}+4\left(x+\frac{1}{4}\right)(3c_2\ln(2)+c_1)$$

Mathematica DSolve solution

Solving time : 0.532 (sec)
Leaf size : 130

```
DSolve[{x*(1-x)*D[y[x],{x,2}]+(1/2+2*x)*D[y[x],x]-2*y[x]==0,{}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{1}{4}(4x+1)\exp\left(\int_1^x \frac{4K[1]+1}{4K[1]-4K[1]^2}dK[1] - \frac{1}{2}\int_1^x \frac{4K[2]+1}{2K[2]-2K[2]^2}dK[2]\right)\left(c_2\int_1^x \frac{16\exp\left(-2\int_1^{K[3]}\frac{4K[1]+1}{4K[1]-4K[1]^2}dK[1]\right)}{(4K[3]+1)^2}dK[3] + c_1\right)$$

2.1.346 Problem 353

Internal problem ID [9518]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 353
Date solved : Monday, January 27, 2025 at 06:03:41 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$4\left(t^2-3t+2\right)y''-2y'+y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.243 (sec)

Writing the ode as

$$\left(4t^2-12t+8\right)y''-2y'+y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4t^2-12t+8 \\ B &= -2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{-4t^2+20t-19}{16\left(t^2-3t+2\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4t^2+20t-19 \\ t &= 16\left(t^2-3t+2\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t)=\left(\frac{-4t^2+20t-19}{16\left(t^2-3t+2\right)^2}\right)z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.654: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(t^2 - 3t + 2)^2$. There is a pole at $t = 2$ of order 2. There is a pole at $t = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16\,(t-2)^2} - \frac{3}{8\,(t-2)} - \frac{3}{16\,(t-1)^2} + \frac{3}{8\,(t-1)}$$

For the pole at $t = 2$ let b be the coefficient of $\frac{1}{(t-2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

For the pole at $t = 1$ let b be the coefficient of $\frac{1}{(t-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-4t^2 + 20t - 19}{16\left(t^2 - 3t + 2\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-4t^2 + 20t - 19}{16\left(t^2 - 3t + 2\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
2	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
1	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{t - c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{4\left(t - 2\right)} + \frac{3}{4\left(t - 1\right)} + (-)\left(0\right) \\
&= -\frac{1}{4\left(t - 2\right)} + \frac{3}{4\left(t - 1\right)} \\
&= \frac{2t - 5}{4\left(t - 1\right)\left(t - 2\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{4(t-2)} + \frac{3}{4(t-1)}\right)(0) + \left(\left(\frac{1}{4(t-2)^2} - \frac{3}{4(t-1)^2}\right) + \left(-\frac{1}{4(t-2)} + \frac{3}{4(t-1)}\right)^2 - \left(\frac{-4t^2 +}{16(t^2}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= e^{\int \left(-\frac{1}{4(t-2)} + \frac{3}{4(t-1)}\right) dt} \\ &= \frac{(t-1)^{3/4}}{(t-2)^{1/4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2}{4t^2-12t+8} \, dt} \\ &= z_1 e^{-\frac{\ln(t-1)}{4} + \frac{\ln(t-2)}{4}} \\ &= z_1 \left(\frac{(t-2)^{1/4}}{(t-1)^{1/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{t-1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{4t^2-12t+8} \, dt}}{(y_1)^2} \, dt \\ &= y_1 \int \frac{e^{-\frac{\ln(t-1)}{2} + \frac{\ln(t-2)}{2}}}{(y_1)^2} \, dt \\ &= y_1 \left(-\frac{2\sqrt{t-2}}{\sqrt{t-1}} + \frac{\ln\left(-\frac{3}{2} + t + \sqrt{t^2-3t+2}\right)\sqrt{(t-1)(t-2)}}{\sqrt{t-2}\,\sqrt{t-1}}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\sqrt{t-1}\right) + c_2\left(\sqrt{t-1}\left(-\frac{2\sqrt{t-2}}{\sqrt{t-1}} + \frac{\ln\left(-\frac{3}{2} + t + \sqrt{t^2-3t+2}\right)\sqrt{(t-1)(t-2)}}{\sqrt{t-2}\,\sqrt{t-1}}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4(t^2-3t+2)\left(\frac{d^2}{dt^2}y(t)\right)-2\frac{d}{dt}y(t)+y(t)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dt^2}y(t)$

- Isolate 2nd derivative

$\frac{d^2}{dt^2}y(t)=-\frac{y(t)}{4(t^2-3t+2)}+\frac{\frac{d}{dt}y(t)}{2(t^2-3t+2)}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dt^2}y(t)-\frac{\frac{d}{dt}y(t)}{2(t^2-3t+2)}+\frac{y(t)}{4(t^2-3t+2)}=0$

□ Check to see if t_0 is a regular singular point

- ◦ Define functions

$\left[P_2(t)=-\frac{1}{2(t^2-3t+2)}, P_3(t)=\frac{1}{4(t^2-3t+2)}\right]$

- ◦ $(t-1)\cdot P_2(t)$ is analytic at $t=1$

$\left((t-1)\cdot P_2(t)\right)\Big|_{t=1}=\frac{1}{2}$

- ◦ $(t-1)^2\cdot P_3(t)$ is analytic at $t=1$

$\left((t-1)^2\cdot P_3(t)\right)\Big|_{t=1}=0$

- ◦ $t=1$is a regular singular point

Check to see if t_0 is a regular singular point

$t_0=1$

- Multiply by denominators

$(4t^2-12t+8)\left(\frac{d^2}{dt^2}y(t)\right)-2\frac{d}{dt}y(t)+y(t)=0$

- Change variables using $t=u+1$ so that the regular singular point is at $u=0$

$(4u^2-4u)\left(\frac{d^2}{du^2}y(u)\right)-2\frac{d}{du}y(u)+y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $\frac{d}{du}y(u)$ to series expansion

$\frac{d}{du}y(u)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1}$

- ◦ Shift index using $k->k+1$

$\frac{d}{du}y(u)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,u^{k+r}$

- ◦ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- ◦ Shift index using $k->k+2-m$

$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

Rewrite ODE with series expansions

$-2a_0r(-1+2r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)\,(2k+1+2r)+a_k(2k+2r-1)^2\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2r(-1+2r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,\frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(2k+2r-1)^2-4(k+1+r)\,a_{k+1}\left(k+r+\frac{1}{2}\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{a_k(2k+2r-1)^2}{2(k+1+r)(2k+1+2r)}$$

- Recursion relation for $r=0$

$$a_{k+1}=\frac{a_k(2k-1)^2}{2(k+1)(2k+1)}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty}a_k u^k, a_{k+1}=\frac{a_k(2k-1)^2}{2(k+1)(2k+1)}\right]$$

- Revert the change of variables $u=t-1$

$$\left[y(t)=\sum_{k=0}^{\infty}a_k(t-1)^k, a_{k+1}=\frac{a_k(2k-1)^2}{2(k+1)(2k+1)}\right]$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+1}=\frac{2a_k k^2}{\left(k+\frac{3}{2}\right)(2k+2)}$$

- Solution for $r=\frac{1}{2}$

$$\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+\frac{1}{2}}, a_{k+1}=\frac{2a_k k^2}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$

- Revert the change of variables $u=t-1$

$$\left[y(t)=\sum_{k=0}^{\infty}a_k(t-1)^{k+\frac{1}{2}}, a_{k+1}=\frac{2a_k k^2}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y(t)=\left(\sum_{k=0}^{\infty}a_k(t-1)^k\right)+\left(\sum_{k=0}^{\infty}b_k(t-1)^{k+\frac{1}{2}}\right), a_{k+1}=\frac{a_k(2k-1)^2}{2(k+1)(2k+1)}, b_{k+1}=\frac{2b_k k^2}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.019 (sec)
Leaf size : 56

```
dsolve(4*(t^2-3*t+2)*diff(diff(y(t),t),t)-2*diff(y(t),t)+y(t) = 0,y(t),singsol=all)
```

$$y = c_1\sqrt{t-1} + \frac{c_2\left(-\frac{\sqrt{t^2-3t+2}\left(-\ln(2)+\ln\left(-3+2t+2\sqrt{(t-1)(t-2)}\right)\right)}{2} + t - 2\right)}{\sqrt{t-2}}$$

Mathematica DSolve solution

Solving time : 0.176 (sec)
Leaf size : 112

```
DSolve[{4*(t^2-3*t+2)*D[y[t],{t,2}]-2*D[y[t],t]+y[t]==0,{}},y[t],t,IncludeSingularSolutions-
```

$$y(t) \to \exp\left(\int_1^t \frac{2K[1]-5}{4\left(K[1]^2-3K[1]+2\right)}dK[1] - \frac{1}{2}\int_1^t -\frac{1}{2\left(K[2]^2-3K[2]+2\right)}dK[2]\right)\left(c_2\int_1^t \exp\left(-2\int_1^{K[3]} \frac{2K[1]-5}{4\left(K[1]^2-3K[1]+2\right)}dK[1]\right)dK[3] + c_1\right)$$

2.1.347 Problem 354

Internal problem ID [9519]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 354
Date solved : `Monday, January 27, 2025 at 06:03:41 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2\left(t^2-5t+6\right)y''+(2t-3)\,y'-8y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.236 (sec)

Writing the ode as

$$\left(2t^2-10t+12\right)y''+(2t-3)\,y'-8y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2t^2-10t+12 \\ B &= 2t-3 \\ C &= -8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{60t^2-308t+381}{16\left(t^2-5t+6\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 60t^2-308t+381 \\ t &= 16\left(t^2-5t+6\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t)=\left(\frac{60t^2-308t+381}{16\left(t^2-5t+6\right)^2}\right)z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.656: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(t^2 - 5t + 6)^2$. There is a pole at $t = 3$ of order 2. There is a pole at $t = 2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{29}{8\,(t-3)} - \frac{3}{16\,(t-3)^2} + \frac{5}{16\,(t-2)^2} - \frac{29}{8\,(t-2)}$$

For the pole at $t = 3$ let b be the coefficient of $\frac{1}{(t-3)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $t = 2$ let b be the coefficient of $\frac{1}{(t-2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{60t^2 - 308t + 381}{16\left(t^2 - 5t + 6\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{60t^2 - 308t + 381}{16\left(t^2 - 5t + 6\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
3	2	0	$\frac{3}{4}$	$\frac{1}{4}$
2	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= \frac{5}{2} - \left(\frac{3}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{t-c_2}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{4t-12} + \frac{5}{4(t-2)} + (0) \\
&= \frac{1}{4t-12} + \frac{5}{4(t-2)} \\
&= \frac{6t-17}{4(t-2)(t-3)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 1$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{4t-12} + \frac{5}{4(t-2)}\right)(1) + \left(\left(-\frac{1}{4(t-3)^2} - \frac{5}{4(t-2)^2}\right) + \left(\frac{1}{4t-12} + \frac{5}{4(t-2)}\right)^2 - \left(\frac{60t^2 - \cdots}{16(t^2 - \cdots}\right.\right.$$

$$\frac{-6\cdots}{2t^2 - \cdots}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{17}{6}\right\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = t - \frac{17}{6}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega\, dt} \\ &= \left(t - \frac{17}{6}\right) e^{\int \left(\frac{1}{4t-12} + \frac{5}{4(t-2)}\right) dt} \\ &= \left(t - \frac{17}{6}\right) e^{\frac{5\ln(t-2)}{4} + \frac{\ln(t-3)}{4}} \\ &= \left(t - \frac{17}{6}\right) (t-2)^{5/4} (t-3)^{1/4} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2t-3}{2t^2-10t+12}\, dt} \\ &= z_1 e^{\frac{\ln(t-2)}{4} - \frac{3\ln(t-3)}{4}} \\ &= z_1 \left(\frac{(t-2)^{1/4}}{(t-3)^{3/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(t-2)^{3/2}(6t-17)}{6\sqrt{t-3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2t-3}{2t^2-10t+12}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{\frac{\ln(t-2)}{2} - \frac{3\ln(t-3)}{2}}}{(y_1)^2}\, dt \\ &= y_1 \left(\frac{24(t-3)^2 (24t^2 - 104t + 111)\, e^{\frac{\ln(t-2)}{2} - \frac{3\ln(t-3)}{2}}}{5(6t-17)(t-2)^2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(t-2)^{3/2} (6t-17)}{6\sqrt{t-3}} \right) \\
&\quad + c_2 \left(\frac{(t-2)^{3/2} (6t-17)}{6\sqrt{t-3}} \left(\frac{24(t-3)^2 (24t^2 - 104t + 111) \mathrm{e}^{\frac{\ln(t-2)}{2} - \frac{3\ln(t-3)}{2}}}{5(6t-17)(t-2)^2} \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2(t^2-5t+6)\left(\frac{d^2}{dt^2}y(t)\right) + (2t-3)\left(\frac{d}{dt}y(t)\right) - 8y(t) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dt^2}y(t)$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = \frac{4y(t)}{t^2-5t+6} - \frac{(2t-3)\left(\frac{d}{dt}y(t)\right)}{2(t^2-5t+6)}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d^2}{dt^2}y(t) + \frac{(2t-3)\left(\frac{d}{dt}y(t)\right)}{2(t^2-5t+6)} - \frac{4y(t)}{t^2-5t+6} = 0$

□ Check to see if t_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(t) = \frac{2t-3}{2(t^2-5t+6)}, P_3(t) = -\frac{4}{t^2-5t+6}\right]$

  ◦ $(t-2) \cdot P_2(t)$ is analytic at $t=2$

  $\left((t-2) \cdot P_2(t)\right)\Big|_{t=2} = -\frac{1}{2}$

  ◦ $(t-2)^2 \cdot P_3(t)$ is analytic at $t=2$

  $\left((t-2)^2 \cdot P_3(t)\right)\Big|_{t=2} = 0$

  ◦ $t=2$is a regular singular point

  Check to see if t_0 is a regular singular point

  $t_0 = 2$

- Multiply by denominators

  $(2t^2-10t+12)\left(\frac{d^2}{dt^2}y(t)\right) + (2t-3)\left(\frac{d}{dt}y(t)\right) - 8y(t) = 0$

- Change variables using $t = u+2$ so that the regular singular point is at $u=0$

  $(2u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (2u+1)\left(\frac{d}{du}y(u)\right) - 8y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum\limits_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum\limits_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum\limits_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-a_0 r(-3+2r)\, u^{-1+r} + \left(\sum\limits_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(2k-1+2r) + 2a_k(k+r+2)\,(k+r-2)\right) u^{k+r}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $-2(k+1+r)\left(k+r-\frac{1}{2}\right) a_{k+1} + 2a_k(k+r+2)\,(k+r-2) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k(k+r+2)(k+r-2)}{(k+1+r)(2k-1+2r)}$

- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+1} = \frac{2a_k(k+2)(k-2)}{(k+1)(2k-1)}$

- Apply recursion relation for $k = 0$

  $a_1 = 8a_0$

- Apply recursion relation for $k = 1$

  $a_2 = -3a_1$

- Express in terms of a_0

  $a_2 = -24a_0$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot (-24u^2 + 8u + 1)$

- Revert the change of variables $u = t - 2$

  $[y(t) = a_0(-24t^2 + 104t - 111)]$

- Recursion relation for $r = \frac{3}{2}$

  $a_{k+1} = \frac{2a_k\left(k+\frac{7}{2}\right)\left(k-\frac{1}{2}\right)}{\left(k+\frac{5}{2}\right)(2k+2)}$

- Solution for $r = \frac{3}{2}$

  $\left[y(u) = \sum\limits_{k=0}^{\infty} a_k u^{k+\frac{3}{2}}, a_{k+1} = \frac{2a_k\left(k+\frac{7}{2}\right)\left(k-\frac{1}{2}\right)}{\left(k+\frac{5}{2}\right)(2k+2)}\right]$

- Revert the change of variables $u = t - 2$

  $\left[y(t) = \sum\limits_{k=0}^{\infty} a_k(t-2)^{k+\frac{3}{2}}, a_{k+1} = \frac{2a_k\left(k+\frac{7}{2}\right)\left(k-\frac{1}{2}\right)}{\left(k+\frac{5}{2}\right)(2k+2)}\right]$

- Combine solutions and rename parameters

  $\left[y(t) = a_0(-24t^2 + 104t - 111) + \left(\sum\limits_{k=0}^{\infty} b_k(t-2)^{k+\frac{3}{2}}\right), b_{k+1} = \frac{2b_k\left(k+\frac{7}{2}\right)\left(k-\frac{1}{2}\right)}{\left(k+\frac{5}{2}\right)(2k+2)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.026 (sec)
Leaf size : 35

```
dsolve(2*(t^2-5*t+6)*diff(diff(y(t),t),t)+(2*t-3)*diff(y(t),t)-8*y(t) = 0,y(t),singsol=a
```

$$y = \frac{c_1(24t^2 - 104t + 111)}{24} + \frac{c_2(6t - 17)(t - 2)^{3/2}}{\sqrt{t - 3}}$$

Mathematica DSolve solution

Solving time : 0.512 (sec)
Leaf size : 130

```
DSolve[{2*(t^2-5*t+6)*D[y[t],{t,2}]+(2*t-3)*D[y[t],t]-8*y[t]==0,{}},y[t],t,IncludeSingularSolu
```

$$y(t) \to \frac{1}{6}(6t - 17) \exp\left(\int_1^t \frac{1}{4}\left(\frac{5}{K[1] - 2} + \frac{1}{K[1] - 3}\right) dK[1] - \frac{1}{2}\int_1^t \frac{2K[2] - 3}{2(K[2]^2 - 5K[2] + 6)} dK[2]\right) \left(c_2 \int_1^t \frac{36 \exp\left(-2\int_1^{K[3]} \frac{1}{4}\left(\frac{5}{K[1]-2} + \frac{1}{K[1]-3}\right) dK[1]\right)}{(17 - 6K[3])^2} dK[3] + c_1\right)$$

2.1.348 Problem 355

Internal problem ID [9520]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 355
Date solved : `Monday, January 27, 2025 at 06:03:42 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$3t(1+t)\,y''+ty'-y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.280 (sec)

Writing the ode as

$$\left(3t^2+3t\right)y''+ty'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3t^2+3t \\ B &= t \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{7t+12}{36t\left(1+t\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 7t+12 \\ t &= 36t(1+t)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t)=\left(\frac{7t+12}{36t\left(1+t\right)^2}\right)z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.658: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36t(1+t)^2$. There is a pole at $t = 0$ of order 1. There is a pole at $t = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $t = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{3t} - \frac{5}{36\,(1+t)^2} - \frac{1}{3\,(1+t)}$$

For the pole at $t = -1$ let b be the coefficient of $\frac{1}{(1+t)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{7t+12}{36t\,(1+t)^2}$$

Since the gcd$(s,t) = 1$. This gives $b = \frac{7}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6}\\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{7t+12}{36t\,(1+t)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1
-1	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{6}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right)\\
&= \frac{7}{6} - \left(\frac{7}{6}\right)\\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{t-c_2}\right) + (+)[\sqrt{r}]_\infty\\
&= \frac{1}{t} + \frac{1}{6+6t} + (0)\\
&= \frac{1}{t} + \frac{1}{6+6t}\\
&= \frac{1}{t} + \frac{1}{6+6t}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{t} + \frac{1}{6+6t}\right)(0) + \left(\left(-\frac{1}{t^2} - \frac{1}{6(1+t)^2}\right) + \left(\frac{1}{t} + \frac{1}{6+6t}\right)^2 - \left(\frac{7t+12}{36t(1+t)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= e^{\int \left(\frac{1}{t} + \frac{1}{6+6t}\right) dt} \\ &= (1+t)^{1/6} t \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\ &= z_1 e^{-\int \frac{1}{2} \frac{t}{3t^2+3t} \, dt} \\ &= z_1 e^{-\frac{\ln(1+t)}{6}} \\ &= z_1 \left(\frac{1}{(1+t)^{1/6}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{t}{3t^2+3t} \, dt}}{(y_1)^2} \, dt \\ &= y_1 \int \frac{e^{-\frac{\ln(1+t)}{3}}}{(y_1)^2} \, dt \\ &= y_1 \left(\frac{-2(1+t)^{1/3} - 1}{3(1+t)^{2/3} + 3(1+t)^{1/3} + 3} + \frac{\ln\left((1+t)^{2/3} + (1+t)^{1/3} + 1\right)}{6} \right. \\ &\quad \left. - \frac{\sqrt{3} \arctan\left(\frac{\left(1+2(1+t)^{1/3}\right)\sqrt{3}}{3}\right)}{3} - \frac{1}{3\left((1+t)^{1/3} - 1\right)} - \frac{\ln\left((1+t)^{1/3} - 1\right)}{3} \right) \end{aligned}$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1(t) + c_2\left(t\left(\frac{-2(1+t)^{1/3}-1}{3(1+t)^{2/3}+3(1+t)^{1/3}+3} + \frac{\ln\left((1+t)^{2/3}+(1+t)^{1/3}+1\right)}{6} - \frac{\sqrt{3}\arctan\left(\frac{\left(1+2(1+t)^{1/3}\right)\sqrt{3}}{3}\right)}{3} - \frac{1}{3\left((1+t)^{1/3}-1\right)} - \frac{\ln\left((1+t)^{1/3}-1\right)}{3}\right)\right)$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$3t(t+1)\left(\frac{d^2}{dt^2}y(t)\right) + t\left(\frac{d}{dt}y(t)\right) - y(t) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dt^2}y(t)$
- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = \frac{y(t)}{3t(t+1)} - \frac{\frac{d}{dt}y(t)}{3(t+1)}$
- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dt^2}y(t) + \frac{\frac{d}{dt}y(t)}{3(t+1)} - \frac{y(t)}{3t(t+1)} = 0$

□ Check to see if t_0 is a regular singular point
  - Define functions

    $\left[P_2(t) = \frac{1}{3(t+1)}, P_3(t) = -\frac{1}{3t(t+1)}\right]$
  - $(t+1)\cdot P_2(t)$ is analytic at $t=-1$

    $\left((t+1)\cdot P_2(t)\right)\Big|_{t=-1} = \frac{1}{3}$
  - $(t+1)^2\cdot P_3(t)$ is analytic at $t=-1$

    $\left((t+1)^2\cdot P_3(t)\right)\Big|_{t=-1} = 0$
  - $t=-1$is a regular singular point

  Check to see if t_0 is a regular singular point

  $t_0 = -1$
- Multiply by denominators

  $3t(t+1)\left(\frac{d^2}{dt^2}y(t)\right) + t\left(\frac{d}{dt}y(t)\right) - y(t) = 0$
- Change variables using $t = u-1$ so that the regular singular point is at $u=0$

  $(3u^2-3u)\left(\frac{d^2}{du^2}y(u)\right) + (u-1)\left(\frac{d}{du}y(u)\right) - y(u) = 0$
- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- ∘ Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- ∘ Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-a_0r(-2+3r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} (-a_{k+1}(k+1+r)\,(3k+3r+1) + a_k(3k+3r+1)\,(k+r-1))\, u^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-r(-2+3r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{0, \frac{2}{3}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $3\left(k+r+\frac{1}{3}\right)\left((-k-r-1)\, a_{k+1} + a_k(k+r-1)\right) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_k(k+r-1)}{k+1+r}$
- Recursion relation for $r = 0$; series terminates at $k = 1$
  $a_{k+1} = \frac{a_k(k-1)}{k+1}$
- Apply recursion relation for $k = 0$
  $a_1 = -a_0$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second linea
  $y(u) = a_0 \cdot (-u+1)$
- Revert the change of variables $u = t+1$
  $[y(t) = -a_0t]$
- Recursion relation for $r = \frac{2}{3}$
  $a_{k+1} = \frac{a_k\left(k-\frac{1}{3}\right)}{k+\frac{5}{3}}$
- Solution for $r = \frac{2}{3}$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{2}{3}}, a_{k+1} = \frac{a_k\left(k-\frac{1}{3}\right)}{k+\frac{5}{3}}\right]$
- Revert the change of variables $u = t+1$
  $\left[y(t) = \sum_{k=0}^{\infty} a_k(t+1)^{k+\frac{2}{3}}, a_{k+1} = \frac{a_k\left(k-\frac{1}{3}\right)}{k+\frac{5}{3}}\right]$
- Combine solutions and rename parameters
  $\left[y(t) = -a_0t + \left(\sum_{k=0}^{\infty} b_k(t+1)^{k+\frac{2}{3}}\right), b_{k+1} = \frac{b_k\left(k-\frac{1}{3}\right)}{k+\frac{5}{3}}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returr
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.043 (sec)
Leaf size : 67

```
dsolve(3*t*(t+1)*diff(diff(y(t),t),t)+t*diff(y(t),t)-y(t) = 0,y(t),singsol=all)
```

$$y = c_1 t + 2\sqrt{3}\,\arctan\left(\frac{\left(2(t+1)^{1/3}+1\right)\sqrt{3}}{3}\right) t c_2 + 2\ln\left((t+1)^{1/3}-1\right) t c_2 + 6(t+1)^{2/3} c_2 - \ln\left((t+1)^{2/3}+(t+1)^{1/3}+1\right) t c_2$$

Mathematica DSolve solution

Solving time : 0.424 (sec)
Leaf size : 78

```
DSolve[{3*t*(1+t)*D[y[t],{t,2}]+t*D[y[t],t]-y[t]==0,{}},y[t],t,IncludeSingularSolutions->Tru
```

$$y(t) \rightarrow \frac{\exp\left(\int_1^t \left(\frac{1}{6K[1]+6}+\frac{1}{K[1]}\right) dK[1]\right)\left(c_2 \int_1^t \exp\left(-2\int_1^{K[2]}\left(\frac{1}{6K[1]+6}+\frac{1}{K[1]}\right) dK[1]\right) dK[2] + c_1\right)}{\sqrt[6]{3}\sqrt[6]{t+1}}$$

2.1.349 Problem 356

Internal problem ID [9521]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 356
Date solved : `Monday, January 27, 2025 at 06:03:43 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + \frac{\left(x + \frac{3}{4}\right) y}{4} = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.200 (sec)

Writing the ode as

$$x^2y'' + \left(\frac{x}{4} + \frac{3}{16}\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 0 \\ C &= \frac{x}{4} + \frac{3}{16} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4x - 3}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4x - 3 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-4x-3}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.660: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x} - \frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1, \; e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(1 - (1)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x - c} \\ &= \frac{1}{2}\left(\frac{1}{(x - (0))}\right) \\ &= \frac{1}{2x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right) p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right) p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2x} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2x} + \frac{1 + 4x}{16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{1 + 2\sqrt{-x}}{4x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{1+2\sqrt{-x}}{4x} dx} \\
&= x^{1/4}\mathrm{e}^{\sqrt{-x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= x^{1/4}\mathrm{e}^{\sqrt{-x}}
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^{1/4}\mathrm{e}^{\sqrt{-x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\
&= x^{1/4}\mathrm{e}^{\sqrt{-x}} \int \frac{1}{\sqrt{x}\,\mathrm{e}^{2\sqrt{-x}}} \, dx \\
&= x^{1/4}\mathrm{e}^{\sqrt{-x}} \left(-\frac{\sqrt{-x}\left(1-\mathrm{e}^{-2\sqrt{-x}}\right)}{\sqrt{x}} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{1/4}\mathrm{e}^{\sqrt{-x}} \right) + c_2 \left(x^{1/4}\mathrm{e}^{\sqrt{-x}} \left(-\frac{\sqrt{-x}\left(1-\mathrm{e}^{-2\sqrt{-x}}\right)}{\sqrt{x}} \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2 \left(\frac{d^2}{dx^2} y(x) \right) + \frac{\left(\frac{3}{4}+x\right) y(x)}{4} = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2} y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2} y(x) = -\frac{(3+4x) y(x)}{16x^2}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) + \frac{(3+4x)y(x)}{16x^2} = 0$
- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = 0, P_3(x) = \frac{3+4x}{16x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $$\left(x \cdot P_2(x)\right)\Big|_{x=0} = 0$$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{3}{16}$$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $16x^2\left(\frac{d^2}{dx^2}y(x)\right) + (3+4x)\,y(x) = 0$
- Assume series solution for $y(x)$
  $$y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$$
- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$
    $$x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$
  - Shift index using $k- >k-m$
    $$x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$
  - Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion
    $$x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$$
    Rewrite ODE with series expansions
    $$a_0(-1+4r)\,(-3+4r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(4k+4r-1)\,(4k+4r-3) + 4a_{k-1}\right) x^{k+r}\right) = 0$$
- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+4r)\,(-3+4r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{\frac{1}{4}, \frac{3}{4}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $16\left(k+r-\frac{3}{4}\right)\left(k+r-\frac{1}{4}\right)a_k + 4a_{k-1} = 0$
- Shift index using $k- >k+1$
  $16\left(k+\frac{1}{4}+r\right)\left(k+\frac{3}{4}+r\right)a_{k+1} + 4a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = -\frac{4a_k}{(4k+1+4r)(4k+3+4r)}$
- Recursion relation for $r = \frac{1}{4}$
  $a_{k+1} = -\frac{4a_k}{(4k+2)(4k+4)}$
- Solution for $r = \frac{1}{4}$
  $$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+1} = -\frac{4a_k}{(4k+2)(4k+4)}\right]$$

- Recursion relation for $r = \frac{3}{4}$

  $a_{k+1} = -\frac{4a_k}{(4k+4)(4k+6)}$
- Solution for $r = \frac{3}{4}$

  $$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{4}}, a_{k+1} = -\frac{4a_k}{(4k+4)(4k+6)}\right]$$
- Combine solutions and rename parameters

  $$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{4}}\right), a_{k+1} = -\frac{4a_k}{(4k+2)(4k+4)}, b_{k+1} = -\frac{4b_k}{(4k+4)(4k+6)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 21

```
dsolve(x^2*diff(diff(y(x),x),x)+1/4*(x+3/4)*y(x) = 0,y(x),singsol=all)
```

$$y = x^{1/4}\left(c_1 \sin\left(\sqrt{x}\right) + c_2 \cos\left(\sqrt{x}\right)\right)$$

Mathematica DSolve solution

Solving time : 0.06 (sec)
Leaf size : 43

```
DSolve[{x^2*D[y[x],{x,2}]+1/4*(x+3/4)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-i\sqrt{x}}\sqrt[4]{x}\left(c_1 e^{2i\sqrt{x}} + ic_2\right)$$

2.1.350 Problem 357

Internal problem ID [9522]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 357
Date solved : `Monday, January 27, 2025 at 06:03:43 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \frac{(x^2-1)\,y}{4} = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.148 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(\frac{x^2}{4} - \frac{1}{4}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= \frac{x^2}{4} - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.662: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos\left(\frac{x}{2}\right)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\, dx \\ &= y_1\left(2 \tan\left(\frac{x}{2}\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\cos\left(\frac{x}{2}\right)}{\sqrt{x}}\right) + c_2\left(\frac{\cos\left(\frac{x}{2}\right)}{\sqrt{x}}\left(2 \tan\left(\frac{x}{2}\right)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x\left(\frac{d}{dx}y(x)\right) + \frac{(x^2-1)y(x)}{4} = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{(x^2-1)y(x)}{4x^2} - \frac{\frac{d}{dx}y(x)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) + \frac{\frac{d}{dx}y(x)}{x} + \frac{(x^2-1)y(x)}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{x^2-1}{4x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $(x \cdot P_2(x))\Big|_{x=0} = 1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{1}{4}$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x\left(\frac{d}{dx}y(x)\right) + (x^2-1)\, y(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$
  - Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) + a_{k-2}\right.$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + a_{k-2} = 0$
- Shift index using $k->k+2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{a_k}{4k^2+12k+8}$
- Solution for $r = -\frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{a_k}{4k^2+12k+8}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{a_k}{4k^2+20k+24}$
- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{a_k}{4k^2+20k+24}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.037 (sec)
Leaf size : 21

```
dsolve(x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x+1/4*(x^2-1)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_1 \sin\left(\frac{x}{2}\right) + c_2 \cos\left(\frac{x}{2}\right)}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.035 (sec)
Leaf size : 36

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+1/4*(x^2-1)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->
```

$$y(x) \to \frac{e^{-\frac{ix}{2}}\left(c_1 - ic_2 e^{ix}\right)}{\sqrt{x}}$$

2.1.351 Problem 358

Internal problem ID [9523]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 358
Date solved : `Monday, January 27, 2025 at 06:03:44 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + (1-2x)\,y' + (x-1)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.146 (sec)

Writing the ode as

$$xy'' + (1-2x)\,y' + (x-1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 1-2x \\ C &= x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.664: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the gcd$(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\,(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{1}{2x}\right)(0)+\left(\left(-\frac{1}{2x^2}\right)+\left(\frac{1}{2x}\right)^2-\left(-\frac{1}{4x^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}z_1(x)&=pe^{\int \omega\,dx}\\&=e^{\int \frac{1}{2x}dx}\\&=\sqrt{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1&=z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\&=z_1e^{-\int \frac{1}{2}\frac{1-2x}{x}\,dx}\\&=z_1e^{x-\frac{\ln(x)}{2}}\\&=z_1\left(\frac{e^x}{\sqrt{x}}\right)\end{aligned}$$

Which simplifies to

$$y_1=e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2&=y_1\int \frac{e^{\int -\frac{1-2x}{x}\,dx}}{(y_1)^2}\,dx\\&=y_1\int \frac{e^{2x-\ln(x)}}{(y_1)^2}\,dx\\&=y_1(\ln(x))\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y&=c_1y_1+c_2y_2\\&=c_1(e^x)+c_2(e^x(\ln(x)))\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dx^2}y(x)\right)x+(-2x+1)\left(\frac{d}{dx}y(x)\right)+(x-1)y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(x-1)y(x)}{x}+\frac{(2x-1)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)-\frac{(2x-1)\left(\frac{d}{dx}y(x)\right)}{x}+\frac{(x-1)y(x)}{x}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=-\frac{2x-1}{x},P_3(x)=\frac{x-1}{x}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\Big|_{x=0}=1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $\left(\frac{d^2}{dx^2}y(x)\right)x+(-2x+1)\left(\frac{d}{dx}y(x)\right)+(x-1)y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

    $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  ◦ Shift index using $k->k-m$

    $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..1$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-1}$

  ◦ Shift index using $k->k+1$

    $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$a_0r^2x^{-1+r} + \left(a_1(1+r)^2 - a_0(1+2r)\right)x^r + \left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+1+r)^2 - a_k(2k+2r+1) + a_{k-1}\right)x^{k+r}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 0$
- Each term must be 0
  $a_1(1+r)^2 - a_0(1+2r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1)^2 + (-2k-1)\,a_k + a_{k-1} = 0$
- Shift index using $k- >k+1$
  $a_{k+2}(k+2)^2 + (-2k-3)\,a_{k+1} + a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{2ka_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}$
- Recursion relation for $r = 0$
  $a_{k+2} = \frac{2ka_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}$
- Solution for $r = 0$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2ka_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}, a_1 - a_0 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 13

```
dsolve(x*diff(diff(y(x),x),x)+(1-2*x)*diff(y(x),x)+(x-1)*y(x) = 0,y(x),singsol=all)
```

$$y = e^x(c_2 \ln(x) + c_1)$$

Mathematica DSolve solution

Solving time : 0.025 (sec)
Leaf size : 17

```
DSolve[{x*D[y[x],{x,2}]+(1-2*x)*D[y[x],x]+(x-1)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to e^x(c_2 \log(x) + c_1)$$

2.1.352 Problem 359

Internal problem ID [9524]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 359
Date solved : `Monday, January 27, 2025 at 06:03:44 PM`
CAS classification : `[_Laguerre]`

Solve

$$xy'' - (x+1)\,y' + y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.220 (sec)

Writing the ode as

$$xy'' + (-x-1)\,y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -x-1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 2x + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 2x + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 2x + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.666: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4x^2} - \frac{1}{2x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{2x^2} + \frac{1}{2x^3} + \frac{1}{4x^4} - \frac{1}{4x^5} - \frac{3}{4x^6} - \frac{3}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 2x + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 3}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-2x + 3}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 2x + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2} - \frac{1}{2x} \\ &= \frac{x-1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2} - \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(\frac{1}{2} - \frac{1}{2x}\right)^2 - \left(\frac{x^2 - 2x + 3}{4x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int (\frac{1}{2} - \frac{1}{2x}) dx} \\ &= \frac{e^{\frac{x}{2}}}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x-1}{x}\,dx} \\
&= z_1 e^{\frac{x}{2}+\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\,\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x-1}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x+\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{(x+1)\,\mathrm{e}^{x+\ln(x)}\mathrm{e}^{-2x}}{x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(-\frac{(x+1)\,\mathrm{e}^{x+\ln(x)}\mathrm{e}^{-2x}}{x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dx^2}y(x)\right)x-(x+1)\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{y(x)}{x}+\frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)-\frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{x}+\frac{y(x)}{x}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=-\frac{x+1}{x}, P_3(x)=\frac{1}{x}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=-1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$\left(\frac{d^2}{dx^2}y(x)\right)x + (-x-1)\left(\frac{d}{dx}y(x)\right) + y(x) = 0$$

- Assume series solution for $y(x)$

  $$y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..1$

  $$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $$x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

  $$x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-2+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)(a_{k+1}(k+1+r) - a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1+r}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 0$

  $$\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+3}$

- Solution for $r = 2$

  $$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

  $$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 13

```
dsolve(x*diff(diff(y(x),x),x)-(x+1)*diff(y(x),x)+y(x) = 0,y(x),singsol=all)
```

$$y = e^{x} c_2 + c_1 x + c_1$$

Mathematica DSolve solution

Solving time : 0.4 (sec)
Leaf size : 78

```
DSolve[{x*D[y[x],{x,2}]-(x+1)*D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \sqrt{x} \exp \left(\frac{1}{2}\left(2 \int_1^x \frac{K[1]-1}{2K[1]} dK[1] + x + 1\right)\right) \left(c_2 \int_1^x \exp \left(-2 \int_1^{K[2]} \frac{K[1]-1}{2K[1]} dK[1]\right) dK[2] + c_1\right)$$

2.1.353 Problem 360

Internal problem ID [9525]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 360
Date solved : `Monday, January 27, 2025 at 06:03:45 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$xy'' + 3y' + 4x^3y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.279 (sec)

Writing the ode as

$$xy'' + 3y' + 4x^3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 3 \\ C &= 4x^3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16x^4 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -16x^4 + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-16x^4 + 3}{4x^2} \right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.668: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -4x^2 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 2ix - \frac{3i}{16x^3} - \frac{9i}{1024x^7} - \frac{27i}{32768x^{11}} - \frac{405i}{4194304x^{15}} - \frac{1701i}{134217728x^{19}} - \frac{15309i}{8589934592x^{23}} - \frac{72171i}{274877906944x^{27}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 2i$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= 2ix \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $([\sqrt{r}]_\infty)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$([\sqrt{r}]_\infty)^2 = -4x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-16x^4 + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (-4x^2) + \left(\frac{3}{4x^2}\right) \\ &= -4x^2 + \frac{3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 2ix \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{0}{2i} - 1\right) = -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{0}{2i} - 1\right) = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-16x^4 + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$2ix$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)\left(2ix\right) \\ &= -\frac{1}{2x} - 2ix \\ &= -\frac{1}{2x} - 2ix \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{2x} - 2ix\right)(0) + \left(\left(\frac{1}{2x^2} - 2i\right) + \left(-\frac{1}{2x} - 2ix\right)^2 - \left(\frac{-16x^4+3}{4x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{2x} - 2ix\right) dx} \\ &= \frac{e^{-ix^2}}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3}{x}\,dx} \\
&= z_1 e^{-\frac{3\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{x^{3/2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{e^{-i x^2}}{x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{i e^{2 i x^2}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{-i x^2}}{x^2}\right) + c_2 \left(\frac{e^{-i x^2}}{x^2}\left(-\frac{i e^{2 i x^2}}{4}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dx^2}y(x)\right) x + 3\frac{d}{dx}y(x) + 4x^3 y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -4x^2 y(x) - \frac{3\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) + \frac{3\left(\frac{d}{dx}y(x)\right)}{x} + 4x^2 y(x) = 0$
- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{3}{x}, P_3(x) = 4x^2\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

$(x \cdot P_2(x))\Big|_{x=0} = 3$

- ∘ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

- ∘ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $\left(\frac{d^2}{dx^2}y(x)\right)x + 3\frac{d}{dx}y(x) + 4x^3y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - ∘ Convert $x^3 \cdot y(x)$ to series expansion

    $x^3 \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+3}$

  - ∘ Shift index using $k- >k-3$

    $x^3 \cdot y(x) = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$

  - ∘ Convert $\frac{d}{dx}y(x)$ to series expansion

    $\frac{d}{dx}y(x) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

  - ∘ Shift index using $k- >k+1$

    $\frac{d}{dx}y(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$

  - ∘ Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-1}$

  - ∘ Shift index using $k- >k+1$

    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(2+r)\, x^{-1+r} + a_1(1+r)(3+r)\, x^r + a_2(2+r)(4+r)\, x^{1+r} + a_3(3+r)(5+r)\, x^{2+r} + \left(\sum_{k=3}^{\infty} (\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 0\}$

- The coefficients of each power of x must be 0

  $[a_1(1+r)(3+r) = 0, a_2(2+r)(4+r) = 0, a_3(3+r)(5+r) = 0]$

- Solve for the dependent coefficient(s)

  $\{a_1 = 0, a_2 = 0, a_3 = 0\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+r+3) + 4a_{k-3} = 0$

- Shift index using $k- >k+3$

  $a_{k+4}(k+4+r)(k+6+r) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+4} = -\frac{4a_k}{(k+4+r)(k+6+r)}$

- Recursion relation for $r = -2$

  $a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}$
- Solution for $r = -2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$
- Recursion relation for $r = 0$

  $a_{k+4} = -\frac{4a_k}{(k+4)(k+6)}$
- Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k}, a_{k+4} = -\frac{4a_k}{(k+4)(k+6)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k}\right), a_{4+k} = -\frac{4a_k}{(k+2)(4+k)}, a_1 = 0, a_2 = 0, a_3 = 0, b_{4+k} = -\frac{4b_k}{(4+k)(k+6)}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 21

```
dsolve(x*diff(diff(y(x),x),x)+3*diff(y(x),x)+4*x^3*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_1 \sin\left(x^2\right) + c_2 \cos\left(x^2\right)}{x^2}$$

Mathematica DSolve solution

Solving time : 0.05 (sec)
Leaf size : 41

```
DSolve[{x*D[y[x],{x,2}]+3*D[y[x],x]+4*x^3*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{4c_1 e^{-ix^2} - ic_2 e^{ix^2}}{4x^2}$$

2.1.354 Problem 361

Internal problem ID [9526]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 361
Date solved : `Monday, January 27, 2025 at 06:03:46 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(-x^2+1\right)y''+2x\left(-x^2+1\right)y'-2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.198 (sec)

Writing the ode as

$$\left(-x^4+x^2\right)y''+\left(-2x^3+2x\right)y'-2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-x^4+x^2\\B&=-2x^3+2x\\C&=-2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-2}{x^2\left(x^2-1\right)} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=-2\\t&=x^2\left(x^2-1\right)\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{2}{x^2\left(x^2-1\right)}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.670: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2(x^2 - 1)$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 1. There is a pole at $x = -1$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 1$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2} - \frac{1}{x - 1} + \frac{1}{x + 1}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= 0\\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{2}{x^2\left(x^2-1\right)}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	1	0	0	1
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right)\\ &= 1 - (0)\\ &= 1\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (-)[\sqrt{r}]_\infty\\ &= \frac{1}{x-1} - \frac{1}{x} + (-)(0)\\ &= \frac{1}{x-1} - \frac{1}{x}\\ &= \frac{1}{x^2-x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x-1} - \frac{1}{x}\right)(1) + \left(\left(-\frac{1}{(x-1)^2} + \frac{1}{x^2}\right) + \left(\frac{1}{x-1} - \frac{1}{x}\right)^2 - \left(-\frac{2}{x^2\left(x^2-1\right)}\right)\right) = 0$$

$$\frac{-2a_0 + 2}{x^3 - x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega \, dx} \\ &= (x+1)\, \mathrm{e}^{\int \left(\frac{1}{x-1} - \frac{1}{x}\right) dx} \\ &= (x+1)\, \mathrm{e}^{\ln(x-1) - \ln(x)} \\ &= \frac{x^2-1}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x^3+2x}{-x^4+x^2}\, dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1 \left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2-1}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^3+2x}{-x^4+x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{1}{4(x+1)} - \frac{\ln(x+1)}{4} - \frac{1}{4(x-1)} + \frac{\ln(x-1)}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{x^2-1}{x^2}\right) + c_2 \left(\frac{x^2-1}{x^2}\left(-\frac{1}{4(x+1)} - \frac{\ln(x+1)}{4} - \frac{1}{4(x-1)} + \frac{\ln(x-1)}{4}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 47

```
dsolve(x^2*(-x^2+1)*diff(diff(y(x),x),x)+2*x*(-x^2+1)*diff(y(x),x)-2*y(x) = 0,y(x),si
```

$$y = \frac{c_2(x^2-1)\ln(x-1)+(-x^2+1)\,c_2\ln(x+1)+2c_1x^2-2c_2x-2c_1}{2x^2}$$

Mathematica DSolve solution

Solving time : 0.311 (sec)
Leaf size : 81

```
DSolve[{x^2*(1-x^2)*D[y[x],{x,2}]+2*x*(1-x^2)*D[y[x],x]-2*y[x]==0,{}},y[x],x,IncludeSingular
```

$$y(x) \to \frac{\exp\left(\int_1^x -\frac{K[1]^2+1}{K[1]-K[1]^3}dK[1]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[2]} -\frac{K[1]^2+1}{K[1]-K[1]^3}dK[1]\right)dK[2]+c_1\right)}{x}$$

2.1.355 Problem 362

Internal problem ID [9527]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 362
Date solved : `Monday, January 27, 2025 at 06:03:46 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2xy'' + (x-2)\,y' - y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.219 (sec)

Writing the ode as

$$2xy'' + (x-2)\,y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x \\ B &= x-2 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2+4x+12}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2+4x+12 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2+4x+12}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.671: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16} + \frac{3}{4x^2} + \frac{1}{4x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{4} + \frac{1}{2x} + \frac{1}{x^2} - \frac{2}{x^3} + \frac{2}{x^4} + \frac{4}{x^5} - \frac{24}{x^6} + \frac{48}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{4}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{16}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 + 4x + 12}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{1}{16}\right) + \left(\frac{4x + 12}{16x^2}\right) \\ &= \frac{1}{16} + \frac{4x + 12}{16x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 16 gives $\frac{1}{4}$. Now b can be found.

$$\begin{aligned}b &= \left(\frac{1}{4}\right) - (0) \\ &= \frac{1}{4}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{4} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{4}}{\frac{1}{4}} - 0\right) &&= \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{4}}{\frac{1}{4}} - 0\right) &&= -\frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 4x + 12}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)\left(\frac{1}{4}\right) \\ &= -\frac{1}{2x} - \frac{1}{4} \\ &= -\frac{x+2}{4x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{2x} - \frac{1}{4}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x} - \frac{1}{4}\right)^2 - \left(\frac{x^2+4x+12}{16x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{2x} - \frac{1}{4}\right) dx} \\ &= \frac{e^{-\frac{x}{4}}}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x-2}{2x}\,dx} \\
&= z_1 e^{-\frac{x}{4}+\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\,\mathrm{e}^{-\frac{x}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x-2}{2x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x}{2}+\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{2(x-2)\,\mathrm{e}^{-\frac{x}{2}+\ln(x)}\mathrm{e}^{x}}{x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x}{2}}\right) + c_2\left(\mathrm{e}^{-\frac{x}{2}}\left(\frac{2(x-2)\,\mathrm{e}^{-\frac{x}{2}+\ln(x)}\mathrm{e}^{x}}{x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2\left(\frac{d^2}{dx^2}y(x)\right)x + (x-2)\left(\frac{d}{dx}y(x)\right) - y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{y(x)}{2x} - \frac{(x-2)\left(\frac{d}{dx}y(x)\right)}{2x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(x-2)\left(\frac{d}{dx}y(x)\right)}{2x} - \frac{y(x)}{2x} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{x-2}{2x}, P_3(x) = -\frac{1}{2x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -1$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2\left(\frac{d^2}{dx^2}y(x)\right)x + (x-2)\left(\frac{d}{dx}y(x)\right) - y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..1$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  - ◦ Shift index using $k- >k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - ◦ Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-1}$

  - ◦ Shift index using $k- >k+1$

    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $2a_0r(-2+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(2a_{k+1}(k+1+r)(k+r-1) + a_k(k+r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $2r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(a_{k+1}(k+1+r) + \frac{a_k}{2}\right)(k+r-1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{2(k+1+r)}$

- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{a_k}{2(k+1)}$

- Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k}{2(k+1)}\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = -\frac{a_k}{2(k+3)}$

- Solution for $r = 2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = -\frac{a_k}{2(k+3)}\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+1} = -\frac{a_k}{2(k+1)}, b_{k+1} = -\frac{b_k}{2(k+3)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 16

```
dsolve(2*x*diff(diff(y(x),x),x)+(x-2)*diff(y(x),x)-y(x) = 0,y(x),singsol=all)
```

$$y = c_1(x-2) + \mathrm{e}^{-\frac{x}{2}} c_2$$

Mathematica DSolve solution

Solving time : 0.291 (sec)
Leaf size : 43

```
DSolve[{2*x*D[y[x],{x,2}]+(x-2)*D[y[x],x]-y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-\frac{x}{2}-\frac{1}{2}} \left(c_2 \int_1^x e^{\frac{K[1]}{2}+1} K[1] dK[1] + c_1 \right)$$

2.1.356 Problem 363

Internal problem ID [9528]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 363
Date solved : Monday, January 27, 2025 at 06:03:47 PM
CAS classification : [_Lienard]

Solve

$$xy'' + 2y' + xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.119 (sec)

Writing the ode as

$$xy'' + 2y' + xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.673: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\, dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1 \left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\cos(x)}{x}\right) + c_2\left(\frac{\cos(x)}{x}(\tan(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dx^2}y(x)\right)x + 2\frac{d}{dx}y(x) + xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -y(x) - \frac{2\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{2\left(\frac{d}{dx}y(x)\right)}{x} + y(x) = 0$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = 1\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $\left(\frac{d^2}{dx^2}y(x)\right)x + 2\frac{d}{dx}y(x) + xy(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x \cdot y(x)$ to series expansion

    $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

  - Shift index using $k- > k - 1$

$x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

- Convert $\frac{d}{dx}y(x)$ to series expansion

  $\frac{d}{dx}y(x) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

- Shift index using $k- >k+1$

  $\frac{d}{dx}y(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$

- Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k- >k+1$

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $a_1(1+r)\,(2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1} = 0$

- Shift index using $k- >k+1$

  $a_{k+2}(k+2+r)\,(k+3+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{(k+1)(k+2)}$

- Solution for $r = -1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+3)}$

- Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0, b_{k+2} = -\frac{b_k}{(k+2)(k+3)}, 2b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 17

```
dsolve(x*diff(diff(y(x),x),x)+2*diff(y(x),x)+x*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{\sin(x) c_1 + \cos(x) c_2}{x}$$

Mathematica DSolve solution

Solving time : 0.036 (sec)
Leaf size : 37

```
DSolve[{x*D[y[x],{x,2}]+2*D[y[x],x]+x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2c_1 e^{-ix} - ic_2 e^{ix}}{2x}$$

2.1.357 Problem 364

Internal problem ID [9529]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 364
Date solved : `Monday, January 27, 2025 at 06:03:47 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + 2x^2 y' + \left(x^4 + 2x - 1\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.094 (sec)

Writing the ode as

$$y'' + 2x^2 y' + \left(x^4 + 2x - 1\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2x^2 \\ C &= x^4 + 2x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.675: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x^2}{1}\, dx} \\ &= z_1 e^{-\frac{x^3}{3}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x^3}{3}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x(x^2+3)}{3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2}{1} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{-\frac{2x^3}{3}}}{(y_1)^2} dx \\ &= y_1 \left(\frac{e^{-\frac{2x^3}{3}} e^{\frac{2x(x^2+3)}{3}}}{2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(e^{-\frac{x(x^2+3)}{3}} \right) + c_2 \left(e^{-\frac{x(x^2+3)}{3}} \left(\frac{e^{-\frac{2x^3}{3}} e^{\frac{2x(x^2+3)}{3}}}{2} \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + 2x^2\left(\frac{d}{dx}y(x)\right) + (x^4 + 2x - 1)\, y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..4$

  $x^m \cdot y(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert $x^2 \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^{k+1}$

  ◦ Shift index using $k- >k - 1$

  $x^2 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=1}^{\infty} a_{k-1}(k-1)\, x^k$

  ◦ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

  $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k + 2$

  $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

$$2a_2 - a_0 + (6a_3 - a_1 + 2a_0)\, x + (12a_4 - a_2 + 4a_1)\, x^2 + (20a_5 - a_3 + 6a_2)\, x^3 + \left(\sum_{k=4}^{\infty} (a_{k+2}(k+2)\right.$$

- The coefficients of each power of x must be 0
  $[2a_2 - a_0 = 0, 6a_3 - a_1 + 2a_0 = 0, 12a_4 - a_2 + 4a_1 = 0, 20a_5 - a_3 + 6a_2 = 0]$
- Solve for the dependent coefficient(s)
  $\left\{a_2 = \frac{a_0}{2}, a_3 = \frac{a_1}{6} - \frac{a_0}{3}, a_4 = \frac{a_0}{24} - \frac{a_1}{3}, a_5 = \frac{a_1}{120} - \frac{a_0}{6}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $(k^2 + 3k + 2)\, a_{k+2} + 2a_{k-1}k - a_k + a_{k-4} = 0$
- Shift index using $k- >k+4$
  $\left((k+4)^2 + 3k + 14\right) a_{k+6} + 2a_{k+3}(k+4) - a_{k+4} + a_k = 0$
- Recursion relation that defines the series solution to the ODE
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+6} = -\frac{2ka_{k+3}+a_k+8a_{k+3}-a_{k+4}}{k^2+11k+30}, a_2 = \frac{a_0}{2}, a_3 = \frac{a_1}{6} - \frac{a_0}{3}, a_4 = \frac{a_0}{24} - \frac{a_1}{3}, a_5 = \frac{a_1}{120} - \right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 27

```
dsolve(diff(diff(y(x),x),x)+2*diff(y(x),x)*x^2+(x^4+2*x-1)*y(x) = 0,y(x),singsol=all)
```

$$y = c_1 e^{-\frac{x(x^2-3)}{3}} + c_2 e^{-\frac{x(x^2+3)}{3}}$$

Mathematica DSolve solution

Solving time : 0.037 (sec)
Leaf size : 34

```
DSolve[{D[y[x],{x,2}]+2*x^2*D[y[x],x]+(x^4+2*x-1)*y[x]==0,{}},y[x],x,IncludeSingularSolution
```

$$y(x) \to \frac{1}{2} e^{-\frac{1}{3}x(x^2+3)} \left(c_2 e^{2x} + 2c_1\right)$$

2.1.358 Problem 365

Internal problem ID [9530]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 365
Date solved : `Monday, January 27, 2025 at 06:03:48 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$u''+\frac{u}{x^2}=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.232 (sec)

Writing the ode as

$$u''+\frac{u}{x^2}=0 \tag{1}$$

$$Au''+Bu'+Cu=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=1\\ B&=0 \\ C&=\frac{1}{x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ue^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-1}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=-1\\ t&=x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{1}{x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.677: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -1$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} - \frac{i\sqrt{3}}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{x^2}$$

Since the gcd$(s,t)=1$. This gives $b=-1$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2}+\frac{i\sqrt{3}}{2}\\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}-\frac{i\sqrt{3}}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=-\frac{1}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}+\frac{i\sqrt{3}}{2}$	$\frac{1}{2}-\frac{i\sqrt{3}}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}+\frac{i\sqrt{3}}{2}$	$\frac{1}{2}-\frac{i\sqrt{3}}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=\frac{1}{2}-\frac{i\sqrt{3}}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right)\\
&= \frac{1}{2}-\frac{i\sqrt{3}}{2}-\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)\\
&= 0
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^-}{x-c_1}\right)+(-)[\sqrt{r}]_\infty\\
&= \frac{\frac{1}{2}-\frac{i\sqrt{3}}{2}}{x}+(-)(0)\\
&= \frac{\frac{1}{2}-\frac{i\sqrt{3}}{2}}{x}\\
&= \frac{1-i\sqrt{3}}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{\frac{1}{2}-\frac{i\sqrt{3}}{2}}{x}\right)(0)+\left(\left(-\frac{\frac{1}{2}-\frac{i\sqrt{3}}{2}}{x^2}\right)+\left(\frac{\frac{1}{2}-\frac{i\sqrt{3}}{2}}{x}\right)^2-\left(-\frac{1}{x^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}z_1(x)&=pe^{\int\omega\,dx}\\&=e^{\int\frac{\frac{1}{2}-\frac{i\sqrt{3}}{2}}{x}dx}\\&=x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}\end{aligned}$$

The first solution to the original ode in u is found from

$$u_1=z_1e^{\int-\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B=0$ then the above reduces to

$$\begin{aligned}u_1&=z_1\\&=x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}\end{aligned}$$

Which simplifies to

$$u_1=x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2=u_1\int\frac{e^{\int-\frac{B}{A}\,dx}}{u_1^2}\,dx$$

Since $B=0$ then the above becomes

$$\begin{aligned}u_2&=u_1\int\frac{1}{u_1^2}\,dx\\&=x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}\int\frac{1}{x^{1-i\sqrt{3}}}\,dx\\&=x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}\left(-\frac{ix\sqrt{3}\,x^{i\sqrt{3}-1}}{3}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}u&=c_1u_1+c_2u_2\\&=c_1\left(x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}\right)+c_2\left(x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}\left(-\frac{ix\sqrt{3}\,x^{i\sqrt{3}-1}}{3}\right)\right)\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}u(x) + \frac{u(x)}{x^2} = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}u(x)$

- Multiply by denominators of the ODE

  $\left(\frac{d^2}{dx^2}u(x)\right)x^2 + u(x) = 0$

- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of u with respect to x , using the chain rule

  $\frac{d}{dx}u(x) = \left(\frac{d}{dt}u(t)\right)\left(\frac{d}{dx}t(x)\right)$

  ◦ Compute derivative

  $\frac{d}{dx}u(x) = \frac{\frac{d}{dt}u(t)}{x}$

  ◦ Calculate the 2nd derivative of u with respect to x , using the chain rule

  $\frac{d^2}{dx^2}u(x) = \left(\frac{d^2}{dt^2}u(t)\right)\left(\frac{d}{dx}t(x)\right)^2 + \left(\frac{d^2}{dx^2}t(x)\right)\left(\frac{d}{dt}u(t)\right)$

  ◦ Compute derivative

  $\frac{d^2}{dx^2}u(x) = \frac{\frac{d^2}{dt^2}u(t)}{x^2} - \frac{\frac{d}{dt}u(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{dt^2}u(t)}{x^2} - \frac{\frac{d}{dt}u(t)}{x^2}\right)x^2 + u(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}u(t) - \frac{d}{dt}u(t) + u(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{1 \pm (\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(\frac{1}{2} - \frac{I\sqrt{3}}{2}, \frac{1}{2} + \frac{I\sqrt{3}}{2}\right)$

- 1st solution of the ODE

  $u_1(t) = e^{\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)$

- 2nd solution of the ODE

  $u_2(t) = e^{\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right)$

- General solution of the ODE

  $u(t) = C1 u_1(t) + C2 u_2(t)$

- Substitute in solutions

  $u(t) = C1\, e^{\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) + C2\, e^{\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right)$

- Change variables back using $t = \ln(x)$

  $u(x) = C1\sqrt{x}\cos\left(\frac{\sqrt{3}\ln(x)}{2}\right) + C2\sqrt{x}\sin\left(\frac{\sqrt{3}\ln(x)}{2}\right)$

- Simplify

  $u(x) = \sqrt{x}\left(C1\cos\left(\frac{\sqrt{3}\ln(x)}{2}\right) + C2\sin\left(\frac{\sqrt{3}\ln(x)}{2}\right)\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 29

```
dsolve(diff(diff(u(x),x),x)+1/x^2*u(x) = 0,u(x),singsol=all)
```

$$u = \sqrt{x}\left(c_1 \sin\left(\frac{\sqrt{3}\,\ln\left(x\right)}{2}\right)+c_2 \cos\left(\frac{\sqrt{3}\,\ln\left(x\right)}{2}\right)\right)$$

Mathematica DSolve solution

Solving time : 0.026 (sec)
Leaf size : 42

```
DSolve[{D[u[x],{x,2}]+1/x^2*u[x]==0,{}},u[x],x,IncludeSingularSolutions->True]
```

$$u(x)\to \sqrt{x}\left(c_1 \cos\left(\frac{1}{2}\sqrt{3}\log(x)\right)+c_2 \sin\left(\frac{1}{2}\sqrt{3}\log(x)\right)\right)$$

2.1.359 Problem 366

Internal problem ID [9531]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 366
Date solved : `Monday, January 27, 2025 at 06:03:48 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$u'' - (2x+1)\,u' + \left(x^2+x-1\right)u = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.076 (sec)

Writing the ode as

$$u'' + (-2x-1)\,u' + \left(x^2+x-1\right)u = 0 \tag{1}$$
$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2x-1 \\ C &= x^2+x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ue^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.679: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in u is found from

$$\begin{aligned} u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x-1}{1}\, dx} \\ &= z_1 e^{\frac{1}{2}x^2+\frac{1}{2}x} \\ &= z_1\left(\mathrm{e}^{\frac{x(x+1)}{2}}\right) \end{aligned}$$

Which simplifies to

$$u_1 = \mathrm{e}^{\frac{x^2}{2}}$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{u_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
u_2 &= u_1 \int \frac{e^{\int -\frac{-2x-1}{1}\,dx}}{(u_1)^2}\,dx \\
&= u_1 \int \frac{e^{x^2+x}}{(u_1)^2}\,dx \\
&= u_1 \left(e^{x^2+x} e^{-x^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1 \left(e^{\frac{x^2}{2}}\right) + c_2 \left(e^{\frac{x^2}{2}} \left(e^{x^2+x} e^{-x^2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}u(x) - (2x+1)\left(\frac{d}{dx}u(x)\right) + (x^2+x-1)\,u(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}u(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}u(x) = (-x^2-x+1)\,u(x) + (2x+1)\left(\frac{d}{dx}u(x)\right)$

- Group terms with $u(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}u(x) + (-2x-1)\left(\frac{d}{dx}u(x)\right) + (x^2+x-1)\,u(x) = 0$

- Assume series solution for $u(x)$

  $u(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot u(x)$ to series expansion for $m = 0..2$

    $x^m \cdot u(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  - Shift index using $k- >k-m$

    $x^m \cdot u(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  - Convert $x^m \cdot \left(\frac{d}{dx}u(x)\right)$ to series expansion for $m = 0..1$

    $x^m \cdot \left(\frac{d}{dx}u(x)\right) = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$

  - Shift index using $k- >k+1-m$

    $x^m \cdot \left(\frac{d}{dx}u(x)\right) = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, x^k$

  - Convert $\frac{d^2}{dx^2}u(x)$ to series expansion

    $\frac{d^2}{dx^2}u(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k+2$

    $\frac{d^2}{dx^2}u(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite ODE with series expansions

$$2a_2 - a_1 - a_0 + (6a_3 - 2a_2 - 3a_1 + a_0)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_{k+1}(k+1) - a_k(2k+1\right.\right.$$

- The coefficients of each power of x must be 0

  $[2a_2 - a_1 - a_0 = 0, 6a_3 - 2a_2 - 3a_1 + a_0 = 0]$
- Solve for the dependent coefficient(s)

  $\left\{a_2 = \frac{a_1}{2} + \frac{a_0}{2}, a_3 = \frac{2a_1}{3}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $k^2 a_{k+2} + (-2a_k - a_{k+1} + 3a_{k+2})\, k - a_k + a_{k-2} + a_{k-1} - a_{k+1} + 2a_{k+2} = 0$
- Shift index using $k- >k+2$

  $(k+2)^2\, a_{k+4} + (-2a_{k+2} - a_{k+3} + 3a_{k+4})\,(k+2) - a_{k+2} + a_k + a_{k+1} - a_{k+3} + 2a_{k+4} = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[u(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{2ka_{k+2} + ka_{k+3} - a_k - a_{k+1} + 5a_{k+2} + 3a_{k+3}}{k^2 + 7k + 12}, a_2 = \frac{a_1}{2} + \frac{a_0}{2}, a_3 = \frac{2a_1}{3}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 22

```
dsolve(diff(diff(u(x),x),x)-(2*x+1)*diff(u(x),x)+(x^2+x-1)*u(x) = 0,u(x),singsol=all)
```

$$u = e^{\frac{x^2}{2}} c_1 + c_2 e^{\frac{x(x+2)}{2}}$$

Mathematica DSolve solution

Solving time : 0.031 (sec)
Leaf size : 24

```
DSolve[{D[u[x],{x,2}]-(2*x+1)*D[u[x],x]+(x^2+x-1)*u[x]==0,{}},u[x],x,IncludeSingularSolution
```

$$u(x) \to e^{\frac{x^2}{2}} (c_2 e^x + c_1)$$

2.1.360 Problem 367

Internal problem ID [9532]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 367
Date solved : `Monday, January 27, 2025 at 06:03:49 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + 2y' + \left(1 + \frac{2}{(1+3x)^2}\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.135 (sec)

Writing the ode as

$$y'' + 2y' + \left(1 + \frac{2}{(1+3x)^2}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 + \frac{2}{(1+3x)^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{(1+3x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -2 \\ t &= (1+3x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{2}{(1+3x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.681: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (1+3x)^2$. There is a pole at $x = -\frac{1}{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1,2,4,6,12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{9\left(x+\frac{1}{3}\right)^2}$$

For the pole at $x = -\frac{1}{3}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{2}{9}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{2}{3} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{3} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{2}{(1+3x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{2}{9}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{2}{3} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{3}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{2}{(1+3x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{3}$	2	0	$\frac{2}{3}$	$\frac{1}{3}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{2}{3}$	$\frac{1}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{3}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{3} - \left(\frac{1}{3}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{1+3x} + (-)(0) \\
&= \frac{1}{1+3x} \\
&= \frac{1}{1+3x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{1+3x}\right)(0) + \left(\left(-\frac{1}{3\left(x+\frac{1}{3}\right)^2}\right) + \left(\frac{1}{1+3x}\right)^2 - \left(-\frac{2}{(1+3x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \frac{1}{1+3x} dx} \\ &= (1+3x)^{1/3} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\, dx} \\ &= z_1 e^{-x} \\ &= z_1 \left(\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}(1+3x)^{1/3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2}\, dx \\ &= y_1 \left((1+3x)^{1/3}\, \mathrm{e}^{-2x}\mathrm{e}^{2x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-x}(1+3x)^{1/3}\right) + c_2\left(\mathrm{e}^{-x}(1+3x)^{1/3}\left((1+3x)^{1/3}\,\mathrm{e}^{-2x}\mathrm{e}^{2x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x)+2\frac{d}{dx}y(x)+\left(1+\frac{2}{(3x+1)^2}\right)y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{3(3x^2+2x+1)y(x)}{(3x+1)^2}-2\frac{d}{dx}y(x)$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x)+2\frac{d}{dx}y(x)+\frac{3(3x^2+2x+1)y(x)}{(3x+1)^2}=0$

□ Check to see if $x_0=-\frac{1}{3}$ is a regular singular point

- ○ Define functions

  $\left[P_2(x)=2, P_3(x)=\frac{3(3x^2+2x+1)}{(3x+1)^2}\right]$

- ○ $\left(x+\frac{1}{3}\right)\cdot P_2(x)$ is analytic at $x=-\frac{1}{3}$

  $\left(\left(x+\frac{1}{3}\right)\cdot P_2(x)\right)\bigg|_{x=-\frac{1}{3}}=0$

- ○ $\left(x+\frac{1}{3}\right)^2\cdot P_3(x)$ is analytic at $x=-\frac{1}{3}$

  $\left(\left(x+\frac{1}{3}\right)^2\cdot P_3(x)\right)\bigg|_{x=-\frac{1}{3}}=\frac{2}{9}$

- ○ $x=-\frac{1}{3}$is a regular singular point

Check to see if $x_0=-\frac{1}{3}$ is a regular singular point

$x_0=-\frac{1}{3}$

- Multiply by denominators

$(3x+1)^2\left(\frac{d^2}{dx^2}y(x)\right)+2(3x+1)^2\left(\frac{d}{dx}y(x)\right)+(9x^2+6x+3)\,y(x)=0$

- Change variables using $x=u-\frac{1}{3}$ so that the regular singular point is at $u=0$

$9u^2\left(\frac{d^2}{du^2}y(u)\right)+18u^2\left(\frac{d}{du}y(u)\right)+(9u^2+2)\,y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot y(u)$ to series expansion for $m=0..2$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

- ○ Shift index using $k->k-m$

  $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

- ○ Convert $u^2\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion

  $u^2\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r+1}$

- ○ Shift index using $k->k-1$

  $u^2\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=1}^{\infty}a_{k-1}(k-1+r)\,u^{k+r}$

- ○ Convert $u^2\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u^2\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k-1+r)\,u^{k+r}$

Rewrite ODE with series expansions

$$a_0(-1+3r)(-2+3r)u^r + (a_1(2+3r)(1+3r)+18a_0r)u^{1+r} + \left(\sum_{k=2}^{\infty}(a_k(3k+3r-1)(3k+3r$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+3r)(-2+3r)=0$
- Values of r that satisfy the indicial equation
  $r \in \left\{\frac{1}{3}, \frac{2}{3}\right\}$
- Each term must be 0
  $a_1(2+3r)(1+3r)+18a_0r=0$
- Solve for the dependent coefficient(s)
  $a_1 = -\frac{18a_0r}{9r^2+9r+2}$
- Each term in the series must be 0, giving the recursion relation
  $9\left(k-\frac{1}{3}+r\right)\left(k+r-\frac{2}{3}\right)a_k + 18a_{k-1}k + 18a_{k-1}r + 9a_{k-2} - 18a_{k-1} = 0$
- Shift index using $k->k+2$
  $9\left(k+\frac{5}{3}+r\right)\left(k+\frac{4}{3}+r\right)a_{k+2} + 18a_{k+1}(k+2) + 18a_{k+1}r + 9a_k - 18a_{k+1} = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{9(2ka_{k+1}+2a_{k+1}r+a_k+2a_{k+1})}{(3k+5+3r)(3k+4+3r)}$
- Recursion relation for $r=\frac{1}{3}$
  $a_{k+2} = -\frac{9\left(2ka_{k+1}+a_k+\frac{8}{3}a_{k+1}\right)}{(3k+6)(3k+5)}$
- Solution for $r=\frac{1}{3}$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{3}}, a_{k+2} = -\frac{9\left(2ka_{k+1}+a_k+\frac{8}{3}a_{k+1}\right)}{(3k+6)(3k+5)}, a_1 = -a_0\right]$
- Revert the change of variables $u = x+\frac{1}{3}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k \left(x+\frac{1}{3}\right)^{k+\frac{1}{3}}, a_{k+2} = -\frac{9\left(2ka_{k+1}+a_k+\frac{8}{3}a_{k+1}\right)}{(3k+6)(3k+5)}, a_1 = -a_0\right]$
- Recursion relation for $r=\frac{2}{3}$
  $a_{k+2} = -\frac{9\left(2ka_{k+1}+a_k+\frac{10}{3}a_{k+1}\right)}{(3k+7)(3k+6)}$
- Solution for $r=\frac{2}{3}$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{2}{3}}, a_{k+2} = -\frac{9\left(2ka_{k+1}+a_k+\frac{10}{3}a_{k+1}\right)}{(3k+7)(3k+6)}, a_1 = -a_0\right]$
- Revert the change of variables $u = x+\frac{1}{3}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k \left(x+\frac{1}{3}\right)^{k+\frac{2}{3}}, a_{k+2} = -\frac{9\left(2ka_{k+1}+a_k+\frac{10}{3}a_{k+1}\right)}{(3k+7)(3k+6)}, a_1 = -a_0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k \left(x+\frac{1}{3}\right)^{k+\frac{1}{3}}\right) + \left(\sum_{k=0}^{\infty} b_k \left(x+\frac{1}{3}\right)^{k+\frac{2}{3}}\right), a_{k+2} = -\frac{9\left(2ka_{k+1}+a_k+\frac{8}{3}a_{k+1}\right)}{(3k+6)(3k+5)}, a_1 = -a_0, b_{k+2}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
```

```
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 27

```
dsolve(diff(diff(y(x),x),x)+2*diff(y(x),x)+(1+2/(3*x+1)^2)*y(x) = 0,y(x),singsol=all)
```

$$y = e^{-x}(3x+1)^{1/3}\left((3x+1)^{1/3} c_2 + c_1\right)$$

Mathematica DSolve solution

Solving time : 0.051 (sec)
Leaf size : 35

```
DSolve[{D[y[x],{x,2}]+2*D[y[x],x]+(1+2/(1+3*x)^2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->
```

$$y(x) \to e^{-x}\sqrt[3]{3x+1}\left(c_2\sqrt[3]{3x+1} + c_1\right)$$

2.1.361 Problem 368

Internal problem ID [9533]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 368
Date solved : `Monday, January 27, 2025 at 06:03:49 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - 2xy' + \left(x^2+2\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.125 (sec)

Writing the ode as

$$x^2y'' - 2xy' + \left(x^2+2\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.683: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2x}{x^2} dx} \\ &= z_1 e^{\ln(x)} \\ &= z_1(x) \end{aligned}$$

Which simplifies to

$$y_1 = x \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\, dx \\ &= y_1(\tan(x))\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1(x\cos(x)) + c_2(x\cos(x)(\tan(x)))\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) - 2x\left(\frac{d}{dx}y(x)\right) + (x^2+2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(x^2+2)y(x)}{x^2} + \frac{2\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) - \frac{2\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(x^2+2)y(x)}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) - 2x\left(\frac{d}{dx}y(x)\right) + (x^2+2)\,y(x) = 0$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k-m$

$x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)\,(-2+r)\, x^r + a_1 r(-1+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)\,(k+r-2) + a_{k-2}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$

- Each term must be 0

  $a_1 r(-1+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)\,(k+r-2) + a_{k-2} = 0$

- Shift index using $k- > k+2$

  $a_{k+2}(k+1+r)\,(k+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+1+r)(k+r)}$

- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$

- Solution for $r = 1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$

- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$

- Solution for $r = 2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+2)(k+3)}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
```

```
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 15

```
dsolve(x^2*diff(diff(y(x),x),x)-2*diff(y(x),x)*x+(x^2+2)*y(x) = 0,y(x),singsol=all)
```

$$y = x(\sin(x) c_1 + \cos(x) c_2)$$

Mathematica DSolve solution

Solving time : 0.028 (sec)
Leaf size : 33

```
DSolve[{x^2*D[y[x],{x,2}]-2*x*D[y[x],x]+(x^2+2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to c_1 e^{-ix} x - \frac{1}{2} i c_2 e^{ix} x$$

2.1.362 Problem 369

Internal problem ID [9534]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 369
Date solved : `Monday, January 27, 2025 at 06:03:50 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + \frac{2y'}{x} - \frac{2y}{(1+x)^2} = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.115 (sec)

Writing the ode as

$$y'' + \frac{2y'}{x} - \frac{2y}{(1+x)^2} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{2}{x} \\ C &= -\frac{2}{(1+x)^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{(1+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= (1+x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{(1+x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.685: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (1+x)^2$. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{(1+x)^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{(1+x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{(1+x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= -1 - (-1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{1+x} + (-)(0) \\
&= -\frac{1}{1+x} \\
&= -\frac{1}{1+x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{1+x}\right)(0) + \left(\left(\frac{1}{(1+x)^2}\right) + \left(-\frac{1}{1+x}\right)^2 - \left(\frac{2}{(1+x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\, dx} \\
&= e^{\int -\frac{1}{1+x} dx} \\
&= \frac{1}{1+x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{\frac{2}{x}}{1}\, dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1\left(\frac{1}{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^2 + x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{\frac{2}{x}}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\, dx \\
&= y_1\left(\frac{(1+x)^3}{3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x^2+x}\right) + c_2\left(\frac{1}{x^2+x}\left(\frac{(1+x)^3}{3}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x)+\frac{2\left(\frac{d}{dx}y(x)\right)}{x}-\frac{2y(x)}{(x+1)^2}=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

□ Check to see if x_0 is a regular singular point

◦ Define functions

$\left[P_2(x)=\frac{2}{x}, P_3(x)=-\frac{2}{(x+1)^2}\right]$

◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

$\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1}=0$

◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

$\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1}=-2$

◦ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1$

- Multiply by denominators

$x(x+1)^2\left(\frac{d^2}{dx^2}y(x)\right)+2(x+1)^2\left(\frac{d}{dx}y(x)\right)-2xy(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(u^3-u^2)\left(\frac{d^2}{du^2}y(u)\right)+2u^2\left(\frac{d}{du}y(u)\right)+(-2u+2)\,y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

◦ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

$u^m\cdot y(u)=\sum_{k=0}^{\infty} a_k u^{k+r+m}$

◦ Shift index using $k->k-m$

$u^m\cdot y(u)=\sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

◦ Convert $u^2\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion

$u^2\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r+1}$

◦ Shift index using $k->k-1$

$u^2\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=1}^{\infty} a_{k-1}(k-1+r)\,u^{k+r}$

◦ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=2..3$

$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\,u^{k+r-2+m}$

◦ Shift index using $k->k+2-m$

$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

Rewrite ODE with series expansions

$-a_0(1+r)\,(-2+r)\,u^r+\left(\sum_{k=1}^{\infty}\left(-a_k(k+r+1)\,(k+r-2)+a_{k-1}(k+r+1)\,(k+r-2)\right)u^{k+r}\right)=$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-(1+r)(-2+r)=0$
- Values of r that satisfy the indicial equation
  $r\in\{-1,2\}$
- Each term in the series must be 0, giving the recursion relation
  $-(k+r+1)(k+r-2)(a_k-a_{k-1})=0$
- Shift index using $k->k+1$
  $-(k+r+2)(k-1+r)(a_{k+1}-a_k)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=a_k$
- Recursion relation for $r=-1$
  $a_{k+1}=a_k$
- Solution for $r=-1$
  $$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k-1}, a_{k+1}=a_k\right]$$
- Revert the change of variables $u=x+1$
  $$\left[y(x)=\sum_{k=0}^{\infty} a_k (x+1)^{k-1}, a_{k+1}=a_k\right]$$
- Recursion relation for $r=2$
  $a_{k+1}=a_k$
- Solution for $r=2$
  $$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1}=a_k\right]$$
- Revert the change of variables $u=x+1$
  $$\left[y(x)=\sum_{k=0}^{\infty} a_k (x+1)^{k+2}, a_{k+1}=a_k\right]$$
- Combine solutions and rename parameters
  $$\left[y(x)=\left(\sum_{k=0}^{\infty} a_k (x+1)^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k (x+1)^{k+2}\right), a_{k+1}=a_k, b_{k+1}=b_k\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 29

```
dsolve(diff(diff(y(x),x),x)+2/x*diff(y(x),x)-2/(x+1)^2*y(x) = 0,y(x),singsol=all)
```

$$y=\frac{(x^3+3x^2+3x)\,c_2+c_1}{x\,(x+1)}$$

Mathematica DSolve solution

Solving time : 0.033 (sec)
Leaf size : 34

```
DSolve[{D[y[x],{x,2}]+2/x*D[y[x],x]-2/(1+x)^2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True
```

$$y(x) \to \frac{c_2 x(x^2 + 3x + 3) + 3c_1}{3x(x+1)}$$

2.1.363 Problem 370

Internal problem ID [9535]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 370
Date solved : `Monday, January 27, 2025 at 06:03:50 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$y'' + \frac{y}{2x^4} = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.238 (sec)

Writing the ode as

$$y'' + \frac{y}{2x^4} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= \frac{1}{2x^4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{2x^4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 2x^4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{2x^4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.687: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 2x^4$. There is a pole at $x = 0$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned} \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) \end{aligned}$$

The partial fraction decomposition of r is

$$r = -\frac{1}{2x^4}$$

There is pole in r at $x = 0$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{i\sqrt{2}}{2x^2} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{i\sqrt{2}}{2x^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^2}$ is

$$a = \frac{i\sqrt{2}}{2}$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be 0. Therefore

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{i\sqrt{2}}{2x^2} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &= \frac{1}{2}\left(\frac{0}{\frac{i\sqrt{2}}{2}} + 2\right) &= 1 \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &= \frac{1}{2}\left(-\frac{0}{\frac{i\sqrt{2}}{2}} + 2\right) &= 1 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{2x^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	4	$\frac{i\sqrt{2}}{2x^2}$	1	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{i\sqrt{2}}{2x^2} + \frac{1}{x} + (-)(0) \\ &= -\frac{i\sqrt{2}}{2x^2} + \frac{1}{x} \\ &= \frac{-i\sqrt{2} + 2x}{2x^2}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{i\sqrt{2}}{2x^2} + \frac{1}{x}\right)(0) + \left(\left(\frac{i\sqrt{2}}{x^3} - \frac{1}{x^2}\right) + \left(-\frac{i\sqrt{2}}{2x^2} + \frac{1}{x}\right)^2 - \left(-\frac{1}{2x^4}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{i\sqrt{2}}{2x^2} + \frac{1}{x}\right) dx} \\ &= x\,\mathrm{e}^{\frac{i\sqrt{2}}{2x}}\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1 \\ &= x\,\mathrm{e}^{\frac{i\sqrt{2}}{2x}}\end{aligned}$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{\frac{i\sqrt{2}}{2x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= x\,\mathrm{e}^{\frac{i\sqrt{2}}{2x}} \int \frac{1}{x^2 \mathrm{e}^{\frac{i\sqrt{2}}{x}}}\,dx \\
&= x\,\mathrm{e}^{\frac{i\sqrt{2}}{2x}} \left(-\frac{i\sqrt{2}\,\mathrm{e}^{-\frac{i\sqrt{2}}{x}}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x\,\mathrm{e}^{\frac{i\sqrt{2}}{2x}}\right) + c_2 \left(x\,\mathrm{e}^{\frac{i\sqrt{2}}{2x}} \left(-\frac{i\sqrt{2}\,\mathrm{e}^{-\frac{i\sqrt{2}}{x}}}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 29

```
dsolve(diff(diff(y(x),x),x)+1/2/x^4*y(x) = 0,y(x),singsol=all)
```

$$
y = x\left(c_1 \sin\left(\frac{\sqrt{2}}{2x}\right) + c_2 \cos\left(\frac{\sqrt{2}}{2x}\right)\right)
$$

Mathematica DSolve solution

Solving time : 0.096 (sec)
Leaf size : 50

```
DSolve[{D[y[x],{x,2}]+1/(2*x^4)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$
y(x) \to c_1 e^{\frac{i}{\sqrt{2}x}} x - \frac{i c_2 e^{-\frac{i}{\sqrt{2}x}} x}{\sqrt{2}}
$$

2.1.364 Problem 371

Internal problem ID [9536]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 371
Date solved : `Monday, January 27, 2025 at 06:03:51 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.229 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.688: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $([\sqrt{r}]_\infty)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$([\sqrt{r}]_\infty)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x - 2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2} + 1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-1-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1-\frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) = 0$$
$$-2 + a_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (2+x)\, e^{\int \left(-1-\frac{x}{2}\right)dx} \\ &= (2+x)\, e^{-x-\frac{1}{4}x^2} \\ &= (2+x)\, e^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1\left(e^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2+x)\,\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\,dx \\ &= y_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) - x\left(\frac{d}{dx}y(x)\right) - xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ○ Convert $x \cdot y(x)$ to series expansion

  $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}$

- ○ Shift index using $k->k-1$

  $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^k$

- ○ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

- ○ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

  $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ○ Shift index using $k->k+2$

$$\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\,x^k$$

Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $2a_2 = 0$
- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)\,a_{k+2} - a_k k - a_{k-1} = 0$
- Shift index using $k- >k+1$

  $\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 51

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x-x*y(x) = 0,y(x),singsol=all)
```

$$y = ic_2 e^{-x-2}\sqrt{2}\,(x+2)\sqrt{\pi}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2c_2 e^{\frac{x(x+2)}{2}} + c_1 e^{-x}(x+2)$$

Mathematica DSolve solution

Solving time : 0.132 (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2 e^{\frac{1}{2}(x+2)^2}\right)$$

2.1.365 Problem 372

Internal problem ID [9537]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 372
Date solved : `Monday, January 27, 2025 at 06:03:52 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.236 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.690: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x - 2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2} + 1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2} + 1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= (2 + x)\,\mathrm{e}^{\int \left(-1-\frac{x}{2}\right)dx} \\ &= (2 + x)\,\mathrm{e}^{-x-\frac{1}{4}x^2} \\ &= (2 + x)\,\mathrm{e}^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1\left(\mathrm{e}^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2+x)\,\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) - x\left(\frac{d}{dx}y(x)\right) - xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite ODE with series expansions
  - Convert $x \cdot y(x)$ to series expansion

    $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k- >k-1$

    $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k+2$

$\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

Rewrite ODE with series expansions

$2a_2 + \left(\sum_{k=1}^{\infty} (a_{k+2}(k+2)(k+1) - a_k k - a_{k-1})\, x^k\right) = 0$

- Each term must be 0

  $2a_2 = 0$
- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)\, a_{k+2} - a_k k - a_{k-1} = 0$
- Shift index using $k- >k+1$

  $\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 51

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x-x*y(x) = 0,y(x),singsol=all)
```

$$y = i c_2 \mathrm{e}^{-x-2} \sqrt{2}\, (x+2) \sqrt{\pi}\, \mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2c_2 \mathrm{e}^{\frac{x(x+2)}{2}} + c_1 \mathrm{e}^{-x}(x+2)$$

Mathematica DSolve solution

Solving time : 0.09 (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2} e^{-x}\left(-\sqrt{2\pi} c_2 \sqrt{(x+2)^2} \mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2} c_1 (x+2) + 2c_2 e^{\frac{1}{2}(x+2)^2}\right)$$

2.1.366 Problem 373

Internal problem ID [9538]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 373
Date solved : `Monday, January 27, 2025 at 06:03:52 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.233 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.692: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $([\sqrt{r}]_\infty)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$([\sqrt{r}]_\infty)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x - 2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2} + 1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2} + 1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (2 + x)\, e^{\int \left(-1 - \frac{x}{2}\right) dx} \\ &= (2 + x)\, e^{-x - \frac{1}{4}x^2} \\ &= (2 + x)\, e^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1 \left(e^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2 + x)\,\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) - x\left(\frac{d}{dx}y(x)\right) - xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

- ☐ Rewrite ODE with series expansions
  - Convert $x \cdot y(x)$ to series expansion

    $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}$
  - Shift index using $k- >k-1$

    $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^k$
  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$
  - Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$
  - Shift index using $k- >k+2$

$\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$

Rewrite ODE with series expansions

$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$

- Each term must be 0

  $2a_2 = 0$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2) a_{k+2} - a_k k - a_{k-1} = 0$
- Shift index using $k- >k+1$

  $\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 51

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x-x*y(x) = 0,y(x),singsol=all)
```

$$y = i c_2 \mathrm{e}^{-x-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2c_2 \mathrm{e}^{\frac{x(x+2)}{2}} + c_1 \mathrm{e}^{-x}(x+2)$$

Mathematica DSolve solution

Solving time : 0.09 (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2 e^{\frac{1}{2}(x+2)^2}\right)$$

2.1.367 Problem 374

Internal problem ID [9539]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 374
Date solved : `Monday, January 27, 2025 at 06:03:53 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.232 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.694: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x - 2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2} + 1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_{\infty}^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_{\infty}^{-} = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_{\infty}^{-} \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_{\infty} \\ &= 0 + (-)\left(\frac{x}{2} + 1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= (2 + x)\,\mathrm{e}^{\int \left(-1-\frac{x}{2}\right)dx} \\ &= (2 + x)\,\mathrm{e}^{-x-\frac{1}{4}x^2} \\ &= (2 + x)\,\mathrm{e}^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1\left(\mathrm{e}^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2+x)\,\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) - x\left(\frac{d}{dx}y(x)\right) - xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y(x)$ to series expansion

    $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ◦ Shift index using $k- >k-1$

    $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ◦ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

$$\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$$

Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $2a_2 = 0$
- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)\, a_{k+2} - a_k k - a_{k-1} = 0$
- Shift index using $k- >k+1$

  $\left((k+1)^2+3k+5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1}+a_k+a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 51

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x-x*y(x) = 0,y(x),singsol=all)
```

$$y = i c_2 \mathrm{e}^{-x-2}\sqrt{2}\,(x+2)\sqrt{\pi}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2c_2 \mathrm{e}^{\frac{x(x+2)}{2}} + c_1 \mathrm{e}^{-x}(x+2)$$

Mathematica DSolve solution

Solving time : 0.112 (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2 e^{\frac{1}{2}(x+2)^2}\right)$$

2.1.368 Problem 375

Internal problem ID [9540]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 375
Date solved : `Monday, January 27, 2025 at 06:03:54 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.220 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.696: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x - 2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2} + 1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1-\frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (2+x)\, e^{\int (-1-\frac{x}{2})dx} \\ &= (2+x)\, e^{-x-\frac{1}{4}x^2} \\ &= (2+x)\, e^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1\left(e^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2+x)\,\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) - x\left(\frac{d}{dx}y(x)\right) - xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

- ☐ Rewrite ODE with series expansions
  - ○ Convert $x \cdot y(x)$ to series expansion

    $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}$
  - ○ Shift index using $k->k-1$

    $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^k$
  - ○ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$
  - ○ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$
  - ○ Shift index using $k->k+2$

$\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

Rewrite ODE with series expansions

$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$

- Each term must be 0

  $2a_2 = 0$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} - a_k k - a_{k-1} = 0$
- Shift index using $k- >k+1$

  $\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 51

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x-x*y(x) = 0,y(x),singsol=all)
```

$$y = i c_2 \mathrm{e}^{-x-2}\sqrt{2}\,(x+2)\sqrt{\pi}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2c_2 \mathrm{e}^{\frac{x(x+2)}{2}} + c_1 \mathrm{e}^{-x}(x+2)$$

Mathematica DSolve solution

Solving time : 0.091 (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2 e^{\frac{1}{2}(x+2)^2}\right)$$

2.1.369 Problem 376

Internal problem ID [9541]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 376
Date solved : `Monday, January 27, 2025 at 06:03:54 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.236 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.698: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x - 2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2} + 1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1-\frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= (2+x)\, e^{\int \left(-1-\frac{x}{2}\right)dx} \\ &= (2+x)\, e^{-x-\frac{1}{4}x^2} \\ &= (2+x)\, e^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1\left(e^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2+x)\,e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\,dx \\ &= y_1 \left(-\frac{e^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,e^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left((2+x)\,e^{-x}\right) + c_2\left((2+x)\,e^{-x}\left(-\frac{e^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,e^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) - x\left(\frac{d}{dx}y(x)\right) - xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y(x)$ to series expansion

    $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ◦ Shift index using $k- >k-1$

    $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ◦ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

$$\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$$

Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_k k - a_{k-1}\right)x^k\right) = 0$$

- Each term must be 0

  $2a_2 = 0$
- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)a_{k+2} - a_k k - a_{k-1} = 0$
- Shift index using $k->k+1$

  $\left((k+1)^2 + 3k + 5\right)a_{k+3} - a_{k+1}(k+1) - a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 51

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x-x*y(x) = 0,y(x),singsol=all)
```

$$y = ic_2 e^{-x-2}\sqrt{2}\,(x+2)\sqrt{\pi}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2c_2 e^{\frac{x(x+2)}{2}} + c_1 e^{-x}(x+2)$$

Mathematica DSolve solution

Solving time : 0.092 (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2 e^{\frac{1}{2}(x+2)^2}\right)$$

2.1.370 Problem 377

Internal problem ID [9542]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 377
Date solved : `Monday, January 27, 2025 at 06:03:55 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.224 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.700: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x - 2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2} + 1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1-\frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (2+x)\, e^{\int \left(-1-\frac{x}{2}\right) dx} \\ &= (2+x)\, e^{-x-\frac{1}{4}x^2} \\ &= (2+x)\, e^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1\left(e^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2+x)\,\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1y_1 + c_2y_2 \\ &= c_1\big((2+x)\,\mathrm{e}^{-x}\big) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) - x\left(\frac{d}{dx}y(x)\right) - xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite ODE with series expansions
  - Convert $x \cdot y(x)$ to series expansion

    $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k->k-1$

    $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k->k+2$

$$\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$$

Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} - a_k k - a_{k-1} = 0$

- Shift index using $k- > k+1$

  $\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 51

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x-x*y(x) = 0,y(x),singsol=all)
```

$$y = ic_2 \mathrm{e}^{-x-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2c_2\mathrm{e}^{\frac{x(x+2)}{2}} + c_1\mathrm{e}^{-x}(x+2)$$

Mathematica DSolve solution

Solving time : 0.092 (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

2.1.371 Problem 378

Internal problem ID [9543]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 378
Date solved : `Monday, January 27, 2025 at 06:03:56 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.227 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.702: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x - 2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2} + 1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-1-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1-\frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) = 0$$
$$-2 + a_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (2+x)\,\mathrm{e}^{\int (-1-\frac{x}{2})dx} \\ &= (2+x)\,\mathrm{e}^{-x-\frac{1}{4}x^2} \\ &= (2+x)\,\mathrm{e}^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1\left(\mathrm{e}^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2+x)\,e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{e^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,e^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,e^{-x}\right) + c_2\left((2+x)\,e^{-x}\left(-\frac{e^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,e^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) - x\left(\frac{d}{dx}y(x)\right) - xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite ODE with series expansions
  - Convert $x \cdot y(x)$ to series expansion

    $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}$
  - Shift index using $k->k-1$

    $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^k$
  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$
  - Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$
  - Shift index using $k->k+2$

$$\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$$

Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} - a_k k - a_{k-1} = 0$

- Shift index using $k- >k+1$

  $\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 51

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x-x*y(x) = 0,y(x),singsol=all)
```

$$y = i c_2 \mathrm{e}^{-x-2}\sqrt{2}\,(x+2)\sqrt{\pi}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2c_2\mathrm{e}^{\frac{x(x+2)}{2}} + c_1\mathrm{e}^{-x}(x+2)$$

Mathematica DSolve solution

Solving time : 0.092 (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

2.1.372 Problem 379

Internal problem ID [9544]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 379
Date solved : `Monday, January 27, 2025 at 06:03:56 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.220 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.704: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x - 2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) = -2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) = 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2} + 1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2} + 1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered} (0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) = 0 \\ -2 + a_0 = 0 \end{gathered}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (2 + x)\, e^{\int \left(-1 - \frac{x}{2}\right) dx} \\ &= (2 + x)\, e^{-x - \frac{1}{4}x^2} \\ &= (2 + x)\, e^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1} \, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1 \left(e^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2+x)\,e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\,dx \\ &= y_1 \left(-\frac{e^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,e^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left((2+x)\,e^{-x}\right) + c_2\left((2+x)\,e^{-x}\left(-\frac{e^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,e^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2}y(x) - x\left(\frac{d}{dx}y(x)\right) - xy(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^k$
- □ Rewrite ODE with series expansions
  - Convert $x \cdot y(x)$ to series expansion
    $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}$
  - Shift index using $k->k-1$
    $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^k$
  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion
    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$
  - Convert $\frac{d^2}{dx^2}y(x)$ to series expansion
    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$
  - Shift index using $k->k+2$

$\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

Rewrite ODE with series expansions

$2a_2 + \left(\sum_{k=1}^{\infty} (a_{k+2}(k+2)(k+1) - a_k k - a_{k-1})\, x^k\right) = 0$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} - a_k k - a_{k-1} = 0$

- Shift index using $k- >k+1$

  $\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)

Leaf size : 51

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x-x*y(x) = 0,y(x),singsol=all)
```

$$y = i c_2 \mathrm{e}^{-x-2} \sqrt{2}\,(x+2)\sqrt{\pi}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2c_2 \mathrm{e}^{\frac{x(x+2)}{2}} + c_1 \mathrm{e}^{-x}(x+2)$$

Mathematica DSolve solution

Solving time : 0.092 (sec)

Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\text{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2 e^{\frac{1}{2}(x+2)^2}\right)$$

2.1.373 Problem 380

Internal problem ID [9545]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 380
Date solved : `Monday, January 27, 2025 at 06:03:57 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.223 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.706: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x - 2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2} + 1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2} + 1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (2 + x)\, e^{\int \left(-1 - \frac{x}{2}\right) dx} \\ &= (2 + x)\, e^{-x - \frac{1}{4}x^2} \\ &= (2 + x)\, e^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1\left(e^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2 + x)\,e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{e^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,e^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,e^{-x}\right) + c_2\left((2+x)\,e^{-x}\left(-\frac{e^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,e^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) - x\left(\frac{d}{dx}y(x)\right) - xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

- ☐ Rewrite ODE with series expansions
  - Convert $x \cdot y(x)$ to series expansion

    $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k->k-1$

    $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k->k+2$

$\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$

Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2) a_{k+2} - a_k k - a_{k-1} = 0$

- Shift index using $k- >k+1$

  $\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 51

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x-x*y(x) = 0,y(x),singsol=all)
```

$$y = i c_2 \mathrm{e}^{-x-2}\sqrt{2}\,(x+2)\sqrt{\pi}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2c_2\mathrm{e}^{\frac{x(x+2)}{2}} + c_1\mathrm{e}^{-x}(x+2)$$

Mathematica DSolve solution

Solving time : 0.092 (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

2.1.374 Problem 381

Internal problem ID [9546]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 381
Date solved : `Monday, January 27, 2025 at 06:03:57 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.233 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.708: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x - 2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2} + 1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2} + 1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (2 + x)\, e^{\int \left(-1 - \frac{x}{2}\right) dx} \\ &= (2 + x)\, e^{-x - \frac{1}{4}x^2} \\ &= (2 + x)\, e^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1} \, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1 \left(e^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2+x)\,e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\,dx \\ &= y_1 \left(-\frac{e^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,e^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left((2+x)\,e^{-x}\right) + c_2\left((2+x)\,e^{-x}\left(-\frac{e^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,e^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) - x\left(\frac{d}{dx}y(x)\right) - xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite ODE with series expansions
  - Convert $x \cdot y(x)$ to series expansion

    $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k->k-1$

    $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k->k+2$

$$\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$$

Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $2a_2 = 0$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2) a_{k+2} - a_k k - a_{k-1} = 0$
- Shift index using $k- >k+1$

  $((k+1)^2 + 3k + 5) a_{k+3} - a_{k+1}(k+1) - a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 51

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x-x*y(x) = 0,y(x),singsol=all)
```

$$y = ic_2 e^{-x-2}\sqrt{2}\,(x+2)\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2c_2 e^{\frac{x(x+2)}{2}} + c_1 e^{-x}(x+2)$$

Mathematica DSolve solution

Solving time : 0.105 (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\operatorname{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2 e^{\frac{1}{2}(x+2)^2}\right)$$

2.1.375 Problem 382

Internal problem ID [9547]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 382
Date solved : `Monday, January 27, 2025 at 06:03:58 PM`
CAS classification : `[_Lienard]`

Solve

$$xy'' + 2y' + xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.121 (sec)

Writing the ode as

$$xy'' + 2y' + xy = 0 \quad (1)$$
$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= x \end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \quad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \quad (6)$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \quad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.710: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\, dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1 \left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\, dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\cos(x)}{x}\right) + c_2\left(\frac{\cos(x)}{x}(\tan(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dx^2}y(x)\right)x + 2\frac{d}{dx}y(x) + xy(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -y(x) - \frac{2\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) + \frac{2\left(\frac{d}{dx}y(x)\right)}{x} + y(x) = 0$
- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{2}{x}, P_3(x) = 1\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $\left(\frac{d^2}{dx^2}y(x)\right)x + 2\frac{d}{dx}y(x) + xy(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x \cdot y(x)$ to series expansion
    $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+1}$
  - Shift index using $k- >k-1$

$x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

- Convert $\frac{d}{dx} y(x)$ to series expansion

  $\frac{d}{dx} y(x) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $\frac{d}{dx} y(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$

- Convert $x \cdot \left(\frac{d^2}{dx^2} y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d^2}{dx^2} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $x \cdot \left(\frac{d^2}{dx^2} y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $a_1(1+r)\,(2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1} = 0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)\,(k+3+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{(k+1)(k+2)}$

- Solution for $r = -1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+3)}$

- Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0, b_{k+2} = -\frac{b_k}{(k+2)(k+3)}, 2b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 17

```
dsolve(x*diff(diff(y(x),x),x)+2*diff(y(x),x)+x*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{\sin(x) c_1 + \cos(x) c_2}{x}$$

Mathematica DSolve solution

Solving time : 0.025 (sec)
Leaf size : 37

```
DSolve[{x*D[y[x],{x,2}]+2*D[y[x],x]+x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2c_1 e^{-ix} - ic_2 e^{ix}}{2x}$$

2.1.376 Problem 383

Internal problem ID [9548]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 383
Date solved : `Monday, January 27, 2025 at 06:03:59 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$2x^2y'' + 3xy' - xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.180 (sec)

Writing the ode as

$$2x^2y'' + 3xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= 3x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{8x - 3}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 8x - 3 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{8x - 3}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.712: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2} + \frac{1}{2x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1,2,3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1, \; e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(1 - (1)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x - c} \\ &= \frac{1}{2}\left(\frac{1}{(x - (0))}\right) \\ &= \frac{1}{2x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2x} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2x} + \frac{1 - 8x}{16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{1 + 2\sqrt{2}\,\sqrt{x}}{4x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \frac{1+2\sqrt{2}\,\sqrt{x}}{4x} dx} \\ &= x^{1/4}\mathrm{e}^{\sqrt{2}\,\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x}{2x^2}\,dx} \\
&= z_1 e^{-\frac{3\ln(x)}{4}} \\
&= z_1 \left(\frac{1}{x^{3/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{\sqrt{2}\,\sqrt{x}}}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{2x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{3\ln(x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{e^{-2\sqrt{2}\,\sqrt{x}}\sqrt{2}}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{\sqrt{2}\,\sqrt{x}}}{\sqrt{x}}\right) + c_2 \left(\frac{e^{\sqrt{2}\,\sqrt{x}}}{\sqrt{x}}\left(-\frac{e^{-2\sqrt{2}\,\sqrt{x}}\sqrt{2}}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2\left(\frac{d^2}{dx^2}y(x)\right) + 3x\left(\frac{d}{dx}y(x)\right) - xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{y(x)}{2x} - \frac{3\left(\frac{d}{dx}y(x)\right)}{2x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{3\left(\frac{d}{dx}y(x)\right)}{2x} - \frac{y(x)}{2x} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{3}{2x}, P_3(x) = -\frac{1}{2x}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{3}{2}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2\left(\frac{d^2}{dx^2}y(x)\right)x + 3\frac{d}{dx}y(x) - y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $\frac{d}{dx}y(x)$ to series expansion

  $\frac{d}{dx}y(x) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

- Shift index using $k- >k+1$

  $\frac{d}{dx}y(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$

- Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k- >k+1$

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(1+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k+3+2r) - a_k\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, -\frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k+r+\frac{3}{2}\right)(k+1+r)\, a_{k+1} - a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{(2k+3+2r)(k+1+r)}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{(2k+3)(k+1)}$

- Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k}{(2k+3)(k+1)}\right]$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+1} = \frac{a_k}{(2k+2)\left(k+\frac{1}{2}\right)}$

- Solution for $r = -\frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = \frac{a_k}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$

- Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty}a_k x^k\right)+\left(\sum_{k=0}^{\infty}b_k x^{k-\frac{1}{2}}\right), a_{k+1}=\frac{a_k}{(2k+3)(k+1)}, b_{k+1}=\frac{b_k}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.040 (sec)
Leaf size : 29

```
dsolve(2*x^2*diff(diff(y(x),x),x)+3*diff(y(x),x)*x-x*y(x) = 0,y(x),singsol=all)
```

$$y=\frac{c_1\sinh\left(\sqrt{x}\,\sqrt{2}\right)+c_2\cosh\left(\sqrt{x}\,\sqrt{2}\right)}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.077 (sec)
Leaf size : 56

```
DSolve[{2*x^2*D[y[x],{x,2}]+3*x*D[y[x],x]-x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to\frac{e^{-\sqrt{2}\sqrt{x}}\left(2c_1e^{2\sqrt{2}\sqrt{x}}-\sqrt{2}c_2\right)}{2\sqrt{x}}$$

2.1.377 Problem 384

Internal problem ID [9549]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 384
Date solved : `Monday, January 27, 2025 at 06:03:59 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + \left(3x^2 + 2x\right) y' - 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.213 (sec)

Writing the ode as

$$x^2y'' + \left(3x^2 + 2x\right) y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 3x^2 + 2x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9x^2 + 12x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9x^2 + 12x + 8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{9x^2 + 12x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.714: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{4} + \frac{3}{x} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3}{2} + \frac{1}{x} + \frac{1}{3x^2} - \frac{2}{9x^3} + \frac{1}{9x^4} - \frac{2}{81x^5} - \frac{2}{81x^6} + \frac{28}{729x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{3}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{9x^2 + 12x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{9}{4}\right) + \left(\frac{12x + 8}{4x^2}\right) \\ &= \frac{9}{4} + \frac{12x + 8}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 12. Dividing this by leading coefficient in t which is 4 gives 3. Now b can be found.

$$\begin{aligned}b &= (3) - (0) \\ &= 3\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{3}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{3}{\frac{3}{2}} - 0\right) &&= 1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{3}{\frac{3}{2}} - 0\right) &&= -1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{9x^2 + 12x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{3}{2}$	1	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + (-)\left(\frac{3}{2}\right) \\ &= -\frac{1}{x} - \frac{3}{2} \\ &= -\frac{1}{x} - \frac{3}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{x} - \frac{3}{2}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - \frac{3}{2}\right)^2 - \left(\frac{9x^2+12x+8}{4x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{x} - \frac{3}{2}\right) dx} \\ &= \frac{e^{-\frac{3x}{2}}}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x^2+2x}{x^2}\,dx} \\
&= z_1 e^{-\frac{3x}{2}-\ln(x)} \\
&= z_1\left(\frac{e^{-\frac{3x}{2}}}{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{e^{-3x}}{x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^2+2x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-3x-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{(9x^2-6x+2)\,x^2 e^{-3x-2\ln(x)} e^{6x}}{27}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{e^{-3x}}{x^2}\right) + c_2\left(\frac{e^{-3x}}{x^2}\left(\frac{(9x^2-6x+2)\,x^2 e^{-3x-2\ln(x)} e^{6x}}{27}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) + (3x^2+2x)\left(\frac{d}{dx}y(x)\right) - 2y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{2y(x)}{x^2} - \frac{(3x+2)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(3x+2)\left(\frac{d}{dx}y(x)\right)}{x} - \frac{2y(x)}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{3x+2}{x}, P_3(x) = -\frac{2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$$(x \cdot P_2(x))\Big|_{x=0} = 2$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$(x^2 \cdot P_3(x))\Big|_{x=0} = -2$$

- $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(3x+2)\left(\frac{d}{dx}y(x)\right) - 2y(x) = 0$$

- Assume series solution for $y(x)$

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

$$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

$$x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(2+r)(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+2)(k+r-1) + 3a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(2+r)(-1+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{-2, 1\}$

- Each term in the series must be 0, giving the recursion relation

$(k+r-1)(a_k(k+r+2) + 3a_{k-1}) = 0$

- Shift index using $k->k+1$

$(k+r)(a_{k+1}(k+3+r) + 3a_k) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = -\frac{3a_k}{k+3+r}$

- Recursion relation for $r = -2$

$a_{k+1} = -\frac{3a_k}{k+1}$

- Solution for $r = -2$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+1} = -\frac{3a_k}{k+1}\right]$$

- Recursion relation for $r = 1$

$a_{k+1} = -\frac{3a_k}{k+4}$

- Solution for $r = 1$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{3a_k}{k+4}\right]$$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+1} = -\frac{3a_k}{k+1}, b_{k+1} = -\frac{3b_k}{4+k}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 27

```
dsolve(x^2*diff(diff(y(x),x),x)+(3*x^2+2*x)*diff(y(x),x)-2*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_1 \mathrm{e}^{-3x} + c_2(9x^2 - 6x + 2)}{x^2}$$

Mathematica DSolve solution

Solving time : 0.856 (sec)
Leaf size : 52

```
DSolve[{x^2*D[y[x],{x,2}]+(2*x+3*x^2)*D[y[x],x]-2*y[x]==0,{}},y[x],x,IncludeSingularSolution
```

$$y(x) \to \frac{e^{-3x}\left(\int_1^x e^{3K[1]} c_1 K[1]^2 dK[1] + c_2\right)}{x^2}$$

$$y(x) \to \frac{c_2 e^{-3x}}{x^2}$$

2.1.378 Problem 385

Internal problem ID [9550]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 385
Date solved : Monday, January 27, 2025 at 06:04:00 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$2x^2\left(x^2+x+1\right)y''+x\left(11x^2+11x+9\right)y'+\left(7x^2+10x+6\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 1.044 (sec)

Writing the ode as

$$\left(2x^4+2x^3+2x^2\right)y''+\left(11x^3+11x^2+9x\right)y'+\left(7x^2+10x+6\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^4+2x^3+2x^2 \\ B &= 11x^3+11x^2+9x \\ C &= 7x^2+10x+6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{21x^4+18x^3+27x^2-2x-3}{16\left(x^3+x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 21x^4+18x^3+27x^2-2x-3 \\ t &= 16\left(x^3+x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{21x^4+18x^3+27x^2-2x-3}{16\left(x^3+x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.716: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^3 + x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ of order 2. There is a pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{-\frac{5}{24} + \frac{i\sqrt{3}}{24}}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{5}{24} - \frac{i\sqrt{3}}{24}}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{1}{8} - \frac{43i\sqrt{3}}{72}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{-\frac{1}{8} + \frac{43i\sqrt{3}}{72}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} - \frac{3}{16x^2} + \frac{1}{4x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{24} + \frac{i\sqrt{3}}{24}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{6+6i\sqrt{3}}}{12}
\end{aligned}
$$

For the pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{24} - \frac{i\sqrt{3}}{24}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{6-6i\sqrt{3}}}{12}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{21x^4 + 18x^3 + 27x^2 - 2x - 3}{16\left(x^3 + x^2 + x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{21}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{21x^4 + 18x^3 + 27x^2 - 2x - 3}{16\left(x^3 + x^2 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
$-\frac{1}{2} + \frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}$	$\frac{1}{2} - \frac{\sqrt{6+6i\sqrt{3}}}{12}$
$-\frac{1}{2} - \frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}$	$\frac{1}{2} - \frac{\sqrt{6-6i\sqrt{3}}}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying

$\alpha_{\infty}^{+} = \frac{7}{4}$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+} + \alpha_{c_3}^{+}\right) \\
&= \frac{7}{4} - \left(\frac{7}{4}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2}\right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{+}}{x - c_3}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{4x} + \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} + (0) \\
&= \frac{1}{4x} + \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} \\
&= \frac{7x^2 + 3x + 1}{4x\left(x^2 + x + 1\right)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{4x} + \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}}\right)(0) + \left(\left(-\frac{1}{4x^2} - \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} - \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2}\right) + \left(
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{4x} + \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}}\right) dx} \\
&= 2\left(x^2 + x + 1\right)^{3/4} \sqrt{2}\, x^{1/4} \mathrm{e}^{-\frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{11x^3 + 11x^2 + 9x}{2x^4 + 2x^3 + 2x^2} \, dx} \\
&= z_1 e^{-\frac{\ln\left(x^2 + x + 1\right)}{4} - \frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6} - \frac{9\ln(x)}{4}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}}{\left(x^2 + x + 1\right)^{1/4} x^{9/4}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{2\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}\sqrt{2}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{11x^3+11x^2+9x}{2x^4+2x^3+2x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{\mathrm{e}^{-\frac{\ln(x^2+x+1)}{2}-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}-\frac{9\ln(x)}{2}}}{(y_1)^2}\,dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{-\frac{\ln(x^2+x+1)}{2}-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}-\frac{9\ln(x)}{2}}\,x^4\,\mathrm{e}^{\frac{2\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{8x^2+8x+8}\,dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{2\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}\sqrt{2}}{x^2}\right) \\ &\quad + c_2\left(\frac{2\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}\sqrt{2}}{x^2}\left(\int \frac{\mathrm{e}^{-\frac{\ln(x^2+x+1)}{2}-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}-\frac{9\ln(x)}{2}}\,x^4\,\mathrm{e}^{\frac{2\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{8x^2+8x+8}\,dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2(x^2+x+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(11x^2+11x+9)\left(\frac{d}{dx}y(x)\right)+(7x^2+10x+6)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(7x^2+10x+6)y(x)}{2x^2(x^2+x+1)} - \frac{(11x^2+11x+9)\left(\frac{d}{dx}y(x)\right)}{2x(x^2+x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(11x^2+11x+9)\left(\frac{d}{dx}y(x)\right)}{2x(x^2+x+1)} + \frac{(7x^2+10x+6)y(x)}{2x^2(x^2+x+1)} = 0$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{11x^2+11x+9}{2x(x^2+x+1)}, P_3(x) = \frac{7x^2+10x+6}{2x^2(x^2+x+1)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x=0$

$(x \cdot P_2(x))\Big|_{x=0} = \frac{9}{2}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 3$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x^2(x^2 + x + 1)\left(\frac{d^2}{dx^2}y(x)\right) + x(11x^2 + 11x + 9)\left(\frac{d}{dx}y(x)\right) + (7x^2 + 10x + 6)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,x^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)(3+2r)\,x^r + (a_1(3+r)(5+2r) + a_0(5+2r)(2+r))\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+2)(2k+\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)(3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-2, -\frac{3}{2}\right\}$

- Each term must be 0

  $a_1(3+r)(5+2r) + a_0(5+2r)(2+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = -\frac{(2+r)a_0}{3+r}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k+r+\frac{3}{2}\right)\left((a_k + a_{k-2} + a_{k-1})\,k + (a_k + a_{k-2} + a_{k-1})\,r + 2a_k - a_{k-2} + a_{k-1}\right) = 0$

- Shift index using $k->k+2$

  $2\left(k+\frac{7}{2}+r\right)\left((a_{k+2} + a_k + a_{k+1})(k+2) + (a_{k+2} + a_k + a_{k+1})\,r + 2a_{k+2} - a_k + a_{k+1}\right) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+2} = -\frac{k a_k + k a_{k+1} + r a_k + r a_{k+1} + a_k + 3 a_{k+1}}{k+4+r}$

- Recursion relation for $r = -2$

$a_{k+2} = -\frac{k a_k + k a_{k+1} - a_k + a_{k+1}}{k+2}$

- Solution for $r = -2$

$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{k a_k + k a_{k+1} - a_k + a_{k+1}}{k+2}, a_1 = 0\right]$

- Recursion relation for $r = -\frac{3}{2}$

$a_{k+2} = -\frac{k a_k + k a_{k+1} - \frac{1}{2} a_k + \frac{3}{2} a_{k+1}}{k+\frac{5}{2}}$

- Solution for $r = -\frac{3}{2}$

$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}, a_{k+2} = -\frac{k a_k + k a_{k+1} - \frac{1}{2} a_k + \frac{3}{2} a_{k+1}}{k+\frac{5}{2}}, a_1 = -\frac{a_0}{3}\right]$

- Combine solutions and rename parameters

$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{3}{2}}\right), a_{k+2} = -\frac{k a_k + k a_{k+1} - a_k + a_{k+1}}{k+2}, a_1 = 0, b_{k+2} = -\frac{k b_k + k b_{k+1} - \frac{1}{2} b_k +}{k+\frac{5}{2}}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Mo
      <- Heun successful: received ODE is equivalent to the  HeunG  ODE, case  a <> 0, e
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.692 (sec)
Leaf size : 231

```
dsolve(2*x^2*(x^2+x+1)*diff(diff(y(x),x),x)+x*(11*x^2+11*x+9)*diff(y(x),x)+(7*x^2+10*x
```

$$y = \frac{\left(2x+i\sqrt{3}+1\right)^{\frac{5\sqrt{3}+3i}{6\sqrt{3}+6i}} \left(-2x+i\sqrt{3}-1\right)^{\frac{64i\sqrt{3}+2368}{\left(\sqrt{3}+i\right)^{3}\left(i-\sqrt{3}\right)^{4}\left(13\sqrt{3}+9i\right)}} e^{-\frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}} \left(\mathrm{HeunG}\left(\frac{\sqrt{3}+i}{i-\sqrt{3}}, 0, 0, \frac{5}{2}, \frac{1}{2}\right.\right.}{x^{5/2}\left(x^{2}+x+1\right)}$$

Mathematica DSolve solution

Solving time : 0.394 (sec)
Leaf size : 135

```
DSolve[{2*x^2*(1+x+x^2)*D[y[x],{x,2}] + x*(9+11*x+11*x^2)*D[y[x],x] + (6+10*x+7*x^2)*y[x] ==
```

$$y(x) \to \exp\left(\int_1^x \frac{K[1](7K[1]+3)+1}{4K[1]\left(K[1]^2+K[1]+1\right)}dK[1] - \frac{1}{2}\int_1^x \left(\frac{K[2]+1}{K[2]^2+K[2]+1} + \frac{9}{2K[2]}\right)dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{7K[1]^2+3K[1]+1}{4K[1]\left(K[1]^2+K[1]+1\right)}dK[1]\right)dK[3] + c_1\right)$$

2.1.379 Problem 388

Internal problem ID [9551]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 388
Date solved : Monday, January 27, 2025 at 06:04:01 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$xy'' + (1+x)\,y' + 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.256 (sec)

Writing the ode as

$$xy'' + (1+x)\,y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 1+x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 6x - 1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 6x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.718: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{4x^2} - \frac{3}{2x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{3}{2x} - \frac{5}{2x^2} - \frac{15}{2x^3} - \frac{115}{4x^4} - \frac{495}{4x^5} - \frac{2285}{4x^6} - \frac{11055}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 6x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-6x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-6x - 1}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -6. Dividing this by leading coefficient in t which is 4 gives $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{2}\right) - (0) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &= -\frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &= \frac{3}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 6x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{3}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\
&= \frac{3}{2} - \left(\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{1}{2x} - \frac{1}{2} \\
&= -\frac{-1+x}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{2}\right)(1) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^2-6x-1}{4x^2}\right) \right) = 0$$

$$\frac{1+a_0}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = -1 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (-1+x)\, e^{\int \left(\frac{1}{2x}-\frac{1}{2}\right) dx} \\ &= (-1+x)\, e^{-\frac{x}{2}+\frac{\ln(x)}{2}} \\ &= (-1+x)\sqrt{x}\, e^{-\frac{x}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1+x}{x} \, dx} \\ &= z_1 e^{-\frac{x}{2}-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{e^{-\frac{x}{2}}}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{-x}(-1+x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1+x}{x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-x-\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{e^{x}}{-1+x} - \operatorname{Ei}_1(-x)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(e^{-x}(-1+x)\right) + c_2\left(e^{-x}(-1+x)\left(-\frac{e^{x}}{-1+x} - \operatorname{Ei}_1(-x)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dx^2}y(x)\right) x + (x+1)\left(\frac{d}{dx}y(x)\right) + 2y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{2y(x)}{x} - \frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x) + \frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{x} + \frac{2y(x)}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{x+1}{x}, P_3(x) = \frac{2}{x}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $\left(\frac{d^2}{dx^2}y(x)\right)x + (x+1)\left(\frac{d}{dx}y(x)\right) + 2y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..1$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-1}$

- Shift index using $k- >k+1$

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r^2 x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)^2 + a_k(k+r+2)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 + a_k(k+2) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+2)}{(k+1)^2}$

- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{a_k(k+2)}{(k+1)^2}$

- Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k(k+2)}{(k+1)^2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 29

```
dsolve(x*diff(diff(y(x),x),x)+(x+1)*diff(y(x),x)+2*y(x) = 0,y(x),singsol=all)
```

$$y = c_2 e^{-x}(x-1)\,\mathrm{Ei}_1(-x) + c_1 e^{-x}(x-1) + c_2$$

Mathematica DSolve solution

Solving time : 0.197 (sec)
Leaf size : 42

```
DSolve[{x*D[y[x],{x,2}] +(1+x)*D[y[x],x]+2*y[x] == 0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-x}(x-1)\left(c_2\int_1^x \frac{e^{K[1]}}{(K[1]-1)^2K[1]}dK[1] + c_1\right)$$

2.1.380 Problem 389

Internal problem ID [9552]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 389
Date solved : `Monday, January 27, 2025 at 06:04:02 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(x^2-2x+1\right)y''-x(3+x)\,y'+(4+x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.300 (sec)

Writing the ode as

$$x^2(x-1)^2\,y''+\left(-x^2-3x\right)y'+(4+x)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=x^2(x-1)^2\\ B&=-x^2-3x\\ C&=4+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{7x^2+10x-1}{4x^2\left(x-1\right)^4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=7x^2+10x-1\\ t&=4x^2(x-1)^4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{7x^2+10x-1}{4x^2\left(x-1\right)^4}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.720: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2(x-1)^4$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{2\,(x-1)} + \frac{3}{2x} - \frac{1}{4x^2} - \frac{2}{(x-1)^3} + \frac{7}{4\,(x-1)^2} + \frac{4}{(x-1)^4}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\alpha_c^+ = \frac{1}{2}\left(\frac{b}{a}+v\right)$$
$$\alpha_c^- = \frac{1}{2}\left(-\frac{b}{a}+v\right)$$

The partial fraction decomposition of r is

$$r = -\frac{3}{2(x-1)} + \frac{3}{2x} - \frac{1}{4x^2} - \frac{2}{(x-1)^3} + \frac{7}{4(x-1)^2} + \frac{4}{(x-1)^4}$$

There is pole in r at $x = 1$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 1$ gives

$$[\sqrt{r}]_c \approx \frac{2}{(x-1)^2} - \frac{1}{2(x-1)} + \frac{21}{32} - \frac{9x}{32} + \frac{53(x-1)^2}{256} - \frac{149(x-1)^3}{1024} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{2}{(x-1)^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-1)^2}$ is

$$a = 2$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 1$. This term becomes $\frac{1}{(x-1)^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be -2. Therefore

$$\begin{aligned} b &= (-2) - (0) \\ &= -2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{2}{(x-1)^2} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a}+v\right) = \frac{1}{2}\left(\frac{-2}{2}+2\right) = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a}+v\right) = \frac{1}{2}\left(-\frac{-2}{2}+2\right) = \frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{7x^2+10x-1}{4x^2(x-1)^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
1	4	$\frac{2}{(x-1)^2}$	$\frac{1}{2}$	$\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2} + (-)\,(0) \\ &= \frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2} \\ &= \frac{2x^2+x+1}{2x\,(x-1)^2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{4}{(x-1)^3} - \frac{1}{2\,(x-1)^2}\right) + \left(\frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2}\right)\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2}\right) dx} \\ &= \sqrt{x-1}\,\sqrt{x}\,\mathrm{e}^{-\frac{2}{x-1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2-3x}{x^2(x-1)^2}\,dx} \\
&= z_1 e^{-\frac{2}{x-1}-\frac{3\ln(x-1)}{2}+\frac{3\ln(x)}{2}} \\
&= z_1 \left(\frac{x^{3/2}\mathrm{e}^{-\frac{2}{x-1}}}{(x-1)^{3/2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^{3/2}\mathrm{e}^{-\frac{4}{x-1}}\sqrt{x\,(x-1)}}{(x-1)^{3/2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-3x}{x^2(x-1)^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{4}{x-1}-3\ln(x-1)+3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\mathrm{e}^{-4}\,\mathrm{Ei}_1\left(-\frac{4}{x-1}-4\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^{3/2}\mathrm{e}^{-\frac{4}{x-1}}\sqrt{x\,(x-1)}}{(x-1)^{3/2}}\right) + c_2\left(\frac{x^{3/2}\mathrm{e}^{-\frac{4}{x-1}}\sqrt{x\,(x-1)}}{(x-1)^{3/2}}\left(\mathrm{e}^{-4}\,\mathrm{Ei}_1\left(-\frac{4}{x-1}-4\right)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2(x^2-2x+1)\left(\frac{d^2}{dx^2}y(x)\right) - x(x+3)\left(\frac{d}{dx}y(x)\right) + (x+4)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{(x+4)y(x)}{x^2(x^2-2x+1)} + \frac{(x+3)\left(\frac{d}{dx}y(x)\right)}{x(x^2-2x+1)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x) - \frac{(x+3)\left(\frac{d}{dx}y(x)\right)}{x(x^2-2x+1)} + \frac{(x+4)y(x)}{x^2(x^2-2x+1)} = 0$

□ Check to see if x_0 is a regular singular point
  - Define functions

$\left[P_2(x) = -\frac{x+3}{x(x^2-2x+1)}, P_3(x) = \frac{x+4}{x^2(x^2-2x+1)}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = -3$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 4$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2-2x+1)\left(\frac{d^2}{dx^2}y(x)\right) - x(x+3)\left(\frac{d}{dx}y(x)\right) + (x+4)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- Rewrite ODE with series expansions

  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k- >k + 1 - m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - Shift index using $k- >k + 2 - m$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)^2 x^r + \left(a_1(-1+r)^2 - a_0(1+2r)\,(-1+r)\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-2)^2 - a_{k-1}(2k-1+\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 2$

- Each term must be 0

  $a_1(-1+r)^2 - a_0(1+2r)\,(-1+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = \frac{a_0(1+2r)}{-1+r}$

- Each term in the series must be 0, giving the recursion relation

  $((a_k + a_{k-2} - 2a_{k-1})\,k + (a_k + a_{k-2} - 2a_{k-1})\,r - 2a_k - 3a_{k-2} + a_{k-1})\,(k+r-2) = 0$

- Shift index using $k- >k + 2$

$$((a_{k+2}+a_k-2a_{k+1})(k+2)+(a_{k+2}+a_k-2a_{k+1})r-2a_{k+2}-3a_k+a_{k+1})(k+r)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{ka_k-2ka_{k+1}+ra_k-2ra_{k+1}-a_k-3a_{k+1}}{k+r}$$

- Recursion relation for $r=2$

$$a_{k+2}=-\frac{ka_k-2ka_{k+1}+a_k-7a_{k+1}}{k+2}$$

- Solution for $r=2$

$$\left[y(x)=\sum_{k=0}^{\infty}a_kx^{k+2},a_{k+2}=-\frac{ka_k-2ka_{k+1}+a_k-7a_{k+1}}{k+2},a_1=5a_0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 45

```
dsolve(x^2*(x^2-2*x+1)*diff(diff(y(x),x),x)-x*(x+3)*diff(y(x),x)+(x+4)*y(x) = 0,y(x),s
```

$$y=\frac{x^2\left(\mathrm{Ei}_1\left(-\frac{4x}{x-1}\right)\mathrm{e}^{-\frac{4x}{x-1}}c_2+\mathrm{e}^{-\frac{4}{x-1}}c_1\right)}{x-1}$$

Mathematica DSolve solution

Solving time : 0.273 (sec)
Leaf size : 116

```
DSolve[{x^2*(1-2*x+x^2)*D[y[x],{x,2}] -x*(3+x)*D[y[x],x]+(4+x)*y[x] == 0,{}},y[x],x,IncludeS
```

$$y(x)\to\exp\left(\int_1^x\frac{2K[1]^2+K[1]+1}{2(K[1]-1)^2K[1]}dK[1]-\frac{1}{2}\int_1^x-\frac{K[2]+3}{(K[2]-1)^2K[2]}dK[2]\right)\left(c_2\int_1^x\exp\left(-2\int_1^{K[3]}\frac{2K[1]^2+K[1]+1}{2(K[1]-1)^2K[1]}dK[1]\right)dK[3]+c_1\right)$$

2.1.381 Problem 390

Internal problem ID [9553]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 390
Date solved : `Monday, January 27, 2025 at 06:04:03 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(2+x)\,y''+5x^2y'+(1+x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.194 (sec)

Writing the ode as

$$\left(2x^3+4x^2\right)y''+5x^2y'+(1+x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x^3+4x^2\\B&=5x^2\\C&=1+x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-3x^2-24x-16}{16\left(x^2+2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=-3x^2-24x-16\\t&=16\left(x^2+2x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-3x^2-24x-16}{16\left(x^2+2x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.722: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2+2x)^2$. There is a pole at $x=0$ of order 2. There is a pole at $x=-2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16\,(2+x)^2} - \frac{1}{8x} + \frac{1}{16+8x} - \frac{1}{4x^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(2+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 - 24x - 16}{16\left(x^2 + 2x\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x^2 - 24x - 16}{16\left(x^2 + 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= \frac{1}{4} - \left(\frac{1}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{4(2 + x)} + \frac{1}{2x} + (-)(0) \\
&= -\frac{1}{4(2 + x)} + \frac{1}{2x} \\
&= \frac{x + 4}{4x(2 + x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{4(2+x)} + \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{4(2+x)^2} - \frac{1}{2x^2}\right) + \left(-\frac{1}{4(2+x)} + \frac{1}{2x}\right)^2 - \left(\frac{-3x^2 - 24x - 16}{16(x^2+2x)^2}\right)\right.$$
$$0 =$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{1}{4(2+x)} + \frac{1}{2x}\right) dx} \\
&= \frac{\sqrt{x}}{(2+x)^{1/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{5x^2}{2x^3+4x^2} \, dx} \\
&= z_1 e^{-\frac{5\ln(2+x)}{4}} \\
&= z_1 \left(\frac{1}{(2+x)^{5/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{(2+x)^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5x^2}{2x^3+4x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{5\ln(2+x)}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(2\sqrt{2+x} - 2\sqrt{2}\, \operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x}}{(2+x)^{3/2}}\right) + c_2 \left(\frac{\sqrt{x}}{(2+x)^{3/2}} \left(2\sqrt{2+x} - 2\sqrt{2}\, \operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right)\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$2x^2(x+2)\left(\frac{d^2}{dx^2}y(x)\right)+5x^2\left(\frac{d}{dx}y(x)\right)+(x+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
$\frac{d^2}{dx^2}y(x)=-\frac{(x+1)y(x)}{2(x+2)x^2}-\frac{5\left(\frac{d}{dx}y(x)\right)}{2(x+2)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
$\frac{d^2}{dx^2}y(x)+\frac{5\left(\frac{d}{dx}y(x)\right)}{2(x+2)}+\frac{(x+1)y(x)}{2(x+2)x^2}=0$

☐ Check to see if x_0 is a regular singular point
  - Define functions
  $\left[P_2(x)=\frac{5}{2(x+2)},P_3(x)=\frac{x+1}{2(x+2)x^2}\right]$
  - $(x+2)\cdot P_2(x)$ is analytic at $x=-2$
  $\left.((x+2)\cdot P_2(x))\right|_{x=-2}=\frac{5}{2}$
  - $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$
  $\left.((x+2)^2\cdot P_3(x))\right|_{x=-2}=0$
  - $x=-2$is a regular singular point
  Check to see if x_0 is a regular singular point
  $x_0=-2$
- Multiply by denominators
$2x^2(x+2)\left(\frac{d^2}{dx^2}y(x)\right)+5x^2\left(\frac{d}{dx}y(x)\right)+(x+1)\,y(x)=0$
- Change variables using $x=u-2$ so that the regular singular point is at $u=0$
$(2u^3-8u^2+8u)\left(\frac{d^2}{du^2}y(u)\right)+(5u^2-20u+20)\left(\frac{d}{du}y(u)\right)+(u-1)\,y(u)=0$
- Assume series solution for $y(u)$
$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions
  - Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$
  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$
  - Shift index using $k->k-m$
  $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$
  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$
  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$
  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$
  - Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$4a_0r(3+2r)\,u^{-1+r} + \left(4a_1(1+r)\,(5+2r) - a_0(8r^2+12r+1)\right)u^r + \left(\sum_{k=1}^{\infty} \left(4a_{k+1}(k+r+1)\,(2k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $4r(3+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{0, -\frac{3}{2}\right\}$
- Each term must be 0
  $4a_1(1+r)\,(5+2r) - a_0(8r^2+12r+1) = 0$
- Each term in the series must be 0, giving the recursion relation
  $2(-4a_k + a_{k-1} + 4a_{k+1})\,k^2 + (4(-4a_k + a_{k-1} + 4a_{k+1})\,r - 12a_k - a_{k-1} + 28a_{k+1})\,k + 2(-4a_k +$
- Shift index using $k- >k+1$
  $2(-4a_{k+1} + a_k + 4a_{k+2})\,(k+1)^2 + (4(-4a_{k+1} + a_k + 4a_{k+2})\,r - 12a_{k+1} - a_k + 28a_{k+2})\,(k+1)$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 4kra_k - 16kra_{k+1} + 2r^2a_k - 8r^2a_{k+1} + 3ka_k - 28ka_{k+1} + 3ra_k - 28ra_{k+1} + a_k - 21a_{k+1}}{4(2k^2+4kr+2r^2+11k+11r+14)}$
- Recursion relation for $r = 0$
  $a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 3ka_k - 28ka_{k+1} + a_k - 21a_{k+1}}{4(2k^2+11k+14)}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 3ka_k - 28ka_{k+1} + a_k - 21a_{k+1}}{4(2k^2+11k+14)}, 20a_1 - a_0 = 0\right]$
- Revert the change of variables $u = x+2$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+2)^k, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 3ka_k - 28ka_{k+1} + a_k - 21a_{k+1}}{4(2k^2+11k+14)}, 20a_1 - a_0 = 0\right]$
- Recursion relation for $r = -\frac{3}{2}$
  $a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} - 3ka_k - 4ka_{k+1} + a_k + 3a_{k+1}}{4(2k^2+5k+2)}$
- Solution for $r = -\frac{3}{2}$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{3}{2}}, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} - 3ka_k - 4ka_{k+1} + a_k + 3a_{k+1}}{4(2k^2+5k+2)}, -4a_1 - a_0 = 0\right]$
- Revert the change of variables $u = x+2$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+2)^{k-\frac{3}{2}}, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} - 3ka_k - 4ka_{k+1} + a_k + 3a_{k+1}}{4(2k^2+5k+2)}, -4a_1 - a_0 = 0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k(x+2)^k\right) + \left(\sum_{k=0}^{\infty} b_k(x+2)^{k-\frac{3}{2}}\right), a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 3ka_k - 28ka_{k+1} + a_k - 21a_{k+1}}{4(2k^2+11k+14)}, 2\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
```

```
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.041 (sec)
Leaf size : 39

```
dsolve(2*x^2*(x+2)*diff(diff(y(x),x),x)+5*diff(y(x),x)*x^2+(x+1)*y(x) = 0,y(x),singsol=a
```

$$y = \frac{\sqrt{x}\left(\sqrt{x+2}\,\sqrt{2}\,c_2 - 2\,\mathrm{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)c_2 + c_1\right)}{(x+2)^{3/2}}$$

Mathematica DSolve solution

Solving time : 0.378 (sec)
Leaf size : 83

```
DSolve[{2*x^2*(2+x)*D[y[x],{x,2}] +5*x^2*D[y[x],x]+(1+x)*y[x] == 0,{}},y[x],x,IncludeSingularS
```

$$y(x) \to \frac{\exp\left(\int_1^x \frac{K[1]+4}{4K[1]^2+8K[1]}dK[1]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[2]} \frac{K[1]+4}{4K[1]^2+8K[1]}dK[1]\right)dK[2] + c_1\right)}{(x+2)^{5/4}}$$

2.1.382 Problem 391

Internal problem ID [9554]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 391
Date solved : `Monday, January 27, 2025 at 06:04:03 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + 4xy' + \left(x^2 + 2\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.127 (sec)

Writing the ode as

$$x^2y'' + 4xy' + \left(x^2 + 2\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 4x \\ C &= x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.724: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x}{x^2}\, dx} \\ &= z_1 e^{-2\ln(x)} \\ &= z_1 \left(\frac{1}{x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-4\ln(x)}}{(y_1)^2}\, dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\cos(x)}{x^2}\right) + c_2 \left(\frac{\cos(x)}{x^2}(\tan(x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x\left(\frac{d}{dx}y(x)\right) + (x^2+2)\, y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(x^2+2)y(x)}{x^2} - \frac{4\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{4\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(x^2+2)y(x)}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{4}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 4$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x\left(\frac{d}{dx}y(x)\right) + (x^2+2)\, y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$
- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)\,(1+r)\, x^r + a_1(3+r)\,(2+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+2)\,(k+r+1) + a_{k-2}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-2, -1\}$
- Each term must be 0

  $a_1(3+r)\,(2+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)\,(k+r+1) + a_{k-2} = 0$
- Shift index using $k->k+2$

  $a_{k+2}(k+4+r)\,(k+3+r) + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+4+r)(k+3+r)}$
- Recursion relation for $r = -2$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$
- Solution for $r = -2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$
- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$
- Solution for $r = -1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+2)(k+3)}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+4*diff(y(x),x)*x+(x^2+2)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{\sin(x) c_1 + \cos(x) c_2}{x^2}$$

Mathematica DSolve solution

Solving time : 0.039 (sec)
Leaf size : 37

```
DSolve[{x^2*D[y[x],{x,2}]+4*x*D[y[x],x]+(x^2+2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to \frac{2c_1 e^{-ix} - ic_2 e^{ix}}{2x^2}$$

2.1.383 Problem 392

Internal problem ID [9555]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 392
Date solved : `Monday, January 27, 2025 at 06:04:04 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.135 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.726: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2\left(\frac{\cos(x)}{\sqrt{x}}(\tan(x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x\left(\frac{d}{dx}y(x)\right) + \left(x^2 - \frac{1}{4}\right)y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{(4x^2-1)y(x)}{4x^2} - \frac{\frac{d}{dx}y(x)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) + \frac{\frac{d}{dx}y(x)}{x} + \frac{(4x^2-1)y(x)}{4x^2} = 0$
- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x\left(\frac{d}{dx}y(x)\right) + (4x^2 - 1)y(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- □ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- > k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$
- Shift index using $k- > k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$
- Solution for $r = -\frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$
- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.036 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x+(x^2-1/4)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{\sin(x) c_1 + \cos(x) c_2}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.033 (sec)
Leaf size : 39

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-1/4)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to \frac{e^{-ix}(2c_1 - ic_2e^{2ix})}{2\sqrt{x}}$$

2.1.384 Problem 394

Internal problem ID [9556]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 394
Date solved : `Monday, January 27, 2025 at 06:04:04 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - xy' - \left(x^2 + \frac{5}{4}\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.234 (sec)

Writing the ode as

$$x^2y'' - xy' + \left(-x^2 - \frac{5}{4}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x \\ C &= -x^2 - \frac{5}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2+2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.728: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{1}{x^2} - \frac{1}{2x^4} + \frac{1}{2x^6} - \frac{5}{8x^8} + \frac{7}{8x^{10}} - \frac{21}{16x^{12}} + \frac{33}{16x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2+2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (1) + \left(\frac{2}{x^2}\right) \\ &= 1 + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{0}{1} - 0\right) = 0 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{0}{1} - 0\right) = 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2+2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + (-)(1) \\ &= -\frac{1}{x} - 1 \\ &= -\frac{1+x}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x} - 1\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - 1\right)^2 - \left(\frac{x^2+2}{x^2}\right) \right) = 0$$

$$\frac{-2+2a_0}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 1 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega \, dx} \\ &= (1+x)\, e^{\int (-\frac{1}{x}-1) dx} \\ &= (1+x)\, e^{-x-\ln(x)} \\ &= \frac{(1+x)\, e^{-x}}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{x^2} \, dx} \\ &= z_1 e^{\frac{\ln(x)}{2}} \\ &= z_1\left(\sqrt{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(1+x)\, e^{-x}}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{(-1+x)\, e^{2x}}{2+2x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{(1+x)\, e^{-x}}{\sqrt{x}}\right) + c_2 \left(\frac{(1+x)\, e^{-x}}{\sqrt{x}} \left(\frac{(-1+x)\, e^{2x}}{2+2x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) - x\left(\frac{d}{dx}y(x)\right) - \left(x^2+\frac{5}{4}\right)y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{(4x^2+5)y(x)}{4x^2} + \frac{\frac{d}{dx}y(x)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) - \frac{\frac{d}{dx}y(x)}{x} - \frac{(4x^2+5)y(x)}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = -\frac{1}{x}, P_3(x) = -\frac{4x^2+5}{4x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -1$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{5}{4}$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) - 4x\left(\frac{d}{dx}y(x)\right) + (-4x^2-5)y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ○ Shift index using $k- >k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ○ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r}$

- ○ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)(-5+2r)\,x^r + a_1(3+2r)(-3+2r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)(2k+2r-5) - 4a_{k-2}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)(-5+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{5}{2}\right\}$

- Each term must be 0
  $a_1(3+2r)(-3+2r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $4\left(k+r+\frac{1}{2}\right)\left(k-\frac{5}{2}+r\right)a_k - 4a_{k-2} = 0$
- Shift index using $k- >k+2$
  $4\left(k+\frac{5}{2}+r\right)\left(k-\frac{1}{2}+r\right)a_{k+2} - 4a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{4a_k}{(2k+5+2r)(2k-1+2r)}$
- Recursion relation for $r = -\frac{1}{2}$
  $a_{k+2} = \frac{4a_k}{(2k+4)(2k-2)}$
- Solution for $r = -\frac{1}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = \frac{4a_k}{(2k+4)(2k-2)}, a_1 = 0\right]$
- Recursion relation for $r = \frac{5}{2}$
  $a_{k+2} = \frac{4a_k}{(2k+10)(2k+4)}$
- Solution for $r = \frac{5}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{2}}, a_{k+2} = \frac{4a_k}{(2k+10)(2k+4)}, a_1 = 0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{5}{2}}\right), a_{k+2} = \frac{4a_k}{(2k+4)(2k-2)}, a_1 = 0, b_{k+2} = \frac{4b_k}{(2k+10)(2k+4)}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.040 (sec)
Leaf size : 25

```
dsolve(x^2*diff(diff(y(x),x),x)-diff(y(x),x)*x-(x^2+5/4)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{(x+1)\, c_2 \mathrm{e}^{-x} + c_1 \mathrm{e}^{x}(x-1)}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.063 (sec)
Leaf size : 53

```
DSolve[{x^2*D[y[x],{x,2}]-x*D[y[x],x]-(x^2+5/4)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tr
```

$$y(x) \to \frac{\sqrt{\frac{2}{\pi}}((ic_2x + c_1)\sinh(x) - (c_1x + ic_2)\cosh(x))}{\sqrt{-ix}}$$

2.1.385 Problem 395

Internal problem ID [9557]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 395
Date solved : `Monday, January 27, 2025 at 06:04:05 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.142 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.730: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\, dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2\left(\frac{\cos(x)}{\sqrt{x}}(\tan(x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x\left(\frac{d}{dx}y(x)\right) + \left(x^2 - \frac{1}{4}\right)y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(4x^2-1)y(x)}{4x^2} - \frac{\frac{d}{dx}y(x)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{\frac{d}{dx}y(x)}{x} + \frac{(4x^2-1)y(x)}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$
- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$
- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\frac{1}{4}$
- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x\left(\frac{d}{dx}y(x)\right) + (4x^2 - 1)y(x) = 0$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-2})\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$

- Shift index using $k- >k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$

- Solution for $r = -\frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$

- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.037 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x+(x^2-1/4)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{\sin(x)\, c_1 + \cos(x)\, c_2}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.028 (sec)
Leaf size : 39

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-1/4)*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to \frac{e^{-ix}(2c_1 - ic_2e^{2ix})}{2\sqrt{x}}$$

2.1.386 Problem 396

Internal problem ID [9558]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 396
Date solved : Monday, January 27, 2025 at 06:04:05 PM
CAS classification : [[_Emden, _Fowler]]

Solve

$$x^2y'' + 3xy' + 4x^4y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.265 (sec)

Writing the ode as

$$x^2y'' + 3xy' + 4x^4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 3x \\ C &= 4x^4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16x^4 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -16x^4 + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-16x^4 + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.732: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -4x^2 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 2ix - \frac{3i}{16x^3} - \frac{9i}{1024x^7} - \frac{27i}{32768x^{11}} - \frac{405i}{4194304x^{15}} - \frac{1701i}{134217728x^{19}} - \frac{15309i}{8589934592x^{23}} - \frac{72171i}{274877906944x^{27}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 2i$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= 2ix \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -4x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-16x^4 + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (-4x^2) + \left(\frac{3}{4x^2}\right) \\ &= -4x^2 + \frac{3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 2ix \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{0}{2i} - 1\right) = -\frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{0}{2i} - 1\right) = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-16x^4 + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$2ix$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)(2ix) \\ &= -\frac{1}{2x} - 2ix \\ &= -\frac{1}{2x} - 2ix \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{2x} - 2ix\right)(0) + \left(\left(\frac{1}{2x^2} - 2i\right) + \left(-\frac{1}{2x} - 2ix\right)^2 - \left(\frac{-16x^4+3}{4x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{2x} - 2ix\right) dx} \\ &= \frac{e^{-ix^2}}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{3x}{x^2}\,dx}\\ &= z_1 e^{-\frac{3\ln(x)}{2}}\\ &= z_1\left(\frac{1}{x^{3/2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-ix^2}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{x^2}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{-3\ln(x)}}{(y_1)^2}\,dx\\ &= y_1\left(-\frac{i\mathrm{e}^{2ix^2}}{4}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2\\ &= c_1\left(\frac{\mathrm{e}^{-ix^2}}{x^2}\right) + c_2\left(\frac{\mathrm{e}^{-ix^2}}{x^2}\left(-\frac{i\mathrm{e}^{2ix^2}}{4}\right)\right)\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) + 3x\left(\frac{d}{dx}y(x)\right) + 4x^4 y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -4x^2 y(x) - \frac{3\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{3\left(\frac{d}{dx}y(x)\right)}{x} + 4x^2 y(x) = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{3}{x}, P_3(x) = 4x^2\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$(x \cdot P_2(x)) \Big|_{x=0} = 3$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x)) \Big|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $\left(\frac{d^2}{dx^2}y(x)\right) x + 3\frac{d}{dx}y(x) + 4x^3y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^3 \cdot y(x)$ to series expansion

  $x^3 \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+3}$

- Shift index using $k- >k-3$

  $x^3 \cdot y(x) = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$

- Convert $\frac{d}{dx}y(x)$ to series expansion

  $\frac{d}{dx}y(x) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

- Shift index using $k- >k+1$

  $\frac{d}{dx}y(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$

- Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k- >k+1$

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(2+r)\, x^{-1+r} + a_1(1+r)\,(3+r)\, x^r + a_2(2+r)\,(4+r)\, x^{1+r} + a_3(3+r)\,(5+r)\, x^{2+r} + \left(\sum_{k=3}^{\infty} (\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 0\}$

- The coefficients of each power of x must be 0

  $[a_1(1+r)\,(3+r) = 0, a_2(2+r)\,(4+r) = 0, a_3(3+r)\,(5+r) = 0]$

- Solve for the dependent coefficient(s)

  $\{a_1 = 0, a_2 = 0, a_3 = 0\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k+r+3) + 4a_{k-3} = 0$

- Shift index using $k- >k+3$

  $a_{k+4}(k+4+r)\,(k+6+r) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+4} = -\frac{4a_k}{(k+4+r)(k+6+r)}$

- Recursion relation for $r=-2$

  $a_{k+4}=-\frac{4a_k}{(k+2)(k+4)}$

- Solution for $r=-2$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+4}=-\frac{4a_k}{(k+2)(k+4)}, a_1=0, a_2=0, a_3=0\right]$

- Recursion relation for $r=0$

  $a_{k+4}=-\frac{4a_k}{(k+4)(k+6)}$

- Solution for $r=0$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k}, a_{k+4}=-\frac{4a_k}{(k+4)(k+6)}, a_1=0, a_2=0, a_3=0\right]$

- Combine solutions and rename parameters

  $\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^{k-2}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k}\right), a_{4+k}=-\frac{4a_k}{(k+2)(4+k)}, a_1=0, a_2=0, a_3=0, b_{4+k}=-\frac{4b_k}{(4+k)(k+6)}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 21

```
dsolve(x^2*diff(diff(y(x),x),x)+3*diff(y(x),x)*x+4*x^4*y(x) = 0,y(x),singsol=all)
```

$$y=\frac{c_1 \sin\left(x^2\right)+c_2 \cos\left(x^2\right)}{x^2}$$

Mathematica DSolve solution

Solving time : `0.05` (sec)
Leaf size : 41

```
DSolve[{x^2*D[y[x],{x,2}]+3*x*D[y[x],x]+4*x^4*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tru
```

$$y(x)\to \frac{4c_1 e^{-ix^2}-ic_2 e^{ix^2}}{4x^2}$$

2.1.387 Problem 398

Internal problem ID [9559]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 398
Date solved : `Monday, January 27, 2025 at 06:04:06 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' = (x^2+3)\, y$$

Solved as second order ode using Kovacic algorithm

Time used: 0.207 (sec)

Writing the ode as

$$y'' + (-x^2-3)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -x^2-3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2+3}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2+3 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = (x^2+3)\, z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.734: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x + \frac{3}{2x} - \frac{9}{8x^3} + \frac{27}{16x^5} - \frac{405}{128x^7} + \frac{1701}{256x^9} - \frac{15309}{1024x^{11}} + \frac{72171}{2048x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2+3}{1} \\ &= Q + \frac{R}{1} \\ &= (x^2+3) + (0) \\ &= x^2 + 3 \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is 3. Now b can be found.

$$\begin{aligned} b &= (3) - (0) \\ &= 3 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= x \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{3}{1} - 1\right) &&= 1 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{3}{1} - 1\right) &&= -2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = x^2 + 3$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	x	1	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 1$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^{+} \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + (x) \\ &= x \\ &= x\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2(x)\,(1) + \left((1) + (x)^2 - (x^2 + 3)\right) &= 0 \\ -2a_0 &= 0\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= (x)\, e^{\int x dx} \\ &= (x)\, e^{\frac{x^2}{2}} \\ &= x\, e^{\frac{x^2}{2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1 \\ &= x\, e^{\frac{x^2}{2}}\end{aligned}$$

Which simplifies to

$$y_1 = x\, e^{\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B=0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= x\,\mathrm{e}^{\frac{x^2}{2}} \int \frac{1}{x^2 \mathrm{e}^{x^2}}\,dx \\ &= x\,\mathrm{e}^{\frac{x^2}{2}} \left(-\frac{\mathrm{e}^{-x^2}}{x} - \sqrt{\pi}\,\mathrm{erf}\left(x\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(x\,\mathrm{e}^{\frac{x^2}{2}}\right) + c_2 \left(x\,\mathrm{e}^{\frac{x^2}{2}} \left(-\frac{\mathrm{e}^{-x^2}}{x} - \sqrt{\pi}\,\mathrm{erf}\left(x\right)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2}y(x) = (x^2+3)\,y(x)$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dx^2}y(x)$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
$\frac{d^2}{dx^2}y(x) + (-x^2-3)\,y(x) = 0$
- Assume series solution for $y(x)$
$y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions
  - ○ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$
    $x^m \cdot y(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$
  - ○ Shift index using $k->k-m$
    $x^m \cdot y(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$
  - ○ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion
    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\,x^{k-2}$
  - ○ Shift index using $k->k+2$
    $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\,x^k$

  Rewrite ODE with series expansions
  $2a_2 - 3a_0 + (6a_3 - 3a_1)\,x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - 3a_k - a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0
$[2a_2 - 3a_0 = 0, 6a_3 - 3a_1 = 0]$
- Solve for the dependent coefficient(s)
$\left\{a_2 = \frac{3a_0}{2}, a_3 = \frac{a_1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation
  $(k^2 + 3k + 2)\, a_{k+2} - 3a_k - a_{k-2} = 0$
- Shift index using $k- >k+2$
  $\left((k+2)^2 + 3k + 8\right) a_{k+4} - 3a_{k+2} - a_k = 0$
- Recursion relation that defines the series solution to the ODE
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{3a_{k+2}+a_k}{k^2+7k+12}, a_2 = \frac{3a_0}{2}, a_3 = \frac{a_1}{2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 30

```
dsolve(diff(diff(y(x),x),x) = (x^2+3)*y(x),y(x),singsol=all)
```

$$y = x\left(c_2 \operatorname{erf}(x)\sqrt{\pi} + c_1\right) e^{\frac{x^2}{2}} + e^{-\frac{x^2}{2}} c_2$$

Mathematica DSolve solution

Solving time : 0.075 (sec)
Leaf size : 46

```
DSolve[{D[y[x],{x,2}]==(x^2+3)*y[x],{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-\frac{x^2}{2}}\left(-\sqrt{\pi}c_2 e^{x^2} x \text{erf}(x) + c_1 e^{x^2} x - c_2\right)$$

2.1.388 Problem 399

Internal problem ID [9560]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 399
Date solved : `Monday, January 27, 2025 at 06:04:07 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + 2xy' + \left(x^2 + 1\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.063 (sec)

Writing the ode as

$$y'' + 2xy' + \left(x^2 + 1\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2x \\ C &= x^2 + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.736: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x}{1}\, dx} \\ &= z_1 e^{-\frac{x^2}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x^2}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-x^2}}{(y_1)^2}\, dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x^2}{2}}\right) + c_2 \left(\mathrm{e}^{-\frac{x^2}{2}}(x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + 2x\left(\frac{d}{dx}y(x)\right) + (x^2+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- >k-m$

  $x^m \cdot y(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

  $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

  $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 + a_0 + (6a_3 + 3a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(2k+1) + a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 + a_0 = 0, 6a_3 + 3a_1 = 0]$
- Solve for the dependent coefficient(s)

  $\left\{a_2 = -\frac{a_0}{2}, a_3 = -\frac{a_1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} + 2a_k k + a_k + a_{k-2} = 0$

- Shift index using $k->k+2$

  $$\left((k+2)^2+3k+8\right)a_{k+4}+2a_{k+2}(k+2)+a_{k+2}+a_k=0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y(x)=\sum_{k=0}^{\infty}a_k x^k, a_{k+4}=-\frac{2ka_{k+2}+a_k+5a_{k+2}}{k^2+7k+12}, a_2=-\frac{a_0}{2}, a_3=-\frac{a_1}{2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 16

```
dsolve(diff(diff(y(x),x),x)+2*diff(y(x),x)*x+(x^2+1)*y(x) = 0,y(x),singsol=all)
```

$$y=e^{-\frac{x^2}{2}}(c_2x+c_1)$$

Mathematica DSolve solution

Solving time : 0.024 (sec)
Leaf size : 22

```
DSolve[{D[y[x],{x,2}]+2*x*D[y[x],x]+(x^2+1)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to e^{-\frac{x^2}{2}}(c_2x+c_1)$$

2.1.389 Problem 400

Internal problem ID [9561]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 400
Date solved : Monday, January 27, 2025 at 06:04:07 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$x^3y'' + y' - \frac{y}{x} = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.211 (sec)

Writing the ode as

$$x^3y'' + y' - \frac{y}{x} = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^3 \\ B &= 1 \\ C &= -\frac{1}{x} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2x^2 + 1}{4x^6} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -2x^2 + 1 \\ t &= 4x^6 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-2x^2+1}{4x^6}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.738: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^6$. There is a pole at $x = 0$ of order 6. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned} \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) \end{aligned}$$

The partial fraction decomposition of r is

$$r = \frac{1}{4x^6} - \frac{1}{2x^4}$$

There is pole in r at $x=0$ of order 6, hence $v=3$. Expanding $\sqrt{r}$ as Laurent series about this pole $c=0$ gives

$$[\sqrt{r}]_c \approx \frac{1}{2x^3} - \frac{1}{2x} - \frac{x}{4} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v=3$ the above becomes

$$[\sqrt{r}]_c = \frac{1}{2x^3} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^3}$ is

$$a = \frac{1}{2}$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c=0$. This term becomes $\frac{1}{x^4}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be $-\frac{1}{2}$. Therefore

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{1}{2x^3} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} + 3\right) &&= 1 \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} + 3\right) &&= 2 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-2x^2+1}{4x^6}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	6	$\frac{1}{2x^3}$	1	2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x^3} + \frac{1}{x} + (-)(0) \\
&= \frac{1}{2x^3} + \frac{1}{x} \\
&= \frac{1}{2x^3} + \frac{1}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2x^3} + \frac{1}{x}\right)(0) + \left(\left(-\frac{3}{2x^4} - \frac{1}{x^2} \right) + \left(\frac{1}{2x^3} + \frac{1}{x} \right)^2 - \left(\frac{-2x^2+1}{4x^6} \right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{2x^3} + \frac{1}{x} \right) dx} \\
&= x\, e^{-\frac{1}{4x^2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{x^3}\, dx} \\
&= z_1 e^{\frac{1}{4x^2}} \\
&= z_1 \left(e^{\frac{1}{4x^2}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{x^3}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{1}{2x^2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}}{2x}\right)}{2} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2 \left(x \left(\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}}{2x}\right)}{2} \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 19

```
dsolve(x^3*diff(diff(y(x),x),x)+diff(y(x),x)-y(x)/x = 0,y(x),singsol=all)
```

$$
y = x \left(c_1 + c_2\,\mathrm{erf}\left(\frac{i\sqrt{2}}{2x}\right) \right)
$$

Mathematica DSolve solution

Solving time : 0.082 (sec)
Leaf size : 34

```
DSolve[{x^3*D[y[x],{x,2}]+ D[y[x],x]-1/x*y[x] == 0,{}},y[x],x,IncludeSingularSolutions->True
```

$$
y(x) \to c_1 x - \sqrt{\frac{\pi}{2}} c_2 x \mathrm{erfi}\left(\frac{1}{\sqrt{2}x}\right)
$$

2.1.390 Problem 401

Internal problem ID [9562]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 401
Date solved : `Monday, January 27, 2025 at 06:04:08 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.139 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.739: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\, dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos(x)}{\sqrt{x}}(\tan(x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x\left(\frac{d}{dx}y(x)\right) + \left(x^2 - \frac{1}{4}\right)y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(4x^2-1)y(x)}{4x^2} - \frac{\frac{d}{dx}y(x)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{\frac{d}{dx}y(x)}{x} + \frac{(4x^2-1)y(x)}{4x^2} = 0$
- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\frac{1}{4}$
  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$
- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x\left(\frac{d}{dx}y(x)\right) + (4x^2 - 1)y(x) = 0$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- Shift index using $k- >k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$
- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\,x^r + a_1(3+2r)\,(1+2r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2+8kr+4r^2-1) + 4a_{k-2} = 0$
- Shift index using $k- >k+2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\,r + 4r^2 - 1\right) + 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$
- Solution for $r = -\frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$
- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.038 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x+(x^2-1/4)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{\sin(x) c_1 + \cos(x) c_2}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.031 (sec)
Leaf size : 39

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-1/4)*y[x] == 0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to \frac{e^{-ix}(2c_1 - ic_2 e^{2ix})}{2\sqrt{x}}$$

2.1.391 Problem 402

Internal problem ID [9563]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 402
Date solved : `Monday, January 27, 2025 at 06:04:08 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' + \left(-8x^2+4x\right)y' + \left(4x^2-4x-1\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.092 (sec)

Writing the ode as

$$4x^2y'' + \left(-8x^2+4x\right)y' + \left(4x^2-4x-1\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -8x^2+4x \\ C &= 4x^2-4x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.741: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-8x^2+4x}{4x^2}\, dx} \\ &= z_1 e^{x - \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{e^x}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^x}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-8x^2+4x}{4x^2} dx}}{(y_1)^2} dx \\
&= y_1 \int \frac{e^{2x-\ln(x)}}{(y_1)^2} dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^x}{\sqrt{x}}\right) + c_2 \left(\frac{e^x}{\sqrt{x}}(x)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2\left(\frac{d^2}{dx^2}y(x)\right) + (-8x^2+4x)\left(\frac{d}{dx}y(x)\right) + (4x^2-4x-1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(4x^2-4x-1)y(x)}{4x^2} + \frac{(2x-1)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) - \frac{(2x-1)\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(4x^2-4x-1)y(x)}{4x^2} = 0$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{2x-1}{x}, P_3(x) = \frac{4x^2-4x-1}{4x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) - 4x(2x-1)\left(\frac{d}{dx}y(x)\right) + (4x^2-4x-1)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\, x^r + (a_1(3+2r)\,(1+2r) - 4a_0(1+2r))\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0

  $a_1(3+2r)\,(1+2r) - 4a_0(1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = \frac{4a_0}{3+2r}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2+8kr+4r^2-1) + (-8k-8r+4)\, a_{k-1} + 4a_{k-2} = 0$
- Shift index using $k->k+2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + (-8k-12-8r)\, a_{k+1} + 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{4(2ka_{k+1}+2ra_{k+1}-a_k+3a_{k+1})}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = \frac{4(2ka_{k+1}-a_k+2a_{k+1})}{4k^2+12k+8}$
- Solution for $r = -\frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = \frac{4(2ka_{k+1}-a_k+2a_{k+1})}{4k^2+12k+8}, a_1 = 2a_0\right]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = \frac{4(2ka_{k+1}-a_k+4a_{k+1})}{4k^2+20k+24}$
- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{4(2ka_{k+1}-a_k+4a_{k+1})}{4k^2+20k+24}, a_1 = a_0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = \frac{4(2ka_{k+1}-a_k+2a_{k+1})}{4k^2+12k+8}, a_1 = 2a_0, b_{k+2} = \frac{4(2kb_{k+1}-b_k+4b_{k+1})}{4k^2+20k+24}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.022 (sec)
Leaf size : 15

```
dsolve(4*x^2*diff(diff(y(x),x),x)+(-8*x^2+4*x)*diff(y(x),x)+(4*x^2-4*x-1)*y(x) = 0,y(x
```

$$y = \frac{e^x (c_2 x + c_1)}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.031 (sec)
Leaf size : 21

```
DSolve[{4*x^2*D[y[x],{x,2}]+(-8*x^2+4*x)*D[y[x],x]+(4*x^2-4*x-1)*y[x] == 0,{}},y[x],x,Includ
```

$$y(x) \to \frac{e^x (c_2 x + c_1)}{\sqrt{x}}$$

2.1.392 Problem 404

Internal problem ID [9564]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 404
Date solved : `Monday, January 27, 2025 at 06:04:09 PM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$y'' - y' + y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.184 (sec)

Writing the ode as

$$y'' - y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{3z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.743: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{3}\, x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\, dx} \\ &= z_1 e^{\frac{x}{2}} \\ &= z_1 \left(e^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right) + c_2 \left(\mathrm{e}^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

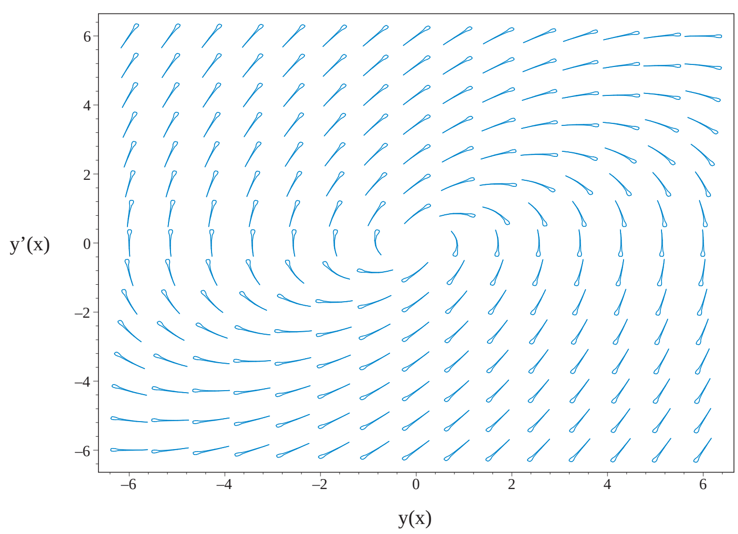

Figure 2.1: Slope field plot
$y'' - y' + y = 0$

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2}y(x) - \frac{d}{dx}y(x) + y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Characteristic polynomial of ODE
  $r^2 - r + 1 = 0$
- Use quadratic formula to solve for r
  $r = \frac{1 \pm (\sqrt{-3})}{2}$
- Roots of the characteristic polynomial
  $r = \left(\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, \frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right)$
- 1st solution of the ODE
  $y_1(x) = \mathrm{e}^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$
- 2nd solution of the ODE

$$y_2(x) = e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)$$

- General solution of the ODE
$$y(x) = C1 y_1(x) + C2 y_2(x)$$
- Substitute in solutions
$$y(x) = C1\, e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + C2\, e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

Maple dsolve solution

Solving time : 0.000 (sec)
Leaf size : 28

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)+y(x) = 0,y(x),singsol=all)
```

$$y = e^{\frac{x}{2}} \left(c_1 \sin\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

Mathematica DSolve solution

Solving time : 0.028 (sec)
Leaf size : 42

```
DSolve[{D[y[x],{x,2}]-D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{x/2} \left(c_1 \cos\left(\frac{\sqrt{3}x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}x}{2}\right)\right)$$

2.1.393 Problem 405

Internal problem ID [9565]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 405
Date solved : `Monday, January 27, 2025 at 06:04:10 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(x^2-1\right) y''-2xy'+2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.191 (sec)

Writing the ode as

$$\left(x^2-1\right) y''-2xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2-1 \\ B &= -2x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{3}{\left(x^2-1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= \left(x^2-1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{3}{\left(x^2-1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.745: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x+1)^2} + \frac{3}{4(x-1)^2} + \frac{3}{4(x+1)} - \frac{3}{4(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3}{(x^2-1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\,(x-1)} + \frac{3}{2\,(x+1)} + (-)\,(0) \\
&= -\frac{1}{2\,(x-1)} + \frac{3}{2\,(x+1)} \\
&= \frac{x-2}{x^2-1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2\left(x-1\right)}+\frac{3}{2\left(x+1\right)}\right)(0)+\left(\left(\frac{1}{2\left(x-1\right)^{2}}-\frac{3}{2\left(x+1\right)^{2}}\right)+\left(-\frac{1}{2\left(x-1\right)}+\frac{3}{2\left(x+1\right)}\right)^{2}-\left(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x-1)}+\frac{3}{2(x+1)}\right)dx} \\
&= \frac{(x+1)^{3/2}}{\sqrt{x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2-1}\, dx} \\
&= z_1 e^{\frac{\ln(x-1)}{2}+\frac{\ln(x+1)}{2}} \\
&= z_1\left(\sqrt{x-1}\,\sqrt{x+1}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = (x+1)^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2-1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\ln(x-1)+\ln(x+1)}}{(y_1)^2}\, dx \\
&= y_1\left(-\frac{x\,\mathrm{e}^{\ln(x-1)+\ln(x+1)}}{(x+1)^3\,(x-1)}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((x+1)^2\right) + c_2\left((x+1)^2\left(-\frac{x\,\mathrm{e}^{\ln(x-1)+\ln(x+1)}}{(x+1)^3\,(x-1)}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x^2-1)\left(\frac{d^2}{dx^2}y(x)\right)-2x\left(\frac{d}{dx}y(x)\right)+2y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{2\left(\frac{d}{dx}y(x)\right)x}{x^2-1}-\frac{2y(x)}{x^2-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{2y(x)}{x^2-1}-\frac{2\left(\frac{d}{dx}y(x)\right)x}{x^2-1}=0$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x)=-\frac{2x}{x^2-1},P_3(x)=\frac{2}{x^2-1}\right]$
  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=-1$
  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$
  - $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $(x^2-1)\left(\frac{d^2}{dx^2}y(x)\right)-2x\left(\frac{d}{dx}y(x)\right)+2y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(-2u+2)\left(\frac{d}{du}y(u)\right)+2y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$
  - Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)(k+r-1)+a_k(k+r-1)(k+r-2)\right)u^{k+r}\right)=$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-2r(-2+r)=0$

- Values of r that satisfy the indicial equation
$r\in\{0,2\}$
- Each term in the series must be 0, giving the recursion relation
$((-2k-2r-2)\,a_{k+1}+a_k(k+r-2))\,(k+r-1)=0$
- Recursion relation that defines series solution to ODE
$a_{k+1}=\frac{a_k(k+r-2)}{2(k+1+r)}$
- Recursion relation for $r=0$; series terminates at $k=2$
$a_{k+1}=\frac{a_k(k-2)}{2(k+1)}$
- Apply recursion relation for $k=0$
$a_1=-a_0$
- Apply recursion relation for $k=1$
$a_2=-\frac{a_1}{4}$
- Express in terms of a_0
$a_2=\frac{a_0}{4}$
- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li
$y(u)=a_0\cdot\left(1-u+\frac{1}{4}u^2\right)$
- Revert the change of variables $u=x+1$
$\left[y(x)=\frac{a_0(x-1)^2}{4}\right]$
- Recursion relation for $r=2$
$a_{k+1}=\frac{a_k k}{2(k+3)}$
- Solution for $r=2$
$\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+2}, a_{k+1}=\frac{a_k k}{2(k+3)}\right]$
- Revert the change of variables $u=x+1$
$\left[y(x)=\sum_{k=0}^{\infty}a_k(x+1)^{k+2}, a_{k+1}=\frac{a_k k}{2(k+3)}\right]$
- Combine solutions and rename parameters
$\left[y(x)=\frac{a_0(x-1)^2}{4}+\left(\sum_{k=0}^{\infty}b_k(x+1)^{k+2}\right), b_{k+1}=\frac{b_k k}{2(k+3)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 14

```
dsolve((x^2-1)*diff(diff(y(x),x),x)-2*diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y = c_2x^2 + c_1x + c_2$$

Mathematica DSolve solution

Solving time : 0.376 (sec)
Leaf size : 75

```
DSolve[{(x^2-1)*D[y[x],{x,2}]-2*x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True
```

$$y(x) \to \sqrt{x^2-1}\exp\left(\int_1^x \frac{K[1]+2}{K[1]^2-1}dK[1]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[2]} \frac{K[1]+2}{K[1]^2-1}dK[1]\right)dK[2] + c_1\right)$$

2.1.394 Problem 406

Internal problem ID [9566]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 406
Date solved : `Monday, January 27, 2025 at 06:04:10 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x(x+2)\,y' + (x+2)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.073 (sec)

Writing the ode as

$$x^2y'' + \left(-x^2 - 2x\right)y' + (x+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 - 2x \\ C &= x + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.747: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x^2-2x}{x^2}\, dx} \\ &= z_1 e^{\frac{x}{2}+\ln(x)} \\ &= z_1 \left(x\, \mathrm{e}^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-2x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x+2\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{e^{x+2\ln(x)}}{x^2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(x) + c_2\left(x\left(\frac{e^{x+2\ln(x)}}{x^2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) - x(x+2)\left(\frac{d}{dx}y(x)\right) + (x+2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(x+2)y(x)}{x^2} + \frac{(x+2)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) - \frac{(x+2)\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(x+2)y(x)}{x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{x+2}{x}, P_3(x) = \frac{x+2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) - x(x+2)\left(\frac{d}{dx}y(x)\right) + (x+2)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)\,(k+r-2) - a_{k-1}(k+r-2)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)\,(a_k(k+r-1) - a_{k-1}) = 0$
- Shift index using $k->k+1$

  $(k+r-1)\,(a_{k+1}(k+r) - a_k) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+r}$
- Recursion relation for $r = 1$

  $a_{k+1} = \frac{a_k}{k+1}$
- Solution for $r = 1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = \frac{a_k}{k+1}\right]$
- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+2}$
- Solution for $r = 2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k}{k+2}\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
```

```
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 12

```
dsolve(x^2*diff(diff(y(x),x),x)-x*(x+2)*diff(y(x),x)+(x+2)*y(x) = 0,y(x),singsol=all)
```

$$y = x(c_1 + e^x c_2)$$

Mathematica DSolve solution

Solving time : 0.039 (sec)
Leaf size : 17

```
DSolve[{x^2*D[y[x],{x,2}]-x*(x+2)*D[y[x],x]+(x+2)*y[x]==0,{}},y[x],x,IncludeSingularSolution
```

$$y(x) \to ex(c_2 e^x + c_1)$$

2.1.395 Problem 407

Internal problem ID [9567]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 407
Date solved : `Monday, January 27, 2025 at 06:04:11 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(x+1)\,y''-(x+2)\,y'+y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.225 (sec)

Writing the ode as

$$(x+1)\,y''+(-x-2)\,y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x+1 \\ B &= -x-2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{x^2+2}{4\,(x+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2+2 \\ t &= 4(x+1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x^2+2}{4\,(x+1)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.749: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x+1)^2$. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4\,(x+1)^2} - \frac{1}{2\,(x+1)}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^2} - \frac{1}{x^3} + \frac{3}{4x^4} - \frac{3}{4x^5} + \frac{1}{x^6} - \frac{1}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2+2}{4x^2+8x+4} \\ &= Q + \frac{R}{4x^2+8x+4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x+1}{4x^2+8x+4}\right) \\ &= \frac{1}{4} + \frac{-2x+1}{4x^2+8x+4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 + 2}{4\left(x + 1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(x + 1\right)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2\left(x + 1\right)} + \frac{1}{2} \\
&= \frac{x}{2x + 2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x+1)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2(x+1)^2}\right) + \left(-\frac{1}{2(x+1)} + \frac{1}{2}\right)^2 - \left(\frac{x^2+2}{4(x+1)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(-\frac{1}{2(x+1)} + \frac{1}{2}\right) dx} \\ &= \frac{e^{\frac{x}{2}}}{\sqrt{x+1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x-2}{x+1}\, dx} \\ &= z_1 e^{\frac{x}{2} + \frac{\ln(x+1)}{2}} \\ &= z_1 \left(\sqrt{x+1}\, e^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x-2}{x+1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x+\ln(x+1)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{(x+2)\, e^{x+\ln(x+1)} e^{-2x}}{x+1}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(e^x) + c_2\left(e^x\left(-\frac{(x+2)\, e^{x+\ln(x+1)} e^{-2x}}{x+1}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x+1)\left(\frac{d^2}{dx^2}y(x)\right)-(x+2)\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{y(x)}{x+1}+\frac{(x+2)\left(\frac{d}{dx}y(x)\right)}{x+1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)-\frac{(x+2)\left(\frac{d}{dx}y(x)\right)}{x+1}+\frac{y(x)}{x+1}=0$

□ Check to see if $x_0=-1$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=-\frac{x+2}{x+1}, P_3(x)=\frac{1}{x+1}\right]$

- ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=-1$

- ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

- ◦ $x=-1$is a regular singular point

  Check to see if $x_0=-1$ is a regular singular point

  $x_0=-1$

- Multiply by denominators

$(x+1)\left(\frac{d^2}{dx^2}y(x)\right)+(-x-2)\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$u\left(\frac{d^2}{du^2}y(u)\right)+(-u-1)\left(\frac{d}{du}y(u)\right)+y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ◦ Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-1}$

- ◦ Shift index using $k->k+1$

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)(k+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)(k+r-1)-a_k(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\,(a_{k+1}(k+1+r)-a_k)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k}{k+1+r}$
- Recursion relation for $r=0$

  $a_{k+1}=\frac{a_k}{k+1}$
- Solution for $r=0$

  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+1}=\frac{a_k}{k+1}\right]$
- Revert the change of variables $u=x+1$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k (x+1)^k, a_{k+1}=\frac{a_k}{k+1}\right]$
- Recursion relation for $r=2$

  $a_{k+1}=\frac{a_k}{k+3}$
- Solution for $r=2$

  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$
- Revert the change of variables $u=x+1$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k (x+1)^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$
- Combine solutions and rename parameters

  $\left[y(x)=\left(\sum_{k=0}^{\infty} a_k (x+1)^k\right)+\left(\sum_{k=0}^{\infty} b_k (x+1)^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+3}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 14

```
dsolve((x+1)*diff(diff(y(x),x),x)-(x+2)*diff(y(x),x)+y(x) = 0,y(x),singsol=all)
```

$$y=c_1(x+2)+e^x c_2$$

Mathematica DSolve solution

Solving time : 0.208 (sec)
Leaf size : 84

```
DSolve[{(x+1)*D[y[x],{x,2}]-(x+2)*D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tru
```

$$y(x) \to \exp\left(\int_1^x \frac{K[1]}{2K[1]+2}dK[1] - \frac{1}{2}\int_1^x \left(-1-\frac{1}{K[2]+1}\right)dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{K[1]}{2K[1]+2}dK[1]\right)dK[3] + c_1\right)$$

2.1.396 Problem 408

Internal problem ID [9568]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 408
Date solved : `Monday, January 27, 2025 at 06:04:11 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-x^2+1\right) y''+2xy'-2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.190 (sec)

Writing the ode as

$$\left(-x^2+1\right) y''+2xy'-2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+1 \\ B &= 2x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{3}{\left(x^2-1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= \left(x^2-1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{3}{\left(x^2-1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.751: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x+1)} - \frac{3}{4(x-1)} + \frac{3}{4(x-1)^2} + \frac{3}{4(x+1)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4>2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= 0\\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{(x^2-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right)\\ &= 1 - (1)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (-)[\sqrt{r}]_\infty\\ &= -\frac{1}{2\left(x-1\right)} + \frac{3}{2\left(x+1\right)} + (-)\left(0\right)\\ &= -\frac{1}{2\left(x-1\right)} + \frac{3}{2\left(x+1\right)}\\ &= \frac{x-2}{x^2-1}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2(x-1)}+\frac{3}{2(x+1)}\right)(0)+\left(\left(\frac{1}{2(x-1)^2}-\frac{3}{2(x+1)^2}\right)+\left(-\frac{1}{2(x-1)}+\frac{3}{2(x+1)}\right)^2-\Bigg(\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\,dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2(x-1)}+\frac{3}{2(x+1)}\right)dx} \\ &= \frac{(x+1)^{3/2}}{\sqrt{x-1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x}{-x^2+1}\,dx} \\ &= z_1 e^{\frac{\ln(x-1)}{2}+\frac{\ln(x+1)}{2}} \\ &= z_1\left(\sqrt{x-1}\,\sqrt{x+1}\right) \end{aligned}$$

Which simplifies to

$$y_1=(x+1)^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1\int \frac{e^{\int -\frac{2x}{-x^2+1}\,dx}}{(y_1)^2}\,dx \\ &= y_1\int \frac{e^{\ln(x-1)+\ln(x+1)}}{(y_1)^2}\,dx \\ &= y_1\left(-\frac{x\,\mathrm{e}^{\ln(x-1)+\ln(x+1)}}{(x+1)^3(x-1)}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1y_1+c_2y_2 \\ &= c_1\left((x+1)^2\right)+c_2\left((x+1)^2\left(-\frac{x\,\mathrm{e}^{\ln(x-1)+\ln(x+1)}}{(x+1)^3(x-1)}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(-x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)+2x\left(\frac{d}{dx}y(x)\right)-2y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=\frac{2\left(\frac{d}{dx}y(x)\right)x}{x^2-1}-\frac{2y(x)}{x^2-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x)+\frac{2y(x)}{x^2-1}-\frac{2\left(\frac{d}{dx}y(x)\right)x}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x)=-\frac{2x}{x^2-1}, P_3(x)=\frac{2}{x^2-1}\right]$

- $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=-1$

- $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

- $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1$

- Multiply by denominators

$(x^2-1)\left(\frac{d^2}{dx^2}y(x)\right)-2x\left(\frac{d}{dx}y(x)\right)+2y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(-2u+2)\left(\frac{d}{du}y(u)\right)+2y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

Rewrite ODE with series expansions

$-2a_0r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)(k+r-1)+a_k(k+r-1)(k+r-2)\right)u^{k+r}\right)=$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-2r(-2+r)=0$

- Values of r that satisfy the indicial equation
  $r\in\{0,2\}$
- Each term in the series must be 0, giving the recursion relation
  $((-2k-2r-2)\,a_{k+1}+a_k(k+r-2))\,(k+r-1)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{a_k(k+r-2)}{2(k+1+r)}$
- Recursion relation for $r=0$; series terminates at $k=2$
  $a_{k+1}=\frac{a_k(k-2)}{2(k+1)}$
- Apply recursion relation for $k=0$
  $a_1=-a_0$
- Apply recursion relation for $k=1$
  $a_2=-\frac{a_1}{4}$
- Express in terms of a_0
  $a_2=\frac{a_0}{4}$
- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li
  $y(u)=a_0\cdot\left(1-u+\frac{1}{4}u^2\right)$
- Revert the change of variables $u=x+1$
  $\left[y(x)=\frac{a_0(x-1)^2}{4}\right]$
- Recursion relation for $r=2$
  $a_{k+1}=\frac{a_k k}{2(k+3)}$
- Solution for $r=2$
  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+2}, a_{k+1}=\frac{a_k k}{2(k+3)}\right]$
- Revert the change of variables $u=x+1$
  $\left[y(x)=\sum_{k=0}^{\infty}a_k(x+1)^{k+2}, a_{k+1}=\frac{a_k k}{2(k+3)}\right]$
- Combine solutions and rename parameters
  $\left[y(x)=\frac{a_0(x-1)^2}{4}+\left(\sum_{k=0}^{\infty}b_k(x+1)^{k+2}\right), b_{k+1}=\frac{b_k k}{2(k+3)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 14

```
dsolve((-x^2+1)*diff(diff(y(x),x),x)+2*diff(y(x),x)*x-2*y(x) = 0,y(x),singsol=all)
```

$$y = c_2 x^2 + c_1 x + c_2$$

Mathematica DSolve solution

Solving time : 0.351 (sec)
Leaf size : 75

```
DSolve[{(1-x^2)*D[y[x],{x,2}]+2*x*D[y[x],x]-2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True
```

$$y(x) \to \sqrt{x^2-1}\exp\left(\int_1^x \frac{K[1]+2}{K[1]^2-1}dK[1]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[2]} \frac{K[1]+2}{K[1]^2-1}dK[1]\right)dK[2] + c_1\right)$$

2.1.397 Problem 409

Internal problem ID [9569]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 409
Date solved : `Monday, January 27, 2025 at 06:04:12 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-x^2+1\right) y''-2xy'+2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.250 (sec)

Writing the ode as

$$\left(-x^2+1\right) y''-2xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+1 \\ B &= -2x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{2x^2-3}{\left(x^2-1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2x^2-3 \\ t &= \left(x^2-1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{2x^2-3}{\left(x^2-1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.753: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(x-1)^2} + \frac{5}{4(x-1)} - \frac{5}{4(x+1)} - \frac{1}{4(x+1)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^2 - 3}{(x^2 - 1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2x^2 - 3}{(x^2 - 1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$\begin{aligned}d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right)\\ &= 2 - (1)\\ &= 1\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (+)[\sqrt{r}]_\infty\\ &= \frac{1}{2x-2} + \frac{1}{2x+2} + (0)\\ &= \frac{1}{2x-2} + \frac{1}{2x+2}\\ &= \frac{x}{x^2-1}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x-2} + \frac{1}{2x+2}\right)(1) + \left(\left(-\frac{1}{2(x-1)^2} - \frac{1}{2(x+1)^2}\right) + \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right)^2 - \left(\frac{2x^2-3}{(x^2-1)^2}\right)\right) = -\frac{2a_0}{x^2-1} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega \, dx} \\ &= (x)\, e^{\int \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right)dx} \\ &= (x)\sqrt{(x-1)(x+1)} \\ &= x\sqrt{x^2-1} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{-x^2+1}\, dx} \\ &= z_1 e^{-\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x-1}\,\sqrt{x+1}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{x+1}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{-x^2+1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\ln(x-1)-\ln(x+1)}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{1}{x} - \frac{\ln(x+1)}{2} + \frac{\ln(x-1)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{x\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{x+1}}\right) + c_2\left(\frac{x\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{x+1}}\left(\frac{1}{x} - \frac{\ln(x+1)}{2} + \frac{\ln(x-1)}{2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(-x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)-2x\left(\frac{d}{dx}y(x)\right)+2y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=\frac{2y(x)}{x^2-1}-\frac{2\left(\frac{d}{dx}y(x)\right)x}{x^2-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)+\frac{2\left(\frac{d}{dx}y(x)\right)x}{x^2-1}-\frac{2y(x)}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

◦ Define functions

$\left[P_2(x)=\frac{2x}{x^2-1}, P_3(x)=-\frac{2}{x^2-1}\right]$

◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

$\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=1$

◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

$\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

◦ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1$

- Multiply by denominators

$(x^2-1)\left(\frac{d^2}{dx^2}y(x)\right)+2x\left(\frac{d}{dx}y(x)\right)-2y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(2u-2)\left(\frac{d}{du}y(u)\right)-2y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

◦ Shift index using $k->k+1-m$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

◦ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

◦ Shift index using $k->k+2-m$

$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

Rewrite ODE with series expansions

$-2a_0r^2u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)^2+a_k(k+r+2)(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-2r^2 = 0$

- Values of r that satisfy the indicial equation
  $r = 0$
- Each term in the series must be 0, giving the recursion relation
  $-2a_{k+1}(k+1)^2 + a_k(k+2)(k-1) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$
- Recursion relation for $r = 0$; series terminates at $k = 1$
  $a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$
- Apply recursion relation for $k = 0$
  $a_1 = -a_0$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second linea
  $y(u) = a_0 \cdot (-u + 1)$
- Revert the change of variables $u = x + 1$
  $[y(x) = -a_0 x]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 25

```
dsolve((-x^2+1)*diff(diff(y(x),x),x)-2*diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y = -\frac{\ln(x+1) c_2 x}{2} + \frac{c_2 \ln(x-1) x}{2} + c_1 x + c_2$$

Mathematica DSolve solution

Solving time : 0.021 (sec)
Leaf size : 33

```
DSolve[{(1-x^2)*D[y[x],{x,2}]-2*x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tru
```

$$y(x) \to c_1 x - \frac{1}{2} c_2 (x \log(1-x) - x \log(x+1) + 2)$$

2.1.398 Problem 410

Internal problem ID [9570]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 410
Date solved : `Monday, January 27, 2025 at 06:04:13 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.142 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.755: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\, dx \\
&= y_1(\tan(x))
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2\left(\frac{\cos(x)}{\sqrt{x}}(\tan(x))\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x\left(\frac{d}{dx}y(x)\right) + \left(x^2 - \frac{1}{4}\right)y(x) = 0$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
$\frac{d^2}{dx^2}y(x) = -\frac{(4x^2-1)y(x)}{4x^2} - \frac{\frac{d}{dx}y(x)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
$\frac{d^2}{dx^2}y(x) + \frac{\frac{d}{dx}y(x)}{x} + \frac{(4x^2-1)y(x)}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
  $\left.(x \cdot P_2(x))\right|_{x=0} = 1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$
  - $x = 0$is a regular singular point
  Check to see if $x_0 = 0$ is a regular singular point
  $x_0 = 0$
- Multiply by denominators
$4x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x\left(\frac{d}{dx}y(x)\right) + (4x^2 - 1)y(x) = 0$
- Assume series solution for $y(x)$
$y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x \cdot \left(\frac{d}{dx} y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$
- Convert $x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\,x^r + a_1(3+2r)\,(1+2r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-2})\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$
- Shift index using $k- >k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\,r + 4r^2 - 1\right) + 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$
- Solution for $r = -\frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$
- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 1.592 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x+(x^2-1/4)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{\sin(x) c_1 + \cos(x) c_2}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.02 (sec)
Leaf size : 33

```
DSolve[{(1-x^2)*D[y[x],{x,2}]-2*x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to c_1 x - \frac{1}{2} c_2 (x \log(1-x) - x \log(x+1) + 2)$$

2.1.399 Problem 411

Internal problem ID [9571]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 411
Date solved : `Monday, January 27, 2025 at 06:04:13 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(x^{2}-1\right) y''-6 x y'+12 y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.196 (sec)

Writing the ode as

$$\left(x^{2}-1\right) y''-6 x y'+12 y = 0 \tag{1}$$
$$A y'' + B y' + C y = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^{2}-1 \\ B &= -6 x \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15}{\left(x^{2}-1\right)^{2}} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15 \\ t &= \left(x^{2}-1\right)^{2} \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15}{\left(x^{2}-1\right)^{2}}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.757: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4\,(x+1)^2} + \frac{15}{4\,(x-1)^2} + \frac{15}{4\,(x+1)} - \frac{15}{4\,(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{15}{(x^2-1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
-1	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{2\,(x-1)} + \frac{5}{2\,(x+1)} + (-)\,(0) \\
&= -\frac{3}{2\,(x-1)} + \frac{5}{2\,(x+1)} \\
&= \frac{x-4}{x^2-1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{3}{2(x-1)}+\frac{5}{2(x+1)}\right)(0)+\left(\left(\frac{3}{2(x-1)^2}-\frac{5}{2(x+1)^2}\right)+\left(-\frac{3}{2(x-1)}+\frac{5}{2(x+1)}\right)^2-\left(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}z_1(x)&=pe^{\int \omega\,dx}\\&=\mathrm{e}^{\int\left(-\frac{3}{2(x-1)}+\frac{5}{2(x+1)}\right)dx}\\&=\frac{(x+1)^{5/2}}{(x-1)^{3/2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1&=z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\&=z_1e^{-\int \frac{1}{2}\frac{-6x}{x^2-1}\,dx}\\&=z_1e^{\frac{3\ln(x-1)}{2}+\frac{3\ln(x+1)}{2}}\\&=z_1\left((x-1)^{3/2}(x+1)^{3/2}\right)\end{aligned}$$

Which simplifies to

$$y_1=(x+1)^4$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2&=y_1\int \frac{e^{\int -\frac{-6x}{x^2-1}\,dx}}{(y_1)^2}\,dx\\&=y_1\int \frac{\mathrm{e}^{3\ln(x-1)+3\ln(x+1)}}{(y_1)^2}\,dx\\&=y_1\left(-\frac{x(x^2+1)\,\mathrm{e}^{3\ln(x-1)+3\ln(x+1)}}{(x+1)^7(x-1)^3}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y&=c_1y_1+c_2y_2\\&=c_1\left((x+1)^4\right)+c_2\left((x+1)^4\left(-\frac{x(x^2+1)\,\mathrm{e}^{3\ln(x-1)+3\ln(x+1)}}{(x+1)^7(x-1)^3}\right)\right)\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x^2-1)\left(\frac{d^2}{dx^2}y(x)\right)-6x\left(\frac{d}{dx}y(x)\right)+12y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{12y(x)}{x^2-1}+\frac{6\left(\frac{d}{dx}y(x)\right)x}{x^2-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)-\frac{6\left(\frac{d}{dx}y(x)\right)x}{x^2-1}+\frac{12y(x)}{x^2-1}=0$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x)=-\frac{6x}{x^2-1},P_3(x)=\frac{12}{x^2-1}\right]$

  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=-3$

  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

  - $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $(x^2-1)\left(\frac{d^2}{dx^2}y(x)\right)-6x\left(\frac{d}{dx}y(x)\right)+12y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(-6u+6)\left(\frac{d}{du}y(u)\right)+12y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r(-4+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)(k+r-3)+a_k(k+r-3)(k+r-4)\right)u^{k+r}\right)=$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-2r(-4+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{0,4\}$
- Each term in the series must be 0, giving the recursion relation

  $((-2k-2r-2)\,a_{k+1}+a_k(k+r-4))\,(k+r-3)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k(k+r-4)}{2(k+1+r)}$
- Recursion relation for $r=0$; series terminates at $k=4$

  $a_{k+1}=\frac{a_k(k-4)}{2(k+1)}$
- Apply recursion relation for $k=0$

  $a_1=-2a_0$
- Apply recursion relation for $k=1$

  $a_2=-\frac{3a_1}{4}$
- Express in terms of a_0

  $a_2=\frac{3a_0}{2}$
- Apply recursion relation for $k=2$

  $a_3=-\frac{a_2}{3}$
- Express in terms of a_0

  $a_3=-\frac{a_0}{2}$
- Apply recursion relation for $k=3$

  $a_4=-\frac{a_3}{8}$
- Express in terms of a_0

  $a_4=\frac{a_0}{16}$
- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li

  $y(u)=a_0\cdot\left(1-2u+\frac{3}{2}u^2-\frac{1}{2}u^3+\frac{1}{16}u^4\right)$
- Revert the change of variables $u=x+1$

  $\left[y(x)=\frac{a_0(x-1)^4}{16}\right]$
- Recursion relation for $r=4$

  $a_{k+1}=\frac{a_k k}{2(k+5)}$
- Solution for $r=4$

  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+4}, a_{k+1}=\frac{a_k k}{2(k+5)}\right]$
- Revert the change of variables $u=x+1$

  $\left[y(x)=\sum_{k=0}^{\infty}a_k(x+1)^{k+4}, a_{k+1}=\frac{a_k k}{2(k+5)}\right]$
- Combine solutions and rename parameters

  $\left[y(x)=\frac{a_0(x-1)^4}{16}+\left(\sum_{k=0}^{\infty}b_k(x+1)^{4+k}\right), b_{k+1}=\frac{b_k k}{2(5+k)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 25

```
dsolve((x^2-1)*diff(diff(y(x),x),x)-6*diff(y(x),x)*x+12*y(x) = 0,y(x),singsol=all)
```

$$y = c_2x^4 + c_1x^3 + 6c_2x^2 + c_1x + c_2$$

Mathematica DSolve solution

Solving time : 0.366 (sec)
Leaf size : 75

```
DSolve[{(x^2-1)*D[y[x],{x,2}]-6*x*D[y[x],x]+12*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tru
```

$$y(x) \to (x^2 - 1)^{3/2} \exp\left(\int_1^x \frac{K[1]+4}{K[1]^2-1} dK[1]\right) \left(c_2 \int_1^x \exp\left(-2\int_1^{K[2]} \frac{K[1]+4}{K[1]^2-1} dK[1]\right) dK[2] + c_1\right)$$

2.1.400 Problem 412

Internal problem ID [9572]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 412
Date solved : `Monday, January 27, 2025 at 06:04:14 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+3\right) y''-7xy'+16y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.420 (sec)

Writing the ode as

$$\left(x^2+3\right) y''-7xy'+16y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+3\\ B &= -7x \\ C &= 16 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-x^2-234}{4\left(x^2+3\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2-234 \\ t &= 4\left(x^2+3\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-x^2-234}{4\left(x^2+3\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.759: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+3)^2$. There is a pole at $x = i\sqrt{3}$ of order 2. There is a pole at $x = -i\sqrt{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{77}{16\left(x - i\sqrt{3}\right)^2} + \frac{77}{16\left(x + i\sqrt{3}\right)^2} + \frac{79i\sqrt{3}}{48\left(x - i\sqrt{3}\right)} - \frac{79i\sqrt{3}}{48\left(x + i\sqrt{3}\right)}$$

For the pole at $x = i\sqrt{3}$ let b be the coefficient of $\frac{1}{\left(x - i\sqrt{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{77}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{7}{4} \end{aligned}$$

For the pole at $x = -i\sqrt{3}$ let b be the coefficient of $\frac{1}{\left(x + i\sqrt{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{77}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{7}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 234}{4\left(x^2+3\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 - 234}{4\left(x^2+3\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$i\sqrt{3}$	2	0	$\frac{11}{4}$	$-\frac{7}{4}$
$-i\sqrt{3}$	2	0	$\frac{11}{4}$	$-\frac{7}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{1}{2} - \left(-\frac{7}{2}\right) \\
&= 4
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{7}{4\left(x - i\sqrt{3}\right)} - \frac{7}{4\left(x + i\sqrt{3}\right)} + (-)\,(0) \\
&= -\frac{7}{4\left(x - i\sqrt{3}\right)} - \frac{7}{4\left(x + i\sqrt{3}\right)} \\
&= -\frac{7x}{2x^2+6}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(12x^2 + 6xa_3 + 2a_2) + 2\left(-\frac{7}{4\,(x - i\sqrt{3})} - \frac{7}{4\,(x + i\sqrt{3})}\right)(4x^3 + 3x^2a_3 + 2xa_2 + a_1) + \left(\left(\frac{7}{4\,(x - i\sqrt{3})^2} + \right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{27}{8}, a_1 = 0, a_2 = -9, a_3 = 0\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 - 9x^2 + \frac{27}{8}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \left(x^4 - 9x^2 + \frac{27}{8}\right) e^{\int \left(-\frac{7}{4\left(x - i\sqrt{3}\right)} - \frac{7}{4\left(x + i\sqrt{3}\right)}\right) dx} \\ &= \left(x^4 - 9x^2 + \frac{27}{8}\right) \frac{1}{\left((i\sqrt{3} - x)\,(x + i\sqrt{3})\right)^{7/4}} \\ &= \frac{8x^4 - 72x^2 + 27}{8\,(-x^2 - 3)^{7/4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-7x}{x^2+3}\, dx} \\ &= z_1 e^{\frac{7\ln\left(x^2+3\right)}{4}} \\ &= z_1 \left((x^2 + 3)^{7/4}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \left(\frac{1}{2} + \frac{i}{2}\right)\sqrt{2}\left(x^4 - 9x^2 + \frac{27}{8}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-7x}{x^2+3}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\frac{7\ln\left(x^2+3\right)}{2}}}{(y_1)^2}\,dx \\ &= y_1(\text{Expression too large to display}) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\left(\frac{1}{2}+\frac{i}{2}\right)\sqrt{2}\left(x^4-9x^2+\frac{27}{8}\right)\right) \\ &\quad + c_2\left(\left(\frac{1}{2}+\frac{i}{2}\right)\sqrt{2}\left(x^4-9x^2+\frac{27}{8}\right)(\text{Expression too large to display})\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 65

```
dsolve((x^2+3)*diff(diff(y(x),x),x)-7*diff(y(x),x)*x+16*y(x) = 0,y(x),singsol=all)
```

$$\begin{aligned} y &= 4c_2\left(x^4-9x^2+\frac{27}{8}\right)\ln\left(\sqrt{x^2+3}-x\right) \\ &\quad + \frac{5(10x^3-33x)\,c_2\sqrt{x^2+3}}{6} + \left(x^4-9x^2+\frac{27}{8}\right)\left(c_1+\frac{25c_2}{3}\right) \end{aligned}$$

Mathematica DSolve solution

Solving time : 0.131 (sec)
Leaf size : 69

```
DSolve[{(x^2+3)*D[y[x],{x,2}]-7*x*D[y[x],x]+16*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tru
```

$$y(x) \to \frac{1}{24}c_2\left(3\left(8x^4-72x^2+27\right)\operatorname{arcsinh}\left(\frac{x}{\sqrt{3}}\right)+5x\sqrt{x^2+3}\left(33-10x^2\right)\right) + c_1\left(x^4-9x^2+\frac{27}{8}\right)$$

2.1.401 Problem 413

Internal problem ID [9573]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 413
Date solved : `Monday, January 27, 2025 at 06:04:15 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(x^2-1\right) y''+8xy'+12y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.181 (sec)

Writing the ode as

$$\left(x^2-1\right) y''+8xy'+12y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2-1 \\ B &= 8x \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{8}{\left(x^2-1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 8 \\ t &= \left(x^2-1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{8}{\left(x^2-1\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.760: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x+1} - \frac{2}{x-1} + \frac{2}{(x+1)^2} + \frac{2}{(x-1)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{8}{(x^2-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	2	-1
-1	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x-1} + \frac{2}{x+1} + (-)(0) \\
&= -\frac{1}{x-1} + \frac{2}{x+1} \\
&= \frac{x-3}{x^2-1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x-1} + \frac{2}{x+1}\right)(0) + \left(\left(\frac{1}{(x-1)^2} - \frac{2}{(x+1)^2} \right) + \left(-\frac{1}{x-1} + \frac{2}{x+1} \right)^2 - \left(\frac{8}{(x^2-1)^2} \right) \right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{x-1}+\frac{2}{x+1}\right)dx} \\ &= \frac{(x+1)^2}{x-1} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{8x}{x^2-1} \, dx} \\ &= z_1 e^{-2\ln(x-1)-2\ln(x+1)} \\ &= z_1 \left(\frac{1}{(x-1)^2 (x+1)^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{(x-1)^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{8x}{x^2-1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-4\ln(x-1)-4\ln(x+1)}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{(x+1)(3x^2+1)(x-1)^4 \mathrm{e}^{-4\ln(x-1)-4\ln(x+1)}}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{1}{(x-1)^3}\right) + c_2 \left(\frac{1}{(x-1)^3}\left(-\frac{(x+1)(3x^2+1)(x-1)^4 \mathrm{e}^{-4\ln(x-1)-4\ln(x+1)}}{3}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x^2-1)\left(\frac{d^2}{dx^2}y(x)\right)+8x\left(\frac{d}{dx}y(x)\right)+12y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{12y(x)}{x^2-1}-\frac{8\left(\frac{d}{dx}y(x)\right)x}{x^2-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)+\frac{8\left(\frac{d}{dx}y(x)\right)x}{x^2-1}+\frac{12y(x)}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

$\left[P_2(x)=\frac{8x}{x^2-1},P_3(x)=\frac{12}{x^2-1}\right]$

- ○ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

$\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=4$

- ○ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

$\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

- ○ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1$

- Multiply by denominators

$(x^2-1)\left(\frac{d^2}{dx^2}y(x)\right)+8x\left(\frac{d}{dx}y(x)\right)+12y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(8u-8)\left(\frac{d}{du}y(u)\right)+12y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ○ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- ○ Shift index using $k->k+2-m$

$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

Rewrite ODE with series expansions

$-2a_0r(3+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)\,(k+r+4)+a_k(k+r+4)\,(k+r+3)\right)u^{k+r}\right)=$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-2r(3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-3, 0\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r+4)\left((-2k-2r-2)\, a_{k+1} + a_k(k+r+3)\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r+3)}{2(k+1+r)}$

- Recursion relation for $r = -3$

  $a_{k+1} = \frac{a_k k}{2(k-2)}$

- Series not valid for $r = -3$, division by 0 in the recursion relation at $k = 2$

  $a_{k+1} = \frac{a_k k}{2(k-2)}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k(k+3)}{2(k+1)}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(k+3)}{2(k+1)}\right]$

- Revert the change of variables $u = x+1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x+1)^k, a_{k+1} = \frac{a_k(k+3)}{2(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 29

```
dsolve((x^2-1)*diff(diff(y(x),x),x)+8*diff(y(x),x)*x+12*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_2 x^3 + 3c_1 x^2 + 3c_2 x + c_1}{(x^2-1)^3}$$

Mathematica DSolve solution

Solving time : `0.311` (sec)
Leaf size : 73

```
DSolve[{(x^2-1)*D[y[x],{x,2}]+8*x*D[y[x],x]+12*y[x]==0,{}},y[x],x,IncludeSingularSolutions->
```

$$y(x) \to \frac{\exp\left(\int_1^x \frac{K[1]+3}{K[1]^2-1} dK[1]\right)\left(c_2 \int_1^x \exp\left(-2\int_1^{K[2]} \frac{K[1]+3}{K[1]^2-1} dK[1]\right) dK[2] + c_1\right)}{(x^2-1)^2}$$

2.1.402 Problem 414

Internal problem ID [9574]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 414
Date solved : `Monday, January 27, 2025 at 06:04:15 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$3y'' + xy' - 4y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.218 (sec)

Writing the ode as

$$3y'' + xy' - 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3 \\ B &= x \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 54}{36} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 54 \\ t &= 36 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{36} + \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.762: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{6} + \frac{9}{2x} - \frac{243}{4x^3} + \frac{6561}{4x^5} - \frac{885735}{16x^7} + \frac{33480783}{16x^9} - \frac{2711943423}{32x^{11}} + \frac{115063885233}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{6} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{36}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 54}{36} \\ &= Q + \frac{R}{36} \\ &= \left(\frac{x^2}{36} + \frac{3}{2}\right) + (0) \\ &= \frac{x^2}{36} + \frac{3}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{3}{2}\right) - (0) \\ &= \frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{6} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{3}{2}}{\frac{1}{6}} - 1\right) &&= 4 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{3}{2}}{\frac{1}{6}} - 1\right) &&= -5 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{36} + \frac{3}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{6}$	4	-5

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 4$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 4 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{x}{6}\right) \\ &= \frac{x}{6} \\ &= \frac{x}{6} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(12x^2 + 6xa_3 + 2a_2\right) + 2\left(\frac{x}{6}\right)\left(4x^3 + 3x^2a_3 + 2xa_2 + a_1\right) + \left(\left(\frac{1}{6}\right) + \left(\frac{x}{6}\right)^2 - \left(\frac{x^2}{36} + \frac{3}{2}\right)\right) = 0$$

$$-\frac{a_3x^3}{3} + \frac{2(18 - a_2)\,x^2}{3} + (-a_1 + 6a_3)\,x - \frac{4a_0}{3} + 2a_2 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 27, a_1 = 0, a_2 = 18, a_3 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 + 18x^2 + 27$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= \left(x^4 + 18x^2 + 27\right) e^{\int \frac{x}{6} dx} \\ &= \left(x^4 + 18x^2 + 27\right) e^{\frac{x^2}{12}} \\ &= \left(x^4 + 18x^2 + 27\right) e^{\frac{x^2}{12}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{3}\, dx} \\ &= z_1 e^{-\frac{x^2}{12}} \\ &= z_1 \left(e^{-\frac{x^2}{12}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^4 + 18x^2 + 27$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{3}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{6}}}{(y_1)^2}\,dx \\ &= y_1 \left(\int \frac{e^{-\frac{x^2}{6}}}{(x^4 + 18x^2 + 27)^2}dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (x^4 + 18x^2 + 27) + c_2 \left(x^4 + 18x^2 + 27 \left(\int \frac{e^{-\frac{x^2}{6}}}{(x^4 + 18x^2 + 27)^2}dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.040 (sec)
Leaf size : 47

```
dsolve(3*diff(diff(y(x),x),x)+diff(y(x),x)*x-4*y(x) = 0,y(x),singsol=all)
```

$$y = xc_1\sqrt{6}\left(x^2+15\right)\mathrm{e}^{-\frac{x^2}{6}} + \left(x^4+18x^2+27\right)\left(\sqrt{\pi}\,\mathrm{erf}\left(\frac{\sqrt{6}\,x}{6}\right)c_1+c_2\right)$$

Mathematica DSolve solution

Solving time : 0.022 (sec)
Leaf size : 43

```
DSolve[{3*D[y[x],{x,2}]+x*D[y[x],x]-4*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{-\frac{x^2}{6}}\,\mathrm{HermiteH}\left(-5,\frac{x}{\sqrt{6}}\right)+\frac{1}{27}c_2\left(x^4+18x^2+27\right)$$

2.1.403 Problem 415

Internal problem ID [9575]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 415
Date solved : `Monday, January 27, 2025 at 06:04:16 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$5y'' - 2xy' + 10y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.263 (sec)

Writing the ode as

$$5y'' - 2xy' + 10y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 5 \\ B &= -2x \\ C &= 10 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 55}{25} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 55$$
$$t = 25$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{25} - \frac{11}{5}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.763: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{5} - \frac{11}{2x} - \frac{605}{8x^3} - \frac{33275}{16x^5} - \frac{9150625}{128x^7} - \frac{704598125}{256x^9} - \frac{116258690625}{1024x^{11}} - \frac{10048072546875}{2048x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{5}$$

From Eq. (9) the sum up to $v=1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{5} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1}=x^0=1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{25}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v=0$ or not. Since $v=1$ which is not zero, then starting $r=\frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2-55}{25} \\ &= Q + \frac{R}{25} \\ &= \left(\frac{x^2}{25} - \frac{11}{5}\right) + (0) \\ &= \frac{x^2}{25} - \frac{11}{5} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{11}{5}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{11}{5}\right) - (0) \\ &= -\frac{11}{5} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{5} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{11}{5}}{\frac{1}{5}} - 1\right) &&= -6 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{11}{5}}{\frac{1}{5}} - 1\right) &&= 5 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{25} - \frac{11}{5}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{5}$	-6	5

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 5$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 5 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{5}\right) \\ &= -\frac{x}{5} \\ &= -\frac{x}{5} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 5$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(20x^3 + 12x^2a_4 + 6xa_3 + 2a_2) + 2\left(-\frac{x}{5}\right)(5x^4 + 4x^3a_4 + 3x^2a_3 + 2xa_2 + a_1) + \left(\left(-\frac{1}{5}\right) + \left(-\frac{x}{5}\right)^2 - \left(\frac{x^2}{25}\right.\right.$$

$$\frac{2a_4x^4}{5} + \frac{4(25 + a_3)\,x^3}{5} + \frac{6(a_2 + 10a_4)\,x^2}{5} + \frac{2(4a_1 + 15a_3)\,x}{5} + 2a_0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = 0, a_1 = \frac{375}{4}, a_2 = 0, a_3 = -25, a_4 = 0\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^5 - 25x^3 + \frac{375}{4}x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= \left(x^5 - 25x^3 + \frac{375}{4}x\right) e^{\int -\frac{x}{5} dx} \\ &= \left(x^5 - 25x^3 + \frac{375}{4}x\right) e^{-\frac{x^2}{10}} \\ &= \frac{(4x^5 - 100x^3 + 375x)\, e^{-\frac{x^2}{10}}}{4} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{5}\, dx} \\ &= z_1 e^{\frac{x^2}{10}} \\ &= z_1\left(e^{\frac{x^2}{10}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^5 - 25x^3 + \frac{375}{4}x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{5}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\frac{x^2}{5}}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{e^{\frac{x^2}{5}}}{\left(x^5 - 25x^3 + \frac{375}{4}x\right)^2} dx \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(x^5 - 25x^3 + \frac{375}{4}x \right) + c_2 \left(x^5 - 25x^3 + \frac{375}{4}x \left(\int \frac{e^{\frac{x^2}{5}}}{\left(x^5 - 25x^3 + \frac{375}{4}x\right)^2} dx \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$5\frac{d^2}{dx^2}y(x) - 2x\left(\frac{d}{dx}y(x)\right) + 10y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{2x\left(\frac{d}{dx}y(x)\right)}{5} - 2y(x)$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) - \frac{2x\left(\frac{d}{dx}y(x)\right)}{5} + 2y(x) = 0$

- Multiply by denominators

  $5\frac{d^2}{dx^2}y(x) - 2x\left(\frac{d}{dx}y(x)\right) + 10y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite DE with series expansions
  - ◦ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  - ◦ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - ◦ Shift index using $k- >k+2$

$\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

Rewrite DE with series expansions

$\sum_{k=0}^{\infty} (5a_{k+2}(k+2)(k+1) - 2a_k(k-5))\, x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $5(k^2+3k+2)\, a_{k+2} - 2a_k(k-5) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2a_k(k-5)}{5(k^2+3k+2)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric sol
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returr
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.021 (sec)
Leaf size : 31

```
dsolve(5*diff(diff(y(x),x),x)-2*diff(y(x),x)*x+10*y(x) = 0,y(x),singsol=all)
```

$$y = c_2 \operatorname{hypergeom}\left(\left[-\frac{5}{2}\right], \left[\frac{1}{2}\right], \frac{x^2}{5}\right) + \frac{4x\left(x^4 - 25x^2 + \frac{375}{4}\right) c_1}{375}$$

Mathematica DSolve solution

Solving time : 0.115 (sec)
Leaf size : 138

```
DSolve[{5*D[y[x],{x,2}]-2*x*D[y[x],x]+10*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{1}{200}\sqrt{\frac{\pi}{5}}c_2\sqrt{x^2}\left(4x^4-100x^2+375\right)\operatorname{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{5}}\right)+\frac{32c_1x^5}{25\sqrt{5}} - \frac{32c_1x^3}{\sqrt{5}}-\frac{9}{20}c_2e^{\frac{x^2}{5}}x^2+c_2e^{\frac{x^2}{5}}+\frac{1}{50}c_2e^{\frac{x^2}{5}}x^4+24\sqrt{5}c_1x$$

2.1.404 Problem 416

Internal problem ID [9576]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 416
Date solved : `Monday, January 27, 2025 at 06:04:16 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - x^2 y' - 3xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.250 (sec)

Writing the ode as

$$y'' - x^2 y' - 3xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x^2 \\ C &= -3x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x(x^3 + 8)}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x(x^3 + 8) \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x(x^3 + 8)}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.765: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 4 \\ &= -4 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{2} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x^2}{2} + \frac{2}{x} - \frac{4}{x^4} + \frac{16}{x^7} - \frac{80}{x^{10}} + \frac{448}{x^{13}} - \frac{2688}{x^{16}} + \frac{16896}{x^{19}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 2$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i x^i \\ &= \frac{x^2}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^1 = x$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^4}{4}$$

This shows that the coefficient of x in the above is 0. Now we need to find the coefficient of x in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x(x^3+8)}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^4 + 2x\right) + (0) \\ &= \frac{1}{4}x^4 + 2x \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is 2. Now b can be found.

$$\begin{aligned} b &= (2) - (0) \\ &= 2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x^2}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{2}{\frac{1}{2}} - 2\right) &= 1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{2}{\frac{1}{2}} - 2\right) &= -3 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x(x^3+8)}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-4	$\frac{x^2}{2}$	1	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 1$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{x^2}{2}\right) \\ &= \frac{x^2}{2} \\ &= \frac{x^2}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{x^2}{2}\right)(1) + \left((x) + \left(\frac{x^2}{2}\right)^2 - \left(\frac{x(x^3+8)}{4}\right) \right) = 0$$
$$-xa_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x)\, e^{\int \frac{x^2}{2} dx} \\ &= (x)\, e^{\frac{x^3}{6}} \\ &= x\, e^{\frac{x^3}{6}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x^2}{1} \, dx} \\ &= z_1 e^{\frac{x^3}{6}} \\ &= z_1 \left(e^{\frac{x^3}{6}} \right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\frac{x^3}{3}} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\frac{x^3}{3}}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{-\frac{x^3}{3}}}{x^2} dx \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{\frac{x^3}{3}} x\right) + c_2 \left(\mathrm{e}^{\frac{x^3}{3}} x \left(\int \frac{\mathrm{e}^{-\frac{x^3}{3}}}{x^2} dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2} y(x) - x^2 \left(\frac{d}{dx} y(x)\right) - 3xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2} y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y(x)$ to series expansion

  $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ◦ Shift index using $k- >k-1$

  $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ◦ Convert $x^2 \cdot \left(\frac{d}{dx} y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^{k+1}$

  ◦ Shift index using $k- >k-1$

  $x^2 \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=1}^{\infty} a_{k-1} (k-1)\, x^k$

  ◦ Convert $\frac{d^2}{dx^2} y(x)$ to series expansion

$\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- Shift index using $k->k+2$

  $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_{k-1}(k+2)\right) x^k\right) = 0$

- Each term must be 0

  $2a_2 = 0$
- Each term in the series must be 0, giving the recursion relation

  $(k+2)\,(ka_{k+2} - a_{k-1} + a_{k+2}) = 0$
- Shift index using $k->k+1$

  $(k+3)\,((k+1)\,a_{k+3} - a_k + a_{k+3}) = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{a_k}{k+2}, 2a_2 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 58

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x^2-3*x*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{9\,\mathrm{e}^{\frac{x^3}{6}}\,\mathrm{WhittakerM}\left(\frac{1}{3}, \frac{5}{6}, \frac{x^3}{3}\right) c_2 x^3 + 9c_1 x^2 \mathrm{e}^{\frac{x^3}{3}} + 5\, 3^{2/3} c_2 (x^3)^{1/3}\,(x^3+2)}{9x}$$

Mathematica DSolve solution

Solving time : 0.063 (sec)
Leaf size : 51

```
DSolve[{D[y[x],{x,2}]-x^2*D[y[x],x]-3*x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{9} e^{\frac{x^3}{3}} \left(9c_1 x - 3^{2/3} c_2 \sqrt[3]{x^3}\,\Gamma\left(-\frac{1}{3}, \frac{x^3}{3}\right)\right)$$

2.1.405 Problem 417

Internal problem ID [9577]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 417
Date solved : `Monday, January 27, 2025 at 06:04:17 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+1\right)y''+2xy'-2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.266 (sec)

Writing the ode as

$$\left(x^2+1\right)y''+2xy'-2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= 2x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{2x^2+3}{\left(x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2x^2+3 \\ t &= \left(x^2+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{2x^2+3}{\left(x^2+1\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.767: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(x-i)^2} - \frac{1}{4(x+i)^2} - \frac{5i}{4(x-i)} + \frac{5i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^2+3}{(x^2+1)^2}$$

Since the gcd$(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2x^2+3}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-i$	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$\begin{aligned}d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right)\\ &= 2 - (1)\\ &= 1\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (+)[\sqrt{r}]_\infty\\ &= \frac{1}{2x-2i} + \frac{1}{2x+2i} + (0)\\ &= \frac{1}{2x-2i} + \frac{1}{2x+2i}\\ &= \frac{x}{x^2+1}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x-2i} + \frac{1}{2x+2i}\right)(1) + \left(\left(-\frac{1}{2(x-i)^2} - \frac{1}{2(x+i)^2}\right) + \left(\frac{1}{2x-2i} + \frac{1}{2x+2i}\right)^2 - \left(\frac{2x^2+3}{(x^2+1)^2}\right)\right.$$

$$-\frac{2(x^2+1)\,a_0}{(-x+i)^2\,(x+i)^2} =$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (x)\, e^{\int \left(\frac{1}{2x-2i} + \frac{1}{2x+2i}\right) dx} \\ &= (x)\, \sqrt{(-x+i)\,(x+i)} \\ &= x\sqrt{-x^2-1} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x}{x^2+1}\, dx} \\ &= z_1 e^{-\frac{\ln(x^2+1)}{2}} \\ &= z_1\left(\frac{1}{\sqrt{x^2+1}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = ix$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{x^2+1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\ln(x^2+1)}}{(y_1)^2}\, dx \\ &= y_1\left(\frac{1}{x} + \arctan(x)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(ix) + c_2\left(ix\left(\frac{1}{x} + \arctan(x)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 14

```
dsolve((x^2+1)*diff(diff(y(x),x),x)+2*diff(y(x),x)*x-2*y(x) = 0,y(x),singsol=all)
```

$$y = c_1 x + \arctan(x) x c_2 + c_2$$

Mathematica DSolve solution

Solving time : 0.021 (sec)
Leaf size : 48

```
DSolve[{(1+x^2)*D[y[x],{x,2}]+2*x*D[y[x],x]-2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to \frac{1}{2} i (2 c_1 x - c_2 x \log(1 - i x) + c_2 x \log(1 + i x) + 2 i c_2)$$

2.1.406 Problem 418

Internal problem ID [9578]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 418
Date solved : `Monday, January 27, 2025 at 06:04:18 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + xy' - 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.214 (sec)

Writing the ode as

$$y'' + xy' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 10}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 10 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} + \frac{5}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.768: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + \frac{5}{2x} - \frac{25}{4x^3} + \frac{125}{4x^5} - \frac{3125}{16x^7} + \frac{21875}{16x^9} - \frac{328125}{32x^{11}} + \frac{2578125}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 10}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} + \frac{5}{2}\right) + (0) \\ &= \frac{x^2}{4} + \frac{5}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{5}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{5}{2}\right) - (0) \\ &= \frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{5}{2}}{\frac{1}{2}} - 1\right) &= 2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{5}{2}}{\frac{1}{2}} - 1\right) &= -3 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} + \frac{5}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	2	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$, and since there are no poles, then

$$\begin{aligned}d &= \alpha_\infty^+\\ &= 2\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= (+)[\sqrt{r}]_\infty\\ &= 0 + \left(\frac{x}{2}\right)\\ &= \frac{x}{2}\\ &= \frac{x}{2}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(2) + 2\left(\frac{x}{2}\right)(2x + a_1) + \left(\left(\frac{1}{2}\right) + \left(\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} + \frac{5}{2}\right)\right) &= 0\\ -a_1 x - 2a_0 + 2 &= 0\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx}\\ &= \left(x^2 + 1\right) e^{\int \frac{x}{2} dx}\\ &= \left(x^2 + 1\right) e^{\frac{x^2}{4}}\\ &= \left(x^2 + 1\right) e^{\frac{x^2}{4}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{1}\, dx}\\ &= z_1 e^{-\frac{x^2}{4}}\\ &= z_1\left(e^{-\frac{x^2}{4}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = x^2 + 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{e^{-\frac{x^2}{2}}}{(x^2+1)^2} dx \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (x^2+1) + c_2 \left(x^2 + 1 \left(\int \frac{e^{-\frac{x^2}{2}}}{(x^2+1)^2} dx \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + x\left(\frac{d}{dx}y(x)\right) - 2y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite DE with series expansions
  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$
  - Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$
  - Shift index using $k- >k+2$

    $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k-2)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)\, a_{k+2} + a_k(k-2) = 0$

- Recursion relation; series terminates at $k = 2$

$a_{k+2} = -\frac{a_k(k-2)}{k^2+3k+2}$

- Apply recursion relation for $k = 0$

  $a_2 = a_0$

- Terminating series solution of the ODE. Use reduction of order to find the second linearly ind

  $y(x) = A_2x^2 + A_1x + a_0$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special functio
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.033 (sec)
Leaf size : 37

```
dsolve(diff(diff(y(x),x),x)+diff(y(x),x)*x-2*y(x) = 0,y(x),singsol=all)
```

$$y = \sqrt{2}\,\mathrm{e}^{-\frac{x^2}{2}} c_1 x + (x^2+1)\left(\sqrt{\pi}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right) c_1 + c_2\right)$$

Mathematica DSolve solution

Solving time : 0.02 (sec)
Leaf size : 35

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]-2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{-\frac{x^2}{2}}\,\mathrm{HermiteH}\left(-3, \frac{x}{\sqrt{2}}\right) + c_2(x^2+1)$$

2.1.407 Problem 419

Internal problem ID [9579]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 419
Date solved : `Monday, January 27, 2025 at 06:04:18 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2-6x+10\right)y''-4(x-3)\,y'+6y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.276 (sec)

Writing the ode as

$$\left(x^2-6x+10\right)y''+(-4x+12)\,y'+6y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2-6x+10\\ B &= -4x+12\\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t}\\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-8}{\left(x^2-6x+10\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -8\\ t &= \left(x^2-6x+10\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{8}{\left(x^2-6x+10\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.770: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 6x + 10)^2$. There is a pole at $x = 3 + i$ of order 2. There is a pole at $x = 3 - i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{(x - 3 - i)^2} + \frac{2}{(x - 3 + i)^2} + \frac{2i}{x - 3 - i} - \frac{2i}{x - 3 + i}$$

For the pole at $x = 3 + i$ let b be the coefficient of $\frac{1}{(x-3-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

For the pole at $x = 3 - i$ let b be the coefficient of $\frac{1}{(x-3+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{8}{(x^2-6x+10)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$3+i$	2	0	2	-1
$3-i$	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x-3-i} + \frac{2}{x-3+i} + (-)(0) \\
&= -\frac{1}{x-3-i} + \frac{2}{x-3+i} \\
&= \frac{x-3-3i}{x^2-6x+10}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{x-3-i} + \frac{2}{x-3+i}\right)(0) + \left(\left(\frac{1}{(x-3-i)^2} - \frac{2}{(x-3+i)^2}\right) + \left(-\frac{1}{x-3-i} + \frac{2}{x-3+i}\right)^2 - \right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{x-3-i}+\frac{2}{x-3+i}\right)dx} \\ &= \frac{\left(x^2-6x+10\right)^2}{\left(ix-3i+1\right)^3} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x+12}{x^2-6x+10} \, dx} \\ &= z_1 e^{\ln\left(x^2-6x+10\right)} \\ &= z_1\left(x^2-6x+10\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(x^2-6x+10\right)^3}{\left(ix-3i+1\right)^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x+12}{x^2-6x+10} \, dx}}{\left(y_1\right)^2} \, dx \\ &= y_1 \int \frac{e^{2\ln\left(x^2-6x+10\right)}}{\left(y_1\right)^2} \, dx \\ &= y_1 \left(\frac{x^2-6x+\frac{26}{3}}{\left(x-3+i\right)^3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\left(x^2-6x+10\right)^3}{\left(ix-3i+1\right)^3}\right) + c_2\left(\frac{\left(x^2-6x+10\right)^3}{\left(ix-3i+1\right)^3}\left(\frac{x^2-6x+\frac{26}{3}}{\left(x-3+i\right)^3}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 31

```
dsolve((x^2-6*x+10)*diff(diff(y(x),x),x)-4*(x-3)*diff(y(x),x)+6*y(x) = 0,y(x),singsol=al
```

$$y = c_1 x^3 + c_2 x^2 + 6(-5c_1 - c_2)\, x + 60c_1 + \frac{26c_2}{3}$$

Mathematica DSolve solution

Solving time : `0.297` (sec)
Leaf size : 84

```
DSolve[{(x^2-6*x+10)*D[y[x],{x,2}]-4*(x-3)*D[y[x],x]+6*y[x]==0,{}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to (x^2 - 6x + 10) \exp\left(\int_1^x \frac{K[1] - (3-3i)}{(K[1]-6)K[1]+10} dK[1]\right) \left(c_2 \int_1^x \exp\left(-2\int_1^{K[2]} \frac{K[1]-(3-3i)}{(K[1]-6)K[1]+10} dK[1]\right) dK[2] + c_1\right)$$

2.1.408 Problem 420

Internal problem ID [9580]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 420
Date solved : `Monday, January 27, 2025 at 06:04:19 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+6x\right)y''+(3x+9)\,y'-3y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.203 (sec)

Writing the ode as

$$\left(x^2+6x\right)y''+(3x+9)\,y'-3y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=x^2+6x\\ B&=3x+9\\ C&=-3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{15x^2+90x-27}{4\left(x^2+6x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=15x^2+90x-27\\ t&=4\left(x^2+6x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{15x^2+90x-27}{4\left(x^2+6x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.771: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+6x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -6$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{11}{16\,(x+6)} - \frac{3}{16\,(x+6)^2} - \frac{3}{16x^2} + \frac{11}{16x}$$

For the pole at $x = -6$ let b be the coefficient of $\frac{1}{(x+6)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15x^2 + 90x - 27}{4\left(x^2 + 6x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15x^2 + 90x - 27}{4\left(x^2 + 6x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-6	2	0	$\frac{3}{4}$	$\frac{1}{4}$
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\
&= \frac{5}{2} - \left(\frac{3}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{3}{4(x+6)} + \frac{3}{4x} + (0) \\
&= \frac{3}{4(x+6)} + \frac{3}{4x} \\
&= \frac{\frac{3x}{2} + \frac{9}{2}}{x(x+6)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{4(x+6)} + \frac{3}{4x}\right)(1) + \left(\left(-\frac{3}{4(x+6)^2} - \frac{3}{4x^2}\right) + \left(\frac{3}{4(x+6)} + \frac{3}{4x}\right)^2 - \left(\frac{15x^2 + 90x - 27}{4(x^2+6x)^2}\right)\right) = 0$$

$$\frac{9 - 3a_0}{x(x+6)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 3\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 3$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (x+3)\, e^{\int \left(\frac{3}{4(x+6)} + \frac{3}{4x}\right) dx} \\ &= (x+3)\,(x(x+6))^{3/4} \\ &= (x+3)\,(x(x+6))^{3/4} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3x+9}{x^2+6x}\, dx} \\ &= z_1 e^{-\frac{3\ln(x(x+6))}{4}} \\ &= z_1 \left(\frac{1}{(x(x+6))^{3/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x + 3$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3x+9}{x^2+6x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{3\ln(x(x+6))}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{(x+6)\, x(2x^2+12x+9)}{81\,(x+3)\,(x(x+6))^{3/2}}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1(x+3) + c_2\left(x + 3\left(-\frac{(x+6)\,x(2x^2+12x+9)}{81\,(x+3)\,(x\,(x+6))^{3/2}}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$(x^2+6x)\left(\frac{d^2}{dx^2}y(x)\right) + (3x+9)\left(\frac{d}{dx}y(x)\right) - 3y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = \frac{3y(x)}{x(6+x)} - \frac{3(x+3)\left(\frac{d}{dx}y(x)\right)}{x(6+x)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x) + \frac{3(x+3)\left(\frac{d}{dx}y(x)\right)}{x(6+x)} - \frac{3y(x)}{x(6+x)} = 0$
- ☐ Check to see if x_0 is a regular singular point
  - ◦ Define functions
    $\left[P_2(x) = \frac{3(x+3)}{x(6+x)}, P_3(x) = -\frac{3}{x(6+x)}\right]$
  - ◦ $(6+x)\cdot P_2(x)$ is analytic at $x=-6$
    $\left((6+x)\cdot P_2(x)\right)\Big|_{x=-6} = \frac{3}{2}$
  - ◦ $(6+x)^2\cdot P_3(x)$ is analytic at $x=-6$
    $\left((6+x)^2\cdot P_3(x)\right)\Big|_{x=-6} = 0$
  - ◦ $x=-6$is a regular singular point
    Check to see if x_0 is a regular singular point
    $x_0 = -6$
- Multiply by denominators
  $x(6+x)\left(\frac{d^2}{dx^2}y(x)\right) + (3x+9)\left(\frac{d}{dx}y(x)\right) - 3y(x) = 0$
- Change variables using $x = u-6$ so that the regular singular point is at $u=0$
  $(u^2-6u)\left(\frac{d^2}{du^2}y(u)\right) + (3u-9)\left(\frac{d}{du}y(u)\right) - 3y(u) = 0$
- Assume series solution for $y(u)$
  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$
- ☐ Rewrite ODE with series expansions
  - ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$
    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
  - ◦ Shift index using $k->k+1-m$
    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - ◦ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$-3a_0r(1+2r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-3a_{k+1}(k+1+r)(2k+3+2r) + a_k(k+r+3)(k+r-1)\right)u^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-3r(1+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{0, -\frac{1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $-6\left(k+r+\frac{3}{2}\right)(k+1+r)\,a_{k+1} + a_k(k+r+3)(k+r-1) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_k(k+r+3)(k+r-1)}{3(2k+3+2r)(k+1+r)}$
- Recursion relation for $r=0$; series terminates at $k=1$
  $a_{k+1} = \frac{a_k(k+3)(k-1)}{3(2k+3)(k+1)}$
- Apply recursion relation for $k=0$
  $a_1 = -\frac{a_0}{3}$
- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second linea
  $y(u) = a_0 \cdot \left(1-\frac{u}{3}\right)$
- Revert the change of variables $u = 6+x$
  $\left[y(x) = a_0\left(-1-\frac{x}{3}\right)\right]$
- Recursion relation for $r = -\frac{1}{2}$
  $a_{k+1} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{3(2k+2)\left(k+\frac{1}{2}\right)}$
- Solution for $r = -\frac{1}{2}$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{1}{2}}, a_{k+1} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{3(2k+2)\left(k+\frac{1}{2}\right)}\right]$
- Revert the change of variables $u = 6+x$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k(6+x)^{k-\frac{1}{2}}, a_{k+1} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{3(2k+2)\left(k+\frac{1}{2}\right)}\right]$
- Combine solutions and rename parameters
  $\left[y(x) = a_0\left(-1-\frac{x}{3}\right) + \left(\sum_{k=0}^{\infty} b_k(6+x)^{k-\frac{1}{2}}\right), b_{k+1} = \frac{b_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{3(2k+2)\left(k+\frac{1}{2}\right)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
```

```
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.027 (sec)
Leaf size : 30

```
dsolve((x^2+6*x)*diff(diff(y(x),x),x)+(3*x+9)*diff(y(x),x)-3*y(x) = 0,y(x),singsol=all
```

$$y = c_1(x+3) + \frac{c_2(2x^2+12x+9)}{\sqrt{x}\,\sqrt{6+x}}$$

Mathematica DSolve solution

Solving time : 0.673 (sec)
Leaf size : 103

```
DSolve[{(x^2+6*x)*D[y[x],{x,2}]+(3*x+9)*D[y[x],x]-3*y[x]==0,{}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to \frac{\left(9\sqrt{\pi}c_2\sqrt[4]{-x(x+6)}Q_{\frac{3}{2}}^{\frac{1}{2}}\left(\frac{x}{3}+1\right)+\sqrt{6}c_1(2x^2+12x+9)\right)\exp\left(\int_1^x -\frac{K[1]+3}{2K[1](K[1]+6)}dK[1]\right)}{9\sqrt{\pi}\sqrt[4]{-x(x+6)}}$$

2.1.409 Problem 421

Internal problem ID [9581]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 421
Date solved : `Monday, January 27, 2025 at 06:04:20 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$ty'' + (t^2 - 1)\, y' + t^3 y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.345 (sec)

Writing the ode as

$$ty'' + (t^2 - 1)\, y' + t^3 y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= t^2 - 1 \\ C &= t^3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3t^4 + 3}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3t^4 + 3 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{-3t^4 + 3}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.773: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3t^2}{4} + \frac{3}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{1} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{i\sqrt{3}\,t}{2} - \frac{i\sqrt{3}}{4t^3} - \frac{i\sqrt{3}}{16t^7} - \frac{i\sqrt{3}}{32t^{11}} - \frac{5i\sqrt{3}}{256t^{15}} - \frac{7i\sqrt{3}}{512t^{19}} - \frac{21i\sqrt{3}}{2048t^{23}} - \frac{33i\sqrt{3}}{4096t^{27}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{i\sqrt{3}}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i t^i \\ &= \frac{i\sqrt{3}\,t}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -\frac{3t^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-3t^4 + 3}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(-\frac{3t^2}{4}\right) + \left(\frac{3}{4t^2}\right) \\ &= -\frac{3t^2}{4} + \frac{3}{4t^2} \end{aligned}$$

We see that the coefficient of the term t in the quotient is 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{i\sqrt{3}\,t}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{\frac{i\sqrt{3}}{2}} - 1\right) &= -\frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{\frac{i\sqrt{3}}{2}} - 1\right) &= -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3t^4 + 3}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{i\sqrt{3}\,t}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2t} + (-)\left(\frac{i\sqrt{3}\,t}{2}\right) \\ &= -\frac{1}{2t} - \frac{i\sqrt{3}\,t}{2} \\ &= \frac{-i\sqrt{3}\,t^2 - 1}{2t} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2t} - \frac{i\sqrt{3}\,t}{2}\right)(0) + \left(\left(\frac{1}{2t^2} - \frac{i\sqrt{3}}{2}\right) + \left(-\frac{1}{2t} - \frac{i\sqrt{3}\,t}{2}\right)^2 - \left(\frac{-3t^4+3}{4t^2}\right) \right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega\, dt} \\ &= e^{\int \left(-\frac{1}{2t} - \frac{i\sqrt{3}\,t}{2}\right) dt} \\ &= \frac{e^{-\frac{i\sqrt{3}\,t^2}{4}}}{\sqrt{t}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{t^2-1}{t}\,dt} \\
&= z_1 e^{-\frac{t^2}{4}+\frac{\ln(t)}{2}} \\
&= z_1\left(\sqrt{t}\,\mathrm{e}^{-\frac{t^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{t^2\left(1+i\sqrt{3}\right)}{4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{t^2-1}{t}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-\frac{t^2}{2}+\ln(t)}}{(y_1)^2}\,dt \\
&= y_1\left(-\frac{i\sqrt{3}\,\mathrm{e}^{-\frac{t^2}{2}+\ln(t)}\mathrm{e}^{\frac{t^2\left(1+i\sqrt{3}\right)}{2}}}{3t}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{t^2\left(1+i\sqrt{3}\right)}{4}}\right) + c_2\left(\mathrm{e}^{-\frac{t^2\left(1+i\sqrt{3}\right)}{4}}\left(-\frac{i\sqrt{3}\,\mathrm{e}^{-\frac{t^2}{2}+\ln(t)}\mathrm{e}^{\frac{t^2\left(1+i\sqrt{3}\right)}{2}}}{3t}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dt^2}y(t)\right)t + (t^2-1)\left(\frac{d}{dt}y(t)\right) + t^3 y(t) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dt^2}y(t)$
- Isolate 2nd derivative
  $\frac{d^2}{dt^2}y(t) = -t^2 y(t) - \frac{(t^2-1)\left(\frac{d}{dt}y(t)\right)}{t}$
- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d^2}{dt^2}y(t) + \frac{(t^2-1)\left(\frac{d}{dt}y(t)\right)}{t} + t^2 y(t) = 0$
- ☐ Check to see if $t_0 = 0$ is a regular singular point
  - ◦ Define functions

$\left[P_2(t) = \frac{t^2-1}{t}, P_3(t) = t^2\right]$

- ◦ $t \cdot P_2(t)$ is analytic at $t = 0$

  $\left(t \cdot P_2(t)\right)\Big|_{t=0} = -1$

- ◦ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $\left(t^2 \cdot P_3(t)\right)\Big|_{t=0} = 0$

- ◦ $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $\left(\frac{d^2}{dt^2}y(t)\right)t + (t^2-1)\left(\frac{d}{dt}y(t)\right) + t^3y(t) = 0$

- Assume series solution for $y(t)$

  $y(t) = \sum_{k=0}^{\infty} a_k t^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $t^3 \cdot y(t)$ to series expansion

  $t^3 \cdot y(t) = \sum_{k=0}^{\infty} a_k t^{k+r+3}$

- ◦ Shift index using $k- >k-3$

  $t^3 \cdot y(t) = \sum_{k=3}^{\infty} a_{k-3} t^{k+r}$

- ◦ Convert $t^m \cdot \left(\frac{d}{dt}y(t)\right)$ to series expansion for $m = 0..2$

  $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$

- ◦ Shift index using $k- >k+1-m$

  $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$

- ◦ Convert $t \cdot \left(\frac{d^2}{dt^2}y(t)\right)$ to series expansion

  $t \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, t^{k+r-1}$

- ◦ Shift index using $k- >k+1$

  $t \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-2+r)\, t^{-1+r} + a_1(1+r)\,(-1+r)\, t^r + (a_2(2+r)\, r + a_0 r)\, t^{1+r} + (a_3(3+r)\,(1+r) + a_1(1+$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- The coefficients of each power of t must be 0

  $[a_1(1+r)\,(-1+r) = 0, a_2(2+r)\, r + a_0 r = 0, a_3(3+r)\,(1+r) + a_1(1+r) = 0]$

- Solve for the dependent coefficient(s)

  $\left\{a_1 = 0, a_2 = -\frac{a_0}{2+r}, a_3 = 0\right\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k+r-1) + a_{k-1}(k+r-1) + a_{k-3} = 0$

- Shift index using $k- >k+3$

$$a_{k+4}(k+4+r)(k+2+r)+a_{k+2}(k+2+r)+a_k=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+4}=-\frac{ka_{k+2}+ra_{k+2}+a_k+2a_{k+2}}{(k+4+r)(k+2+r)}$$

- Recursion relation for $r=0$

$$a_{k+4}=-\frac{ka_{k+2}+a_k+2a_{k+2}}{(k+4)(k+2)}$$

- Solution for $r=0$

$$\left[y(t)=\sum_{k=0}^{\infty}a_k t^k, a_{k+4}=-\frac{ka_{k+2}+a_k+2a_{k+2}}{(k+4)(k+2)}, a_1=0, a_2=-\frac{a_0}{2}, a_3=0\right]$$

- Recursion relation for $r=2$

$$a_{k+4}=-\frac{ka_{k+2}+a_k+4a_{k+2}}{(k+6)(k+4)}$$

- Solution for $r=2$

$$\left[y(t)=\sum_{k=0}^{\infty}a_k t^{k+2}, a_{k+4}=-\frac{ka_{k+2}+a_k+4a_{k+2}}{(k+6)(k+4)}, a_1=0, a_2=-\frac{a_0}{4}, a_3=0\right]$$

- Combine solutions and rename parameters

$$\left[y(t)=\left(\sum_{k=0}^{\infty}a_k t^k\right)+\left(\sum_{k=0}^{\infty}b_k t^{k+2}\right), a_{4+k}=-\frac{ka_{k+2}+a_k+2a_{k+2}}{(4+k)(k+2)}, a_1=0, a_2=-\frac{a_0}{2}, a_3=0, b_{4+k}=-\frac{kb_k}{(}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.013` (sec)
Leaf size : 34

```
dsolve(t*diff(diff(y(t),t),t)+(t^2-1)*diff(y(t),t)+t^3*y(t) = 0,y(t),singsol=all)
```

$$y=e^{-\frac{t^2}{4}}\left(c_1\cos\left(\frac{\sqrt{3}\,t^2}{4}\right)+c_2\sin\left(\frac{\sqrt{3}\,t^2}{4}\right)\right)$$

Mathematica DSolve solution

Solving time : `0.03` (sec)
Leaf size : 48

```
DSolve[{t*D[y[t],{t,2}]+(t^2-1)*D[y[t],t]+t^3*y[t]==0,{}},y[t],t,IncludeSingularSolutions->Tru
```

$$y(t)\to e^{-\frac{t^2}{4}}\left(c_2\cos\left(\frac{\sqrt{3}t^2}{4}\right)+c_1\sin\left(\frac{\sqrt{3}t^2}{4}\right)\right)$$

2.1.410 Problem 422

Internal problem ID [9582]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 422
Date solved : Monday, January 27, 2025 at 06:04:20 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$t^2y'' - t(t+2)\, y' + (t+2)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.073 (sec)

Writing the ode as

$$t^2y'' + (-t^2 - 2t)\, y' + (t+2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= -t^2 - 2t \\ C &= t+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.775: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = e^{-\frac{t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-t^2-2t}{t^2}\, dt} \\ &= z_1 e^{\frac{t}{2}+\ln(t)} \\ &= z_1 \left(t\, e^{\frac{t}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-t^2-2t}{t^2}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{t+2\ln(t)}}{(y_1)^2}\,dt \\
&= y_1 \left(\frac{e^{t+2\ln(t)}}{t^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(t) + c_2\left(t\left(\frac{e^{t+2\ln(t)}}{t^2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dt^2}y(t)\right)t^2 - t(t+2)\left(\frac{d}{dt}y(t)\right) + (t+2)\,y(t) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dt^2}y(t)$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = -\frac{(t+2)y(t)}{t^2} + \frac{(t+2)\left(\frac{d}{dt}y(t)\right)}{t}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dt^2}y(t) - \frac{(t+2)\left(\frac{d}{dt}y(t)\right)}{t} + \frac{(t+2)y(t)}{t^2} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(t) = -\frac{t+2}{t}, P_3(t) = \frac{t+2}{t^2}\right]$

  ◦ $t \cdot P_2(t)$ is analytic at $t = 0$

    $(t \cdot P_2(t))\Big|_{t=0} = -2$

  ◦ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

    $(t^2 \cdot P_3(t))\Big|_{t=0} = 2$

  ◦ $t = 0$is a regular singular point

    Check to see if $t_0 = 0$ is a regular singular point

    $t_0 = 0$

- Multiply by denominators

  $\left(\frac{d^2}{dt^2}y(t)\right)t^2 - t(t+2)\left(\frac{d}{dt}y(t)\right) + (t+2)\,y(t) = 0$

- Assume series solution for $y(t)$

  $y(t) = \sum_{k=0}^{\infty} a_k t^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $t^m \cdot y(t)$ to series expansion for $m = 0..1$

    $t^m \cdot y(t) = \sum_{k=0}^{\infty} a_k t^{k+r+m}$

- Shift index using $k->k-m$

  $t^m \cdot y(t) = \sum_{k=m}^{\infty} a_{k-m} t^{k+r}$
- Convert $t^m \cdot \left(\frac{d}{dt}y(t)\right)$ to series expansion for $m = 1..2$

  $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$
- Shift index using $k->k+1-m$

  $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$
- Convert $t^2 \cdot \left(\frac{d^2}{dt^2}y(t)\right)$ to series expansion

  $t^2 \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)(-2+r)\, t^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)(k+r-2) - a_{k-1}(k+r-2)\right) t^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)\left(a_k(k+r-1) - a_{k-1}\right) = 0$
- Shift index using $k->k+1$

  $(k+r-1)\left(a_{k+1}(k+r) - a_k\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+r}$
- Recursion relation for $r = 1$

  $a_{k+1} = \frac{a_k}{k+1}$
- Solution for $r = 1$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = \frac{a_k}{k+1}\right]$
- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+2}$
- Solution for $r = 2$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k+2}, a_{k+1} = \frac{a_k}{k+2}\right]$
- Combine solutions and rename parameters

  $\left[y(t) = \left(\sum_{k=0}^{\infty} a_k t^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
```

```
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 12

```
dsolve(t^2*diff(diff(y(t),t),t)-t*(t+2)*diff(y(t),t)+(t+2)*y(t) = 0,y(t),singsol=all)
```

$$y = t\left(c_1 + c_2 e^t\right)$$

Mathematica DSolve solution

Solving time : 0.048 (sec)
Leaf size : 17

```
DSolve[{t^2*D[y[t],{t,2}]-t*(t+2)*D[y[t],t]+(t+2)*y[t]==0,{}},y[t],t,IncludeSingularSolution
```

$$y(t) \to et\left(c_2 e^t + c_1\right)$$

2.1.411 Problem 423

Internal problem ID [9583]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 423
Date solved : `Monday, January 27, 2025 at 06:04:21 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(x-1)\,y''-xy'+y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.231 (sec)

Writing the ode as

$$(x-1)\,y''-xy'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x-1\\B&=-x\\C&=1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{x^2-4x+6}{4\,(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=x^2-4x+6\\t&=4(x-1)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x^2-4x+6}{4\,(x-1)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.777: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2\,(x-1)} + \frac{3}{4\,(x-1)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a = \frac{1}{2}
$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}
$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\
&= Q + \frac{R}{4x^2 - 8x + 4} \\
&= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\
&= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4}
\end{aligned}
$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (0) \\
&= -\frac{1}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 4x + 6}{4(x-1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(x-1)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2(x-1)} + \frac{1}{2} \\
&= \frac{x-2}{2x-2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2(x-1)}+\frac{1}{2}\right)(0)+\left(\left(\frac{1}{2(x-1)^2}\right)+\left(-\frac{1}{2(x-1)}+\frac{1}{2}\right)^2-\left(\frac{x^2-4x+6}{4(x-1)^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x-1)}+\frac{1}{2}\right)dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{x-1}\, dx} \\
&= z_1 e^{\frac{x}{2}+\frac{\ln(x-1)}{2}} \\
&= z_1\left(\sqrt{x-1}\,\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1\int \frac{e^{\int -\frac{-x}{x-1}\, dx}}{(y_1)^2}\, dx \\
&= y_1\int \frac{e^{x+\ln(x-1)}}{(y_1)^2}\, dx \\
&= y_1\left(-\frac{x\,\mathrm{e}^{x+\ln(x-1)}\mathrm{e}^{-2x}}{x-1}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x)+c_2\left(\mathrm{e}^x\left(-\frac{x\,\mathrm{e}^{x+\ln(x-1)}\mathrm{e}^{-2x}}{x-1}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x-1)\left(\frac{d^2}{dx^2}y(x)\right)-x\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{y(x)}{x-1}+\frac{\left(\frac{d}{dx}y(x)\right)x}{x-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)-\frac{\left(\frac{d}{dx}y(x)\right)x}{x-1}+\frac{y(x)}{x-1}=0$

□ Check to see if $x_0=1$ is a regular singular point

- ○ Define functions

  $\left[P_2(x)=-\frac{x}{x-1}, P_3(x)=\frac{1}{x-1}\right]$

- ○ $(x-1)\cdot P_2(x)$ is analytic at $x=1$

  $\left.((x-1)\cdot P_2(x))\right|_{x=1}=-1$

- ○ $(x-1)^2\cdot P_3(x)$ is analytic at $x=1$

  $\left.\left((x-1)^2\cdot P_3(x)\right)\right|_{x=1}=0$

- ○ $x=1$is a regular singular point

  Check to see if $x_0=1$ is a regular singular point

  $x_0=1$

- Multiply by denominators

$(x-1)\left(\frac{d^2}{dx^2}y(x)\right)-x\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Change variables using $x=u+1$ so that the regular singular point is at $u=0$

$u\left(\frac{d^2}{du^2}y(u)\right)+(-u-1)\left(\frac{d}{du}y(u)\right)+y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ○ Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-1}$

- ○ Shift index using $k->k+1$

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(k+r-1)-a_k(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r)=0$

- Values of r that satisfy the indicial equation
  $r\in\{0,2\}$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-1)\left(a_{k+1}(k+1+r)-a_k\right)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{a_k}{k+1+r}$
- Recursion relation for $r=0$
  $a_{k+1}=\frac{a_k}{k+1}$
- Solution for $r=0$
  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+1}=\frac{a_k}{k+1}\right]$
- Revert the change of variables $u=x-1$
  $\left[y(x)=\sum_{k=0}^{\infty} a_k(x-1)^k, a_{k+1}=\frac{a_k}{k+1}\right]$
- Recursion relation for $r=2$
  $a_{k+1}=\frac{a_k}{k+3}$
- Solution for $r=2$
  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$
- Revert the change of variables $u=x-1$
  $\left[y(x)=\sum_{k=0}^{\infty} a_k(x-1)^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$
- Combine solutions and rename parameters
  $\left[y(x)=\left(\sum_{k=0}^{\infty} a_k(x-1)^k\right)+\left(\sum_{k=0}^{\infty} b_k(x-1)^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+3}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 12

```
dsolve((x-1)*diff(diff(y(x),x),x)-diff(y(x),x)*x+y(x) = 0,y(x),singsol=all)
```

$$y=c_1x+e^x c_2$$

Mathematica DSolve solution

Solving time : 0.157 (sec)
Leaf size : 90

```
DSolve[{(x-1)*D[y[x],{x,2}]-x*D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \exp\left(\int_1^x \frac{K[1]-2}{2(K[1]-1)}dK[1] - \frac{1}{2}\int_1^x -\frac{K[2]}{K[2]-1}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{K[1]-2}{2(K[1]-1)}dK[1]\right)dK[3] + c_1\right)$$

2.1.412 Problem 424

Internal problem ID [9584]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 424
Date solved : `Monday, January 27, 2025 at 06:04:21 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - \left(x - \frac{3}{16}\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.150 (sec)

Writing the ode as

$$x^2y'' + \left(-x + \frac{3}{16}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 0 \\ C &= -x + \frac{3}{16} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{16x - 3}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 16x - 3 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{16x-3}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.779: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{x} - \frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{1,2,3\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1, \; e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(1 - (1)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{1}{(x-(0))}\right) \\ &= \frac{1}{2x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2x} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2x} + \frac{1-16x}{16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{1+4\sqrt{x}}{4x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= e^{\int \omega \, dx} \\ &= e^{\int \frac{1+4\sqrt{x}}{4x} dx} \\ &= x^{1/4} e^{2\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= x^{1/4} e^{2\sqrt{x}} \end{aligned}$$

Which simplifies to

$$y_1 = x^{1/4} e^{2\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= x^{1/4} e^{2\sqrt{x}} \int \frac{1}{\sqrt{x}\, e^{4\sqrt{x}}} \, dx \\ &= x^{1/4} e^{2\sqrt{x}} \left(-\frac{e^{-4\sqrt{x}}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(x^{1/4} e^{2\sqrt{x}}\right) + c_2 \left(x^{1/4} e^{2\sqrt{x}} \left(-\frac{e^{-4\sqrt{x}}}{2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2 \left(\frac{d^2}{dx^2} y(x)\right) - \left(x - \frac{3}{16}\right) y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2} y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2} y(x) = \frac{(16x-3) y(x)}{16x^2}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2} y(x) - \frac{(16x-3) y(x)}{16x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - ◦ Define functions

    $\left[P_2(x) = 0, P_3(x) = -\frac{16x-3}{16x^2}\right]$
  - ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 0$
  - ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{3}{16}$
  - ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $16x^2\left(\frac{d^2}{dx^2}y(x)\right) + (-16x + 3)\, y(x) = 0$
- Assume series solution for $y(x)$

  $y(x) = \sum\limits_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

    $x^m \cdot y(x) = \sum\limits_{k=0}^{\infty} a_k x^{k+r+m}$
  - ◦ Shift index using $k- >k - m$

    $x^m \cdot y(x) = \sum\limits_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - ◦ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum\limits_{k=0}^{\infty} a_k(k + r)(k + r - 1)\, x^{k+r}$

    Rewrite ODE with series expansions

    $a_0(-1 + 4r)(-3 + 4r)\, x^r + \left(\sum\limits_{k=1}^{\infty} \left(a_k(4k + 4r - 1)(4k + 4r - 3) - 16a_{k-1}\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1 + 4r)(-3 + 4r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{4}, \frac{3}{4}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $16\left(k + r - \frac{3}{4}\right)\left(k + r - \frac{1}{4}\right) a_k - 16a_{k-1} = 0$
- Shift index using $k- >k + 1$

  $16\left(k + \frac{1}{4} + r\right)\left(k + \frac{3}{4} + r\right) a_{k+1} - 16a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{16a_k}{(4k+1+4r)(4k+3+4r)}$
- Recursion relation for $r = \frac{1}{4}$

  $a_{k+1} = \frac{16a_k}{(4k+2)(4k+4)}$
- Solution for $r = \frac{1}{4}$

  $\left[y(x) = \sum\limits_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+1} = \frac{16a_k}{(4k+2)(4k+4)}\right]$
- Recursion relation for $r = \frac{3}{4}$

  $a_{k+1} = \frac{16a_k}{(4k+4)(4k+6)}$

- Solution for $r = \frac{3}{4}$

  $$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{4}}, a_{k+1} = \frac{16a_k}{(4k+4)(4k+6)}\right]$$

- Combine solutions and rename parameters

  $$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{4}}\right), a_{k+1} = \frac{16a_k}{(4k+2)(4k+4)}, b_{k+1} = \frac{16b_k}{(4k+4)(4k+6)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 25

```
dsolve(x^2*diff(diff(y(x),x),x)-(x-3/16)*y(x) = 0,y(x),singsol=all)
```

$$y = x^{1/4}\left(c_1 \sinh\left(2\sqrt{x}\right) + c_2 \cosh\left(2\sqrt{x}\right)\right)$$

Mathematica DSolve solution

Solving time : 0.047 (sec)
Leaf size : 41

```
DSolve[{x^2*D[y[x],{x,2}]-(x-1875/10000)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2} e^{-2\sqrt{x}} \sqrt[4]{x}\left(2c_1 e^{4\sqrt{x}} - c_2\right)$$

2.1.413 Problem 425

Internal problem ID [9585]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 425
Date solved : `Monday, January 27, 2025 at 06:04:22 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.135 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.781: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x}{x^2} dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\, dx \\
&= y_1(\tan(x))
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2\left(\frac{\cos(x)}{\sqrt{x}}(\tan(x))\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x\left(\frac{d}{dx}y(x)\right) + \left(x^2 - \frac{1}{4}\right)y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(4x^2-1)y(x)}{4x^2} - \frac{\frac{d}{dx}y(x)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{\frac{d}{dx}y(x)}{x} + \frac{(4x^2-1)y(x)}{4x^2} = 0$
- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$
  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$
- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x\left(\frac{d}{dx}y(x)\right) + (4x^2-1)y(x) = 0$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- □ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- Shift index using $k->k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$
- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$
- Shift index using $k->k+2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$
- Solution for $r = -\frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$
- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.038 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x+(x^2-1/4)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{\sin(x)\, c_1 + \cos(x)\, c_2}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.034 (sec)
Leaf size : 39

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-25/100)*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to \frac{e^{-ix}(2c_1 - ic_2e^{2ix})}{2\sqrt{x}}$$

2.1.414 Problem 426

Internal problem ID [9586]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 426
Date solved : `Monday, January 27, 2025 at 06:04:23 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$t^2y'' - t(t+2)\,y' + (t+2)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.069 (sec)

Writing the ode as

$$t^2y'' + \left(-t^2 - 2t\right)y' + (t+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= -t^2 - 2t \\ C &= t+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.783: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\frac{t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-t^2-2t}{t^2}\, dt} \\ &= z_1 e^{\frac{t}{2}+\ln(t)} \\ &= z_1 \left(t\, \mathrm{e}^{\frac{t}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-t^2-2t}{t^2}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{t+2\ln(t)}}{(y_1)^2}\,dt \\
&= y_1 \left(\frac{e^{t+2\ln(t)}}{t^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(t) + c_2\left(t\left(\frac{e^{t+2\ln(t)}}{t^2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dt^2}y(t)\right)t^2 - t(t+2)\left(\frac{d}{dt}y(t)\right) + (t+2)\,y(t) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dt^2}y(t)$
- Isolate 2nd derivative
  $\frac{d^2}{dt^2}y(t) = -\frac{(t+2)y(t)}{t^2} + \frac{(t+2)\left(\frac{d}{dt}y(t)\right)}{t}$
- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dt^2}y(t) - \frac{(t+2)\left(\frac{d}{dt}y(t)\right)}{t} + \frac{(t+2)y(t)}{t^2} = 0$
- ☐ Check to see if $t_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(t) = -\frac{t+2}{t}, P_3(t) = \frac{t+2}{t^2}\right]$
  - $t \cdot P_2(t)$ is analytic at $t = 0$
    $\left.(t \cdot P_2(t))\right|_{t=0} = -2$
  - $t^2 \cdot P_3(t)$ is analytic at $t = 0$
    $\left.(t^2 \cdot P_3(t))\right|_{t=0} = 2$
  - $t = 0$is a regular singular point
    Check to see if $t_0 = 0$ is a regular singular point
    $t_0 = 0$
- Multiply by denominators
  $\left(\frac{d^2}{dt^2}y(t)\right)t^2 - t(t+2)\left(\frac{d}{dt}y(t)\right) + (t+2)\,y(t) = 0$
- Assume series solution for $y(t)$
  $y(t) = \sum_{k=0}^{\infty} a_k t^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $t^m \cdot y(t)$ to series expansion for $m = 0..1$
    $t^m \cdot y(t) = \sum_{k=0}^{\infty} a_k t^{k+r+m}$

- Shift index using $k->k-m$

  $t^m \cdot y(t) = \sum_{k=m}^{\infty} a_{k-m} t^{k+r}$

- Convert $t^m \cdot \left(\frac{d}{dt}y(t)\right)$ to series expansion for $m = 1..2$

  $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$

- Convert $t^2 \cdot \left(\frac{d^2}{dt^2}y(t)\right)$ to series expansion

  $t^2 \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)(-2+r)\, t^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)(k+r-2) - a_{k-1}(k+r-2)\right) t^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)\left(a_k(k+r-1) - a_{k-1}\right) = 0$

- Shift index using $k->k+1$

  $(k+r-1)\left(a_{k+1}(k+r) - a_k\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+r}$

- Recursion relation for $r = 1$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 1$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = \frac{a_k}{k+1}\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+2}$

- Solution for $r = 2$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k+2}, a_{k+1} = \frac{a_k}{k+2}\right]$

- Combine solutions and rename parameters

  $\left[y(t) = \left(\sum_{k=0}^{\infty} a_k t^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
```

```
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 12

```
dsolve(t^2*diff(diff(y(t),t),t)-t*(t+2)*diff(y(t),t)+(t+2)*y(t) = 0,y(t),singsol=all)
```

$$y = t\left(c_1 + c_2 \mathrm{e}^{t}\right)$$

Mathematica DSolve solution

Solving time : 0.037 (sec)
Leaf size : 17

```
DSolve[{t^2*D[y[t],{t,2}]-t*(t+2)*D[y[t],t]+(t+2)*y[t] == 0,{}},y[t],t,IncludeSingularSoluti
```

$$y(t) \to et\left(c_2 e^t + c_1\right)$$

2.1.415 Problem 427

Internal problem ID [9587]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 427
Date solved : `Monday, January 27, 2025 at 06:04:23 PM`
CAS classification : `[_Laguerre]`

Solve

$$ty'' - (1+t)\,y' + y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.210 (sec)

Writing the ode as

$$ty'' + (-1-t)\,y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= -1-t \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 - 2t + 3}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^2 - 2t + 3 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 - 2t + 3}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.785: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2t} + \frac{3}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2t} + \frac{1}{2t^2} + \frac{1}{2t^3} + \frac{1}{4t^4} - \frac{1}{4t^5} - \frac{3}{4t^6} - \frac{3}{4t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 - 2t + 3}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2t + 3}{4t^2}\right) \\ &= \frac{1}{4} + \frac{-2t + 3}{4t^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 - 2t + 3}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right)\\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right)\\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (+)[\sqrt{r}]_\infty\\
&= -\frac{1}{2t} + \left(\frac{1}{2}\right)\\
&= \frac{1}{2} - \frac{1}{2t}\\
&= \frac{t-1}{2t}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d=0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2} - \frac{1}{2t}\right)(0) + \left(\left(\frac{1}{2t^2}\right) + \left(\frac{1}{2} - \frac{1}{2t}\right)^2 - \left(\frac{t^2-2t+3}{4t^2}\right) \right) &= 0\\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt}\\
&= e^{\int \left(\frac{1}{2} - \frac{1}{2t}\right) dt}\\
&= \frac{e^{\frac{t}{2}}}{\sqrt{t}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1-t}{t}\,dt} \\
&= z_1 e^{\frac{t}{2}+\frac{\ln(t)}{2}} \\
&= z_1 \left(\sqrt{t}\,\mathrm{e}^{\frac{t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{t}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1-t}{t}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{t+\ln(t)}}{(y_1)^2}\,dt \\
&= y_1 \left(-\frac{(1+t)\,\mathrm{e}^{t+\ln(t)}\mathrm{e}^{-2t}}{t}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{t}\right) + c_2\left(\mathrm{e}^{t}\left(-\frac{(1+t)\,\mathrm{e}^{t+\ln(t)}\mathrm{e}^{-2t}}{t}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dt^2}y(t)\right)t-(t+1)\left(\frac{d}{dt}y(t)\right)+y(t)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dt^2}y(t)$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t)=-\frac{y(t)}{t}+\frac{(t+1)\left(\frac{d}{dt}y(t)\right)}{t}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d^2}{dt^2}y(t)-\frac{(t+1)\left(\frac{d}{dt}y(t)\right)}{t}+\frac{y(t)}{t}=0$

- ☐ Check to see if $t_0=0$ is a regular singular point
  - Define functions

    $\left[P_2(t)=-\frac{t+1}{t}, P_3(t)=\frac{1}{t}\right]$
  - $t\cdot P_2(t)$ is analytic at $t=0$

$$\left(t \cdot P_2(t)\right)\bigg|_{t=0} = -1$$

- ◦ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

$$\left(t^2 \cdot P_3(t)\right)\bigg|_{t=0} = 0$$

- ◦ $t = 0$is a regular singular point

Check to see if $t_0 = 0$ is a regular singular point

$t_0 = 0$

- Multiply by denominators

$$\left(\frac{d^2}{dt^2}y(t)\right)t + (-t-1)\left(\frac{d}{dt}y(t)\right) + y(t) = 0$$

- Assume series solution for $y(t)$

$$y(t) = \sum_{k=0}^{\infty} a_k t^{k+r}$$

- □ Rewrite ODE with series expansions
  - ◦ Convert $t^m \cdot \left(\frac{d}{dt}y(t)\right)$ to series expansion for $m = 0..1$

$$t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$$

  - ◦ Shift index using $k- >k+1-m$

$$t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$$

  - ◦ Convert $t \cdot \left(\frac{d^2}{dt^2}y(t)\right)$ to series expansion

$$t \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, t^{k+r-1}$$

  - ◦ Shift index using $k- >k+1$

$$t \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, t^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-2+r)\, t^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right) t^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

$(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = \frac{a_k}{k+1+r}$

- Recursion relation for $r = 0$

$a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 0$

$$\left[y(t) = \sum_{k=0}^{\infty} a_k t^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 2$

$a_{k+1} = \frac{a_k}{k+3}$

- Solution for $r = 2$

$$\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y(t)=\left(\sum_{k=0}^{\infty}a_k t^k\right)+\left(\sum_{k=0}^{\infty}b_k t^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+3}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 13

```
dsolve(t*diff(diff(y(t),t),t)-(t+1)*diff(y(t),t)+y(t) = 0,y(t),singsol=all)
```

$$y=c_2 e^{t}+c_1 t+c_1$$

Mathematica DSolve solution

Solving time : 0.406 (sec)
Leaf size : 78

```
DSolve[{t*D[y[t],{t,2}]-(1+t)*D[y[t],t]+y[t] == 0,{}},y[t],t,IncludeSingularSolutions->True]
```

$$y(t)\to \sqrt{t}\exp\left(\frac{1}{2}\left(2\int_1^t \frac{K[1]-1}{2K[1]}dK[1]+t+1\right)\right)\left(c_2\int_1^t \exp\left(-2\int_1^{K[2]}\frac{K[1]-1}{2K[1]}dK[1]\right)dK[2]+c_1\right)$$

2.1.416 Problem 428

Internal problem ID [9588]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 428
Date solved : `Monday, January 27, 2025 at 06:04:24 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(1-t)\,y''+ty'-y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.224 (sec)

Writing the ode as

$$(1-t)\,y''+ty'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1-t \\ B &= t \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=y e^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{t^2-4t+6}{4\left(-1+t\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^2-4t+6 \\ t &= 4(-1+t)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t)=\left(\frac{t^2-4t+6}{4\left(-1+t\right)^2}\right)z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.787: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(-1+t)^2$. There is a pole at $t = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4(-1+t)^2} - \frac{1}{2(-1+t)}$$

For the pole at $t = 1$ let b be the coefficient of $\frac{1}{(-1+t)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2t} + \frac{1}{t^3} + \frac{11}{4t^4} + \frac{21}{4t^5} + \frac{15}{2t^6} + \frac{6}{t^7} - \frac{117}{16t^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 - 4t + 6}{4t^2 - 8t + 4} \\ &= Q + \frac{R}{4t^2 - 8t + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2t + 5}{4t^2 - 8t + 4}\right) \\ &= \frac{1}{4} + \frac{-2t + 5}{4t^2 - 8t + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{t^2 - 4t + 6}{4\left(-1+t\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(-1+t\right)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2\left(-1+t\right)} + \frac{1}{2} \\
&= \frac{t-2}{2t-2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0)+2\left(-\frac{1}{2\left(-1+t\right)}+\frac{1}{2}\right)(0)+\left(\left(\frac{1}{2\left(-1+t\right)^{2}}\right)+\left(-\frac{1}{2\left(-1+t\right)}+\frac{1}{2}\right)^{2}-\left(\frac{t^{2}-4t+6}{4\left(-1+t\right)^{2}}\right)\right)=0
$$
$$
0=0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(-1+t)}+\frac{1}{2}\right)dt} \\
&= \frac{\mathrm{e}^{\frac{t}{2}}}{\sqrt{-1+t}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{t}{1-t}\, dt} \\
&= z_1 e^{\frac{t}{2}+\frac{\ln(-1+t)}{2}} \\
&= z_1\left(\sqrt{-1+t}\,\mathrm{e}^{\frac{t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{t}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{t}{1-t}\, dt}}{\left(y_1\right)^2}\, dt \\
&= y_1 \int \frac{e^{t+\ln(-1+t)}}{\left(y_1\right)^2}\, dt \\
&= y_1\left(-\frac{t\,\mathrm{e}^{t+\ln(-1+t)}\mathrm{e}^{-2t}}{-1+t}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{t}\right)+c_2\left(\mathrm{e}^{t}\left(-\frac{t\,\mathrm{e}^{t+\ln(-1+t)}\mathrm{e}^{-2t}}{-1+t}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(1-t)\left(\frac{d^2}{dt^2}y(t)\right)+t\left(\frac{d}{dt}y(t)\right)-y(t)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dt^2}y(t)$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t)=-\frac{y(t)}{t-1}+\frac{\left(\frac{d}{dt}y(t)\right)t}{t-1}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d^2}{dt^2}y(t)-\frac{\left(\frac{d}{dt}y(t)\right)t}{t-1}+\frac{y(t)}{t-1}=0$

- □ Check to see if $t_0=1$ is a regular singular point
  - Define functions

    $\left[P_2(t)=-\frac{t}{t-1},P_3(t)=\frac{1}{t-1}\right]$
  - $(t-1)\cdot P_2(t)$ is analytic at $t=1$

    $\left((t-1)\cdot P_2(t)\right)\Big|_{t=1}=-1$
  - $(t-1)^2\cdot P_3(t)$ is analytic at $t=1$

    $\left((t-1)^2\cdot P_3(t)\right)\Big|_{t=1}=0$
  - $t=1$is a regular singular point

  Check to see if $t_0=1$ is a regular singular point

  $t_0=1$

- Multiply by denominators

  $(t-1)\left(\frac{d^2}{dt^2}y(t)\right)-t\left(\frac{d}{dt}y(t)\right)+y(t)=0$

- Change variables using $t=u+1$ so that the regular singular point is at $u=0$

  $u\left(\frac{d^2}{du^2}y(u)\right)+(-u-1)\left(\frac{d}{du}y(u)\right)+y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

    $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-1}$
  - Shift index using $k->k+1$

    $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)(k+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)(k+r-1)-a_k(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r)=0$

- Values of r that satisfy the indicial equation

$r\in\{0,2\}$

- Each term in the series must be 0, giving the recursion relation

$(k+r-1)\left(a_{k+1}(k+1+r)-a_k\right)=0$

- Recursion relation that defines series solution to ODE

$a_{k+1}=\frac{a_k}{k+1+r}$

- Recursion relation for $r=0$

$a_{k+1}=\frac{a_k}{k+1}$

- Solution for $r=0$

$\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+1}=\frac{a_k}{k+1}\right]$

- Revert the change of variables $u=t-1$

$\left[y(t)=\sum_{k=0}^{\infty} a_k(t-1)^k, a_{k+1}=\frac{a_k}{k+1}\right]$

- Recursion relation for $r=2$

$a_{k+1}=\frac{a_k}{k+3}$

- Solution for $r=2$

$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$

- Revert the change of variables $u=t-1$

$\left[y(t)=\sum_{k=0}^{\infty} a_k(t-1)^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$

- Combine solutions and rename parameters

$\left[y(t)=\left(\sum_{k=0}^{\infty} a_k(t-1)^k\right)+\left(\sum_{k=0}^{\infty} b_k(t-1)^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+3}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 12

```
dsolve((1-t)*diff(diff(y(t),t),t)+t*diff(y(t),t)-y(t) = 0,y(t),singsol=all)
```

$$y=c_1t+c_2\mathrm{e}^t$$

Mathematica DSolve solution

Solving time : 0.151 (sec)
Leaf size : 90

```
DSolve[{(1-t)*D[y[t],{t,2}]+t*D[y[t],t]-y[t] == 0,{}},y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \exp\left(\int_1^t \frac{K[1]-2}{2(K[1]-1)}dK[1] - \frac{1}{2}\int_1^t -\frac{K[2]}{K[2]-1}dK[2]\right)\left(c_2\int_1^t \exp\left(-2\int_1^{K[3]} \frac{K[1]-2}{2(K[1]-1)}dK[1]\right)dK[3] + c_1\right)$$

2.1.417 Problem 429

Internal problem ID [9589]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 429
Date solved : `Monday, January 27, 2025 at 06:04:24 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.142 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.789: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\, dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos(x)}{\sqrt{x}}(\tan(x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x\left(\frac{d}{dx}y(x)\right) + \left(x^2 - \frac{1}{4}\right)y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(4x^2-1)y(x)}{4x^2} - \frac{\frac{d}{dx}y(x)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{\frac{d}{dx}y(x)}{x} + \frac{(4x^2-1)y(x)}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x\left(\frac{d}{dx}y(x)\right) + (4x^2 - 1)y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx} y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-2})\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$

- Shift index using $k- >k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$

- Solution for $r = -\frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$

- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.039 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x+(x^2-1/4)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{\sin(x) c_1 + \cos(x) c_2}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.032 (sec)
Leaf size : 39

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-25/100)*y[x] == 0,{}},y[x],x,IncludeSingularSolut
```

$$y(x) \to \frac{e^{-ix}(2c_1 - ic_2 e^{2ix})}{2\sqrt{x}}$$

2.1.418 Problem 430

Internal problem ID [9590]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 430
Date solved : `Monday, January 27, 2025 at 06:04:25 PM`
CAS classification : `[_Laguerre]`

Solve

$$
ty'' - (1+t)\,y' + y = 0
$$

Solved as second order ode using Kovacic algorithm

Time used: 0.215 (sec)

Writing the ode as

$$
\begin{align*}
ty'' + (-1-t)\,y' + y &= 0 \tag{1}\\
Ay'' + By' + Cy &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= t \\
B &= -1-t \tag{3}\\
C &= 1
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(t) = ye^{\int \frac{B}{2A}\,dt}
$$

Then (2) becomes

$$
z''(t) = rz(t) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5}\\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{t^2 - 2t + 3}{4t^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align*}
s &= t^2 - 2t + 3\\
t &= 4t^2
\end{align*}
$$

Therefore eq. (4) becomes

$$
z''(t) = \left(\frac{t^2 - 2t + 3}{4t^2}\right) z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.791: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2t} + \frac{3}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2t} + \frac{1}{2t^2} + \frac{1}{2t^3} + \frac{1}{4t^4} - \frac{1}{4t^5} - \frac{3}{4t^6} - \frac{3}{4t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 - 2t + 3}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2t + 3}{4t^2}\right) \\ &= \frac{1}{4} + \frac{-2t + 3}{4t^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 - 2t + 3}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{2t} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2} - \frac{1}{2t} \\ &= \frac{t-1}{2t} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2} - \frac{1}{2t}\right)(0) + \left(\left(\frac{1}{2t^2}\right) + \left(\frac{1}{2} - \frac{1}{2t}\right)^2 - \left(\frac{t^2-2t+3}{4t^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= e^{\int \left(\frac{1}{2} - \frac{1}{2t}\right) dt} \\ &= \frac{e^{\frac{t}{2}}}{\sqrt{t}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-1-t}{t}\,dt} \\ &= z_1 e^{\frac{t}{2}+\frac{\ln(t)}{2}} \\ &= z_1\left(\sqrt{t}\,\mathrm{e}^{\frac{t}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-1-t}{t}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{t+\ln(t)}}{(y_1)^2}\,dt \\ &= y_1\left(-\frac{(1+t)\,\mathrm{e}^{t+\ln(t)}\mathrm{e}^{-2t}}{t}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^t\right) + c_2\left(\mathrm{e}^t\left(-\frac{(1+t)\,\mathrm{e}^{t+\ln(t)}\mathrm{e}^{-2t}}{t}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dt^2}y(t)\right)t-(t+1)\left(\frac{d}{dt}y(t)\right)+y(t)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dt^2}y(t)$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t)=-\frac{y(t)}{t}+\frac{(t+1)\left(\frac{d}{dt}y(t)\right)}{t}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d^2}{dt^2}y(t)-\frac{(t+1)\left(\frac{d}{dt}y(t)\right)}{t}+\frac{y(t)}{t}=0$

- ☐ Check to see if $t_0=0$ is a regular singular point
  - Define functions

    $\left[P_2(t)=-\frac{t+1}{t}, P_3(t)=\frac{1}{t}\right]$

  - $t\cdot P_2(t)$ is analytic at $t=0$

$\left(t \cdot P_2(t)\right)\Big|_{t=0} = -1$

- $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $\left(t^2 \cdot P_3(t)\right)\Big|_{t=0} = 0$

- $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $\left(\frac{d^2}{dt^2}y(t)\right)t + (-t-1)\left(\frac{d}{dt}y(t)\right) + y(t) = 0$

- Assume series solution for $y(t)$

  $y(t) = \sum_{k=0}^{\infty} a_k t^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $t^m \cdot \left(\frac{d}{dt}y(t)\right)$ to series expansion for $m = 0..1$

    $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$

  - Convert $t \cdot \left(\frac{d^2}{dt^2}y(t)\right)$ to series expansion

    $t \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, t^{k+r-1}$

  - Shift index using $k- >k+1$

    $t \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-2+r)\, t^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right) t^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1+r}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 0$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^k, a_{k+1} = \frac{a_k}{k+1}\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+3}$

- Solution for $r = 2$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$

- Combine solutions and rename parameters

$$\left[y(t)=\left(\sum_{k=0}^{\infty} a_k t^k\right)+\left(\sum_{k=0}^{\infty} b_k t^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+3}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 13

```
dsolve(t*diff(diff(y(t),t),t)-(t+1)*diff(y(t),t)+y(t) = 0,y(t),singsol=all)
```

$$y = c_2 \mathrm{e}^{t}+c_1 t+c_1$$

Mathematica DSolve solution

Solving time : 0.388 (sec)
Leaf size : 78

```
DSolve[{t*D[y[t],{t,2}]-(1+t)*D[y[t],t]+y[t] ==0,{}},y[t],t,IncludeSingularSolutions->True]
```

$$y(t)\to \sqrt{t}\exp\left(\frac{1}{2}\left(2\int_1^t \frac{K[1]-1}{2K[1]}dK[1]+t+1\right)\right)\left(c_2\int_1^t \exp\left(-2\int_1^{K[2]}\frac{K[1]-1}{2K[1]}dK[1]\right)dK[2]+c_1\right)$$

2.1.419 Problem 431

Internal problem ID [9591]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 431
Date solved : `Monday, January 27, 2025 at 06:04:25 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(1-t)\, y''+t y'-y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.220 (sec)

Writing the ode as

$$(1-t)\, y''+t y'-y=0 \tag{1}$$
$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1-t \\ B &= t \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=y e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t)=r z(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{t^2-4t+6}{4\left(-1+t\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^2-4t+6 \\ t &= 4(-1+t)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t)=\left(\frac{t^2-4t+6}{4\left(-1+t\right)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.793: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(-1+t)^2$. There is a pole at $t = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4\,(-1+t)^2} - \frac{1}{2\,(-1+t)}$$

For the pole at $t = 1$ let b be the coefficient of $\frac{1}{(-1+t)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\
&= \sum_{i=0}^{0} a_i t^i \qquad (8)
\end{aligned}
$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r} \approx \frac{1}{2} - \frac{1}{2t} + \frac{1}{t^3} + \frac{11}{4t^4} + \frac{21}{4t^5} + \frac{15}{2t^6} + \frac{6}{t^7} - \frac{117}{16t^8} + \dots \qquad (9)
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a = \frac{1}{2}
$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\
&= \frac{1}{2} \qquad (10)
\end{aligned}
$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}
$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{t^2 - 4t + 6}{4t^2 - 8t + 4} \\
&= Q + \frac{R}{4t^2 - 8t + 4} \\
&= \left(\frac{1}{4}\right) + \left(\frac{-2t + 5}{4t^2 - 8t + 4}\right) \\
&= \frac{1}{4} + \frac{-2t + 5}{4t^2 - 8t + 4}
\end{aligned}
$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (0) \\
&= -\frac{1}{2}
\end{aligned}
$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 - 4t + 6}{4\left(-1+t\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1}\right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(-1+t\right)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2\left(-1+t\right)} + \frac{1}{2} \\
&= \frac{t-2}{2t-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(-1+t)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2(-1+t)^2}\right) + \left(-\frac{1}{2(-1+t)} + \frac{1}{2}\right)^2 - \left(\frac{t^2-4t+6}{4(-1+t)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega\, dt} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(-1+t)} + \frac{1}{2}\right) dt} \\
&= \frac{\mathrm{e}^{\frac{t}{2}}}{\sqrt{-1+t}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{t}{1-t}\, dt} \\
&= z_1 e^{\frac{t}{2} + \frac{\ln(-1+t)}{2}} \\
&= z_1 \left(\sqrt{-1+t}\,\mathrm{e}^{\frac{t}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{t}{1-t}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{t+\ln(-1+t)}}{(y_1)^2}\, dt \\
&= y_1 \left(-\frac{t\,\mathrm{e}^{t+\ln(-1+t)}\mathrm{e}^{-2t}}{-1+t}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^t\right) + c_2 \left(\mathrm{e}^t \left(-\frac{t\,\mathrm{e}^{t+\ln(-1+t)}\mathrm{e}^{-2t}}{-1+t}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(1-t)\left(\frac{d^2}{dt^2}y(t)\right)+t\left(\frac{d}{dt}y(t)\right)-y(t)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dt^2}y(t)$

- Isolate 2nd derivative

$\frac{d^2}{dt^2}y(t)=-\frac{y(t)}{t-1}+\frac{\left(\frac{d}{dt}y(t)\right)t}{t-1}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d^2}{dt^2}y(t)-\frac{\left(\frac{d}{dt}y(t)\right)t}{t-1}+\frac{y(t)}{t-1}=0$

□ Check to see if $t_0=1$ is a regular singular point

  ◦ Define functions

  $\left[P_2(t)=-\frac{t}{t-1}, P_3(t)=\frac{1}{t-1}\right]$

  ◦ $(t-1)\cdot P_2(t)$ is analytic at $t=1$

  $\left.((t-1)\cdot P_2(t))\right|_{t=1}=-1$

  ◦ $(t-1)^2\cdot P_3(t)$ is analytic at $t=1$

  $\left.((t-1)^2\cdot P_3(t))\right|_{t=1}=0$

  ◦ $t=1$is a regular singular point

  Check to see if $t_0=1$ is a regular singular point

  $t_0=1$

- Multiply by denominators

$(t-1)\left(\frac{d^2}{dt^2}y(t)\right)-t\left(\frac{d}{dt}y(t)\right)+y(t)=0$

- Change variables using $t=u+1$ so that the regular singular point is at $u=0$

$u\left(\frac{d^2}{du^2}y(u)\right)+(-u-1)\left(\frac{d}{du}y(u)\right)+y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  ◦ Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-1}$

  ◦ Shift index using $k->k+1$

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)(k+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)(k+r-1)-a_k(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-2+r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\{0,2\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)\left(a_{k+1}(k+1+r)-a_k\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{a_k}{k+1+r}$$

- Recursion relation for $r=0$

$$a_{k+1}=\frac{a_k}{k+1}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty}a_k u^k,a_{k+1}=\frac{a_k}{k+1}\right]$$

- Revert the change of variables $u=t-1$

$$\left[y(t)=\sum_{k=0}^{\infty}a_k(t-1)^k,a_{k+1}=\frac{a_k}{k+1}\right]$$

- Recursion relation for $r=2$

$$a_{k+1}=\frac{a_k}{k+3}$$

- Solution for $r=2$

$$\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+2},a_{k+1}=\frac{a_k}{k+3}\right]$$

- Revert the change of variables $u=t-1$

$$\left[y(t)=\sum_{k=0}^{\infty}a_k(t-1)^{k+2},a_{k+1}=\frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y(t)=\left(\sum_{k=0}^{\infty}a_k(t-1)^k\right)+\left(\sum_{k=0}^{\infty}b_k(t-1)^{k+2}\right),a_{k+1}=\frac{a_k}{k+1},b_{k+1}=\frac{b_k}{k+3}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 12

```
dsolve((1-t)*diff(diff(y(t),t),t)+t*diff(y(t),t)-y(t) = 0,y(t),singsol=all)
```

$$y=c_1t+c_2\mathrm{e}^t$$

Mathematica DSolve solution

Solving time : 0.148 (sec)
Leaf size : 90

```
DSolve[{(1-t)*D[y[t],{t,2}]+t*D[y[t],t]-y[t] ==0,{}},y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \exp\left(\int_1^t \frac{K[1]-2}{2(K[1]-1)}dK[1] - \frac{1}{2}\int_1^t -\frac{K[2]}{K[2]-1}dK[2]\right)\left(c_2\int_1^t \exp\left(-2\int_1^{K[3]} \frac{K[1]-2}{2(K[1]-1)}dK[1]\right)dK[3] + c_1\right)$$

2.1.420 Problem 432

Internal problem ID [9592]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 432
Date solved : `Monday, January 27, 2025 at 06:04:26 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + xy' + 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.208 (sec)

Writing the ode as

$$y'' + xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 6 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.795: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{2x} - \frac{9}{4x^3} - \frac{27}{4x^5} - \frac{405}{16x^7} - \frac{1701}{16x^9} - \frac{15309}{32x^{11}} - \frac{72171}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 6}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} - \frac{3}{2}\right) + (0) \\ &= \frac{x^2}{4} - \frac{3}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{2}\right) - (0) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{3}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned}d &= \alpha_\infty^-\\ &= 1\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= (-)[\sqrt{r}]_\infty\\ &= 0 + (-)\left(\frac{x}{2}\right)\\ &= -\frac{x}{2}\\ &= -\frac{x}{2}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{3}{2}\right)\right) &= 0\\ a_0 &= 0\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= pe^{\int \omega\,dx}\\ &= (x)\,e^{\int -\frac{x}{2}dx}\\ &= (x)\,e^{-\frac{x^2}{4}}\\ &= x\,e^{-\frac{x^2}{4}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{1}\,dx}\\ &= z_1 e^{-\frac{x^2}{4}}\\ &= z_1\left(e^{-\frac{x^2}{4}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^2}{2}} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x^2}{2}} x\right) + c_2 \left(\mathrm{e}^{-\frac{x^2}{2}} x \left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} \right) \right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + x\left(\frac{d}{dx}y(x)\right) + 2y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

  $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k->k+2$

  $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+2)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

$$(k+2)\left(k a_{k+2}+a_k+a_{k+2}\right)=0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y(x)=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=-\frac{a_k}{k+1}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 34

```
dsolve(diff(diff(y(x),x),x)+diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y=x\left(i c_2\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)+c_1\right)\mathrm{e}^{-\frac{x^2}{2}}+2c_2$$

Mathematica DSolve solution

Solving time : `0.049` (sec)
Leaf size : 69

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to-\sqrt{\frac{\pi}{2}}c_2 e^{-\frac{x^2}{2}}\sqrt{x^2}\operatorname{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{2}}\right)+\sqrt{2}c_1 e^{-\frac{x^2}{2}}x+c_2$$

2.1.421 Problem 433

Internal problem ID [9593]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 433
Date solved : Monday, January 27, 2025 at 06:04:27 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$\left(x^2+1\right) y''-4xy'+6y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.247 (sec)

Writing the ode as

$$\left(x^2+1\right) y''-4xy'+6y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= -4x \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-8}{\left(x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -8 \\ t &= \left(x^2+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{8}{\left(x^2+1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.797: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 + 1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{(x - i)^2} + \frac{2}{(x + i)^2} + \frac{2i}{x - i} - \frac{2i}{x + i}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $4>2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{8}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
i	2	0	2	-1
$-i$	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=1$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x-i} + \frac{2}{x+i} + (-)\,(0) \\
&= -\frac{1}{x-i} + \frac{2}{x+i} \\
&= \frac{x-3i}{x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x-i} + \frac{2}{x+i}\right)(0) + \left(\left(\frac{1}{(x-i)^2} - \frac{2}{(x+i)^2}\right) + \left(-\frac{1}{x-i} + \frac{2}{x+i}\right)^2 - \left(-\frac{8}{(x^2+1)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{x-i}+\frac{2}{x+i}\right) dx} \\
&= \frac{(x^2+1)^2}{(ix+1)^3}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x}{x^2+1} \, dx} \\
&= z_1 e^{\ln(x^2+1)} \\
&= z_1 (x^2+1)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(x^2+1)^3}{(ix+1)^3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{x^2+1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{2\ln(x^2+1)}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{x^2 - \frac{1}{3}}{(x+i)^3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x^2+1)^3}{(ix+1)^3}\right) + c_2 \left(\frac{(x^2+1)^3}{(ix+1)^3} \left(\frac{x^2 - \frac{1}{3}}{(x+i)^3}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 21

```
dsolve((x^2+1)*diff(diff(y(x),x),x)-4*diff(y(x),x)*x+6*y(x) = 0,y(x),singsol=all)
```

$$y = c_2x^3 - 3c_1x^2 - 3c_2x + c_1$$

Mathematica DSolve solution

Solving time : 0.253 (sec)
Leaf size : 75

```
DSolve[{(1+x^2)*D[y[x],{x,2}]-4*x*D[y[x],x]+6*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to (x^2+1) \exp\left(\int_1^x \frac{K[1]+3i}{K[1]^2+1}dK[1]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[2]} \frac{K[1]+3i}{K[1]^2+1}dK[1]\right)dK[2] + c_1\right)$$

2.1.422 Problem 434

Internal problem ID [9594]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 434
Date solved : `Monday, January 27, 2025 at 06:04:27 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(1-x)\,y''+xy'-y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.234 (sec)

Writing the ode as

$$(1-x)\,y''+xy'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=1-x\\ B&=x\\ C&=-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{x^2-4x+6}{4\,(-1+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=x^2-4x+6\\ t&=4(-1+x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x^2-4x+6}{4\,(-1+x)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.798: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(-1+x)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4(-1+x)^2} - \frac{1}{2(-1+x)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 6}{4(-1+x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(-1+x)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2(-1+x)} + \frac{1}{2} \\
&= \frac{x-2}{2x-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2(-1+x)}+\frac{1}{2}\right)(0)+\left(\left(\frac{1}{2(-1+x)^2}\right)+\left(-\frac{1}{2(-1+x)}+\frac{1}{2}\right)^2-\left(\frac{x^2-4x+6}{4(-1+x)^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(-1+x)}+\frac{1}{2}\right)dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{-1+x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{1-x}\, dx} \\
&= z_1 e^{\frac{x}{2}+\frac{\ln(-1+x)}{2}} \\
&= z_1\left(\sqrt{-1+x}\,\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1\int \frac{e^{\int -\frac{x}{1-x}\, dx}}{(y_1)^2}\, dx \\
&= y_1\int \frac{e^{x+\ln(-1+x)}}{(y_1)^2}\, dx \\
&= y_1\left(-\frac{x\,\mathrm{e}^{x+\ln(-1+x)}\mathrm{e}^{-2x}}{-1+x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1y_1+c_2y_2 \\
&= c_1(\mathrm{e}^x)+c_2\left(\mathrm{e}^x\left(-\frac{x\,\mathrm{e}^{x+\ln(-1+x)}\mathrm{e}^{-2x}}{-1+x}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(1-x)\left(\frac{d^2}{dx^2}y(x)\right)+x\left(\frac{d}{dx}y(x)\right)-y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{y(x)}{x-1}+\frac{\left(\frac{d}{dx}y(x)\right)x}{x-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)-\frac{\left(\frac{d}{dx}y(x)\right)x}{x-1}+\frac{y(x)}{x-1}=0$

□ Check to see if $x_0=1$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{x}{x-1},P_3(x)=\frac{1}{x-1}\right]$

  ◦ $(x-1)\cdot P_2(x)$ is analytic at $x=1$

  $\left((x-1)\cdot P_2(x)\right)\Big|_{x=1}=-1$

  ◦ $(x-1)^2\cdot P_3(x)$ is analytic at $x=1$

  $\left((x-1)^2\cdot P_3(x)\right)\Big|_{x=1}=0$

  ◦ $x=1$is a regular singular point

  Check to see if $x_0=1$ is a regular singular point

  $x_0=1$

- Multiply by denominators

$(x-1)\left(\frac{d^2}{dx^2}y(x)\right)-x\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Change variables using $x=u+1$ so that the regular singular point is at $u=0$

$u\left(\frac{d^2}{du^2}y(u)\right)+(-u-1)\left(\frac{d}{du}y(u)\right)+y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  ◦ Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-1}$

  ◦ Shift index using $k->k+1$

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(k+r-1)-a_k(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{0,2\}$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_{k+1}(k+1+r)-a_k\right)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k}{k+1+r}$
- Recursion relation for $r=0$

  $a_{k+1}=\frac{a_k}{k+1}$
- Solution for $r=0$

  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+1}=\frac{a_k}{k+1}\right]$
- Revert the change of variables $u=x-1$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k (x-1)^k, a_{k+1}=\frac{a_k}{k+1}\right]$
- Recursion relation for $r=2$

  $a_{k+1}=\frac{a_k}{k+3}$
- Solution for $r=2$

  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$
- Revert the change of variables $u=x-1$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k (x-1)^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$
- Combine solutions and rename parameters

  $\left[y(x)=\left(\sum_{k=0}^{\infty} a_k (x-1)^k\right)+\left(\sum_{k=0}^{\infty} b_k (x-1)^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+3}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 12

```
dsolve((1-x)*diff(diff(y(x),x),x)+diff(y(x),x)*x-y(x) = 0,y(x),singsol=all)
```

$$y = c_1 x + e^{x} c_2$$

Mathematica DSolve solution

Solving time : 0.15 (sec)
Leaf size : 90

```
DSolve[{(1-x)*D[y[x],{x,2}]+x*D[y[x],x]-y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \exp\left(\int_1^x \frac{K[1]-2}{2(K[1]-1)}dK[1] - \frac{1}{2}\int_1^x -\frac{K[2]}{K[2]-1}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{K[1]-2}{2(K[1]-1)}dK[1]\right)dK[3] + c_1\right)$$

2.1.423 Problem 435

Internal problem ID [9595]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 435
Date solved : `Monday, January 27, 2025 at 06:04:28 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2y'' + xy' + 3y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.233 (sec)

Writing the ode as

$$2y'' + xy' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= x \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 20}{16} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 20 \\ t &= 16 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{16} - \frac{5}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.800: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{4} - \frac{5}{2x} - \frac{25}{2x^3} - \frac{125}{x^5} - \frac{3125}{2x^7} - \frac{21875}{x^9} - \frac{328125}{x^{11}} - \frac{5156250}{x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{4}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{16}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 20}{16} \\ &= Q + \frac{R}{16} \\ &= \left(\frac{x^2}{16} - \frac{5}{4}\right) + (0) \\ &= \frac{x^2}{16} - \frac{5}{4}\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{5}{4}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{5}{4}\right) - (0) \\ &= -\frac{5}{4}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{x}{4} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{5}{4}}{\frac{1}{4}} - 1\right) &= -3 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{5}{4}}{\frac{1}{4}} - 1\right) &= 2\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{16} - \frac{5}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{4}$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$, and since there are no poles then

$$
\begin{aligned}
d &= \alpha_\infty^- \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= (-)[\sqrt{r}]_\infty \\
&= 0 + (-)\left(\frac{x}{4}\right) \\
&= -\frac{x}{4} \\
&= -\frac{x}{4}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(2) + 2\left(-\frac{x}{4}\right)(2x + a_1) + \left(\left(-\frac{1}{4}\right) + \left(-\frac{x}{4}\right)^2 - \left(\frac{x^2}{16} - \frac{5}{4}\right)\right) &= 0 \\
2 + \frac{a_1 x}{2} + a_0 &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -2, a_1 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 - 2
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (x^2 - 2)\, e^{\int -\frac{x}{4} dx} \\
&= (x^2 - 2)\, e^{-\frac{x^2}{8}} \\
&= (x^2 - 2)\, e^{-\frac{x^2}{8}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{2}\, dx} \\
&= z_1 e^{-\frac{x^2}{8}} \\
&= z_1 \left(e^{-\frac{x^2}{8}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = e^{-\frac{x^2}{4}}(x^2 - 2)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{4}}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{e^{\frac{x^2}{4}}}{(x^2-2)^2} dx \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(e^{-\frac{x^2}{4}}(x^2-2) \right) + c_2 \left(e^{-\frac{x^2}{4}}(x^2-2) \left(\int \frac{e^{\frac{x^2}{4}}}{(x^2-2)^2} dx \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2\frac{d^2}{dx^2}y(x) + x\left(\frac{d}{dx}y(x)\right) + 3y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{x\left(\frac{d}{dx}y(x)\right)}{2} - \frac{3y(x)}{2}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d^2}{dx^2}y(x) + \frac{x\left(\frac{d}{dx}y(x)\right)}{2} + \frac{3y(x)}{2} = 0$

- Multiply by denominators

  $2\frac{d^2}{dx^2}y(x) + x\left(\frac{d}{dx}y(x)\right) + 3y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

  $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

$\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$

Rewrite DE with series expansions

$\sum_{k=0}^{\infty} (2a_{k+2}(k+2)(k+1) + a_k(k+3))x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(2k^2 + 6k + 4)a_{k+2} + a_k(k+3) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(k+3)}{2(k^2+3k+2)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.026 (sec)
Leaf size : 32

```
dsolve(2*diff(diff(y(x),x),x)+diff(y(x),x)*x+3*y(x) = 0,y(x),singsol=all)
```

$$y = (x^2 - 2)\left(c_1 \operatorname{erfi}\left(\frac{x}{2}\right)\sqrt{\pi} + c_2\right) e^{-\frac{x^2}{4}} - 2c_1 x$$

Mathematica DSolve solution

Solving time : 0.202 (sec)
Leaf size : 52

```
DSolve[{2*D[y[x],{x,2}]+x*D[y[x],x]+3*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-\frac{x^2}{4}}(x^2 - 2)\left(c_2 \int_1^x \frac{e^{\frac{K[1]^2}{4}}}{(K[1]^2 - 2)^2} dK[1] + c_1\right)$$

2.1.424 Problem 436

Internal problem ID [9596]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 436
Date solved : `Monday, January 27, 2025 at 06:04:29 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + xy' + 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.208 (sec)

Writing the ode as

$$y'' + xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 6 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.802: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 2\\ &= -2\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{1} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{2x} - \frac{9}{4x^3} - \frac{27}{4x^5} - \frac{405}{16x^7} - \frac{1701}{16x^9} - \frac{15309}{32x^{11}} - \frac{72171}{32x^{13}} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $([\sqrt{r}]_\infty)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$([\sqrt{r}]_\infty)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 6}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} - \frac{3}{2}\right) + (0) \\ &= \frac{x^2}{4} - \frac{3}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{2}\right) - (0) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{3}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{3}{2}\right)\right) &= 0 \\ a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= (x)\,e^{\int -\frac{x}{2}dx} \\ &= (x)\,e^{-\frac{x^2}{4}} \\ &= x\,e^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{1}\, dx} \\ &= z_1 e^{-\frac{x^2}{4}} \\ &= z_1\left(e^{-\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^2}{2}} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-\frac{x^2}{2}} x\right) + c_2 \left(\mathrm{e}^{-\frac{x^2}{2}} x \left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} \right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + x\left(\frac{d}{dx}y(x)\right) + 2y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

- ☐ Rewrite DE with series expansions
  - ∘ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  - ∘ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - ∘ Shift index using $k- >k+2$

    $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+2)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

$$(k+2)(ka_{k+2}+a_k+a_{k+2})=0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y(x)=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=-\frac{a_k}{k+1}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 34

```
dsolve(diff(diff(y(x),x),x)+diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y=x\left(ic_2\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)+c_1\right)\mathrm{e}^{-\frac{x^2}{2}}+2c_2$$

Mathematica DSolve solution

Solving time : 0.049 (sec)
Leaf size : 69

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to-\sqrt{\frac{\pi}{2}}c_2e^{-\frac{x^2}{2}}\sqrt{x^2}\operatorname{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{2}}\right)+\sqrt{2}c_1e^{-\frac{x^2}{2}}x+c_2$$

2.1.425 Problem 437

Internal problem ID [9597]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 437
Date solved : `Monday, January 27, 2025 at 06:04:29 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(1-x)\,y''+xy'-y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.223 (sec)

Writing the ode as

$$(1-x)\,y''+xy'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=1-x\\ B&=x \\ C&=-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{x^2-4x+6}{4\left(-1+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=x^2-4x+6\\ t&=4(-1+x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x^2-4x+6}{4\left(-1+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.804: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(-1+x)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2(-1+x)} + \frac{3}{4(-1+x)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a = \frac{1}{2}
$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}
$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\
&= Q + \frac{R}{4x^2 - 8x + 4} \\
&= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\
&= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4}
\end{aligned}
$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (0) \\
&= -\frac{1}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 4x + 6}{4\left(-1 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(-1 + x\right)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2\left(-1 + x\right)} + \frac{1}{2} \\
&= \frac{x - 2}{2x - 2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2(-1+x)}+\frac{1}{2}\right)(0)+\left(\left(\frac{1}{2(-1+x)^2}\right)+\left(-\frac{1}{2(-1+x)}+\frac{1}{2}\right)^2-\left(\frac{x^2-4x+6}{4(-1+x)^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2(-1+x)}+\frac{1}{2}\right)dx} \\ &= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{-1+x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{1-x}\, dx} \\ &= z_1 e^{\frac{x}{2}+\frac{\ln(-1+x)}{2}} \\ &= z_1\left(\sqrt{-1+x}\,\mathrm{e}^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1-x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x+\ln(-1+x)}}{(y_1)^2}\, dx \\ &= y_1\left(-\frac{x\,\mathrm{e}^{x+\ln(-1+x)}\mathrm{e}^{-2x}}{-1+x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(-\frac{x\,\mathrm{e}^{x+\ln(-1+x)}\mathrm{e}^{-2x}}{-1+x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(1-x)\left(\frac{d^2}{dx^2}y(x)\right)+x\left(\frac{d}{dx}y(x)\right)-y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{y(x)}{x-1}+\frac{\left(\frac{d}{dx}y(x)\right)x}{x-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)-\frac{\left(\frac{d}{dx}y(x)\right)x}{x-1}+\frac{y(x)}{x-1}=0$

□ Check to see if $x_0=1$ is a regular singular point

- ◦ Define functions

$\left[P_2(x)=-\frac{x}{x-1},P_3(x)=\frac{1}{x-1}\right]$

- ◦ $(x-1)\cdot P_2(x)$ is analytic at $x=1$

$\left((x-1)\cdot P_2(x)\right)\Big|_{x=1}=-1$

- ◦ $(x-1)^2\cdot P_3(x)$ is analytic at $x=1$

$\left((x-1)^2\cdot P_3(x)\right)\Big|_{x=1}=0$

- ◦ $x=1$is a regular singular point

Check to see if $x_0=1$ is a regular singular point

$x_0=1$

- Multiply by denominators

$(x-1)\left(\frac{d^2}{dx^2}y(x)\right)-x\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Change variables using $x=u+1$ so that the regular singular point is at $u=0$

$u\left(\frac{d^2}{du^2}y(u)\right)+(-u-1)\left(\frac{d}{du}y(u)\right)+y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ◦ Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-1}$

- ◦ Shift index using $k->k+1$

$u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)(k+r)\,u^{k+r}$

Rewrite ODE with series expansions

$a_0r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)(k+r-1)-a_k(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\,(a_{k+1}(k+1+r) - a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1+r}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$

- Revert the change of variables $u = x - 1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x-1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+3}$

- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$

- Revert the change of variables $u = x - 1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x-1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k (x-1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x-1)^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 12

```
dsolve((1-x)*diff(diff(y(x),x),x)+diff(y(x),x)*x-y(x) = 0,y(x),singsol=all)
```

$$y = c_1 x + e^x c_2$$

Mathematica DSolve solution

Solving time : 0.15 (sec)
Leaf size : 90

```
DSolve[{(1-x)*D[y[x],{x,2}]+x*D[y[x],x]-y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \exp\left(\int_1^x \frac{K[1]-2}{2(K[1]-1)}dK[1] - \frac{1}{2}\int_1^x -\frac{K[2]}{K[2]-1}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{K[1]-2}{2(K[1]-1)}dK[1]\right)dK[3] + c_1\right)$$

2.1.426 Problem 438

Internal problem ID [9598]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 438
Date solved : `Monday, January 27, 2025 at 06:04:30 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + xy' + 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.220 (sec)

Writing the ode as

$$y'' + xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 6 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.806: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{2x} - \frac{9}{4x^3} - \frac{27}{4x^5} - \frac{405}{16x^7} - \frac{1701}{16x^9} - \frac{15309}{32x^{11}} - \frac{72171}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{2}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}
$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 6}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} - \frac{3}{2}\right) + (0) \\
&= \frac{x^2}{4} - \frac{3}{2}
\end{aligned}
$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{3}{2}\right) - (0) \\
&= -\frac{3}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2}{4} - \frac{3}{2}
$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{3}{2}\right)\right) &= 0 \\ a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (x)\, \mathrm{e}^{\int -\frac{x}{2} dx} \\ &= (x)\, \mathrm{e}^{-\frac{x^2}{4}} \\ &= x\, \mathrm{e}^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{1}\, dx} \\ &= z_1 e^{-\frac{x^2}{4}} \\ &= z_1\left(\mathrm{e}^{-\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^2}{2}} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x^2}{2}} x\right) + c_2 \left(\mathrm{e}^{-\frac{x^2}{2}} x \left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + x\left(\frac{d}{dx}y(x)\right) + 2y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k->k+2$

    $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+2)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

$$(k+2)\left(k a_{k+2}+a_k+a_{k+2}\right)=0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y(x)=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=-\frac{a_k}{k+1}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 34

```
dsolve(diff(diff(y(x),x),x)+diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y=x\left(i c_2\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)+c_1\right)\mathrm{e}^{-\frac{x^2}{2}}+2c_2$$

Mathematica DSolve solution

Solving time : 0.045 (sec)
Leaf size : 69

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to-\sqrt{\frac{\pi}{2}}c_2e^{-\frac{x^2}{2}}\sqrt{x^2}\operatorname{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{2}}\right)+\sqrt{2}c_1e^{-\frac{x^2}{2}}x+c_2$$

2.1.427 Problem 439

Internal problem ID [9599]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 439
Date solved : `Monday, January 27, 2025 at 06:04:30 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(-x^2+4)\,y''+xy'+2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 1.028 (sec)

Writing the ode as

$$(-x^2+4)\,y''+xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+4\\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{11x^2-24}{4\,(x^2-4)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 11x^2-24\\ t &= 4\big(x^2-4\big)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{11x^2-24}{4\,(x^2-4)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.808: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 4)^2$. There is a pole at $x = 2$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{17}{32\,(x+2)} + \frac{17}{32\,(x-2)} + \frac{5}{16\,(x+2)^2} + \frac{5}{16\,(x-2)^2}$$

For the pole at $x = 2$ let b be the coefficient of $\frac{1}{(x-2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{11x^2 - 24}{4\left(x^2 - 4\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{11}{4}$. Hence

$$\begin{aligned}E_\infty &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{2\}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
2	2	$\{-1, 2, 5\}$
-2	2	$\{-1, 2, 5\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = -1,\ e_2 = -1,\ e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned}d &= \frac{1}{2}\left(e_\infty - \sum_{c \in \Gamma} e_c\right) \\ &= \frac{1}{2}(2 - (-1 + (-1))) \\ &= 2\end{aligned}$$

We now form the following rational function

$$\begin{aligned}\theta &= \frac{1}{2} \sum_{c \in \Gamma} \frac{e_c}{x - c} \\ &= \frac{1}{2}\left(\frac{-1}{(x - (2))} + \frac{-1}{(x - (-2))}\right) \\ &= -\frac{1}{2\left(x - 2\right)} - \frac{1}{2\left(x + 2\right)}\end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 2$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 2$, then letting

$$p = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$\frac{11x^2 a_1 + 16(6 + a_0)\,x + 36a_1}{\left(x^2 - 4\right)^2} = 0$$

And solving for p gives

$$p = x^2 - 6$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p} \\ &= \frac{2x}{x^2-6} - \frac{1}{2(x-2)} - \frac{1}{2(x+2)}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{2x}{x^2-6} - \frac{1}{2(x-2)} - \frac{1}{2(x+2)}\right)w + \frac{-11x^4+74x^2-128}{4x^6-56x^4+256x^2-384} = 0$$

Solving for ω gives

$$\omega = \frac{2\sqrt{3}\,x^2\sqrt{x^2-4}+x^3-8\sqrt{3}\,\sqrt{x^2-4}-2x}{2(x^2-6)(x-2)(x+2)}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \frac{2\sqrt{3}\,x^2\sqrt{x^2-4}+x^3-8\sqrt{3}\,\sqrt{x^2-4}-2x}{2(x^2-6)(x-2)(x+2)}dx} \\ &= \frac{\sqrt{x^2-6}\left(x+\sqrt{x^2-4}\right)^{\sqrt{3}}\mathrm{e}^{-\frac{\operatorname{arctanh}\left(\frac{(\sqrt{2}\sqrt{3}\,x-4)\sqrt{2}}{2\sqrt{x^2-4}}\right)}{2}-\frac{\operatorname{arctanh}\left(\frac{(4+\sqrt{2}\sqrt{3}\,x)\sqrt{2}}{2\sqrt{x^2-4}}\right)}{2}}}{(x-2)^{1/4}(x+2)^{1/4}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{-x^2+4}\, dx} \\ &= z_1 e^{\frac{\ln(x^2-4)}{4}} \\ &= z_1\left((x^2-4)^{1/4}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x^2-6}\left(x+\sqrt{x^2-4}\right)^{\sqrt{3}}\mathrm{e}^{-\frac{\operatorname{arctanh}\left(\frac{x\sqrt{6}-4}{\sqrt{2x^2-8}}\right)}{2}-\frac{\operatorname{arctanh}\left(\frac{4+x\sqrt{6}}{\sqrt{2x^2-8}}\right)}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{x}{-x^2+4}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\frac{\ln(x^2-4)}{2}}}{(y_1)^2}\, dx \\ &= y_1\left(\int \frac{\sqrt{x^2-4}\left(x+\sqrt{x^2-4}\right)^{-2\sqrt{3}}\mathrm{e}^{\operatorname{arctanh}\left(\frac{x\sqrt{6}-4}{\sqrt{2x^2-8}}\right)+\operatorname{arctanh}\left(\frac{4+x\sqrt{6}}{\sqrt{2x^2-8}}\right)}}{x^2-6}dx\right)\end{aligned}$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1\left(\sqrt{x^2-6}\left(x+\sqrt{x^2-4}\right)^{\sqrt{3}} e^{-\frac{\operatorname{arctanh}\left(\frac{x\sqrt{6}-4}{\sqrt{2x^2-8}}\right)}{2}-\frac{\operatorname{arctanh}\left(\frac{4+x\sqrt{6}}{\sqrt{2x^2-8}}\right)}{2}}\right) + c_2\left(\sqrt{x^2-6}\left(x\right.\right.$$

$$\left.+\sqrt{x^2-4}\right)^{\sqrt{3}} e^{-\frac{\operatorname{arctanh}\left(\frac{x\sqrt{6}-4}{\sqrt{2x^2-8}}\right)}{2}-\frac{\operatorname{arctanh}\left(\frac{4+x\sqrt{6}}{\sqrt{2x^2-8}}\right)}{2}}\left(\int \frac{\sqrt{x^2-4}\left(x+\sqrt{x^2-4}\right)^{-2\sqrt{3}} e^{\operatorname{arctanh}\left(\frac{x\sqrt{6}-4}{\sqrt{2x^2-8}}\right)+\operatorname{arctanh}\left(\frac{4}{\sqrt{}}\right.}}{x^2-6}\right.$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$(-x^2+4)\left(\frac{d^2}{dx^2}y(x)\right)+x\left(\frac{d}{dx}y(x)\right)+2y(x)=0$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
$\frac{d^2}{dx^2}y(x)=\frac{2y(x)}{x^2-4}+\frac{x\left(\frac{d}{dx}y(x)\right)}{x^2-4}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
$\frac{d^2}{dx^2}y(x)-\frac{x\left(\frac{d}{dx}y(x)\right)}{x^2-4}-\frac{2y(x)}{x^2-4}=0$
- □ Check to see if x_0 is a regular singular point
  - Define functions
  $\left[P_2(x)=-\frac{x}{x^2-4}, P_3(x)=-\frac{2}{x^2-4}\right]$
  - $(x+2)\cdot P_2(x)$ is analytic at $x=-2$
  $\left.\left((x+2)\cdot P_2(x)\right)\right|_{x=-2}=-\frac{1}{2}$
  - $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$
  $\left.\left((x+2)^2\cdot P_3(x)\right)\right|_{x=-2}=0$
  - $x=-2$is a regular singular point
  Check to see if x_0 is a regular singular point
  $x_0=-2$
- Multiply by denominators
$(x^2-4)\left(\frac{d^2}{dx^2}y(x)\right)-x\left(\frac{d}{dx}y(x)\right)-2y(x)=0$
- Change variables using $x=u-2$ so that the regular singular point is at $u=0$
$(u^2-4u)\left(\frac{d^2}{du^2}y(u)\right)+(-u+2)\left(\frac{d}{du}y(u)\right)-2y(u)=0$
- Assume series solution for $y(u)$
$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$
- □ Rewrite ODE with series expansions
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$
  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0r(-3+2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)\,(2k-1+2r) + a_k(k^2+2kr+r^2-2k-2r-\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
$$-2r(-3+2r) = 0$$
- Values of r that satisfy the indicial equation
$$r \in \left\{0, \frac{3}{2}\right\}$$
- Each term in the series must be 0, giving the recursion relation
$$-4(k+1+r)\left(k+r-\frac{1}{2}\right) a_{k+1} + a_k(k^2+(2r-2)\,k+r^2-2r-2) = 0$$
- Recursion relation that defines series solution to ODE
$$a_{k+1} = \frac{a_k\,(k^2+2kr+r^2-2k-2r-2)}{2(k+1+r)(2k-1+2r)}$$
- Recursion relation for $r = 0$
$$a_{k+1} = \frac{a_k\,(k^2-2k-2)}{2(k+1)(2k-1)}$$
- Solution for $r = 0$
$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k\,(k^2-2k-2)}{2(k+1)(2k-1)}\right]$$
- Revert the change of variables $u = x+2$
$$\left[y(x) = \sum_{k=0}^{\infty} a_k(x+2)^k, a_{k+1} = \frac{a_k\,(k^2-2k-2)}{2(k+1)(2k-1)}\right]$$
- Recursion relation for $r = \frac{3}{2}$
$$a_{k+1} = \frac{a_k\,(k^2+k-\frac{11}{4})}{2(k+\frac{5}{2})(2k+2)}$$
- Solution for $r = \frac{3}{2}$
$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{3}{2}}, a_{k+1} = \frac{a_k\,(k^2+k-\frac{11}{4})}{2(k+\frac{5}{2})(2k+2)}\right]$$
- Revert the change of variables $u = x+2$
$$\left[y(x) = \sum_{k=0}^{\infty} a_k(x+2)^{k+\frac{3}{2}}, a_{k+1} = \frac{a_k\,(k^2+k-\frac{11}{4})}{2(k+\frac{5}{2})(2k+2)}\right]$$
- Combine solutions and rename parameters
$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k(x+2)^k\right) + \left(\sum_{k=0}^{\infty} b_k(x+2)^{k+\frac{3}{2}}\right), a_{k+1} = \frac{a_k\,(k^2-2k-2)}{2(k+1)(2k-1)}, b_{k+1} = \frac{b_k\,(k^2+k-\frac{11}{4})}{2(k+\frac{5}{2})(2k+2)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Legendre successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special fu
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.010 (sec)
Leaf size : 37

```
dsolve((-x^2+4)*diff(diff(y(x),x),x)+diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y = (x^2-4)^{3/4}\left(\operatorname{LegendreP}\left(\sqrt{3}-\frac{1}{2}, \frac{3}{2}, \frac{x}{2}\right)c_1 + \operatorname{LegendreQ}\left(\sqrt{3}-\frac{1}{2}, \frac{3}{2}, \frac{x}{2}\right)c_2\right)$$

Mathematica DSolve solution

Solving time : 0.05 (sec)
Leaf size : 58

```
DSolve[{(4-x^2)*D[y[x],{x,2}]+x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to (x^2-4)^{3/4}\left(c_1 P_{-\frac{1}{2}+\sqrt{3}}^{\frac{3}{2}}\left(\frac{x}{2}\right) + c_2 Q_{-\frac{1}{2}+\sqrt{3}}^{\frac{3}{2}}\left(\frac{x}{2}\right)\right)$$

2.1.428 Problem 440

Internal problem ID [9600]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 440
Date solved : Monday, January 27, 2025 at 06:04:32 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$4x^2y'' - 4xy' + \left(-16x^2 + 3\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.094 (sec)

Writing the ode as

$$4x^2y'' - 4xy' + \left(-16x^2 + 3\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x \\ C &= -16x^2 + 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.810: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{4x^2}\, dx} \\ &= z_1 e^{\frac{\ln(x)}{2}} \\ &= z_1\left(\sqrt{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}\, \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{e^{4x}}{4}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x}\,e^{-2x}\right) + c_2\left(\sqrt{x}\,e^{-2x}\left(\frac{e^{4x}}{4}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2\left(\frac{d^2}{dx^2}y(x)\right) - 4x\left(\frac{d}{dx}y(x)\right) + \left(-16x^2+3\right)y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{(16x^2-3)y(x)}{4x^2} + \frac{\frac{d}{dx}y(x)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) - \frac{\frac{d}{dx}y(x)}{x} - \frac{(16x^2-3)y(x)}{4x^2} = 0$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{1}{x}, P_3(x) = -\frac{16x^2-3}{4x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{3}{4}$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) - 4x\left(\frac{d}{dx}y(x)\right) + \left(-16x^2+3\right)y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m}x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)\,(-3+2r)\, x^r + a_1(1+2r)\,(-1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r-1)\,(2k+2r-3) - 16a\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)\,(-3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$

- Each term must be 0

  $a_1(1+2r)\,(-1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r-\frac{3}{2}\right)\left(k+r-\frac{1}{2}\right) a_k - 16a_{k-2} = 0$

- Shift index using $k->k+2$

  $4\left(k+\frac{1}{2}+r\right)\left(k+\frac{3}{2}+r\right) a_{k+2} - 16a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{16a_k}{(2k+1+2r)(2k+3+2r)}$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = \frac{16a_k}{(2k+2)(2k+4)}$

- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{16a_k}{(2k+2)(2k+4)}, a_1 = 0\right]$

- Recursion relation for $r = \frac{3}{2}$

  $a_{k+2} = \frac{16a_k}{(2k+4)(2k+6)}$

- Solution for $r = \frac{3}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2} = \frac{16a_k}{(2k+4)(2k+6)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+2} = \frac{16a_k}{(2k+2)(2k+4)}, a_1 = 0, b_{k+2} = \frac{16b_k}{(2k+4)(2k+6)}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 21

```
dsolve(4*x^2*diff(diff(y(x),x),x)-4*diff(y(x),x)*x+(-16*x^2+3)*y(x) = 0,y(x),singsol=a
```

$$y = \sqrt{x}\,(c_1 \sinh(2x) + c_2 \cosh(2x))$$

Mathematica DSolve solution

Solving time : 0.036 (sec)
Leaf size : 32

```
DSolve[{4*x^2*D[y[x],{x,2}]-4*x*D[y[x],x]+(3-16*x^2)*y[x]==0,{}},y[x],x,IncludeSingularSolut
```

$$y(x) \to \frac{1}{4}e^{-2x}\sqrt{x}\left(c_2e^{4x} + 4c_1\right)$$

2.1.429 Problem 441

Internal problem ID [9601]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 441
Date solved : `Monday, January 27, 2025 at 06:04:32 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(x-1)\,y''-xy'+y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.227 (sec)

Writing the ode as

$$(x-1)\,y''-xy'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x-1 \\ B &= -x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{x^2-4x+6}{4\,(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2-4x+6 \\ t &= 4(x-1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x^2-4x+6}{4\,(x-1)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.812: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2\,(x-1)} + \frac{3}{4\,(x-1)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 4x + 6}{4\left(x - 1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(x - 1\right)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2\left(x - 1\right)} + \frac{1}{2} \\
&= \frac{x - 2}{2x - 2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2(x-1)^2}\right) + \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)^2 - \left(\frac{x^2-4x+6}{4(x-1)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right) dx} \\ &= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x-1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{x-1}\, dx} \\ &= z_1 e^{\frac{x}{2} + \frac{\ln(x-1)}{2}} \\ &= z_1\left(\sqrt{x-1}\,\mathrm{e}^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x-1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x+\ln(x-1)}}{(y_1)^2}\, dx \\ &= y_1\left(-\frac{x\,\mathrm{e}^{x+\ln(x-1)}\mathrm{e}^{-2x}}{x-1}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(-\frac{x\,\mathrm{e}^{x+\ln(x-1)}\mathrm{e}^{-2x}}{x-1}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x-1)\left(\frac{d^2}{dx^2}y(x)\right)-x\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{y(x)}{x-1}+\frac{\left(\frac{d}{dx}y(x)\right)x}{x-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)-\frac{\left(\frac{d}{dx}y(x)\right)x}{x-1}+\frac{y(x)}{x-1}=0$

☐ Check to see if $x_0=1$ is a regular singular point

- ◦ Define functions

$\left[P_2(x)=-\frac{x}{x-1},P_3(x)=\frac{1}{x-1}\right]$

- ◦ $(x-1)\cdot P_2(x)$ is analytic at $x=1$

$\left((x-1)\cdot P_2(x)\right)\bigg|_{x=1}=-1$

- ◦ $(x-1)^2\cdot P_3(x)$ is analytic at $x=1$

$\left((x-1)^2\cdot P_3(x)\right)\bigg|_{x=1}=0$

- ◦ $x=1$is a regular singular point

Check to see if $x_0=1$ is a regular singular point

$x_0=1$

- Multiply by denominators

$(x-1)\left(\frac{d^2}{dx^2}y(x)\right)-x\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Change variables using $x=u+1$ so that the regular singular point is at $u=0$

$u\left(\frac{d^2}{du^2}y(u)\right)+(-u-1)\left(\frac{d}{du}y(u)\right)+y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ◦ Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-1}$

- ◦ Shift index using $k->k+1$

$u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$

Rewrite ODE with series expansions

$a_0r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(k+r-1)-a_k(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{0,2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\,(a_{k+1}(k+1+r)-a_k)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k}{k+1+r}$

- Recursion relation for $r=0$

  $a_{k+1}=\frac{a_k}{k+1}$

- Solution for $r=0$

  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+1}=\frac{a_k}{k+1}\right]$

- Revert the change of variables $u=x-1$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k (x-1)^k, a_{k+1}=\frac{a_k}{k+1}\right]$

- Recursion relation for $r=2$

  $a_{k+1}=\frac{a_k}{k+3}$

- Solution for $r=2$

  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$

- Revert the change of variables $u=x-1$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k (x-1)^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$

- Combine solutions and rename parameters

  $\left[y(x)=\left(\sum_{k=0}^{\infty} a_k (x-1)^k\right)+\left(\sum_{k=0}^{\infty} b_k (x-1)^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+3}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 12

```
dsolve((x-1)*diff(diff(y(x),x),x)-diff(y(x),x)*x+y(x) = 0,y(x),singsol=all)
```

$$y = c_1 x + e^{x} c_2$$

Mathematica DSolve solution

Solving time : 0.145 (sec)
Leaf size : 90

```
DSolve[{(x-1)*D[y[x],{x,2}]-x*D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \exp\left(\int_1^x \frac{K[1]-2}{2(K[1]-1)}dK[1] - \frac{1}{2}\int_1^x -\frac{K[2]}{K[2]-1}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{K[1]-2}{2(K[1]-1)}dK[1]\right)dK[3] + c_1\right)$$

2.1.430 Problem 442

Internal problem ID [9602]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 442
Date solved : `Monday, January 27, 2025 at 06:04:33 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - 2xy' + \left(x^2+2\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.124 (sec)

Writing the ode as

$$x^2y'' - 2xy' + \left(x^2+2\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.814: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\, dx} \\ &= z_1 e^{\ln(x)} \\ &= z_1(x) \end{aligned}$$

Which simplifies to

$$y_1 = x\cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(x\cos(x)) + c_2(x\cos(x)(\tan(x))) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) - 2x\left(\frac{d}{dx}y(x)\right) + (x^2+2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(x^2+2)y(x)}{x^2} + \frac{2\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) - \frac{2\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(x^2+2)y(x)}{x^2} = 0$
- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{2}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$
  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$
- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) - 2x\left(\frac{d}{dx}y(x)\right) + (x^2+2)\,y(x) = 0$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k - m$

$$x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

$$x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

$$x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+r)\,(-2+r)\,x^r + a_1 r(-1+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)\,(k+r-2) + a_{k-2}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+r)\,(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{1, 2\}$$

- Each term must be 0

$$a_1 r(-1+r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+r-1)\,(k+r-2) + a_{k-2} = 0$$

- Shift index using $k- > k+2$

$$a_{k+2}(k+1+r)\,(k+r) + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k}{(k+1+r)(k+r)}$$

- Recursion relation for $r = 1$

$$a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$$

- Solution for $r = 1$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$$

- Recursion relation for $r = 2$

$$a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$$

- Solution for $r = 2$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+2)(k+3)}, b_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
```

```
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 15

```
dsolve(x^2*diff(diff(y(x),x),x)-2*diff(y(x),x)*x+(x^2+2)*y(x) = 0,y(x),singsol=all)
```

$$y = x(\sin(x)\, c_1 + \cos(x)\, c_2)$$

Mathematica DSolve solution

Solving time : 0.028 (sec)
Leaf size : 33

```
DSolve[{x^2*D[y[x],{x,2}]-2*x*D[y[x],x]+(x^2+2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tr
```

$$y(x) \to c_1 e^{-ix} x - \frac{1}{2} i c_2 e^{ix} x$$

2.1.431 Problem 444

Internal problem ID [9603]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 444
Date solved : `Monday, January 27, 2025 at 06:04:34 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2-2x\right)y''+\left(-x^2+2\right)y'+(2x-2)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.308 (sec)

Writing the ode as

$$\left(x^2-2x\right)y''+\left(-x^2+2\right)y'+(2x-2)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2-2x\\B&=-x^2+2\\C&=2x-2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{x^4-8x^3+24x^2-24x+12}{4\left(x^2-2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=x^4-8x^3+24x^2-24x+12\\t&=4\left(x^2-2x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x^4-8x^3+24x^2-24x+12}{4\left(x^2-2x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.816: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4x^2} - \frac{3}{4x} + \frac{3}{4(x-2)^2} - \frac{1}{4(x-2)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = 2$ let b be the coefficient of $\frac{1}{(x-2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{2}{x^3} + \frac{11}{x^4} + \frac{42}{x^5} + \frac{132}{x^6} + \frac{348}{x^7} + \frac{711}{x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4x^4 - 16x^3 + 16x^2} \\ &= Q + \frac{R}{4x^4 - 16x^3 + 16x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-4x^3 + 20x^2 - 24x + 12}{4x^4 - 16x^3 + 16x^2}\right) \\ &= \frac{1}{4} + \frac{-4x^3 + 20x^2 - 24x + 12}{4x^4 - 16x^3 + 16x^2} \end{aligned}$$

Since the degree of t is 4, then we see that the coefficient of the term x^3 in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 0\right) &&= -1 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 0\right) &&= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4\left(x^2 - 2x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
2	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-}\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} - \frac{1}{2(x-2)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2x} - \frac{1}{2(x-2)} + \frac{1}{2} \\
&= -\frac{1}{2x} - \frac{1}{2x-4} + \frac{1}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2x}-\frac{1}{2(x-2)}+\frac{1}{2}\right)(0)+\left(\left(\frac{1}{2x^2}+\frac{1}{2(x-2)^2}\right)+\left(-\frac{1}{2x}-\frac{1}{2(x-2)}+\frac{1}{2}\right)^2-\left(\frac{x^4-8x^3+}{4(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\,dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2x}-\frac{1}{2(x-2)}+\frac{1}{2}\right)dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x-2}\,\sqrt{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2+2}{x^2-2x}\,dx} \\
&= z_1 e^{\frac{x}{2}+\frac{\ln(x-2)}{2}+\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x-2}\,\sqrt{x}\,\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\frac{\sqrt{x-2}\,\sqrt{x}\,\mathrm{e}^{x}}{\sqrt{x(x-2)}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1\int \frac{e^{\int -\frac{-x^2+2}{x^2-2x}\,dx}}{(y_1)^2}\,dx \\
&= y_1\int \frac{e^{x+\ln(x-2)+\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{x\,\mathrm{e}^{x+\ln(x-2)+\ln(x)}\mathrm{e}^{-2x}}{x-2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1y_1+c_2y_2 \\
&= c_1\left(\frac{\sqrt{x-2}\,\sqrt{x}\,\mathrm{e}^{x}}{\sqrt{x(x-2)}}\right)+c_2\left(\frac{\sqrt{x-2}\,\sqrt{x}\,\mathrm{e}^{x}}{\sqrt{x(x-2)}}\left(-\frac{x\,\mathrm{e}^{x+\ln(x-2)+\ln(x)}\mathrm{e}^{-2x}}{x-2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x^2-2x)\left(\frac{d^2}{dx^2}y(x)\right)+(-x^2+2)\left(\frac{d}{dx}y(x)\right)+(2x-2)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{2(x-1)y(x)}{x(x-2)}+\frac{(x^2-2)\left(\frac{d}{dx}y(x)\right)}{x(x-2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)-\frac{(x^2-2)\left(\frac{d}{dx}y(x)\right)}{x(x-2)}+\frac{2(x-1)y(x)}{x(x-2)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=-\frac{x^2-2}{x(x-2)},P_3(x)=\frac{2(x-1)}{x(x-2)}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\Big|_{x=0}=-1$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\Big|_{x=0}=0$

- ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x(x-2)\left(\frac{d^2}{dx^2}y(x)\right)+(-x^2+2)\left(\frac{d}{dx}y(x)\right)+(2x-2)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ◦ Shift index using $k->k-m$

  $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..2$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ◦ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=1..2$

  $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- ◦ Shift index using $k->k+2-m$

  $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$-2a_0r(-2+r)\,x^{-1+r}+(-2a_1(1+r)\,(-1+r)+a_0(1+r)\,(-2+r))\,x^r+\left(\sum_{k=1}^{\infty}(-2a_{k+1}(k+r+\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $$-2r(-2+r)=0$$
- Values of r that satisfy the indicial equation
  $$r\in\{0,2\}$$
- Each term must be 0
  $$-2a_1(1+r)\,(-1+r)+a_0(1+r)\,(-2+r)=0$$
- Each term in the series must be 0, giving the recursion relation
  $$a_k(k+r+1)\,(k+r-2)-2k^2a_{k+1}+(-4ra_{k+1}-a_{k-1})\,k-2r^2a_{k+1}-a_{k-1}r+3a_{k-1}+2a_{k+1}=$$
- Shift index using $k->k+1$
  $$a_{k+1}(k+2+r)\,(k+r-1)-2(k+1)^2\,a_{k+2}+(-4ra_{k+2}-a_k)\,(k+1)-2r^2a_{k+2}-ra_k+3a_k+$$
- Recursion relation that defines series solution to ODE
  $$a_{k+2}=\frac{k^2a_{k+1}+2kra_{k+1}+r^2a_{k+1}-ka_k+ka_{k+1}-ra_k+ra_{k+1}+2a_k-2a_{k+1}}{2(k^2+2kr+r^2+2k+2r)}$$
- Recursion relation for $r=0$
  $$a_{k+2}=\frac{k^2a_{k+1}-ka_k+ka_{k+1}+2a_k-2a_{k+1}}{2(k^2+2k)}$$
- Series not valid for $r=0$, division by 0 in the recursion relation at $k=0$
  $$a_{k+2}=\frac{k^2a_{k+1}-ka_k+ka_{k+1}+2a_k-2a_{k+1}}{2(k^2+2k)}$$
- Recursion relation for $r=2$
  $$a_{k+2}=\frac{k^2a_{k+1}-ka_k+5ka_{k+1}+4a_{k+1}}{2(k^2+6k+8)}$$
- Solution for $r=2$
  $$\left[y(x)=\sum_{k=0}^{\infty}a_kx^{k+2},a_{k+2}=\frac{k^2a_{k+1}-ka_k+5ka_{k+1}+4a_{k+1}}{2(k^2+6k+8)},-6a_1=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 14

```
dsolve((x^2-2*x)*diff(diff(y(x),x),x)+(-x^2+2)*diff(y(x),x)+(2*x-2)*y(x) = 0,y(x),sing
```

$$y=c_1x^2+e^xc_2$$

Mathematica DSolve solution

Solving time : `0.288` (sec)
Leaf size : 115

```
DSolve[{(x^2-2*x)*D[y[x],{x,2}]+(2-x^2)*D[y[x],x]+(2*x-2)*y[x]==0,{}},y[x],x,IncludeSingularSol
```

$$y(x) \to \exp\left(\int_1^x \frac{(K[1]-4)K[1]+2}{2(K[1]-2)K[1]}dK[1] - \frac{1}{2}\int_1^x \left(-\frac{1}{K[2]} - 1 + \frac{1}{2-K[2]}\right) dK[2]\right)\left(c_2 \int_1^x \exp\left(-2\int_1^{K[3]} \frac{K[1]^2-4K[1]+2}{2(K[1]-2)K[1]}dK[1]\right) dK[3] + c_1\right)$$

2.1.432 Problem 445

Internal problem ID [9604]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 445
Date solved : `Monday, January 27, 2025 at 06:04:34 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(2x+1)\,y''-2y'-(2x+3)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.220 (sec)

Writing the ode as

$$(2x+1)\,y''-2y'+(-2x-3)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=2x+1\\ B&=-2\\ C&=-2x-3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{4x^2+8x+6}{(2x+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=4x^2+8x+6\\ t&=(2x+1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{4x^2+8x+6}{(2x+1)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.818: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x+1)^2$. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{3}{4\left(x+\frac{1}{2}\right)^2} + \frac{1}{x+\frac{1}{2}}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{1}{2x} - \frac{1}{4x^3} + \frac{11}{32x^4} - \frac{21}{64x^5} + \frac{15}{64x^6} - \frac{3}{32x^7} - \frac{117}{2048x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 + 8x + 6}{4x^2 + 4x + 1} \\ &= Q + \frac{R}{4x^2 + 4x + 1} \\ &= (1) + \left(\frac{4x + 5}{4x^2 + 4x + 1}\right) \\ &= 1 + \frac{4x + 5}{4x^2 + 4x + 1} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$\begin{aligned} b &= (1) - (0) \\ &= 1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{1}{1} - 0\right) &&= \frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{1}{1} - 0\right) &&= -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^2+8x+6}{(2x+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(x+\frac{1}{2}\right)} + (-)(1) \\
&= -\frac{1}{2\left(x+\frac{1}{2}\right)} - 1 \\
&= -\frac{2(x+1)}{2x+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right)(0) + \left(\left(\frac{1}{2\left(x+\frac{1}{2}\right)^2}\right) + \left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right)^2 - \left(\frac{4x^2+8x+6}{(2x+1)^2}\right) \right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2\left(x+\frac{1}{2}\right)}-1\right) dx} \\ &= \frac{\mathrm{e}^{-x}}{\sqrt{2x+1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{2x+1} \, dx} \\ &= z_1 e^{\frac{\ln(2x+1)}{2}} \\ &= z_1\left(\sqrt{2x+1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{2x+1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\ln(2x+1)}}{(y_1)^2} \, dx \\ &= y_1\left(x\,\mathrm{e}^{2x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}\left(x\,\mathrm{e}^{2x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$(2x+1)\left(\frac{d^2}{dx^2}y(x)\right) - 2\frac{d}{dx}y(x) - (2x+3)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = \frac{(2x+3)y(x)}{2x+1} + \frac{2\left(\frac{d}{dx}y(x)\right)}{2x+1}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lir

$\frac{d^2}{dx^2}y(x) - \frac{2\left(\frac{d}{dx}y(x)\right)}{2x+1} - \frac{(2x+3)y(x)}{2x+1} = 0$

- ☐ Check to see if $x_0 = -\frac{1}{2}$ is a regular singular point
  - ◦ Define functions
    $\left[P_2(x) = -\frac{2}{2x+1}, P_3(x) = -\frac{2x+3}{2x+1}\right]$
  - ◦ $\left(x+\frac{1}{2}\right) \cdot P_2(x)$ is analytic at $x = -\frac{1}{2}$
    $\left(\left(x+\frac{1}{2}\right) \cdot P_2(x)\right)\bigg|_{x=-\frac{1}{2}} = -1$
  - ◦ $\left(x+\frac{1}{2}\right)^2 \cdot P_3(x)$ is analytic at $x = -\frac{1}{2}$
    $\left(\left(x+\frac{1}{2}\right)^2 \cdot P_3(x)\right)\bigg|_{x=-\frac{1}{2}} = 0$
  - ◦ $x = -\frac{1}{2}$is a regular singular point
    Check to see if $x_0 = -\frac{1}{2}$ is a regular singular point
    $x_0 = -\frac{1}{2}$
- • Multiply by denominators
  $(2x+1)\left(\frac{d^2}{dx^2}y(x)\right) - 2\frac{d}{dx}y(x) + (-2x-3)\,y(x) = 0$
- • Change variables using $x = u - \frac{1}{2}$ so that the regular singular point is at $u = 0$
  $2u\left(\frac{d^2}{du^2}y(u)\right) - 2\frac{d}{du}y(u) + (-2u-2)\,y(u) = 0$
- • Assume series solution for $y(u)$
  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$
- ☐ Rewrite ODE with series expansions
  - ◦ Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$
    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$
  - ◦ Shift index using $k- >k-m$
    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
  - ◦ Convert $\frac{d}{du}y(u)$ to series expansion
    $\frac{d}{du}y(u) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1}$
  - ◦ Shift index using $k- >k+1$
    $\frac{d}{du}y(u) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,u^{k+r}$
  - ◦ Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion
    $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-1}$
  - ◦ Shift index using $k- >k+1$
    $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\,u^{k+r}$

  Rewrite ODE with series expansions
  $2a_0 r(-2+r)\,u^{-1+r} + (2a_1(1+r)(-1+r) - 2a_0)\,u^r + \left(\sum_{k=1}^{\infty} (2a_{k+1}(k+1+r)(k+r-1) - 2a_k -\right.$
- • $a_0$cannot be 0 by assumption, giving the indicial equation
  $2r(-2+r) = 0$
- • Values of r that satisfy the indicial equation
  $r \in \{0, 2\}$

- Each term must be 0
  $2a_1(1+r)(-1+r)-2a_0=0$
- Each term in the series must be 0, giving the recursion relation
  $2a_{k+1}(k+1+r)(k+r-1)-2a_k-2a_{k-1}=0$
- Shift index using $k->k+1$
  $2a_{k+2}(k+2+r)(k+r)-2a_{k+1}-2a_k=0$
- Recursion relation that defines series solution to ODE
  $a_{k+2}=\frac{a_{k+1}+a_k}{(k+2+r)(k+r)}$
- Recursion relation for $r=0$
  $a_{k+2}=\frac{a_{k+1}+a_k}{(k+2)k}$
- Series not valid for $r=0$, division by 0 in the recursion relation at $k=0$
  $a_{k+2}=\frac{a_{k+1}+a_k}{(k+2)k}$
- Recursion relation for $r=2$
  $a_{k+2}=\frac{a_{k+1}+a_k}{(k+4)(k+2)}$
- Solution for $r=2$
  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2}=\frac{a_{k+1}+a_k}{(k+4)(k+2)}, 6a_1-2a_0=0\right]$
- Revert the change of variables $u=x+\frac{1}{2}$
  $\left[y(x)=\sum_{k=0}^{\infty} a_k \left(x+\frac{1}{2}\right)^{k+2}, a_{k+2}=\frac{a_{k+1}+a_k}{(k+4)(k+2)}, 6a_1-2a_0=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 16

```
dsolve((2*x+1)*diff(diff(y(x),x),x)-2*diff(y(x),x)-(2*x+3)*y(x) = 0,y(x),singsol=all)
```

$$y=c_1\mathrm{e}^{-x}+c_2\mathrm{e}^{x}x$$

Mathematica DSolve solution

Solving time : 0.333 (sec)
Leaf size : 69

```
DSolve[{(2*x+1)*D[y[x],{x,2}]-2*D[y[x],x]-(2*x+3)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->
```

$$y(x) \to \sqrt{2x+1}\exp\left(\int_1^x\left(\frac{1}{-2K[1]-1}-1\right)dK[1]\right)\left(c_2\int_1^x\exp\left(-2\int_1^{K[2]}\left(\frac{1}{-2K[1]-1}-1\right)dK[1]\right)dK[2]+c_1\right)$$

2.1.433 Problem 446

Internal problem ID [9605]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 446
Date solved : Monday, January 27, 2025 at 06:04:35 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$4x^2y'' + \left(-8x^2 + 4x\right) y' + \left(4x^2 - 4x - 1\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.084 (sec)

Writing the ode as

$$4x^2y'' + \left(-8x^2 + 4x\right) y' + \left(4x^2 - 4x - 1\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -8x^2 + 4x \\ C &= 4x^2 - 4x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.820: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-8x^2+4x}{4x^2}\, dx} \\ &= z_1 e^{x - \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{e^x}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^x}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-8x^2+4x}{4x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{2x-\ln(x)}}{(y_1)^2}\, dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{e^x}{\sqrt{x}}\right) + c_2 \left(\frac{e^x}{\sqrt{x}}(x)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2 \left(\frac{d^2}{dx^2} y(x)\right) + (-8x^2 + 4x) \left(\frac{d}{dx} y(x)\right) + (4x^2 - 4x - 1)\, y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2} y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2} y(x) = -\frac{(4x^2-4x-1)y(x)}{4x^2} + \frac{(2x-1)\left(\frac{d}{dx} y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2} y(x) - \frac{(2x-1)\left(\frac{d}{dx} y(x)\right)}{x} + \frac{(4x^2-4x-1)y(x)}{4x^2} = 0$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{2x-1}{x}, P_3(x) = \frac{4x^2-4x-1}{4x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{1}{4}$
  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2 \left(\frac{d^2}{dx^2} y(x)\right) - 4x(2x-1) \left(\frac{d}{dx} y(x)\right) + (4x^2 - 4x - 1)\, y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- > k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k- > k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)(-1+2r)\, x^r + (a_1(3+2r)(1+2r) - 4a_0(1+2r))\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)(2k+2\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3+2r)(1+2r) - 4a_0(1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = \frac{4a_0}{3+2r}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + (-8k - 8r + 4)\, a_{k-1} + 4a_{k-2} = 0$

- Shift index using $k- > k+2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + (-8k - 12 - 8r)\, a_{k+1} + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{4(2ka_{k+1} + 2ra_{k+1} - a_k + 3a_{k+1})}{4k^2 + 8kr + 4r^2 + 16k + 16r + 15}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = \frac{4(2ka_{k+1} - a_k + 2a_{k+1})}{4k^2 + 12k + 8}$

- Solution for $r = -\frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = \frac{4(2ka_{k+1} - a_k + 2a_{k+1})}{4k^2 + 12k + 8}, a_1 = 2a_0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = \frac{4(2ka_{k+1} - a_k + 4a_{k+1})}{4k^2 + 20k + 24}$

- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{4(2ka_{k+1} - a_k + 4a_{k+1})}{4k^2 + 20k + 24}, a_1 = a_0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = \frac{4(2ka_{k+1} - a_k + 2a_{k+1})}{4k^2 + 12k + 8}, a_1 = 2a_0, b_{k+2} = \frac{4(2kb_{k+1} - b_k + 4b_{k+1}}{4k^2 + 20k + 24}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.019 (sec)
Leaf size : 15

```
dsolve(4*x^2*diff(diff(y(x),x),x)+(-8*x^2+4*x)*diff(y(x),x)+(4*x^2-4*x-1)*y(x) = 0,y(x
```

$$y = \frac{e^{x} (c_2 x + c_1)}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.031 (sec)
Leaf size : 21

```
DSolve[{4*x^2*D[y[x],{x,2}]+(4*x-8*x^2)*D[y[x],x]+(4*x^2-4*x-1)*y[x]==0,{}},y[x],x,IncludeSi
```

$$y(x) \to \frac{e^{x} (c_2 x + c_1)}{\sqrt{x}}$$

2.1.434 Problem 447

Internal problem ID [9606]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 447
Date solved : Monday, January 27, 2025 at 06:04:35 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$y'' + 4xy' + \left(4x^2 + 2\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.059 (sec)

Writing the ode as

$$y'' + 4xy' + \left(4x^2 + 2\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4x \\ C &= 4x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.822: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x}{1}\,dx} \\
&= z_1 e^{-x^2} \\
&= z_1\left(\mathrm{e}^{-x^2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2x^2}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-x^2}\right) + c_2\left(\mathrm{e}^{-x^2}(x)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + 4x\left(\frac{d}{dx}y(x)\right) + (4x^2+2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$
  - Shift index using $k- >k-m$

    $x^m \cdot y(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$
  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$
  - Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$
  - Shift index using $k- >k+2$

    $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 + 2a_0 + (6a_3 + 6a_1)\,x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 2a_k(2k+1) + 4a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 + 2a_0 = 0, 6a_3 + 6a_1 = 0]$
- Solve for the dependent coefficient(s)

  $\{a_2 = -a_0, a_3 = -a_1\}$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\,a_{k+2} + 4a_k k + 2a_k + 4a_{k-2} = 0$

- Shift index using $k->k+2$

  $$\left((k+2)^2+3k+8\right)a_{k+4}+4a_{k+2}(k+2)+2a_{k+2}+4a_k=0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y(x)=\sum_{k=0}^{\infty}a_k x^k, a_{k+4}=-\frac{2(2ka_{k+2}+2a_k+5a_{k+2})}{k^2+7k+12}, a_2=-a_0, a_3=-a_1\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 16

```
dsolve(diff(diff(y(x),x),x)+4*diff(y(x),x)*x+(4*x^2+2)*y(x) = 0,y(x),singsol=all)
```

$$y=\mathrm{e}^{-x^2}(c_2x+c_1)$$

Mathematica DSolve solution

Solving time : 0.025 (sec)
Leaf size : 20

```
DSolve[{D[y[x],{x,2}]+4*x*D[y[x],x]+(4*x^2+2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x)\to e^{-x^2}(c_2x+c_1)$$

2.1.435 Problem 448

Internal problem ID [9607]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 448
Date solved : `Monday, January 27, 2025 at 06:04:36 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + 2x(x-1)\,y' + \left(x^2-2x+2\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.069 (sec)

Writing the ode as

$$x^2y'' + \left(2x^2-2x\right)y' + \left(x^2-2x+2\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 2x^2-2x \\ C &= x^2-2x+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.824: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - -\infty\\ &= \infty\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{2x^2-2x}{x^2}\, dx}\\ &= z_1 e^{-x+\ln(x)}\\ &= z_1\left(x\,\mathrm{e}^{-x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2-2x}{x^2} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{-2x+2\ln(x)}}{(y_1)^2} dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(x \,\mathrm{e}^{-x}\right) + c_2 \left(x \,\mathrm{e}^{-x}(x)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right) + 2x(x-1)\left(\frac{d}{dx}y(x)\right) + (x^2-2x+2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{(x^2-2x+2)y(x)}{x^2} - \frac{2\left(\frac{d}{dx}y(x)\right)(x-1)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) + \frac{2\left(\frac{d}{dx}y(x)\right)(x-1)}{x} + \frac{(x^2-2x+2)y(x)}{x^2} = 0$
- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{2(x-1)}{x}, P_3(x) = \frac{x^2-2x+2}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 2$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $x^2\left(\frac{d^2}{dx^2}y(x)\right) + 2x(x-1)\left(\frac{d}{dx}y(x)\right) + (x^2-2x+2)\,y(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$
    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- > k-m$

$$x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx} y(x)\right)$ to series expansion for $m = 1..2$

$$x^m \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right)$ to series expansion

$$x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+r)(-2+r)\, x^r + (a_1 r(-1+r) + 2a_0(-1+r))\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-1)(k+r-2) +$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+r)(-2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{1, 2\}$
- Each term must be 0
  $a_1 r(-1+r) + 2a_0(-1+r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = -\frac{2a_0}{r}$
- Each term in the series must be 0, giving the recursion relation
  $a_k(k+r-1)(k+r-2) + 2a_{k-1}k + 2a_{k-1}r + a_{k-2} - 4a_{k-1} = 0$
- Shift index using $k- >k+2$
  $a_{k+2}(k+1+r)(k+r) + 2a_{k+1}(k+2) + 2a_{k+1}r + a_k - 4a_{k+1} = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{2ka_{k+1} + 2a_{k+1}r + a_k}{(k+1+r)(k+r)}$
- Recursion relation for $r = 1$
  $a_{k+2} = -\frac{2ka_{k+1} + a_k + 2a_{k+1}}{(k+2)(k+1)}$
- Solution for $r = 1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{2ka_{k+1} + a_k + 2a_{k+1}}{(k+2)(k+1)}, a_1 = -2a_0\right]$
- Recursion relation for $r = 2$
  $a_{k+2} = -\frac{2ka_{k+1} + a_k + 4a_{k+1}}{(k+3)(k+2)}$
- Solution for $r = 2$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{2ka_{k+1} + a_k + 4a_{k+1}}{(k+3)(k+2)}, a_1 = -a_0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = -\frac{2ka_{k+1} + a_k + 2a_{k+1}}{(k+2)(k+1)}, a_1 = -2a_0, b_{k+2} = -\frac{2kb_{k+1} + b_k +}{(k+3)(k+}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 15

```
dsolve(x^2*diff(diff(y(x),x),x)+2*x*(x-1)*diff(y(x),x)+(x^2-2*x+2)*y(x) = 0,y(x),singsol
```

$$y = e^{-x} x (c_2 x + c_1)$$

Mathematica DSolve solution

Solving time : 0.032 (sec)
Leaf size : 19

```
DSolve[{x^2*D[y[x],{x,2}]+2*x*(x-1)*D[y[x],x]+(x^2-2*x+2)*y[x]==0,{}},y[x],x,IncludeSingularSo
```

$$y(x) \to e^{-x} x (c_2 x + c_1)$$

2.1.436 Problem 449

Internal problem ID [9608]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 449
Date solved : `Monday, January 27, 2025 at 06:04:36 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x(2x-1)\,y' + \left(x^2-x-1\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.150 (sec)

Writing the ode as

$$x^2y'' + \left(-2x^2+x\right)y' + \left(x^2-x-1\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x^2+x \\ C &= x^2-x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.826: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=-\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2}-\left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^-}{x-c_1}\right)+(-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x}+(-)\,(0) \\
&= -\frac{1}{2x} \\
&= -\frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}
$$

Let

$$
p(x)=1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2x}\right)(0)+\left(\left(\frac{1}{2x^2}\right)+\left(-\frac{1}{2x}\right)^2-\left(\frac{3}{4x^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx}\\ &= \mathrm{e}^{\int -\frac{1}{2x} dx}\\ &= \frac{1}{\sqrt{x}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x^2+x}{x^2} \, dx}\\ &= z_1 e^{x-\frac{\ln(x)}{2}}\\ &= z_1\left(\frac{\mathrm{e}^x}{\sqrt{x}}\right)\end{aligned}$$

Which simplifies to

$$y_1=\frac{\mathrm{e}^x}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1\int \frac{e^{\int -\frac{-2x^2+x}{x^2} \, dx}}{(y_1)^2} \, dx\\ &= y_1\int \frac{e^{2x-\ln(x)}}{(y_1)^2} \, dx\\ &= y_1\left(\frac{x^3\mathrm{e}^{2x-\ln(x)}\mathrm{e}^{-2x}}{2}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2\\ &= c_1\left(\frac{\mathrm{e}^x}{x}\right)+c_2\left(\frac{\mathrm{e}^x}{x}\left(\frac{x^3\mathrm{e}^{2x-\ln(x)}\mathrm{e}^{-2x}}{2}\right)\right)\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right)-x(2x-1)\left(\frac{d}{dx}y(x)\right)+\left(x^2-x-1\right)y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{(x^2-x-1)y(x)}{x^2}+\frac{(2x-1)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)-\frac{(2x-1)\left(\frac{d}{dx}y(x)\right)}{x}+\frac{(x^2-x-1)y(x)}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

- ○ Define functions

$\left[P_2(x)=-\frac{2x-1}{x},P_3(x)=\frac{x^2-x-1}{x^2}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

$\left(x\cdot P_2(x)\right)\bigg|_{x=0}=1$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=-1$

- ○ $x=0$is a regular singular point

Check to see if $x_0=0$ is a regular singular point

$x_0=0$

- Multiply by denominators

$x^2\left(\frac{d^2}{dx^2}y(x)\right)-x(2x-1)\left(\frac{d}{dx}y(x)\right)+\left(x^2-x-1\right)y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

$x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ○ Shift index using $k->k-m$

$x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ○ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ○ Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

$x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

Rewrite ODE with series expansions

$a_0(1+r)\,(-1+r)\,x^r+\left(a_1(2+r)\,r-a_0(1+2r)\right)x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_k(k+r+1)\,(k+r-1)-a_{k-1}\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(1 + r)(-1 + r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 1\}$
- Each term must be 0

  $a_1(2 + r)r - a_0(1 + 2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = \frac{a_0(1+2r)}{r(2+r)}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k + r + 1)(k + r - 1) + (1 - 2k - 2r)a_{k-1} + a_{k-2} = 0$
- Shift index using $k- >k + 2$

  $a_{k+2}(k + 3 + r)(k + r + 1) + (-3 - 2k - 2r)a_{k+1} + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2ka_{k+1}+2ra_{k+1}-a_k+3a_{k+1}}{(k+3+r)(k+r+1)}$
- Recursion relation for $r = -1$

  $a_{k+2} = \frac{2ka_{k+1}-a_k+a_{k+1}}{(k+2)k}$
- Series not valid for $r = -1$, division by 0 in the recursion relation at $k = 0$

  $a_{k+2} = \frac{2ka_{k+1}-a_k+a_{k+1}}{(k+2)k}$
- Recursion relation for $r = 1$

  $a_{k+2} = \frac{2ka_{k+1}-a_k+5a_{k+1}}{(k+4)(k+2)}$
- Solution for $r = 1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = \frac{2ka_{k+1}-a_k+5a_{k+1}}{(k+4)(k+2)}, a_1 = a_0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)-x*(-1+2*x)*diff(y(x),x)+(x^2-x-1)*y(x) = 0,y(x),singsol=
```

$$y = \frac{e^x(c_2x^2 + c_1)}{x}$$

Mathematica DSolve solution

Solving time : 0.031 (sec)
Leaf size : 23

```
DSolve[{x^2*D[y[x],{x,2}]-x*(2*x-1)*D[y[x],x]+(x^2-x-1)*y[x]==0,{}},y[x],x,IncludeSingularSo
```

$$y(x) \to e^x \left(\frac{c_1}{x} + \frac{c_2 x}{2}\right)$$

2.1.437 Problem 450

Internal problem ID [9609]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 450
Date solved : `Monday, January 27, 2025 at 06:04:37 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(1-2x)\,y''+2y'+(2x-3)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.225 (sec)

Writing the ode as

$$(1-2x)\,y''+2y'+(2x-3)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=1-2x\\ B&=2\\ C&=2x-3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{4x^2-8x+6}{(-1+2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=4x^2-8x+6\\ t&=(-1+2x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{4x^2-8x+6}{(-1+2x)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.828: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (-1+2x)^2$. There is a pole at $x = \frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{3}{4\left(x-\frac{1}{2}\right)^2} - \frac{1}{x-\frac{1}{2}}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 - \frac{1}{2x} + \frac{1}{4x^3} + \frac{11}{32x^4} + \frac{21}{64x^5} + \frac{15}{64x^6} + \frac{3}{32x^7} - \frac{117}{2048x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 - 8x + 6}{4x^2 - 4x + 1} \\ &= Q + \frac{R}{4x^2 - 4x + 1} \\ &= (1) + \left(\frac{-4x + 5}{4x^2 - 4x + 1}\right) \\ &= 1 + \frac{-4x + 5}{4x^2 - 4x + 1} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{1} - 0\right) &&= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{1} - 0\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^2 - 8x + 6}{(-1 + 2x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{1}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(x - \frac{1}{2}\right)} + (1) \\
&= -\frac{1}{2\left(x - \frac{1}{2}\right)} + 1 \\
&= \frac{-2 + 2x}{-1 + 2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\left(x - \frac{1}{2}\right)} + 1\right)(0) + \left(\left(\frac{1}{2\left(x - \frac{1}{2}\right)^2}\right) + \left(-\frac{1}{2\left(x - \frac{1}{2}\right)} + 1\right)^2 - \left(\frac{4x^2 - 8x + 6}{(-1 + 2x)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2\left(x-\frac{1}{2}\right)}+1\right) dx} \\
&= \frac{\mathrm{e}^{x}}{\sqrt{-1+2x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1-2x}\, dx} \\
&= z_1 e^{\frac{\ln(1-2x)}{2}} \\
&= z_1\left(\sqrt{1-2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\sqrt{1-2x}\,\mathrm{e}^{x}}{\sqrt{-1+2x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1-2x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\ln(1-2x)}}{(y_1)^2}\, dx \\
&= y_1\left(-x\,\mathrm{e}^{-2x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\sqrt{1-2x}\,\mathrm{e}^{x}}{\sqrt{-1+2x}}\right) + c_2\left(\frac{\sqrt{1-2x}\,\mathrm{e}^{x}}{\sqrt{-1+2x}}\left(-x\,\mathrm{e}^{-2x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(-2x+1)\left(\frac{d^2}{dx^2}y(x)\right) + 2\frac{d}{dx}y(x) + (2x-3)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{(2x-3)y(x)}{2x-1} + \frac{2\left(\frac{d}{dx}y(x)\right)}{2x-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x) - \frac{2\left(\frac{d}{dx}y(x)\right)}{2x-1} - \frac{(2x-3)y(x)}{2x-1} = 0$

- ☐ Check to see if $x_0 = \frac{1}{2}$ is a regular singular point
  - ◦ Define functions
    $\left[P_2(x) = -\frac{2}{2x-1}, P_3(x) = -\frac{2x-3}{2x-1}\right]$
  - ◦ $\left(x-\frac{1}{2}\right) \cdot P_2(x)$ is analytic at $x = \frac{1}{2}$
    $\left(\left(x-\frac{1}{2}\right) \cdot P_2(x)\right)\bigg|_{x=\frac{1}{2}} = -1$
  - ◦ $\left(x-\frac{1}{2}\right)^2 \cdot P_3(x)$ is analytic at $x = \frac{1}{2}$
    $\left(\left(x-\frac{1}{2}\right)^2 \cdot P_3(x)\right)\bigg|_{x=\frac{1}{2}} = 0$
  - ◦ $x = \frac{1}{2}$is a regular singular point
    Check to see if $x_0 = \frac{1}{2}$ is a regular singular point
    $x_0 = \frac{1}{2}$
- • Multiply by denominators
  $(2x-1)\left(\frac{d^2}{dx^2}y(x)\right) - 2\frac{d}{dx}y(x) + (3-2x)\,y(x) = 0$
- • Change variables using $x = u + \frac{1}{2}$ so that the regular singular point is at $u = 0$
  $2u\left(\frac{d^2}{du^2}y(u)\right) - 2\frac{d}{du}y(u) + (2-2u)\,y(u) = 0$
- • Assume series solution for $y(u)$
  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$
- ☐ Rewrite ODE with series expansions
  - ◦ Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$
    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$
  - ◦ Shift index using $k- >k-m$
    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
  - ◦ Convert $\frac{d}{du}y(u)$ to series expansion
    $\frac{d}{du}y(u) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1}$
  - ◦ Shift index using $k- >k+1$
    $\frac{d}{du}y(u) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,u^{k+r}$
  - ◦ Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion
    $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-1}$
  - ◦ Shift index using $k- >k+1$
    $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$

    Rewrite ODE with series expansions
    $2a_0r(-2+r)\,u^{-1+r} + (2a_1(1+r)\,(-1+r) + 2a_0)\,u^r + \left(\sum_{k=1}^{\infty} (2a_{k+1}(k+1+r)\,(k+r-1) + 2a_k\right.$
- • $a_0$cannot be 0 by assumption, giving the indicial equation
  $2r(-2+r) = 0$
- • Values of r that satisfy the indicial equation
  $r \in \{0, 2\}$

- Each term must be 0

  $2a_1(1+r)(-1+r)+2a_0=0$
- Each term in the series must be 0, giving the recursion relation

  $2a_{k+1}(k+1+r)(k+r-1)+2a_k-2a_{k-1}=0$
- Shift index using $k->k+1$

  $2a_{k+2}(k+2+r)(k+r)+2a_{k+1}-2a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{-a_{k+1}+a_k}{(k+2+r)(k+r)}$
- Recursion relation for $r=0$

  $a_{k+2}=\frac{-a_{k+1}+a_k}{(k+2)k}$
- Series not valid for $r=0$, division by 0 in the recursion relation at $k=0$

  $a_{k+2}=\frac{-a_{k+1}+a_k}{(k+2)k}$
- Recursion relation for $r=2$

  $a_{k+2}=\frac{-a_{k+1}+a_k}{(k+4)(k+2)}$
- Solution for $r=2$

  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2}=\frac{-a_{k+1}+a_k}{(k+4)(k+2)}, 6a_1+2a_0=0\right]$
- Revert the change of variables $u=x-\frac{1}{2}$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k\left(x-\frac{1}{2}\right)^{k+2}, a_{k+2}=\frac{-a_{k+1}+a_k}{(k+4)(k+2)}, 6a_1+2a_0=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 16

```
dsolve((1-2*x)*diff(diff(y(x),x),x)+2*diff(y(x),x)+(2*x-3)*y(x) = 0,y(x),singsol=all)
```

$$y=e^{x} c_1+c_2 x\,e^{-x}$$

Mathematica DSolve solution

Solving time : 0.35 (sec)
Leaf size : 69

```
DSolve[{(1-2*x)*D[y[x],{x,2}]+2*D[y[x],x]+(2*x-3)*y[x]==0,{}},y[x],x,IncludeSingularSolution
```

$$y(x) \to \sqrt{1-2x} \exp\left(\int_1^x \left(1 + \frac{1}{1-2K[1]}\right) dK[1]\right) \left(c_2 \int_1^x \exp\left(-2\int_1^{K[2]} \left(1+\frac{1}{1-2K[1]}\right) dK[1]\right) dK[2] + c_1\right)$$

2.1.438 Problem 451

Internal problem ID [9610]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 451
Date solved : Monday, January 27, 2025 at 06:04:38 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$2xy'' + (4x+1)\,y' + (2x+1)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.142 (sec)

Writing the ode as

$$2xy'' + (4x+1)\,y' + (2x+1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x \\ B &= 4x+1 \\ C &= 2x+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{3}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.830: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{3}{16x^2}$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{4}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=-\frac{3}{16x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}\right)\\
&= \frac{1}{4}-\left(\frac{1}{4}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^{-}}{x-c_1}\right)+(-)[\sqrt{r}]_\infty\\
&= \frac{1}{4x}+(-)(0)\\
&= \frac{1}{4x}\\
&= \frac{1}{4x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}
$$

Let

$$
p(x)=1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0)+2\left(\frac{1}{4x}\right)(0)+\left(\left(-\frac{1}{4x^2}\right)+\left(\frac{1}{4x}\right)^2-\left(-\frac{3}{16x^2}\right)\right)=0
$$
$$
0=0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\,dx} \\
&= \mathrm{e}^{\int \frac{1}{4x}dx} \\
&= x^{1/4}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x+1}{2x}\,dx} \\
&= z_1 e^{-x-\frac{\ln(x)}{4}} \\
&= z_1\left(\frac{\mathrm{e}^{-x}}{x^{1/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1=\mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1\int \frac{e^{\int -\frac{4x+1}{2x}\,dx}}{(y_1)^2}\,dx \\
&= y_1\int \frac{e^{-2x-\frac{\ln(x)}{2}}}{(y_1)^2}\,dx \\
&= y_1\left(2x\,\mathrm{e}^{-2x-\frac{\ln(x)}{2}}\mathrm{e}^{2x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1+c_2y_2 \\
&= c_1\left(\mathrm{e}^{-x}\right)+c_2\left(\mathrm{e}^{-x}\left(2x\,\mathrm{e}^{-2x-\frac{\ln(x)}{2}}\mathrm{e}^{2x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$2\left(\frac{d^2}{dx^2}y(x)\right)x+(4x+1)\left(\frac{d}{dx}y(x)\right)+(2x+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
$\frac{d^2}{dx^2}y(x)=-\frac{(2x+1)y(x)}{2x}-\frac{(4x+1)\left(\frac{d}{dx}y(x)\right)}{2x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
$\frac{d^2}{dx^2}y(x)+\frac{(4x+1)\left(\frac{d}{dx}y(x)\right)}{2x}+\frac{(2x+1)y(x)}{2x}=0$
- ☐ Check to see if $x_0=0$ is a regular singular point
  - Define functions
  $\left[P_2(x)=\frac{4x+1}{2x},P_3(x)=\frac{2x+1}{2x}\right]$
  - $x\cdot P_2(x)$ is analytic at $x=0$
  $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{1}{2}$
  - $x^2\cdot P_3(x)$ is analytic at $x=0$
  $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$
  - $x=0$is a regular singular point
  Check to see if $x_0=0$ is a regular singular point
  $x_0=0$
- Multiply by denominators
$2\left(\frac{d^2}{dx^2}y(x)\right)x+(4x+1)\left(\frac{d}{dx}y(x)\right)+(2x+1)\,y(x)=0$
- Assume series solution for $y(x)$
$y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$
  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$
  - Shift index using $k->k-m$
  $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$
  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..1$
  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k->k+1-m$
  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion
  $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-1}$
  - Shift index using $k->k+1$
  $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$
  Rewrite ODE with series expansions

$$a_0 r(-1+2r)\, x^{-1+r} + (a_1(1+r)\,(1+2r) + a_0(1+4r))\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(2k+1+2r)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
$$r(-1+2r) = 0$$
- Values of r that satisfy the indicial equation
$$r \in \left\{0, \frac{1}{2}\right\}$$
- Each term must be 0
$$a_1(1+r)\,(1+2r) + a_0(1+4r) = 0$$
- Each term in the series must be 0, giving the recursion relation
$$2(k+1+r)\left(k+r+\frac{1}{2}\right) a_{k+1} + 4a_k k + 4a_k r + a_k + 2a_{k-1} = 0$$
- Shift index using $k->k+1$
$$2(k+2+r)\left(k+\frac{3}{2}+r\right) a_{k+2} + 4a_{k+1}(k+1) + 4r a_{k+1} + a_{k+1} + 2a_k = 0$$
- Recursion relation that defines series solution to ODE
$$a_{k+2} = -\frac{4k a_{k+1} + 4r a_{k+1} + 2a_k + 5a_{k+1}}{(k+2+r)(2k+3+2r)}$$
- Recursion relation for $r=0$
$$a_{k+2} = -\frac{4k a_{k+1} + 2a_k + 5a_{k+1}}{(k+2)(2k+3)}$$
- Solution for $r=0$
$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{4k a_{k+1} + 2a_k + 5a_{k+1}}{(k+2)(2k+3)}, a_1 + a_0 = 0\right]$$
- Recursion relation for $r=\frac{1}{2}$
$$a_{k+2} = -\frac{4k a_{k+1} + 2a_k + 7a_{k+1}}{\left(k+\frac{5}{2}\right)(2k+4)}$$
- Solution for $r=\frac{1}{2}$
$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4k a_{k+1} + 2a_k + 7a_{k+1}}{\left(k+\frac{5}{2}\right)(2k+4)}, 3a_1 + 3a_0 = 0\right]$$
- Combine solutions and rename parameters
$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4k a_{k+1} + 2a_k + 5a_{k+1}}{(k+2)(2k+3)}, a_1 + a_0 = 0, b_{k+2} = -\frac{4k b_{k+1} + 2b_k}{\left(k+\frac{5}{2}\right)(2k}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 16

```
dsolve(2*x*diff(diff(y(x),x),x)+(4*x+1)*diff(y(x),x)+(2*x+1)*y(x) = 0,y(x),singsol=all
```

$$y = e^{-x}\left(c_1 + c_2\sqrt{x}\right)$$

Mathematica DSolve solution

Solving time : 0.036 (sec)
Leaf size : 23

```
DSolve[{2*x*D[y[x],{x,2}]+(4*x+1)*D[y[x],x]+(2*x+1)*y[x]==0,{}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to e^{-x}\left(2c_2\sqrt{x} + c_1\right)$$

2.1.439 Problem 452

Internal problem ID [9611]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 452
Date solved : `Monday, January 27, 2025 at 06:04:38 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' - (2x+1)\,y' + (x+1)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.149 (sec)

Writing the ode as

$$xy'' + (-2x-1)\,y' + (x+1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -2x-1 \\ C &= x+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.832: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the gcd$(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)\,(0) \\
&= -\frac{1}{2x} \\
&= -\frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2x}\right)(0)+\left(\left(\frac{1}{2x^2}\right)+\left(-\frac{1}{2x}\right)^2-\left(\frac{3}{4x^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\,dx} \\
&= \mathrm{e}^{\int -\frac{1}{2x}dx} \\
&= \frac{1}{\sqrt{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x-1}{x}\,dx} \\
&= z_1 e^{x+\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\,\mathrm{e}^{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1\int \frac{e^{\int -\frac{-2x-1}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1\int \frac{\mathrm{e}^{2x+\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{x\,\mathrm{e}^{2x+\ln(x)}\mathrm{e}^{-2x}}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1y_1+c_2y_2 \\
&= c_1(\mathrm{e}^{x})+c_2\left(\mathrm{e}^{x}\left(\frac{x\,\mathrm{e}^{2x+\ln(x)}\mathrm{e}^{-2x}}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dx^2}y(x)\right)x-(2x+1)\left(\frac{d}{dx}y(x)\right)+(x+1)y(x)=0$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
$\frac{d^2}{dx^2}y(x)=-\frac{(x+1)y(x)}{x}+\frac{(2x+1)\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
$\frac{d^2}{dx^2}y(x)-\frac{(2x+1)\left(\frac{d}{dx}y(x)\right)}{x}+\frac{(x+1)y(x)}{x}=0$

□ Check to see if $x_0=0$ is a regular singular point
- ○ Define functions
$\left[P_2(x)=-\frac{2x+1}{x},P_3(x)=\frac{x+1}{x}\right]$
- ○ $x\cdot P_2(x)$ is analytic at $x=0$
$\left.(x\cdot P_2(x))\right|_{x=0}=-1$
- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$
$\left.(x^2\cdot P_3(x))\right|_{x=0}=0$
- ○ $x=0$is a regular singular point
Check to see if $x_0=0$ is a regular singular point
$x_0=0$

- Multiply by denominators
$\left(\frac{d^2}{dx^2}y(x)\right)x+(-2x-1)\left(\frac{d}{dx}y(x)\right)+(x+1)y(x)=0$
- Assume series solution for $y(x)$
$y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions
- ○ Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$
$x^m\cdot y(x)=\sum_{k=0}^{\infty}a_k x^{k+r+m}$
- ○ Shift index using $k->k-m$
$x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$
- ○ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..1$
$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$
- ○ Shift index using $k->k+1-m$
$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$
- ○ Convert $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion
$x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-1}$
- ○ Shift index using $k->k+1$
$x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

Rewrite ODE with series expansions

$$a_0 r(-2+r)\, x^{-1+r} + (a_1(1+r)(-1+r) - a_0(-1+2r))\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)(k+r-1) - c$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 2\}$
- Each term must be 0
  $a_1(1+r)(-1+r) - a_0(-1+2r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)(k+r-1) + a_k(-2k-2r+1) + a_{k-1} = 0$
- Shift index using $k- >k+1$
  $a_{k+2}(k+2+r)(k+r) + a_{k+1}(-2k-1-2r) + a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{2k a_{k+1} + 2r a_{k+1} - a_k + a_{k+1}}{(k+2+r)(k+r)}$
- Recursion relation for $r = 0$
  $a_{k+2} = \frac{2k a_{k+1} - a_k + a_{k+1}}{(k+2)k}$
- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$
  $a_{k+2} = \frac{2k a_{k+1} - a_k + a_{k+1}}{(k+2)k}$
- Recursion relation for $r = 2$
  $a_{k+2} = \frac{2k a_{k+1} - a_k + 5a_{k+1}}{(k+4)(k+2)}$
- Solution for $r = 2$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{2k a_{k+1} - a_k + 5a_{k+1}}{(k+4)(k+2)}, 3a_1 - 3a_0 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 14

```
dsolve(x*diff(diff(y(x),x),x)-(2*x+1)*diff(y(x),x)+(x+1)*y(x) = 0,y(x),singsol=all)
```

$$y = e^x (c_2 x^2 + c_1)$$

Mathematica DSolve solution

Solving time : 0.026 (sec)
Leaf size : 23

```
DSolve[{x*D[y[x],{x,2}]-(2*x+1)*D[y[x],x]+(x+1)*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to \frac{1}{2}e^x\left(c_2x^2 + 2c_1\right)$$

2.1.440 Problem 453

Internal problem ID [9612]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 453
Date solved : `Monday, January 27, 2025 at 06:04:39 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' - 4x(x+1)\,y' + (2x+3)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.098 (sec)

Writing the ode as

$$4x^2y'' + \left(-4x^2 - 4x\right)y' + (2x+3)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x^2 - 4x \\ C &= 2x+3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.834: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x^2-4x}{4x^2}\, dx} \\ &= z_1 e^{\frac{x}{2}+\frac{\ln(x)}{2}} \\ &= z_1 \left(\sqrt{x}\, e^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^2-4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{x+\ln(x)}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{e^{x+\ln(x)}}{x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\sqrt{x}\right) + c_2\left(\sqrt{x}\left(\frac{e^{x+\ln(x)}}{x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2\left(\frac{d^2}{dx^2}y(x)\right) - 4x(x+1)\left(\frac{d}{dx}y(x)\right) + (2x+3)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(2x+3)y(x)}{4x^2} + \frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) - \frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(2x+3)y(x)}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{x+1}{x}, P_3(x) = \frac{2x+3}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{3}{4}$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) - 4x(x+1)\left(\frac{d}{dx}y(x)\right) + (2x+3)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$
  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$
  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
- Shift index using $k->k+1-m$
  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion
  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions
  $a_0(-1+2r)\,(-3+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)\,(2k+2r-3) - 2a_{k-1}(2k+2r-3)\right) x^{k+r}\right) =$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+2r)\,(-3+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $4\left(\left(k+r-\frac{1}{2}\right) a_k - a_{k-1}\right)\left(k+r-\frac{3}{2}\right) = 0$
- Shift index using $k->k+1$
  $4\left(\left(k+\frac{1}{2}+r\right) a_{k+1} - a_k\right)\left(k+r-\frac{1}{2}\right) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{2a_k}{2k+1+2r}$
- Recursion relation for $r = \frac{1}{2}$
  $a_{k+1} = \frac{2a_k}{2k+2}$
- Solution for $r = \frac{1}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{2a_k}{2k+2}\right]$
- Recursion relation for $r = \frac{3}{2}$
  $a_{k+1} = \frac{2a_k}{2k+4}$
- Solution for $r = \frac{3}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1} = \frac{2a_k}{2k+4}\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+1} = \frac{2a_k}{2k+2}, b_{k+1} = \frac{2b_k}{2k+4}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 14

```
dsolve(4*x^2*diff(diff(y(x),x),x)-4*x*(x+1)*diff(y(x),x)+(2*x+3)*y(x) = 0,y(x),singsol=a
```

$$y = (c_1 + e^x c_2)\sqrt{x}$$

Mathematica DSolve solution

Solving time : 0.047 (sec)
Leaf size : 25

```
DSolve[{4*x^2*D[y[x],{x,2}]-4*x*(x+1)*D[y[x],x]+(2*x+3)*y[x]==0,{}},y[x],x,IncludeSingularSolu
```

$$y(x) \to \sqrt{e}\sqrt{x}(c_2 e^x + c_1)$$

2.1.441 Problem 454

Internal problem ID [9613]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 454
Date solved : `Monday, January 27, 2025 at 06:04:39 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + (2-2x)\,y' + (x-2)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.062 (sec)

Writing the ode as

$$xy'' + (2-2x)\,y' + (x-2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2-2x \\ C &= x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.836: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2-2x}{x}\, dx} \\ &= z_1 e^{x-\ln(x)} \\ &= z_1 \left(\frac{e^x}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^x}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2-2x}{x} dx}}{(y_1)^2} dx \\
&= y_1 \int \frac{e^{2x-2\ln(x)}}{(y_1)^2} dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^x}{x}\right) + c_2 \left(\frac{e^x}{x}(x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dx^2} y(x)\right) x + (-2x+2)\left(\frac{d}{dx} y(x)\right) + (x-2)\, y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2} y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2} y(x) = -\frac{(x-2)y(x)}{x} + \frac{2\left(\frac{d}{dx} y(x)\right)(x-1)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2} y(x) - \frac{2\left(\frac{d}{dx} y(x)\right)(x-1)}{x} + \frac{(x-2)y(x)}{x} = 0$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{2(x-1)}{x}, P_3(x) = \frac{x-2}{x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $\left(\frac{d^2}{dx^2} y(x)\right) x + (-2x+2)\left(\frac{d}{dx} y(x)\right) + (x-2)\, y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..1$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(1+r)\, x^{-1+r} + (a_1(1+r)\,(2+r) - 2a_0(1+r))\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(k+2+r) - 2a_k(k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $a_1(1+r)\,(2+r) - 2a_0(1+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k+2+r) - 2a_k k - 2a_k r - 2a_k + a_{k-1} = 0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)\,(k+3+r) - 2a_{k+1}(k+1) - 2ra_{k+1} - 2a_{k+1} + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2ka_{k+1}+2ra_{k+1}-a_k+4a_{k+1}}{(k+2+r)(k+3+r)}$

- Recursion relation for $r = -1$

  $a_{k+2} = \frac{2ka_{k+1}-a_k+2a_{k+1}}{(k+1)(k+2)}$

- Solution for $r = -1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = \frac{2ka_{k+1}-a_k+2a_{k+1}}{(k+1)(k+2)}, 0 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{2ka_{k+1}-a_k+4a_{k+1}}{(k+2)(k+3)}$

- Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2ka_{k+1}-a_k+4a_{k+1}}{(k+2)(k+3)}, 2a_1 - 2a_0 = 0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = \frac{2ka_{k+1}-a_k+2a_{k+1}}{(k+1)(k+2)}, 0 = 0, b_{k+2} = \frac{2kb_{k+1}-b_k+4b_{k+1}}{(k+2)(k+3)}, 2b_1 - 2b\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 15

```
dsolve(x*diff(diff(y(x),x),x)+(-2*x+2)*diff(y(x),x)+(x-2)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{e^{x}(c_1 x + c_2)}{x}$$

Mathematica DSolve solution

Solving time : 0.034 (sec)
Leaf size : 19

```
DSolve[{x*D[y[x],{x,2}]+(2-2*x)*D[y[x],x]+(x-2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to \frac{e^{x}(c_2 x + c_1)}{x}$$

2.1.442 Problem 455

Internal problem ID [9614]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 455
Date solved : Monday, January 27, 2025 at 06:04:40 PM
CAS classification :
[[_Emden, _Fowler], [_2nd_order, _linear, '_with_symmetry_[0,F(x)]']]

Solve

$$x^2y'' - 2xy' + 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.049 (sec)

Writing the ode as

$$x^2y'' - 2xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.838: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\, dx} \\ &= z_1 e^{\ln(x)} \\ &= z_1(x) \end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2} dx}}{(y_1)^2} dx \\
&= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2} dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2(x(x))
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right) - 2x\left(\frac{d}{dx}y(x)\right) + 2y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{2y(x)}{x^2} + \frac{2\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) - \frac{2\left(\frac{d}{dx}y(x)\right)}{x} + \frac{2y(x)}{x^2} = 0$
- Multiply by denominators of the ODE
  $x^2\left(\frac{d^2}{dx^2}y(x)\right) - 2x\left(\frac{d}{dx}y(x)\right) + 2y(x) = 0$
- Make a change of variables
  $t = \ln(x)$
- ☐ Substitute the change of variables back into the ODE
  - Calculate the 1st derivative of y with respect to x , using the chain rule
    $\frac{d}{dx}y(x) = \left(\frac{d}{dt}y(t)\right)\left(\frac{d}{dx}t(x)\right)$
  - Compute derivative
    $\frac{d}{dx}y(x) = \frac{\frac{d}{dt}y(t)}{x}$
  - Calculate the 2nd derivative of y with respect to x , using the chain rule
    $\frac{d^2}{dx^2}y(x) = \left(\frac{d^2}{dt^2}y(t)\right)\left(\frac{d}{dx}t(x)\right)^2 + \left(\frac{d^2}{dx^2}t(x)\right)\left(\frac{d}{dt}y(t)\right)$
  - Compute derivative
    $\frac{d^2}{dx^2}y(x) = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE
  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - 2\frac{d}{dt}y(t) + 2y(t) = 0$
- Simplify
  $\frac{d^2}{dt^2}y(t) - 3\frac{d}{dt}y(t) + 2y(t) = 0$
- Characteristic polynomial of ODE
  $r^2 - 3r + 2 = 0$
- Factor the characteristic polynomial

$(r-1)(r-2)=0$

- Roots of the characteristic polynomial
  $r=(1,2)$
- 1st solution of the ODE
  $y_1(t)=e^t$
- 2nd solution of the ODE
  $y_2(t)=e^{2t}$
- General solution of the ODE
  $y(t)=C1y_1(t)+C2y_2(t)$
- Substitute in solutions
  $y(t)=C1\,e^t+C2\,e^{2t}$
- Change variables back using $t=\ln(x)$
  $y(x)=C2\,x^2+C1x$
- Simplify
  $y(x)=x(C2x+C1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 11

```
dsolve(x^2*diff(diff(y(x),x),x)-2*diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y=x(c_1x+c_2)$$

Mathematica DSolve solution

Solving time : 0.012 (sec)
Leaf size : 14

```
DSolve[{x^2*D[y[x],{x,2}]-2*x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to x(c_2x+c_1)$$

2.1.443 Problem 456

Internal problem ID [9615]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 456
Date solved : `Monday, January 27, 2025 at 06:04:40 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' - (2x+2)\,y' + (x+2)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.126 (sec)

Writing the ode as

$$xy'' + (-2x-2)\,y' + (x+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -2x-2 \\ C &= x+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.840: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the gcd$(s,t)=1$. This gives $b=2$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=2\\ \alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=-1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=-1$ then

$$\begin{aligned}d&=\alpha_\infty^- - \left(\alpha_{c_1}^-\right)\\ &=-1-(-1)\\ &=0\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega&=\left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^-}{x-c_1}\right)+(-)[\sqrt{r}]_\infty\\ &=-\frac{1}{x}+(-)(0)\\ &=-\frac{1}{x}\\ &=-\frac{1}{x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0)+2\left(-\frac{1}{x}\right)(0)+\left(\left(\frac{1}{x^2}\right)+\left(-\frac{1}{x}\right)^2-\left(\frac{2}{x^2}\right)\right)&=0\\ 0&=0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2x-2}{x} \, dx} \\ &= z_1 e^{x+\ln(x)} \\ &= z_1 (x \, \mathrm{e}^{x}) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-2}{x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{2x+2\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{x \, \mathrm{e}^{2x+2\ln(x)} \mathrm{e}^{-2x}}{3} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (\mathrm{e}^{x}) + c_2 \left(\mathrm{e}^{x} \left(\frac{x \, \mathrm{e}^{2x+2\ln(x)} \mathrm{e}^{-2x}}{3} \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dx^2} y(x)\right) x - (2x+2) \left(\frac{d}{dx} y(x)\right) + (x+2) y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2} y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2} y(x) = -\frac{(x+2)y(x)}{x} + \frac{2(x+1)\left(\frac{d}{dx} y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x) - \frac{2(x+1)\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(x+2)y(x)}{x} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{2(x+1)}{x}, P_3(x) = \frac{x+2}{x}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $(x \cdot P_2(x))\Big|_{x=0} = -2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $\left(\frac{d^2}{dx^2}y(x)\right)x + (-2-2x)\left(\frac{d}{dx}y(x)\right) + (x+2)\,y(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$
    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k - m$
    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..1$
    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k- >k + 1 - m$
    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion
    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,x^{k+r-1}$
  - Shift index using $k- >k + 1$
    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions
  $a_0 r(-3+r)\,x^{-1+r} + (a_1(1+r)(-2+r) - 2a_0(-1+r))\,x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)(k-2+r) - 2\right.$
- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-3+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 3\}$
- Each term must be 0
  $a_1(1+r)(-2+r) - 2a_0(-1+r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)(k-2+r) - 2a_k k - 2a_k r + 2a_k + a_{k-1} = 0$

- Shift index using $k- >k+1$

  $a_{k+2}(k+2+r)(k+r-1)-2a_{k+1}(k+1)-2ra_{k+1}+2a_{k+1}+a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{2ka_{k+1}+2ra_{k+1}-a_k}{(k+2+r)(k+r-1)}$
- Recursion relation for $r=0$

  $a_{k+2}=\frac{2ka_{k+1}-a_k}{(k+2)(k-1)}$
- Series not valid for $r=0$, division by 0 in the recursion relation at $k=1$

  $a_{k+2}=\frac{2ka_{k+1}-a_k}{(k+2)(k-1)}$
- Recursion relation for $r=3$

  $a_{k+2}=\frac{2ka_{k+1}-a_k+6a_{k+1}}{(k+5)(k+2)}$
- Solution for $r=3$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2}=\frac{2ka_{k+1}-a_k+6a_{k+1}}{(k+5)(k+2)}, 4a_1-4a_0=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 14

```
dsolve(x*diff(diff(y(x),x),x)-(2+2*x)*diff(y(x),x)+(x+2)*y(x) = 0,y(x),singsol=all)
```

$$y=\mathrm{e}^{x}\left(c_2 x^3+c_1\right)$$

Mathematica DSolve solution

Solving time : 0.045 (sec)
Leaf size : 25

```
DSolve[{x*D[y[x],{x,2}]-(2*x+2)*D[y[x],x]+(x+2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x)\to \frac{1}{3}e^{x+1}\left(c_2 x^3+3c_1\right)$$

2.1.444 Problem 457

Internal problem ID [9616]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 457
Date solved : Monday, January 27, 2025 at 06:04:41 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$x^2y'' - 2xy' + \left(x^2+2\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.126 (sec)

Writing the ode as

$$x^2y'' - 2xy' + \left(x^2+2\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.842: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\, dx} \\ &= z_1 e^{\ln(x)} \\ &= z_1(x) \end{aligned}$$

Which simplifies to

$$y_1 = x\cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x\cos(x)) + c_2(x\cos(x)(\tan(x)))
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) - 2x\left(\frac{d}{dx}y(x)\right) + (x^2+2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(x^2+2)y(x)}{x^2} + \frac{2\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) - \frac{2\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(x^2+2)y(x)}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{2}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) - 2x\left(\frac{d}{dx}y(x)\right) + (x^2+2)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

$$x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

$$x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

$$x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+r)\,(-2+r)\,x^r + a_1 r(-1+r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)\,(k+r-2) + a_{k-2}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+r)\,(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{1, 2\}$$

- Each term must be 0

$$a_1 r(-1+r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+r-1)\,(k+r-2) + a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$a_{k+2}(k+1+r)\,(k+r) + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k}{(k+1+r)(k+r)}$$

- Recursion relation for $r = 1$

$$a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$$

- Solution for $r = 1$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$$

- Recursion relation for $r = 2$

$$a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$$

- Solution for $r = 2$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+2)(k+3)}, b_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
```

```
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 15

```
dsolve(x^2*diff(diff(y(x),x),x)-2*diff(y(x),x)*x+(x^2+2)*y(x) = 0,y(x),singsol=all)
```

$$y = x(\sin(x) c_1 + \cos(x) c_2)$$

Mathematica DSolve solution

Solving time : 0.028 (sec)
Leaf size : 33

```
DSolve[{x^2*D[y[x],{x,2}]-2*x*D[y[x],x]+(x^2+2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tr
```

$$y(x) \to c_1 e^{-ix} x - \frac{1}{2} i c_2 e^{ix} x$$

2.1.445 Problem 458

Internal problem ID [9617]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 458
Date solved : `Monday, January 27, 2025 at 06:04:41 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' - (4x+1)\,y' + (4x+2)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.158 (sec)

Writing the ode as

$$xy'' + (-4x-1)\,y' + (4x+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -4x-1 \\ C &= 4x+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.844: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the gcd$(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)\,(0) \\
&= -\frac{1}{2x} \\
&= -\frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2x}\right)(0)+\left(\left(\frac{1}{2x^2}\right)+\left(-\frac{1}{2x}\right)^2-\left(\frac{3}{4x^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}z_1(x)&=pe^{\int \omega\,dx}\\&=e^{\int -\frac{1}{2x}dx}\\&=\frac{1}{\sqrt{x}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1&=z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\&=z_1e^{-\int \frac{1}{2}\frac{-4x-1}{x}\,dx}\\&=z_1e^{2x+\frac{\ln(x)}{2}}\\&=z_1\left(\sqrt{x}\,e^{2x}\right)\end{aligned}$$

Which simplifies to

$$y_1=e^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int\frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2&=y_1\int\frac{e^{\int -\frac{-4x-1}{x}\,dx}}{(y_1)^2}\,dx\\&=y_1\int\frac{e^{4x+\ln(x)}}{(y_1)^2}\,dx\\&=y_1\left(\frac{x\,e^{4x+\ln(x)}e^{-4x}}{2}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y&=c_1y_1+c_2y_2\\&=c_1\left(e^{2x}\right)+c_2\left(e^{2x}\left(\frac{x\,e^{4x+\ln(x)}e^{-4x}}{2}\right)\right)\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dx^2}y(x)\right)x-(4x+1)\left(\frac{d}{dx}y(x)\right)+(2+4x)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{2(2x+1)y(x)}{x}+\frac{(4x+1)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)-\frac{(4x+1)\left(\frac{d}{dx}y(x)\right)}{x}+\frac{2(2x+1)y(x)}{x}=0$

☐ Check to see if $x_0=0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x)=-\frac{4x+1}{x},P_3(x)=\frac{2(2x+1)}{x}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\Big|_{x=0}=-1$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\Big|_{x=0}=0$

- ○ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $\left(\frac{d^2}{dx^2}y(x)\right)x+(-4x-1)\left(\frac{d}{dx}y(x)\right)+(2+4x)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ○ Shift index using $k->k-m$

  $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ○ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..1$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ○ Convert $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-1}$

- ○ Shift index using $k->k+1$

  $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$a_0 r(-2+r)\, x^{-1+r} + (a_1(1+r)(-1+r) - 2a_0(-1+2r))\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)(k+r-1) -\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 2\}$
- Each term must be 0
  $a_1(1+r)(-1+r) - 2a_0(-1+2r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)(k+r-1) + a_k(-4k-4r+2) + 4a_{k-1} = 0$
- Shift index using $k- > k+1$
  $a_{k+2}(k+2+r)(k+r) + a_{k+1}(-4k-2-4r) + 4a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{2(2ka_{k+1}+2ra_{k+1}-2a_k+a_{k+1})}{(k+2+r)(k+r)}$
- Recursion relation for $r = 0$
  $a_{k+2} = \frac{2(2ka_{k+1}-2a_k+a_{k+1})}{(k+2)k}$
- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$
  $a_{k+2} = \frac{2(2ka_{k+1}-2a_k+a_{k+1})}{(k+2)k}$
- Recursion relation for $r = 2$
  $a_{k+2} = \frac{2(2ka_{k+1}-2a_k+5a_{k+1})}{(k+4)(k+2)}$
- Solution for $r = 2$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{2(2ka_{k+1}-2a_k+5a_{k+1})}{(k+4)(k+2)}, 3a_1 - 6a_0 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 16

```
dsolve(x*diff(diff(y(x),x),x)-(4*x+1)*diff(y(x),x)+(4*x+2)*y(x) = 0,y(x),singsol=all)
```

$$y = e^{2x}\left(c_2 x^2 + c_1\right)$$

Mathematica DSolve solution

Solving time : 0.037 (sec)
Leaf size : 25

```
DSolve[{x*D[y[x],{x,2}]-(4*x+1)*D[y[x],x]+(4*x+2)*y[x]==0,{}},y[x],x,IncludeSingularSolution
```

$$y(x) \to \frac{1}{2}e^{2x}\left(c_2x^2 + 2c_1\right)$$

2.1.446 Problem 460

Internal problem ID [9618]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 460
Date solved : Monday, January 27, 2025 at 06:04:42 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$4x^2y'' - 4xy' + \left(-16x^2 + 3\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.088 (sec)

Writing the ode as

$$4x^2y'' - 4xy' + \left(-16x^2 + 3\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x \\ C &= -16x^2 + 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.846: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{4x^2}\, dx} \\ &= z_1 e^{\frac{\ln(x)}{2}} \\ &= z_1\left(\sqrt{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}\,\mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-4x}{4x^2}\,dx}}{(y_1)^2}\,dx$$
$$= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx$$
$$= y_1\left(\frac{e^{4x}}{4}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1\left(\sqrt{x}\,e^{-2x}\right) + c_2\left(\sqrt{x}\,e^{-2x}\left(\frac{e^{4x}}{4}\right)\right)$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2\left(\frac{d^2}{dx^2}y(x)\right) - 4x\left(\frac{d}{dx}y(x)\right) + \left(-16x^2+3\right)y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{(16x^2-3)y(x)}{4x^2} + \frac{\frac{d}{dx}y(x)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) - \frac{\frac{d}{dx}y(x)}{x} - \frac{(16x^2-3)y(x)}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{1}{x}, P_3(x) = -\frac{16x^2-3}{4x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{3}{4}$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) - 4x\left(\frac{d}{dx}y(x)\right) + \left(-16x^2+3\right)y(x) = 0$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)\,(-3+2r)\, x^r + a_1(1+2r)\,(-1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r-1)\,(2k+2r-3) - 1$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)\,(-3+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$
- Each term must be 0

  $a_1(1+2r)\,(-1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r-\frac{3}{2}\right)\left(k+r-\frac{1}{2}\right) a_k - 16a_{k-2} = 0$
- Shift index using $k- >k+2$

  $4\left(k+\frac{1}{2}+r\right)\left(k+\frac{3}{2}+r\right) a_{k+2} - 16a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{16a_k}{(2k+1+2r)(2k+3+2r)}$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = \frac{16a_k}{(2k+2)(2k+4)}$
- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{16a_k}{(2k+2)(2k+4)}, a_1 = 0\right]$
- Recursion relation for $r = \frac{3}{2}$

  $a_{k+2} = \frac{16a_k}{(2k+4)(2k+6)}$
- Solution for $r = \frac{3}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2} = \frac{16a_k}{(2k+4)(2k+6)}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+2} = \frac{16a_k}{(2k+2)(2k+4)}, a_1 = 0, b_{k+2} = \frac{16b_k}{(2k+4)(2k+6)}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 21

```
dsolve(4*x^2*diff(diff(y(x),x),x)-4*diff(y(x),x)*x+(-16*x^2+3)*y(x) = 0,y(x),singsol=all
```

$$y = \sqrt{x}\, (c_1 \sinh (2x) + c_2 \cosh (2x))$$

Mathematica DSolve solution

Solving time : 0.035 (sec)
Leaf size : 32

```
DSolve[{4*x^2*D[y[x],{x,2}]-4*x*D[y[x],x]+(3-16*x^2)*y[x]==0,{}},y[x],x,IncludeSingularSolutio
```

$$y(x) \to \frac{1}{4} e^{-2x} \sqrt{x} (c_2 e^{4x} + 4c_1)$$

2.1.447 Problem 461

Internal problem ID [9619]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 461
Date solved : `Monday, January 27, 2025 at 06:04:42 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(2x+1)\,xy''-2\left(2x^2-1\right)y'-4(x+1)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.235 (sec)

Writing the ode as

$$\left(2x^2+x\right)y''+\left(-4x^2+2\right)y'+(-4x-4)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2+x \\ B &= -4x^2+2 \\ C &= -4x-4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{4x^2+8x+6}{(2x+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2+8x+6 \\ t &= (2x+1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{4x^2+8x+6}{(2x+1)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.848: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x+1)^2$. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{3}{4\left(x+\frac{1}{2}\right)^2} + \frac{1}{x+\frac{1}{2}}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{1}{2x} - \frac{1}{4x^3} + \frac{11}{32x^4} - \frac{21}{64x^5} + \frac{15}{64x^6} - \frac{3}{32x^7} - \frac{117}{2048x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 + 8x + 6}{4x^2 + 4x + 1} \\ &= Q + \frac{R}{4x^2 + 4x + 1} \\ &= (1) + \left(\frac{4x + 5}{4x^2 + 4x + 1}\right) \\ &= 1 + \frac{4x + 5}{4x^2 + 4x + 1} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$\begin{aligned} b &= (1) - (0) \\ &= 1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{1}{1} - 0\right) = \frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{1}{1} - 0\right) = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^2 + 8x + 6}{(2x+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(x+\frac{1}{2}\right)} + (-)(1) \\
&= -\frac{1}{2\left(x+\frac{1}{2}\right)} - 1 \\
&= -\frac{2(x+1)}{2x+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right)(0) + \left(\left(\frac{1}{2\left(x+\frac{1}{2}\right)^2}\right) + \left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right)^2 - \left(\frac{4x^2+8x+6}{(2x+1)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2\left(x+\frac{1}{2}\right)}-1\right)dx} \\ &= \frac{\mathrm{e}^{-x}}{\sqrt{2x+1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x^2+2}{2x^2+x} \, dx} \\ &= z_1 e^{x-\ln(x)+\frac{\ln(2x+1)}{2}} \\ &= z_1 \left(\frac{\sqrt{2x+1}\,\mathrm{e}^{x}}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^2+2}{2x^2+x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{2x-2\ln(x)+\ln(2x+1)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{x^3 \mathrm{e}^{2x-2\ln(x)+\ln(2x+1)}}{2x+1}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x} \left(\frac{x^3 \mathrm{e}^{2x-2\ln(x)+\ln(2x+1)}}{2x+1}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(2x+1)\,x\left(\frac{d^2}{dx^2}y(x)\right)-2(2x^2-1)\left(\frac{d}{dx}y(x)\right)-4(x+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{4(x+1)y(x)}{(2x+1)x}+\frac{2(2x^2-1)\left(\frac{d}{dx}y(x)\right)}{(2x+1)x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)-\frac{2(2x^2-1)\left(\frac{d}{dx}y(x)\right)}{(2x+1)x}-\frac{4(x+1)y(x)}{(2x+1)x}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=-\frac{2(2x^2-1)}{(2x+1)x},P_3(x)=-\frac{4(x+1)}{(2x+1)x}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=2$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=0$

- ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $(2x+1)\,x\left(\frac{d^2}{dx^2}y(x)\right)+(-4x^2+2)\left(\frac{d}{dx}y(x)\right)+(-4-4x)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ◦ Shift index using $k->k-m$

  $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..2$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ◦ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=1..2$

  $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- ◦ Shift index using $k->k+2-m$

  $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$a_0r(1+r)\,x^{-1+r}+(a_1(1+r)\,(2+r)+2a_0(1+r)\,(-2+r))\,x^r+\left(\sum_{k=1}^{\infty}(a_{k+1}(k+r+1)\,(k+2+$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(1+r)=0$
- Values of r that satisfy the indicial equation
  $r\in\{-1,0\}$
- Each term must be 0
  $a_1(1+r)\,(2+r)+2a_0(1+r)\,(-2+r)=0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+r+1)\,(k+2+r)+2a_k(k+r+1)\,(k+r-2)-4a_{k-1}(k+r)=0$
- Shift index using $k->k+1$
  $a_{k+2}(k+2+r)\,(k+3+r)+2a_{k+1}(k+2+r)\,(k+r-1)-4a_k(k+r+1)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+2}=-\frac{2(k^2a_{k+1}+2kra_{k+1}+r^2a_{k+1}-2ka_k+ka_{k+1}-2ra_k+ra_{k+1}-2a_k-2a_{k+1})}{(k+2+r)(k+3+r)}$
- Recursion relation for $r=-1$
  $a_{k+2}=-\frac{2(k^2a_{k+1}-2ka_k-ka_{k+1}-2a_{k+1})}{(k+1)(k+2)}$
- Solution for $r=-1$
  $\left[y(x)=\sum_{k=0}^{\infty}a_kx^{k-1},a_{k+2}=-\frac{2(k^2a_{k+1}-2ka_k-ka_{k+1}-2a_{k+1})}{(k+1)(k+2)},0=0\right]$
- Recursion relation for $r=0$
  $a_{k+2}=-\frac{2(k^2a_{k+1}-2ka_k+ka_{k+1}-2a_k-2a_{k+1})}{(k+2)(k+3)}$
- Solution for $r=0$
  $\left[y(x)=\sum_{k=0}^{\infty}a_kx^k,a_{k+2}=-\frac{2(k^2a_{k+1}-2ka_k+ka_{k+1}-2a_k-2a_{k+1})}{(k+2)(k+3)},2a_1-4a_0=0\right]$
- Combine solutions and rename parameters
  $\left[y(x)=\left(\sum_{k=0}^{\infty}a_kx^{k-1}\right)+\left(\sum_{k=0}^{\infty}b_kx^k\right),a_{k+2}=-\frac{2(k^2a_{k+1}-2ka_k-ka_{k+1}-2a_{k+1})}{(k+1)(k+2)},0=0,b_{k+2}=-\frac{2(k^2b_{k+1}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 17

```
dsolve((2*x+1)*x*diff(diff(y(x),x),x)-2*(2*x^2-1)*diff(y(x),x)-4*(x+1)*y(x) = 0,y(x),s
```

$$y=\frac{c_2\mathrm{e}^{2x}x+c_1}{x}$$

Mathematica DSolve solution

Solving time : 0.242 (sec)
Leaf size : 92

```
DSolve[{(2*x+1)*x*D[y[x],{x,2}]-2*(2*x^2-1)*D[y[x],x]-4*(x+1)*y[x]==0,{}},y[x],x,IncludeSingula
```

$$y(x)\to \exp\left(\int_1^x\left(\frac{1}{-2K[1]-1}-1\right)dK[1]-\frac{1}{2}\int_1^x\frac{2-4K[2]^2}{2K[2]^2+K[2]}dK[2]\right)\left(c_2\int_1^x\exp\left(-2\int_1^{K[3]}\left(\frac{1}{-2K[1]-1}-1\right)dK[1]\right)dK[3]+c_1\right)$$

2.1.448 Problem 462

Internal problem ID [9620]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 462
Date solved : Monday, January 27, 2025 at 06:04:43 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$\left(x^{2}-2x\right)y''+\left(-x^{2}+2\right)y'+\left(2x-2\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.312 (sec)

Writing the ode as

$$\left(x^{2}-2x\right)y''+\left(-x^{2}+2\right)y'+\left(2x-2\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^{2}-2x\\B&=-x^{2}+2\\C&=2x-2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^{2}-4AC}{4A^{2}}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{x^{4}-8x^{3}+24x^{2}-24x+12}{4\left(x^{2}-2x\right)^{2}} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=x^{4}-8x^{3}+24x^{2}-24x+12\\t&=4\left(x^{2}-2x\right)^{2}\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x^{4}-8x^{3}+24x^{2}-24x+12}{4\left(x^{2}-2x\right)^{2}}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.850: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4x^2} + \frac{3}{4(x-2)^2} - \frac{3}{4x} - \frac{1}{4(x-2)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = 2$ let b be the coefficient of $\frac{1}{(x-2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{2}{x^3} + \frac{11}{x^4} + \frac{42}{x^5} + \frac{132}{x^6} + \frac{348}{x^7} + \frac{711}{x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4x^4 - 16x^3 + 16x^2} \\ &= Q + \frac{R}{4x^4 - 16x^3 + 16x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-4x^3 + 20x^2 - 24x + 12}{4x^4 - 16x^3 + 16x^2}\right) \\ &= \frac{1}{4} + \frac{-4x^3 + 20x^2 - 24x + 12}{4x^4 - 16x^3 + 16x^2} \end{aligned}$$

Since the degree of t is 4, then we see that the coefficient of the term x^3 in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 0\right) &&= -1 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 0\right) &&= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4\left(x^2 - 2x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
2	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-}\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} - \frac{1}{2\left(x-2\right)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2x} - \frac{1}{2\left(x-2\right)} + \frac{1}{2} \\
&= -\frac{1}{2x} - \frac{1}{2x-4} + \frac{1}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x} - \frac{1}{2(x-2)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2x^2} + \frac{1}{2(x-2)^2}\right) + \left(-\frac{1}{2x} - \frac{1}{2(x-2)} + \frac{1}{2}\right)^2 - \left(\frac{x^4 - 8x^3 +}{4(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= e^{\int \left(-\frac{1}{2x} - \frac{1}{2(x-2)} + \frac{1}{2}\right) dx} \\ &= \frac{e^{\frac{x}{2}}}{\sqrt{x}\,\sqrt{x-2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x^2+2}{x^2-2x}\, dx} \\ &= z_1 e^{\frac{x}{2} + \frac{\ln(x)}{2} + \frac{\ln(x-2)}{2}} \\ &= z_1 \left(\sqrt{x}\,\sqrt{x-2}\, e^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}\,\sqrt{x-2}\, e^{x}}{\sqrt{x(x-2)}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+2}{x^2-2x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x+\ln(x)+\ln(x-2)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{x\, e^{x+\ln(x)+\ln(x-2)} e^{-2x}}{x-2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\sqrt{x}\,\sqrt{x-2}\, e^{x}}{\sqrt{x(x-2)}}\right) + c_2 \left(\frac{\sqrt{x}\,\sqrt{x-2}\, e^{x}}{\sqrt{x(x-2)}} \left(-\frac{x\, e^{x+\ln(x)+\ln(x-2)} e^{-2x}}{x-2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x^2-2x)\left(\frac{d^2}{dx^2}y(x)\right)+(-x^2+2)\left(\frac{d}{dx}y(x)\right)+(2x-2)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{2(x-1)y(x)}{x(x-2)}+\frac{(x^2-2)\left(\frac{d}{dx}y(x)\right)}{x(x-2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)-\frac{(x^2-2)\left(\frac{d}{dx}y(x)\right)}{x(x-2)}+\frac{2(x-1)y(x)}{x(x-2)}=0$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x)=-\frac{x^2-2}{x(x-2)},P_3(x)=\frac{2(x-1)}{x(x-2)}\right]$
  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=-1$
  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$
  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x(x-2)\left(\frac{d^2}{dx^2}y(x)\right)+(-x^2+2)\left(\frac{d}{dx}y(x)\right)+(2x-2)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

    $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_k x^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$
  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..2$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=1..2$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$
  - Shift index using $k->k+2-m$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$-2a_0r(-2+r)\,x^{-1+r}+(-2a_1(1+r)\,(-1+r)+a_0(1+r)\,(-2+r))\,x^r+\left(\sum_{k=1}^{\infty}(-2a_{k+1}(k+r+\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-2r(-2+r)=0$
- Values of r that satisfy the indicial equation
  $r\in\{0,2\}$
- Each term must be 0
  $-2a_1(1+r)\,(-1+r)+a_0(1+r)\,(-2+r)=0$
- Each term in the series must be 0, giving the recursion relation
  $a_k(k+r+1)\,(k+r-2)-2k^2a_{k+1}+(-4ra_{k+1}-a_{k-1})\,k-2r^2a_{k+1}-a_{k-1}r+3a_{k-1}+2a_{k+1}=$
- Shift index using $k->k+1$
  $a_{k+1}(k+2+r)\,(k+r-1)-2(k+1)^2\,a_{k+2}+(-4ra_{k+2}-a_k)\,(k+1)-2r^2a_{k+2}-ra_k+3a_k+$
- Recursion relation that defines series solution to ODE
  $a_{k+2}=\frac{k^2a_{k+1}+2kra_{k+1}+r^2a_{k+1}-ka_k+ka_{k+1}-ra_k+ra_{k+1}+2a_k-2a_{k+1}}{2(k^2+2kr+r^2+2k+2r)}$
- Recursion relation for $r=0$
  $a_{k+2}=\frac{k^2a_{k+1}-ka_k+ka_{k+1}+2a_k-2a_{k+1}}{2(k^2+2k)}$
- Series not valid for $r=0$, division by 0 in the recursion relation at $k=0$
  $a_{k+2}=\frac{k^2a_{k+1}-ka_k+ka_{k+1}+2a_k-2a_{k+1}}{2(k^2+2k)}$
- Recursion relation for $r=2$
  $a_{k+2}=\frac{k^2a_{k+1}-ka_k+5ka_{k+1}+4a_{k+1}}{2(k^2+6k+8)}$
- Solution for $r=2$
  $\left[y(x)=\sum_{k=0}^{\infty}a_kx^{k+2}, a_{k+2}=\frac{k^2a_{k+1}-ka_k+5ka_{k+1}+4a_{k+1}}{2(k^2+6k+8)}, -6a_1=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 14

```
dsolve((x^2-2*x)*diff(diff(y(x),x),x)+(-x^2+2)*diff(y(x),x)+(2*x-2)*y(x) = 0,y(x),sing
```

$$y=c_1x^2+e^xc_2$$

Mathematica DSolve solution

Solving time : 0.273 (sec)
Leaf size : 115

```
DSolve[{(x^2-2*x)*D[y[x],{x,2}]+(2-x^2)*D[y[x],x]+(2*x-2)*y[x]==0,{}},y[x],x,IncludeSingularSol
```

$$y(x) \to \exp\left(\int_1^x \frac{(K[1]-4)K[1]+2}{2(K[1]-2)K[1]}dK[1] - \frac{1}{2}\int_1^x \left(-\frac{1}{K[2]} - 1 + \frac{1}{2-K[2]}\right) dK[2]\right)\left(c_2 \int_1^x \exp\left(-2\int_1^{K[3]} \frac{K[1]^2-4K[1]+2}{2(K[1]-2)K[1]}dK[1]\right) dK[3] + c_1\right)$$

2.1.449 Problem 463

Internal problem ID [9621]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 463
Date solved : `Monday, January 27, 2025 at 06:04:44 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' - (4x+1)\,y' + (4x+2)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.151 (sec)

Writing the ode as

$$xy'' + (-4x-1)\,y' + (4x+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -4x-1 \\ C &= 4x+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.852: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=-\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2}-\left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^{-}}{x-c_1}\right)+(-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x}+(-)(0) \\
&= -\frac{1}{2x} \\
&= -\frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2x}\right)(0)+\left(\left(\frac{1}{2x^2}\right)+\left(-\frac{1}{2x}\right)^2-\left(\frac{3}{4x^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\,dx}\\
&= \mathrm{e}^{\int -\frac{1}{2x}dx}\\
&= \frac{1}{\sqrt{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x-1}{x}\,dx}\\
&= z_1 e^{2x+\frac{\ln(x)}{2}}\\
&= z_1\left(\sqrt{x}\,\mathrm{e}^{2x}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\mathrm{e}^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1\int \frac{e^{\int -\frac{-4x-1}{x}\,dx}}{(y_1)^2}\,dx\\
&= y_1\int \frac{e^{4x+\ln(x)}}{(y_1)^2}\,dx\\
&= y_1\left(\frac{x\,\mathrm{e}^{4x+\ln(x)}\mathrm{e}^{-4x}}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1y_1+c_2y_2\\
&= c_1\left(\mathrm{e}^{2x}\right)+c_2\left(\mathrm{e}^{2x}\left(\frac{x\,\mathrm{e}^{4x+\ln(x)}\mathrm{e}^{-4x}}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dx^2}y(x)\right)x-(4x+1)\left(\frac{d}{dx}y(x)\right)+(2+4x)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{2(2x+1)y(x)}{x}+\frac{(4x+1)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)-\frac{(4x+1)\left(\frac{d}{dx}y(x)\right)}{x}+\frac{2(2x+1)y(x)}{x}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=-\frac{4x+1}{x},P_3(x)=\frac{2(2x+1)}{x}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=-1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $\left(\frac{d^2}{dx^2}y(x)\right)x+(-4x-1)\left(\frac{d}{dx}y(x)\right)+(2+4x)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

    $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  ◦ Shift index using $k->k-m$

    $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..1$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-1}$

  ◦ Shift index using $k->k+1$

    $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$a_0 r(-2+r)\, x^{-1+r} + (a_1(1+r)\,(-1+r) - 2a_0(-1+2r))\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(k+r-1) -\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 2\}$
- Each term must be 0
  $a_1(1+r)\,(-1+r) - 2a_0(-1+2r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)\,(k+r-1) + a_k(-4k-4r+2) + 4a_{k-1} = 0$
- Shift index using $k- >k+1$
  $a_{k+2}(k+2+r)\,(k+r) + a_{k+1}(-4k-2-4r) + 4a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{2(2ka_{k+1}+2ra_{k+1}-2a_k+a_{k+1})}{(k+2+r)(k+r)}$
- Recursion relation for $r = 0$
  $a_{k+2} = \frac{2(2ka_{k+1}-2a_k+a_{k+1})}{(k+2)k}$
- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$
  $a_{k+2} = \frac{2(2ka_{k+1}-2a_k+a_{k+1})}{(k+2)k}$
- Recursion relation for $r = 2$
  $a_{k+2} = \frac{2(2ka_{k+1}-2a_k+5a_{k+1})}{(k+4)(k+2)}$
- Solution for $r = 2$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{2(2ka_{k+1}-2a_k+5a_{k+1})}{(k+4)(k+2)}, 3a_1 - 6a_0 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 16

```
dsolve(x*diff(diff(y(x),x),x)-(4*x+1)*diff(y(x),x)+(4*x+2)*y(x) = 0,y(x),singsol=all)
```

$$y = e^{2x}\left(c_2 x^2 + c_1\right)$$

Mathematica DSolve solution

Solving time : 0.029 (sec)
Leaf size : 25

```
DSolve[{x*D[y[x],{x,2}]-(4*x+1)*D[y[x],x]+(4*x+2)*y[x]==0,{}},y[x],x,IncludeSingularSolution
```

$$y(x) \to \frac{1}{2}e^{2x}\left(c_2x^2 + 2c_1\right)$$

2.1.450 Problem 464

Internal problem ID [9622]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 464
Date solved : `Monday, January 27, 2025 at 06:04:44 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(3x-1)\,y''-(3x+2)\,y'-(6x-8)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.242 (sec)

Writing the ode as

$$(3x-1)\,y''+(-3x-2)\,y'+(-6x+8)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=3x-1\\B&=-3x-2\\C&=-6x+8\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{81x^2-108x+54}{4\,(3x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=81x^2-108x+54\\t&=4(3x-1)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{81x^2-108x+54}{4\,(3x-1)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.854: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(3x-1)^2$. There is a pole at $x = \frac{1}{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{4} + \frac{3}{4\left(x - \frac{1}{3}\right)^2} - \frac{3}{2\left(x - \frac{1}{3}\right)}$$

For the pole at $x = \frac{1}{3}$ let b be the coefficient of $\frac{1}{\left(x-\frac{1}{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3}{2} - \frac{1}{2x} + \frac{1}{9x^3} + \frac{11}{108x^4} + \frac{7}{108x^5} + \frac{5}{162x^6} + \frac{2}{243x^7} - \frac{13}{3888x^8} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{3}{2} \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{81x^2 - 108x + 54}{36x^2 - 24x + 4} \\
&= Q + \frac{R}{36x^2 - 24x + 4} \\
&= \left(\frac{9}{4}\right) + \left(\frac{-54x + 45}{36x^2 - 24x + 4}\right) \\
&= \frac{9}{4} + \frac{-54x + 45}{36x^2 - 24x + 4}
\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -54. Dividing this by leading coefficient in t which is 36 gives $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{3}{2}\right) - (0) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{3}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{3}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{3}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{81x^2 - 108x + 54}{4\left(3x - 1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$\frac{1}{3}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(x - \frac{1}{3}\right)} + \left(\frac{3}{2}\right) \\
&= -\frac{1}{2\left(x - \frac{1}{3}\right)} + \frac{3}{2} \\
&= \frac{9x - 6}{6x - 2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2\left(x-\frac{1}{3}\right)}+\frac{3}{2}\right)(0)+\left(\left(\frac{1}{2\left(x-\frac{1}{3}\right)^{2}}\right)+\left(-\frac{1}{2\left(x-\frac{1}{3}\right)}+\frac{3}{2}\right)^{2}-\left(\frac{81x^{2}-108x+54}{4\left(3x-1\right)^{2}}\right)\right)=0$$

$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\,dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2\left(x-\frac{1}{3}\right)}+\frac{3}{2}\right)dx} \\
&= \frac{\mathrm{e}^{\frac{3x}{2}}}{\sqrt{3x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3x-2}{3x-1}\,dx} \\
&= z_1 e^{\frac{x}{2}+\frac{\ln(3x-1)}{2}} \\
&= z_1\left(\sqrt{3x-1}\,\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\mathrm{e}^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1\int \frac{e^{\int -\frac{-3x-2}{3x-1}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1\int \frac{\mathrm{e}^{x+\ln(3x-1)}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(-\frac{x\,\mathrm{e}^{x+\ln(3x-1)}\mathrm{e}^{-4x}}{3x-1}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{2x}\right)+c_2\left(\mathrm{e}^{2x}\left(-\frac{x\,\mathrm{e}^{x+\ln(3x-1)}\mathrm{e}^{-4x}}{3x-1}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(3x-1)\left(\frac{d^2}{dx^2}y(x)\right)-(3x+2)\left(\frac{d}{dx}y(x)\right)-(6x-8)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{2(3x-4)y(x)}{3x-1}+\frac{(3x+2)\left(\frac{d}{dx}y(x)\right)}{3x-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)-\frac{(3x+2)\left(\frac{d}{dx}y(x)\right)}{3x-1}-\frac{2(3x-4)y(x)}{3x-1}=0$

□ Check to see if $x_0=\frac{1}{3}$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=-\frac{3x+2}{3x-1},P_3(x)=-\frac{2(3x-4)}{3x-1}\right]$

- ◦ $\left(x-\frac{1}{3}\right)\cdot P_2(x)$ is analytic at $x=\frac{1}{3}$

  $\left(\left(x-\frac{1}{3}\right)\cdot P_2(x)\right)\bigg|_{x=\frac{1}{3}}=-1$

- ◦ $\left(x-\frac{1}{3}\right)^2\cdot P_3(x)$ is analytic at $x=\frac{1}{3}$

  $\left(\left(x-\frac{1}{3}\right)^2\cdot P_3(x)\right)\bigg|_{x=\frac{1}{3}}=0$

- ◦ $x=\frac{1}{3}$is a regular singular point

  Check to see if $x_0=\frac{1}{3}$ is a regular singular point

  $x_0=\frac{1}{3}$

- Multiply by denominators

  $(3x-1)\left(\frac{d^2}{dx^2}y(x)\right)+(-3x-2)\left(\frac{d}{dx}y(x)\right)+(-6x+8)\,y(x)=0$

- Change variables using $x=u+\frac{1}{3}$ so that the regular singular point is at $u=0$

  $3u\left(\frac{d^2}{du^2}y(u)\right)+(-3u-3)\left(\frac{d}{du}y(u)\right)+(-6u+6)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

- ◦ Shift index using $k->k-m$

  $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ◦ Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-1}$

- Shift index using $k- > k+1$

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $3a_0r(-2+r)\, u^{-1+r} + (3a_1(1+r)(-1+r) - 3a_0(-2+r))\, u^r + \left(\sum_{k=1}^{\infty} (3a_{k+1}(k+1+r)(k+r-1)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $3r(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$
- Each term must be 0

  $3a_1(1+r)(-1+r) - 3a_0(-2+r) = 0$
- Each term in the series must be 0, giving the recursion relation

  $3a_{k+1}(k+1+r)(k+r-1) + a_k(-3k-3r+6) - 6a_{k-1} = 0$
- Shift index using $k- > k+1$

  $3a_{k+2}(k+2+r)(k+r) + a_{k+1}(-3k+3-3r) - 6a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{ka_{k+1}+ra_{k+1}+2a_k-a_{k+1}}{(k+2+r)(k+r)}$
- Recursion relation for $r = 0$

  $a_{k+2} = \frac{ka_{k+1}+2a_k-a_{k+1}}{(k+2)k}$
- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$

  $a_{k+2} = \frac{ka_{k+1}+2a_k-a_{k+1}}{(k+2)k}$
- Recursion relation for $r = 2$

  $a_{k+2} = \frac{ka_{k+1}+2a_k+a_{k+1}}{(k+4)(k+2)}$
- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2} = \frac{ka_{k+1}+2a_k+a_{k+1}}{(k+4)(k+2)}, 9a_1 = 0\right]$
- Revert the change of variables $u = x - \frac{1}{3}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k \left(x-\frac{1}{3}\right)^{k+2}, a_{k+2} = \frac{ka_{k+1}+2a_k+a_{k+1}}{(k+4)(k+2)}, 9a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 18

```
dsolve((3*x-1)*diff(diff(y(x),x),x)-(2+3*x)*diff(y(x),x)-(6*x-8)*y(x) = 0,y(x),singsol
```

$$y = e^{2x}c_1 + c_2 x\, e^{-x}$$

Mathematica DSolve solution

Solving time : 0.222 (sec)
Leaf size : 94

```
DSolve[{(3*x-1)*D[y[x],{x,2}]-(3*x+2)*D[y[x],x]-(6*x-8)*y[x]==0,{}},y[x],x,IncludeSingularSo
```

$$y(x) \to \exp\left(\int_1^x \frac{6-9K[1]}{2-6K[1]}dK[1] - \frac{1}{2}\int_1^x \left(\frac{3}{1-3K[2]}-1\right)dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{6-9K[1]}{2-6K[1]}dK[1]\right)dK[3] + c_1\right)$$

2.1.451 Problem 465

Internal problem ID [9623]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 465
Date solved : Monday, January 27, 2025 at 06:04:45 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$(x+1)^2 y'' - 2(x+1) y' - \left(x^2+2x-1\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.074 (sec)

Writing the ode as

$$(x+1)^2 y'' + (-2x-2) y' + \left(-x^2-2x+1\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= (x+1)^2 \\ B &= -2x-2 \\ C &= -x^2-2x+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.856: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x-2}{(x+1)^2}\, dx} \\ &= z_1 e^{\ln(x+1)} \\ &= z_1(x+1) \end{aligned}$$

Which simplifies to

$$y_1 = (x+1)\, e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-2}{(x+1)^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2\ln(x+1)}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{e^{2x}}{2}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left((x+1)\,e^{-x}\right) + c_2\left((x+1)\,e^{-x}\left(\frac{e^{2x}}{2}\right)\right)\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x+1)^2\left(\frac{d^2}{dx^2}y(x)\right) - 2(x+1)\left(\frac{d}{dx}y(x)\right) - (x^2+2x-1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{(x^2+2x-1)y(x)}{(x+1)^2} + \frac{2\left(\frac{d}{dx}y(x)\right)}{x+1}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) - \frac{2\left(\frac{d}{dx}y(x)\right)}{x+1} - \frac{(x^2+2x-1)y(x)}{(x+1)^2} = 0$
- ☐ Check to see if $x_0 = -1$ is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{2}{x+1}, P_3(x) = -\frac{x^2+2x-1}{(x+1)^2}\right]$
  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1} = -2$
  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1} = 2$
  - $x=-1$is a regular singular point

  Check to see if $x_0 = -1$ is a regular singular point

  $x_0 = -1$
- Multiply by denominators

  $(x+1)^2\left(\frac{d^2}{dx^2}y(x)\right) + (-2-2x)\left(\frac{d}{dx}y(x)\right) + (-x^2-2x+1)\,y(x) = 0$
- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $u^2\left(\frac{d^2}{du^2}y(u)\right) - 2u\left(\frac{d}{du}y(u)\right) + (-u^2+2)\,y(u) = 0$
- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..2$

    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$
  - Shift index using $k->k-m$

    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
  - Convert $u \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion

    $u \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r}$
  - Convert $u^2 \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

    $u^2 \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)\,(-2+r)\, u^r + a_1 r(-1+r)\, u^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)\,(k+r-2) - a_{k-2}\right) u^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$
- Each term must be 0

  $a_1 r(-1+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)\,(k+r-2) - a_{k-2} = 0$
- Shift index using $k->k+2$

  $a_{k+2}(k+1+r)\,(k+r) - a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k}{(k+1+r)(k+r)}$
- Recursion relation for $r = 1$

  $a_{k+2} = \frac{a_k}{(k+2)(k+1)}$
- Solution for $r = 1$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+1}, a_{k+2} = \frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$
- Revert the change of variables $u = x+1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^{k+1}, a_{k+2} = \frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$
- Recursion relation for $r = 2$

  $a_{k+2} = \frac{a_k}{(k+3)(k+2)}$
- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2} = \frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$
- Revert the change of variables $u = x+1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^{k+2}, a_{k+2} = \frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k(x+1)^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k(x+1)^{k+2}\right), a_{k+2} = \frac{a_k}{(k+1)(k+2)}, a_1 = 0, b_{k+2} = \frac{b_k}{(k+2)(k+3)}, b\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 17

```
dsolve((x+1)^2*diff(diff(y(x),x),x)-2*(x+1)*diff(y(x),x)-(x^2+2*x-1)*y(x) = 0,y(x),sings
```

$$y = (x+1)\left(c_1 \sinh(x) + c_2 \cosh(x)\right)$$

Mathematica DSolve solution

Solving time : 0.246 (sec)
Leaf size : 146

```
DSolve[{(x+1)^2*D[y[x],{x,2}]-2*(x+1)*x*D[y[x],x]-(x^2+2*x-1)*y[x]==0,{}},y[x],x,IncludeSingula
```

$$y(x) \to \left(c_1 \operatorname{HypergeometricU}\left(\frac{1}{2}\left(1-\sqrt{2}+i\sqrt{7}\right), 1+i\sqrt{7}, 2\sqrt{2}(x+1)\right) + c_2 L_{\frac{1}{2}\left(-1+\sqrt{2}-i\sqrt{7}\right)}^{i\sqrt{7}}\left(2\sqrt{2}(x+1)\right)\right) \exp\left(\int_1^x \frac{-2\sqrt{2}K[1]+2K[1]+i\sqrt{7}-2\sqrt{2}+1}{2K[1]+2} dK[1]\right)$$

2.1.452 Problem 466

Internal problem ID [9624]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 466
Date solved : Monday, January 27, 2025 at 06:04:45 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$4x^2y'' + \left(-8x^2+4x\right)y' + \left(4x^2-4x-1\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.084 (sec)

Writing the ode as

$$4x^2y'' + \left(-8x^2+4x\right)y' + \left(4x^2-4x-1\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -8x^2+4x \\ C &= 4x^2-4x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.858: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-8x^2+4x}{4x^2}\, dx} \\ &= z_1 e^{x-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{e^x}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^x}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-8x^2+4x}{4x^2} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{2x-\ln(x)}}{(y_1)^2} dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{e^x}{\sqrt{x}}\right) + c_2 \left(\frac{e^x}{\sqrt{x}}(x)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2\left(\frac{d^2}{dx^2}y(x)\right) + (-8x^2+4x)\left(\frac{d}{dx}y(x)\right) + (4x^2-4x-1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(4x^2-4x-1)y(x)}{4x^2} + \frac{(2x-1)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) - \frac{(2x-1)\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(4x^2-4x-1)y(x)}{4x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - ○ Define functions

    $\left[P_2(x) = -\frac{2x-1}{x}, P_3(x) = \frac{4x^2-4x-1}{4x^2}\right]$
  - ○ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$
  - ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\frac{1}{4}$
  - ○ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) - 4x(2x-1)\left(\frac{d}{dx}y(x)\right) + (4x^2-4x-1)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - ○ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y(x)=\sum_{k=m}^{\infty} a_{k-m}x^{k+r}$
- Convert $x^{m}\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

  $x^{m}\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
- Shift index using $k->k+1-m$

  $x^{m}\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
- Convert $x^{2}\cdot\left(\frac{d^{2}}{dx^{2}}y(x)\right)$ to series expansion

  $x^{2}\cdot\left(\frac{d^{2}}{dx^{2}}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\,x^{r}+\left(a_1(3+2r)\,(1+2r)-4a_0(1+2r)\right)x^{1+r}+\left(\sum_{k=2}^{\infty}(a_k(2k+2r+1)\,(2k+2\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r)=0$
- Values of r that satisfy the indicial equation

  $r\in\left\{-\frac{1}{2},\frac{1}{2}\right\}$
- Each term must be 0

  $a_1(3+2r)\,(1+2r)-4a_0(1+2r)=0$
- Solve for the dependent coefficient(s)

  $a_1=\frac{4a_0}{3+2r}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^{2}+8kr+4r^{2}-1)+(-8k-8r+4)\,a_{k-1}+4a_{k-2}=0$
- Shift index using $k->k+2$

  $a_{k+2}\left(4(k+2)^{2}+8(k+2)\,r+4r^{2}-1\right)+(-8k-12-8r)\,a_{k+1}+4a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{4(2ka_{k+1}+2ra_{k+1}-a_k+3a_{k+1})}{4k^{2}+8kr+4r^{2}+16k+16r+15}$
- Recursion relation for $r=-\frac{1}{2}$

  $a_{k+2}=\frac{4(2ka_{k+1}-a_k+2a_{k+1})}{4k^{2}+12k+8}$
- Solution for $r=-\frac{1}{2}$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}},a_{k+2}=\frac{4(2ka_{k+1}-a_k+2a_{k+1})}{4k^{2}+12k+8},a_1=2a_0\right]$
- Recursion relation for $r=\frac{1}{2}$

  $a_{k+2}=\frac{4(2ka_{k+1}-a_k+4a_{k+1})}{4k^{2}+20k+24}$
- Solution for $r=\frac{1}{2}$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}},a_{k+2}=\frac{4(2ka_{k+1}-a_k+4a_{k+1})}{4k^{2}+20k+24},a_1=a_0\right]$
- Combine solutions and rename parameters

  $\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right),a_{k+2}=\frac{4(2ka_{k+1}-a_k+2a_{k+1})}{4k^{2}+12k+8},a_1=2a_0,b_{k+2}=\frac{4(2kb_{k+1}-b_k+4b_{k+}}{4k^{2}+20k+24}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.022 (sec)
Leaf size : 15

```
dsolve(4*x^2*diff(diff(y(x),x),x)+(-8*x^2+4*x)*diff(y(x),x)+(4*x^2-4*x-1)*y(x) = 0,y(x
```

$$y = \frac{e^{x}(c_2 x + c_1)}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.03 (sec)
Leaf size : 21

```
DSolve[{4*x^2*D[y[x],{x,2}]+(4*x-8*x^2)*D[y[x],x]+(4*x^2-4*x-1)*y[x]==0,{}},y[x],x,IncludeSi
```

$$y(x) \to \frac{e^{x}(c_2 x + c_1)}{\sqrt{x}}$$

2.1.453 Problem 467

Internal problem ID [9625]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 467
Date solved : `Monday, January 27, 2025 at 06:04:46 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + 4xy' + \left(4x^2 + 2\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.056 (sec)

Writing the ode as

$$y'' + 4xy' + \left(4x^2 + 2\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4x \\ C &= 4x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.860: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x}{1}\, dx} \\ &= z_1 e^{-x^2} \\ &= z_1\left(\mathrm{e}^{-x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x^2}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(e^{-x^2}\right) + c_2\left(e^{-x^2}(x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2}y(x) + 4x\left(\frac{d}{dx}y(x)\right) + (4x^2+2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^k$
- ☐ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$
    $x^m \cdot y(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$
  - ◦ Shift index using $k- >k-m$
    $x^m \cdot y(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$
  - ◦ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion
    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$
  - ◦ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion
    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$
  - ◦ Shift index using $k- >k+2$
    $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions
  $2a_2 + 2a_0 + (6a_3 + 6a_1)\,x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 2a_k(2k+1) + 4a_{k-2}\right) x^k\right) = 0$
- The coefficients of each power of x must be 0
  $[2a_2 + 2a_0 = 0, 6a_3 + 6a_1 = 0]$
- Solve for the dependent coefficient(s)
  $\{a_2 = -a_0, a_3 = -a_1\}$
- Each term in the series must be 0, giving the recursion relation
  $(k^2 + 3k + 2)\,a_{k+2} + 4a_k k + 2a_k + 4a_{k-2} = 0$

- Shift index using $k->k+2$

  $$\left((k+2)^2+3k+8\right)a_{k+4}+4a_{k+2}(k+2)+2a_{k+2}+4a_k=0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y(x)=\sum_{k=0}^{\infty}a_k x^k, a_{k+4}=-\frac{2(2ka_{k+2}+2a_k+5a_{k+2})}{k^2+7k+12}, a_2=-a_0, a_3=-a_1\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 16

```
dsolve(diff(diff(y(x),x),x)+4*diff(y(x),x)*x+(4*x^2+2)*y(x) = 0,y(x),singsol=all)
```

$$y=e^{-x^2}(c_2x+c_1)$$

Mathematica DSolve solution

Solving time : 0.027 (sec)
Leaf size : 21

```
DSolve[{4*x^2*D[y[x],{x,2}]+(4*x-8*x^2)*D[y[x],x]+(4*x^2-4*x-1)*y[x]==0,{}},y[x],x,IncludeSi
```

$$y(x)\to\frac{e^x(c_2x+c_1)}{\sqrt{x}}$$

2.1.454 Problem 468

Internal problem ID [9626]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 468
Date solved : `Monday, January 27, 2025 at 06:04:46 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(2x+1)\,y''-2y'-(2x+3)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.218 (sec)

Writing the ode as

$$(2x+1)\,y''-2y'+(-2x-3)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x+1 \\ B &= -2 \\ C &= -2x-3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{4x^2+8x+6}{(2x+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2+8x+6 \\ t &= (2x+1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{4x^2+8x+6}{(2x+1)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.862: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x+1)^2$. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{3}{4\left(x+\frac{1}{2}\right)^2} + \frac{1}{x+\frac{1}{2}}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{1}{2x} - \frac{1}{4x^3} + \frac{11}{32x^4} - \frac{21}{64x^5} + \frac{15}{64x^6} - \frac{3}{32x^7} - \frac{117}{2048x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 + 8x + 6}{4x^2 + 4x + 1} \\ &= Q + \frac{R}{4x^2 + 4x + 1} \\ &= (1) + \left(\frac{4x + 5}{4x^2 + 4x + 1}\right) \\ &= 1 + \frac{4x + 5}{4x^2 + 4x + 1} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$\begin{aligned} b &= (1) - (0) \\ &= 1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{1}{1} - 0\right) = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{1}{1} - 0\right) = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^2 + 8x + 6}{(2x + 1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2\left(x + \frac{1}{2}\right)} + (-)(1) \\ &= -\frac{1}{2\left(x + \frac{1}{2}\right)} - 1 \\ &= -\frac{2(x + 1)}{2x + 1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\left(x + \frac{1}{2}\right)} - 1\right)(0) + \left(\left(\frac{1}{2\left(x + \frac{1}{2}\right)^2}\right) + \left(-\frac{1}{2\left(x + \frac{1}{2}\right)} - 1\right)^2 - \left(\frac{4x^2 + 8x + 6}{(2x + 1)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2\left(x+\frac{1}{2}\right)}-1\right) dx} \\ &= \frac{\mathrm{e}^{-x}}{\sqrt{2x+1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{2x+1} \, dx} \\ &= z_1 e^{\frac{\ln(2x+1)}{2}} \\ &= z_1 \left(\sqrt{2x+1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{2x+1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\ln(2x+1)}}{(y_1)^2} \, dx \\ &= y_1 \left(x \, \mathrm{e}^{2x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(x \, \mathrm{e}^{2x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$(2x+1)\left(\frac{d^2}{dx^2}y(x)\right) - 2\frac{d}{dx}y(x) - (2x+3)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = \frac{(2x+3)y(x)}{2x+1} + \frac{2\left(\frac{d}{dx}y(x)\right)}{2x+1}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x) - \frac{2\left(\frac{d}{dx}y(x)\right)}{2x+1} - \frac{(2x+3)y(x)}{2x+1} = 0$

- ☐ Check to see if $x_0 = -\frac{1}{2}$ is a regular singular point
  - ∘ Define functions

    $\left[P_2(x) = -\frac{2}{2x+1}, P_3(x) = -\frac{2x+3}{2x+1}\right]$
  - ∘ $\left(x+\frac{1}{2}\right) \cdot P_2(x)$ is analytic at $x = -\frac{1}{2}$

    $\left(\left(x+\frac{1}{2}\right) \cdot P_2(x)\right)\bigg|_{x=-\frac{1}{2}} = -1$
  - ∘ $\left(x+\frac{1}{2}\right)^2 \cdot P_3(x)$ is analytic at $x = -\frac{1}{2}$

    $\left(\left(x+\frac{1}{2}\right)^2 \cdot P_3(x)\right)\bigg|_{x=-\frac{1}{2}} = 0$
  - ∘ $x = -\frac{1}{2}$is a regular singular point

    Check to see if $x_0 = -\frac{1}{2}$ is a regular singular point

    $x_0 = -\frac{1}{2}$
- • Multiply by denominators

  $(2x+1)\left(\frac{d^2}{dx^2}y(x)\right) - 2\frac{d}{dx}y(x) + (-2x-3)\,y(x) = 0$
- • Change variables using $x = u - \frac{1}{2}$ so that the regular singular point is at $u = 0$

  $2u\left(\frac{d^2}{du^2}y(u)\right) - 2\frac{d}{du}y(u) + (-2u-2)\,y(u) = 0$
- • Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$
- ☐ Rewrite ODE with series expansions
  - ∘ Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$
  - ∘ Shift index using $k- >k-m$

    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
  - ∘ Convert $\frac{d}{du}y(u)$ to series expansion

    $\frac{d}{du}y(u) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1}$
  - ∘ Shift index using $k- >k+1$

    $\frac{d}{du}y(u) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, u^{k+r}$
  - ∘ Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

    $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r-1}$
  - ∘ Shift index using $k- >k+1$

    $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, u^{k+r}$

    Rewrite ODE with series expansions

    $2a_0 r(-2+r)\, u^{-1+r} + (2a_1(1+r)\,(-1+r) - 2a_0)\, u^r + \left(\sum_{k=1}^{\infty} (2a_{k+1}(k+1+r)\,(k+r-1) - 2a_k\right.$
- • $a_0$cannot be 0 by assumption, giving the indicial equation

  $2r(-2+r) = 0$
- • Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term must be 0
  $2a_1(1+r)(-1+r) - 2a_0 = 0$
- Each term in the series must be 0, giving the recursion relation
  $2a_{k+1}(k+1+r)(k+r-1) - 2a_k - 2a_{k-1} = 0$
- Shift index using $k- > k+1$
  $2a_{k+2}(k+2+r)(k+r) - 2a_{k+1} - 2a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{a_{k+1}+a_k}{(k+2+r)(k+r)}$
- Recursion relation for $r = 0$
  $a_{k+2} = \frac{a_{k+1}+a_k}{(k+2)k}$
- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$
  $a_{k+2} = \frac{a_{k+1}+a_k}{(k+2)k}$
- Recursion relation for $r = 2$
  $a_{k+2} = \frac{a_{k+1}+a_k}{(k+4)(k+2)}$
- Solution for $r = 2$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2} = \frac{a_{k+1}+a_k}{(k+4)(k+2)}, 6a_1 - 2a_0 = 0\right]$
- Revert the change of variables $u = x + \frac{1}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k \left(x+\frac{1}{2}\right)^{k+2}, a_{k+2} = \frac{a_{k+1}+a_k}{(k+4)(k+2)}, 6a_1 - 2a_0 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 16

```
dsolve((2*x+1)*diff(diff(y(x),x),x)-2*diff(y(x),x)-(2*x+3)*y(x) = 0,y(x),singsol=all)
```

$$y = c_1 e^{-x} + c_2 e^{x} x$$

Mathematica DSolve solution

Solving time : 0.333 (sec)
Leaf size : 69

```
DSolve[{(2*x+1)*D[y[x],{x,2}]-2*D[y[x],x]-(2*x+3)*y[x]==0,{}},y[x],x,IncludeSingularSolution
```

$$y(x) \to \sqrt{2x+1}\exp\left(\int_1^x \left(\frac{1}{-2K[1]-1} - 1\right) dK[1]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[2]}\left(\frac{1}{-2K[1]-1}-1\right)dK[1]\right)dK[2]+c_1\right)$$

2.1.455 Problem 469

Internal problem ID [9627]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 469
Date solved : `Monday, January 27, 2025 at 06:04:47 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' - (2x+2)\,y' + (x+2)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.121 (sec)

Writing the ode as

$$xy'' + (-2x-2)\,y' + (x+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -2x-2 \\ C &= x+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.864: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the gcd$(s,t)=1$. This gives $b=2$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=2\\ \alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=-1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=-1$ then

$$\begin{aligned}d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right)\\ &= -1-(-1)\\ &= 0\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^-}{x-c_1}\right)+(-)[\sqrt{r}]_\infty\\ &= -\frac{1}{x}+(-)(0)\\ &= -\frac{1}{x}\\ &= -\frac{1}{x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0)+2\left(-\frac{1}{x}\right)(0)+\left(\left(\frac{1}{x^2}\right)+\left(-\frac{1}{x}\right)^2-\left(\frac{2}{x^2}\right)\right) &= 0\\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{1}{x} dx} \\
&= \frac{1}{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x-2}{x} \, dx} \\
&= z_1 e^{x+\ln(x)} \\
&= z_1 (x\, \mathrm{e}^{x})
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-2}{x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{2x+2\ln(x)}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{x\, \mathrm{e}^{2x+2\ln(x)} \mathrm{e}^{-2x}}{3} \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^{x}) + c_2 \left(\mathrm{e}^{x} \left(\frac{x\, \mathrm{e}^{2x+2\ln(x)} \mathrm{e}^{-2x}}{3} \right) \right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dx^2} y(x)\right) x - (2x+2)\left(\frac{d}{dx} y(x)\right) + (x+2)\, y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2} y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2} y(x) = -\frac{(x+2)y(x)}{x} + \frac{2(x+1)\left(\frac{d}{dx} y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x) - \frac{2(x+1)\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(x+2)y(x)}{x} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{2(x+1)}{x}, P_3(x) = \frac{x+2}{x}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = -2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$
  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$
- Multiply by denominators
  $\left(\frac{d^2}{dx^2}y(x)\right)x + (-2-2x)\left(\frac{d}{dx}y(x)\right) + (x+2)\,y(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$
    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k-m$
    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..1$
    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k- >k+1-m$
    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion
    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$
  - Shift index using $k- >k+1$
    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-3+r)\,x^{-1+r} + (a_1(1+r)\,(-2+r) - 2a_0(-1+r))\,x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(k-2+r) - 2\right.$
- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-3+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 3\}$
- Each term must be 0
  $a_1(1+r)\,(-2+r) - 2a_0(-1+r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)\,(k-2+r) - 2a_kk - 2a_kr + 2a_k + a_{k-1} = 0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)(k+r-1)-2a_{k+1}(k+1)-2ra_{k+1}+2a_{k+1}+a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{2ka_{k+1}+2ra_{k+1}-a_k}{(k+2+r)(k+r-1)}$
- Recursion relation for $r=0$

  $a_{k+2}=\frac{2ka_{k+1}-a_k}{(k+2)(k-1)}$
- Series not valid for $r=0$, division by 0 in the recursion relation at $k=1$

  $a_{k+2}=\frac{2ka_{k+1}-a_k}{(k+2)(k-1)}$
- Recursion relation for $r=3$

  $a_{k+2}=\frac{2ka_{k+1}-a_k+6a_{k+1}}{(k+5)(k+2)}$
- Solution for $r=3$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2}=\frac{2ka_{k+1}-a_k+6a_{k+1}}{(k+5)(k+2)}, 4a_1-4a_0=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 14

```
dsolve(x*diff(diff(y(x),x),x)-(2+2*x)*diff(y(x),x)+(x+2)*y(x) = 0,y(x),singsol=all)
```

$$y=e^x(c_2x^3+c_1)$$

Mathematica DSolve solution

Solving time : `0.062` (sec)
Leaf size : 31

```
DSolve[{x*D[y[x],{x,2}]-(2*x+2)*D[y[x],x]+(x+2)*y[x]==6*x^3*Exp[x],{}},y[x],x,IncludeSingula
```

$$y(x)\to\frac{1}{6}e^x(9x^4+2ec_2x^3+6ec_1)$$

2.1.456 Problem 470

Internal problem ID [9628]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 470
Date solved : `Monday, January 27, 2025 at 06:04:47 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - 2xy' + (x^2 + 2)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.125 (sec)

Writing the ode as

$$x^2y'' - 2xy' + (x^2 + 2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.866: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\, dx} \\ &= z_1 e^{\ln(x)} \\ &= z_1(x) \end{aligned}$$

Which simplifies to

$$y_1 = x\cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\, dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(x\cos(x)) + c_2(x\cos(x)(\tan(x))) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) - 2x\left(\frac{d}{dx}y(x)\right) + (x^2+2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(x^2+2)y(x)}{x^2} + \frac{2\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) - \frac{2\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(x^2+2)y(x)}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{2}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) - 2x\left(\frac{d}{dx}y(x)\right) + (x^2+2)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

$x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)\,(-2+r)\, x^r + a_1 r(-1+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)\,(k+r-2) + a_{k-2}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$
- Each term must be 0

  $a_1 r(-1+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)\,(k+r-2) + a_{k-2} = 0$
- Shift index using $k- >k+2$

  $a_{k+2}(k+1+r)\,(k+r) + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+1+r)(k+r)}$
- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$
- Solution for $r = 1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$
- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$
- Solution for $r = 2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+2)(k+3)}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
```

```
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 15

```
dsolve(x^2*diff(diff(y(x),x),x)-2*diff(y(x),x)*x+(x^2+2)*y(x) = 0,y(x),singsol=all)
```

$$y = x(\sin(x) c_1 + \cos(x) c_2)$$

Mathematica DSolve solution

Solving time : 0.029 (sec)
Leaf size : 33

```
DSolve[{x^2*D[y[x],{x,2}]-2*x*D[y[x],x]+(x^2+2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tr
```

$$y(x) \to c_1 e^{-ix} x - \frac{1}{2} i c_2 e^{ix} x$$

2.1.457 Problem 472

Internal problem ID [9629]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 472
Date solved : `Monday, January 27, 2025 at 06:04:48 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' - 4xy' + \left(-16x^2+3\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.090 (sec)

Writing the ode as

$$4x^2y'' - 4xy' + \left(-16x^2+3\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x \\ C &= -16x^2+3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.868: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{4x^2}\, dx} \\ &= z_1 e^{\frac{\ln(x)}{2}} \\ &= z_1\left(\sqrt{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}\, \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{4x^2}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx\\ &= y_1\left(\frac{e^{4x}}{4}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2\\ &= c_1\left(\sqrt{x}\,e^{-2x}\right) + c_2\left(\sqrt{x}\,e^{-2x}\left(\frac{e^{4x}}{4}\right)\right)\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$4x^2\left(\frac{d^2}{dx^2}y(x)\right) - 4x\left(\frac{d}{dx}y(x)\right) + \left(-16x^2+3\right)y(x) = 0$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
$\frac{d^2}{dx^2}y(x) = \frac{(16x^2-3)y(x)}{4x^2} + \frac{\frac{d}{dx}y(x)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
$\frac{d^2}{dx^2}y(x) - \frac{\frac{d}{dx}y(x)}{x} - \frac{(16x^2-3)y(x)}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
  $\left[P_2(x) = -\frac{1}{x}, P_3(x) = -\frac{16x^2-3}{4x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
  $\left.(x \cdot P_2(x))\right|_{x=0} = -1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{3}{4}$
  - $x = 0$is a regular singular point
  Check to see if $x_0 = 0$ is a regular singular point
  $x_0 = 0$

- Multiply by denominators
$4x^2\left(\frac{d^2}{dx^2}y(x)\right) - 4x\left(\frac{d}{dx}y(x)\right) + \left(-16x^2+3\right)y(x) = 0$
- Assume series solution for $y(x)$
$y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$
  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$
- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)\,(-3+2r)\,x^r + a_1(1+2r)\,(-1+2r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r-1)\,(2k+2r-3) - 16a\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)\,(-3+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$
- Each term must be 0

  $a_1(1+2r)\,(-1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r-\frac{3}{2}\right)\left(k+r-\frac{1}{2}\right)a_k - 16a_{k-2} = 0$
- Shift index using $k->k+2$

  $4\left(k+\frac{1}{2}+r\right)\left(k+\frac{3}{2}+r\right)a_{k+2} - 16a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{16a_k}{(2k+1+2r)(2k+3+2r)}$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = \frac{16a_k}{(2k+2)(2k+4)}$
- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{16a_k}{(2k+2)(2k+4)}, a_1 = 0\right]$
- Recursion relation for $r = \frac{3}{2}$

  $a_{k+2} = \frac{16a_k}{(2k+4)(2k+6)}$
- Solution for $r = \frac{3}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2} = \frac{16a_k}{(2k+4)(2k+6)}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+2} = \frac{16a_k}{(2k+2)(2k+4)}, a_1 = 0, b_{k+2} = \frac{16b_k}{(2k+4)(2k+6)}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 21

```
dsolve(4*x^2*diff(diff(y(x),x),x)-4*diff(y(x),x)*x+(-16*x^2+3)*y(x) = 0,y(x),singsol=a
```

$$y = \sqrt{x}\,(c_1 \sinh(2x) + c_2 \cosh(2x))$$

Mathematica DSolve solution

Solving time : 0.035 (sec)
Leaf size : 32

```
DSolve[{4*x^2*D[y[x],{x,2}]-4*x*D[y[x],x]+(3-16*x^2)*y[x]==0,{}},y[x],x,IncludeSingularSolut
```

$$y(x) \to \frac{1}{4} e^{-2x} \sqrt{x} \left(c_2 e^{4x} + 4c_1\right)$$

2.1.458 Problem 473

Internal problem ID [9630]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 473
Date solved : `Monday, January 27, 2025 at 06:04:48 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' - 4xy' + (4x^2+3)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.147 (sec)

Writing the ode as

$$4x^2y'' - 4xy' + (4x^2+3)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x \\ C &= 4x^2+3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.870: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{4x^2}\, dx} \\ &= z_1 e^{\frac{\ln(x)}{2}} \\ &= z_1\left(\sqrt{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}\, \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\sqrt{x}\,\cos(x)\right) + c_2\left(\sqrt{x}\,\cos(x)\,(\tan(x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2\left(\frac{d^2}{dx^2}y(x)\right) - 4x\left(\frac{d}{dx}y(x)\right) + (4x^2+3)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(4x^2+3)y(x)}{4x^2} + \frac{\frac{d}{dx}y(x)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) - \frac{\frac{d}{dx}y(x)}{x} + \frac{(4x^2+3)y(x)}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = -\frac{1}{x}, P_3(x) = \frac{4x^2+3}{4x^2}\right]$
- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -1$
- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{3}{4}$
- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) - 4x\left(\frac{d}{dx}y(x)\right) + (4x^2+3)\,y(x) = 0$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- ◦ Shift index using $k- >k - m$

$$x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

◦ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

$$x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$$

◦ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

$$x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+2r)\,(-3+2r)\, x^r + a_1(1+2r)\,(-1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r-1)\,(2k+2r-3) + 4$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+2r)\,(-3+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$
- Each term must be 0
  $a_1(1+2r)\,(-1+2r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $4\left(k+r-\frac{3}{2}\right)\left(k+r-\frac{1}{2}\right) a_k + 4a_{k-2} = 0$
- Shift index using $k- > k+2$
  $4\left(k+\frac{1}{2}+r\right)\left(k+\frac{3}{2}+r\right) a_{k+2} + 4a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{4a_k}{(2k+1+2r)(2k+3+2r)}$
- Recursion relation for $r = \frac{1}{2}$
  $a_{k+2} = -\frac{4a_k}{(2k+2)(2k+4)}$
- Solution for $r = \frac{1}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{(2k+2)(2k+4)}, a_1 = 0\right]$
- Recursion relation for $r = \frac{3}{2}$
  $a_{k+2} = -\frac{4a_k}{(2k+4)(2k+6)}$
- Solution for $r = \frac{3}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2} = -\frac{4a_k}{(2k+4)(2k+6)}, a_1 = 0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+2} = -\frac{4a_k}{(2k+2)(2k+4)}, a_1 = 0, b_{k+2} = -\frac{4b_k}{(2k+4)(2k+6)}, b_1 =\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 17

```
dsolve(4*x^2*diff(diff(y(x),x),x)-4*diff(y(x),x)*x+(4*x^2+3)*y(x) = 0,y(x),singsol=all)
```

$$y = \sqrt{x}\,(\sin(x)\,c_1 + \cos(x)\,c_2)$$

Mathematica DSolve solution

Solving time : 0.032 (sec)
Leaf size : 39

```
DSolve[{4*x^2*D[y[x],{x,2}]-4*x*D[y[x],x]+(4*x^2+3)*y[x]==0,{}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{1}{2}e^{-ix}\sqrt{x}\left(2c_1 - ic_2e^{2ix}\right)$$

2.1.459 Problem 474

Internal problem ID [9631]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 474
Date solved : `Monday, January 27, 2025 at 06:04:49 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - 2xy' - \left(x^2 - 2\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.067 (sec)

Writing the ode as

$$x^2y'' - 2xy' + \left(-x^2 + 2\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= -x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.872: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\, dx} \\ &= z_1 e^{\ln(x)} \\ &= z_1(x) \end{aligned}$$

Which simplifies to

$$y_1 = x\, \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{e^{2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x\, e^{-x}\right) + c_2 \left(x\, e^{-x} \left(\frac{e^{2x}}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2 \left(\frac{d^2}{dx^2} y(x)\right) - 2x \left(\frac{d}{dx} y(x)\right) - (x^2 - 2)\, y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2} y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2} y(x) = \frac{(x^2-2) y(x)}{x^2} + \frac{2\left(\frac{d}{dx} y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2} y(x) - \frac{2\left(\frac{d}{dx} y(x)\right)}{x} - \frac{(x^2-2) y(x)}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2}{x}, P_3(x) = -\frac{x^2-2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 \left(\frac{d^2}{dx^2} y(x)\right) - 2x \left(\frac{d}{dx} y(x)\right) + (-x^2 + 2)\, y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

$x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)\,(-2+r)\, x^r + a_1 r(-1+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)\,(k+r-2) - a_{k-2}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$

- Each term must be 0

  $a_1 r(-1+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)\,(k+r-2) - a_{k-2} = 0$

- Shift index using $k- >k+2$

  $a_{k+2}(k+1+r)\,(k+r) - a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k}{(k+1+r)(k+r)}$

- Recursion relation for $r = 1$

  $a_{k+2} = \frac{a_k}{(k+2)(k+1)}$

- Solution for $r = 1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = \frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$

- Recursion relation for $r = 2$

  $a_{k+2} = \frac{a_k}{(k+3)(k+2)}$

- Solution for $r = 2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = \frac{a_k}{(k+1)(k+2)}, a_1 = 0, b_{k+2} = \frac{b_k}{(k+2)(k+3)}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 15

```
dsolve(x^2*diff(diff(y(x),x),x)-2*diff(y(x),x)*x-(x^2-2)*y(x) = 0,y(x),singsol=all)
```

$$y = x(c_1 \sinh(x) + c_2 \cosh(x))$$

Mathematica DSolve solution

Solving time : 0.028 (sec)
Leaf size : 25

```
DSolve[{x^2*D[y[x],{x,2}]-2*x*D[y[x],x]-(x^2-2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to c_1 e^{-x} x + \frac{1}{2} c_2 e^x x$$

2.1.460 Problem 475

Internal problem ID [9632]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 475
Date solved : `Monday, January 27, 2025 at 06:04:49 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - 2x(x+1)\,y' + \left(x^2+2x+2\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.071 (sec)

Writing the ode as

$$x^2y'' + \left(-2x^2-2x\right)y' + \left(x^2+2x+2\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x^2-2x \\ C &= x^2+2x+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.874: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x^2-2x}{x^2}\, dx} \\ &= z_1 e^{x+\ln(x)} \\ &= z_1 (x\, \mathrm{e}^{x}) \end{aligned}$$

Which simplifies to

$$y_1 = x\, \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^2-2x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2x+2\ln(x)}}{(y_1)^2}\, dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 (x\, e^x) + c_2 (x\, e^x(x))
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) - 2x(x+1)\left(\frac{d}{dx}y(x)\right) + (x^2+2x+2)\, y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(x^2+2x+2)y(x)}{x^2} + \frac{2(x+1)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) - \frac{2(x+1)\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(x^2+2x+2)y(x)}{x^2} = 0$

- □ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2(x+1)}{x}, P_3(x) = \frac{x^2+2x+2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 2$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) - 2x(x+1)\left(\frac{d}{dx}y(x)\right) + (x^2+2x+2)\, y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k-m$

$x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)\,(-2+r)\, x^r + (a_1 r(-1+r) - 2a_0(-1+r))\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-1)\,(k+r-2) -\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$

- Each term must be 0

  $a_1 r(-1+r) - 2a_0(-1+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = \frac{2a_0}{r}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)\,(k+r-2) - 2a_{k-1}k - 2a_{k-1}r + a_{k-2} + 4a_{k-1} = 0$

- Shift index using $k- >k+2$

  $a_{k+2}(k+1+r)\,(k+r) - 2a_{k+1}(k+2) - 2a_{k+1}r + a_k + 4a_{k+1} = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2ka_{k+1} + 2a_{k+1}r - a_k}{(k+1+r)(k+r)}$

- Recursion relation for $r = 1$

  $a_{k+2} = \frac{2ka_{k+1} - a_k + 2a_{k+1}}{(k+2)(k+1)}$

- Solution for $r = 1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = \frac{2ka_{k+1} - a_k + 2a_{k+1}}{(k+2)(k+1)}, a_1 = 2a_0\right]$

- Recursion relation for $r = 2$

  $a_{k+2} = \frac{2ka_{k+1} - a_k + 4a_{k+1}}{(k+3)(k+2)}$

- Solution for $r = 2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{2ka_{k+1} - a_k + 4a_{k+1}}{(k+3)(k+2)}, a_1 = a_0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = \frac{2ka_{k+1} - a_k + 2a_{k+1}}{(k+1)(k+2)}, a_1 = 2a_0, b_{k+2} = \frac{2kb_{k+1} - b_k + 4b_{k+1}}{(k+2)(k+3)},\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 13

```
dsolve(x^2*diff(diff(y(x),x),x)-2*x*(x+1)*diff(y(x),x)+(x^2+2*x+2)*y(x) = 0,y(x),singsol
```

$$y = e^x x(c_2 x + c_1)$$

Mathematica DSolve solution

Solving time : 0.132 (sec)
Leaf size : 41

```
DSolve[{x^2*D[y[x],{x,2}]-2*x*D[y[x],x]+(x^2+2*x+2)*y[x]==0,{}},y[x],x,IncludeSingularSolution
```

$$y(x) \to e^{ix} x(c_1 \operatorname{HypergeometricU}(-i, 0, -2ix) + c_2 L_i^{-1}(-2ix))$$

2.1.461 Problem 476

Internal problem ID [9633]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 476
Date solved : `Monday, January 27, 2025 at 06:04:50 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - 2x(x+2)\,y' + \left(x^2+4x+6\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.066 (sec)

Writing the ode as

$$x^2y'' + \left(-2x^2-4x\right)y' + \left(x^2+4x+6\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x^2-4x \\ C &= x^2+4x+6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.876: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x^2-4x}{x^2}\, dx} \\ &= z_1 e^{x+2\ln(x)} \\ &= z_1 \left(x^2 \mathrm{e}^x\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^2 \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^2-4x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x+4\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(x^2 e^x\right) + c_2 \left(x^2 e^x (x)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) - 2x(x+2)\left(\frac{d}{dx}y(x)\right) + (x^2+4x+6)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(x^2+4x+6)y(x)}{x^2} + \frac{2(x+2)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) - \frac{2(x+2)\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(x^2+4x+6)y(x)}{x^2} = 0$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{2(x+2)}{x}, P_3(x) = \frac{x^2+4x+6}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -4$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 6$
  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) - 2x(x+2)\left(\frac{d}{dx}y(x)\right) + (x^2+4x+6)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k-m$

$$x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx} y(x)\right)$ to series expansion for $m = 1..2$

$$x^m \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- > k+1-m$

$$x^m \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right)$ to series expansion

$$x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-2+r)\,(-3+r)\, x^r + (a_1(-1+r)\,(-2+r) - 2a_0(-2+r))\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-2)\,(k+r-\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-2+r)\,(-3+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{2, 3\}$
- Each term must be 0
  $a_1(-1+r)\,(-2+r) - 2a_0(-2+r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = \frac{2a_0}{-1+r}$
- Each term in the series must be 0, giving the recursion relation
  $a_k(k+r-2)\,(k+r-3) - 2a_{k-1}k - 2a_{k-1}r + a_{k-2} + 6a_{k-1} = 0$
- Shift index using $k- > k+2$
  $a_{k+2}(k+r)\,(k+r-1) - 2a_{k+1}(k+2) - 2a_{k+1}r + a_k + 6a_{k+1} = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{2ka_{k+1}+2a_{k+1}r-a_k-2a_{k+1}}{(k+r)(k+r-1)}$
- Recursion relation for $r = 2$
  $a_{k+2} = \frac{2ka_{k+1}-a_k+2a_{k+1}}{(k+2)(k+1)}$
- Solution for $r = 2$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{2ka_{k+1}-a_k+2a_{k+1}}{(k+2)(k+1)}, a_1 = 2a_0\right]$
- Recursion relation for $r = 3$
  $a_{k+2} = \frac{2ka_{k+1}-a_k+4a_{k+1}}{(k+3)(k+2)}$
- Solution for $r = 3$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = \frac{2ka_{k+1}-a_k+4a_{k+1}}{(k+3)(k+2)}, a_1 = a_0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), a_{k+2} = \frac{2ka_{k+1}-a_k+2a_{k+1}}{(k+1)(k+2)}, a_1 = 2a_0, b_{k+2} = \frac{2kb_{k+1}-b_k+4b_{k+1}}{(k+2)(k+3)}, b_1\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 15

```
dsolve(x^2*diff(diff(y(x),x),x)-2*x*(x+2)*diff(y(x),x)+(x^2+4*x+6)*y(x) = 0,y(x),sings
```

$$y = e^x x^2 (c_2 x + c_1)$$

Mathematica DSolve solution

Solving time : 0.049 (sec)
Leaf size : 21

```
DSolve[{x^2*D[y[x],{x,2}]-2*x*(x+2)*D[y[x],x]+(x^2+4*x+6)*y[x]==0,{}},y[x],x,IncludeSingular
```

$$y(x) \to e^{x+2} x^2 (c_2 x + c_1)$$

2.1.462 Problem 477

Internal problem ID [9634]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 477
Date solved : `Monday, January 27, 2025 at 06:04:50 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - 4xy' + (x^2+6)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.131 (sec)

Writing the ode as

$$x^2y'' - 4xy' + (x^2+6)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -4x \\ C &= x^2+6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.878: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{x^2}\, dx} \\ &= z_1 e^{2\ln(x)} \\ &= z_1 \left(x^2\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^2 \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{4\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^2\cos(x)\right) + c_2\left(x^2\cos(x)\left(\tan(x)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) - 4x\left(\frac{d}{dx}y(x)\right) + (x^2+6)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(x^2+6)y(x)}{x^2} + \frac{4\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) - \frac{4\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(x^2+6)y(x)}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ∘ Define functions

  $\left[P_2(x) = -\frac{4}{x}, P_3(x) = \frac{x^2+6}{x^2}\right]$
- ∘ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -4$
- ∘ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 6$
- ∘ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$
- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) - 4x\left(\frac{d}{dx}y(x)\right) + (x^2+6)\,y(x) = 0$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ∘ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- ∘ Shift index using $k- >k-m$

$x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)\,(-3+r)\,x^r + a_1(-1+r)\,(-2+r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-2)\,(k+r-3) + a_{k-2})\, x^{k}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)\,(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{2, 3\}$

- Each term must be 0

  $a_1(-1+r)\,(-2+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-2)\,(k+r-3) + a_{k-2} = 0$

- Shift index using $k- >k+2$

  $a_{k+2}(k+r)\,(k+r-1) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+r)(k+r-1)}$

- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$

- Solution for $r = 2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$

- Recursion relation for $r = 3$

  $a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$

- Solution for $r = 3$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+2)(k+3)}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
```

```
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)-4*diff(y(x),x)*x+(x^2+6)*y(x) = 0,y(x),singsol=all)
```

$$y = x^2(\sin(x)\, c_1 + \cos(x)\, c_2)$$

Mathematica DSolve solution

Solving time : 0.036 (sec)
Leaf size : 37

```
DSolve[{x^2*D[y[x],{x,2}]-4*x*D[y[x],x]+(x^2+6)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tr
```

$$y(x) \to \frac{1}{2} e^{-ix} x^2 \left(2c_1 - ic_2 e^{2ix}\right)$$

2.1.463 Problem 478

Internal problem ID [9635]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 478
Date solved : `Monday, January 27, 2025 at 06:04:51 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(x-1)\, y''-x y'+y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.232 (sec)

Writing the ode as

$$(x-1)\, y''-x y'+y=0 \tag{1}$$
$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x-1 \\ B &= -x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{x^2-4x+6}{4\left(x-1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2-4x+6 \\ t &= 4(x-1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x^2-4x+6}{4\left(x-1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.880: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2(x-1)} + \frac{3}{4(x-1)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(x-1)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2(x-1)} + \frac{1}{2} \\
&= \frac{x-2}{2x-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2(x-1)}+\frac{1}{2}\right)(0)+\left(\left(\frac{1}{2(x-1)^2}\right)+\left(-\frac{1}{2(x-1)}+\frac{1}{2}\right)^2-\left(\frac{x^2-4x+6}{4(x-1)^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\,dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x-1)}+\frac{1}{2}\right)dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{x-1}\,dx} \\
&= z_1 e^{\frac{x}{2}+\frac{\ln(x-1)}{2}} \\
&= z_1\left(\sqrt{x-1}\,\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x-1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x+\ln(x-1)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{x\,\mathrm{e}^{x+\ln(x-1)}\mathrm{e}^{-2x}}{x-1}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^{x}) + c_2\left(\mathrm{e}^{x}\left(-\frac{x\,\mathrm{e}^{x+\ln(x-1)}\mathrm{e}^{-2x}}{x-1}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x-1)\left(\frac{d^2}{dx^2}y(x)\right)-x\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{y(x)}{x-1}+\frac{\left(\frac{d}{dx}y(x)\right)x}{x-1}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)-\frac{\left(\frac{d}{dx}y(x)\right)x}{x-1}+\frac{y(x)}{x-1}=0$

□ Check to see if $x_0=1$ is a regular singular point

  - Define functions

    $\left[P_2(x)=-\frac{x}{x-1}, P_3(x)=\frac{1}{x-1}\right]$
  - $(x-1)\cdot P_2(x)$ is analytic at $x=1$

    $\left((x-1)\cdot P_2(x)\right)\Big|_{x=1}=-1$
  - $(x-1)^2\cdot P_3(x)$ is analytic at $x=1$

    $\left((x-1)^2\cdot P_3(x)\right)\Big|_{x=1}=0$
  - $x=1$is a regular singular point

  Check to see if $x_0=1$ is a regular singular point

  $x_0=1$

- Multiply by denominators

  $(x-1)\left(\frac{d^2}{dx^2}y(x)\right)-x\left(\frac{d}{dx}y(x)\right)+y(x)=0$
- Change variables using $x=u+1$ so that the regular singular point is at $u=0$

  $u\left(\frac{d^2}{du^2}y(u)\right)+(-u-1)\left(\frac{d}{du}y(u)\right)+y(u)=0$
- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

    $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-1}$
  - Shift index using $k->k+1$

    $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)(k+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)(k+r-1)-a_k(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{0,2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_{k+1}(k+1+r)-a_k\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k}{k+1+r}$

- Recursion relation for $r=0$

  $a_{k+1}=\frac{a_k}{k+1}$

- Solution for $r=0$

  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+1}=\frac{a_k}{k+1}\right]$

- Revert the change of variables $u=x-1$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k (x-1)^k, a_{k+1}=\frac{a_k}{k+1}\right]$

- Recursion relation for $r=2$

  $a_{k+1}=\frac{a_k}{k+3}$

- Solution for $r=2$

  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$

- Revert the change of variables $u=x-1$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k (x-1)^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$

- Combine solutions and rename parameters

  $\left[y(x)=\left(\sum_{k=0}^{\infty} a_k (x-1)^k\right)+\left(\sum_{k=0}^{\infty} b_k (x-1)^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+3}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 12

```
dsolve((x-1)*diff(diff(y(x),x),x)-diff(y(x),x)*x+y(x) = 0,y(x),singsol=all)
```

$$y = c_1 x + e^x c_2$$

Mathematica DSolve solution

Solving time : 0.153 (sec)
Leaf size : 90

```
DSolve[{(x-1)*D[y[x],{x,2}]-x*D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \exp\left(\int_1^x \frac{K[1]-2}{2(K[1]-1)}dK[1] - \frac{1}{2}\int_1^x -\frac{K[2]}{K[2]-1}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{K[1]-2}{2(K[1]-1)}dK[1]\right)dK[3] + c_1\right)$$

2.1.464 Problem 479

Internal problem ID [9636]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 479
Date solved : `Monday, January 27, 2025 at 06:04:51 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' - 4x(x+1)\,y' + (2x+3)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.093 (sec)

Writing the ode as

$$4x^2y'' + \left(-4x^2 - 4x\right) y' + (2x+3)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x^2 - 4x \\ C &= 2x+3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.882: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x^2-4x}{4x^2}\, dx} \\ &= z_1 e^{\frac{x}{2}+\frac{\ln(x)}{2}} \\ &= z_1\left(\sqrt{x}\, \mathrm{e}^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^2-4x}{4x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x+\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{e^{x+\ln(x)}}{x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x}\right) + c_2\left(\sqrt{x}\left(\frac{e^{x+\ln(x)}}{x}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$4x^2\left(\frac{d^2}{dx^2}y(x)\right) - 4x(x+1)\left(\frac{d}{dx}y(x)\right) + (2x+3)\, y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{(2x+3)y(x)}{4x^2} + \frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x) - \frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(2x+3)y(x)}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{x+1}{x}, P_3(x) = \frac{2x+3}{4x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = -1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{3}{4}$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$

- Multiply by denominators
  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) - 4x(x+1)\left(\frac{d}{dx}y(x)\right) + (2x+3)\, y(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$
    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y(x)=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)\,(-3+2r)\, x^r+\left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)\,(2k+2r-3)-2a_{k-1}(2k+2r-3)\right) x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)\,(-3+2r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{\frac{1}{2}, \frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $4\left(\left(k+r-\frac{1}{2}\right) a_k-a_{k-1}\right)\left(k+r-\frac{3}{2}\right)=0$

- Shift index using $k->k+1$

  $4\left(\left(k+\frac{1}{2}+r\right) a_{k+1}-a_k\right)\left(k+r-\frac{1}{2}\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{2a_k}{2k+1+2r}$

- Recursion relation for $r=\frac{1}{2}$

  $a_{k+1}=\frac{2a_k}{2k+2}$

- Solution for $r=\frac{1}{2}$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1}=\frac{2a_k}{2k+2}\right]$

- Recursion relation for $r=\frac{3}{2}$

  $a_{k+1}=\frac{2a_k}{2k+4}$

- Solution for $r=\frac{3}{2}$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1}=\frac{2a_k}{2k+4}\right]$

- Combine solutions and rename parameters

  $\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+1}=\frac{2a_k}{2k+2}, b_{k+1}=\frac{2b_k}{2k+4}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 14

```
dsolve(4*x^2*diff(diff(y(x),x),x)-4*x*(x+1)*diff(y(x),x)+(2*x+3)*y(x) = 0,y(x),singsol
```

$$y = (c_1 + e^x c_2)\sqrt{x}$$

Mathematica DSolve solution

Solving time : 0.047 (sec)
Leaf size : 25

```
DSolve[{4*x^2*D[y[x],{x,2}]-4*x*(x+1)*D[y[x],x]+(2*x+3)*y[x]==0,{}},y[x],x,IncludeSingularSo
```

$$y(x) \to \sqrt{e}\sqrt{x}(c_2 e^x + c_1)$$

2.1.465 Problem 480

Internal problem ID [9637]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 480
Date solved : `Monday, January 27, 2025 at 06:04:52 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(3x-1)\,y''-(3x+2)\,y'-(6x-8)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.238 (sec)

Writing the ode as

$$(3x-1)\,y''+(-3x-2)\,y'+(-6x+8)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x-1 \\ B &= -3x-2 \\ C &= -6x+8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{81x^2-108x+54}{4\,(3x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 81x^2-108x+54 \\ t &= 4(3x-1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{81x^2-108x+54}{4\,(3x-1)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.884: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(3x-1)^2$. There is a pole at $x = \frac{1}{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{4} + \frac{3}{4\left(x - \frac{1}{3}\right)^2} - \frac{3}{2\left(x - \frac{1}{3}\right)}$$

For the pole at $x = \frac{1}{3}$ let b be the coefficient of $\frac{1}{\left(x-\frac{1}{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3}{2} - \frac{1}{2x} + \frac{1}{9x^3} + \frac{11}{108x^4} + \frac{7}{108x^5} + \frac{5}{162x^6} + \frac{2}{243x^7} - \frac{13}{3888x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{3}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{81x^2 - 108x + 54}{36x^2 - 24x + 4} \\
&= Q + \frac{R}{36x^2 - 24x + 4} \\
&= \left(\frac{9}{4}\right) + \left(\frac{-54x + 45}{36x^2 - 24x + 4}\right) \\
&= \frac{9}{4} + \frac{-54x + 45}{36x^2 - 24x + 4}
\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -54. Dividing this by leading coefficient in t which is 36 gives $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{3}{2}\right) - (0) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{3}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a}-v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{3}{2}}-0\right) &&= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a}-v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{3}{2}}-0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{81x^2-108x+54}{4\left(3x-1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{1}{3}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(x-\frac{1}{3}\right)} + \left(\frac{3}{2}\right) \\
&= -\frac{1}{2\left(x-\frac{1}{3}\right)} + \frac{3}{2} \\
&= \frac{9x-6}{6x-2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2\left(x-\frac{1}{3}\right)}+\frac{3}{2}\right)(0)+\left(\left(\frac{1}{2\left(x-\frac{1}{3}\right)^2}\right)+\left(-\frac{1}{2\left(x-\frac{1}{3}\right)}+\frac{3}{2}\right)^2-\left(\frac{81x^2-108x+54}{4\left(3x-1\right)^2}\right)\right)=0$$

$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2\left(x-\frac{1}{3}\right)}+\frac{3}{2}\right) dx} \\ &= \frac{\mathrm{e}^{\frac{3x}{2}}}{\sqrt{3x-1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-3x-2}{3x-1}\, dx} \\ &= z_1 e^{\frac{x}{2}+\frac{\ln(3x-1)}{2}} \\ &= z_1\left(\sqrt{3x-1}\,\mathrm{e}^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3x-2}{3x-1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x+\ln(3x-1)}}{(y_1)^2}\, dx \\ &= y_1\left(-\frac{x\,\mathrm{e}^{x+\ln(3x-1)}\mathrm{e}^{-4x}}{3x-1}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{2x}\right) + c_2\left(\mathrm{e}^{2x}\left(-\frac{x\,\mathrm{e}^{x+\ln(3x-1)}\mathrm{e}^{-4x}}{3x-1}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(3x-1)\left(\frac{d^2}{dx^2}y(x)\right)-(3x+2)\left(\frac{d}{dx}y(x)\right)-(6x-8)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{2(3x-4)y(x)}{3x-1}+\frac{(3x+2)\left(\frac{d}{dx}y(x)\right)}{3x-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)-\frac{(3x+2)\left(\frac{d}{dx}y(x)\right)}{3x-1}-\frac{2(3x-4)y(x)}{3x-1}=0$

□ Check to see if $x_0=\frac{1}{3}$ is a regular singular point

  - Define functions

    $\left[P_2(x)=-\frac{3x+2}{3x-1},P_3(x)=-\frac{2(3x-4)}{3x-1}\right]$

  - $\left(x-\frac{1}{3}\right)\cdot P_2(x)$ is analytic at $x=\frac{1}{3}$

    $\left(\left(x-\frac{1}{3}\right)\cdot P_2(x)\right)\bigg|_{x=\frac{1}{3}}=-1$

  - $\left(x-\frac{1}{3}\right)^2\cdot P_3(x)$ is analytic at $x=\frac{1}{3}$

    $\left(\left(x-\frac{1}{3}\right)^2\cdot P_3(x)\right)\bigg|_{x=\frac{1}{3}}=0$

  - $x=\frac{1}{3}$is a regular singular point

  Check to see if $x_0=\frac{1}{3}$ is a regular singular point

  $x_0=\frac{1}{3}$

- Multiply by denominators

  $(3x-1)\left(\frac{d^2}{dx^2}y(x)\right)+(-3x-2)\left(\frac{d}{dx}y(x)\right)+(-6x+8)\,y(x)=0$

- Change variables using $x=u+\frac{1}{3}$ so that the regular singular point is at $u=0$

  $3u\left(\frac{d^2}{du^2}y(u)\right)+(-3u-3)\left(\frac{d}{du}y(u)\right)+(-6u+6)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

    $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

  - Shift index using $k->k-m$

    $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

    $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-1}$

- Shift index using $k->k+1$
  $$u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$$
  Rewrite ODE with series expansions
  $$3a_0r(-2+r)\,u^{-1+r}+\left(3a_1(1+r)\,(-1+r)-3a_0(-2+r)\right)u^r+\left(\sum_{k=1}^{\infty}\left(3a_{k+1}(k+1+r)\,(k+r-1)\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $3r(-2+r)=0$
- Values of r that satisfy the indicial equation
  $r\in\{0,2\}$
- Each term must be 0
  $3a_1(1+r)\,(-1+r)-3a_0(-2+r)=0$
- Each term in the series must be 0, giving the recursion relation
  $3a_{k+1}(k+1+r)\,(k+r-1)+a_k(-3k-3r+6)-6a_{k-1}=0$
- Shift index using $k->k+1$
  $3a_{k+2}(k+2+r)\,(k+r)+a_{k+1}(-3k+3-3r)-6a_k=0$
- Recursion relation that defines series solution to ODE
  $a_{k+2}=\frac{ka_{k+1}+ra_{k+1}+2a_k-a_{k+1}}{(k+2+r)(k+r)}$
- Recursion relation for $r=0$
  $a_{k+2}=\frac{ka_{k+1}+2a_k-a_{k+1}}{(k+2)k}$
- Series not valid for $r=0$, division by 0 in the recursion relation at $k=0$
  $a_{k+2}=\frac{ka_{k+1}+2a_k-a_{k+1}}{(k+2)k}$
- Recursion relation for $r=2$
  $a_{k+2}=\frac{ka_{k+1}+2a_k+a_{k+1}}{(k+4)(k+2)}$
- Solution for $r=2$
  $$\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k+2},a_{k+2}=\frac{ka_{k+1}+2a_k+a_{k+1}}{(k+4)(k+2)},9a_1=0\right]$$
- Revert the change of variables $u=x-\frac{1}{3}$
  $$\left[y(x)=\sum_{k=0}^{\infty}a_k\left(x-\frac{1}{3}\right)^{k+2},a_{k+2}=\frac{ka_{k+1}+2a_k+a_{k+1}}{(k+4)(k+2)},9a_1=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 18

```
dsolve((3*x-1)*diff(diff(y(x),x),x)-(2+3*x)*diff(y(x),x)-(6*x-8)*y(x) = 0,y(x),singsol
```

$$y = e^{2x}c_1 + c_2 x\,e^{-x}$$

Mathematica DSolve solution

Solving time : 0.205 (sec)
Leaf size : 94

```
DSolve[{(3*x-1)*D[y[x],{x,2}]-(3*x+2)*D[y[x],x]-(6*x-8)*y[x]==0,{}},y[x],x,IncludeSingularSo
```

$$y(x) \to \exp\left(\int_1^x \frac{6-9K[1]}{2-6K[1]}dK[1] - \frac{1}{2}\int_1^x \left(\frac{3}{1-3K[2]}-1\right)dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]}\frac{6-9K[1]}{2-6K[1]}dK[1]\right)dK[3] + c_1\right)$$

2.1.466 Problem 481

Internal problem ID [9638]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 481
Date solved : `Monday, January 27, 2025 at 06:04:53 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(2+x)\,y''+xy'+3y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.301 (sec)

Writing the ode as

$$(2+x)\,y''+xy'+3y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2+x \\ B &= x \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{x^2-12x-20}{4\,(2+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2-12x-20 \\ t &= 4(2+x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x^2-12x-20}{4\,(2+x)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.886: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2 + x)^2$. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{(2 + x)^2} - \frac{4}{2 + x}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(2+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{4}{x} - \frac{6}{x^2} - \frac{72}{x^3} - \frac{556}{x^4} - \frac{5440}{x^5} - \frac{55088}{x^6} - \frac{586688}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 12x - 20}{4x^2 + 16x + 16} \\ &= Q + \frac{R}{4x^2 + 16x + 16} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-16x - 24}{4x^2 + 16x + 16}\right) \\ &= \frac{1}{4} + \frac{-16x - 24}{4x^2 + 16x + 16} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -16. Dividing this by leading coefficient in t which is 4 gives -4. Now b can be found.

$$\begin{aligned} b &= (-4) - (0) \\ &= -4 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-4}{\frac{1}{2}} - 0\right) &&= -4 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-4}{\frac{1}{2}} - 0\right) &&= 4 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 12x - 20}{4\left(2+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	-4	4

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 4$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= 4 - (2) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{2}{2+x} + (-)\left(\frac{1}{2}\right) \\ &= \frac{2}{2+x} - \frac{1}{2} \\ &= -\frac{x-2}{2\left(2+x\right)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{2}{2+x} - \frac{1}{2}\right)(2x + a_1) + \left(\left(-\frac{2}{\left(2+x\right)^2} \right) + \left(\frac{2}{2+x} - \frac{1}{2} \right)^2 - \left(\frac{x^2 - 12x - 20}{4\left(2+x\right)^2} \right) \right) = 0$$

$$\frac{\left(a_1 + 6\right)x + 2a_0 + 2a_1 + 4}{2+x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 4, a_1 = -6\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 6x + 4$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x^2 - 6x + 4\right) e^{\int \left(\frac{2}{2+x} - \frac{1}{2}\right) dx} \\ &= \left(x^2 - 6x + 4\right) e^{-\frac{x}{2} + 2\ln(2+x)} \\ &= \left(x^2 - 6x + 4\right) (2+x)^2 e^{-\frac{x}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x}{2+x} \, dx} \\ &= z_1 e^{-\frac{x}{2} + \ln(2+x)} \\ &= z_1 \left((2+x) e^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2+x)^3 e^{-x} \left(x^2 - 6x + 4\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{2+x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-x + 2\ln(2+x)}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{e^x (x^4 - x^3 - 18x^2 - 22x + 8)}{240 \left(x^2 - 6x + 4\right) (2+x)^3} - \frac{e^{-2} \operatorname{Ei}_1(-2 - x)}{240} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left((2+x)^3 e^{-x} \left(x^2 - 6x + 4\right)\right) \\ &\quad + c_2 \left((2+x)^3 e^{-x} \left(x^2 - 6x + 4\right) \left(-\frac{e^x (x^4 - x^3 - 18x^2 - 22x + 8)}{240 \left(x^2 - 6x + 4\right) (2+x)^3} - \frac{e^{-2} \operatorname{Ei}_1(-2 - x)}{240} \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x+2)\left(\frac{d^2}{dx^2}y(x)\right)+x\left(\frac{d}{dx}y(x)\right)+3y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{3y(x)}{x+2}-\frac{\left(\frac{d}{dx}y(x)\right)x}{x+2}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)+\frac{\left(\frac{d}{dx}y(x)\right)x}{x+2}+\frac{3y(x)}{x+2}=0$

□ Check to see if $x_0=-2$ is a regular singular point

- ○ Define functions

  $\left[P_2(x)=\frac{x}{x+2},P_3(x)=\frac{3}{x+2}\right]$

- ○ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

  $\left((x+2)\cdot P_2(x)\right)\bigg|_{x=-2}=-2$

- ○ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

  $\left((x+2)^2\cdot P_3(x)\right)\bigg|_{x=-2}=0$

- ○ $x=-2$is a regular singular point

Check to see if $x_0=-2$ is a regular singular point

$x_0=-2$

- Multiply by denominators

$(x+2)\left(\frac{d^2}{dx^2}y(x)\right)+x\left(\frac{d}{dx}y(x)\right)+3y(x)=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

$u\left(\frac{d^2}{du^2}y(u)\right)+(u-2)\left(\frac{d}{du}y(u)\right)+3y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ○ Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-1}$

- ○ Shift index using $k->k+1$

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$

Rewrite ODE with series expansions

$a_0r(-3+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(k-2+r)+a_k(k+r+3)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-3+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{0,3\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k-2+r)+a_k(k+r+3)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{a_k(k+r+3)}{(k+1+r)(k-2+r)}$

- Recursion relation for $r=0$

  $a_{k+1}=-\frac{a_k(k+3)}{(k+1)(k-2)}$

- Series not valid for $r=0$, division by 0 in the recursion relation at $k=2$

  $a_{k+1}=-\frac{a_k(k+3)}{(k+1)(k-2)}$

- Recursion relation for $r=3$

  $a_{k+1}=-\frac{a_k(k+6)}{(k+4)(k+1)}$

- Solution for $r=3$

  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+3}, a_{k+1}=-\frac{a_k(k+6)}{(k+4)(k+1)}\right]$

- Revert the change of variables $u=x+2$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k(x+2)^{k+3}, a_{k+1}=-\frac{a_k(k+6)}{(k+4)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 72

```
dsolve((x+2)*diff(diff(y(x),x),x)+diff(y(x),x)*x+3*y(x) = 0,y(x),singsol=all)
```

$$y = \mathrm{e}^{-x-2} c_2 (x^2-6x+4)(x+2)^3 \operatorname{Ei}_1(-x-2) + c_1 \mathrm{e}^{-x}(x^2-6x+4)(x+2)^3 + c_2(x^4-x^3-18x^2-22x+8)$$

Mathematica DSolve solution

Solving time : 0.493 (sec)
Leaf size : 106

```
DSolve[{(2+x)*D[y[x],{x,2}]+x*D[y[x],x]+3*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to (x^2 - 6x + 4) \exp\left(\int_1^x \left(\frac{2}{K[1]+2} - \frac{1}{2}\right) dK[1] - \frac{1}{2}\int_1^x \frac{K[2]}{K[2]+2} dK[2]\right) \left(c_2 \int_1^x \frac{\exp\left(-2\int_1^{K[3]} \left(\frac{2}{K[1]+2} - \frac{1}{2}\right) dK[1]\right)}{(K[3]^2 - 6K[3] + 4)^2} dK[3] + c_1\right)$$

2.1.467 Problem 482

Internal problem ID [9639]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 482
Date solved : Monday, January 27, 2025 at 06:04:53 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$x^2(1-x)\,y''+x(4+x)\,y'+(2-x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.223 (sec)

Writing the ode as

$$\left(-x^3+x^2\right)y''+\left(x^2+4x\right)y'+(2-x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^3+x^2 \\ B &= x^2+4x \\ C &= 2-x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-x+36}{4x\left(-1+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x+36 \\ t &= 4x(-1+x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-x+36}{4x\left(-1+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.888: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x(-1+x)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{9}{-1+x} + \frac{35}{4\,(-1+x)^2} + \frac{9}{x}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x+36}{4x\left(-1+x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x+36}{4x\left(-1+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1
1	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{1}{2} - \left(-\frac{3}{2}\right) \\
&= 2
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{x} - \frac{5}{2\left(-1+x\right)} + (-)\left(0\right) \\
&= \frac{1}{x} - \frac{5}{2\left(-1+x\right)} \\
&= \frac{1}{x} - \frac{5}{-2+2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{1}{x} - \frac{5}{2(-1+x)}\right)(2x + a_1) + \left(\left(-\frac{1}{x^2} + \frac{5}{2(-1+x)^2}\right) + \left(\frac{1}{x} - \frac{5}{2(-1+x)}\right)^2 - \left(\frac{-x+36}{4x(-1+x)^2}\right.\right.$$
$$\frac{(a_1 - 6)x + 4a_0 - 2a_1}{x(-1+x)}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 3, a_1 = 6\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + 6x + 3$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x^2 + 6x + 3) e^{\int \left(\frac{1}{x} - \frac{5}{2(-1+x)}\right) dx} \\
&= (x^2 + 6x + 3) e^{\ln(x) - \frac{5\ln(-1+x)}{2}} \\
&= \frac{(x^2 + 6x + 3) x}{(-1+x)^{5/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2+4x}{-x^3+x^2} \, dx} \\
&= z_1 e^{-2\ln(x) + \frac{5\ln(-1+x)}{2}} \\
&= z_1 \left(\frac{(-1+x)^{5/2}}{x^2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2 + 6x + 3}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+4x}{-x^3+x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-4\ln(x) + 5\ln(-1+x)}}{(y_1)^2} \, dx \\
&= y_1 \left(-\frac{4\left(-38x - \frac{69}{2}\right)}{9(x^2 + 6x + 3)} + \ln(x) + \frac{1}{9x}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^2+6x+3}{x}\right) + c_2\left(\frac{x^2+6x+3}{x}\left(-\frac{4\left(-38x-\frac{69}{2}\right)}{9\left(x^2+6x+3\right)} + \ln\left(x\right) + \frac{1}{9x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 49

```
dsolve((1-x)*x^2*diff(diff(y(x),x),x)+x*(x+4)*diff(y(x),x)+(-x+2)*y(x) = 0,y(x),singsol=
```

$$
y = \frac{3xc_2\left(x^2+6x+3\right)\ln\left(x\right) + c_1x^3 + \left(6c_1+51c_2\right)x^2 + \left(3c_1+48c_2\right)x + c_2}{x^2}
$$

Mathematica DSolve solution

Solving time : 0.593 (sec)
Leaf size : 119

```
DSolve[{x^2*(1-x)*D[y[x],{x,2}]+x*(4+x)*D[y[x],x]+(2-x)*y[x]==0,{}},y[x],x,IncludeSingularSolu
```

$$
\begin{aligned}
y(x) \to \left(x^2+6x+3\right)\exp\Bigg(\int_1^x\left(\frac{1}{K[1]}+\frac{5}{2-2K[1]}\right)dK[1] \\
-\frac{1}{2}\int_1^x\frac{K[2]+4}{K[2]-K[2]^2}dK[2]\Bigg)\left(c_2\int_1^x\frac{\exp\left(-2\int_1^{K[3]}\left(\frac{1}{K[1]}+\frac{5}{2-2K[1]}\right)dK[1]\right)}{\left(K[3]^2+6K[3]+3\right)^2}dK[3]\right. \\
\left.+c_1\right)
\end{aligned}
$$

2.1.468 Problem 483

Internal problem ID [9640]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 483
Date solved : `Monday, January 27, 2025 at 06:04:54 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+x)\,y''+x(1+2x)\,y'-(4+6x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.220 (sec)

Writing the ode as

$$x^2(1+x)\,y''+\left(2x^2+x\right)y'+(-6x-4)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2(1+x)\\B&=2x^2+x\\C&=-6x-4\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{24x^2+40x+15}{4\left(x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=24x^2+40x+15\\t&=4\left(x^2+x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{24x^2+40x+15}{4\left(x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.889: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{2x} - \frac{1}{4\,(1+x)^2} - \frac{5}{2\,(1+x)} + \frac{15}{4x^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{24x^2 + 40x + 15}{4\left(x^2 + x\right)^2}$$

Since the gcd$(s, t) = 1$. This gives $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{24x^2 + 40x + 15}{4\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	3	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 3$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\
&= 3 - (3) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2 + 2x} + \frac{5}{2x} + (0) \\
&= \frac{1}{2 + 2x} + \frac{5}{2x} \\
&= \frac{6x + 5}{2x\,(1 + x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2+2x} + \frac{5}{2x}\right)(0) + \left(\left(-\frac{1}{2(1+x)^2} - \frac{5}{2x^2}\right) + \left(\frac{1}{2+2x} + \frac{5}{2x}\right)^2 - \left(\frac{24x^2+40x+15}{4(x^2+x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{1}{2+2x} + \frac{5}{2x}\right) dx} \\ &= x^{5/2}\sqrt{1+x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x^2+x}{x^2(1+x)} \, dx} \\ &= z_1 e^{-\frac{\ln(x(1+x))}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x(1+x)}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{5/2}\sqrt{1+x}}{\sqrt{x(1+x)}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2+x}{x^2(1+x)} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\ln(x(1+x))}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{1}{4x^4} - \frac{1}{2x^2} + \ln(x) + \frac{1}{3x^3} + \frac{1}{x} - \ln(1+x)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{x^{5/2}\sqrt{1+x}}{\sqrt{x(1+x)}}\right) + c_2 \left(\frac{x^{5/2}\sqrt{1+x}}{\sqrt{x(1+x)}}\left(-\frac{1}{4x^4} - \frac{1}{2x^2} + \ln(x) + \frac{1}{3x^3} + \frac{1}{x} - \ln(1+x)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(2x+1)\left(\frac{d}{dx}y(x)\right)-(4+6x)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{2(3x+2)y(x)}{(x+1)x^2}-\frac{(2x+1)\left(\frac{d}{dx}y(x)\right)}{x(x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{(2x+1)\left(\frac{d}{dx}y(x)\right)}{x(x+1)}-\frac{2(3x+2)y(x)}{(x+1)x^2}=0$

□ Check to see if x_0 is a regular singular point

- ∘ Define functions

  $\left[P_2(x)=\frac{2x+1}{x(x+1)},P_3(x)=-\frac{2(3x+2)}{(x+1)x^2}\right]$

- ∘ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=1$

- ∘ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

- ∘ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(2x+1)\left(\frac{d}{dx}y(x)\right)+(-6x-4)\,y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^3-2u^2+u)\left(\frac{d^2}{du^2}y(u)\right)+(2u^2-3u+1)\left(\frac{d}{du}y(u)\right)+(-6u+2)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ∘ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

- ∘ Shift index using $k->k-m$

  $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

- ∘ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ∘ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ∘ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- ∘ Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 u^{-1+r} + \left(a_1(1+r)^2 - a_0(2r^2+r-2)\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)^2 - a_k(2k^2+4kr+2r^2+k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 0$
- Each term must be 0
  $a_1(1+r)^2 - a_0(2r^2+r-2) = 0$
- Each term in the series must be 0, giving the recursion relation
  $(k^2-k-6)\, a_{k-1} + (-2k^2-k+2)\, a_k + a_{k+1}(k+1)^2 = 0$
- Shift index using $k->k+1$
  $\left((k+1)^2-k-7\right) a_k + \left(-2(k+1)^2-k+1\right) a_{k+1} + a_{k+2}(k+2)^2 = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + k a_k - 5k a_{k+1} - 6a_k - a_{k+1}}{(k+2)^2}$
- Recursion relation for $r = 0$
  $a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + k a_k - 5k a_{k+1} - 6a_k - a_{k+1}}{(k+2)^2}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + k a_k - 5k a_{k+1} - 6a_k - a_{k+1}}{(k+2)^2}, a_1 + 2a_0 = 0\right]$
- Revert the change of variables $u = x+1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x+1)^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + k a_k - 5k a_{k+1} - 6a_k - a_{k+1}}{(k+2)^2}, a_1 + 2a_0 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 46

```
dsolve(x^2*(x+1)*diff(diff(y(x),x),x)+x*(2*x+1)*diff(y(x),x)-(4+6*x)*y(x) = 0,y(x),sings
```

$$y = c_1 x^2 + \frac{c_2(12\ln(x)\, x^4 - 12\ln(x+1)\, x^4 + 12x^3 - 6x^2 + 4x - 3)}{x^2}$$

Mathematica DSolve solution

Solving time : 0.223 (sec)
Leaf size : 103

```
DSolve[{x^2*(1+x)*D[y[x],{x,2}]+x*(1+2*x)*D[y[x],x]-(4+6*x)*y[x]==0,{}},y[x],x,IncludeSingul
```

$$y(x) \to \exp\left(\int_1^x \frac{6K[1]+5}{2K[1]^2+2K[1]}dK[1] - \frac{1}{2}\int_1^x \left(\frac{1}{K[2]+1} + \frac{1}{K[2]}\right) dK[2]\right) \left(c_2 \int_1^x \exp\left(-2\int_1^{K[3]} \frac{6K[1]+5}{2K[1]^2+2K[1]}dK[1]\right) dK[3] + c_1\right)$$

2.1.469 Problem 484

Internal problem ID [9641]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 484
Date solved : `Monday, January 27, 2025 at 06:04:55 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(2x^2+1)\,y''+x(2x^2+4)\,y'+2(-x^2+1)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.412 (sec)

Writing the ode as

$$\left(2x^4+x^2\right)y''+\left(2x^3+4x\right)y'+\left(-2x^2+2\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^4+x^2 \\ B &= 2x^3+4x \\ C &= -2x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{3x^2-9}{\left(2x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3x^2-9 \\ t &= \left(2x^2+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{3x^2-9}{\left(2x^2+1\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.891: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = \left(2x^2+1\right)^2$. There is a pole at $x = \frac{i\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{i\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{21}{16\left(x-\frac{i\sqrt{2}}{2}\right)^2} + \frac{21}{16\left(x+\frac{i\sqrt{2}}{2}\right)^2} + \frac{15i\sqrt{2}}{16\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{15i\sqrt{2}}{16\left(x+\frac{i\sqrt{2}}{2}\right)}$$

For the pole at $x = \frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4} \end{aligned}$$

For the pole at $x = -\frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{3x^2-9}{(2x^2+1)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3x^2-9}{(2x^2+1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{i\sqrt{2}}{2}$	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
$-\frac{i\sqrt{2}}{2}$	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{3}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{4\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{3}{4\left(x+\frac{i\sqrt{2}}{2}\right)} + (-)\,(0) \\
&= -\frac{3}{4\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{3}{4\left(x+\frac{i\sqrt{2}}{2}\right)} \\
&= -\frac{3x}{2x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{4\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{3}{4\left(x+\frac{i\sqrt{2}}{2}\right)}\right)(1) + \left(\left(\frac{3}{4\left(x-\frac{i\sqrt{2}}{2}\right)^2} + \frac{3}{4\left(x+\frac{i\sqrt{2}}{2}\right)^2}\right) + \left(-\frac{3}{4\left(x-\frac{i\sqrt{2}}{2}\right)} - \right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int \left(-\frac{3}{4\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{3}{4\left(x+\frac{i\sqrt{2}}{2}\right)}\right) dx} \\
&= (x)\, \frac{1}{\left(\left(i\sqrt{2}-2x\right)\left(2x+i\sqrt{2}\right)\right)^{3/4}} \\
&= \frac{x}{\left(-4x^2-2\right)^{3/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^3+4x}{2x^4+x^2}\, dx} \\
&= z_1 e^{-2\ln(x) + \frac{3\ln\left(2x^2+1\right)}{4}} \\
&= z_1 \left(\frac{\left(2x^2+1\right)^{3/4}}{x^2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{2^{1/4}\left(4x^2+2\right)^{3/4}}{2x\left(-4x^2-2\right)^{3/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3+4x}{2x^4+x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-4\ln(x)+\frac{3\ln(2x^2+1)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{2i(2x^4-x^2-1)\sqrt{2x^2+1}\,\sqrt{2}\,\sqrt{\frac{(-4x^2-2)(4x^2+2)}{(2x^2+1)^2}}}{x\sqrt{-4x^2-2}\,\sqrt{4x^2+2}} - \frac{6i\,\mathrm{arcsinh}\left(\sqrt{2}\,x\right)\sqrt{\frac{(-4x^2-2)(4x^2+2)}{(2x^2+1)^2}}\,(2x^2+1)}{\sqrt{-4x^2-2}\,\sqrt{4x^2+2}} \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{2^{1/4}(4x^2+2)^{3/4}}{2x\,(-4x^2-2)^{3/4}} \right) + c_2 \left(\frac{2^{1/4}(4x^2+2)^{3/4}}{2x\,(-4x^2-2)^{3/4}} \left(-\frac{2i(2x^4-x^2-1)\sqrt{2x^2+1}\,\sqrt{2}\,\sqrt{\frac{(-4x^2-2)(4x^2+2)}{(2x^2+1)^2}}}{x\sqrt{-4x^2-2}\,\sqrt{4x^2+2}} - 6i\,\mathrm{ar}\ldots \right.\right.
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(2x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + x(2x^2+4)\left(\frac{d}{dx}y(x)\right) + 2(-x^2+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{2(x^2-1)y(x)}{x^2(2x^2+1)} - \frac{2(x^2+2)\left(\frac{d}{dx}y(x)\right)}{x(2x^2+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{2(x^2+2)\left(\frac{d}{dx}y(x)\right)}{x(2x^2+1)} - \frac{2(x^2-1)y(x)}{x^2(2x^2+1)} = 0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2(x^2+2)}{x(2x^2+1)},\ P_3(x) = -\frac{2(x^2-1)}{x^2(2x^2+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 4$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$

  ◦ $x = 0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

$x^2(2x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)+2x(x^2+2)\left(\frac{d}{dx}y(x)\right)+(-2x^2+2)\,y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x)=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k-m$

    $x^m \cdot y(x)=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - Shift index using $k- >k+2-m$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)\,(1+r)\,x^r+a_1(3+r)\,(2+r)\,x^{1+r}+\left(\sum_{k=2}^{\infty} (a_k(k+r+2)\,(k+r+1)+2a_{k-2}(k+r-1$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(2+r)\,(1+r)=0$

- Values of r that satisfy the indicial equation

$r \in \{-2,-1\}$

- Each term must be 0

$a_1(3+r)\,(2+r)=0$

- Solve for the dependent coefficient(s)

$a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

$a_k(k+r+2)\,(k+r+1)+2a_{k-2}(k+r-1)\,(k-3+r)=0$

- Shift index using $k- >k+2$

$a_{k+2}(k+4+r)\,(k+3+r)+2a_k(k+r+1)\,(k+r-1)=0$

- Recursion relation that defines series solution to ODE

$a_{k+2}=-\frac{2a_k(k+r+1)(k+r-1)}{(k+4+r)(k+3+r)}$

- Recursion relation for $r = -2$

$a_{k+2}=-\frac{2a_k(k-1)(k-3)}{(k+2)(k+1)}$

- Solution for $r = -2$

$\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2}=-\frac{2a_k(k-1)(k-3)}{(k+2)(k+1)}, a_1=0\right]$

- Recursion relation for $r = -1$; series terminates at $k = 2$

$a_{k+2} = -\frac{2a_k k(k-2)}{(k+3)(k+2)}$

- Solution for $r = -1$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{2a_k k(k-2)}{(k+3)(k+2)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), a_{k+2} = -\frac{2a_k(k-1)(k-3)}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{2b_k k(k-2)}{(k+3)(k+2)}, b_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 42

```
dsolve(x^2*(2*x^2+1)*diff(diff(y(x),x),x)+x*(2*x^2+4)*diff(y(x),x)+2*(-x^2+1)*y(x) = 0,y
```

$$y = \frac{c_2(x-1)(x+1)\sqrt{2}\sqrt{2x^2+1} + x\left(3c_2 \operatorname{arcsinh}\left(\sqrt{2}\,x\right) + c_1\right)}{x^2}$$

Mathematica DSolve solution

Solving time : 0.683 (sec)
Leaf size : 73

```
DSolve[{x^2*(1+2*x^2)*D[y[x],{x,2}]+x*(4+2*x^2)*D[y[x],x]+2*(1-x^2)*y[x]==0,{}},y[x],x,IncludeS
```

$$y(x) \to \frac{\left(c_1 x - c_2 \text{Hypergeometric2F1}\left(-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, -2x^2\right)\right) \exp\left(-\frac{1}{2}\int_1^x \frac{2(K[1]^2+2)}{2K[1]^3+K[1]}dK[1]\right)}{(2x^2+1)^{3/4}}$$

2.1.470 Problem 485

Internal problem ID [9642]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 485
Date solved : Monday, January 27, 2025 at 06:04:55 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$x^2\left(x^2+2\right)y''+2x\left(x^2+5\right)y'+2\left(-x^2+3\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.464 (sec)

Writing the ode as

$$\left(x^4+2x^2\right)y''+\left(2x^3+10x\right)y'+\left(-2x^2+6\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=x^4+2x^2\\ B&=2x^3+10x\\ C&=-2x^2+6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{2x^4-5x^2+3}{\left(x^3+2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=2x^4-5x^2+3\\ t&=\left(x^3+2x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{2x^4-5x^2+3}{\left(x^3+2x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.893: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^3 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i\sqrt{2}$ of order 2. There is a pole at $x = -i\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2} + \frac{21}{16\left(x - i\sqrt{2}\right)^2} + \frac{21}{16\left(x + i\sqrt{2}\right)^2} + \frac{11i\sqrt{2}}{32\left(x - i\sqrt{2}\right)} - \frac{11i\sqrt{2}}{32\left(x + i\sqrt{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x-i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}
$$

For the pole at $x = -i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x+i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{2x^4 - 5x^2 + 3}{(x^3 + 2x)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{2x^4 - 5x^2 + 3}{(x^3 + 2x)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$i\sqrt{2}$	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
$-i\sqrt{2}$	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= 2 - (0) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x - c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{3}{2x} - \frac{3}{4\left(x - i\sqrt{2}\right)} - \frac{3}{4\left(x + i\sqrt{2}\right)} + (0) \\
&= \frac{3}{2x} - \frac{3}{4\left(x - i\sqrt{2}\right)} - \frac{3}{4\left(x + i\sqrt{2}\right)} \\
&= \frac{3}{x^3 + 2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{3}{2x} - \frac{3}{4\left(x - i\sqrt{2}\right)} - \frac{3}{4\left(x + i\sqrt{2}\right)}\right)(2x + a_1) + \left(\left(-\frac{3}{2x^2} + \frac{3}{4\left(x - i\sqrt{2}\right)^2} + \frac{3}{4\left(x + i\sqrt{2}\right)^2}\right) + \left(\frac{3}{2 \cdots}\right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 8, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + 8$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 + 8\right) e^{\int \left(\frac{3}{2x} - \frac{3}{4\left(x - i\sqrt{2}\right)} - \frac{3}{4\left(x + i\sqrt{2}\right)}\right) dx} \\
&= \left(x^2 + 8\right) e^{-\frac{3\ln\left(x^2+2\right)}{4} + \frac{3\ln(x)}{2}} \\
&= \frac{\left(x^2 + 8\right) x^{3/2}}{\left(x^2 + 2\right)^{3/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{2x^3 + 10x}{x^4 + 2x^2} \, dx} \\
&= z_1 e^{\frac{3\ln\left(x^2+2\right)}{4} - \frac{5\ln(x)}{2}} \\
&= z_1 \left(\frac{\left(x^2 + 2\right)^{3/4}}{x^{5/2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2+8}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3+10x}{x^4+2x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{3\ln(x^2+2)}{2}-5\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{(x^2+2)^{5/2}}{256x^2} + \frac{(x^2+2)^{3/2}}{384} + \frac{\sqrt{x^2+2}}{96} - \frac{\sqrt{2}\,\mathrm{arctanh}\left(\frac{\sqrt{2}}{\sqrt{x^2+2}}\right)}{64} + \frac{3\sqrt{x^2+2}}{64(x^2+8)} + \frac{x^2\sqrt{x^2+2}}{768}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^2+8}{x}\right) + c_2\left(\frac{x^2+8}{x}\left(-\frac{(x^2+2)^{5/2}}{256x^2} + \frac{(x^2+2)^{3/2}}{384} + \frac{\sqrt{x^2+2}}{96} - \frac{\sqrt{2}\,\mathrm{arctanh}\left(\frac{\sqrt{2}}{\sqrt{x^2+2}}\right)}{64} + \frac{3\sqrt{x^2+2}}{64(x^2+8)} + \frac{x^2\sqrt{x^2+2}}{768}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x^2+2)\left(\frac{d^2}{dx^2}y(x)\right) + 2x(x^2+5)\left(\frac{d}{dx}y(x)\right) + 2(-x^2+3)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{2(x^2-3)y(x)}{(x^2+2)x^2} - \frac{2(x^2+5)\left(\frac{d}{dx}y(x)\right)}{x(x^2+2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{2(x^2+5)\left(\frac{d}{dx}y(x)\right)}{x(x^2+2)} - \frac{2(x^2-3)y(x)}{(x^2+2)x^2} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{2(x^2+5)}{x(x^2+2)}, P_3(x) = -\frac{2(x^2-3)}{(x^2+2)x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x=0$

$\left(x \cdot P_2(x)\right)\Big|_{x=0} = 5$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 3$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2+2)\left(\frac{d^2}{dx^2}y(x)\right) + 2x(x^2+5)\left(\frac{d}{dx}y(x)\right) + (-2x^2+6)\, y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k-m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$

  - Shift index using $k- >k+2-m$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $2a_0(3+r)(1+r)\, x^r + 2a_1(4+r)(2+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (2a_k(k+r+3)(k+r+1) + a_{k-2}(k+r)(k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $2(3+r)(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-3, -1\}$

- Each term must be 0

  $2a_1(4+r)(2+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $2a_k(k+r+3)(k+r+1) + a_{k-2}(k+r)(k-3+r) = 0$

- Shift index using $k- >k+2$

  $2a_{k+2}(k+5+r)(k+r+3) + a_k(k+r+2)(k+r-1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r+2)(k+r-1)}{2(k+5+r)(k+r+3)}$

- Recursion relation for $r = -3$; series terminates at $k = 4$

  $a_{k+2} = -\frac{a_k(k-1)(k-4)}{2(k+2)k}$
- Solution for $r = -3$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2} = -\frac{a_k(k-1)(k-4)}{2(k+2)k}, a_1 = 0\right]$
- Recursion relation for $r = -1$; series terminates at $k = 2$

  $a_{k+2} = -\frac{a_k(k+1)(k-2)}{2(k+4)(k+2)}$
- Solution for $r = -1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k(k+1)(k-2)}{2(k+4)(k+2)}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-3}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), a_{k+2} = -\frac{a_k(k-1)(-4+k)}{2(k+2)k}, a_1 = 0, b_{k+2} = -\frac{b_k(k+1)(k-2)}{2(4+k)(k+2)}, b_1 =\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 53

```
dsolve(x^2*(x^2+2)*diff(diff(y(x),x),x)+2*x*(x^2+5)*diff(y(x),x)+2*(-x^2+3)*y(x) = 0,y
```

$$y = \frac{-c_2\sqrt{x^2+2}\,(x+2)\,(x-2)\,\sqrt{2} + x^2\left(\operatorname{arctanh}\left(\frac{\sqrt{2}}{\sqrt{x^2+2}}\right) c_2 + c_1\right)(x^2+8)}{x^3}$$

Mathematica DSolve solution

Solving time : `0.454` (sec)
Leaf size : 112

```
DSolve[{x^2*(2+x^2)*D[y[x],{x,2}]+2*x*(x^2+5)*D[y[x],x]+2*(3-x^2)*y[x]==0,{}},y[x],x,IncludeSi
```

$$y(x) \to (x^2+8)\exp\left(\int_1^x \frac{3}{K[1]^3+2K[1]}dK[1] - \frac{1}{2}\int_1^x \frac{2K[2]^2+10}{K[2]^3+2K[2]}dK[2]\right)\left(c_2\int_1^x \frac{\exp\left(-2\int_1^{K[3]}\frac{3}{K[1]^3+2K[1]}dK[1]\right)}{(K[3]^2+8)^2}dK[3] + c_1\right)$$

2.1.471 Problem 486

Internal problem ID [9643]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 486
Date solved : `Monday, January 27, 2025 at 06:04:56 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^{2}+1\right) y''+6 x y'+6 y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.267 (sec)

Writing the ode as

$$\left(x^{2}+1\right) y''+6 x y'+6 y = 0 \tag{1}$$
$$A y''+B y'+C y = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^{2}+1 \\ B &= 6 x \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2 A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2 A B' - 2 B A' + B^2 - 4 A C}{4 A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{\left(x^{2}+1\right)^{2}} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= \left(x^{2}+1\right)^{2} \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{3}{\left(x^{2}+1\right)^{2}}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.895: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{3i}{4(x-i)} - \frac{3i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= 0\\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right)\\ &= 1 - (1)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (-)[\sqrt{r}]_\infty\\ &= -\frac{1}{2\,(x-i)} + \frac{3}{2\,(x+i)} + (-)\,(0)\\ &= -\frac{1}{2\,(x-i)} + \frac{3}{2\,(x+i)}\\ &= \frac{x-2i}{x^2+1}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2\left(x-i\right)}+\frac{3}{2\left(x+i\right)}\right)(0)+\left(\left(\frac{1}{2\left(x-i\right)^{2}}-\frac{3}{2\left(x+i\right)^{2}}\right)+\left(-\frac{1}{2\left(x-i\right)}+\frac{3}{2\left(x+i\right)}\right)^{2}-\left(-\frac{}{(x^2}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\,dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x-i)}+\frac{3}{2(x+i)}\right)dx} \\
&= \frac{\left(x^2+1\right)^{3/2}}{\left(ix+1\right)^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{6x}{x^2+1}\,dx} \\
&= z_1 e^{-\frac{3\ln\left(x^2+1\right)}{2}} \\
&= z_1\left(\frac{1}{\left(x^2+1\right)^{3/2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{\left(ix+1\right)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{6x}{x^2+1}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-3\ln\left(x^2+1\right)}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(-\frac{x}{\left(x+i\right)^2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{\left(ix+1\right)^2}\right) + c_2\left(\frac{1}{\left(ix+1\right)^2}\left(-\frac{x}{\left(x+i\right)^2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 24

```
dsolve((x^2+1)*diff(diff(y(x),x),x)+6*diff(y(x),x)*x+6*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_2 x^2 + c_1 x - c_2}{\left(x^2+1\right)^2}$$

Mathematica DSolve solution

Solving time : 0.321 (sec)
Leaf size : 79

```
DSolve[{(1+x^2)*D[y[x],{x,2}]+6*x*D[y[x],x]+6*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to \frac{\exp\left(\int_1^x \frac{K[1]+2i}{K[1]^2+1} dK[1]\right)\left(c_2 \int_1^x \exp\left(-2\int_1^{K[2]} \frac{K[1]+2i}{K[1]^2+1} dK[1]\right) dK[2] + c_1\right)}{\left(x^2+1\right)^{3/2}}$$

2.1.472 Problem 487

Internal problem ID [9644]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 487
Date solved : `Monday, January 27, 2025 at 06:04:57 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(x^2+1)\,y''+2xy'-2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.270 (sec)

Writing the ode as

$$(x^2+1)\,y''+2xy'-2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= 2x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{2x^2+3}{(x^2+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2x^2+3 \\ t &= (x^2+1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{2x^2+3}{(x^2+1)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.896: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 + 1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(x-i)^2} - \frac{1}{4(x+i)^2} - \frac{5i}{4(x-i)} + \frac{5i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^2+3}{(x^2+1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2x^2+3}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
i	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-i$	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 2$ then

$$\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-}\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x-2i} + \frac{1}{2x+2i} + (0) \\
&= \frac{1}{2x-2i} + \frac{1}{2x+2i} \\
&= \frac{x}{x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x-2i} + \frac{1}{2x+2i}\right)(1) + \left(\left(-\frac{1}{2\left(x-i\right)^2} - \frac{1}{2\left(x+i\right)^2}\right) + \left(\frac{1}{2x-2i} + \frac{1}{2x+2i}\right)^2 - \left(\frac{2x^2+3}{\left(x^2+1\right)^2}\right.\right.$$

$$-\frac{2(x^2+1)\,a_0}{\left(-x+i\right)^2\left(x+i\right)^2}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (x)\, e^{\int \left(\frac{1}{2x-2i} + \frac{1}{2x+2i}\right) dx} \\ &= (x)\sqrt{\left(-x+i\right)\left(x+i\right)} \\ &= x\sqrt{-x^2-1} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x}{x^2+1}\, dx} \\ &= z_1 e^{-\frac{\ln\left(x^2+1\right)}{2}} \\ &= z_1\left(\frac{1}{\sqrt{x^2+1}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = ix$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{x^2+1}\, dx}}{\left(y_1\right)^2}\, dx \\ &= y_1 \int \frac{e^{-\ln\left(x^2+1\right)}}{\left(y_1\right)^2}\, dx \\ &= y_1\left(\arctan\left(x\right) + \frac{1}{x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(ix) + c_2\left(ix\left(\arctan\left(x\right) + \frac{1}{x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 14

```
dsolve((x^2+1)*diff(diff(y(x),x),x)+2*diff(y(x),x)*x-2*y(x) = 0,y(x),singsol=all)
```

$$y = c_1 x + \arctan(x)\, x c_2 + c_2$$

Mathematica DSolve solution

Solving time : 0.022 (sec)
Leaf size : 48

```
DSolve[{(1+x^2)*D[y[x],{x,2}]+2*x*D[y[x],x]-2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tru
```

$$y(x) \to \frac{1}{2} i (2 c_1 x - c_2 x \log(1 - i x) + c_2 x \log(1 + i x) + 2 i c_2)$$

2.1.473 Problem 488

Internal problem ID [9645]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 488
Date solved : `Monday, January 27, 2025 at 06:04:58 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+1\right) y''-8xy'+20y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.263 (sec)

Writing the ode as

$$\left(x^2+1\right) y''-8xy'+20y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= -8x \\ C &= 20 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-24}{\left(x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -24 \\ t &= \left(x^2+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{24}{\left(x^2+1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.897: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{(x-i)^2} + \frac{6}{(x+i)^2} + \frac{6i}{x-i} - \frac{6i}{x+i}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{24}{(x^2+1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
i	2	0	3	-2
$-i$	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{2}{x-i} + \frac{3}{x+i} + (-)\,(0) \\
&= -\frac{2}{x-i} + \frac{3}{x+i} \\
&= \frac{x-5i}{x^2+1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{2}{x-i} + \frac{3}{x+i}\right)(0) + \left(\left(\frac{2}{(x-i)^2} - \frac{3}{(x+i)^2}\right) + \left(-\frac{2}{x-i} + \frac{3}{x+i}\right)^2 - \left(-\frac{24}{(x^2+1)^2}\right)\right) = 0
$$

$$
0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{2}{x-i}+\frac{3}{x+i}\right) dx} \\ &= \frac{\left(x^2+1\right)^3}{\left(ix+1\right)^5} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-8x}{x^2+1}\, dx} \\ &= z_1 e^{2\ln\left(x^2+1\right)} \\ &= z_1 \left(\left(x^2+1\right)^2\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(x^2+1\right)^5}{\left(ix+1\right)^5}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-8x}{x^2+1}\, dx}}{\left(y_1\right)^2}\, dx \\ &= y_1 \int \frac{e^{4\ln\left(x^2+1\right)}}{\left(y_1\right)^2}\, dx \\ &= y_1 \left(\frac{x^4 - 2x^2 + \frac{1}{5}}{\left(x+i\right)^5}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\left(x^2+1\right)^5}{\left(ix+1\right)^5}\right) + c_2 \left(\frac{\left(x^2+1\right)^5}{\left(ix+1\right)^5} \left(\frac{x^4 - 2x^2 + \frac{1}{5}}{\left(x+i\right)^5}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 33

```
dsolve((x^2+1)*diff(diff(y(x),x),x)-8*diff(y(x),x)*x+20*y(x) = 0,y(x),singsol=all)
```

$$y = c_2 x^5 + 5c_1 x^4 - 10c_2 x^3 - 10c_1 x^2 + 5c_2 x + c_1$$

Mathematica DSolve solution

Solving time : 0.274 (sec)
Leaf size : 77

```
DSolve[{(1+x^2)*D[y[x],{x,2}]-8*x*D[y[x],x]+20*y[x]==0,{}},y[x],x,IncludeSingularSolutions->
```

$$y(x) \to (x^2 + 1)^2 \exp\left(\int_1^x \frac{K[1]+5i}{K[1]^2+1} dK[1]\right) \left(c_2 \int_1^x \exp\left(-2\int_1^{K[2]} \frac{K[1]+5i}{K[1]^2+1} dK[1]\right) dK[2] + c_1\right)$$

2.1.474 Problem 489

Internal problem ID [9646]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 489
Date solved : `Monday, January 27, 2025 at 06:04:58 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-x^2+1\right) y''-8 x y'-12 y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.181 (sec)

Writing the ode as

$$\left(-x^2+1\right) y''-8 x y'-12 y=0 \tag{1}$$
$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+1 \\ B &= -8 x \\ C &= -12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2 A} d x}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2 A B'-2 B A'+B^2-4 A C}{4 A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{8}{\left(x^2-1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 8 \\ t &= \left(x^2-1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{8}{\left(x^2-1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.898: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x+1} + \frac{2}{(x-1)^2} + \frac{2}{(x+1)^2} - \frac{2}{x-1}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{8}{(x^2-1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	2	-1
-1	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x-1} + \frac{2}{x+1} + (-)\,(0) \\
&= -\frac{1}{x-1} + \frac{2}{x+1} \\
&= \frac{x-3}{x^2-1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x-1} + \frac{2}{x+1}\right)(0) + \left(\left(\frac{1}{(x-1)^2} - \frac{2}{(x+1)^2}\right) + \left(-\frac{1}{x-1} + \frac{2}{x+1}\right)^2 - \left(\frac{8}{(x^2-1)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{x-1}+\frac{2}{x+1}\right) dx} \\
&= \frac{(x+1)^2}{x-1}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-8x}{-x^2+1}\, dx} \\
&= z_1 e^{-2\ln(x-1)-2\ln(x+1)} \\
&= z_1 \left(\frac{1}{(x-1)^2 (x+1)^2}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{(x-1)^3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-8x}{-x^2+1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-4\ln(x-1)-4\ln(x+1)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{(x+1)(3x^2+1)(x-1)^4 \,\mathrm{e}^{-4\ln(x-1)-4\ln(x+1)}}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{(x-1)^3}\right) + c_2 \left(\frac{1}{(x-1)^3}\left(-\frac{(x+1)(3x^2+1)(x-1)^4 \,\mathrm{e}^{-4\ln(x-1)-4\ln(x+1)}}{3}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(-x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)-8x\left(\frac{d}{dx}y(x)\right)-12y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{12y(x)}{x^2-1}-\frac{8\left(\frac{d}{dx}y(x)\right)x}{x^2-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x)+\frac{8\left(\frac{d}{dx}y(x)\right)x}{x^2-1}+\frac{12y(x)}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{8x}{x^2-1},P_3(x)=\frac{12}{x^2-1}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=4$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1$

- Multiply by denominators

$(x^2-1)\left(\frac{d^2}{dx^2}y(x)\right)+8x\left(\frac{d}{dx}y(x)\right)+12y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(8u-8)\left(\frac{d}{du}y(u)\right)+12y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  ◦ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

  ◦ Shift index using $k->k+2-m$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

Rewrite ODE with series expansions

$-2a_0r(3+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)(k+r+4)+a_k(k+r+4)(k+r+3)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-2r(3+r)=0$

- Values of r that satisfy the indicial equation
  $r\in\{-3,0\}$
- Each term in the series must be 0, giving the recursion relation
  $(k+r+4)\left((-2k-2r-2)\,a_{k+1}+a_k(k+r+3)\right)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{a_k(k+r+3)}{2(k+1+r)}$
- Recursion relation for $r=-3$
  $a_{k+1}=\frac{a_k k}{2(k-2)}$
- Series not valid for $r=-3$, division by 0 in the recursion relation at $k=2$
  $a_{k+1}=\frac{a_k k}{2(k-2)}$
- Recursion relation for $r=0$
  $a_{k+1}=\frac{a_k(k+3)}{2(k+1)}$
- Solution for $r=0$
  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^k, a_{k+1}=\frac{a_k(k+3)}{2(k+1)}\right]$
- Revert the change of variables $u=x+1$
  $\left[y(x)=\sum_{k=0}^{\infty}a_k(x+1)^k, a_{k+1}=\frac{a_k(k+3)}{2(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 29

```
dsolve((-x^2+1)*diff(diff(y(x),x),x)-8*diff(y(x),x)*x-12*y(x) = 0,y(x),singsol=all)
```

$$y=\frac{c_2x^3+3c_1x^2+3c_2x+c_1}{\left(x^2-1\right)^3}$$

Mathematica DSolve solution

Solving time : `0.314` (sec)
Leaf size : 73

```
DSolve[{(1-x^2)*D[y[x],{x,2}]-8*x*D[y[x],x]-12*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tru
```

$$y(x) \to \frac{\exp\left(\int_1^x \frac{K[1]+3}{K[1]^2-1} dK[1]\right)\left(c_2 \int_1^x \exp\left(-2\int_1^{K[2]} \frac{K[1]+3}{K[1]^2-1} dK[1]\right) dK[2] + c_1\right)}{(x^2-1)^2}$$

2.1.475 Problem 490

Internal problem ID [9647]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 490
Date solved : Monday, January 27, 2025 at 06:04:59 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$\left(2x^2+1\right)y''+7xy'+2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.334 (sec)

Writing the ode as

$$\left(2x^2+1\right)y''+7xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2+1 \\ B &= 7x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{5x^2+6}{4\left(2x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 5x^2+6 \\ t &= 4\left(2x^2+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{5x^2+6}{4\left(2x^2+1\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.900: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 + 1)^2$. There is a pole at $x = \frac{i\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{i\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{64\left(x - \frac{i\sqrt{2}}{2}\right)^2} - \frac{7}{64\left(x + \frac{i\sqrt{2}}{2}\right)^2} - \frac{17i\sqrt{2}}{64\left(x - \frac{i\sqrt{2}}{2}\right)} + \frac{17i\sqrt{2}}{64\left(x + \frac{i\sqrt{2}}{2}\right)}$$

For the pole at $x = \frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{8} \end{aligned}$$

For the pole at $x=-\frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{7}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{7}{8}\\
\alpha_c^- &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{8}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty=0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r=\frac{s}{t}=\frac{5x^2+6}{4\left(2x^2+1\right)^2}
$$

Since the $\gcd(s,t)=1$. This gives $b=\frac{5}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{4}\\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{5x^2+6}{4\left(2x^2+1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{i\sqrt{2}}{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$
$-\frac{i\sqrt{2}}{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=\frac{5}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+-\left(\alpha_{c_1}^-+\alpha_{c_2}^-\right)\\
&= \frac{5}{4}-\left(\frac{1}{4}\right)\\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4i\sqrt{2}} + (0) \\
&= \frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4i\sqrt{2}} \\
&= \frac{x}{4x^2 + 2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4i\sqrt{2}}\right)(1) + \left(\left(-\frac{1}{8\left(x - \frac{i\sqrt{2}}{2}\right)^2} - \frac{1}{8\left(x + \frac{i\sqrt{2}}{2}\right)^2}\right) + \left(\frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4i\sqrt{2}}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, e^{\int \left(\frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4i\sqrt{2}}\right) dx} \\
&= (x)\left(\left(i\sqrt{2} - 2x\right)\left(2x + i\sqrt{2}\right)\right)^{1/8} \\
&= x\left(-4x^2 - 2\right)^{1/8}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7x}{2x^2+1} \, dx} \\
&= z_1 e^{-\frac{7\ln\left(2x^2+1\right)}{8}} \\
&= z_1 \left(\frac{1}{\left(2x^2 + 1\right)^{7/8}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{2^{7/8} x\left(-4x^2 - 2\right)^{1/8}}{\left(4x^2 + 2\right)^{7/8}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{7x}{2x^2+1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{7\ln(2x^2+1)}{4}}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{2^{1/4}(4x^2+2)^{7/4}}{4\,(2x^2+1)^{7/4}\, x^2\, (-4x^2-2)^{1/4}} dx \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{2^{7/8} x(-4x^2-2)^{1/8}}{(4x^2+2)^{7/8}} \right) + c_2 \left(\frac{2^{7/8} x(-4x^2-2)^{1/8}}{(4x^2+2)^{7/8}} \left(\int \frac{2^{1/4}(4x^2+2)^{7/4}}{4\,(2x^2+1)^{7/4}\, x^2\, (-4x^2-2)^{1/4}} dx \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Legendre successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special functio
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.039` (sec)
Leaf size : 37

```
dsolve((2*x^2+1)*diff(diff(y(x),x),x)+7*diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_1 \operatorname{LegendreP}\left(\frac{1}{4}, \frac{3}{4}, i\sqrt{2}\, x\right) + c_2 \operatorname{LegendreQ}\left(\frac{1}{4}, \frac{3}{4}, i\sqrt{2}\, x\right)}{\left(2x^2+1\right)^{3/8}}$$

Mathematica DSolve solution

Solving time : `0.061` (sec)
Leaf size : 66

```
DSolve[{(1+2*x^2)*D[y[x],{x,2}]+7*x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to \frac{c_2 Q_{\frac{1}{4}}^{\frac{3}{4}}\left(i\sqrt{2}x\right)}{\left(2x^2+1\right)^{3/8}} + \frac{2i\sqrt[4]{2}c_1 x}{\left(2x^2+1\right)^{3/4} \operatorname{Gamma}\left(\frac{1}{4}\right)}$$

2.1.476 Problem 491

Internal problem ID [9648]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 491
Date solved : `Monday, January 27, 2025 at 06:05:00 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-x^2+1\right) y''-5xy'-4y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.230 (sec)

Writing the ode as

$$\left(-x^2+1\right) y''-5xy'-4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+1 \\ B &= -5x \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{-x^2+6}{4\left(x^2-1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2+6 \\ t &= 4\left(x^2-1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-x^2+6}{4\left(x^2-1\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.901: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{16 (x - 1)} + \frac{7}{16 (x + 1)} + \frac{5}{16 (x + 1)^2} + \frac{5}{16 (x - 1)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2+6}{4(x^2-1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2+6}{4(x^2-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
-1	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{4(x-1)} - \frac{1}{4(x+1)} + (-)(0) \\
&= -\frac{1}{4(x-1)} - \frac{1}{4(x+1)} \\
&= -\frac{x}{2x^2-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{4(x-1)} - \frac{1}{4(x+1)}\right)(1) + \left(\left(\frac{1}{4(x-1)^2} + \frac{1}{4(x+1)^2}\right) + \left(-\frac{1}{4(x-1)} - \frac{1}{4(x+1)}\right)^2 - \left(\frac{-\cdots}{4(x\cdots}\right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (x)\, e^{\int \left(-\frac{1}{4(x-1)} - \frac{1}{4(x+1)}\right) dx} \\ &= (x) \frac{1}{((x-1)(x+1))^{1/4}} \\ &= \frac{x}{(x^2-1)^{1/4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-5x}{-x^2+1}\, dx} \\ &= z_1 e^{-\frac{5\ln(x-1)}{4} - \frac{5\ln(x+1)}{4}} \\ &= z_1 \left(\frac{1}{(x-1)^{5/4}(x+1)^{5/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x}{(x-1)^{5/4}(x+1)^{5/4}(x^2-1)^{1/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-5x}{-x^2+1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{5\ln(x-1)}{2} - \frac{5\ln(x+1)}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{(x^2-1)^{3/2}}{x} - x\sqrt{x^2-1} + \ln\left(x + \sqrt{x^2-1}\right)\right) \end{aligned}$$

Therefore the solution is

$$y = c_1y_1 + c_2y_2$$
$$= c_1\left(\frac{x}{(x-1)^{5/4}(x+1)^{5/4}(x^2-1)^{1/4}}\right) + c_2\left(\frac{x}{(x-1)^{5/4}(x+1)^{5/4}(x^2-1)^{1/4}}\left(\frac{(x^2-1)^{3/2}}{x} - x\sqrt{x^2-1}+\right.\right.$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$(-x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) - 5x\left(\frac{d}{dx}y(x)\right) - 4y(x) = 0$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
$\frac{d^2}{dx^2}y(x) = -\frac{4y(x)}{x^2-1} - \frac{5\left(\frac{d}{dx}y(x)\right)x}{x^2-1}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
$\frac{d^2}{dx^2}y(x) + \frac{5\left(\frac{d}{dx}y(x)\right)x}{x^2-1} + \frac{4y(x)}{x^2-1} = 0$

☐ Check to see if x_0 is a regular singular point
  - Define functions
  $\left[P_2(x) = \frac{5x}{x^2-1}, P_3(x) = \frac{4}{x^2-1}\right]$
  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$
  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1} = \frac{5}{2}$
  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$
  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$
  - $x=-1$is a regular singular point
  Check to see if x_0 is a regular singular point
  $x_0 = -1$

- Multiply by denominators
$(x^2-1)\left(\frac{d^2}{dx^2}y(x)\right) + 5x\left(\frac{d}{dx}y(x)\right) + 4y(x) = 0$
- Change variables using $x = u-1$ so that the regular singular point is at $u=0$
$(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (5u-5)\left(\frac{d}{du}y(u)\right) + 4y(u) = 0$
- Assume series solution for $y(u)$
$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$
  $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$
  $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-a_0 r(3+2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)(2k+5+2r) + a_k(k+r+2)^2\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-r(3+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{0, -\frac{3}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $a_k(k+r+2)^2 - 2(k+1+r)\left(k+r+\frac{5}{2}\right)a_{k+1} = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_k(k+r+2)^2}{(k+1+r)(2k+5+2r)}$
- Recursion relation for $r = 0$
  $a_{k+1} = \frac{a_k(k+2)^2}{(k+1)(2k+5)}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(k+2)^2}{(k+1)(2k+5)}\right]$
- Revert the change of variables $u = x+1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+1} = \frac{a_k(k+2)^2}{(k+1)(2k+5)}\right]$
- Recursion relation for $r = -\frac{3}{2}$
  $a_{k+1} = \frac{a_k\left(k+\frac{1}{2}\right)^2}{\left(k-\frac{1}{2}\right)(2k+2)}$
- Solution for $r = -\frac{3}{2}$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{3}{2}}, a_{k+1} = \frac{a_k\left(k+\frac{1}{2}\right)^2}{\left(k-\frac{1}{2}\right)(2k+2)}\right]$
- Revert the change of variables $u = x+1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^{k-\frac{3}{2}}, a_{k+1} = \frac{a_k\left(k+\frac{1}{2}\right)^2}{\left(k-\frac{1}{2}\right)(2k+2)}\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k(x+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k(x+1)^{k-\frac{3}{2}}\right), a_{k+1} = \frac{a_k(k+2)^2}{(k+1)(2k+5)}, b_{k+1} = \frac{b_k\left(k+\frac{1}{2}\right)^2}{\left(k-\frac{1}{2}\right)(2k+2)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
```

```
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.033 (sec)
Leaf size : 39

```
dsolve((-x^2+1)*diff(diff(y(x),x),x)-5*diff(y(x),x)*x-4*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{\ln\left(x+\sqrt{x^2-1}\right) c_2 x + c_1 x - \sqrt{x^2-1}\, c_2}{\left(x^2-1\right)^{3/2}}$$

Mathematica DSolve solution

Solving time : 0.091 (sec)
Leaf size : 47

```
DSolve[{(1-x^2)*D[y[x],{x,2}]-5*x*D[y[x],x]-4*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to \frac{c_2 x \arcsin(x)}{(1-x^2)^{3/2}} + \frac{c_1 x}{(x^2-1)^{3/2}} - \frac{c_2}{x^2-1}$$

2.1.477 Problem 492

Internal problem ID [9649]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 492
Date solved : Monday, January 27, 2025 at 06:05:00 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$\left(x^2+1\right)y''-10xy'+28y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.327 (sec)

Writing the ode as

$$\left(x^2+1\right)y''-10xy'+28y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= -10x \\ C &= 28 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{2x^2-33}{\left(x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2x^2-33 \\ t &= \left(x^2+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{2x^2-33}{\left(x^2+1\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.903: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{35}{4(x-i)^2} + \frac{35}{4(x+i)^2} + \frac{31i}{4(x-i)} - \frac{31i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^2 - 33}{(x^2+1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2x^2 - 33}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{7}{2}$	$-\frac{5}{2}$
$-i$	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{5}{2\,(x-i)} + \frac{7}{2\,(x+i)} + (0) \\
&= -\frac{5}{2\,(x-i)} + \frac{7}{2\,(x+i)} \\
&= \frac{x-6i}{x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{5}{2(x-i)} + \frac{7}{2(x+i)}\right)(1) + \left(\left(\frac{5}{2(x-i)^2} - \frac{7}{2(x+i)^2}\right) + \left(-\frac{5}{2(x-i)} + \frac{7}{2(x+i)}\right)^2 - \left(\frac{2x}{(x^2 \right.\right.$$
$$- \frac{2(x^2+1)(6i}{(-x+i)^2(x}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -6i\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - 6i$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x - 6i)\, e^{\int \left(-\frac{5}{2(x-i)} + \frac{7}{2(x+i)}\right) dx} \\
&= (x - 6i)\, e^{\frac{\ln(x^2+1)}{2} - 6i \arctan(x)} \\
&= \frac{(-x+6i)(x^2+1)^{7/2}}{(-x+i)^6}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-10x}{x^2+1}\, dx} \\
&= z_1 e^{\frac{5\ln(x^2+1)}{2}} \\
&= z_1\left((x^2+1)^{5/2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2+1)^6(-x+6i)}{(-x+i)^6}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-10x}{x^2+1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{5\ln(x^2+1)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{724i}{2401(x+i)^4} - \frac{16i}{147(x+i)^6} - \frac{3125i}{117649(x+i)^2} + \frac{496}{1715(x+i)^5} - \frac{7432}{50421(x+i)^3} \right. \\
&\qquad \left. - \frac{3125}{823543(x+i)} + \frac{3125}{823543(x-6i)}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\left(x^2+1\right)^6(-x+6i)}{(-x+i)^6}\right) \\
&\quad + c_2\left(\frac{\left(x^2+1\right)^6(-x+6i)}{(-x+i)^6}\left(\frac{724i}{2401\,(x+i)^4}-\frac{16i}{147\,(x+i)^6}-\frac{3125i}{117649\,(x+i)^2}\right.\right. \\
&\quad \left.\left.+\frac{496}{1715\,(x+i)^5}-\frac{7432}{50421\,(x+i)^3}-\frac{3125}{823543\,(x+i)}+\frac{3125}{823543\,(x-6i)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 39

```
dsolve((x^2+1)*diff(diff(y(x),x),x)-10*diff(y(x),x)*x+28*y(x) = 0,y(x),singsol=all)
```

$$
y = c_1 + \frac{35}{3}c_1 x^4 - 14c_1 x^2 + c_2 x^7 + 21c_2 x^5 - 105c_2 x^3 + 35c_2 x
$$

Mathematica DSolve solution

Solving time : `0.334` (sec)
Leaf size : 93

```
DSolve[{(1+x^2)*D[y[x],{x,2}]-10*x*D[y[x],x]+28*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$
y(x) \to (x+6i)\left(x^2+1\right)^{5/2}\exp\left(\int_1^x \frac{K[1]+6i}{K[1]^2+1}dK[1]\right)\left(c_2\int_1^x \frac{\exp\left(-2\int_1^{K[2]}\frac{K[1]+6i}{K[1]^2+1}dK[1]\right)}{(K[2]+6i)^2}dK[2] + c_1\right)
$$

2.1.478 Problem 493

Internal problem ID [9650]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 493
Date solved : `Monday, January 27, 2025 at 06:05:01 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + xy' + 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.208 (sec)

Writing the ode as

$$y'' + xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 6 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.904: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{2x} - \frac{9}{4x^3} - \frac{27}{4x^5} - \frac{405}{16x^7} - \frac{1701}{16x^9} - \frac{15309}{32x^{11}} - \frac{72171}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 6}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} - \frac{3}{2}\right) + (0) \\ &= \frac{x^2}{4} - \frac{3}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{2}\right) - (0) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{3}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{3}{2}\right)\right) &= 0 \\ a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (x)\, e^{\int -\frac{x}{2} dx} \\ &= (x)\, e^{-\frac{x^2}{4}} \\ &= x\, e^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{1}\, dx} \\ &= z_1 e^{-\frac{x^2}{4}} \\ &= z_1\left(e^{-\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^2}{2}} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-\frac{x^2}{2}} x\right) + c_2 \left(\mathrm{e}^{-\frac{x^2}{2}} x \left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} \right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + x\left(\frac{d}{dx}y(x)\right) + 2y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k+2$

    $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+2)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

$$(k+2)(ka_{k+2}+a_k+a_{k+2})=0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y(x)=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=-\frac{a_k}{k+1}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 34

```
dsolve(diff(diff(y(x),x),x)+diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y=x\left(ic_2\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)+c_1\right)\mathrm{e}^{-\frac{x^2}{2}}+2c_2$$

Mathematica DSolve solution

Solving time : 0.047 (sec)
Leaf size : 69

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to -\sqrt{\frac{\pi}{2}}c_2e^{-\frac{x^2}{2}}\sqrt{x^2}\mathrm{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{2}}\right)+\sqrt{2}c_1e^{-\frac{x^2}{2}}x+c_2$$

2.1.479 Problem 495

Internal problem ID [9651]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 495
Date solved : `Monday, January 27, 2025 at 06:05:02 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(2x^2-8x+11\right)y''-16(x-2)\,y'+36y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.606 (sec)

Writing the ode as

$$\left(2x^2-8x+11\right)y''+(-16x+32)\,y'+36y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2-8x+11 \\ B &= -16x+32 \\ C &= 36 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{8x^2-32x-100}{\left(2x^2-8x+11\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 8x^2-32x-100 \\ t &= \left(2x^2-8x+11\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{8x^2-32x-100}{\left(2x^2-8x+11\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.906: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x^2 - 8x + 11)^2$. There is a pole at $x = 2 + \frac{i\sqrt{6}}{2}$ of order 2. There is a pole at $x = 2 - \frac{i\sqrt{6}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{\left(x - 2 - \frac{i\sqrt{6}}{2}\right)^2} + \frac{6}{\left(x - 2 + \frac{i\sqrt{6}}{2}\right)^2} + \frac{5i\sqrt{6}}{3\left(x - 2 - \frac{i\sqrt{6}}{2}\right)} - \frac{5i\sqrt{6}}{3\left(x - 2 + \frac{i\sqrt{6}}{2}\right)}$$

For the pole at $x = 2 + \frac{i\sqrt{6}}{2}$ let b be the coefficient of $\frac{1}{\left(x-2-\frac{i\sqrt{6}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2 \end{aligned}$$

For the pole at $x = 2 - \frac{i\sqrt{6}}{2}$ let b be the coefficient of $\frac{1}{\left(x-2+\frac{i\sqrt{6}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{8x^2 - 32x - 100}{(2x^2 - 8x + 11)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{8x^2 - 32x - 100}{(2x^2 - 8x + 11)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$2 + \frac{i\sqrt{6}}{2}$	2	0	3	-2
$2 - \frac{i\sqrt{6}}{2}$	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{2}{x-2-\frac{i\sqrt{6}}{2}} + \frac{3}{x-2+\frac{i\sqrt{6}}{2}} + (0) \\
&= -\frac{2}{x-2-\frac{i\sqrt{6}}{2}} + \frac{3}{x-2+\frac{i\sqrt{6}}{2}} \\
&= \frac{-5i\sqrt{6}+2x-4}{2x^2-8x+11}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{2}{x-2-\frac{i\sqrt{6}}{2}} + \frac{3}{x-2+\frac{i\sqrt{6}}{2}}\right)(1) + \left(\left(\frac{2}{\left(x-2-\frac{i\sqrt{6}}{2}\right)^2} - \frac{3}{\left(x-2+\frac{i\sqrt{6}}{2}\right)^2}\right) + \left(-\frac{2}{x-2-\frac{i\sqrt{6}}{2}} + \right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{5i\sqrt{6}}{2} - 2\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - 2 - \frac{5i\sqrt{6}}{2}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - 2 - \frac{5i\sqrt{6}}{2}\right) \mathrm{e}^{\int \left(-\frac{2}{x-2-\frac{i\sqrt{6}}{2}} + \frac{3}{x-2+\frac{i\sqrt{6}}{2}}\right) dx} \\
&= \left(x - 2 - \frac{5i\sqrt{6}}{2}\right) \mathrm{e}^{\frac{\ln\left(4x^2-16x+22\right)}{2} - 5i \arctan\left(\frac{(2x-4)\sqrt{6}}{6}\right)} \\
&= \frac{9\left(5\sqrt{6}+2ix-4i\right)\left(2x^2-8x+11\right)^3 \sqrt{6}}{2\left(-x\sqrt{6}+2\sqrt{6}+3i\right)^5}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-16x+32}{2x^2-8x+11} \, dx} \\
&= z_1 e^{2 \ln(2x^2-8x+11)} \\
&= z_1 \left(\left(2x^2 - 8x + 11\right)^2\right)
\end{aligned}$$

Which simplifies to

$$y_1 = -\frac{9(2x^2-8x+11)^5 \left(5\sqrt{6}+2ix-4i\right)\sqrt{6}}{2\left(x\sqrt{6}-2\sqrt{6}-3i\right)^5}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-16x+32}{2x^2-8x+11}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{4\ln(2x^2-8x+11)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{10i\sqrt{6}}{27\left(2x-4+i\sqrt{6}\right)^4} + \frac{8i\sqrt{6}}{729\left(2x-4+i\sqrt{6}\right)^2} - \frac{16}{15\left(2x-4+i\sqrt{6}\right)^5} \right. \\
&\quad \left. + \frac{22}{81\left(2x-4+i\sqrt{6}\right)^3} + \frac{4}{2187\left(2x-4+i\sqrt{6}\right)} - \frac{4}{2187\left(-5i\sqrt{6}+2x-4\right)}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(-\frac{9(2x^2-8x+11)^5 \left(5\sqrt{6}+2ix-4i\right)\sqrt{6}}{2\left(x\sqrt{6}-2\sqrt{6}-3i\right)^5}\right) \\
&\quad + c_2 \left(-\frac{9(2x^2-8x+11)^5 \left(5\sqrt{6}+2ix-4i\right)\sqrt{6}}{2\left(x\sqrt{6}-2\sqrt{6}-3i\right)^5}\left(-\frac{10i\sqrt{6}}{27\left(2x-4+i\sqrt{6}\right)^4} \right.\right. \\
&\quad + \frac{8i\sqrt{6}}{729\left(2x-4+i\sqrt{6}\right)^2} - \frac{16}{15\left(2x-4+i\sqrt{6}\right)^5} + \frac{22}{81\left(2x-4+i\sqrt{6}\right)^3} \\
&\quad \left.\left. + \frac{4}{2187\left(2x-4+i\sqrt{6}\right)} - \frac{4}{2187\left(-5i\sqrt{6}+2x-4\right)}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(2x^2-8x+11)\left(\frac{d^2}{dx^2}y(x)\right) - 16(x-2)\left(\frac{d}{dx}y(x)\right) + 36y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{36y(x)}{2x^2-8x+11} + \frac{16(x-2)\left(\frac{d}{dx}y(x)\right)}{2x^2-8x+11}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) - \frac{16(x-2)\left(\frac{d}{dx}y(x)\right)}{2x^2-8x+11} + \frac{36y(x)}{2x^2-8x+11} = 0$

□ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = -\frac{16(x-2)}{2x^2-8x+11}, P_3(x) = \frac{36}{2x^2-8x+11}\right]$
- $\left(x-2+\frac{I\sqrt{6}}{2}\right) \cdot P_2(x)$ is analytic at $x = 2-\frac{I\sqrt{6}}{2}$

  $\left(\left(x-2+\frac{I\sqrt{6}}{2}\right) \cdot P_2(x)\right)\Big|_{x=2-\frac{I\sqrt{6}}{2}} = 0$
- $\left(x-2+\frac{I\sqrt{6}}{2}\right)^2 \cdot P_3(x)$ is analytic at $x = 2-\frac{I\sqrt{6}}{2}$

  $\left(\left(x-2+\frac{I\sqrt{6}}{2}\right)^2 \cdot P_3(x)\right)\Big|_{x=2-\frac{I\sqrt{6}}{2}} = 0$
- $x = 2-\frac{I\sqrt{6}}{2}$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 2-\frac{I\sqrt{6}}{2}$

- Multiply by denominators

  $(2x^2-8x+11)\left(\frac{d^2}{dx^2}y(x)\right) + (-16x+32)\left(\frac{d}{dx}y(x)\right) + 36y(x) = 0$
- Change variables using $x = u+2-\frac{I\sqrt{6}}{2}$ so that the regular singular point is at $u = 0$

  $(2u^2-2\,\mathrm{I}u\sqrt{6})\left(\frac{d^2}{du^2}y(u)\right) + (-16u+8\,\mathrm{I}\sqrt{6})\left(\frac{d}{du}y(u)\right) + 36y(u) = 0$
- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$
- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$
- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2\,\mathrm{I}\sqrt{6}\,r(r-5)\,a_0u^{-1+r} + \left(\sum_{k=0}^{\infty}\left(-2\,\mathrm{I}\sqrt{6}\,(k+1+r)(k-4+r)\,a_{k+1} + 2a_k(k+r-3)(k+r-6)\right)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2\,\mathrm{I}\sqrt{6}\,r(r-5) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 5\}$
- Each term in the series must be 0, giving the recursion relation

  $-2\,\mathrm{I}\sqrt{6}\,(k+1+r)(k-4+r)\,a_{k+1} + 2a_k(k+r-3)(k+r-6) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{-\frac{I}{6}a_k(k^2+2kr+r^2-9k-9r+18)\sqrt{6}}{k^2+2kr+r^2-3k-3r-4}$
- Recursion relation for $r = 0$; series terminates at $k = 3$

  $a_{k+1} = \frac{-\frac{I}{6}a_k(k^2-9k+18)\sqrt{6}}{k^2-3k-4}$

- Apply recursion relation for $k = 0$

  $a_1 = \frac{3\mathrm{I}}{4} a_0 \sqrt{6}$
- Apply recursion relation for $k = 1$

  $a_2 = \frac{5\mathrm{I}}{18} a_1 \sqrt{6}$
- Express in terms of a_0

  $a_2 = -\frac{5a_0}{4}$
- Apply recursion relation for $k = 2$

  $a_3 = \frac{\mathrm{I}}{9} a_2 \sqrt{6}$
- Express in terms of a_0

  $a_3 = -\frac{5\mathrm{I}}{36} a_0 \sqrt{6}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 + \frac{3\mathrm{I}\sqrt{6}\, u}{4} - \frac{5u^2}{4} - \frac{5\mathrm{I}\sqrt{6}\, u^3}{36}\right)$
- Revert the change of variables $u = x - 2 + \frac{\mathrm{I}\sqrt{6}}{2}$

  $\left[y(x) = -\frac{\mathrm{I}}{72} a_0 \sqrt{6}\,(10x^3 - 60x^2 + 111x - 62)\right]$
- Recursion relation for $r = 5$; series terminates at $k = 1$

  $a_{k+1} = \frac{-\frac{\mathrm{I}}{6} a_k (k^2+k-2)\sqrt{6}}{k^2+7k+6}$
- Apply recursion relation for $k = 0$

  $a_1 = \frac{\mathrm{I}}{18} a_0 \sqrt{6}$
- Terminating series solution of the ODE for $r = 5$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 + \frac{\mathrm{I}\sqrt{6}\, u}{18}\right)$
- Revert the change of variables $u = x - 2 + \frac{\mathrm{I}\sqrt{6}}{2}$

  $\left[y(x) = a_0 \left(\frac{5}{6} + \frac{\mathrm{I}(x-2)\sqrt{6}}{18}\right)\right]$
- Combine solutions and rename parameters

  $\left[y(x) = -\frac{\mathrm{I} a_0 \sqrt{6}\,(10x^3-60x^2+111x-62)}{72} + b_0 \left(\frac{5}{6} + \frac{\mathrm{I}(x-2)\sqrt{6}}{18}\right)\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 55

```
dsolve((2*x^2-8*x+11)*diff(diff(y(x),x),x)-16*(x-2)*diff(y(x),x)+36*y(x) = 0,y(x),singso
```

$$y = c_2x^6 - 12c_2x^5 + \frac{165c_2x^4}{2} + c_1x^3 + \frac{3(-8c_1 - 1815c_2)\,x^2}{4} + \frac{3(37c_1 + 10890c_2)\,x}{10} - \frac{31c_1}{5} - \frac{16577c_2}{8}$$

Mathematica DSolve solution

Solving time : 0.587 (sec)
Leaf size : 146

```
DSolve[{(11-8*x+2*x^2)*D[y[x],{x,2}]-16*(x-2)*D[y[x],x]+36*y[x]==0,{}},y[x],x,IncludeSingularSo
```

$$y(x) \to \frac{1}{2}\left(2x+5i\sqrt{6}-4\right)\left(2x^2-8x+11\right)^2\exp\left(\int_1^x \frac{2K[1]+5i\sqrt{6}-4}{2(K[1]-4)K[1]+11}dK[1]\right)\left(c_2\int_1^x -\frac{4\exp\left(-2\int_1^{K[2]}\frac{2K[1]+5i\sqrt{6}-4}{2(K[1]-4)K[1]+11}dK[1]\right)}{\left(-2iK[2]+5\sqrt{6}+4i\right)^2}dK[2]+c_1\right)$$

2.1.480 Problem 496

Internal problem ID [9652]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 496
Date solved : `Monday, January 27, 2025 at 06:05:03 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + (x-3)\,y' + 3y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.278 (sec)

Writing the ode as

$$y'' + (x-3)\,y' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x-3 \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6x - 1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 6x - 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4} + \frac{1}{4}x^2 - \frac{3}{2}x\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.908: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{2} - \frac{5}{2x} - \frac{15}{2x^2} - \frac{115}{4x^3} - \frac{495}{4x^4} - \frac{2285}{4x^5} - \frac{11055}{4x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} - \frac{3}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 - \frac{3}{2}x + \frac{9}{4}$$

This shows that the coefficient of 1 in the above is $\frac{9}{4}$. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 6x - 1}{4} \\ &= Q + \frac{R}{4} \\ &= \left(-\frac{1}{4} + \frac{1}{4}x^2 - \frac{3}{2}x\right) + (0) \\ &= -\frac{1}{4} + \frac{1}{4}x^2 - \frac{3}{2}x\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{4}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{1}{4}\right) - \left(\frac{9}{4}\right) \\ &= -\frac{5}{2}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{x}{2} - \frac{3}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &= -3 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &= 2\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4} + \frac{1}{4}x^2 - \frac{3}{2}x$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2} - \frac{3}{2}$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 2 \end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0+(-)\left(\frac{x}{2}-\frac{3}{2}\right) \\ &= \frac{3}{2}-\frac{x}{2} \\ &= \frac{3}{2}-\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=x^2+a_1x+a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2)+2\left(\frac{3}{2}-\frac{x}{2}\right)(2x+a_1)+\left(\left(-\frac{1}{2}\right)+\left(\frac{3}{2}-\frac{x}{2}\right)^2-\left(-\frac{1}{4}+\frac{1}{4}x^2-\frac{3}{2}x\right)\right)=0$$
$$(x+3)\,a_1+6x+2a_0+2=0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0=8, a_1=-6\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x)=x^2-6x+8$$

Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\,dx} \\ &= \left(x^2-6x+8\right)e^{\int\left(\frac{3}{2}-\frac{x}{2}\right)dx} \\ &= \left(x^2-6x+8\right)e^{\frac{3}{2}x-\frac{1}{4}x^2} \\ &= \left(x^2-6x+8\right)e^{-\frac{x(-6+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1e^{-\int \frac{1}{2}\frac{x-3}{1}\,dx} \\ &= z_1e^{\frac{3}{2}x-\frac{1}{4}x^2} \\ &= z_1\left(e^{-\frac{x(-6+x)}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x(-6+x)}{2}} \left(x^2 - 6x + 8\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x-3}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{1}{2}x^2+3x}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{-\frac{1}{2}x^2+3x} \mathrm{e}^{x(-6+x)}}{\left(x^2-6x+8\right)^2} dx \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-\frac{x(-6+x)}{2}} \left(x^2 - 6x + 8\right)\right) + c_2 \left(\mathrm{e}^{-\frac{x(-6+x)}{2}} \left(x^2 - 6x + 8\right) \left(\int \frac{\mathrm{e}^{-\frac{1}{2}x^2+3x} \mathrm{e}^{x(-6+x)}}{\left(x^2-6x+8\right)^2} dx \right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + (x-3)\left(\frac{d}{dx}y(x)\right) + 3y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite DE with series expansions

  ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..1$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$

  ◦ Shift index using $k->k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, x^k$

  ◦ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

  $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k->k+2$

  $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite DE with series expansions

$$\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\left(k+1\right)-3a_{k+1}(k+1)+a_{k}(k+3)\right)x^{k}=0$$

- Each term in the series must be 0, giving the recursion relation

$$k^{2}a_{k+2}+\left(a_{k}-3a_{k+1}+3a_{k+2}\right)k+3a_{k}-3a_{k+1}+2a_{k+2}=0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y(x)=\sum_{k=0}^{\infty} a_{k}x^{k}, a_{k+2}=-\frac{a_{k}k-3a_{k+1}k+3a_{k}-3a_{k+1}}{k^{2}+3k+2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special fu
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.020 (sec)
Leaf size : 73

```
dsolve(diff(diff(y(x),x),x)+(x-3)*diff(y(x),x)+3*y(x) = 0,y(x),singsol=all)
```

$$y=(x-4)\left(\operatorname{erf}\left(\frac{\sqrt{2}\sqrt{-\left(x-3\right)^{2}}}{2}\right)-1\right)c_{2}\mathrm{e}^{-\frac{\left(x-3\right)^{2}}{2}}\left(x-2\right)\sqrt{\pi}$$
$$-\sqrt{2}\sqrt{-\left(x-3\right)^{2}}\,c_{2}-c_{1}\mathrm{e}^{-\frac{\left(x-3\right)^{2}}{2}}\left(x-2\right)\left(x-4\right)$$

Mathematica DSolve solution

Solving time : `0.261` (sec)
Leaf size : 62

```
DSolve[{D[y[x],{x,2}]+(x-3)*D[y[x],x]+3*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-\frac{1}{2}(x-6)x}\left(x^2-6x+8\right)\left(c_2\int_1^x \frac{e^{\frac{1}{2}(K[1]-6)K[1]}}{(K[1]-4)^2(K[1]-2)^2}dK[1]+c_1\right)$$

2.1.481 Problem 497

Internal problem ID [9653]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 497
Date solved : `Monday, January 27, 2025 at 06:05:03 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2-8x+14\right)y''-8(x-4)\,y'+20y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.260 (sec)

Writing the ode as

$$\left(x^2-8x+14\right)y''+(-8x+32)\,y'+20y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2-8x+14\\ B &= -8x+32 \\ C &= 20 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{48}{\left(x^2-8x+14\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 48 \\ t &= \left(x^2-8x+14\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{48}{\left(x^2-8x+14\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.910: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 8x + 14)^2$. There is a pole at $x = 4 + \sqrt{2}$ of order 2. There is a pole at $x = 4 - \sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{\left(x - 4 + \sqrt{2}\right)^2} + \frac{6}{\left(x - 4 - \sqrt{2}\right)^2} + \frac{3\sqrt{2}}{x - 4 + \sqrt{2}} - \frac{3\sqrt{2}}{x - 4 - \sqrt{2}}$$

For the pole at $x = 4 + \sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x-4-\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2 \end{aligned}$$

For the pole at $x = 4 - \sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x-4+\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{48}{(x^2 - 8x + 14)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$4+\sqrt{2}$	2	0	3	-2
$4-\sqrt{2}$	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{2}{x-4-\sqrt{2}} + \frac{3}{x-4+\sqrt{2}} + (-)(0) \\
&= -\frac{2}{x-4-\sqrt{2}} + \frac{3}{x-4+\sqrt{2}} \\
&= \frac{x-4-5\sqrt{2}}{x^2-8x+14}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{2}{x-4-\sqrt{2}}+\frac{3}{x-4+\sqrt{2}}\right)(0)+\left(\left(\frac{2}{\left(x-4-\sqrt{2}\right)^{2}}-\frac{3}{\left(x-4+\sqrt{2}\right)^{2}}\right)+\left(-\frac{2}{x-4-\sqrt{2}}+\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx}\\
&= \mathrm{e}^{\int \left(-\frac{2}{x-4-\sqrt{2}}+\frac{3}{x-4+\sqrt{2}}\right)dx}\\
&= \frac{\left(x-4+\sqrt{2}\right)^{3}}{\left(-x+4+\sqrt{2}\right)^{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}\\
&= z_1 e^{-\int \frac{1}{2}\frac{-8x+32}{x^2-8x+14}\, dx}\\
&= z_1 e^{2\ln(x^2-8x+14)}\\
&= z_1\left(\left(x^2-8x+14\right)^{2}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\frac{\left(x^2-8x+14\right)^{2}\left(x-4+\sqrt{2}\right)^{3}}{\left(-x+4+\sqrt{2}\right)^{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1\int \frac{e^{\int -\frac{-8x+32}{x^2-8x+14}\, dx}}{\left(y_1\right)^{2}}\, dx\\
&= y_1\int \frac{e^{4\ln(x^2-8x+14)}}{\left(y_1\right)^{2}}\, dx\\
&= y_1\left(-\frac{64}{5\left(x-4+\sqrt{2}\right)^{5}}+\frac{4\sqrt{2}}{\left(x-4+\sqrt{2}\right)^{2}}+\frac{16\sqrt{2}}{\left(x-4+\sqrt{2}\right)^{4}}-\frac{1}{x-4+\sqrt{2}}\right.\\
&\qquad\left.-\frac{16}{\left(x-4+\sqrt{2}\right)^{3}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1y_1+c_2y_2\\
&= c_1\left(\frac{\left(x^2-8x+14\right)^{2}\left(x-4+\sqrt{2}\right)^{3}}{\left(-x+4+\sqrt{2}\right)^{2}}\right)\\
&\quad + c_2\left(\frac{\left(x^2-8x+14\right)^{2}\left(x-4+\sqrt{2}\right)^{3}}{\left(-x+4+\sqrt{2}\right)^{2}}\left(-\frac{64}{5\left(x-4+\sqrt{2}\right)^{5}}+\frac{4\sqrt{2}}{\left(x-4+\sqrt{2}\right)^{2}}\right.\right.\\
&\qquad\left.\left.+\frac{16\sqrt{2}}{\left(x-4+\sqrt{2}\right)^{4}}-\frac{1}{x-4+\sqrt{2}}-\frac{16}{\left(x-4+\sqrt{2}\right)^{3}}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x^2-8x+14)\left(\frac{d^2}{dx^2}y(x)\right)-8(-4+x)\left(\frac{d}{dx}y(x)\right)+20y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{20y(x)}{x^2-8x+14}+\frac{8(-4+x)\left(\frac{d}{dx}y(x)\right)}{x^2-8x+14}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x)-\frac{8(-4+x)\left(\frac{d}{dx}y(x)\right)}{x^2-8x+14}+\frac{20y(x)}{x^2-8x+14}=0$

- ☐ Check to see if x_0 is a regular singular point
  - ○ Define functions

    $\left[P_2(x)=-\frac{8(-4+x)}{x^2-8x+14}, P_3(x)=\frac{20}{x^2-8x+14}\right]$
  - ○ $(x-4+\sqrt{2})\cdot P_2(x)$ is analytic at $x=4-\sqrt{2}$

    $\left((x-4+\sqrt{2})\cdot P_2(x)\right)\Big|_{x=4-\sqrt{2}}=0$
  - ○ $(x-4+\sqrt{2})^2\cdot P_3(x)$ is analytic at $x=4-\sqrt{2}$

    $\left((x-4+\sqrt{2})^2\cdot P_3(x)\right)\Big|_{x=4-\sqrt{2}}=0$
  - ○ $x=4-\sqrt{2}$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=4-\sqrt{2}$

- Multiply by denominators

$(x^2-8x+14)\left(\frac{d^2}{dx^2}y(x)\right)+(-8x+32)\left(\frac{d}{dx}y(x)\right)+20y(x)=0$

- Change variables using $x=u+4-\sqrt{2}$ so that the regular singular point is at $u=0$

$(u^2-2u\sqrt{2})\left(\frac{d^2}{du^2}y(u)\right)+(-8u+8\sqrt{2})\left(\frac{d}{du}y(u)\right)+20y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

- ☐ Rewrite ODE with series expansions
  - ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$
  - ○ Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - ○ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$
  - ○ Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2\sqrt{2}\,(r-5)\,ra_0u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2\sqrt{2}\,(k+r-4)(k+1+r)\,a_{k+1}+a_k(k+r-4)(k+r-5)\right)u^{k+}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-2\sqrt{2}\,(r-5)\,r = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 5\}$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-4)\left(-2a_{k+1}(k+1+r)\sqrt{2}+a_k(k+r-5)\right) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_k(k+r-5)\sqrt{2}}{4(k+1+r)}$
- Recursion relation for $r = 0$; series terminates at $k = 5$
  $a_{k+1} = \frac{a_k(k-5)\sqrt{2}}{4(k+1)}$
- Apply recursion relation for $k = 0$
  $a_1 = -\frac{5a_0\sqrt{2}}{4}$
- Apply recursion relation for $k = 1$
  $a_2 = -\frac{a_1\sqrt{2}}{2}$
- Express in terms of a_0
  $a_2 = \frac{5a_0}{4}$
- Apply recursion relation for $k = 2$
  $a_3 = -\frac{a_2\sqrt{2}}{4}$
- Express in terms of a_0
  $a_3 = -\frac{5a_0\sqrt{2}}{16}$
- Apply recursion relation for $k = 3$
  $a_4 = -\frac{a_3\sqrt{2}}{8}$
- Express in terms of a_0
  $a_4 = \frac{5a_0}{64}$
- Apply recursion relation for $k = 4$
  $a_5 = -\frac{a_4\sqrt{2}}{20}$
- Express in terms of a_0
  $a_5 = -\frac{a_0\sqrt{2}}{256}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li
  $y(u) = a_0 \cdot \left(1 - \frac{5u\sqrt{2}}{4} + \frac{5u^2}{4} - \frac{5\sqrt{2}\,u^3}{16} + \frac{5u^4}{64} - \frac{\sqrt{2}\,u^5}{256}\right)$
- Revert the change of variables $u = x - 4 + \sqrt{2}$
  $\left[y(x) = a_0\left(\frac{(-x^5+20x^4-180x^3+880x^2-2260x+2384)\sqrt{2}}{256} + \frac{5x^4}{128} - \frac{5x^3}{8} + \frac{125x^2}{32} - \frac{45x}{4} + \frac{401}{32}\right)\right]$
- Recursion relation for $r = 5$
  $a_{k+1} = \frac{a_k k\sqrt{2}}{4(k+6)}$
- Solution for $r = 5$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+5}, a_{k+1} = \frac{a_k k\sqrt{2}}{4(k+6)}\right]$
- Revert the change of variables $u = x - 4 + \sqrt{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k\left(x-4+\sqrt{2}\right)^{k+5}, a_{k+1} = \frac{a_k k\sqrt{2}}{4(k+6)}\right]$
- Combine solutions and rename parameters
  $\left[y(x) = a_0\left(\frac{(-x^5+20x^4-180x^3+880x^2-2260x+2384)\sqrt{2}}{256} + \frac{5x^4}{128} - \frac{5x^3}{8} + \frac{125x^2}{32} - \frac{45x}{4} + \frac{401}{32}\right) + \left(\sum_{k=0}^{\infty} b_k(x-4\right.\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 55

```
dsolve((x^2-8*x+14)*diff(diff(y(x),x),x)-8*(x-4)*diff(y(x),x)+20*y(x) = 0,y(x),singsol=a
```

$$y = c_1 x^5 + c_2 x^4 + 4(-35c_1 - 4c_2)\, x^3 + 20(56c_1 + 5c_2)\, x^2 + 4(-875c_1 - 72c_2)\, x + 4032c_1 + \frac{1604c_2}{5}$$

Mathematica DSolve solution

Solving time : 0.089 (sec)
Leaf size : 77

```
DSolve[{(x^2-8*x+14)*D[y[x],{x,2}]+8*(x-4)*D[y[x],x]+20*y[x]==0,{}},y[x],x,IncludeSingularSolu
```

$$y(x) \to \frac{c_1 P^3_{\frac{1}{2}i\left(i+\sqrt{31}\right)}\left(\frac{x-4}{\sqrt{2}}\right) + c_2 Q^3_{\frac{1}{2}i\left(i+\sqrt{31}\right)}\left(\frac{x-4}{\sqrt{2}}\right)}{(x^2-8x+14)^{3/2}}$$

2.1.482 Problem 498

Internal problem ID [9654]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 498
Date solved : `Monday, January 27, 2025 at 06:05:04 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(2x^2+4x+5\right)y''-20(x+1)\,y'+60y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.573 (sec)

Writing the ode as

$$\left(2x^2+4x+5\right)y''+(-20x-20)\,y'+60y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x^2+4x+5\\B&=-20x-20\\C&=60\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-210}{\left(2x^2+4x+5\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=-210\\t&=\left(2x^2+4x+5\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{210}{\left(2x^2+4x+5\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.912: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x^2 + 4x + 5)^2$. There is a pole at $x = -1 + \frac{i\sqrt{6}}{2}$ of order 2. There is a pole at $x = -1 - \frac{i\sqrt{6}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{35}{4\left(x + 1 - \frac{i\sqrt{6}}{2}\right)^2} + \frac{35}{4\left(x + 1 + \frac{i\sqrt{6}}{2}\right)^2} + \frac{35i\sqrt{6}}{12\left(x + 1 - \frac{i\sqrt{6}}{2}\right)} - \frac{35i\sqrt{6}}{12\left(x + 1 + \frac{i\sqrt{6}}{2}\right)}$$

For the pole at $x = -1 + \frac{i\sqrt{6}}{2}$ let b be the coefficient of $\frac{1}{\left(x+1-\frac{i\sqrt{6}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{5}{2} \end{aligned}$$

For the pole at $x = -1 - \frac{i\sqrt{6}}{2}$ let b be the coefficient of $\frac{1}{\left(x+1+\frac{i\sqrt{6}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{210}{(2x^2+4x+5)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-1+\frac{i\sqrt{6}}{2}$	2	0	$\frac{7}{2}$	$-\frac{5}{2}$
$-1-\frac{i\sqrt{6}}{2}$	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{5}{2\left(x+1-\frac{i\sqrt{6}}{2}\right)} + \frac{7}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)} + (-)(0) \\
&= -\frac{5}{2\left(x+1-\frac{i\sqrt{6}}{2}\right)} + \frac{7}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)} \\
&= \frac{-6i\sqrt{6}+2x+2}{2x^2+4x+5}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{5}{2\left(x+1-\frac{i\sqrt{6}}{2}\right)} + \frac{7}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)}\right)(0) + \left(\left(\frac{5}{2\left(x+1-\frac{i\sqrt{6}}{2}\right)^2} - \frac{7}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)^2}\right) + \left(-\frac{}{2}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(-\frac{5}{2\left(x+1-\frac{i\sqrt{6}}{2}\right)} + \frac{7}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)}\right) dx} \\ &= \frac{27\sqrt{2}\,(2x^2+4x+5)^{7/2}}{\left(3+i(x+1)\sqrt{6}\right)^6} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-20x-20}{2x^2+4x+5}\, dx} \\ &= z_1 e^{\frac{5\ln(2x^2+4x+5)}{2}} \\ &= z_1\left((2x^2+4x+5)^{5/2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = -\frac{(2x^2+4x+5)^6\sqrt{2}}{27\left(i - \frac{(x+1)\sqrt{2}\sqrt{3}}{3}\right)^6}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-20x-20}{2x^2+4x+5}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{5\ln(2x^2+4x+5)}}{(y_1)^2}\, dx \\ &= y_1\left(-\frac{\frac{1}{2}x^5 + \frac{5}{2}x^4 + \frac{5}{2}x^3 - \frac{5}{2}x^2 - \frac{31}{8}x - \frac{7}{8}}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)^6}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(-\frac{(2x^2+4x+5)^6\sqrt{2}}{27\left(i-\frac{(x+1)\sqrt{2}\sqrt{3}}{3}\right)^6}\right) \\
&\quad + c_2\left(-\frac{(2x^2+4x+5)^6\sqrt{2}}{27\left(i-\frac{(x+1)\sqrt{2}\sqrt{3}}{3}\right)^6}\left(-\frac{\frac{1}{2}x^5+\frac{5}{2}x^4+\frac{5}{2}x^3-\frac{5}{2}x^2-\frac{31}{8}x-\frac{7}{8}}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)^6}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(2x^2+4x+5)\left(\frac{d^2}{dx^2}y(x)\right)-20(x+1)\left(\frac{d}{dx}y(x)\right)+60y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{60y(x)}{2x^2+4x+5}+\frac{20(x+1)\left(\frac{d}{dx}y(x)\right)}{2x^2+4x+5}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)-\frac{20(x+1)\left(\frac{d}{dx}y(x)\right)}{2x^2+4x+5}+\frac{60y(x)}{2x^2+4x+5}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=-\frac{20(x+1)}{2x^2+4x+5}, P_3(x)=\frac{60}{2x^2+4x+5}\right]$

  ◦ $\left(x+1+\frac{\mathrm{I}\sqrt{6}}{2}\right)\cdot P_2(x)$ is analytic at $x=-1-\frac{\mathrm{I}\sqrt{6}}{2}$

    $\left(\left(x+1+\frac{\mathrm{I}\sqrt{6}}{2}\right)\cdot P_2(x)\right)\bigg|_{x=-1-\frac{\mathrm{I}\sqrt{6}}{2}}=0$

  ◦ $\left(x+1+\frac{\mathrm{I}\sqrt{6}}{2}\right)^2\cdot P_3(x)$ is analytic at $x=-1-\frac{\mathrm{I}\sqrt{6}}{2}$

    $\left(\left(x+1+\frac{\mathrm{I}\sqrt{6}}{2}\right)^2\cdot P_3(x)\right)\bigg|_{x=-1-\frac{\mathrm{I}\sqrt{6}}{2}}=0$

  ◦ $x=-1-\frac{\mathrm{I}\sqrt{6}}{2}$ is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1-\frac{\mathrm{I}\sqrt{6}}{2}$

- Multiply by denominators

  $(2x^2+4x+5)\left(\frac{d^2}{dx^2}y(x)\right)+(-20x-20)\left(\frac{d}{dx}y(x)\right)+60y(x)=0$

- Change variables using $x=u-1-\frac{\mathrm{I}\sqrt{6}}{2}$ so that the regular singular point is at $u=0$

  $(2u^2-2\mathrm{I}u\sqrt{6})\left(\frac{d^2}{du^2}y(u)\right)+(-20u+10\mathrm{I}\sqrt{6})\left(\frac{d}{du}y(u)\right)+60y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-2\,\mathrm{I}\sqrt{6}\, r(r-6)\, a_0 u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2\,\mathrm{I}\sqrt{6}\,(k+1+r)(k+r-5)\, a_{k+1} + 2a_k(k+r-5)(k+r-6)\right)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2\,\mathrm{I}\sqrt{6}\, r(r-6) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 6\}$$

- Each term in the series must be 0, giving the recursion relation

$$-2\left(\mathrm{I}a_{k+1}(k+1+r)\sqrt{6} - a_k(k+r-6)\right)(k+r-5) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{-\frac{\mathrm{I}}{6}a_k(k+r-6)\sqrt{6}}{k+1+r}$$

- Recursion relation for $r = 0$; series terminates at $k = 6$

$$a_{k+1} = \frac{-\frac{\mathrm{I}}{6}a_k(k-6)\sqrt{6}}{k+1}$$

- Recursion relation that defines the terminating series solution of the ODE for $r = 0$

$$\left[y(u) = \sum_{k=0}^{5} a_k u^k, a_{k+1} = \frac{-\frac{\mathrm{I}}{6}a_k(k-6)\sqrt{6}}{k+1}\right]$$

- Revert the change of variables $u = x + 1 + \frac{\mathrm{I}\sqrt{6}}{2}$

$$\left[y(x) = \sum_{k=0}^{5} a_k \left(x+1+\frac{\mathrm{I}\sqrt{6}}{2}\right)^k, a_{k+1} = \frac{-\frac{\mathrm{I}}{6}a_k(k-6)\sqrt{6}}{k+1}\right]$$

- Recursion relation for $r = 6$

$$a_{k+1} = \frac{-\frac{\mathrm{I}}{6}a_k k\sqrt{6}}{k+7}$$

- Solution for $r = 6$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+6}, a_{k+1} = \frac{-\frac{\mathrm{I}}{6}a_k k\sqrt{6}}{k+7}\right]$$

- Revert the change of variables $u = x + 1 + \frac{\mathrm{I}\sqrt{6}}{2}$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k \left(x+1+\frac{\mathrm{I}\sqrt{6}}{2}\right)^{k+6}, a_{k+1} = \frac{-\frac{\mathrm{I}}{6}a_k k\sqrt{6}}{k+7}\right]$$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{5} a_k \left(x+1+\frac{\mathrm{I}\sqrt{6}}{2}\right)^k\right) + \left(\sum_{k=0}^{\infty} b_k \left(x+1+\frac{\mathrm{I}\sqrt{6}}{2}\right)^{k+6}\right), a_{k+1} = \frac{-\frac{\mathrm{I}}{6}a_k(k-6)\sqrt{6}}{k+1}, b_{k+1} = \frac{-\frac{\mathrm{I}}{6}b_k k\sqrt{6}}{k+7}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 65

```
dsolve((2*x^2+4*x+5)*diff(diff(y(x),x),x)-20*(x+1)*diff(y(x),x)+60*y(x) = 0,y(x),sings
```

$$y = c_2 x^6 + c_1 x^5 + \frac{5(2c_1 - 15c_2)\, x^4}{2} + 5(c_1 - 20c_2)\, x^3 + \frac{5(-4c_1 - 45c_2)\, x^2}{4} + \frac{(-31c_1 + 120c_2)\, x}{4} - \frac{7c_1}{4} + \frac{155c_2}{8}$$

Mathematica DSolve solution

Solving time : 0.569 (sec)
Leaf size : 108

```
DSolve[{(2*x^2+4*x+5)*D[y[x],{x,2}]-20*(x+1)*D[y[x],x]+60*y[x]==0,{}},y[x],x,IncludeSingular
```

$$y(x) \to \left(2x^2 + 4x + 5\right)^{5/2} \exp\left(\int_1^x \frac{2\left(K[1] + 3i\sqrt{6} + 1\right)}{2K[1](K[1]+2)+5} dK[1]\right) \left(c_2 \int_1^x \exp\left(-2\int_1^{K[2]} \frac{2\left(K[1] + 3i\sqrt{6} + 1\right)}{2K[1](K[1]+2)+5} dK[1]\right) dK[+ c_1\right)$$

2.1.483 Problem 499

Internal problem ID [9655]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 499
Date solved : `Monday, January 27, 2025 at 06:05:05 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^3+1\right)y''+7x^2y'+9xy=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.378 (sec)

Writing the ode as

$$\left(x^3+1\right)y''+7x^2y'+9xy=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= x^3+1\\ B &= 7x^2\\ C &= 9x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r &= \frac{s}{t}\\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-x(x^3+8)}{4\left(x^3+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s &= -x\left(x^3+8\right)\\ t &= 4\left(x^3+1\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{x(x^3+8)}{4\left(x^3+1\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.914: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3+1)^2$. There is a pole at $x = -1$ of order 2. There is a pole at $x = \frac{1}{2} - \frac{i\sqrt{3}}{2}$ of order 2. There is a pole at $x = \frac{1}{2} + \frac{i\sqrt{3}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$\begin{aligned} r = {} & \frac{5}{18\,(x+1)} + \frac{7}{36\,(x+1)^2} + \frac{7}{36\left(x - \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} \\ & + \frac{7}{36\left(x - \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{5}{36} + \frac{5i\sqrt{3}}{36}}{x - \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{-\frac{5}{36} - \frac{5i\sqrt{3}}{36}}{x - \frac{1}{2} + \frac{i\sqrt{3}}{2}} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6} \end{aligned}$$

For the pole at $x = \frac{1}{2} - \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6}
\end{aligned}
$$

For the pole at $x = \frac{1}{2} + \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{x(x^3+8)}{4\left(x^3+1\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{x(x^3+8)}{4\left(x^3+1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{7}{6}$	$-\frac{1}{6}$
$\frac{1}{2} - \frac{i\sqrt{3}}{2}$	2	0	$\frac{7}{6}$	$-\frac{1}{6}$
$\frac{1}{2} + \frac{i\sqrt{3}}{2}$	2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{6\,(x+1)} - \frac{1}{6\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)} - \frac{1}{6\left(x-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)} + (-)\,(0) \\
&= -\frac{1}{6\,(x+1)} - \frac{1}{6\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)} - \frac{1}{6\left(x-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)} \\
&= -\frac{x^2}{2x^3+2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{6\,(x+1)} - \frac{1}{6\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)} - \frac{1}{6\left(x-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)}\right)(1) + \left(\left(\frac{1}{6\,(x+1)^2} + \frac{1}{6\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2} + \right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\, dx} \\
&= (x)\,\mathrm{e}^{\int \left(-\frac{1}{6(x+1)} - \frac{1}{6\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)} - \frac{1}{6\left(x-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)}\right) dx} \\
&= (x)\,\frac{1}{\left((x+1)\left(2x-1+i\sqrt{3}\right)\left(i\sqrt{3}-2x+1\right)\right)^{1/6}} \\
&= \frac{x}{\left((x+1)\left(2x-1+i\sqrt{3}\right)\left(i\sqrt{3}-2x+1\right)\right)^{1/6}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7x^2}{x^3+1}\, dx} \\
&= z_1 e^{-\frac{7\ln\left(x^3+1\right)}{6}} \\
&= z_1\left(\frac{1}{\left(x^3+1\right)^{7/6}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x}{(x^3+1)^{7/6}(-4x^3-4)^{1/6}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x^2}{x^3+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{7\ln(x^3+1)}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{(-4x^3-4)^{1/3}}{x^2}\,dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x}{(x^3+1)^{7/6}(-4x^3-4)^{1/6}}\right) + c_2\left(\frac{x}{(x^3+1)^{7/6}(-4x^3-4)^{1/6}}\left(\int \frac{(-4x^3-4)^{1/3}}{x^2}\,dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$(x^3+1)\left(\frac{d^2}{dx^2}y(x)\right) + 7x^2\left(\frac{d}{dx}y(x)\right) + 9xy(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{9xy(x)}{x^3+1} - \frac{7x^2\left(\frac{d}{dx}y(x)\right)}{x^3+1}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) + \frac{7x^2\left(\frac{d}{dx}y(x)\right)}{x^3+1} + \frac{9xy(x)}{x^3+1} = 0$
- ☐ Check to see if x_0 is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{7x^2}{x^3+1}, P_3(x) = \frac{9x}{x^3+1}\right]$
  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$
    $\left.\left((x+1)\cdot P_2(x)\right)\right|_{x=-1} = \frac{7}{3}$
  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$
    $\left.\left((x+1)^2\cdot P_3(x)\right)\right|_{x=-1} = 0$
  - $x=-1$is a regular singular point
    Check to see if x_0 is a regular singular point
    $x_0 = -1$

- Multiply by denominators

  $(x^3+1)\left(\frac{d^2}{dx^2}y(x)\right)+7x^2\left(\frac{d}{dx}y(x)\right)+9xy(x)=0$
- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^3-3u^2+3u)\left(\frac{d^2}{du^2}y(u)\right)+(7u^2-14u+7)\left(\frac{d}{du}y(u)\right)+(9u-9)\,y(u)=0$
- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

    $u^m\cdot y(u)=\sum_{k=0}^{\infty} a_k u^{k+r+m}$
  - Shift index using $k->k-m$

    $u^m\cdot y(u)=\sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$
  - Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(4+3r)\,u^{-1+r}+\left(a_1(1+r)\,(7+3r)-a_0(3r^2+11r+9)\right)u^r+\left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+1+r)\,(3k+7\right.\right.$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(4+3r)=0$
- Values of r that satisfy the indicial equation

  $r\in\left\{0,-\frac{4}{3}\right\}$
- Each term must be 0

  $a_1(1+r)\,(7+3r)-a_0(3r^2+11r+9)=0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(3k+7+3r)-a_k(3k^2+6kr+3r^2+11k+11r+9)+a_{k-1}(k+2+r)^2=0$
- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)\,(3k+10+3r)-a_{k+1}\left(3(k+1)^2+6(k+1)\,r+3r^2+11k+20+11r\right)+a_k(k+$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{k^2a_k-3k^2a_{k+1}+2kra_k-6kra_{k+1}+r^2a_k-3r^2a_{k+1}+6ka_k-17ka_{k+1}+6ra_k-17ra_{k+1}+9a_k-23a_{k+1}}{(k+2+r)(3k+10+3r)}$
- Recursion relation for $r=0$

  $a_{k+2}=-\frac{k^2a_k-3k^2a_{k+1}+6ka_k-17ka_{k+1}+9a_k-23a_{k+1}}{(k+2)(3k+10)}$
- Solution for $r=0$

  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+2}=-\frac{k^2a_k-3k^2a_{k+1}+6ka_k-17ka_{k+1}+9a_k-23a_{k+1}}{(k+2)(3k+10)}, 7a_1-9a_0=0\right]$
- Revert the change of variables $u=x+1$

$$\left[y(x)=\sum_{k=0}^{\infty} a_k (x+1)^k, a_{k+2}=-\frac{k^2 a_k-3k^2 a_{k+1}+6k a_k-17k a_{k+1}+9a_k-23a_{k+1}}{(k+2)(3k+10)}, 7a_1-9a_0=0\right]$$

- Recursion relation for $r=-\frac{4}{3}$

$$a_{k+2}=-\frac{k^2 a_k-3k^2 a_{k+1}+\frac{10}{3}k a_k-9k a_{k+1}+\frac{25}{9}a_k-\frac{17}{3}a_{k+1}}{\left(k+\frac{2}{3}\right)(3k+6)}$$

- Solution for $r=-\frac{4}{3}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k-\frac{4}{3}}, a_{k+2}=-\frac{k^2 a_k-3k^2 a_{k+1}+\frac{10}{3}k a_k-9k a_{k+1}+\frac{25}{9}a_k-\frac{17}{3}a_{k+1}}{\left(k+\frac{2}{3}\right)(3k+6)}, -a_1+\frac{a_0}{3}=0\right]$$

- Revert the change of variables $u=x+1$

$$\left[y(x)=\sum_{k=0}^{\infty} a_k (x+1)^{k-\frac{4}{3}}, a_{k+2}=-\frac{k^2 a_k-3k^2 a_{k+1}+\frac{10}{3}k a_k-9k a_{k+1}+\frac{25}{9}a_k-\frac{17}{3}a_{k+1}}{\left(k+\frac{2}{3}\right)(3k+6)}, -a_1+\frac{a_0}{3}=0\right]$$

- Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty} a_k (x+1)^k\right)+\left(\sum_{k=0}^{\infty} b_k (x+1)^{k-\frac{4}{3}}\right), a_{k+2}=-\frac{k^2 a_k-3k^2 a_{k+1}+6k a_k-17k a_{k+1}+9a_k-23a_{k+1}}{(k+2)(3k+10)}, 7a_1\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returnin
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.179 (sec)
Leaf size : 28

```
dsolve((x^3+1)*diff(diff(y(x),x),x)+7*diff(y(x),x)*x^2+9*x*y(x) = 0,y(x),singsol=all)
```

$$y=c_1 \operatorname{hypergeom}\left([1,1],\left[\frac{2}{3}\right],-x^3\right)+\frac{c_2 x}{\left(x^3+1\right)^{4/3}}$$

Mathematica DSolve solution

Solving time : `0.562` (sec)
Leaf size : 39

```
DSolve[{(1+x^3)*D[y[x],{x,2}]+7*x^2*D[y[x],x]+9*x*y[x]==0,{}},y[x],x,IncludeSingularSolution
```

$$y(x) \to \frac{c_1 x - c_2 \text{Hypergeometric2F1}\left(-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}, -x^3\right)}{(x^3+1)^{4/3}}$$

2.1.484 Problem 500

Internal problem ID [9656]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 500
Date solved : Monday, January 27, 2025 at 06:05:06 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$\left(2x^5+1\right)y''+14x^4y'+10x^3y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 1.023 (sec)

Writing the ode as

$$\left(2x^5+1\right)y''+14x^4y'+10x^3y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=2x^5+1\\ B&=14x^4\\ C&=10x^3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{3x^3(5x^5+6)}{\left(2x^5+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=3x^3\left(5x^5+6\right)\\ t&=\left(2x^5+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{3x^3(5x^5+6)}{\left(2x^5+1\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.916: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 10 - 8 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x^5+1)^2$. There is a pole at $x = \frac{2^{4/5}\sqrt{5}}{8} + \frac{2^{4/5}}{8} + \frac{i 2^{3/10}\sqrt{5-\sqrt{5}}}{4}$ of order 2. There is a pole at $x = \frac{2^{4/5}}{8} - \frac{2^{4/5}\sqrt{5}}{8} + \frac{i 2^{3/10}\sqrt{5-\sqrt{5}}}{8} + \frac{i 2^{3/10}\sqrt{5-\sqrt{5}}\,\sqrt{5}}{8}$ of order 2. There is a pole at $x = -\frac{2^{4/5}}{2}$ of order 2. There is a pole at $x = \frac{2^{4/5}}{8} - \frac{2^{4/5}\sqrt{5}}{8} - \frac{i 2^{3/10}\sqrt{5-\sqrt{5}}\,\sqrt{5}}{8} - \frac{i 2^{3/10}\sqrt{5-\sqrt{5}}}{8}$ of order 2. There is a pole at $x = \frac{2^{4/5}\sqrt{5}}{8} + \frac{2^{4/5}}{8} - \frac{i 2^{3/10}\sqrt{5-\sqrt{5}}}{4}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \text{Expression too large to display}$$

For the pole at $x = \frac{2^{4/5}\sqrt{5}}{8} + \frac{2^{4/5}}{8} + \frac{i 2^{3/10}\sqrt{5-\sqrt{5}}}{4}$ let b be the coefficient of $\frac{1}{\left(x - \frac{2^{4/5}\sqrt{5}}{8} - \frac{2^{4/5}}{8} - \frac{i 2^{3/10}\sqrt{5-\sqrt{5}}}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{21}{100}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{10} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{10} \end{aligned}$$

For the pole at $x=\frac{2^{4/5}}{8}-\frac{2^{4/5}\sqrt{5}}{8}+\frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8}+\frac{i2^{3/10}\sqrt{5-\sqrt{5}}\,\sqrt{5}}{8}$ let b be the coefficient of $\frac{1}{\left(x-\frac{2^{4/5}}{8}+\frac{2^{4/5}\sqrt{5}}{8}-\frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8}-\frac{i2^{3/10}\sqrt{5-\sqrt{5}}\,\sqrt{5}}{8}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{21}{100}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{7}{10} \\
\alpha_c^- &= \frac{1}{2}-\sqrt{1+4b}=\frac{3}{10}
\end{aligned}
$$

For the pole at $x=-\frac{2^{4/5}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{2^{4/5}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{21}{100}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{7}{10} \\
\alpha_c^- &= \frac{1}{2}-\sqrt{1+4b}=\frac{3}{10}
\end{aligned}
$$

For the pole at $x=\frac{2^{4/5}}{8}-\frac{2^{4/5}\sqrt{5}}{8}-\frac{i2^{3/10}\sqrt{5-\sqrt{5}}\,\sqrt{5}}{8}-\frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8}$ let b be the coefficient of $\frac{1}{\left(x-\frac{2^{4/5}}{8}+\frac{2^{4/5}\sqrt{5}}{8}+\frac{i2^{3/10}\sqrt{5-\sqrt{5}}\,\sqrt{5}}{8}+\frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{21}{100}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{7}{10} \\
\alpha_c^- &= \frac{1}{2}-\sqrt{1+4b}=\frac{3}{10}
\end{aligned}
$$

For the pole at $x=\frac{2^{4/5}\sqrt{5}}{8}+\frac{2^{4/5}}{8}-\frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}$ let b be the coefficient of $\frac{1}{\left(x-\frac{2^{4/5}\sqrt{5}}{8}-\frac{2^{4/5}}{8}+\frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{21}{100}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{7}{10} \\
\alpha_c^- &= \frac{1}{2}-\sqrt{1+4b}=\frac{3}{10}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty=0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r=\frac{s}{t}=\frac{3x^3(5x^5+6)}{(2x^5+1)^2}
$$

Since the $\gcd(s,t)=1$. This gives $b=\frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=-\frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{3x^3(5x^5+6)}{(2x^5+1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{2^{4/5}\sqrt{5}}{8}+\frac{2^{4/5}}{8}+\frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}$	2	0	$\frac{7}{10}$	$\frac{3}{10}$
$\frac{2^{4/5}}{8}-\frac{2^{4/5}\sqrt{5}}{8}+\frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8}+\frac{i2^{3/10}\sqrt{5-\sqrt{5}}\,\sqrt{5}}{8}$	2	0	$\frac{7}{10}$	$\frac{3}{10}$
$-\frac{2^{4/5}}{2}$	2	0	$\frac{7}{10}$	$\frac{3}{10}$
$\frac{2^{4/5}}{8}-\frac{2^{4/5}\sqrt{5}}{8}-\frac{i2^{3/10}\sqrt{5-\sqrt{5}}\,\sqrt{5}}{8}-\frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8}$	2	0	$\frac{7}{10}$	$\frac{3}{10}$
$\frac{2^{4/5}\sqrt{5}}{8}+\frac{2^{4/5}}{8}-\frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}$	2	0	$\frac{7}{10}$	$\frac{3}{10}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=\frac{5}{2}$ then

$$\begin{aligned}
d&=\alpha_\infty^+-\left(\alpha_{c_1}^-+\alpha_{c_2}^-+\alpha_{c_3}^-+\alpha_{c_4}^-+\alpha_{c_5}^-\right)\\
&=\frac{5}{2}-\left(\frac{3}{2}\right)\\
&=1
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega&=\left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^-}{x-c_1}\right)+\left((-)[\sqrt{r}]_{c_2}+\frac{\alpha_{c_2}^-}{x-c_2}\right)+\left((-)[\sqrt{r}]_{c_3}+\frac{\alpha_{c_3}^-}{x-c_3}\right)+\left((-)[\sqrt{r}]_{c_4}+\frac{\alpha_{c_4}^-}{x-c_4}\right)\\
&=\frac{3}{10\left(x-\frac{2^{4/5}\sqrt{5}}{8}-\frac{2^{4/5}}{8}-\frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}\right)}+\frac{3}{10\left(x-\frac{2^{4/5}}{8}+\frac{2^{4/5}\sqrt{5}}{8}-\frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8}-\frac{i2^{3/10}\sqrt{5-\sqrt{5}}\,\sqrt{5}}{8}\right)}+\\
&=\frac{3}{10\left(x-\frac{2^{4/5}\sqrt{5}}{8}-\frac{2^{4/5}}{8}-\frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}\right)}+\frac{3}{10\left(x-\frac{2^{4/5}}{8}+\frac{2^{4/5}\sqrt{5}}{8}-\frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8}-\frac{i2^{3/10}\sqrt{5-\sqrt{5}}\,\sqrt{5}}{8}\right)}+\\
&=\frac{3x^4}{2x^5+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=x+a_0 \tag{2A}$$

Substituting the above in eq. (1A) and Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0=0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (x)\, \mathrm{e}^{\int \left(\frac{3}{10\left(x-\frac{2^{4/5}\sqrt{5}}{8}-\frac{2^{4/5}}{8}-\frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}\right)}+\frac{3}{10\left(x-\frac{2^{4/5}}{8}+\frac{2^{4/5}\sqrt{5}}{8}-\frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8}-\frac{i2^{3/10}\sqrt{5-\sqrt{5}}\sqrt{5}}{8}\right)}+\frac{3}{10\left(x+\frac{2^{4/5}}{2}\right)}+\frac{}{10\left(x-\frac{2^{4/5}}{8}+2^{4/}\right.}\right.} \\
&= (x)\left(\left(2^{4/5}\sqrt{5}+2i2^{3/10}\sqrt{5-\sqrt{5}}+2^{4/5}-8x\right)\left(-i2^{3/10}\sqrt{5-\sqrt{5}}\sqrt{5}+2^{4/5}\sqrt{5}-i2^{3/10}\sqrt{5-\sqrt{5}}-\right.\right. \\
&= x8^{3/10}\left(\left(x+\frac{2^{4/5}}{2}\right)\left(i2^{3/10}\sqrt{5-\sqrt{5}}+\frac{\left(-\sqrt{5}-1\right)2^{4/5}}{2}+4x\right)\left(i2^{3/10}\sqrt{5-\sqrt{5}}+\frac{\left(\sqrt{5}+1\right)2^{4/5}}{2}-\right.\right.
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{14x^4}{2x^5+1}\, dx} \\
&= z_1 e^{-\frac{7\ln\left(2x^5+1\right)}{10}} \\
&= z_1\left(\frac{1}{\left(2x^5+1\right)^{7/10}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x8^{3/10}\left(1024x^5+512\right)^{3/10}}{\left(2x^5+1\right)^{7/10}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{14x^4}{2x^5+1}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{7\ln\left(2x^5+1\right)}{5}}}{\left(y_1\right)^2}\, dx \\
&= y_1\left(\int \frac{8^{2/5}}{8x^2\left(1024x^5+512\right)^{3/5}}\, dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x8^{3/10}\left(1024x^5+512\right)^{3/10}}{\left(2x^5+1\right)^{7/10}}\right)+c_2\left(\frac{x8^{3/10}\left(1024x^5+512\right)^{3/10}}{\left(2x^5+1\right)^{7/10}}\left(\int \frac{8^{2/5}}{8x^2\left(1024x^5+512\right)^{3/5}}\, dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - return
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.100 (sec)
Leaf size : 30

```
dsolve((2*x^5+1)*diff(diff(y(x),x),x)+14*x^4*diff(y(x),x)+10*x^3*y(x) = 0,y(x),singsol
```

$$y = \frac{c_1 x}{\left(2x^5+1\right)^{2/5}} + c_2 \,\text{hypergeom}\left(\left[\frac{1}{5}, 1\right], \left[\frac{4}{5}\right], -2x^5\right)$$

Mathematica DSolve solution

Solving time : 0.0 (sec)
Leaf size : 0

```
DSolve[{(1+2*x^5)*D[y[x],{x,2}]+14*x^4*D[y[x],x]+10*x^3*y[x]==0,{}},y[x],x,IncludeSingularSo
```

Timed out

2.1.485 Problem 501

Internal problem ID [9657]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 501
Date solved : Monday, January 27, 2025 at 06:05:07 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$y'' + x^6 y' + 7x^5 y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.410 (sec)

Writing the ode as

$$y'' + x^6 y' + 7x^5 y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x^6 \\ C &= 7x^5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^5(x^7 - 16)}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^5\left(x^7 - 16\right) \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^5(x^7 - 16)}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.917: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 12 \\ &= -12 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -12 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -12$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{12}{2} = 6$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{6} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^6$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x^6}{2} - \frac{4}{x} - \frac{16}{x^8} - \frac{128}{x^{15}} - \frac{1280}{x^{22}} - \frac{14336}{x^{29}} - \frac{172032}{x^{36}} - \frac{2162688}{x^{43}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 6$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{6} a_i x^i \\ &= \frac{x^6}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^5 = x^5$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^{12}}{4}$$

This shows that the coefficient of x^5 in the above is 0. Now we need to find the coefficient of x^5 in r. How this is done depends on if $v = 0$ or not. Since $v = 6$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x^5 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^5(x^7 - 16)}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^{12} - 4x^5\right) + (0) \\ &= \frac{1}{4}x^{12} - 4x^5\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is -4. Now b can be found.

$$\begin{aligned}b &= (-4) - (0) \\ &= -4\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{x^6}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-4}{\frac{1}{2}} - 6\right) &&= -7 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-4}{\frac{1}{2}} - 6\right) &&= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^5(x^7 - 16)}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-12	$\frac{x^6}{2}$	-7	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x^6}{2}\right) \\ &= -\frac{x^6}{2} \\ &= -\frac{x^6}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{x^6}{2}\right)(1) + \left((-3x^5) + \left(-\frac{x^6}{2}\right)^2 - \left(\frac{x^5(x^7 - 16)}{4}\right)\right) &= 0 \\ x^5 a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (x)\, \mathrm{e}^{\int -\frac{x^6}{2} dx} \\ &= (x)\, \mathrm{e}^{-\frac{x^7}{14}} \\ &= x\, \mathrm{e}^{-\frac{x^7}{14}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x^6}{1} dx} \\ &= z_1 e^{-\frac{x^7}{14}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x^7}{14}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^7}{7}} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^6}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{x^7}{7}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{7^{6/7}(-1)^{1/7}\left(-\frac{7x^6(-1)^{6/7}\Gamma\left(\frac{6}{7}\right)}{(-x^7)^{6/7}} + \frac{7\, 7^{1/7}(-1)^{6/7}\mathrm{e}^{\frac{x^7}{7}}}{x} + \frac{7x^6(-1)^{6/7}\Gamma\left(\frac{6}{7}, -\frac{x^7}{7}\right)}{(-x^7)^{6/7}}\right)}{49} \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x^7}{7}} x\right) \\
&\quad + c_2 \left(\mathrm{e}^{-\frac{x^7}{7}} x \left(\frac{7^{6/7}(-1)^{1/7}\left(-\frac{7x^6(-1)^{6/7}\Gamma\left(\frac{6}{7}\right)}{(-x^7)^{6/7}} + \frac{7\, 7^{1/7}(-1)^{6/7}\mathrm{e}^{\frac{x^7}{7}}}{x} + \frac{7x^6(-1)^{6/7}\Gamma\left(\frac{6}{7}, -\frac{x^7}{7}\right)}{(-x^7)^{6/7}}\right)}{49} \right) \right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2} y(x) + x^6 \left(\frac{d}{dx} y(x)\right) + 7x^5 y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2} y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^5 \cdot y(x)$ to series expansion

  $x^5 \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+5}$

  ◦ Shift index using $k- >k-5$

  $x^5 \cdot y(x) = \sum_{k=5}^{\infty} a_{k-5} x^k$

  ◦ Convert $x^6 \cdot \left(\frac{d}{dx} y(x)\right)$ to series expansion

  $x^6 \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^{k+5}$

- Shift index using $k- >k-5$

  $x^6 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=5}^{\infty} a_{k-5}(k-5)\, x^k$

- Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

  $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- Shift index using $k- >k+2$

  $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $30a_6x^4 + 20a_5x^3 + 12a_4x^2 + 6a_3x + 2a_2 + \left(\sum_{k=5}^{\infty} (a_{k+2}(k+2)\,(k+1) + a_{k-5}(k+2))\, x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 = 0, 6a_3 = 0, 12a_4 = 0, 20a_5 = 0, 30a_6 = 0]$

- Solve for the dependent coefficient(s)

  $\{a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0, a_6 = 0\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+2)\,(ka_{k+2} + a_{k-5} + a_{k+2}) = 0$

- Shift index using $k- >k+5$

  $(k+7)\,((k+5)\, a_{k+7} + a_k + a_{k+7}) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+7} = -\frac{a_k}{k+6}, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0, a_6 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 62

```
dsolve(diff(diff(y(x),x),x)+x^6*diff(y(x),x)+7*x^5*y(x) = 0,y(x),singsol=all)
```

$$y = -\frac{\left(-c_1 e^{-\frac{x^7}{7}} x - c_2 7^{1/7}\right) \left(-x^7\right)^{6/7} + x^7 c_2 e^{-\frac{x^7}{7}} \left(\Gamma\left(\frac{6}{7}\right) - \Gamma\left(\frac{6}{7}, -\frac{x^7}{7}\right)\right)}{\left(-x^7\right)^{6/7}}$$

Mathematica DSolve solution

Solving time : 0.08 (sec)
Leaf size : 53

```
DSolve[{D[y[x],{x,2}]+x^6*D[y[x],x]+7*x^5*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{49} e^{-\frac{x^7}{7}} \left(49 c_1 x - 7^{6/7} c_2 \sqrt[7]{-x^7} \Gamma\left(-\frac{1}{7}, -\frac{x^7}{7}\right)\right)$$

2.1.486 Problem 502

Internal problem ID [9658]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 502
Date solved : `Tuesday, January 28, 2025 at 04:07:53 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^8+1\right)y''-16x^7y'+72x^6y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 450.727 (sec)

Writing the ode as

$$\left(x^8+1\right)y''-16x^7y'+72x^6y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=x^8+1\\ B&=-16x^7\\ C&=72x^6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-128x^6}{\left(x^8+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=-128x^6\\ t&=\left(x^8+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{128x^6}{\left(x^8+1\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.919: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 16 - 6 \\ &= 10 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^8 + 1)^2$. There is a pole at $x = \frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2} + \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = -\frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = -\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = -\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = -\frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = \frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2} + \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 10 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 10 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
\begin{aligned}
r = {} & \frac{2}{\left(x-\frac{\sqrt{2+\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^{2}}+\frac{2}{\left(x-\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2+\sqrt{2}}}{2}\right)^{2}}+\frac{2}{\left(x+\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2+\sqrt{2}}}{2}\right)^{2}} \\
& +\frac{2}{\left(x+\frac{\sqrt{2+\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^{2}}+\frac{2}{\left(x+\frac{\sqrt{2+\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^{2}} \\
& +\frac{2}{\left(x+\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2+\sqrt{2}}}{2}\right)^{2}}+\frac{2}{\left(x-\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2+\sqrt{2}}}{2}\right)^{2}} \\
& +\frac{2}{\left(x-\frac{\sqrt{2+\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^{2}}+\frac{2\left(\frac{\sqrt{2+\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^{7}}{x-\frac{\sqrt{2+\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}}+\frac{2\left(\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2+\sqrt{2}}}{2}\right)^{7}}{x-\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2+\sqrt{2}}}{2}} \\
& +\frac{2\left(-\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2+\sqrt{2}}}{2}\right)^{7}}{x+\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2+\sqrt{2}}}{2}}+\frac{2\left(-\frac{\sqrt{2+\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^{7}}{x+\frac{\sqrt{2+\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}}+\frac{2\left(-\frac{\sqrt{2+\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^{7}}{x+\frac{\sqrt{2+\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}} \\
& +\frac{2\left(-\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2+\sqrt{2}}}{2}\right)^{7}}{x+\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2+\sqrt{2}}}{2}}+\frac{2\left(\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2+\sqrt{2}}}{2}\right)^{7}}{x-\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2+\sqrt{2}}}{2}}+\frac{2\left(\frac{\sqrt{2+\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^{7}}{x-\frac{\sqrt{2+\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}}
\end{aligned}
$$

For the pole at $x=\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2}+\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2}-\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b=2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b}=2 \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b}=-1
\end{aligned}
$$

For the pole at $x=\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b=2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b}=2 \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b}=-1
\end{aligned}
$$

For the pole at $x=-\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b=2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b}=2 \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b}=-1
\end{aligned}
$$

For the pole at $x=-\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2}-\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2}+\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b=2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b}=2 \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b}=-1
\end{aligned}
$$

For the pole at $x = -\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2}+\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

For the pole at $x = -\frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

For the pole at $x = \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

For the pole at $x = \frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2} + \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2}-\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

Since the order of r at ∞ is $10 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{128x^6}{(x^8+1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2} + \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$\frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$-\frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$-\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$-\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$-\frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$\frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2} + \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
10	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^- + \alpha_{c_4}^- + \alpha_{c_5}^- + \alpha_{c_6}^- + \alpha_{c_7}^- + \alpha_{c_8}^+\right) \\ &= 1 - (-5) \\ &= 6 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + \left((-)[\sqrt{r}]_{c_4} + \frac{\alpha_{c_4}^-}{x-c_4}\right) \\ &= -\frac{1}{x - \frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}} - \frac{1}{x - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}} - \frac{1}{x + \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2} - \cdots} \\ &= -\frac{1}{x - \frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}} - \frac{1}{x - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}} - \frac{1}{x + \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2} - \cdots} \\ &= \frac{\left((3x^6 - 3ix^4 - 3ix^2 - 3)\sqrt{2} - 3(x^2+1)\left((-1+i)x^4 + 1 + i\right)\right)\sqrt{2-\sqrt{2}} - 3\left(\left((-1+i)x^4 + 1 + i\right) \cdots\right.}{2\left(-x\left(1+\sqrt{2}\right)\sqrt{2-\sqrt{2}} + x^2 + 1\right)\left(x\sqrt{2-\sqrt{2}} + x^2 + 1\right)\left(x^2 - x\sqrt{2-\sqrt{2}} + 1\right)\left(x\left(1+\sqrt{2}\right) \cdots\right.} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 6$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^6 + x^5 a_5 + x^4 a_4 + x^3 a_3 + x^2 a_2 + x a_1 + a_0 \tag{2A}$$

Substituting the above in eq. (1A) and Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{ a_0 = -\frac{i\sqrt{2} - 1 + i}{i\sqrt{2} + 1 + i}, a_1 = \frac{\left(\frac{12}{7} - \frac{12i}{7}\right)\sqrt{2}}{\left(i\sqrt{2} + 1 + i\right)\sqrt{2-\sqrt{2}}}, a_2 = -\frac{15\left(-\sqrt{2} - 1 + i\right)}{7\left(i\sqrt{2} + 1 + i\right)}, a_3 = \frac{32}{7\left(i\sqrt{2} + 1 + i\right)\sqrt{2 \cdots}} \right.$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^6 + \frac{\left(\frac{12}{7} + \frac{12i}{7}\right)x^5\sqrt{2}}{\left(i\sqrt{2} + 1 + i\right)\sqrt{2-\sqrt{2}}} + \frac{15x^4\left(\sqrt{2} + 1 + i\right)}{7\left(i\sqrt{2} + 1 + i\right)} + \frac{32x^3}{7\left(i\sqrt{2} + 1 + i\right)\sqrt{2-\sqrt{2}}} - \frac{15x^2\left(-\sqrt{2} - 1 \cdots\right.}{7\left(i\sqrt{2} + 1 + \cdots\right.}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \left(x^6 + \frac{\left(\frac{12}{7} + \frac{12i}{7}\right)x^5\sqrt{2}}{\left(i\sqrt{2} + 1 + i\right)\sqrt{2-\sqrt{2}}} + \frac{15x^4\left(\sqrt{2} + 1 + i\right)}{7\left(i\sqrt{2} + 1 + i\right)} + \frac{32x^3}{7\left(i\sqrt{2} + 1 + i\right)\sqrt{2-\sqrt{2}}} - \frac{15x^2\left(-\sqrt{2} - \cdots\right.}{7\left(i\sqrt{2} + 1 \cdots\right.}\right. \\ &= \left(x^6 + \frac{\left(\frac{12}{7} + \frac{12i}{7}\right)x^5\sqrt{2}}{\left(i\sqrt{2} + 1 + i\right)\sqrt{2-\sqrt{2}}} + \frac{15x^4\left(\sqrt{2} + 1 + i\right)}{7\left(i\sqrt{2} + 1 + i\right)} + \frac{32x^3}{7\left(i\sqrt{2} + 1 + i\right)\sqrt{2-\sqrt{2}}} - \frac{15x^2\left(-\sqrt{2} - \cdots\right.}{7\left(i\sqrt{2} + 1 \cdots\right.}\right. \\ &= \text{Expression too large to display} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-16x^7}{x^8+1}\,dx} \\ &= z_1 e^{\ln(x^8+1)} \\ &= z_1 \left(x^8+1\right) \end{aligned}$$

Which simplifies to

$$y_1 = \text{Expression too large to display}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-16x^7}{x^8+1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2\ln(x^8+1)}}{(y_1)^2}\,dx \\ &= y_1(\text{Expression too large to display}) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\text{Expression too large to display}) \\ &\quad + c_2(\text{Expression too large to display}(\text{Expression too large to display})) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 22

```
dsolve((x^8+1)*diff(diff(y(x),x),x)-16*x^7*diff(y(x),x)+72*y(x)*x^6 = 0,y(x),singsol=all
```

$$y = -\frac{7}{9}c_1 + c_1 x^8 + c_2 x^9 - \frac{9}{7}c_2 x$$

Mathematica DSolve solution

Solving time : 0.0 (sec)
Leaf size : 0

```
DSolve[{(1+x^8)*D[y[x],{x,2}]-16*x^7*D[y[x],x]+72*x^6*y[x]==0,{}},y[x],x,IncludeSingularSolu
```

Timed out

2.1.487 Problem 503

Internal problem ID [9659]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 503
Date solved : `Monday, January 27, 2025 at 06:12:14 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + x^5 y' + 6x^4 y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.410 (sec)

Writing the ode as

$$y'' + x^5 y' + 6x^4 y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x^5 \\ C &= 6x^4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4(x^6 - 14)}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4\left(x^6 - 14\right) \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4(x^6 - 14)}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.920: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 10 \\ &= -10 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -10 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -10$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{10}{2} = 5$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{5} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^5$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x^5}{2} - \frac{7}{2x} - \frac{49}{4x^7} - \frac{343}{4x^{13}} - \frac{12005}{16x^{19}} - \frac{117649}{16x^{25}} - \frac{2470629}{32x^{31}} - \frac{27176919}{32x^{37}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 5$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{5} a_i x^i \\ &= \frac{x^5}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^4 = x^4$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^{10}}{4}$$

This shows that the coefficient of x^4 in the above is 0. Now we need to find the coefficient of x^4 in r. How this is done depends on if $v = 0$ or not. Since $v = 5$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x^4 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^4(x^6 - 14)}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^{10} - \frac{7}{2}x^4\right) + (0) \\ &= \frac{1}{4}x^{10} - \frac{7}{2}x^4 \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{7}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{7}{2}\right) - (0) \\ &= -\frac{7}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x^5}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{7}{2}}{\frac{1}{2}} - 5\right) &&= -6 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{7}{2}}{\frac{1}{2}} - 5\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4(x^6 - 14)}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-10	$\frac{x^5}{2}$	-6	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x^5}{2}\right) \\ &= -\frac{x^5}{2} \\ &= -\frac{x^5}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{x^5}{2}\right)(1) + \left(\left(-\frac{5x^4}{2}\right) + \left(-\frac{x^5}{2}\right)^2 - \left(\frac{x^4(x^6-14)}{4}\right)\right) &= 0 \\ x^4 a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (x)\, e^{\int -\frac{x^5}{2} dx} \\ &= (x)\, e^{-\frac{x^6}{12}} \\ &= x\, e^{-\frac{x^6}{12}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^5}{1} dx} \\ &= z_1 e^{-\frac{x^6}{12}} \\ &= z_1 \left(e^{-\frac{x^6}{12}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^6}{6}} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^5}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{x^6}{6}}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{6^{5/6} (-1)^{1/6} \left(-\frac{6x^5 (-1)^{5/6} \Gamma\left(\frac{5}{6}\right)}{(-x^6)^{5/6}} + \frac{6\, 6^{1/6} (-1)^{5/6} \mathrm{e}^{\frac{x^6}{6}}}{x} + \frac{6x^5 (-1)^{5/6} \Gamma\left(\frac{5}{6}, -\frac{x^6}{6}\right)}{(-x^6)^{5/6}} \right)}{36} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-\frac{x^6}{6}} x \right) \\ &\quad + c_2 \left(\mathrm{e}^{-\frac{x^6}{6}} x \left(\frac{6^{5/6} (-1)^{1/6} \left(-\frac{6x^5 (-1)^{5/6} \Gamma\left(\frac{5}{6}\right)}{(-x^6)^{5/6}} + \frac{6\, 6^{1/6} (-1)^{5/6} \mathrm{e}^{\frac{x^6}{6}}}{x} + \frac{6x^5 (-1)^{5/6} \Gamma\left(\frac{5}{6}, -\frac{x^6}{6}\right)}{(-x^6)^{5/6}} \right)}{36} \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2} y(x) + x^5 \left(\frac{d}{dx} y(x) \right) + 6x^4 y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2} y(x)$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^4 \cdot y(x)$ to series expansion
    $x^4 \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+4}$
  ◦ Shift index using $k- >k-4$
    $x^4 \cdot y(x) = \sum_{k=4}^{\infty} a_{k-4} x^k$
  ◦ Convert $x^5 \cdot \left(\frac{d}{dx} y(x) \right)$ to series expansion
    $x^5 \cdot \left(\frac{d}{dx} y(x) \right) = \sum_{k=0}^{\infty} a_k k\, x^{k+4}$

- Shift index using $k- > k-4$

  $$x^5 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=4}^{\infty} a_{k-4}(k-4)\, x^k$$

- Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

  $$\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- > k+2$

  $$\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

  Rewrite ODE with series expansions

  $$20a_5x^3 + 12a_4x^2 + 6a_3x + 2a_2 + \left(\sum_{k=4}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_{k-4}(k+2)\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

  $[2a_2 = 0, 6a_3 = 0, 12a_4 = 0, 20a_5 = 0]$

- Solve for the dependent coefficient(s)

  $\{a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+2)\,(ka_{k+2} + a_{k-4} + a_{k+2}) = 0$

- Shift index using $k- > k+4$

  $(k+6)\,((k+4)\, a_{k+6} + a_k + a_{k+6}) = 0$

- Recursion relation that defines the series solution to the ODE

  $$\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+6} = -\frac{a_k}{k+5}, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.007` (sec)
Leaf size : 62

```
dsolve(diff(diff(y(x),x),x)+x^5*diff(y(x),x)+6*x^4*y(x) = 0,y(x),singsol=all)
```

$$y = -\frac{\left(-c_1 e^{-\frac{x^6}{6}} x - c_2 6^{1/6}\right) (-x^6)^{5/6} + x^6 c_2 e^{-\frac{x^6}{6}} \left(\Gamma\left(\frac{5}{6}\right) - \Gamma\left(\frac{5}{6}, -\frac{x^6}{6}\right)\right)}{(-x^6)^{5/6}}$$

Mathematica DSolve solution

Solving time : 0.096 (sec)
Leaf size : 53

```
DSolve[{D[y[x],{x,2}]+x^5*D[y[x],x]+6*x^4*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \rightarrow \frac{1}{36} e^{-\frac{x^6}{6}} \left(36 c_1 x - 6^{5/6} c_2 \sqrt[6]{-x^6} \Gamma\left(-\frac{1}{6}, -\frac{x^6}{6}\right)\right)$$

2.1.488 Problem 504

Internal problem ID [9660]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 504
Date solved : `Monday, January 27, 2025 at 06:12:14 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(1+3x)\,y''+xy'+2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 30.653 (sec)

Writing the ode as

$$(1+3x)\,y''+xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=1+3x\\ B&=x \\ C&=2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{x^2-24x-6}{4\,(1+3x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=x^2-24x-6\\ t&=4(1+3x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x^2-24x-6}{4\,(1+3x)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.922: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(1+3x)^2$. There is a pole at $x = -\frac{1}{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{36} + \frac{19}{324\left(x+\frac{1}{3}\right)^2} - \frac{37}{54\left(x+\frac{1}{3}\right)}$$

For the pole at $x = -\frac{1}{3}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{19}{324}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{19}{18} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{18} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{6} - \frac{37}{18x} - \frac{319}{27x^2} - \frac{11831}{81x^3} - \frac{2157901}{972x^4} - \frac{110035199}{2916x^5} - \frac{1501983319}{2187x^6} - \frac{85889060456}{6561x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{6} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{36}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 24x - 6}{36x^2 + 24x + 4} \\ &= Q + \frac{R}{36x^2 + 24x + 4} \\ &= \left(\frac{1}{36}\right) + \left(\frac{-\frac{74x}{3} - \frac{55}{9}}{36x^2 + 24x + 4}\right) \\ &= \frac{1}{36} + \frac{-\frac{74x}{3} - \frac{55}{9}}{36x^2 + 24x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is $-\frac{74}{3}$. Dividing this by leading coefficient in t which is 36 gives $-\frac{37}{54}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{37}{54}\right) - (0) \\ &= -\frac{37}{54} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{6} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a}-v\right) &= \frac{1}{2}\left(\frac{-\frac{37}{54}}{\frac{1}{6}}-0\right) &= -\frac{37}{18} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a}-v\right) = \frac{1}{2}\left(-\frac{-\frac{37}{54}}{\frac{1}{6}}-0\right) = \frac{37}{18}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2-24x-6}{4\left(1+3x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$-\frac{1}{3}$	2	0	$\frac{19}{18}$	$-\frac{1}{18}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{6}$	$-\frac{37}{18}$	$\frac{37}{18}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{37}{18}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{37}{18} - \left(\frac{19}{18}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{19}{18\left(x+\frac{1}{3}\right)} + (-)\left(\frac{1}{6}\right) \\
&= \frac{19}{18\left(x+\frac{1}{3}\right)} - \frac{1}{6} \\
&= -\frac{-6+x}{2\left(1+3x\right)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p''+2\omega p'+\left(\omega'+\omega^2-r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{19}{18\left(x+\frac{1}{3}\right)}-\frac{1}{6}\right)(1)+\left(\left(-\frac{19}{18\left(x+\frac{1}{3}\right)^{2}}\right)+\left(\frac{19}{18\left(x+\frac{1}{3}\right)}-\frac{1}{6}\right)^{2}-\left(\frac{x^{2}-24x-6}{4\left(1+3x\right)^{2}}\right)\right)=0$$

$$\frac{a_0+6}{1+3x}=0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0=-6\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x)=-6+x$$

Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x)&=pe^{\int \omega\,dx}\\
&=(-6+x)\,\mathrm{e}^{\int\left(\frac{19}{18\left(x+\frac{1}{3}\right)}-\frac{1}{6}\right)dx}\\
&=(-6+x)\,\mathrm{e}^{-\frac{x}{6}+\frac{19\ln(1+3x)}{18}}\\
&=(-6+x)\,(1+3x)^{19/18}\,\mathrm{e}^{-\frac{x}{6}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1&=z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\
&=z_1e^{-\int \frac{1}{2}\frac{x}{1+3x}\,dx}\\
&=z_1e^{-\frac{x}{6}+\frac{\ln(1+3x)}{18}}\\
&=z_1\left((1+3x)^{1/18}\,\mathrm{e}^{-\frac{x}{6}}\right)
\end{aligned}$$

Which simplifies to

$$y_1=(1+3x)^{10/9}\,\mathrm{e}^{-\frac{x}{3}}(-6+x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int\frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2&=y_1\int\frac{e^{\int -\frac{x}{1+3x}\,dx}}{(y_1)^2}\,dx\\
&=y_1\int\frac{e^{-\frac{x}{3}+\frac{\ln(1+3x)}{9}}}{(y_1)^2}\,dx\\
&=y_1\left(\int\frac{\mathrm{e}^{-\frac{x}{3}+\frac{\ln(1+3x)}{9}}\,\mathrm{e}^{\frac{2x}{3}}}{(1+3x)^{20/9}\,(-6+x)^2}dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y&=c_1y_1+c_2y_2\\
&=c_1\left((1+3x)^{10/9}\,\mathrm{e}^{-\frac{x}{3}}(-6+x)\right)\\
&\quad+c_2\left((1+3x)^{10/9}\,\mathrm{e}^{-\frac{x}{3}}(-6+x)\left(\int\frac{\mathrm{e}^{-\frac{x}{3}+\frac{\ln(1+3x)}{9}}\,\mathrm{e}^{\frac{2x}{3}}}{(1+3x)^{20/9}\,(-6+x)^2}dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(3x+1)\left(\frac{d^2}{dx^2}y(x)\right)+x\left(\frac{d}{dx}y(x)\right)+2y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{2y(x)}{3x+1}-\frac{x\left(\frac{d}{dx}y(x)\right)}{3x+1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x)+\frac{x\left(\frac{d}{dx}y(x)\right)}{3x+1}+\frac{2y(x)}{3x+1}=0$

□ Check to see if $x_0=-\frac{1}{3}$ is a regular singular point

- ◦ Define functions

$\left[P_2(x)=\frac{x}{3x+1}, P_3(x)=\frac{2}{3x+1}\right]$

- ◦ $\left(x+\frac{1}{3}\right)\cdot P_2(x)$ is analytic at $x=-\frac{1}{3}$

$\left(\left(x+\frac{1}{3}\right)\cdot P_2(x)\right)\bigg|_{x=-\frac{1}{3}}=-\frac{1}{9}$

- ◦ $\left(x+\frac{1}{3}\right)^2\cdot P_3(x)$ is analytic at $x=-\frac{1}{3}$

$\left(\left(x+\frac{1}{3}\right)^2\cdot P_3(x)\right)\bigg|_{x=-\frac{1}{3}}=0$

- ◦ $x=-\frac{1}{3}$is a regular singular point

Check to see if $x_0=-\frac{1}{3}$ is a regular singular point

$x_0=-\frac{1}{3}$

- Multiply by denominators

$(3x+1)\left(\frac{d^2}{dx^2}y(x)\right)+x\left(\frac{d}{dx}y(x)\right)+2y(x)=0$

- Change variables using $x=u-\frac{1}{3}$ so that the regular singular point is at $u=0$

$3u\left(\frac{d^2}{du^2}y(u)\right)+\left(u-\frac{1}{3}\right)\left(\frac{d}{du}y(u)\right)+2y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ◦ Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-1}$

- ◦ Shift index using $k->k+1$

$u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$

Rewrite ODE with series expansions

$\frac{a_0r(-10+9r)u^{-1+r}}{3}+\left(\sum_{k=0}^{\infty}\left(\frac{a_{k+1}(k+1+r)(9k-1+9r)}{3}+a_k(k+r+2)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $\frac{r(-10+9r)}{3} = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{0, \frac{10}{9}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $3(k+1+r)\left(k-\frac{1}{9}+r\right)a_{k+1} + a_k(k+r+2) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = -\frac{3a_k(k+r+2)}{(k+1+r)(9k-1+9r)}$
- Recursion relation for $r = 0$
  $a_{k+1} = -\frac{3a_k(k+2)}{(k+1)(9k-1)}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = -\frac{3a_k(k+2)}{(k+1)(9k-1)}\right]$
- Revert the change of variables $u = x + \frac{1}{3}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k \left(x+\frac{1}{3}\right)^k, a_{k+1} = -\frac{3a_k(k+2)}{(k+1)(9k-1)}\right]$
- Recursion relation for $r = \frac{10}{9}$
  $a_{k+1} = -\frac{3a_k\left(k+\frac{28}{9}\right)}{\left(k+\frac{19}{9}\right)(9k+9)}$
- Solution for $r = \frac{10}{9}$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{10}{9}}, a_{k+1} = -\frac{3a_k\left(k+\frac{28}{9}\right)}{\left(k+\frac{19}{9}\right)(9k+9)}\right]$
- Revert the change of variables $u = x + \frac{1}{3}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k \left(x+\frac{1}{3}\right)^{k+\frac{10}{9}}, a_{k+1} = -\frac{3a_k\left(k+\frac{28}{9}\right)}{\left(k+\frac{19}{9}\right)(9k+9)}\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k \left(x+\frac{1}{3}\right)^k\right) + \left(\sum_{k=0}^{\infty} b_k \left(x+\frac{1}{3}\right)^{k+\frac{10}{9}}\right), a_{k+1} = -\frac{3a_k(k+2)}{(k+1)(9k-1)}, b_{k+1} = -\frac{3b_k\left(k+\frac{28}{9}\right)}{\left(k+\frac{19}{9}\right)(9k+9)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
```

```
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returnin
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.033 (sec)
Leaf size : 62

```
dsolve((3*x+1)*diff(diff(y(x),x),x)+diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{(x-6)\left(x+\frac{1}{3}\right)c_1 \mathrm{e}^{-\frac{x}{3}}\left(\Gamma\left(-\frac{1}{9}\right)+\frac{10\Gamma\left(-\frac{10}{9},-\frac{x}{3}-\frac{1}{9}\right)}{9}\right)\left(-\frac{x}{3}-\frac{1}{9}\right)^{1/9}}{9} + 3c_2(x-6)\left(x+\frac{1}{3}\right)\mathrm{e}^{-\frac{x}{3}}\left(\frac{x}{3}+\frac{1}{9}\right)^{1/9} - \frac{10c_1 \mathrm{e}^{\frac{1}{9}}}{9}$$

Mathematica DSolve solution

Solving time : 0.393 (sec)
Leaf size : 103

```
DSolve[{(1+3*x)*D[y[x],{x,2}]+x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to (x-6)\exp\left(\int_1^x \frac{6-K[1]}{6K[1]+2}dK[1] - \frac{1}{2}\int_1^x \frac{K[2]}{3K[2]+1}dK[2]\right)\left(c_2\int_1^x \frac{\exp\left(-2\int_1^{K[3]}\frac{6-K[1]}{6K[1]+2}dK[1]\right)}{(K[3]-6)^2}dK[3]+c_1\right)$$

2.1.489 Problem 505

Internal problem ID [9661]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 505
Date solved : `Monday, January 27, 2025 at 06:12:46 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(3x^2+x+1\right)y''+(2+15x)\,y'+12y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.753 (sec)

Writing the ode as

$$\left(3x^2+x+1\right)y''+(2+15x)\,y'+12y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^2+x+1 \\ B &= 2+15x \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-9x^2-12x-18}{4\left(3x^2+x+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9x^2-12x-18 \\ t &= 4\left(3x^2+x+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-9x^2-12x-18}{4\left(3x^2+x+1\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.924: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(3x^2 + x + 1)^2$. There is a pole at $x = -\frac{1}{6} + \frac{i\sqrt{11}}{6}$ of order 2. There is a pole at $x = -\frac{1}{6} - \frac{i\sqrt{11}}{6}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{\frac{27}{88} + \frac{3i\sqrt{11}}{88}}{\left(x + \frac{1}{6} - \frac{i\sqrt{11}}{6}\right)^2} + \frac{\frac{27}{88} - \frac{3i\sqrt{11}}{88}}{\left(x + \frac{1}{6} + \frac{i\sqrt{11}}{6}\right)^2} + \frac{57i\sqrt{11}}{242\left(x + \frac{1}{6} - \frac{i\sqrt{11}}{6}\right)} - \frac{57i\sqrt{11}}{242\left(x + \frac{1}{6} + \frac{i\sqrt{11}}{6}\right)}$$

For the pole at $x = -\frac{1}{6} + \frac{i\sqrt{11}}{6}$ let b be the coefficient of $\frac{1}{\left(x + \frac{1}{6} - \frac{i\sqrt{11}}{6}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{27}{88} + \frac{3i\sqrt{11}}{88}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} + \frac{\sqrt{1078 + 66i\sqrt{11}}}{44} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} - \frac{\sqrt{1078 + 66i\sqrt{11}}}{44} \end{aligned}$$

For the pole at $x=-\frac{1}{6}-\frac{i\sqrt{11}}{6}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{6}+\frac{i\sqrt{11}}{6}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{27}{88}-\frac{3i\sqrt{11}}{88}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2}+\frac{\sqrt{1078-66i\sqrt{11}}}{44} \\
\alpha_c^- &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}-\frac{\sqrt{1078-66i\sqrt{11}}}{44}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty=0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r=\frac{s}{t}=\frac{-9x^2-12x-18}{4\left(3x^2+x+1\right)^2}
$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-9x^2-12x-18}{4\left(3x^2+x+1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{6}+\frac{i\sqrt{11}}{6}$	2	0	$\frac{1}{2}+\frac{\sqrt{1078+66i\sqrt{11}}}{44}$	$\frac{1}{2}-\frac{\sqrt{1078+66i\sqrt{11}}}{44}$
$-\frac{1}{6}-\frac{i\sqrt{11}}{6}$	2	0	$\frac{1}{2}+\frac{\sqrt{1078-66i\sqrt{11}}}{44}$	$\frac{1}{2}-\frac{\sqrt{1078-66i\sqrt{11}}}{44}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{1}{2}-\left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{\frac{1}{2}-\frac{\sqrt{1078+66i\sqrt{11}}}{44}}{x+\frac{1}{6}-\frac{i\sqrt{11}}{6}} + \frac{\frac{1}{2}-\frac{\sqrt{1078-66i\sqrt{11}}}{44}}{x+\frac{1}{6}+\frac{i\sqrt{11}}{6}} + (-)(0) \\
&= \frac{\frac{1}{2}-\frac{\sqrt{1078+66i\sqrt{11}}}{44}}{x+\frac{1}{6}-\frac{i\sqrt{11}}{6}} + \frac{\frac{1}{2}-\frac{\sqrt{1078-66i\sqrt{11}}}{44}}{x+\frac{1}{6}+\frac{i\sqrt{11}}{6}} \\
&= -\frac{3x}{6x^2+2x+2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{\frac{1}{2}-\frac{\sqrt{1078+66i\sqrt{11}}}{44}}{x+\frac{1}{6}-\frac{i\sqrt{11}}{6}} + \frac{\frac{1}{2}-\frac{\sqrt{1078-66i\sqrt{11}}}{44}}{x+\frac{1}{6}+\frac{i\sqrt{11}}{6}}\right)(1) + \left(\left(-\frac{\frac{1}{2}-\frac{\sqrt{1078+66i\sqrt{11}}}{44}}{\left(x+\frac{1}{6}-\frac{i\sqrt{11}}{6}\right)^2} - \frac{\frac{1}{2}-\frac{\sqrt{1078-66i\sqrt{11}}}{44}}{\left(x+\frac{1}{6}+\frac{i\sqrt{11}}{6}\right)^2}\right) + \left(\frac{\frac{1}{2}-}{x}\right.\right.
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \,dx} \\
&= (x)\, e^{\int \left(\frac{\frac{1}{2}-\frac{\sqrt{1078+66i\sqrt{11}}}{44}}{x+\frac{1}{6}-\frac{i\sqrt{11}}{6}} + \frac{\frac{1}{2}-\frac{\sqrt{1078-66i\sqrt{11}}}{44}}{x+\frac{1}{6}+\frac{i\sqrt{11}}{6}}\right) dx} \\
&= (x)\, e^{-\frac{\ln(36x^2+12x+12)}{4} + \frac{\sqrt{11}\arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{22}} \\
&= \frac{x\sqrt{2}\,3^{3/4} e^{\frac{\sqrt{11}\arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{22}}}{6\,(3x^2+x+1)^{1/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2+15x}{3x^2+x+1}\,dx} \\
&= z_1 e^{-\frac{5\ln(3x^2+x+1)}{4} + \frac{\sqrt{11}\arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{22}} \\
&= z_1 \left(\frac{e^{\frac{\sqrt{11}\arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{22}}}{(3x^2+x+1)^{5/4}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{\frac{\sqrt{11}\,\arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{11}} x\sqrt{2}\,3^{3/4}}{6\left(3x^2+x+1\right)^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2+15x}{3x^2+x+1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{5\ln\left(3x^2+x+1\right)}{2}+\frac{\sqrt{11}\,\arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{11}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{2\,\mathrm{e}^{-\frac{5\ln\left(3x^2+x+1\right)}{2}+\frac{\sqrt{11}\,\arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{11}}\left(3x^2+x+1\right)^3 \mathrm{e}^{-\frac{2\sqrt{11}\,\arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{11}}\sqrt{3}}{x^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{\frac{\sqrt{11}\,\arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{11}} x\sqrt{2}\,3^{3/4}}{6\left(3x^2+x+1\right)^{3/2}}\right) \\
&\quad + c_2 \left(\frac{\mathrm{e}^{\frac{\sqrt{11}\,\arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{11}} x\sqrt{2}\,3^{3/4}}{6\left(3x^2+x+1\right)^{3/2}} \left(\int \frac{2\,\mathrm{e}^{-\frac{5\ln\left(3x^2+x+1\right)}{2}+\frac{\sqrt{11}\,\arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{11}}\left(3x^2+x+1\right)^3 \mathrm{e}^{-\frac{2\sqrt{11}\,\arctan\left(\frac{(6x-}{11}}}{x^2}\right.\right.
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(3x^2+x+1\right)\left(\frac{d^2}{dx^2}y(x)\right)+(2+15x)\left(\frac{d}{dx}y(x)\right)+12y(x)=0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x)=-\frac{12y(x)}{3x^2+x+1}-\frac{(2+15x)\left(\frac{d}{dx}y(x)\right)}{3x^2+x+1}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x)+\frac{(2+15x)\left(\frac{d}{dx}y(x)\right)}{3x^2+x+1}+\frac{12y(x)}{3x^2+x+1}=0$

☐ Check to see if x_0 is a regular singular point
  - Define functions
    $\left[P_2(x)=\frac{2+15x}{3x^2+x+1}, P_3(x)=\frac{12}{3x^2+x+1}\right]$
  - $\left(\frac{\mathrm{I}\sqrt{11}}{6}+x+\frac{1}{6}\right)\cdot P_2(x)$ is analytic at $x=-\frac{1}{6}-\frac{\mathrm{I}\sqrt{11}}{6}$

$\left(\left(\frac{I\sqrt{11}}{6}+x+\frac{1}{6}\right)\cdot P_2(x)\right)\Big|_{x=-\frac{1}{6}-\frac{I\sqrt{11}}{6}}=0$

- $\left(\frac{I\sqrt{11}}{6}+x+\frac{1}{6}\right)^2\cdot P_3(x)$ is analytic at $x=-\frac{1}{6}-\frac{I\sqrt{11}}{6}$

  $\left(\left(\frac{I\sqrt{11}}{6}+x+\frac{1}{6}\right)^2\cdot P_3(x)\right)\Big|_{x=-\frac{1}{6}-\frac{I\sqrt{11}}{6}}=0$

- $x=-\frac{1}{6}-\frac{I\sqrt{11}}{6}$ is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-\frac{1}{6}-\frac{I\sqrt{11}}{6}$

- Multiply by denominators

  $(3x^2+x+1)\left(\frac{d^2}{dx^2}y(x)\right)+(2+15x)\left(\frac{d}{dx}y(x)\right)+12y(x)=0$

- Change variables using $x=u-\frac{1}{6}-\frac{I\sqrt{11}}{6}$ so that the regular singular point is at $u=0$

  $(3u^2-Iu\sqrt{11})\left(\frac{d^2}{du^2}y(u)\right)+\left(-\frac{1}{2}+15u-\frac{5I\sqrt{11}}{2}\right)\left(\frac{d}{du}y(u)\right)+12y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $\frac{I\sqrt{11}\,r\left(I\sqrt{11}-33-22r\right)a_0u^{-1+r}}{22}+\left(\sum_{k=0}^{\infty}\left(\frac{I\sqrt{11}\,(k+1+r)\left(I\sqrt{11}-22k-55-22r\right)a_{k+1}}{22}+3a_k(k+r+2)^2\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $\frac{I}{22}\sqrt{11}\,r\left(I\sqrt{11}-33-22r\right)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{0,-\frac{3}{2}+\frac{I\sqrt{11}}{22}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $3a_k(k+r+2)^2-(k+1+r)\,a_{k+1}\left(\frac{1}{2}+I\left(k+r+\frac{5}{2}\right)\sqrt{11}\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{6a_k\,(k^2+2kr+r^2+4k+4r+4)}{2I\sqrt{11}\,k^2+4Ikr\sqrt{11}+2I\sqrt{11}\,r^2+7Ik\sqrt{11}+7Ir\sqrt{11}+5I\sqrt{11}+k+r+1}$

- Recursion relation for $r=0$

  $a_{k+1}=\frac{6a_k\,(k^2+4k+4)}{2I\sqrt{11}\,k^2+1+7Ik\sqrt{11}+5I\sqrt{11}+k}$

- Solution for $r=0$

  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^k, a_{k+1}=\frac{6a_k\,(k^2+4k+4)}{2I\sqrt{11}\,k^2+1+7Ik\sqrt{11}+5I\sqrt{11}+k}\right]$

- Revert the change of variables $u = \frac{I\sqrt{11}}{6} + x + \frac{1}{6}$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k \left(\frac{I\sqrt{11}}{6} + x + \frac{1}{6}\right)^k, a_{k+1} = \frac{6a_k(k^2+4k+4)}{2I\sqrt{11}\,k^2+1+7Ik\sqrt{11}+5I\sqrt{11}+k}\right]$$

- Recursion relation for $r = -\frac{3}{2} + \frac{I\sqrt{11}}{22}$

$$a_{k+1} = \frac{6a_k\left(k^2+2k\left(-\frac{3}{2}+\frac{I\sqrt{11}}{22}\right)+\left(-\frac{3}{2}+\frac{I\sqrt{11}}{22}\right)^2+4k-2+\frac{2I\sqrt{11}}{11}\right)}{2I\sqrt{11}\,k^2+4Ik\left(-\frac{3}{2}+\frac{I\sqrt{11}}{22}\right)\sqrt{11}+2I\sqrt{11}\left(-\frac{3}{2}+\frac{I\sqrt{11}}{22}\right)^2+7Ik\sqrt{11}+7I\left(-\frac{3}{2}+\frac{I\sqrt{11}}{22}\right)\sqrt{11}+\frac{111I\sqrt{11}}{22}+k-\frac{1}{2}}$$

- Solution for $r = -\frac{3}{2} + \frac{I\sqrt{11}}{22}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{3}{2}+\frac{I\sqrt{11}}{22}}, a_{k+1} = \frac{6a_k\left(k^2+2k\left(-\frac{3}{2}+\frac{I\sqrt{11}}{22}\right)+\left(-\frac{3}{2}+\frac{I\sqrt{11}}{22}\right)^2+4k-2+\frac{2I\sqrt{11}}{11}\right)}{2I\sqrt{11}\,k^2+4Ik\left(-\frac{3}{2}+\frac{I\sqrt{11}}{22}\right)\sqrt{11}+2I\sqrt{11}\left(-\frac{3}{2}+\frac{I\sqrt{11}}{22}\right)^2+7Ik\sqrt{11}+7I\left(-\frac{3}{2}+\frac{I\sqrt{11}}{22}\right)\sqrt{11}}\right.$$

- Revert the change of variables $u = \frac{I\sqrt{11}}{6} + x + \frac{1}{6}$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k \left(\frac{I\sqrt{11}}{6} + x + \frac{1}{6}\right)^{k-\frac{3}{2}+\frac{I\sqrt{11}}{22}}, a_{k+1} = \frac{6a_k\left(k^2+2k\left(-\frac{3}{2}+\frac{I\sqrt{11}}{22}\right)+\left(-\frac{3}{2}+\frac{I\sqrt{11}}{22}\right)^2+4\right.}{2I\sqrt{11}\,k^2+4Ik\left(-\frac{3}{2}+\frac{I\sqrt{11}}{22}\right)\sqrt{11}+2I\sqrt{11}\left(-\frac{3}{2}+\frac{I\sqrt{11}}{22}\right)^2+7Ik\sqrt{11}+}\right.$$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k \left(\frac{I\sqrt{11}}{6} + x + \frac{1}{6}\right)^k\right) + \left(\sum_{k=0}^{\infty} b_k \left(\frac{I\sqrt{11}}{6} + x + \frac{1}{6}\right)^{k-\frac{3}{2}+\frac{I\sqrt{11}}{22}}\right), a_{k+1} = \frac{6a_k(k^2+4k+}{2I\sqrt{11}\,k^2+1+7Ik\sqrt{11}}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special functio
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.218 (sec)
Leaf size : 163

```
dsolve((3*x^2+x+1)*diff(diff(y(x),x),x)+(2+15*x)*diff(y(x),x)+12*y(x) = 0,y(x),singsol=a
```

$$y = \frac{\left((-36x^2-12x-12)^{-\frac{1}{4}+\frac{i\sqrt{11}}{44}}\left(i\sqrt{11}-6x-1\right)^{3/2} c_1 \operatorname{hypergeom}\left(\left[\frac{1}{2}+\frac{i\sqrt{11}}{22}, \frac{1}{2}+\frac{i\sqrt{11}}{22}\right], \left[-\frac{1}{2}+\frac{i\sqrt{11}}{22}\right], \frac{1}{2}+\frac{i}{}\right.\right.}{}$$

Mathematica DSolve solution

Solving time : 1.073 (sec)
Leaf size : 113

```
DSolve[{(1+x+3*x^2)*D[y[x],{x,2}]+(2+15*x)*D[y[x],x]+12*y[x]==0,{}},y[x],x,IncludeSingularSolu
```

$$y(x) \to x \exp\left(\int_1^x -\frac{3K[1]}{6K[1]^2+2K[1]+2}dK[1] - \frac{1}{2}\int_1^x \frac{15K[2]+2}{3K[2]^2+K[2]+1}dK[2]\right)\left(c_2\int_1^x \frac{\exp\left(-2\int_1^{K[3]} -\frac{3K[1]}{6K[1]^2+2K[1]+2}dK[1]\right)}{K[3]^2}dK[3] + c_1\right)$$

2.1.490 Problem 506

Internal problem ID [9662]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 506
Date solved : `Monday, January 27, 2025 at 06:12:47 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(2+x)\,y''+(1+x)\,y'+3y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.297 (sec)

Writing the ode as

$$(2+x)\,y''+(1+x)\,y'+3y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=2+x\\ B&=1+x\\ C&=3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{x^2-10x-21}{4\left(2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=x^2-10x-21\\ t&=4(2+x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x^2-10x-21}{4\left(2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.926: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2+x)^2$. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4\,(2+x)^2} - \frac{7}{2\,(2+x)}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(2+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{7}{2x} - \frac{9}{2x^2} - \frac{97}{2x^3} - \frac{1291}{4x^4} - \frac{11103}{4x^5} - \frac{98061}{4x^6} - \frac{913053}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 10x - 21}{4x^2 + 16x + 16} \\ &= Q + \frac{R}{4x^2 + 16x + 16} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-14x - 25}{4x^2 + 16x + 16}\right) \\ &= \frac{1}{4} + \frac{-14x - 25}{4x^2 + 16x + 16} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -14. Dividing this by leading coefficient in t which is 4 gives $-\frac{7}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{7}{2}\right) - (0) \\ &= -\frac{7}{2} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{7}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{7}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{7}{2}}{\frac{1}{2}} - 0\right) &&= \frac{7}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 10x - 21}{4\left(2 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{7}{2}$	$\frac{7}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{7}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\
&= \frac{7}{2} - \left(\frac{3}{2}\right) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{3}{2\left(2 + x\right)} + (-)\left(\frac{1}{2}\right) \\
&= \frac{3}{2\left(2 + x\right)} - \frac{1}{2} \\
&= -\frac{-1 + x}{2\left(2 + x\right)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(2)+2\left(\frac{3}{2\,(2+x)}-\frac{1}{2}\right)(2x+a_1)+\left(\left(-\frac{3}{2\,(2+x)^2}\right)+\left(\frac{3}{2\,(2+x)}-\frac{1}{2}\right)^2-\left(\frac{x^2-10x-21}{4\,(2+x)^2}\right)\right)=0$$

$$\frac{(a_1+4)\,x+2a_0+a_1+4}{2+x}=0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0=0,a_1=-4\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x)=x^2-4x$$

Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\,dx} \\ &= \left(x^2-4x\right)\mathrm{e}^{\int \left(\frac{3}{2(2+x)}-\frac{1}{2}\right)dx} \\ &= \left(x^2-4x\right)\mathrm{e}^{-\frac{x}{2}+\frac{3\ln(2+x)}{2}} \\ &= x(x-4)\,(2+x)^{3/2}\,\mathrm{e}^{-\frac{x}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1+x}{2+x}\,dx} \\ &= z_1 e^{-\frac{x}{2}+\frac{\ln(2+x)}{2}} \\ &= z_1\left(\sqrt{2+x}\,\mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1=(2+x)^2\,\mathrm{e}^{-x}x(x-4)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1\int \frac{e^{\int -\frac{1+x}{2+x}\,dx}}{(y_1)^2}\,dx \\ &= y_1\int \frac{e^{-x+\ln(2+x)}}{(y_1)^2}\,dx \\ &= y_1\left(-\frac{11\,\mathrm{e}^{x}}{864\,(2+x)}-\frac{\mathrm{e}^{-2}\,\mathrm{Ei}_1\left(-2-x\right)}{48}-\frac{\mathrm{e}^{x}}{3456\,(x-4)}-\frac{\mathrm{e}^{x}}{128x}-\frac{\mathrm{e}^{x}}{288\,(2+x)^2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1y_1+c_2y_2 \\ &= c_1\left((2+x)^2\,\mathrm{e}^{-x}x(x-4)\right)+c_2\left((2+x)^2\,\mathrm{e}^{-x}x(x-4)\left(-\frac{11\,\mathrm{e}^{x}}{864\,(2+x)}-\frac{\mathrm{e}^{-2}\,\mathrm{Ei}_1\left(-2-x\right)}{48}\right.\right. \\ &\qquad \left.\left.-\frac{\mathrm{e}^{x}}{3456\,(x-4)}-\frac{\mathrm{e}^{x}}{128x}-\frac{\mathrm{e}^{x}}{288\,(2+x)^2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x+2)\left(\frac{d^2}{dx^2}y(x)\right)+(x+1)\left(\frac{d}{dx}y(x)\right)+3y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{3y(x)}{x+2}-\frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{x+2}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{x+2}+\frac{3y(x)}{x+2}=0$

- ☐ Check to see if $x_0=-2$ is a regular singular point
  - Define functions

    $\left[P_2(x)=\frac{x+1}{x+2}, P_3(x)=\frac{3}{x+2}\right]$
  - $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

    $\left((x+2)\cdot P_2(x)\right)\bigg|_{x=-2}=-1$
  - $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

    $\left((x+2)^2\cdot P_3(x)\right)\bigg|_{x=-2}=0$
  - $x=-2$is a regular singular point

  Check to see if $x_0=-2$ is a regular singular point

  $x_0=-2$

- Multiply by denominators

  $(x+2)\left(\frac{d^2}{dx^2}y(x)\right)+(x+1)\left(\frac{d}{dx}y(x)\right)+3y(x)=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

  $u\left(\frac{d^2}{du^2}y(u)\right)+(u-1)\left(\frac{d}{du}y(u)\right)+3y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

    $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-1}$
  - Shift index using $k->k+1$

    $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)(k+r-1)+a_k(k+r+3)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+r-1) + a_k(k+r+3) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r+3)}{(k+1+r)(k+r-1)}$
- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{a_k(k+3)}{(k+1)(k-1)}$
- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 1$

  $a_{k+1} = -\frac{a_k(k+3)}{(k+1)(k-1)}$
- Recursion relation for $r = 2$

  $a_{k+1} = -\frac{a_k(k+5)}{(k+3)(k+1)}$
- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = -\frac{a_k(k+5)}{(k+3)(k+1)}\right]$
- Revert the change of variables $u = x+2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x+2)^{k+2}, a_{k+1} = -\frac{a_k(k+5)}{(k+3)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 59

```
dsolve((x+2)*diff(diff(y(x),x),x)+(x+1)*diff(y(x),x)+3*y(x) = 0,y(x),singsol=all)
```

$$y = x c_2 e^{-x-2}(x-4)(x+2)^2 \operatorname{Ei}_1(-x-2) + c_1 e^{-x} x(x-4)(x+2)^2 + c_2\left(x^3 - x^2 - 10x - 6\right)$$

Mathematica DSolve solution

Solving time : 0.496 (sec)
Leaf size : 106

```
DSolve[{(2+x)*D[y[x],{x,2}]+(1+x)*D[y[x],x]+3*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True
```

$$y(x) \to (x-4)x\exp\left(\int_1^x\left(\frac{3}{2(K[1]+2)}-\frac{1}{2}\right)dK[1] - \frac{1}{2}\int_1^x\frac{K[2]+1}{K[2]+2}dK[2]\right)\left(c_2\int_1^x\frac{\exp\left(-2\int_1^{K[3]}\left(\frac{3}{2(K[1]+2)}-\frac{1}{2}\right)dK[1]\right)}{(K[3]-4)^2K[3]^2}dK[3] + c_1\right)$$

2.1.491 Problem 507

Internal problem ID [9663]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 507
Date solved : `Monday, January 27, 2025 at 06:12:47 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(4+x)\,y''+(2+x)\,y'+2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.286 (sec)

Writing the ode as

$$(4+x)\,y''+(2+x)\,y'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=4+x\\ B&=2+x\\ C&=2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{x^2-4x-24}{4\,(4+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=x^2-4x-24\\ t&=4(4+x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x^2-4x-24}{4\,(4+x)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.928: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(4+x)^2$. There is a pole at $x = -4$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{(4+x)^2} - \frac{3}{4+x}$$

For the pole at $x = -4$ let b be the coefficient of $\frac{1}{(4+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{3}{x} + \frac{5}{x^2} - \frac{34}{x^3} + \frac{59}{x^4} - \frac{586}{x^5} + \frac{370}{x^6} - \frac{12484}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x - 24}{4x^2 + 32x + 64} \\ &= Q + \frac{R}{4x^2 + 32x + 64} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-12x - 40}{4x^2 + 32x + 64}\right) \\ &= \frac{1}{4} + \frac{-12x - 40}{4x^2 + 32x + 64} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -12. Dividing this by leading coefficient in t which is 4 gives -3. Now b can be found.

$$\begin{aligned} b &= (-3) - (0) \\ &= -3 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-3}{\frac{1}{2}} - 0\right) &= -3 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-3}{\frac{1}{2}} - 0\right) &= 3 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x - 24}{4\left(4+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-4	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	-3	3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 3$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= 3 - (2) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{2}{4+x} + (-)\left(\frac{1}{2}\right) \\ &= \frac{2}{4+x} - \frac{1}{2} \\ &= -\frac{x}{2\left(4+x\right)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{2}{4+x} - \frac{1}{2}\right)(1) + \left(\left(-\frac{2}{\left(4+x\right)^2}\right) + \left(\frac{2}{4+x} - \frac{1}{2}\right)^2 - \left(\frac{x^2-4x-24}{4\left(4+x\right)^2}\right)\right) = 0$$

$$\frac{a_0}{4+x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega \, dx} \\
&= (x) \, \mathrm{e}^{\int \left(\frac{2}{4+x} - \frac{1}{2}\right) dx} \\
&= (x) \, \mathrm{e}^{-\frac{x}{2} + 2\ln(4+x)} \\
&= x(4+x)^2 \, \mathrm{e}^{-\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2+x}{4+x} \, dx} \\
&= z_1 e^{-\frac{x}{2} + \ln(4+x)} \\
&= z_1 \left((4+x) \, \mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = (4+x)^3 \, \mathrm{e}^{-x} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2+x}{4+x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-x + 2\ln(4+x)}}{(y_1)^2} \, dx \\
&= y_1 \left(-\frac{\mathrm{e}^{-4} \operatorname{Ei}_1(-4-x)}{24} - \frac{5\,\mathrm{e}^{x}}{192\,(4+x)^2} - \frac{29\,\mathrm{e}^{x}}{768\,(4+x)} - \frac{\mathrm{e}^{x}}{256x} - \frac{\mathrm{e}^{x}}{48\,(4+x)^3} \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left((4+x)^3 \, \mathrm{e}^{-x} x\right) + c_2 \left((4+x)^3 \, \mathrm{e}^{-x} x \left(-\frac{\mathrm{e}^{-4} \operatorname{Ei}_1(-4-x)}{24} - \frac{5\,\mathrm{e}^{x}}{192\,(4+x)^2} - \frac{29\,\mathrm{e}^{x}}{768\,(4+x)} \right.\right. \\
&\qquad \left.\left. - \frac{\mathrm{e}^{x}}{256x} - \frac{\mathrm{e}^{x}}{48\,(4+x)^3} \right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x+4)\left(\frac{d^2}{dx^2}y(x)\right)+(x+2)\left(\frac{d}{dx}y(x)\right)+2y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{2y(x)}{x+4}-\frac{(x+2)\left(\frac{d}{dx}y(x)\right)}{x+4}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{(x+2)\left(\frac{d}{dx}y(x)\right)}{x+4}+\frac{2y(x)}{x+4}=0$

□ Check to see if $x_0=-4$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{x+2}{x+4}, P_3(x)=\frac{2}{x+4}\right]$

  ◦ $(x+4)\cdot P_2(x)$ is analytic at $x=-4$

    $\left((x+4)\cdot P_2(x)\right)\Big|_{x=-4}=-2$

  ◦ $(x+4)^2\cdot P_3(x)$ is analytic at $x=-4$

    $\left((x+4)^2\cdot P_3(x)\right)\Big|_{x=-4}=0$

  ◦ $x=-4$is a regular singular point

  Check to see if $x_0=-4$ is a regular singular point

  $x_0=-4$

- Multiply by denominators

  $(x+4)\left(\frac{d^2}{dx^2}y(x)\right)+(x+2)\left(\frac{d}{dx}y(x)\right)+2y(x)=0$

- Change variables using $x=u-4$ so that the regular singular point is at $u=0$

  $u\left(\frac{d^2}{du^2}y(u)\right)+(u-2)\left(\frac{d}{du}y(u)\right)+2y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  ◦ Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

    $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-1}$

  ◦ Shift index using $k->k+1$

    $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-3+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(k-2+r)+a_k(k+r+2)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 3\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k-2+r) + a_k(k+r+2) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r+2)}{(k+1+r)(k-2+r)}$

- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{a_k(k+2)}{(k+1)(k-2)}$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 2$

  $a_{k+1} = -\frac{a_k(k+2)}{(k+1)(k-2)}$

- Recursion relation for $r = 3$

  $a_{k+1} = -\frac{a_k(k+5)}{(k+4)(k+1)}$

- Solution for $r = 3$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+3}, a_{k+1} = -\frac{a_k(k+5)}{(k+4)(k+1)}\right]$

- Revert the change of variables $u = x + 4$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+4)^{k+3}, a_{k+1} = -\frac{a_k(k+5)}{(k+4)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 53

```
dsolve((x+4)*diff(diff(y(x),x),x)+(x+2)*diff(y(x),x)+2*y(x) = 0,y(x),singsol=all)
```

$$y = xc_2 e^{-x-4}(x+4)^3 \operatorname{Ei}_1(-x-4) + c_1 e^{-x} x(x+4)^3 + c_2(x^3 + 9x^2 + 22x + 6)$$

Mathematica DSolve solution

Solving time : 0.354 (sec)
Leaf size : 93

```
DSolve[{(4+x)*D[y[x],{x,2}]+(2+x)*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tru
```

$$y(x) \to x \exp\left(\int_1^x -\frac{K[1]}{2(K[1]+4)}dK[1] - \frac{1}{2}\int_1^x \frac{K[2]+2}{K[2]+4}dK[2]\right)\left(c_2\int_1^x \frac{\exp\left(-2\int_1^{K[3]} -\frac{K[1]}{2(K[1]+4)}dK[1]\right)}{K[3]^2}dK[3] + c_1\right)$$

2.1.492 Problem 508

Internal problem ID [9664]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 508
Date solved : `Monday, January 27, 2025 at 06:12:48 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(2x^2+3x\right)y''+10(1+x)\,y'+8y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.283 (sec)

Writing the ode as

$$\left(2x^2+3x\right)y''+(10x+10)\,y'+8y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2+3x \\ B &= 10x+10 \\ C &= 8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-x^2+6x+10}{\left(2x^2+3x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2+6x+10 \\ t &= \left(2x^2+3x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-x^2+6x+10}{\left(2x^2+3x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.930: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x^2 + 3x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{3}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{22}{27x} + \frac{10}{9x^2} - \frac{5}{36\left(x + \frac{3}{2}\right)^2} + \frac{22}{27\left(x + \frac{3}{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{10}{9}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{3} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{2}{3} \end{aligned}$$

For the pole at $x = -\frac{3}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{3}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{6} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 + 6x + 10}{(2x^2 + 3x)^2}$$

Since the gcd$(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 + 6x + 10}{(2x^2 + 3x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{3}$	$-\frac{2}{3}$
$-\frac{3}{2}$	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{2}{3x} + \frac{1}{6x + 9} + (-)(0) \\
&= -\frac{2}{3x} + \frac{1}{6x + 9} \\
&= -\frac{x + 2}{x(2x + 3)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{2}{3x} + \frac{1}{6x+9}\right)(1) + \left(\left(\frac{2}{3x^2} - \frac{1}{6\left(x+\frac{3}{2}\right)^2}\right) + \left(-\frac{2}{3x} + \frac{1}{6x+9}\right)^2 - \left(\frac{-x^2+6x+10}{\left(2x^2+3x\right)^2}\right)\right) = 0$$

$$\frac{-4+2a_0}{x\left(2x+3\right)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+2)\, e^{\int \left(-\frac{2}{3x} + \frac{1}{6x+9}\right) dx} \\
&= (x+2)\, e^{\frac{\ln(2x+3)}{6} - \frac{2\ln(x)}{3}} \\
&= \frac{(x+2)\,(2x+3)^{1/6}}{x^{2/3}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{10x+10}{2x^2+3x}\, dx} \\
&= z_1 e^{-\frac{5\ln(2x+3)}{6} - \frac{5\ln(x)}{3}} \\
&= z_1 \left(\frac{1}{\left(2x+3\right)^{5/6} x^{5/3}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x+2}{\left(2x+3\right)^{2/3} x^{7/3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{10x+10}{2x^2+3x}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{5\ln(2x+3)}{3} - \frac{10\ln(x)}{3}}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\int \frac{e^{-\frac{5\ln(2x+3)}{3} - \frac{10\ln(x)}{3}} \left(2x+3\right)^{4/3} x^{14/3}}{\left(x+2\right)^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1\left(\frac{x+2}{(2x+3)^{2/3}\,x^{7/3}}\right) + c_2\left(\frac{x+2}{(2x+3)^{2/3}\,x^{7/3}}\left(\int \frac{\mathrm{e}^{-\frac{5\ln(2x+3)}{3}-\frac{10\ln(x)}{3}}(2x+3)^{4/3}\,x^{14/3}}{(x+2)^2}dx\right)\right)$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(2x^2+3x)\left(\frac{d^2}{dx^2}y(x)\right) + 10(x+1)\left(\frac{d}{dx}y(x)\right) + 8y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{8y(x)}{x(2x+3)} - \frac{10(x+1)\left(\frac{d}{dx}y(x)\right)}{x(2x+3)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{10(x+1)\left(\frac{d}{dx}y(x)\right)}{x(2x+3)} + \frac{8y(x)}{x(2x+3)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{10(x+1)}{x(2x+3)}, P_3(x) = \frac{8}{x(2x+3)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = \frac{10}{3}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x(2x+3)\left(\frac{d^2}{dx^2}y(x)\right) + (10x+10)\left(\frac{d}{dx}y(x)\right) + 8y(x) = 0$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..1$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 1..2$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,x^{k+r-2+m}$

  ◦ Shift index using $k- >k+2-m$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(7+3r)\,x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(3k+10+3r) + 2a_k(k+r+2)^2\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(7+3r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{0, -\frac{7}{3}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)\,(3k+10+3r) + 2a_k(k+r+2)^2 = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = -\frac{2a_k(k+r+2)^2}{(k+1+r)(3k+10+3r)}$
- Recursion relation for $r = 0$
  $a_{k+1} = -\frac{2a_k(k+2)^2}{(k+1)(3k+10)}$
- Solution for $r = 0$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{2a_k(k+2)^2}{(k+1)(3k+10)}\right]$
- Recursion relation for $r = -\frac{7}{3}$
  $a_{k+1} = -\frac{2a_k\left(k-\frac{1}{3}\right)^2}{\left(k-\frac{4}{3}\right)(3k+3)}$
- Solution for $r = -\frac{7}{3}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{7}{3}}, a_{k+1} = -\frac{2a_k\left(k-\frac{1}{3}\right)^2}{\left(k-\frac{4}{3}\right)(3k+3)}\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{7}{3}}\right), a_{k+1} = -\frac{2a_k(k+2)^2}{(k+1)(3k+10)}, b_{k+1} = -\frac{2b_k\left(k-\frac{1}{3}\right)^2}{\left(k-\frac{4}{3}\right)(3k+3)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
```

```
      <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
   <- hypergeometric successful
<- special function solution successful
   -> Trying to convert hypergeometric functions to elementary form...
   <- elementary form for at least one hypergeometric solution is achieved - return
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.171 (sec)
Leaf size : 31

```
dsolve((2*x^2+3*x)*diff(diff(y(x),x),x)+10*(x+1)*diff(y(x),x)+8*y(x) = 0,y(x),singsol=
```

$$y = \frac{c_1(x+2)}{\left(1+\frac{2x}{3}\right)^{2/3} x^{7/3}} + c_2 \,\mathrm{hypergeom}\left([2,2], \left[\frac{10}{3}\right], -\frac{2x}{3}\right)$$

Mathematica DSolve solution

Solving time : 0.424 (sec)
Leaf size : 118

```
DSolve[{(3*x+2*x^2)*D[y[x],{x,2}]+10*(1+x)*D[y[x],x]+8*y[x]==0,{}},y[x],x,IncludeSingularSol
```

$$y(x) \to (x+2)\exp\left(\int_1^x -\frac{K[1]+2}{2K[1]^2+3K[1]}dK[1] - \frac{1}{2}\int_1^x \frac{10(K[2]+1)}{K[2](2K[2]+3)}dK[2]\right)\left(c_2\int_1^x \frac{\exp\left(-2\int_1^{K[3]} -\frac{K[1]+2}{2K[1]^2+3K[1]}dK[1]\right)}{(K[3]+2)^2}dK[3] + c_1\right)$$

2.1.493 Problem 509

Internal problem ID [9665]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 509
Date solved : `Monday, January 27, 2025 at 06:12:49 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - (6 - 7x)\,y' + 8y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.271 (sec)

Writing the ode as

$$x^2y'' + (-6 + 7x)\,y' + 8y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -6 + 7x \\ C &= 8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^2 - 60x + 36}{4x^4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3x^2 - 60x + 36 \\ t &= 4x^4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^2 - 60x + 36}{4x^4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.932: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^4$. There is a pole at $x = 0$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned} \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) \end{aligned}$$

The partial fraction decomposition of r is

$$r = -\frac{15}{x^3} + \frac{9}{x^4} + \frac{3}{4x^2}$$

There is pole in r at $x = 0$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{3}{x^2} - \frac{5}{2x} - \frac{11}{12} - \frac{55x}{72} - \frac{671x^2}{864} - \frac{4565x^3}{5184} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{3}{x^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^2}$ is

$$a = 3$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be -15. Therefore

$$\begin{aligned} b &= (-15) - (0) \\ &= -15 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{3}{x^2} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) = \frac{1}{2}\left(\frac{-15}{3} + 2\right) = -\frac{3}{2} \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) = \frac{1}{2}\left(-\frac{-15}{3} + 2\right) = \frac{7}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x^2 - 60x + 36}{4x^4}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3x^2 - 60x + 36}{4x^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	4	$\frac{3}{x^2}$	$-\frac{3}{2}$	$\frac{7}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= -\frac{1}{2} - \left(-\frac{3}{2}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\ &= \frac{3}{x^2} - \frac{3}{2x} + (-)\,(0) \\ &= \frac{3}{x^2} - \frac{3}{2x} \\ &= -\frac{3(-2+x)}{2x^2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{3}{x^2} - \frac{3}{2x}\right)(1) + \left(\left(-\frac{6}{x^3} + \frac{3}{2x^2}\right) + \left(\frac{3}{x^2} - \frac{3}{2x}\right)^2 - \left(\frac{3x^2 - 60x + 36}{4x^4}\right)\right) &= 0 \\ \frac{6 + 3a_0}{x^2} &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = -2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (-2+x)\,e^{\int \left(\frac{3}{x^2} - \frac{3}{2x}\right)dx} \\ &= (-2+x)\,e^{-\frac{3\ln(x)}{2} - \frac{3}{x}} \\ &= \frac{(-2+x)\,e^{-\frac{3}{x}}}{x^{3/2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-6+7x}{x^2}\,dx} \\ &= z_1 e^{-\frac{7\ln(x)}{2}-\frac{3}{x}} \\ &= z_1\left(\frac{\mathrm{e}^{-\frac{3}{x}}}{x^{7/2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-\frac{6}{x}}(-2+x)}{x^5}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-6+7x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-7\ln(x)-\frac{6}{x}}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{x^2\mathrm{e}^{\frac{6}{x}}}{2} + 7x\,\mathrm{e}^{\frac{6}{x}} + 54\,\mathrm{Ei}_1\left(-\frac{6}{x}\right) + \frac{12\,\mathrm{e}^{\frac{6}{x}}}{\frac{6}{x}-3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\mathrm{e}^{-\frac{6}{x}}(-2+x)}{x^5}\right) + c_2\left(\frac{\mathrm{e}^{-\frac{6}{x}}(-2+x)}{x^5}\left(\frac{x^2\mathrm{e}^{\frac{6}{x}}}{2} + 7x\,\mathrm{e}^{\frac{6}{x}} + 54\,\mathrm{Ei}_1\left(-\frac{6}{x}\right) + \frac{12\,\mathrm{e}^{\frac{6}{x}}}{\frac{6}{x}-3}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 50

```
dsolve(x^2*diff(diff(y(x),x),x)-(6-7*x)*diff(y(x),x)+8*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{108c_2 e^{-\frac{6}{x}}(x-2)\,\mathrm{Ei}_1\left(-\frac{6}{x}\right) + c_1(x-2)\,e^{-\frac{6}{x}} + xc_2(x^2+12x-36)}{x^5}$$

Mathematica DSolve solution

Solving time : 0.296 (sec)
Leaf size : 55

```
DSolve[{x^2*D[y[x],{x,2}]-(6-7*x)*D[y[x],x]+8*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to \frac{e^{5-\frac{6}{x}}(x-2)\left(c_2 \int_1^x \frac{e^{\frac{6}{K[1]}-3}K[1]^3}{(K[1]-2)^2}dK[1] + c_1\right)}{x^5}$$

2.1.494 Problem 510

Internal problem ID [9666]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 510
Date solved : `Monday, January 27, 2025 at 06:12:50 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(2x^2+x+1\right)y''+(1+7x)\,y'+2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.982 (sec)

Writing the ode as

$$\left(2x^2+x+1\right)y''+(1+7x)\,y'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2+x+1\\ B &= 1+7x\\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t}\\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{5x^2-2x+5}{4\left(2x^2+x+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 5x^2-2x+5\\ t &= 4\left(2x^2+x+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{5x^2-2x+5}{4\left(2x^2+x+1\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.933: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 + x + 1)^2$. There is a pole at $x = -\frac{1}{4} + \frac{i\sqrt{7}}{4}$ of order 2. There is a pole at $x = -\frac{1}{4} - \frac{i\sqrt{7}}{4}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{-\frac{29}{224} + \frac{9i\sqrt{7}}{224}}{\left(x + \frac{1}{4} - \frac{i\sqrt{7}}{4}\right)^2} + \frac{-\frac{29}{224} - \frac{9i\sqrt{7}}{224}}{\left(x + \frac{1}{4} + \frac{i\sqrt{7}}{4}\right)^2} - \frac{8i\sqrt{7}}{49\left(x + \frac{1}{4} - \frac{i\sqrt{7}}{4}\right)} + \frac{8i\sqrt{7}}{49\left(x + \frac{1}{4} + \frac{i\sqrt{7}}{4}\right)}$$

For the pole at $x = -\frac{1}{4} + \frac{i\sqrt{7}}{4}$ let b be the coefficient of $\frac{1}{\left(x + \frac{1}{4} - \frac{i\sqrt{7}}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{29}{224} + \frac{9i\sqrt{7}}{224}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} + \frac{3\sqrt{42 + 14i\sqrt{7}}}{56} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} - \frac{3\sqrt{42 + 14i\sqrt{7}}}{56} \end{aligned}$$

For the pole at $x = -\frac{1}{4} - \frac{i\sqrt{7}}{4}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{4}+\frac{i\sqrt{7}}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{29}{224} - \frac{9i\sqrt{7}}{224}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{3\sqrt{42-14i\sqrt{7}}}{56} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{3\sqrt{42-14i\sqrt{7}}}{56}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{5x^2 - 2x + 5}{4\left(2x^2 + x + 1\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{5x^2 - 2x + 5}{4\left(2x^2 + x + 1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{4} + \frac{i\sqrt{7}}{4}$	2	0	$\frac{1}{2} + \frac{3\sqrt{42+14i\sqrt{7}}}{56}$	$\frac{1}{2} - \frac{3\sqrt{42+14i\sqrt{7}}}{56}$
$-\frac{1}{4} - \frac{i\sqrt{7}}{4}$	2	0	$\frac{1}{2} + \frac{3\sqrt{42-14i\sqrt{7}}}{56}$	$\frac{1}{2} - \frac{3\sqrt{42-14i\sqrt{7}}}{56}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{5}{4} - \left(\frac{1}{4}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{\frac{1}{2}-\frac{3\sqrt{42+14i\sqrt{7}}}{56}}{x+\frac{1}{4}-\frac{i\sqrt{7}}{4}} + \frac{\frac{1}{2}-\frac{3\sqrt{42-14i\sqrt{7}}}{56}}{x+\frac{1}{4}+\frac{i\sqrt{7}}{4}} + (0) \\
&= \frac{\frac{1}{2}-\frac{3\sqrt{42+14i\sqrt{7}}}{56}}{x+\frac{1}{4}-\frac{i\sqrt{7}}{4}} + \frac{\frac{1}{2}-\frac{3\sqrt{42-14i\sqrt{7}}}{56}}{x+\frac{1}{4}+\frac{i\sqrt{7}}{4}} \\
&= \frac{x+1}{4x^2+2x+2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{\frac{1}{2}-\frac{3\sqrt{42+14i\sqrt{7}}}{56}}{x+\frac{1}{4}-\frac{i\sqrt{7}}{4}} + \frac{\frac{1}{2}-\frac{3\sqrt{42-14i\sqrt{7}}}{56}}{x+\frac{1}{4}+\frac{i\sqrt{7}}{4}}\right)(1) + \left(\left(-\frac{\frac{1}{2}-\frac{3\sqrt{42+14i\sqrt{7}}}{56}}{\left(x+\frac{1}{4}-\frac{i\sqrt{7}}{4}\right)^2} - \frac{\frac{1}{2}-\frac{3\sqrt{42-14i\sqrt{7}}}{56}}{\left(x+\frac{1}{4}+\frac{i\sqrt{7}}{4}\right)^2}\right) + \left(\frac{\frac{1}{2}-3\sqrt{}}{x+\frac{1}{4}}\right.\right.
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 1\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + 1
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+1)\, e^{\int \left(\frac{\frac{1}{2}-\frac{3\sqrt{42+14i\sqrt{7}}}{56}}{x+\frac{1}{4}-\frac{i\sqrt{7}}{4}} + \frac{\frac{1}{2}-\frac{3\sqrt{42-14i\sqrt{7}}}{56}}{x+\frac{1}{4}+\frac{i\sqrt{7}}{4}}\right) dx} \\
&= (x+1)\, e^{\frac{\ln(16x^2+8x+8)}{8} + \frac{3\sqrt{7}\arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{28}} \\
&= (x+1)\, 2^{3/8} (2x^2+x+1)^{1/8}\, e^{\frac{3\sqrt{7}\arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{28}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1+7x}{2x^2+x+1}\, dx} \\
&= z_1 e^{-\frac{7\ln(2x^2+x+1)}{8} + \frac{3\sqrt{7}\arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{28}} \\
&= z_1 \left(\frac{e^{\frac{3\sqrt{7}\arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{28}}}{(2x^2+x+1)^{7/8}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{\frac{3\sqrt{7}\,\arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{14}}(x+1)\,2^{3/8}}{(2x^2+x+1)^{3/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1+7x}{2x^2+x+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{7\ln(2x^2+x+1)}{4}+\frac{3\sqrt{7}\,\arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{14}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{7\ln(2x^2+x+1)}{4}+\frac{3\sqrt{7}\,\arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{14}}(2x^2+x+1)^{3/2}\,\mathrm{e}^{-\frac{3\sqrt{7}\,\arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{7}}\,2^{1/4}}{2(x+1)^2}dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\mathrm{e}^{\frac{3\sqrt{7}\,\arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{14}}(x+1)\,2^{3/8}}{(2x^2+x+1)^{3/4}}\right) \\
&\quad + c_2\left(\frac{\mathrm{e}^{\frac{3\sqrt{7}\,\arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{14}}(x+1)\,2^{3/8}}{(2x^2+x+1)^{3/4}}\left(\int \frac{\mathrm{e}^{-\frac{7\ln(2x^2+x+1)}{4}+\frac{3\sqrt{7}\,\arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{14}}(2x^2+x+1)^{3/2}\,\mathrm{e}^{-\frac{3\sqrt{7}\,\arctan\left(\frac{(4x+1)}{7}\right.}{7}}}{2(x+1)^2}\right.\right.
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(2x^2+x+1)\left(\frac{d^2}{dx^2}y(x)\right)+(1+7x)\left(\frac{d}{dx}y(x)\right)+2y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{2y(x)}{2x^2+x+1} - \frac{(1+7x)\left(\frac{d}{dx}y(x)\right)}{2x^2+x+1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(1+7x)\left(\frac{d}{dx}y(x)\right)}{2x^2+x+1} + \frac{2y(x)}{2x^2+x+1} = 0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1+7x}{2x^2+x+1}, P_3(x) = \frac{2}{2x^2+x+1}\right]$

  ◦ $\left(\frac{\mathrm{I}\sqrt{7}}{4}+x+\frac{1}{4}\right)\cdot P_2(x)$ is analytic at $x = -\frac{1}{4}-\frac{\mathrm{I}\sqrt{7}}{4}$

$\left(\left(\frac{I\sqrt{7}}{4}+x+\frac{1}{4}\right)\cdot P_2(x)\right)\bigg|_{x=-\frac{1}{4}-\frac{I\sqrt{7}}{4}}=0$

- $\left(\frac{I\sqrt{7}}{4}+x+\frac{1}{4}\right)^2\cdot P_3(x)$ is analytic at $x=-\frac{1}{4}-\frac{I\sqrt{7}}{4}$

  $\left(\left(\frac{I\sqrt{7}}{4}+x+\frac{1}{4}\right)^2\cdot P_3(x)\right)\bigg|_{x=-\frac{1}{4}-\frac{I\sqrt{7}}{4}}=0$

- $x=-\frac{1}{4}-\frac{I\sqrt{7}}{4}$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-\frac{1}{4}-\frac{I\sqrt{7}}{4}$

- Multiply by denominators

  $(2x^2+x+1)\left(\frac{d^2}{dx^2}y(x)\right)+(1+7x)\left(\frac{d}{dx}y(x)\right)+2y(x)=0$

- Change variables using $x=u-\frac{1}{4}-\frac{I\sqrt{7}}{4}$ so that the regular singular point is at $u=0$

  $(2u^2-Iu\sqrt{7})\left(\frac{d^2}{du^2}y(u)\right)+\left(-\frac{3}{4}+7u-\frac{7I\sqrt{7}}{4}\right)\left(\frac{d}{du}y(u)\right)+2y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $\frac{I\sqrt{7}\,r\left(3I\sqrt{7}-21-28r\right)a_0u^{-1+r}}{28}+\left(\sum_{k=0}^{\infty}\left(\frac{I\sqrt{7}\,(k+1+r)\left(3I\sqrt{7}-28k-49-28r\right)a_{k+1}}{28}+a_k(k+r+2)(2k+2r+1)\right)\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $\frac{I}{28}\sqrt{7}\,r\left(3I\sqrt{7}-21-28r\right)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{0,\frac{3I\sqrt{7}}{28}-\frac{3}{4}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $-I\left(k+r+\frac{7}{4}\right)a_{k+1}(k+1+r)\sqrt{7}+\frac{(-3k-3r-3)a_{k+1}}{4}+2(k+r+2)\,a_k\left(k+r+\frac{1}{2}\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{4a_k\left(2k^2+4kr+2r^2+5k+5r+2\right)}{3+4I\sqrt{7}\,k^2+8I\sqrt{7}\,kr+4I\sqrt{7}\,r^2+11I\sqrt{7}\,k+11I\sqrt{7}\,r+7I\sqrt{7}+3k+3r}$

- Recursion relation for $r=0$

  $a_{k+1}=\frac{4a_k\left(2k^2+5k+2\right)}{3+4I\sqrt{7}\,k^2+11I\sqrt{7}\,k+7I\sqrt{7}+3k}$

- Solution for $r=0$

  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^k, a_{k+1}=\frac{4a_k\left(2k^2+5k+2\right)}{3+4I\sqrt{7}\,k^2+11I\sqrt{7}\,k+7I\sqrt{7}+3k}\right]$

- Revert the change of variables $u = \frac{I\sqrt{7}}{4} + x + \frac{1}{4}$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k \left(\frac{I\sqrt{7}}{4} + x + \frac{1}{4}\right)^k, a_{k+1} = \frac{4a_k(2k^2+5k+2)}{3+4I\sqrt{7}\,k^2+11I\sqrt{7}\,k+7I\sqrt{7}+3k}\right]$$

- Recursion relation for $r = \frac{3I\sqrt{7}}{28} - \frac{3}{4}$

$$a_{k+1} = \frac{4a_k\left(2k^2+4k\left(\frac{3I\sqrt{7}}{28}-\frac{3}{4}\right)+2\left(\frac{3I\sqrt{7}}{28}-\frac{3}{4}\right)^2+5k+\frac{15I\sqrt{7}}{28}-\frac{7}{4}\right)}{\frac{3}{4}+4I\sqrt{7}\,k^2+8I\sqrt{7}\,k\left(\frac{3I\sqrt{7}}{28}-\frac{3}{4}\right)+4I\sqrt{7}\left(\frac{3I\sqrt{7}}{28}-\frac{3}{4}\right)^2+11I\sqrt{7}\,k+11I\sqrt{7}\left(\frac{3I\sqrt{7}}{28}-\frac{3}{4}\right)+\frac{205I\sqrt{7}}{28}+3k}$$

- Solution for $r = \frac{3I\sqrt{7}}{28} - \frac{3}{4}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{3I\sqrt{7}}{28}-\frac{3}{4}}, a_{k+1} = \frac{4a_k\left(2k^2+4k\left(\frac{3I\sqrt{7}}{28}-\frac{3}{4}\right)+2\left(\frac{3I\sqrt{7}}{28}-\frac{3}{4}\right)^2+5k+\frac{15I\sqrt{7}}{28}-\frac{7}{4}\right)}{\frac{3}{4}+4I\sqrt{7}\,k^2+8I\sqrt{7}\,k\left(\frac{3I\sqrt{7}}{28}-\frac{3}{4}\right)+4I\sqrt{7}\left(\frac{3I\sqrt{7}}{28}-\frac{3}{4}\right)^2+11I\sqrt{7}\,k+11I\sqrt{7}\left(\frac{3I\sqrt{7}}{28}-\frac{3}{4}\right)+\frac{205I}{28}}\right.$$

- Revert the change of variables $u = \frac{I\sqrt{7}}{4} + x + \frac{1}{4}$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k \left(\frac{I\sqrt{7}}{4} + x + \frac{1}{4}\right)^{k+\frac{3I\sqrt{7}}{28}-\frac{3}{4}}, a_{k+1} = \frac{4a_k\left(2k^2+4k\left(\frac{3I\sqrt{7}}{28}-\frac{3}{4}\right)+2\left(\frac{3I\sqrt{7}}{28}-\frac{3}{4}\right)^2+5k+\frac{15I\sqrt{}}{28}\right.}{\frac{3}{4}+4I\sqrt{7}\,k^2+8I\sqrt{7}\,k\left(\frac{3I\sqrt{7}}{28}-\frac{3}{4}\right)+4I\sqrt{7}\left(\frac{3I\sqrt{7}}{28}-\frac{3}{4}\right)^2+11I\sqrt{7}\,k+11I\sqrt{}}\right.$$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k \left(\frac{I\sqrt{7}}{4} + x + \frac{1}{4}\right)^k\right) + \left(\sum_{k=0}^{\infty} b_k \left(\frac{I\sqrt{7}}{4} + x + \frac{1}{4}\right)^{k+\frac{3I\sqrt{7}}{28}-\frac{3}{4}}\right), a_{k+1} = \frac{4a_k(2k^2+5k+2)}{3+4I\sqrt{7}\,k^2+11I\sqrt{7}\,k+7I\sqrt{}}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returnin
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.067 (sec)
Leaf size : 77

```
dsolve((2*x^2+x+1)*diff(diff(y(x),x),x)+(1+7*x)*diff(y(x),x)+2*y(x) = 0,y(x),singsol=a
```

$$y = c_1 \,\mathrm{hypergeom}\left(\left[\frac{1}{2}, 2\right], \left[-\frac{\left(-7\sqrt{7}+3i\right)\sqrt{7}}{28}\right], \frac{1}{2}+\frac{i(-4x-1)\sqrt{7}}{14}\right) + c_2\left(i\sqrt{7}+4x+1\right)^{-\frac{3}{4}+\frac{3i\sqrt{7}}{28}}\left(i\sqrt{7}-4x-1\right)^{-\frac{3}{4}-\frac{3i\sqrt{7}}{28}}(x+1)$$

Mathematica DSolve solution

Solving time : 0.943 (sec)
Leaf size : 119

```
DSolve[{(1+x+2*x^2)*D[y[x],{x,2}]+(1+7*x)*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolu
```

$$y(x) \to (x+1)\exp\left(\int_1^x \frac{K[1]+1}{4K[1]^2+2K[1]+2}dK[1] - \frac{1}{2}\int_1^x \frac{7K[2]+1}{2K[2]^2+K[2]+1}dK[2]\right)\left(c_2\int_1^x \frac{\exp\left(-2\int_1^{K[3]}\frac{K[1]+1}{4K[1]^2+2K[1]+2}dK[1]\right)}{(K[3]+1)^2}dK[3] + c_1\right)$$

2.1.495 Problem 511

Internal problem ID [9667]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 511
Date solved : `Monday, January 27, 2025 at 06:12:51 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(3+x)\, y'' + (1+2x)\, y' - (2-x)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.163 (sec)

Writing the ode as

$$(3+x)\, y'' + (1+2x)\, y' + (x-2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3+x \\ B &= 1+2x \\ C &= x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{35}{4\,(3+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 35 \\ t &= 4(3+x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{35}{4\,(3+x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.935: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(3+x)^2$. There is a pole at $x = -3$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{35}{4\,(3+x)^2}$$

For the pole at $x = -3$ let b be the coefficient of $\frac{1}{(3+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{35}{4\,(3+x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{35}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{7}{2} \\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=-\frac{5}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{35}{4\,(3+x)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-3	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=-\frac{5}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{5}{2}-\left(-\frac{5}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^-}{x-c_1}\right)+(-)[\sqrt{r}]_\infty \\
&= -\frac{5}{2\,(3+x)}+(-)\,(0) \\
&= -\frac{5}{2\,(3+x)} \\
&= -\frac{5}{2\,(3+x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}
$$

Let

$$
p(x)=1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{5}{2\left(3+x\right)}\right)(0)+\left(\left(\frac{5}{2\left(3+x\right)^{2}}\right)+\left(-\frac{5}{2\left(3+x\right)}\right)^{2}-\left(\frac{35}{4\left(3+x\right)^{2}}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{5}{2(3+x)}dx} \\ &= \frac{1}{\left(3+x\right)^{5/2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1+2x}{3+x}\, dx} \\ &= z_1 e^{-x+\frac{5\ln(3+x)}{2}} \\ &= z_1\left(\left(3+x\right)^{5/2}\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$y_1=\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1\int \frac{e^{\int -\frac{1+2x}{3+x}\, dx}}{\left(y_1\right)^2}\, dx \\ &= y_1\int \frac{e^{-2x+5\ln(3+x)}}{\left(y_1\right)^2}\, dx \\ &= y_1\left(\frac{x\left(x^5+18x^4+135x^3+540x^2+1215x+1458\right)\mathrm{e}^{-2x+5\ln(3+x)}\mathrm{e}^{2x}}{6\left(3+x\right)^5}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1+c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-x}\right)+c_2\left(\mathrm{e}^{-x}\left(\frac{x\left(x^5+18x^4+135x^3+540x^2+1215x+1458\right)\mathrm{e}^{-2x+5\ln(3+x)}\mathrm{e}^{2x}}{6\left(3+x\right)^5}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x+3)\left(\frac{d^2}{dx^2}y(x)\right)+(2x+1)\left(\frac{d}{dx}y(x)\right)-(-x+2)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(x-2)y(x)}{x+3}-\frac{(2x+1)\left(\frac{d}{dx}y(x)\right)}{x+3}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{(2x+1)\left(\frac{d}{dx}y(x)\right)}{x+3}+\frac{(x-2)y(x)}{x+3}=0$

☐ Check to see if $x_0=-3$ is a regular singular point

- ○ Define functions

  $\left[P_2(x)=\frac{2x+1}{x+3}, P_3(x)=\frac{x-2}{x+3}\right]$
- ○ $(x+3)\cdot P_2(x)$ is analytic at $x=-3$

  $\left((x+3)\cdot P_2(x)\right)\Big|_{x=-3}=-5$
- ○ $(x+3)^2\cdot P_3(x)$ is analytic at $x=-3$

  $\left((x+3)^2\cdot P_3(x)\right)\Big|_{x=-3}=0$
- ○ $x=-3$is a regular singular point

  Check to see if $x_0=-3$ is a regular singular point

  $x_0=-3$

- Multiply by denominators

  $(x+3)\left(\frac{d^2}{dx^2}y(x)\right)+(2x+1)\left(\frac{d}{dx}y(x)\right)+(x-2)\,y(x)=0$
- Change variables using $x=u-3$ so that the regular singular point is at $u=0$

  $u\left(\frac{d^2}{du^2}y(u)\right)+(2u-5)\left(\frac{d}{du}y(u)\right)+(u-5)\,y(u)=0$
- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$
- ○ Shift index using $k->k-m$

  $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$
- ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$
- ○ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$
- ○ Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-1}$
- ○ Shift index using $k->k+1$

$$u \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-6+r)\,u^{-1+r} + (a_1(1+r)\,(-5+r) + a_0(-5+2r))\,u^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(k-5+r)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-6+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 6\}$
- Each term must be 0
  $a_1(1+r)\,(-5+r) + a_0(-5+2r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)\,(k-5+r) + 2a_k k + 2a_k r - 5a_k + a_{k-1} = 0$
- Shift index using $k- >k+1$
  $a_{k+2}(k+2+r)\,(k-4+r) + 2a_{k+1}(k+1) + 2ra_{k+1} - 5a_{k+1} + a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{2ka_{k+1}+2ra_{k+1}+a_k-3a_{k+1}}{(k+2+r)(k-4+r)}$
- Recursion relation for $r = 0$
  $a_{k+2} = -\frac{2ka_{k+1}+a_k-3a_{k+1}}{(k+2)(k-4)}$
- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 4$
  $a_{k+2} = -\frac{2ka_{k+1}+a_k-3a_{k+1}}{(k+2)(k-4)}$
- Recursion relation for $r = 6$
  $a_{k+2} = -\frac{2ka_{k+1}+a_k+9a_{k+1}}{(k+8)(k+2)}$
- Solution for $r = 6$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+6}, a_{k+2} = -\frac{2ka_{k+1}+a_k+9a_{k+1}}{(k+8)(k+2)}, 7a_1 + 7a_0 = 0\right]$
- Revert the change of variables $u = x+3$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x+3)^{k+6}, a_{k+2} = -\frac{2ka_{k+1}+a_k+9a_{k+1}}{(k+8)(k+2)}, 7a_1 + 7a_0 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 33

```
dsolve((x+3)*diff(diff(y(x),x),x)+(2*x+1)*diff(y(x),x)-(-x+2)*y(x) = 0,y(x),singsol=all)
```

$$y = \left(x(6+x)\left(x^2+9x+27\right)\left(x^2+3x+9\right)c_2+c_1\right)\mathrm{e}^{-x}$$

Mathematica DSolve solution

Solving time : 0.336 (sec)
Leaf size : 52

```
DSolve[{(3+x)*D[y[x],{x,2}]+(1+2*x)*D[y[x],x]-(2-x)*y[x]==0,{}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{(c_2(x+3)^6+6c_1)\exp\left(-\frac{1}{2}\int_1^x\left(2-\frac{5}{K[1]+3}\right)dK[1]\right)}{6(x+3)^{5/2}}$$

2.1.496 Problem 512

Internal problem ID [9668]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 512
Date solved : Monday, January 27, 2025 at 06:12:52 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$y'' + 3xy' + (2x^2 + 4) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.253 (sec)

Writing the ode as

$$y'' + 3xy' + (2x^2 + 4) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3x \\ C &= 2x^2 + 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 10}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 10 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{5}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.937: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{5}{2x} - \frac{25}{4x^3} - \frac{125}{4x^5} - \frac{3125}{16x^7} - \frac{21875}{16x^9} - \frac{328125}{32x^{11}} - \frac{2578125}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 10}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} - \frac{5}{2}\right) + (0) \\ &= \frac{x^2}{4} - \frac{5}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{5}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{5}{2}\right) - (0) \\ &= -\frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= -3 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= 2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{5}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$, and since there are no poles then

$$
\begin{aligned}
d &= \alpha_\infty^- \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= (-)[\sqrt{r}]_\infty \\
&= 0 + (-)\left(\frac{x}{2}\right) \\
&= -\frac{x}{2} \\
&= -\frac{x}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(2) + 2\left(-\frac{x}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{5}{2}\right)\right) &= 0 \\
a_1 x + 2a_0 + 2 &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -1, a_1 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 - 1
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\,dx} \\
&= (x^2 - 1)\,e^{\int -\frac{x}{2}dx} \\
&= (x^2 - 1)\,e^{-\frac{x^2}{4}} \\
&= (x^2 - 1)\,e^{-\frac{x^2}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x}{1}\,dx} \\
&= z_1 e^{-\frac{3x^2}{4}} \\
&= z_1\left(e^{-\frac{3x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = (x^2 - 1)\,\mathrm{e}^{-x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\frac{3x^2}{2}}}{(y_1)^2}\,dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{-\frac{3x^2}{2}}\,\mathrm{e}^{2x^2}}{(x^2-1)^2}dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left((x^2-1)\,\mathrm{e}^{-x^2}\right) + c_2\left((x^2-1)\,\mathrm{e}^{-x^2}\left(\int \frac{\mathrm{e}^{-\frac{3x^2}{2}}\,\mathrm{e}^{2x^2}}{(x^2-1)^2}dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + 3x\left(\frac{d}{dx}y(x)\right) + (2x^2+4)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- >k-m$

    $x^m \cdot y(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

$\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$

Rewrite ODE with series expansions

$2a_2 + 4a_0 + (6a_3 + 7a_1)x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)(k+1) + a_k(3k+4) + 2a_{k-2})x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 + 4a_0 = 0, 6a_3 + 7a_1 = 0]$
- Solve for the dependent coefficient(s)

  $\{a_2 = -2a_0, a_3 = -\frac{7a_1}{6}\}$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)a_{k+2} + 3a_k k + 4a_k + 2a_{k-2} = 0$
- Shift index using $k- > k+2$

  $((k+2)^2 + 3k + 8)a_{k+4} + 3a_{k+2}(k+2) + 4a_{k+2} + 2a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{3ka_{k+2} + 2a_k + 10a_{k+2}}{k^2 + 7k + 12}, a_2 = -2a_0, a_3 = -\frac{7a_1}{6}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special fu
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.026 (sec)
Leaf size : 48

```
dsolve(diff(diff(y(x),x),x)+3*diff(y(x),x)*x+(2*x^2+4)*y(x) = 0,y(x),singsol=all)
```

$$y = 2\,\mathrm{e}^{-\frac{x^2}{2}} c_1 x - \mathrm{e}^{-x^2}(x-1)(x+1)\left(c_1\sqrt{\pi}\,\mathrm{erfi}\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2} - c_2\right)$$

Mathematica DSolve solution

Solving time : `0.201` (sec)
Leaf size : 50

```
DSolve[{D[y[x],{x,2}]+3*x*D[y[x],x]+(4+2*x^2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to e^{-x^2}\left(x^2-1\right)\left(c_2\int_1^x \frac{e^{\frac{K[1]^2}{2}}}{\left(K[1]^2-1\right)^2}dK[1]+c_1\right)$$

2.1.497 Problem 513

Internal problem ID [9669]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 513
Date solved : Monday, January 27, 2025 at 06:12:52 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$(2+4x)\,y''-4y'-(6+4x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.214 (sec)

Writing the ode as

$$(2+4x)\,y''-4y'+(-4x-6)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2+4x\\ B&=-4\\ C&=-4x-6\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{4x^2+8x+6}{\left(1+2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=4x^2+8x+6\\ t&=\left(1+2x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{4x^2+8x+6}{\left(1+2x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.939: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (1 + 2x)^2$. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{3}{4\left(x + \frac{1}{2}\right)^2} + \frac{1}{x + \frac{1}{2}}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{1}{2x} - \frac{1}{4x^3} + \frac{11}{32x^4} - \frac{21}{64x^5} + \frac{15}{64x^6} - \frac{3}{32x^7} - \frac{117}{2048x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 + 8x + 6}{4x^2 + 4x + 1} \\ &= Q + \frac{R}{4x^2 + 4x + 1} \\ &= (1) + \left(\frac{4x + 5}{4x^2 + 4x + 1}\right) \\ &= 1 + \frac{4x + 5}{4x^2 + 4x + 1} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$\begin{aligned} b &= (1) - (0) \\ &= 1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{1}{1} - 0\right) = \frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{1}{1} - 0\right) = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^2+8x+6}{(1+2x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2\left(x+\frac{1}{2}\right)} + (-)(1) \\ &= -\frac{1}{2\left(x+\frac{1}{2}\right)} - 1 \\ &= -\frac{2(x+1)}{1+2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right)(0) + \left(\left(\frac{1}{2\left(x+\frac{1}{2}\right)^2} \right) + \left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1 \right)^2 - \left(\frac{4x^2+8x+6}{(1+2x)^2} \right) \right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2\left(x+\frac{1}{2}\right)}-1\right) dx} \\ &= \frac{\mathrm{e}^{-x}}{\sqrt{1+2x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4}{2+4x} \, dx} \\ &= z_1 e^{\frac{\ln(1+2x)}{2}} \\ &= z_1 \left(\sqrt{1+2x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{2+4x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\ln(1+2x)}}{(y_1)^2} \, dx \\ &= y_1 \left(x \, \mathrm{e}^{2x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(x \, \mathrm{e}^{2x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(2+4x)\left(\frac{d^2}{dx^2}y(x)\right) - 4\frac{d}{dx}y(x) - (6+4x)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{(2x+3)y(x)}{2x+1} + \frac{2\left(\frac{d}{dx}y(x)\right)}{2x+1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$$\frac{d^2}{dx^2}y(x)-\frac{2\left(\frac{d}{dx}y(x)\right)}{2x+1}-\frac{(2x+3)y(x)}{2x+1}=0$$

- ☐ Check to see if $x_0=-\frac{1}{2}$ is a regular singular point
  - Define functions
    $\left[P_2(x)=-\frac{2}{2x+1}, P_3(x)=-\frac{2x+3}{2x+1}\right]$
  - $\left(x+\frac{1}{2}\right)\cdot P_2(x)$ is analytic at $x=-\frac{1}{2}$
    $\left(\left(x+\frac{1}{2}\right)\cdot P_2(x)\right)\bigg|_{x=-\frac{1}{2}}=-1$
  - $\left(x+\frac{1}{2}\right)^2\cdot P_3(x)$ is analytic at $x=-\frac{1}{2}$
    $\left(\left(x+\frac{1}{2}\right)^2\cdot P_3(x)\right)\bigg|_{x=-\frac{1}{2}}=0$
  - $x=-\frac{1}{2}$is a regular singular point
    Check to see if $x_0=-\frac{1}{2}$ is a regular singular point
    $x_0=-\frac{1}{2}$
- Multiply by denominators
  $(2x+1)\left(\frac{d^2}{dx^2}y(x)\right)-2\frac{d}{dx}y(x)+(-2x-3)\,y(x)=0$
- Change variables using $x=u-\frac{1}{2}$ so that the regular singular point is at $u=0$
  $2u\left(\frac{d^2}{du^2}y(u)\right)-2\frac{d}{du}y(u)+(-2u-2)\,y(u)=0$
- Assume series solution for $y(u)$
  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$
    $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$
  - Shift index using $k->k-m$
    $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$
  - Convert $\frac{d}{du}y(u)$ to series expansion
    $\frac{d}{du}y(u)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1}$
  - Shift index using $k->k+1$
    $\frac{d}{du}y(u)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,u^{k+r}$
  - Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion
    $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-1}$
  - Shift index using $k->k+1$
    $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$
    Rewrite ODE with series expansions
    $2a_0r(-2+r)\,u^{-1+r}+(2a_1(1+r)\,(-1+r)-2a_0)\,u^r+\left(\sum_{k=1}^{\infty}(2a_{k+1}(k+1+r)\,(k+r-1)-2a_k\right.$
- $a_0$cannot be 0 by assumption, giving the indicial equation
  $2r(-2+r)=0$
- Values of r that satisfy the indicial equation
  $r\in\{0,2\}$

- Each term must be 0

  $2a_1(1+r)(-1+r)-2a_0=0$
- Each term in the series must be 0, giving the recursion relation

  $2a_{k+1}(k+1+r)(k+r-1)-2a_k-2a_{k-1}=0$
- Shift index using $k->k+1$

  $2a_{k+2}(k+2+r)(k+r)-2a_{k+1}-2a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{a_{k+1}+a_k}{(k+2+r)(k+r)}$
- Recursion relation for $r=0$

  $a_{k+2}=\frac{a_{k+1}+a_k}{(k+2)k}$
- Series not valid for $r=0$, division by 0 in the recursion relation at $k=0$

  $a_{k+2}=\frac{a_{k+1}+a_k}{(k+2)k}$
- Recursion relation for $r=2$

  $a_{k+2}=\frac{a_{k+1}+a_k}{(k+4)(k+2)}$
- Solution for $r=2$

  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+2}, a_{k+2}=\frac{a_{k+1}+a_k}{(k+4)(k+2)}, 6a_1-2a_0=0\right]$
- Revert the change of variables $u=x+\frac{1}{2}$

  $\left[y(x)=\sum_{k=0}^{\infty}a_k\left(x+\frac{1}{2}\right)^{k+2}, a_{k+2}=\frac{a_{k+1}+a_k}{(k+4)(k+2)}, 6a_1-2a_0=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 16

```
dsolve((4*x+2)*diff(diff(y(x),x),x)-4*diff(y(x),x)-(4*x+6)*y(x) = 0,y(x),singsol=all)
```

$$y=c_1\mathrm{e}^{-x}+c_2\mathrm{e}^{x}x$$

Mathematica DSolve solution

Solving time : 0.342 (sec)
Leaf size : 69

```
DSolve[{(2+4*x)*D[y[x],{x,2}]-4*D[y[x],x]-(6+4*x)*y[x]==0,{}},y[x],x,IncludeSingularSolution
```

$$y(x) \to \sqrt{2x+1}\exp\left(\int_1^x \left(\frac{1}{-2K[1]-1} - 1\right) dK[1]\right)\left(c_2 \int_1^x \exp\left(-2\int_1^{K[2]} \left(\frac{1}{-2K[1]-1} - 1\right) dK[1]\right) dK[2] + c_1\right)$$

2.1.498 Problem 514

Internal problem ID [9670]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 514
Date solved : `Monday, January 27, 2025 at 06:12:53 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - 3xy' + (2x^2 + 5) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.235 (sec)

Writing the ode as

$$y'' - 3xy' + (2x^2 + 5) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3x \\ C &= 2x^2 + 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 26}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 26 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{13}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.941: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{13}{2x} - \frac{169}{4x^3} - \frac{2197}{4x^5} - \frac{142805}{16x^7} - \frac{2599051}{16x^9} - \frac{101362989}{32x^{11}} - \frac{2070701061}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 26}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} - \frac{13}{2}\right) + (0) \\ &= \frac{x^2}{4} - \frac{13}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{13}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{13}{2}\right) - (0) \\ &= -\frac{13}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{13}{2}}{\frac{1}{2}} - 1\right) &&= -7 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{13}{2}}{\frac{1}{2}} - 1\right) &&= 6 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{13}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}$	-7	6

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 6$, and since there are no poles then

$$\begin{aligned}d &= \alpha_\infty^- \\ &= 6\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 6$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(30x^4 + 20x^3a_5 + 12x^2a_4 + 6xa_3 + 2a_2\right) + 2\left(-\frac{x}{2}\right)\left(6x^5 + 5x^4a_5 + 4x^3a_4 + 3x^2a_3 + 2xa_2 + a_1\right) + \left(\left(-\frac{1}{2}\right.\right.$$

$$a_5x^5 + 2(15 + a_4)x^4 + (3a_3 + 20a_5)x^3 + 4(a_2 + 3a_4)x^2 + (5$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -15, a_1 = 0, a_2 = 45, a_3 = 0, a_4 = -15, a_5 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^6 - 15x^4 + 45x^2 - 15$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= pe^{\int \omega\,dx} \\ &= \left(x^6 - 15x^4 + 45x^2 - 15\right)e^{\int -\frac{x}{2}dx} \\ &= \left(x^6 - 15x^4 + 45x^2 - 15\right)e^{-\frac{x^2}{4}} \\ &= \left(x^6 - 15x^4 + 45x^2 - 15\right)e^{-\frac{x^2}{4}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1e^{-\int \frac{1}{2}\frac{-3x}{1}\,dx} \\ &= z_1e^{\frac{3x^2}{4}} \\ &= z_1\left(e^{\frac{3x^2}{4}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = e^{\frac{x^2}{2}}\left(x^6 - 15x^4 + 45x^2 - 15\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\frac{3x^2}{2}}}{(y_1)^2}\,dx \\ &= y_1 \left(\int \frac{e^{\frac{3x^2}{2}} e^{-x^2}}{\left(x^6 - 15x^4 + 45x^2 - 15\right)^2}dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(e^{\frac{x^2}{2}}\left(x^6 - 15x^4 + 45x^2 - 15\right)\right) \\ &\quad + c_2\left(e^{\frac{x^2}{2}}\left(x^6 - 15x^4 + 45x^2 - 15\right)\left(\int \frac{e^{\frac{3x^2}{2}} e^{-x^2}}{\left(x^6 - 15x^4 + 45x^2 - 15\right)^2}dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) - 3x\left(\frac{d}{dx}y(x)\right) + \left(2x^2 + 5\right)y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  - ◦ Shift index using $k- >k - m$

    $x^m \cdot y(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  - ◦ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  - ◦ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- Shift index using $k->k+2$

  $\frac{d^2}{dx^2}y(x)=\sum_{k=0}^{\infty}a_{k+2}(k+2)(k+1)x^k$

  Rewrite ODE with series expansions

  $2a_2+5a_0+(6a_3+2a_1)x+\left(\sum_{k=2}^{\infty}\left(a_{k+2}(k+2)(k+1)-a_k(3k-5)+2a_{k-2}\right)x^k\right)=0$

- The coefficients of each power of x must be 0

  $[2a_2+5a_0=0, 6a_3+2a_1=0]$
- Solve for the dependent coefficient(s)

  $\left\{a_2=-\frac{5a_0}{2}, a_3=-\frac{a_1}{3}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)a_{k+2}-3a_kk+5a_k+2a_{k-2}=0$
- Shift index using $k->k+2$

  $\left((k+2)^2+3k+8\right)a_{k+4}-3a_{k+2}(k+2)+5a_{k+2}+2a_k=0$
- Recursion relation that defines the series solution to the ODE

  $\left[y(x)=\sum_{k=0}^{\infty}a_kx^k, a_{k+4}=\frac{3ka_{k+2}-2a_k+a_{k+2}}{k^2+7k+12}, a_2=-\frac{5a_0}{2}, a_3=-\frac{a_1}{3}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.032` (sec)
Leaf size : 62

```
dsolve(diff(diff(y(x),x),x)-3*diff(y(x),x)*x+(2*x^2+5)*y(x) = 0,y(x),singsol=all)
```

$$y=\left(x^6-15x^4+45x^2-15\right)\left(c_1\sqrt{\pi}\,\mathrm{erfi}\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}+c_2\right)\mathrm{e}^{\frac{x^2}{2}}-2\,\mathrm{e}^{x^2}c_1x\left(x^2-11\right)\left(x^2-3\right)$$

Mathematica DSolve solution

Solving time : 0.267 (sec)
Leaf size : 74

```
DSolve[{D[y[x],{x,2}]-3*x*D[y[x],x]+(5+2*x^2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True
```

$$y(x) \to e^{\frac{x^2}{2}}\left(x^6-15x^4+45x^2-15\right)\left(c_2\int_1^x \frac{e^{\frac{K[1]^2}{2}}}{\left(K[1]^6-15K[1]^4+45K[1]^2-15\right)^2}dK[1]+c_1\right)$$

2.1.499 Problem 515

Internal problem ID [9671]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 515
Date solved : Monday, January 27, 2025 at 06:12:53 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$2y'' + 5xy' + \left(2x^2 + 4\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.186 (sec)

Writing the ode as

$$2y'' + 5xy' + \left(2x^2 + 4\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= 5x \\ C &= 2x^2 + 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9x^2 - 12}{16} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9x^2 - 12 \\ t &= 16 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{9x^2}{16} - \frac{3}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.943: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3x}{4} - \frac{1}{2x} - \frac{1}{6x^3} - \frac{1}{9x^5} - \frac{5}{54x^7} - \frac{7}{81x^9} - \frac{7}{81x^{11}} - \frac{22}{243x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{4}$$

From Eq. (9) the sum up to $v=1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{3x}{4} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1}=x^0=1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9x^2}{16}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v=0$ or not. Since $v=1$ which is not zero, then starting $r=\frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{9x^2-12}{16} \\ &= Q + \frac{R}{16} \\ &= \left(\frac{9x^2}{16} - \frac{3}{4}\right) + (0) \\ &= \frac{9x^2}{16} - \frac{3}{4} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{4}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{4}\right) - (0) \\ &= -\frac{3}{4} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{3x}{4} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{4}}{\frac{3}{4}} - 1\right) &&= -1 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{4}}{\frac{3}{4}} - 1\right) &&= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{9x^2}{16} - \frac{3}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{3x}{4}$	-1	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{3x}{4}\right) \\ &= -\frac{3x}{4} \\ &= -\frac{3x}{4} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{3x}{4}\right)(0) + \left(\left(-\frac{3}{4}\right) + \left(-\frac{3x}{4}\right)^2 - \left(\frac{9x^2}{16} - \frac{3}{4}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\,dx} \\ &= e^{\int -\frac{3x}{4} dx} \\ &= e^{-\frac{3x^2}{8}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{5x}{2}\,dx} \\ &= z_1 e^{-\frac{5x^2}{8}} \\ &= z_1\left(e^{-\frac{5x^2}{8}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{-x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{5x}{2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{5x^2}{4}}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{i\sqrt{\pi}\,\sqrt{3}\,\mathrm{erf}\left(\frac{i\sqrt{3}\,x}{2}\right)}{3} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-x^2}\right) + c_2 \left(\mathrm{e}^{-x^2} \left(-\frac{i\sqrt{\pi}\,\sqrt{3}\,\mathrm{erf}\left(\frac{i\sqrt{3}\,x}{2}\right)}{3} \right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2\frac{d^2}{dx^2}y(x) + 5x\left(\frac{d}{dx}y(x)\right) + (2x^2+4)\, y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = (-x^2-2)\, y(x) - \frac{5x\left(\frac{d}{dx}y(x)\right)}{2}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{5x\left(\frac{d}{dx}y(x)\right)}{2} + (x^2+2)\, y(x) = 0$

- Multiply by denominators

  $2\frac{d^2}{dx^2}y(x) + 5x\left(\frac{d}{dx}y(x)\right) + (2x^2+4)\, y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  - Shift index using $k- >k-m$

    $x^m \cdot y(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

- Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

  $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- Shift index using $k- >k+2$

  $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$

  Rewrite ODE with series expansions

  $4a_2 + 4a_0 + (12a_3 + 9a_1)\, x + \left(\sum_{k=2}^{\infty} (2a_{k+2}(k+2)\,(k+1) + a_k(5k+4) + 2a_{k-2})\, x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[4a_2 + 4a_0 = 0, 12a_3 + 9a_1 = 0]$

- Solve for the dependent coefficient(s)

  $\left\{a_2 = -a_0, a_3 = -\frac{3a_1}{4}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $(2k^2 + 6k + 4)\, a_{k+2} + 5a_k k + 4a_k + 2a_{k-2} = 0$

- Shift index using $k- >k+2$

  $\left(2(k+2)^2 + 6k + 16\right) a_{k+4} + 5a_{k+2}(k+2) + 4a_{k+2} + 2a_k = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{5ka_{k+2}+2a_k+14a_{k+2}}{2(k^2+7k+12)}, a_2 = -a_0, a_3 = -\frac{3a_1}{4}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 22

```
dsolve(2*diff(diff(y(x),x),x)+5*diff(y(x),x)*x+(2*x^2+4)*y(x) = 0,y(x),singsol=all)
```

$$y = e^{-x^2}\left(c_1 + \operatorname{erf}\left(\frac{i\sqrt{3}\,x}{2}\right) c_2\right)$$

Mathematica DSolve solution

Solving time : 0.053 (sec)
Leaf size : 42

```
DSolve[{2*D[y[x],{x,2}]+5*x*D[y[x],x]+(4+2*x^2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to \frac{1}{3}e^{-x^2}\left(\sqrt{3\pi}c_2\text{erfi}\left(\frac{\sqrt{3}x}{2}\right)+3c_1\right)$$

2.1.500 Problem 516

Internal problem ID [9672]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 516
Date solved : `Monday, January 27, 2025 at 06:12:54 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + 4xy' + \left(4x^2 + 2\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.054 (sec)

Writing the ode as

$$y'' + 4xy' + \left(4x^2 + 2\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4x \\ C &= 4x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.945: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x}{1}\, dx} \\ &= z_1 e^{-x^2} \\ &= z_1 \left(\mathrm{e}^{-x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-2x^2}}{(y_1)^2}\, dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-x^2}\right) + c_2\left(\mathrm{e}^{-x^2}(x)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + 4x\left(\frac{d}{dx}y(x)\right) + (4x^2+2)\, y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$
  - Shift index using $k->k-m$

    $x^m \cdot y(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$
  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$
  - Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$
  - Shift index using $k->k+2$

    $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 + 2a_0 + (6a_3 + 6a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 2a_k(2k+1) + 4a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 + 2a_0 = 0, 6a_3 + 6a_1 = 0]$
- Solve for the dependent coefficient(s)

  $\{a_2 = -a_0, a_3 = -a_1\}$
- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)\, a_{k+2} + 4a_k k + 2a_k + 4a_{k-2} = 0$

- Shift index using $k->k+2$
  $$\left((k+2)^2+3k+8\right)a_{k+4}+4a_{k+2}(k+2)+2a_{k+2}+4a_k=0$$
- Recursion relation that defines the series solution to the ODE
  $$\left[y(x)=\sum_{k=0}^{\infty}a_kx^k, a_{k+4}=-\frac{2(2ka_{k+2}+2a_k+5a_{k+2})}{k^2+7k+12}, a_2=-a_0, a_3=-a_1\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 16

```
dsolve(diff(diff(y(x),x),x)+4*diff(y(x),x)*x+(4*x^2+2)*y(x) = 0,y(x),singsol=all)
```

$$y=e^{-x^2}(c_2x+c_1)$$

Mathematica DSolve solution

Solving time : 0.024 (sec)
Leaf size : 20

```
DSolve[{D[y[x],{x,2}]+4*x*D[y[x],x]+(2+4*x^2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x)\to e^{-x^2}(c_2x+c_1)$$

2.1.501 Problem 517

Internal problem ID [9673]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 517
Date solved : `Monday, January 27, 2025 at 06:12:55 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + 4xy' + (4x^2 + 2)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.056 (sec)

Writing the ode as

$$y'' + 4xy' + (4x^2 + 2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4x \\ C &= 4x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.947: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x}{1}\, dx} \\ &= z_1 e^{-x^2} \\ &= z_1 \left(\mathrm{e}^{-x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2x^2}}{(y_1)^2}\, dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x^2}\right) + c_2\left(\mathrm{e}^{-x^2}(x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + 4x\left(\frac{d}{dx}y(x)\right) + (4x^2+2)\, y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$
- ◦ Shift index using $k- >k-m$

  $x^m \cdot y(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$
- ◦ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$
- ◦ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

  $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$
- ◦ Shift index using $k- >k+2$

  $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$2a_2 + 2a_0 + (6a_3 + 6a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 2a_k(2k+1) + 4a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 + 2a_0 = 0, 6a_3 + 6a_1 = 0]$
- Solve for the dependent coefficient(s)

  $\{a_2 = -a_0, a_3 = -a_1\}$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} + 4a_k k + 2a_k + 4a_{k-2} = 0$

- Shift index using $k->k+2$

  $$\left((k+2)^2+3k+8\right)a_{k+4}+4a_{k+2}(k+2)+2a_{k+2}+4a_k=0$$
- Recursion relation that defines the series solution to the ODE

  $$\left[y(x)=\sum_{k=0}^{\infty}a_k x^k, a_{k+4}=-\frac{2(2ka_{k+2}+2a_k+5a_{k+2})}{k^2+7k+12}, a_2=-a_0, a_3=-a_1\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 16

```
dsolve(diff(diff(y(x),x),x)+4*diff(y(x),x)*x+(4*x^2+2)*y(x) = 0,y(x),singsol=all)
```

$$y=\mathrm{e}^{-x^2}(c_2x+c_1)$$

Mathematica DSolve solution

Solving time : 0.033 (sec)
Leaf size : 20

```
DSolve[{D[y[x],{x,2}]+4*x*D[y[x],x]+(2+4*x^2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x)\to e^{-x^2}(c_2x+c_1)$$

2.1.502 Problem 518

Internal problem ID [9674]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 518
Date solved : `Monday, January 27, 2025 at 06:12:55 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2\left(x^2+x+1\right)y''+x\left(11x^2+11x+9\right)y'+\left(7x^2+10x+6\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.983 (sec)

Writing the ode as

$$\left(2x^4+2x^3+2x^2\right)y''+\left(11x^3+11x^2+9x\right)y'+\left(7x^2+10x+6\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=2x^4+2x^3+2x^2\\ B&=11x^3+11x^2+9x\\ C&=7x^2+10x+6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{21x^4+18x^3+27x^2-2x-3}{16\left(x^3+x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=21x^4+18x^3+27x^2-2x-3\\ t&=16\left(x^3+x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{21x^4+18x^3+27x^2-2x-3}{16\left(x^3+x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.949: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^3 + x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ of order 2. There is a pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{-\frac{5}{24} + \frac{i\sqrt{3}}{24}}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{5}{24} - \frac{i\sqrt{3}}{24}}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{1}{8} - \frac{43i\sqrt{3}}{72}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{-\frac{1}{8} + \frac{43i\sqrt{3}}{72}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} - \frac{3}{16x^2} + \frac{1}{4x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{24} + \frac{i\sqrt{3}}{24}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{6+6i\sqrt{3}}}{12}
\end{aligned}$$

For the pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{24} - \frac{i\sqrt{3}}{24}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{6-6i\sqrt{3}}}{12}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{21x^4 + 18x^3 + 27x^2 - 2x - 3}{16\left(x^3 + x^2 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{21x^4 + 18x^3 + 27x^2 - 2x - 3}{16\left(x^3 + x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
$-\frac{1}{2} + \frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}$	$\frac{1}{2} - \frac{\sqrt{6+6i\sqrt{3}}}{12}$
$-\frac{1}{2} - \frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}$	$\frac{1}{2} - \frac{\sqrt{6-6i\sqrt{3}}}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying

$\alpha_{\infty}^{+} = \frac{7}{4}$ then

$$\begin{aligned} d &= \alpha_{\infty}^{+} - \left(\alpha_{c1}^{-} + \alpha_{c2}^{+} + \alpha_{c3}^{+}\right) \\ &= \frac{7}{4} - \left(\frac{7}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2} \right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{+}}{x - c_3} \right) + (+)[\sqrt{r}]_{\infty} \\ &= \frac{1}{4x} + \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} + (0) \\ &= \frac{1}{4x} + \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} \\ &= \frac{7x^2 + 3x + 1}{4x\left(x^2 + x + 1\right)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{4x} + \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} \right)(0) + \left(\left(-\frac{1}{4x^2} - \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} - \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2} \right) + \left(\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{4x} + \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} \right) dx} \\ &= 2\left(x^2 + x + 1\right)^{3/4} \sqrt{2}\, x^{1/4} \mathrm{e}^{-\frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{11x^3 + 11x^2 + 9x}{2x^4 + 2x^3 + 2x^2} \, dx} \\ &= z_1 e^{-\frac{9\ln(x)}{4} - \frac{\ln\left(x^2 + x + 1\right)}{4} - \frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}} \\ &= z_1 \left(\frac{\mathrm{e}^{-\frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}}{x^{9/4}\left(x^2 + x + 1\right)^{1/4}} \right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{2\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}\sqrt{2}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{11x^3+11x^2+9x}{2x^4+2x^3+2x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{\mathrm{e}^{-\frac{9\ln(x)}{2}-\frac{\ln(x^2+x+1)}{2}-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{9\ln(x)}{2}-\frac{\ln(x^2+x+1)}{2}-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}\,x^4\,\mathrm{e}^{\frac{2\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{8x^2+8x+8}\,dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{2\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}\sqrt{2}}{x^2}\right) \\
&\quad + c_2\left(\frac{2\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}\sqrt{2}}{x^2}\left(\int \frac{\mathrm{e}^{-\frac{9\ln(x)}{2}-\frac{\ln(x^2+x+1)}{2}-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}\,x^4\,\mathrm{e}^{\frac{2\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{8x^2+8x+8}\,dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2(x^2+x+1)\left(\frac{d^2}{dx^2}y(x)\right) + x(11x^2+11x+9)\left(\frac{d}{dx}y(x)\right) + (7x^2+10x+6)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(7x^2+10x+6)y(x)}{2x^2(x^2+x+1)} - \frac{(11x^2+11x+9)\left(\frac{d}{dx}y(x)\right)}{2x(x^2+x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(11x^2+11x+9)\left(\frac{d}{dx}y(x)\right)}{2x(x^2+x+1)} + \frac{(7x^2+10x+6)y(x)}{2x^2(x^2+x+1)} = 0$

☐ Check to see if x_0 is a regular singular point

  ○ Define functions

    $\left[P_2(x) = \frac{11x^2+11x+9}{2x(x^2+x+1)}, P_3(x) = \frac{7x^2+10x+6}{2x^2(x^2+x+1)}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = \frac{9}{2}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 3$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x^2(x^2+x+1)\left(\frac{d^2}{dx^2}y(x)\right) + x(11x^2+11x+9)\left(\frac{d}{dx}y(x)\right) + (7x^2+10x+6)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k- >k + 2 - m$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)\,(3+2r)\,x^r + \left(a_1(3+r)\,(5+2r) + a_0(5+2r)\,(2+r)\right)x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+2)\,(2k +\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-2, -\frac{3}{2}\right\}$

- Each term must be 0

  $a_1(3+r)\,(5+2r) + a_0(5+2r)\,(2+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = -\frac{(2+r)a_0}{3+r}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k+r+\frac{3}{2}\right)\left((a_k+a_{k-2}+a_{k-1})\,k + (a_k+a_{k-2}+a_{k-1})\,r + 2a_k - a_{k-2} + a_{k-1}\right) = 0$

- Shift index using $k- >k + 2$

  $2\left(k+\frac{7}{2}+r\right)\left((a_{k+2}+a_k+a_{k+1})\,(k+2) + (a_{k+2}+a_k+a_{k+1})\,r + 2a_{k+2} - a_k + a_{k+1}\right) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+2} = -\frac{ka_k + ka_{k+1} + ra_k + ra_{k+1} + a_k + 3a_{k+1}}{k+4+r}$

- Recursion relation for $r = -2$

$a_{k+2} = -\frac{ka_k + ka_{k+1} - a_k + a_{k+1}}{k+2}$

- Solution for $r = -2$

$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{ka_k + ka_{k+1} - a_k + a_{k+1}}{k+2}, a_1 = 0\right]$

- Recursion relation for $r = -\frac{3}{2}$

$a_{k+2} = -\frac{ka_k + ka_{k+1} - \frac{1}{2}a_k + \frac{3}{2}a_{k+1}}{k+\frac{5}{2}}$

- Solution for $r = -\frac{3}{2}$

$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}, a_{k+2} = -\frac{ka_k + ka_{k+1} - \frac{1}{2}a_k + \frac{3}{2}a_{k+1}}{k+\frac{5}{2}}, a_1 = -\frac{a_0}{3}\right]$

- Combine solutions and rename parameters

$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{3}{2}}\right), a_{k+2} = -\frac{ka_k + ka_{k+1} - a_k + a_{k+1}}{k+2}, a_1 = 0, b_{k+2} = -\frac{kb_k + kb_{k+1} - \frac{1}{2}b_k +}{k+\frac{5}{2}}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Mo
      <- Heun successful: received ODE is equivalent to the  HeunG  ODE, case  a <> 0, e
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.654 (sec)
Leaf size : 231

```
dsolve(2*x^2*(x^2+x+1)*diff(diff(y(x),x),x)+x*(11*x^2+11*x+9)*diff(y(x),x)+(7*x^2+10*x
```

$$y = \frac{\left(2x+i\sqrt{3}+1\right)^{\frac{5\sqrt{3}+3i}{6\sqrt{3}+6i}}\left(-2x+i\sqrt{3}-1\right)^{\frac{64i\sqrt{3}+2368}{\left(\sqrt{3}+i\right)^{3}\left(i-\sqrt{3}\right)^{4}\left(13\sqrt{3}+9i\right)}} \mathrm{e}^{-\frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}\left(\mathrm{HeunG}\left(\frac{\sqrt{3}+i}{i-\sqrt{3}},0,0,\frac{5}{2},\frac{1}{2}\right.\right.}{x^{5/2}\left(x^{2}+x+1\right)}$$

Mathematica DSolve solution

Solving time : 0.367 (sec)
Leaf size : 135

```
DSolve[{2*x^2*(1+x+x^2)*D[y[x],{x,2}]+x*(9+11*x+11*x^2)*D[y[x],x]+(6+10*x+7*x^2)*y[x]==0,{}}
```

$$y(x) \to \exp\left(\int_1^x \frac{K[1](7K[1]+3)+1}{4K[1]\left(K[1]^2+K[1]+1\right)}dK[1]-\frac{1}{2}\int_1^x\left(\frac{K[2]+1}{K[2]^2+K[2]+1}+\frac{9}{2K[2]}\right)dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]}\frac{7K[1]^2+3K[1]+1}{4K[1]\left(K[1]^2+K[1]+1\right)}dK[1]\right)dK[3]+c_1\right)$$

2.1.503 Problem 519

Internal problem ID [9675]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 519
Date solved : `Monday, January 27, 2025 at 06:12:56 PM`
CAS classification :
`[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, '_with_symmetry_[0,F(x)]']`

Solve

$$3x^2y'' + 2x(-2x^2 + x + 1)\, y' + (-8x^2 + 2x)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.349 (sec)

Writing the ode as

$$3x^2y'' + (-4x^3 + 2x^2 + 2x)\, y' + (-8x^2 + 2x)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^2 \\ B &= -4x^3 + 2x^2 + 2x \\ C &= -8x^2 + 2x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^4 - 4x^3 + 15x^2 - 4x - 2}{9x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^4 - 4x^3 + 15x^2 - 4x - 2 \\ t &= 9x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4 - 4x^3 + 15x^2 - 4x - 2}{9x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.951: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 9x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{4x^2}{9} - \frac{4x}{9} + \frac{5}{3} - \frac{2}{9x^2} - \frac{4}{9x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{2}{9}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{2}{3} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{3} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{2x}{3} - \frac{1}{3} + \frac{7}{6x} + \frac{1}{4x^2} - \frac{17}{16x^3} - \frac{31}{32x^4} + \frac{85}{64x^5} + \frac{353}{128x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{2}{3}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= -\frac{1}{3} + \frac{2x}{3} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{9} - \frac{4}{9}x + \frac{4}{9}x^2$$

This shows that the coefficient of 1 in the above is $\frac{1}{9}$. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^4 - 4x^3 + 15x^2 - 4x - 2}{9x^2} \\ &= Q + \frac{R}{9x^2} \\ &= \left(\frac{4}{9}x^2 - \frac{4}{9}x + \frac{5}{3}\right) + \left(\frac{-4x - 2}{9x^2}\right) \\ &= \frac{4x^2}{9} - \frac{4x}{9} + \frac{5}{3} + \frac{-4x - 2}{9x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is $\frac{5}{3}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{5}{3}\right) - \left(\frac{1}{9}\right) \\ &= \frac{14}{9} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= -\frac{1}{3} + \frac{2x}{3} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{14}{9}}{\frac{2}{3}} - 1\right) &&= \frac{2}{3} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{14}{9}}{\frac{2}{3}} - 1\right) &&= -\frac{5}{3} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^4 - 4x^3 + 15x^2 - 4x - 2}{9x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{2}{3}$	$\frac{1}{3}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$-\frac{1}{3} + \frac{2x}{3}$	$\frac{2}{3}$	$-\frac{5}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{2}{3}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+\right) \\ &= \frac{2}{3} - \left(\frac{2}{3}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{2}{3x} + \left(-\frac{1}{3} + \frac{2x}{3} \right) \\ &= \frac{2}{3x} - \frac{1}{3} + \frac{2x}{3} \\ &= \frac{2}{3x} - \frac{1}{3} + \frac{2x}{3} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{2}{3x} - \frac{1}{3} + \frac{2x}{3}\right)(0) + \left(\left(-\frac{2}{3x^2} + \frac{2}{3}\right) + \left(\frac{2}{3x} - \frac{1}{3} + \frac{2x}{3}\right)^2 - \left(\frac{4x^4 - 4x^3 + 15x^2 - 4x - 2}{9x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(\frac{2}{3x} - \frac{1}{3} + \frac{2x}{3}\right) dx} \\ &= x^{2/3} \mathrm{e}^{\frac{x(x-1)}{3}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-4x^3+2x^2+2x}{3x^2} \, dx} \\ &= z_1 e^{\frac{x^2}{3} - \frac{x}{3} - \frac{\ln(x)}{3}} \\ &= z_1 \left(\frac{\mathrm{e}^{\frac{x(x-1)}{3}}}{x^{1/3}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^{1/3} \mathrm{e}^{\frac{2x(x-1)}{3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^3+2x^2+2x}{3x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{2x^2}{3} - \frac{2x}{3} - \frac{2\ln(x)}{3}}}{(y_1)^2} \, dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{\frac{2x^2}{3} - \frac{2x}{3} - \frac{2\ln(x)}{3}} \mathrm{e}^{-\frac{4x(x-1)}{3}}}{x^{2/3}} dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(x^{1/3} \mathrm{e}^{\frac{2x(x-1)}{3}}\right) + c_2 \left(x^{1/3} \mathrm{e}^{\frac{2x(x-1)}{3}} \left(\int \frac{\mathrm{e}^{\frac{2x^2}{3} - \frac{2x}{3} - \frac{2\ln(x)}{3}} \mathrm{e}^{-\frac{4x(x-1)}{3}}}{x^{2/3}} dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$3x^2\left(\frac{d^2}{dx^2}y(x)\right)+2x(-2x^2+x+1)\left(\frac{d}{dx}y(x)\right)+(-8x^2+2x)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{2(4x-1)y(x)}{3x}+\frac{2(2x^2-x-1)\left(\frac{d}{dx}y(x)\right)}{3x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)-\frac{2(2x^2-x-1)\left(\frac{d}{dx}y(x)\right)}{3x}-\frac{2(4x-1)y(x)}{3x}=0$

□ Check to see if $x_0=0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=-\frac{2(2x^2-x-1)}{3x},P_3(x)=-\frac{2(4x-1)}{3x}\right]$
- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=\frac{2}{3}$
- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=0$
- ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $3\left(\frac{d^2}{dx^2}y(x)\right)x+(-4x^2+2x+2)\left(\frac{d}{dx}y(x)\right)+(2-8x)\,y(x)=0$
- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$
- ◦ Shift index using $k->k-m$

  $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$
- ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..2$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$
- ◦ Shift index using $k->k+1-m$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$
- ◦ Convert $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-1}$
- ◦ Shift index using $k->k+1$

  $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$a_0 r(-1+3r)\, x^{-1+r} + (a_1(1+r)\,(2+3r) + 2a_0(1+r))\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(3k+2+3r) + \right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-1+3r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{0, \frac{1}{3}\right\}$
- Each term must be 0
  $a_1(1+r)\,(2+3r) + 2a_0(1+r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $(k+1+r)\,(3ka_{k+1} + 3ra_{k+1} + 2a_k - 4a_{k-1} + 2a_{k+1}) = 0$
- Shift index using $k- >k+1$
  $(k+r+2)\,(3(k+1)\,a_{k+2} + 3ra_{k+2} + 2a_{k+1} - 4a_k + 2a_{k+2}) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{2(-a_{k+1}+2a_k)}{3k+5+3r}$
- Recursion relation for $r = 0$
  $a_{k+2} = \frac{2(-a_{k+1}+2a_k)}{3k+5}$
- Solution for $r = 0$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2(-a_{k+1}+2a_k)}{3k+5}, 2a_1 + 2a_0 = 0\right]$
- Recursion relation for $r = \frac{1}{3}$
  $a_{k+2} = \frac{2(-a_{k+1}+2a_k)}{3k+6}$
- Solution for $r = \frac{1}{3}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = \frac{2(-a_{k+1}+2a_k)}{3k+6}, 4a_1 + \frac{8a_0}{3} = 0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = \frac{2(-a_{k+1}+2a_k)}{3k+5}, 2a_1 + 2a_0 = 0, b_{k+2} = \frac{2(-b_{k+1}+2b_k)}{3k+6}, 4b_1 + \right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
```

```
    -> Mathieu
       -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
    -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @
    <- Heun successful: received ODE is equivalent to the  HeunB  ODE, case  c = 0
 <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 1.974 (sec)
Leaf size : 38

```
dsolve(3*x^2*diff(diff(y(x),x),x)+2*x*(-2*x^2+x+1)*diff(y(x),x)+(-8*x^2+2*x)*y(x) = 0,
```

$$y = c_1 x^{1/3} \mathrm{e}^{\frac{2(x-1)x}{3}} + c_2 \operatorname{HeunB}\left(-\frac{1}{3}, \frac{\sqrt{6}}{3}, -\frac{7}{3}, \frac{4\sqrt{6}}{9}, -\frac{\sqrt{6}\, x}{3}\right)$$

Mathematica DSolve solution

Solving time : 0.309 (sec)
Leaf size : 53

```
DSolve[{3*x^2*D[y[x],{x,2}]+2*x*(1+x-2*x^2)*D[y[x],x]+(2*x-8*x^2)*y[x]==0,{}},y[x],x,Include
```

$$y(x) \to e^{\frac{2}{3}(x-1)x} \sqrt[3]{x} \left(c_2 \int_1^x \frac{e^{-\frac{2}{3}(K[1]-1)K[1]}}{K[1]^{4/3}} dK[1] + c_1\right)$$

2.1.504 Problem 520

Internal problem ID [9676]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 520
Date solved : Monday, January 27, 2025 at 06:12:57 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$12x^2(1+x)\,y''+x\left(3x^2+35x+11\right)y'-\left(-5x^2-10x+1\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.322 (sec)

Writing the ode as

$$\left(12x^3+12x^2\right)y''+\left(3x^3+35x^2+11x\right)y'+\left(5x^2+10x-1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=12x^3+12x^2\\ B&=3x^3+35x^2+11x \\ C&=5x^2+10x-1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{9x^4-30x^3-197x^2-190x-95}{576\left(x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=9x^4-30x^3-197x^2-190x-95\\ t&=576\left(x^2+x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{9x^4-30x^3-197x^2-190x-95}{576\left(x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.953: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 576(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{64} - \frac{7}{64\,(1+x)^2} - \frac{1}{12\,(1+x)} - \frac{95}{576x^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{95}{576}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{19}{24} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{24} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{8} - \frac{1}{3x} - \frac{29}{24x^2} - \frac{193}{72x^3} - \frac{3017}{216x^4} - \frac{40009}{648x^5} - \frac{642029}{1944x^6} - \frac{10350493}{5832x^7} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{8}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{8} \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{64}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{9x^4 - 30x^3 - 197x^2 - 190x - 95}{576x^4 + 1152x^3 + 576x^2} \\
&= Q + \frac{R}{576x^4 + 1152x^3 + 576x^2} \\
&= \left(\frac{1}{64}\right) + \left(\frac{-48x^3 - 206x^2 - 190x - 95}{576x^4 + 1152x^3 + 576x^2}\right) \\
&= \frac{1}{64} + \frac{-48x^3 - 206x^2 - 190x - 95}{576x^4 + 1152x^3 + 576x^2}
\end{aligned}$$

Since the degree of t is 4, then we see that the coefficient of the term x^3 in the remainder R is -48. Dividing this by leading coefficient in t which is 576 gives $-\frac{1}{12}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{12}\right) - (0) \\
&= -\frac{1}{12}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{8} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{12}}{\frac{1}{8}} - 0\right) &= -\frac{1}{3} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{-\frac{1}{12}}{\frac{1}{8}} - 0\right) = \frac{1}{3}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{9x^4 - 30x^3 - 197x^2 - 190x - 95}{576\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{7}{8}$	$\frac{1}{8}$
0	2	0	$\frac{19}{24}$	$\frac{5}{24}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{8}$	$-\frac{1}{3}$	$\frac{1}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{3}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{1}{3} - \left(\frac{1}{3}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{8 + 8x} + \frac{5}{24x} + (-)\left(\frac{1}{8}\right) \\
&= \frac{1}{8 + 8x} + \frac{5}{24x} - \frac{1}{8} \\
&= \frac{1}{8 + 8x} + \frac{5}{24x} - \frac{1}{8}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{1}{8+8x}+\frac{5}{24x}-\frac{1}{8}\right)(0)+\left(\left(-\frac{1}{8\left(1+x\right)^{2}}-\frac{5}{24x^{2}}\right)+\left(\frac{1}{8+8x}+\frac{5}{24x}-\frac{1}{8}\right)^{2}-\left(\frac{9x^{4}-30x^{3}-19}{576\left(x\right.}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\,dx}\\
&= e^{\int \left(\frac{1}{8+8x}+\frac{5}{24x}-\frac{1}{8}\right)dx}\\
&= (1+x)^{1/8}\,x^{5/24}e^{-\frac{x}{8}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\
&= z_1 e^{-\int \frac{1}{2}\frac{3x^3+35x^2+11x}{12x^3+12x^2}\,dx}\\
&= z_1 e^{-\frac{x}{8}-\frac{7\ln(1+x)}{8}-\frac{11\ln(x)}{24}}\\
&= z_1\left(\frac{e^{-\frac{x}{8}}}{(1+x)^{7/8}\,x^{11/24}}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\frac{e^{-\frac{x}{4}}}{(1+x)^{3/4}\,x^{1/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1\int \frac{e^{\int -\frac{3x^3+35x^2+11x}{12x^3+12x^2}\,dx}}{(y_1)^2}\,dx\\
&= y_1\int \frac{e^{-\frac{x}{4}-\frac{7\ln(1+x)}{4}-\frac{11\ln(x)}{12}}}{(y_1)^2}\,dx\\
&= y_1\left(\int e^{-\frac{x}{4}-\frac{7\ln(1+x)}{4}-\frac{11\ln(x)}{12}}(1+x)^{3/2}\sqrt{x}\,e^{\frac{x}{2}}\,dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1y_1+c_2y_2\\
&= c_1\left(\frac{e^{-\frac{x}{4}}}{(1+x)^{3/4}\,x^{1/4}}\right)+c_2\left(\frac{e^{-\frac{x}{4}}}{(1+x)^{3/4}\,x^{1/4}}\left(\int e^{-\frac{x}{4}-\frac{7\ln(1+x)}{4}-\frac{11\ln(x)}{12}}(1+x)^{3/2}\sqrt{x}\,e^{\frac{x}{2}}\,dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$12x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(3x^2+35x+11)\left(\frac{d}{dx}y(x)\right)-(-5x^2-10x+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{(5x^2+10x-1)y(x)}{12(x+1)x^2}-\frac{(3x^2+35x+11)\left(\frac{d}{dx}y(x)\right)}{12x(x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)+\frac{(3x^2+35x+11)\left(\frac{d}{dx}y(x)\right)}{12x(x+1)}+\frac{(5x^2+10x-1)y(x)}{12(x+1)x^2}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=\frac{3x^2+35x+11}{12x(x+1)},P_3(x)=\frac{5x^2+10x-1}{12(x+1)x^2}\right]$

- ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

$\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=\frac{7}{4}$

- ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

$\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

- ◦ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1$

- Multiply by denominators

$12x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(3x^2+35x+11)\left(\frac{d}{dx}y(x)\right)+(5x^2+10x-1)\,y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(12u^3-24u^2+12u)\left(\frac{d^2}{du^2}y(u)\right)+(3u^3+26u^2-50u+21)\left(\frac{d}{du}y(u)\right)+(5u^2-6)\,y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot y(u)$ to series expansion for $m=0..2$

$u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

- ◦ Shift index using $k->k-m$

$u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..3$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ◦ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- ◦ Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$3a_0r(3+4r)\,u^{-1+r} + (3a_1(1+r)(7+4r) - 2a_0(3+4r)(1+3r))\,u^r + (3a_2(2+r)(11+4r) - 2a_1($$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $3r(3+4r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{0, -\frac{3}{4}\right\}$
- The coefficients of each power of u must be 0
  $[3a_1(1+r)(7+4r) - 2a_0(3+4r)(1+3r) = 0, 3a_2(2+r)(11+4r) - 2a_1(7+4r)(4+3r) + 2a_0r($
- Solve for the dependent coefficient(s)
  $\left\{a_1 = \frac{2a_0(12r^2+13r+3)}{3(4r^2+11r+7)}, a_2 = \frac{2a_0(54r^3+135r^2+101r+24)}{9(4r^3+23r^2+41r+22)}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $12(-2a_k + a_{k-1} + a_{k+1})\,k^2 + (24(-2a_k + a_{k-1} + a_{k+1})\,r - 26a_k + 3a_{k-2} - 10a_{k-1} + 33a_{k+1})\,k + 12$
- Shift index using $k- >k+2$
  $12(-2a_{k+2} + a_{k+1} + a_{k+3})\,(k+2)^2 + (24(-2a_{k+2} + a_{k+1} + a_{k+3})\,r - 26a_{k+2} + 3a_k - 10a_{k+1} + 33a_k$
- Recursion relation that defines series solution to ODE
  $a_{k+3} = -\frac{12k^2a_{k+1} - 24k^2a_{k+2} + 24kra_{k+1} - 48kra_{k+2} + 12r^2a_{k+1} - 24r^2a_{k+2} + 3ka_k + 38ka_{k+1} - 122ka_{k+2} + 3ra_k + 38ra_{k+1} - 122ra_{k+}}{3(4k^2+8kr+4r^2+27k+27r+45)}$
- Recursion relation for $r = 0$
  $a_{k+3} = -\frac{12k^2a_{k+1} - 24k^2a_{k+2} + 3ka_k + 38ka_{k+1} - 122ka_{k+2} + 5a_k + 26a_{k+1} - 154a_{k+2}}{3(4k^2+27k+45)}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+3} = -\frac{12k^2a_{k+1} - 24k^2a_{k+2} + 3ka_k + 38ka_{k+1} - 122ka_{k+2} + 5a_k + 26a_{k+1} - 154a_{k+2}}{3(4k^2+27k+45)}, a_1 = \frac{2a_0}{7}, a_2 = \right.$
- Revert the change of variables $u = x+1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+3} = -\frac{12k^2a_{k+1} - 24k^2a_{k+2} + 3ka_k + 38ka_{k+1} - 122ka_{k+2} + 5a_k + 26a_{k+1} - 154a_{k+2}}{3(4k^2+27k+45)}, a_1 = \frac{2a_0}{7}, \right.$
- Recursion relation for $r = -\frac{3}{4}$
  $a_{k+3} = -\frac{12k^2a_{k+1} - 24k^2a_{k+2} + 3ka_k + 20ka_{k+1} - 86ka_{k+2} + \frac{11}{4}a_k + \frac{17}{4}a_{k+1} - 76a_{k+2}}{3(4k^2+21k+27)}$
- Solution for $r = -\frac{3}{4}$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{3}{4}}, a_{k+3} = -\frac{12k^2a_{k+1} - 24k^2a_{k+2} + 3ka_k + 20ka_{k+1} - 86ka_{k+2} + \frac{11}{4}a_k + \frac{17}{4}a_{k+1} - 76a_{k+2}}{3(4k^2+21k+27)}, a_1 = 0, a_2 = \right.$
- Revert the change of variables $u = x+1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^{k-\frac{3}{4}}, a_{k+3} = -\frac{12k^2a_{k+1} - 24k^2a_{k+2} + 3ka_k + 20ka_{k+1} - 86ka_{k+2} + \frac{11}{4}a_k + \frac{17}{4}a_{k+1} - 76a_{k+2}}{3(4k^2+21k+27)}, a_1 = 0, \right.$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k(x+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k(x+1)^{k-\frac{3}{4}}\right), a_{k+3} = -\frac{12k^2a_{k+1} - 24k^2a_{k+2} + 3ka_k + 38ka_{k+1} - 122ka_{k+2} + 5}{3(4k^2+27k+45)}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @
      <- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0,
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.398 (sec)
Leaf size : 43

```
dsolve(12*x^2*(x+1)*diff(diff(y(x),x),x)+x*(3*x^2+35*x+11)*diff(y(x),x)-(-5*x^2-10*x+1
```

$$y = \frac{\mathrm{e}^{-\frac{x}{4}}\left(\operatorname{HeunC}\left(\frac{1}{4}, \frac{7}{12}, -\frac{3}{4}, -\frac{1}{12}, \frac{1}{2}, -x\right) x^{7/12} c_2 + \operatorname{HeunC}\left(\frac{1}{4}, -\frac{7}{12}, -\frac{3}{4}, -\frac{1}{12}, \frac{1}{2}, -x\right) c_1\right)}{(x+1)^{3/4} x^{1/4}}$$

Mathematica DSolve solution

Solving time : 0.352 (sec)
Leaf size : 118

```
DSolve[{12*x^2*(1+x)*D[y[x],{x,2}]+x*(11+35*x+3*x^2)*D[y[x],x]-(1-10*x-5*x^2)*y[x]==0,{}},y[
```

$$y(x) \to \exp\left(\int_1^x \frac{1}{24}\left(\frac{3}{K[1]+1} - 3 + \frac{5}{K[1]}\right) dK[1] - \frac{1}{2}\int_1^x \frac{1}{12}\left(\frac{21}{K[2]+1} + 3 + \frac{11}{K[2]}\right) dK[2]\right)\left(c_2 \int_1^x \exp\left(-2\int_1^{K[3]} \frac{-3K[1]^2 + 5K[1] + 5}{24K[1]^2 + 24K[1]} dK[1]\right) dK[3] + c_1\right)$$

2.1.505 Problem 521

Internal problem ID [9677]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 521
Date solved : Monday, January 27, 2025 at 06:12:58 PM
CAS classification : [[_2nd_order, _missing_x]]

Solve

$$y'' + 3y' + 4y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.187 (sec)

Writing the ode as

$$y'' + 3y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-7}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = -7$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{7z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.955: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{7}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{7}\, x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\, dx} \\ &= z_1 e^{-\frac{3x}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{3x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{3x}{2}} \cos\left(\frac{\sqrt{7}\, x}{2}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-3x}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{2\sqrt{7} \tan\left(\frac{\sqrt{7}\, x}{2}\right)}{7} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{3x}{2}} \cos\left(\frac{\sqrt{7}\, x}{2}\right) \right) + c_2 \left(\mathrm{e}^{-\frac{3x}{2}} \cos\left(\frac{\sqrt{7}\, x}{2}\right) \left(\frac{2\sqrt{7} \tan\left(\frac{\sqrt{7}\, x}{2}\right)}{7} \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

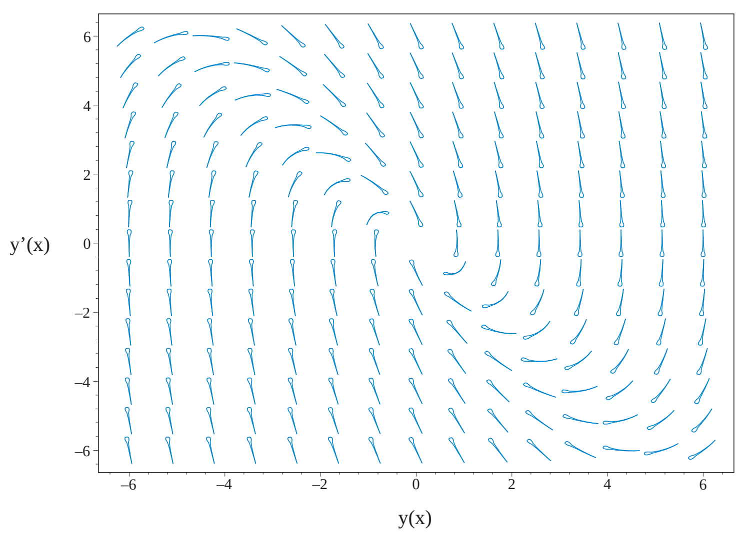

Figure 2.2: Slope field plot
$y'' + 3y' + 4y = 0$

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2} y(x) + 3 \frac{d}{dx} y(x) + 4y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2} y(x)$
- Characteristic polynomial of ODE
  $r^2 + 3r + 4 = 0$
- Use quadratic formula to solve for r
  $r = \frac{(-3) \pm (\sqrt{-7})}{2}$
- Roots of the characteristic polynomial
  $r = \left(-\frac{3}{2} - \frac{\mathrm{I}\sqrt{7}}{2}, -\frac{3}{2} + \frac{\mathrm{I}\sqrt{7}}{2}\right)$
- 1st solution of the ODE
  $y_1(x) = \mathrm{e}^{-\frac{3x}{2}} \cos\left(\frac{\sqrt{7}\, x}{2}\right)$
- 2nd solution of the ODE

$$y_2(x) = e^{-\frac{3x}{2}} \sin\left(\frac{\sqrt{7}\,x}{2}\right)$$

- General solution of the ODE
  $y(x) = C1 y_1(x) + C2 y_2(x)$
- Substitute in solutions
  $y(x) = C1\, e^{-\frac{3x}{2}} \cos\left(\frac{\sqrt{7}\,x}{2}\right) + C2\, e^{-\frac{3x}{2}} \sin\left(\frac{\sqrt{7}\,x}{2}\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 28

```
dsolve(diff(diff(y(x),x),x)+3*diff(y(x),x)+4*y(x) = 0,y(x),singsol=all)
```

$$y = e^{-\frac{3x}{2}} \left(c_1 \sin\left(\frac{\sqrt{7}\,x}{2}\right) + c_2 \cos\left(\frac{\sqrt{7}\,x}{2}\right)\right)$$

Mathematica DSolve solution

Solving time : 0.026 (sec)
Leaf size : 42

```
DSolve[{D[y[x],{x,2}]+3*D[y[x],x]+4*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-3x/2} \left(c_2 \cos\left(\frac{\sqrt{7}x}{2}\right) + c_1 \sin\left(\frac{\sqrt{7}x}{2}\right)\right)$$

2.1.506 Problem 522

Internal problem ID [9678]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 522
Date solved : `Monday, January 27, 2025 at 06:12:59 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$18x^2(1+x)\,y''+3x\left(x^2+11x+5\right)y'-\left(-5x^2-2x+1\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.323 (sec)

Writing the ode as

$$\left(18x^3+18x^2\right)y''+\left(3x^3+33x^2+15x\right)y'+\left(5x^2+2x-1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=18x^3+18x^2\\ B&=3x^3+33x^2+15x\\ C&=5x^2+2x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{x^4-18x^3-45x^2-18x-27}{144\left(x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=x^4-18x^3-45x^2-18x-27\\ t&=144\left(x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x^4-18x^3-45x^2-18x-27}{144\left(x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.957: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 144(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{144} - \frac{7}{18\,(1 + x)} - \frac{35}{144\,(1 + x)^2} + \frac{1}{4x} - \frac{3}{16x^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{5}{12} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{12} - \frac{5}{6x} - \frac{53}{12x^2} - \frac{523}{12x^3} - \frac{6659}{12x^4} - \frac{94267}{12x^5} - \frac{1432421}{12x^6} - \frac{22802941}{12x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{12}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{12} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{144}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^4 - 18x^3 - 45x^2 - 18x - 27}{144x^4 + 288x^3 + 144x^2} \\ &= Q + \frac{R}{144x^4 + 288x^3 + 144x^2} \\ &= \left(\frac{1}{144}\right) + \left(\frac{-20x^3 - 46x^2 - 18x - 27}{144x^4 + 288x^3 + 144x^2}\right) \\ &= \frac{1}{144} + \frac{-20x^3 - 46x^2 - 18x - 27}{144x^4 + 288x^3 + 144x^2} \end{aligned}$$

Since the degree of t is 4, then we see that the coefficient of the term x^3 in the remainder R is -20. Dividing this by leading coefficient in t which is 144 gives $-\frac{5}{36}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{5}{36}\right) - (0) \\ &= -\frac{5}{36} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{12}\\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a}-v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{36}}{\frac{1}{12}}-0\right) &&= -\frac{5}{6}\\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a}-v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{36}}{\frac{1}{12}}-0\right) &&= \frac{5}{6}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^4-18x^3-45x^2-18x-27}{144\left(x^2+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-1	2	0	$\frac{7}{12}$	$\frac{5}{12}$
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{12}$	$-\frac{5}{6}$	$\frac{5}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{5}{6}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}+\alpha_{c_2}^{-}\right)\\
&= \frac{5}{6}-\left(\frac{5}{6}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^{+}}{x-c_1}\right)+\left((-)[\sqrt{r}]_{c_2}+\frac{\alpha_{c_2}^{-}}{x-c_2}\right)+(-)[\sqrt{r}]_\infty\\
&= \frac{7}{12\left(1+x\right)}+\frac{1}{4x}+(-)\left(\frac{1}{12}\right)\\
&= \frac{7}{12\left(1+x\right)}+\frac{1}{4x}-\frac{1}{12}\\
&= \frac{7}{12+12x}+\frac{1}{4x}-\frac{1}{12}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}
$$

Let

$$
p(x)=1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{7}{12\,(1+x)}+\frac{1}{4x}-\frac{1}{12}\right)(0)+\left(\left(-\frac{7}{12\,(1+x)^2}-\frac{1}{4x^2}\right)+\left(\frac{7}{12\,(1+x)}+\frac{1}{4x}-\frac{1}{12}\right)^2-\left(\frac{x^4-18x^3}{1\cdots}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(\frac{7}{12(1+x)}+\frac{1}{4x}-\frac{1}{12}\right) dx} \\ &= x^{1/4}(1+x)^{7/12}\,\mathrm{e}^{-\frac{x}{12}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3x^3+33x^2+15x}{18x^3+18x^2}\, dx} \\ &= z_1 e^{-\frac{x}{12}-\frac{5\ln(x)}{12}-\frac{5\ln(1+x)}{12}} \\ &= z_1\left(\frac{\mathrm{e}^{-\frac{x}{12}}}{x^{5/12}\,(1+x)^{5/12}}\right) \end{aligned}$$

Which simplifies to

$$y_1=\frac{(1+x)^{1/6}\,\mathrm{e}^{-\frac{x}{6}}}{x^{1/6}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1\int \frac{e^{\int -\frac{3x^3+33x^2+15x}{18x^3+18x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1\int \frac{e^{-\frac{x}{6}-\frac{5\ln(x)}{6}-\frac{5\ln(1+x)}{6}}}{(y_1)^2}\, dx \\ &= y_1\left(\int \frac{\mathrm{e}^{-\frac{x}{6}-\frac{5\ln(x)}{6}-\frac{5\ln(1+x)}{6}}\, x^{1/3}\mathrm{e}^{\frac{x}{3}}}{(1+x)^{1/3}}dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1y_1+c_2y_2 \\ &= c_1\left(\frac{(1+x)^{1/6}\,\mathrm{e}^{-\frac{x}{6}}}{x^{1/6}}\right)+c_2\left(\frac{(1+x)^{1/6}\,\mathrm{e}^{-\frac{x}{6}}}{x^{1/6}}\left(\int \frac{\mathrm{e}^{-\frac{x}{6}-\frac{5\ln(x)}{6}-\frac{5\ln(1+x)}{6}}\, x^{1/3}\mathrm{e}^{\frac{x}{3}}}{(1+x)^{1/3}}dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$18x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)+3x(x^2+11x+5)\left(\frac{d}{dx}y(x)\right)-(-5x^2-2x+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{(5x^2+2x-1)y(x)}{18(x+1)x^2}-\frac{(x^2+11x+5)\left(\frac{d}{dx}y(x)\right)}{6x(x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)+\frac{(x^2+11x+5)\left(\frac{d}{dx}y(x)\right)}{6x(x+1)}+\frac{(5x^2+2x-1)y(x)}{18(x+1)x^2}=0$

☐ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=\frac{x^2+11x+5}{6x(x+1)},P_3(x)=\frac{5x^2+2x-1}{18(x+1)x^2}\right]$

- ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

$\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1}=\frac{5}{6}$

- ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

$\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1}=0$

- ◦ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1$

- Multiply by denominators

$18x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)+3x(x^2+11x+5)\left(\frac{d}{dx}y(x)\right)+(5x^2+2x-1)\,y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(18u^3-36u^2+18u)\left(\frac{d^2}{du^2}y(u)\right)+(3u^3+24u^2-42u+15)\left(\frac{d}{du}y(u)\right)+(5u^2-8u+2)\,y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot y(u)$ to series expansion for $m=0..2$

$u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

- ◦ Shift index using $k->k-m$

$u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..3$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ◦ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

- ◦ Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$3a_0r(-1+6r)\,u^{-1+r} + (3a_1(1+r)(5+6r) - 2a_0(1+3r)(-1+6r))\,u^r + (3a_2(2+r)(11+6r) - 2$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $3r(-1+6r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{0, \frac{1}{6}\right\}$
- The coefficients of each power of u must be 0
  $[3a_1(1+r)(5+6r) - 2a_0(1+3r)(-1+6r) = 0, 3a_2(2+r)(11+6r) - 2a_1(4+3r)(5+6r) + 2a_0$
- Solve for the dependent coefficient(s)
  $\left\{a_1 = \frac{2a_0(18r^2+3r-1)}{3(6r^2+11r+5)}, a_2 = \frac{2a_0(81r^3+126r^2+21r+4)}{9(6r^3+29r^2+45r+22)}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $18(-2a_k + a_{k-1} + a_{k+1})\,k^2 + 3(12(-2a_k + a_{k-1} + a_{k+1})\,r - 2a_k + a_{k-2} - 10a_{k-1} + 11a_{k+1})\,k + 18($
- Shift index using $k- >k+2$
  $18(-2a_{k+2} + a_{k+1} + a_{k+3})\,(k+2)^2 + 3(12(-2a_{k+2} + a_{k+1} + a_{k+3})\,r - 2a_{k+2} + a_k - 10a_{k+1} + 11a_{k+}$
- Recursion relation that defines series solution to ODE
  $a_{k+3} = -\frac{18k^2a_{k+1} - 36k^2a_{k+2} + 36kra_{k+1} - 72kra_{k+2} + 18r^2a_{k+1} - 36r^2a_{k+2} + 3ka_k + 42ka_{k+1} - 150ka_{k+2} + 3ra_k + 42ra_{k+1} - 150ra_{k+}}{3(6k^2+12kr+6r^2+35k+35r+51)}$
- Recursion relation for $r = 0$
  $a_{k+3} = -\frac{18k^2a_{k+1} - 36k^2a_{k+2} + 3ka_k + 42ka_{k+1} - 150ka_{k+2} + 5a_k + 16a_{k+1} - 154a_{k+2}}{3(6k^2+35k+51)}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+3} = -\frac{18k^2a_{k+1} - 36k^2a_{k+2} + 3ka_k + 42ka_{k+1} - 150ka_{k+2} + 5a_k + 16a_{k+1} - 154a_{k+2}}{3(6k^2+35k+51)}, a_1 = -\frac{2a_0}{15}, a_2 = \right.$
- Revert the change of variables $u = x+1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+3} = -\frac{18k^2a_{k+1} - 36k^2a_{k+2} + 3ka_k + 42ka_{k+1} - 150ka_{k+2} + 5a_k + 16a_{k+1} - 154a_{k+2}}{3(6k^2+35k+51)}, a_1 = -\frac{2a}{15}\right.$
- Recursion relation for $r = \frac{1}{6}$
  $a_{k+3} = -\frac{18k^2a_{k+1} - 36k^2a_{k+2} + 3ka_k + 48ka_{k+1} - 162ka_{k+2} + \frac{11}{2}a_k + \frac{47}{2}a_{k+1} - 180a_{k+2}}{3(6k^2+37k+57)}$
- Solution for $r = \frac{1}{6}$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{6}}, a_{k+3} = -\frac{18k^2a_{k+1} - 36k^2a_{k+2} + 3ka_k + 48ka_{k+1} - 162ka_{k+2} + \frac{11}{2}a_k + \frac{47}{2}a_{k+1} - 180a_{k+2}}{3(6k^2+37k+57)}, a_1 = 0, a_2 = \right.$
- Revert the change of variables $u = x+1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^{k+\frac{1}{6}}, a_{k+3} = -\frac{18k^2a_{k+1} - 36k^2a_{k+2} + 3ka_k + 48ka_{k+1} - 162ka_{k+2} + \frac{11}{2}a_k + \frac{47}{2}a_{k+1} - 180a_{k+2}}{3(6k^2+37k+57)}, a_1 = \right.$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k(x+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k(x+1)^{k+\frac{1}{6}}\right), a_{k+3} = -\frac{18k^2a_{k+1} - 36k^2a_{k+2} + 3ka_k + 42ka_{k+1} - 150ka_{k+2} + 5}{3(6k^2+35k+51)}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @
      <- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0,
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.378 (sec)
Leaf size : 38

```
dsolve(18*x^2*(x+1)*diff(diff(y(x),x),x)+3*x*(x^2+11*x+5)*diff(y(x),x)-(-5*x^2-2*x+1)*
```

$$y = \frac{e^{-\frac{x}{6}} \left(\sqrt{x}\, \mathrm{HeunC}\left(\frac{1}{6}, \frac{1}{2}, -\frac{1}{6}, -\frac{5}{36}, \frac{1}{4}, -x\right) c_2 + \mathrm{HeunC}\left(\frac{1}{6}, -\frac{1}{2}, -\frac{1}{6}, -\frac{5}{36}, \frac{1}{4}, -x\right) c_1\right)}{x^{1/6}}$$

Mathematica DSolve solution

Solving time : 0.332 (sec)
Leaf size : 118

```
DSolve[{18*x^2*(1+x)*D[y[x],{x,2}]+3*x*(5+11*x+x^2)*D[y[x],x]-(1-2*x-5*x^2)*y[x]==0,{}},y[x]
```

$$y(x) \to \exp\left(\int_1^x \frac{1}{12}\left(\frac{7}{K[1]+1} - 1 + \frac{3}{K[1]}\right) dK[1] - \frac{1}{2}\int_1^x \frac{1}{6}\left(\frac{5}{K[2]+1} + 1 + \frac{5}{K[2]}\right) dK[2]\right) \left(c_2 \int_1^x \exp\left(-2\int_1^{K[3]} \frac{-K[1]^2 + 9K[1] + 3}{12K[1]^2 + 12K[1]} dK[1]\right) dK[3] + c_1\right)$$

2.1.507 Problem 523

Internal problem ID [9679]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 523
Date solved : `Monday, January 27, 2025 at 06:12:59 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2y'' + x(3+2x)\,y' - (1-x)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.240 (sec)

Writing the ode as

$$2x^2y'' + \left(2x^2+3x\right)y' + (x-1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= 2x^2+3x \\ C &= x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2+4x+5}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2+4x+5 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2+4x+5}{16x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.959: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{1}{4x} + \frac{5}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{4x} + \frac{1}{4x^2} - \frac{1}{8x^3} + \frac{1}{16x^5} - \frac{3}{64x^6} - \frac{1}{128x^7} + \frac{11}{256x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{4x^2 + 4x + 5}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{4x + 5}{16x^2}\right) \\ &= \frac{1}{4} + \frac{4x + 5}{16x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 16 gives $\frac{1}{4}$. Now b can be found.

$$\begin{aligned}b &= \left(\frac{1}{4}\right) - (0) \\ &= \frac{1}{4}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{4}}{\frac{1}{2}} - 0\right) &&= \frac{1}{4} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{4}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{4}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^2 + 4x + 5}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{1}{4} - \left(-\frac{1}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{4x} + (-)\left(\frac{1}{2}\right) \\ &= -\frac{1}{4x} - \frac{1}{2} \\ &= -\frac{1}{4x} - \frac{1}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{4x} - \frac{1}{2}\right)(0) + \left(\left(\frac{1}{4x^2}\right) + \left(-\frac{1}{4x} - \frac{1}{2}\right)^2 - \left(\frac{4x^2+4x+5}{16x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(-\frac{1}{4x} - \frac{1}{2}\right) dx} \\ &= \frac{e^{-\frac{x}{2}}}{x^{1/4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^2+3x}{2x^2}\, dx} \\
&= z_1 e^{-\frac{x}{2}-\frac{3\ln(x)}{4}} \\
&= z_1 \left(\frac{e^{-\frac{x}{2}}}{x^{3/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{e^{-x}}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2+3x}{2x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-x-\frac{3\ln(x)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\sqrt{x}\, e^{x} - \frac{\sqrt{\pi}\, \operatorname{erfi}\left(\sqrt{x}\right)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{-x}}{x}\right) + c_2 \left(\frac{e^{-x}}{x}\left(\sqrt{x}\, e^{x} - \frac{\sqrt{\pi}\, \operatorname{erfi}\left(\sqrt{x}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(2x+3)\left(\frac{d}{dx}y(x)\right) - (1-x)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(x-1)y(x)}{2x^2} - \frac{(2x+3)\left(\frac{d}{dx}y(x)\right)}{2x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d^2}{dx^2}y(x) + \frac{(2x+3)\left(\frac{d}{dx}y(x)\right)}{2x} + \frac{(x-1)y(x)}{2x^2} = 0$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - ◦ Define functions

    $\left[P_2(x) = \frac{2x+3}{2x}, P_3(x) = \frac{x-1}{2x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{3}{2}$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\frac{1}{2}$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(2x+3)\left(\frac{d}{dx}y(x)\right) + (x-1)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - ◦ Shift index using $k- >k-m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - ◦ Shift index using $k- >k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - ◦ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-1+2r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(2k+2r-1) + a_{k-1}(2k+2r-1)\right)x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-1, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k+r-\frac{1}{2}\right)\left(a_k(k+r+1) + a_{k-1}\right) = 0$

- Shift index using $k- >k+1$

  $2\left(k+\frac{1}{2}+r\right)\left(a_{k+1}(k+2+r) + a_k\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{k+2+r}$

- Recursion relation for $r = -1$

  $a_{k+1} = -\frac{a_k}{k+1}$

- Solution for $r = -1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = -\frac{a_k}{k+1}\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = -\frac{a_k}{k+\frac{5}{2}}$

- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{a_k}{k+\frac{5}{2}}\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = -\frac{a_k}{k+1}, b_{k+1} = -\frac{b_k}{k+\frac{5}{2}}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Whittaker successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returnin
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.071 (sec)
Leaf size : 52

```
dsolve(2*x^2*diff(diff(y(x),x),x)+x*(2*x+3)*diff(y(x),x)-(1-x)*y(x) = 0,y(x),singsol=all
```

$$y = -\frac{3\left(2c_1(-x)^{3/2} + e^{-x}\left(xc_1\sqrt{\pi}\,\operatorname{erf}\left(\sqrt{-x}\right) - \frac{4c_2\sqrt{x}\,\sqrt{-x}}{3}\right)\right)}{4\sqrt{-x}\,x^{3/2}}$$

Mathematica DSolve solution

Solving time : 0.035 (sec)
Leaf size : 33

```
DSolve[{2*x^2*D[y[x],{x,2}]+x*(3+2*x)*D[y[x],x]-(1-x)*y[x]==0,{}},y[x],x,IncludeSingularSolu
```

$$y(x) \to \frac{e^{-x}\left(c_2 x^{3/2} L_{-\frac{3}{2}}^{\frac{3}{2}}(x) + c_1\right)}{x}$$

2.1.508 Problem 524

Internal problem ID [9680]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 524
Date solved : Monday, January 27, 2025 at 06:13:00 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$2x^2y'' + x(5+x)\,y' - (2-3x)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.253 (sec)

Writing the ode as

$$2x^2y'' + \left(x^2+5x\right)y' + (3x-2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= x^2+5x \\ C &= 3x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2-14x+21}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2-14x+21 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2-14x+21}{16x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.961: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16} + \frac{21}{16x^2} - \frac{7}{8x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{4} - \frac{7}{4x} - \frac{7}{2x^2} - \frac{49}{2x^3} - \frac{196}{x^4} - \frac{1715}{x^5} - \frac{31899}{2x^6} - \frac{309729}{2x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{4} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{16}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 14x + 21}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{1}{16}\right) + \left(\frac{-14x + 21}{16x^2}\right) \\ &= \frac{1}{16} + \frac{-14x + 21}{16x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -14. Dividing this by leading coefficient in t which is 16 gives $-\frac{7}{8}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{7}{8}\right) - (0) \\ &= -\frac{7}{8} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{4} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{7}{8}}{\frac{1}{4}} - 0\right) &&= -\frac{7}{4} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{7}{8}}{\frac{1}{4}} - 0\right) &&= \frac{7}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 14x + 21}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{4}$	$-\frac{7}{4}$	$\frac{7}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{7}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{7}{4} - \left(\frac{7}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{7}{4x} + (-)\left(\frac{1}{4}\right) \\ &= \frac{7}{4x} - \frac{1}{4} \\ &= -\frac{x-7}{4x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{7}{4x} - \frac{1}{4}\right)(0) + \left(\left(-\frac{7}{4x^2}\right) + \left(\frac{7}{4x} - \frac{1}{4}\right)^2 - \left(\frac{x^2-14x+21}{16x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{7}{4x} - \frac{1}{4}\right) dx} \\ &= x^{7/4} e^{-\frac{x}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2+5x}{2x^2}\,dx} \\
&= z_1 e^{-\frac{x}{4}-\frac{5\ln(x)}{4}} \\
&= z_1\left(\frac{\mathrm{e}^{-\frac{x}{4}}}{x^{5/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \sqrt{x}\,\mathrm{e}^{-\frac{x}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+5x}{2x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x}{2}-\frac{5\ln(x)}{2}}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{2\,\mathrm{e}^{\frac{x}{2}}}{5x^{5/2}}-\frac{2\,\mathrm{e}^{\frac{x}{2}}}{15x^{3/2}}-\frac{2\,\mathrm{e}^{\frac{x}{2}}}{15\sqrt{x}}-\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,\sqrt{x}}{2}\right)}{15}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\sqrt{x}\,\mathrm{e}^{-\frac{x}{2}}\right) + c_2\left(\sqrt{x}\,\mathrm{e}^{-\frac{x}{2}}\left(-\frac{2\,\mathrm{e}^{\frac{x}{2}}}{5x^{5/2}}-\frac{2\,\mathrm{e}^{\frac{x}{2}}}{15x^{3/2}}-\frac{2\,\mathrm{e}^{\frac{x}{2}}}{15\sqrt{x}}-\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,\sqrt{x}}{2}\right)}{15}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(5+x)\left(\frac{d}{dx}y(x)\right) - (-3x+2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(3x-2)y(x)}{2x^2} - \frac{(5+x)\left(\frac{d}{dx}y(x)\right)}{2x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(5+x)\left(\frac{d}{dx}y(x)\right)}{2x} + \frac{(3x-2)y(x)}{2x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = \frac{5+x}{2x}, P_3(x) = \frac{3x-2}{2x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{5}{2}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -1$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(5+x)\left(\frac{d}{dx}y(x)\right) + (3x-2)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)\,(-1+2r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+2)\,(2k+2r-1) + a_{k-1}(k+r+2)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-2, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(\left(k+r-\frac{1}{2}\right) a_k + \frac{a_{k-1}}{2}\right)(k+r+2) = 0$

- Shift index using $k- >k+1$

  $2\left(\left(k+\frac{1}{2}+r\right) a_{k+1} + \frac{a_k}{2}\right)(k+r+3) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{2k+1+2r}$

- Recursion relation for $r = -2$

  $a_{k+1} = -\frac{a_k}{2k-3}$

- Solution for $r = -2$

$$\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+1}=-\frac{a_k}{2k-3}\right]$$

- Recursion relation for $r=\frac{1}{2}$

  $a_{k+1}=-\frac{a_k}{2k+2}$
- Solution for $r=\frac{1}{2}$

$$\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1}=-\frac{a_k}{2k+2}\right]$$

- Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^{k-2}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1}=-\frac{a_k}{2k-3}, b_{k+1}=-\frac{b_k}{2k+2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 52

```
dsolve(2*x^2*diff(diff(y(x),x),x)+x*(x+5)*diff(y(x),x)-(2-3*x)*y(x) = 0,y(x),singsol=all
```

$$y=\frac{i e^{-\frac{x}{2}} x^{5/2} \sqrt{2}\, \operatorname{erf}\left(\frac{i \sqrt{2}\, \sqrt{x}}{2}\right) \sqrt{\pi}\, c_2+c_1 x^{5/2} \mathrm{e}^{-\frac{x}{2}}+2 c_2\left(x^2+x+3\right)}{x^2}$$

Mathematica DSolve solution

Solving time : `0.153` (sec)
Leaf size : 94

```
DSolve[{2*x^2*D[y[x],{x,2}]+x*(5+x)*D[y[x],x]-(2-3*x)*y[x]==0,{}},y[x],x,IncludeSingularSolu
```

$$y(x) \to \frac{e^{-\frac{x}{2}-\frac{5}{4}}\left(15c_1x^{5/2} - 2c_2e^{x/2}x^2 - 6c_2e^{x/2} - 2c_2e^{x/2}x + \sqrt{2}c_2(-x)^{5/2}\Gamma\left(\frac{1}{2}, -\frac{x}{2}\right)\right)}{15x^2}$$

2.1.509 Problem 525

Internal problem ID [9681]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 525
Date solved : `Monday, January 27, 2025 at 06:13:01 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$3x^2y'' + x(1+x)\,y' - y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.411 (sec)

Writing the ode as

$$3x^2y'' + \left(x^2+x\right)y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^2 \\ B &= x^2 + x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2+2x+7}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 2x + 7 \\ t &= 36x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2+2x+7}{36x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.963: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{36} + \frac{7}{36x^2} + \frac{1}{18x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{6} + \frac{1}{6x} + \frac{1}{2x^2} - \frac{1}{2x^3} - \frac{1}{4x^4} + \frac{7}{4x^5} - \frac{7}{4x^6} - \frac{17}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{6} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{36}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 2x + 7}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{1}{36}\right) + \left(\frac{2x + 7}{36x^2}\right) \\ &= \frac{1}{36} + \frac{2x + 7}{36x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 2. Dividing this by leading coefficient in t which is 36 gives $\frac{1}{18}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{18}\right) - (0) \\ &= \frac{1}{18} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{6} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{1}{18}}{\frac{1}{6}} - 0\right) &= \frac{1}{6} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{1}{18}}{\frac{1}{6}} - 0\right) &= -\frac{1}{6} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 2x + 7}{36x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{6}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{6} - \left(-\frac{1}{6}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{6x} + (-)\left(\frac{1}{6}\right) \\
&= -\frac{1}{6x} - \frac{1}{6} \\
&= -\frac{1+x}{6x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{6x} - \frac{1}{6}\right)(0) + \left(\left(\frac{1}{6x^2}\right) + \left(-\frac{1}{6x} - \frac{1}{6}\right)^2 - \left(\frac{x^2+2x+7}{36x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{1}{6x} - \frac{1}{6}\right) dx} \\
&= \frac{e^{-\frac{x}{6}}}{x^{1/6}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^2+x}{3x^2}\,dx} \\ &= z_1 e^{-\frac{x}{6}-\frac{\ln(x)}{6}} \\ &= z_1\left(\frac{\mathrm{e}^{-\frac{x}{6}}}{x^{1/6}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-\frac{x}{3}}}{x^{1/3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+x}{3x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{\mathrm{e}^{-\frac{x}{3}-\frac{\ln(x)}{3}}}{(y_1)^2}\,dx \\ &= y_1\left(\int \mathrm{e}^{-\frac{x}{3}-\frac{\ln(x)}{3}} x^{2/3}\mathrm{e}^{\frac{2x}{3}}\,dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\mathrm{e}^{-\frac{x}{3}}}{x^{1/3}}\right) + c_2\left(\frac{\mathrm{e}^{-\frac{x}{3}}}{x^{1/3}}\left(\int \mathrm{e}^{-\frac{x}{3}-\frac{\ln(x)}{3}} x^{2/3}\mathrm{e}^{\frac{2x}{3}}\,dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$3x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(x+1)\left(\frac{d}{dx}y(x)\right) - y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{y(x)}{3x^2} - \frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{3x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{3x} - \frac{y(x)}{3x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{x+1}{3x}, P_3(x) = -\frac{1}{3x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$(x \cdot P_2(x))\Big|_{x=0} = \frac{1}{3}$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{1}{3}$
- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$
- • Multiply by denominators

  $3x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(x+1)\left(\frac{d}{dx}y(x)\right) - y(x) = 0$
- • Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- □ Rewrite ODE with series expansions
  - ○ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
  - ○ Shift index using $k- >k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - ○ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+3r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(3k+3r+1)\,(k+r-1) + a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$
- • $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+3r)\,(-1+r) = 0$
- • Values of r that satisfy the indicial equation

  $r \in \left\{1, -\frac{1}{3}\right\}$
- • Each term in the series must be 0, giving the recursion relation

  $3\left(\left(k+r+\frac{1}{3}\right) a_k + \frac{a_{k-1}}{3}\right)(k+r-1) = 0$
- • Shift index using $k- >k+1$

  $3\left(\left(k+\frac{4}{3}+r\right) a_{k+1} + \frac{a_k}{3}\right)(k+r) = 0$
- • Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{3k+4+3r}$
- • Recursion relation for $r = 1$

  $a_{k+1} = -\frac{a_k}{3k+7}$
- • Solution for $r = 1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{a_k}{3k+7}\right]$
- • Recursion relation for $r = -\frac{1}{3}$

  $a_{k+1} = -\frac{a_k}{3k+3}$
- • Solution for $r = -\frac{1}{3}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}, a_{k+1} = -\frac{a_k}{3k+3}\right]$
- • Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty}a_k x^{k+1}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k-\frac{1}{3}}\right), a_{k+1}=-\frac{a_k}{3k+7}, b_{k+1}=-\frac{b_k}{3k+3}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Whittaker successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returnin
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.081 (sec)
Leaf size : 30

```
dsolve(3*x^2*diff(diff(y(x),x),x)+x*(x+1)*diff(y(x),x)-y(x) = 0,y(x),singsol=all)
```

$$y=\frac{\mathrm{e}^{-\frac{x}{6}}\left(x^{1/6}\,\mathrm{WhittakerM}\left(-\frac{1}{6},\frac{2}{3},\frac{x}{3}\right)c_1+\mathrm{e}^{-\frac{x}{6}}c_2\right)}{x^{1/3}}$$

Mathematica DSolve solution

Solving time : 0.062 (sec)
Leaf size : 56

```
DSolve[{3*x^2*D[y[x],{x,2}]+x*(1+x)*D[y[x],x]-y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tru
```

$$y(x)\to\frac{e^{\frac{1}{3}(-x-1)}\left(c_2x^{2/3}-3\sqrt[3]{3}ec_1(-x)^{2/3}\Gamma\left(\frac{4}{3},-\frac{x}{3}\right)\right)}{x}$$

2.1.510 Problem 526

Internal problem ID [9682]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 526
Date solved : `Monday, January 27, 2025 at 06:13:02 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2y'' - xy' + (1-2x)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.158 (sec)

Writing the ode as

$$2x^2y'' - xy' + (1-2x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= -x \\ C &= 1-2x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3+16x}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3+16x \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3+16x}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.965: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{x} - \frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1,2,3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 1,\ e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(1 - (1)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{1}{(x-(0))}\right) \\ &= \frac{1}{2x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2x} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2x} + \frac{1-16x}{16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{1+4\sqrt{x}}{4x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= e^{\int \omega\, dx} \\ &= e^{\int \frac{1+4\sqrt{x}}{4x} dx} \\ &= x^{1/4} e^{2\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{2x^2}\,dx} \\
&= z_1 e^{\frac{\ln(x)}{4}} \\
&= z_1\left(x^{1/4}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \sqrt{x}\,\mathrm{e}^{2\sqrt{x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{2x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{\ln(x)}{2}}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{\mathrm{e}^{-4\sqrt{x}}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x}\,\mathrm{e}^{2\sqrt{x}}\right) + c_2\left(\sqrt{x}\,\mathrm{e}^{2\sqrt{x}}\left(-\frac{\mathrm{e}^{-4\sqrt{x}}}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2\left(\frac{d^2}{dx^2}y(x)\right) - x\left(\frac{d}{dx}y(x)\right) + (-2x+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{(2x-1)y(x)}{2x^2} + \frac{\frac{d}{dx}y(x)}{2x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) - \frac{\frac{d}{dx}y(x)}{2x} - \frac{(2x-1)y(x)}{2x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{1}{2x}, P_3(x) = -\frac{2x-1}{2x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -\frac{1}{2}$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{1}{2}$
- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$
- • Multiply by denominators

  $2x^2\left(\frac{d^2}{dx^2}y(x)\right) - x\left(\frac{d}{dx}y(x)\right) + (-2x+1)\,y(x) = 0$
- • Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- □ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - ◦ Shift index using $k- >k - m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - ◦ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$
  - ◦ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)\,(k+r-1) - 2a_{k-1}\right) x^{k+r}\right) = 0$
- • $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)\,(-1+r) = 0$
- • Values of r that satisfy the indicial equation

  $r \in \left\{1, \frac{1}{2}\right\}$
- • Each term in the series must be 0, giving the recursion relation

  $2\left(k+r-\frac{1}{2}\right)(k+r-1)\, a_k - 2a_{k-1} = 0$
- • Shift index using $k- >k + 1$

  $2\left(k+\frac{1}{2}+r\right)(k+r)\, a_{k+1} - 2a_k = 0$
- • Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k}{(2k+1+2r)(k+r)}$
- • Recursion relation for $r = 1$

  $a_{k+1} = \frac{2a_k}{(2k+3)(k+1)}$
- • Solution for $r = 1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = \frac{2a_k}{(2k+3)(k+1)}\right]$
- • Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = \frac{2a_k}{(2k+2)\left(k+\frac{1}{2}\right)}$
- • Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{2a_k}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$

- Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^{k+1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1}=\frac{2a_k}{(2k+3)(k+1)}, b_{k+1}=\frac{2b_k}{(2k+2)(k+\frac{1}{2})}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 25

```
dsolve(2*x^2*diff(diff(y(x),x),x)-diff(y(x),x)*x+(1-2*x)*y(x) = 0,y(x),singsol=all)
```

$$y=\sqrt{x}\left(c_1\sinh\left(2\sqrt{x}\right)+c_2\cosh\left(2\sqrt{x}\right)\right)$$

Mathematica DSolve solution

Solving time : 0.05 (sec)
Leaf size : 41

```
DSolve[{2*x^2*D[y[x],{x,2}]-x*D[y[x],x]+(1-2*x)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x)\to\frac{1}{2}e^{-2\sqrt{x}}\sqrt{x}\left(2c_1e^{4\sqrt{x}}-c_2\right)$$

2.1.511 Problem 527

Internal problem ID [9683]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 527
Date solved : `Monday, January 27, 2025 at 06:13:02 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$3x^2y'' + x(1+x)\, y' - (1+3x)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.558 (sec)

Writing the ode as

$$3x^2y'' + \left(x^2+x\right) y' + (-3x-1)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^2 \\ B &= x^2+x \\ C &= -3x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2+38x+7}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2+38x+7 \\ t &= 36x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2+38x+7}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.967: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{36} + \frac{19}{18x} + \frac{7}{36x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{6} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{6} + \frac{19}{6x} - \frac{59}{2x^2} + \frac{1121}{2x^3} - \frac{53041}{4x^4} + \frac{1404613}{4x^5} - \frac{39845827}{4x^6} + \frac{1184064097}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{6} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{36}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 38x + 7}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{1}{36}\right) + \left(\frac{38x + 7}{36x^2}\right) \\ &= \frac{1}{36} + \frac{38x + 7}{36x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 38. Dividing this by leading coefficient in t which is 36 gives $\frac{19}{18}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{19}{18}\right) - (0) \\ &= \frac{19}{18} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{6} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{19}{18}}{\frac{1}{6}} - 0\right) &= \frac{19}{6} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{19}{18}}{\frac{1}{6}} - 0\right) &= -\frac{19}{6} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 38x + 7}{36x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{6}$	$\frac{19}{6}$	$-\frac{19}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{19}{6}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+\right) \\ &= \frac{19}{6} - \left(\frac{7}{6}\right) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{7}{6x} + \left(\frac{1}{6}\right) \\ &= \frac{7}{6x} + \frac{1}{6} \\ &= \frac{7+x}{6x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{7}{6x} + \frac{1}{6}\right)(2x + a_1) + \left(\left(-\frac{7}{6x^2}\right) + \left(\frac{7}{6x} + \frac{1}{6}\right)^2 - \left(\frac{x^2 + 38x + 7}{36x^2}\right)\right) = 0$$
$$\frac{(-a_1 + 20)\,x - 2a_0 + 7a_1}{3x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 70, a_1 = 20\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + 20x + 70$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x^2+20x+70\right) e^{\int \left(\frac{7}{6x}+\frac{1}{6}\right)dx} \\ &= \left(x^2+20x+70\right) e^{\frac{x}{6}+\frac{7\ln(x)}{6}} \\ &= \left(x^2+20x+70\right) x^{7/6} e^{\frac{x}{6}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^2+x}{3x^2}\, dx} \\ &= z_1 e^{-\frac{x}{6}-\frac{\ln(x)}{6}} \\ &= z_1 \left(\frac{e^{-\frac{x}{6}}}{x^{1/6}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \left(x^2+20x+70\right) x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+x}{3x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{x}{3}-\frac{\ln(x)}{3}}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{e^{-\frac{x}{3}-\frac{\ln(x)}{3}}}{\left(x^2+20x+70\right)^2 x^2} dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\left(x^2+20x+70\right) x\right) + c_2\left(\left(x^2+20x+70\right) x \left(\int \frac{e^{-\frac{x}{3}-\frac{\ln(x)}{3}}}{\left(x^2+20x+70\right)^2 x^2} dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$3x^2\left(\frac{d^2}{dx^2}y(x)\right)+x(x+1)\left(\frac{d}{dx}y(x)\right)-(3x+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=\frac{(3x+1)y(x)}{3x^2}-\frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{3x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x)+\frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{3x}-\frac{(3x+1)y(x)}{3x^2}=0$

☐ Check to see if $x_0=0$ is a regular singular point

- Define functions

  $\left[P_2(x)=\frac{x+1}{3x},P_3(x)=-\frac{3x+1}{3x^2}\right]$

- $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{1}{3}$

- $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=-\frac{1}{3}$

- $x=0$is a regular singular point

Check to see if $x_0=0$ is a regular singular point

$x_0=0$

- Multiply by denominators

$3x^2\left(\frac{d^2}{dx^2}y(x)\right)+x(x+1)\left(\frac{d}{dx}y(x)\right)+(-3x-1)\,y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- Shift index using $k->k-m$

  $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

Rewrite ODE with series expansions

$a_0(1+3r)\,(-1+r)\,x^r+\left(\sum_{k=1}^{\infty}\left(a_k(3k+3r+1)\,(k+r-1)+a_{k-1}(k-4+r)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(1+3r)(-1+r)=0$

- Values of r that satisfy the indicial equation
  $r \in \{1, -\frac{1}{3}\}$
- Each term in the series must be 0, giving the recursion relation
  $3\left(k+r+\frac{1}{3}\right)(k+r-1)a_k + a_{k-1}(k-4+r) = 0$
- Shift index using $k->k+1$
  $3\left(k+\frac{4}{3}+r\right)(k+r)a_{k+1} + a_k(k+r-3) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = -\frac{a_k(k+r-3)}{(3k+4+3r)(k+r)}$
- Recursion relation for $r=1$; series terminates at $k=2$
  $a_{k+1} = -\frac{a_k(k-2)}{(3k+7)(k+1)}$
- Apply recursion relation for $k=0$
  $a_1 = \frac{2a_0}{7}$
- Apply recursion relation for $k=1$
  $a_2 = \frac{a_1}{20}$
- Express in terms of a_0
  $a_2 = \frac{a_0}{70}$
- Terminating series solution of the ODE for $r=1$. Use reduction of order to find the second li
  $y(x) = a_0 \cdot \left(1+\frac{2}{7}x+\frac{1}{70}x^2\right)$
- Recursion relation for $r=-\frac{1}{3}$
  $a_{k+1} = -\frac{a_k\left(k-\frac{10}{3}\right)}{(3k+3)\left(k-\frac{1}{3}\right)}$
- Solution for $r=-\frac{1}{3}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}, a_{k+1} = -\frac{a_k\left(k-\frac{10}{3}\right)}{(3k+3)\left(k-\frac{1}{3}\right)}\right]$
- Combine solutions and rename parameters
  $\left[y(x) = a_0 \cdot \left(1+\frac{2}{7}x+\frac{1}{70}x^2\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{3}}\right), b_{k+1} = -\frac{b_k\left(k-\frac{10}{3}\right)}{(3k+3)\left(k-\frac{1}{3}\right)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric sol
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
```

```
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returnin
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.045 (sec)
Leaf size : 41

```
dsolve(3*x^2*diff(diff(y(x),x),x)+x*(x+1)*diff(y(x),x)-(3*x+1)*y(x) = 0,y(x),singsol=all
```

$$y = \frac{c_2 \mathrm{e}^{-\frac{x}{3}} \operatorname{hypergeom}\left([3], \left[-\frac{1}{3}\right], \frac{x}{3}\right) + 70 c_1 \left(x^{4/3} + \frac{2 x^{7/3}}{7} + \frac{x^{10/3}}{70}\right)}{x^{1/3}}$$

Mathematica DSolve solution

Solving time : 0.357 (sec)
Leaf size : 60

```
DSolve[{3*x^2*D[y[x],{x,2}]+x*(1+x)*D[y[x],x]-(1+3*x)*y[x]==0,{}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to ex\left(x^2 + 20x + 70\right) \left(c_2 \int_1^x \frac{e^{-\frac{K[1]}{3} - \frac{7}{3}}}{K[1]^{7/3} \left(K[1]^2 + 20K[1] + 70\right)^2} dK[1] + c_1\right)$$

2.1.512 Problem 528

Internal problem ID [9684]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 528
Date solved : `Monday, January 27, 2025 at 06:13:03 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(3+x)\,y''+x(1+5x)\,y'+(1+x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.265 (sec)

Writing the ode as

$$\left(2x^3+6x^2\right)y''+\left(5x^2+x\right)y'+(1+x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^3+6x^2 \\ B &= 5x^2+x \\ C &= 1+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-3x^2-30x-35}{16\left(x^2+3x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3x^2-30x-35 \\ t &= 16\left(x^2+3x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-3x^2-30x-35}{16\left(x^2+3x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.969: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2+3x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -3$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{108\,(3+x)} - \frac{5}{108x} - \frac{35}{144x^2} + \frac{7}{36\,(3+x)^2}$$

For the pole at $x = -3$ let b be the coefficient of $\frac{1}{(3+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 - 30x - 35}{16\left(x^2 + 3x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x^2 - 30x - 35}{16\left(x^2 + 3x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-3	2	0	$\frac{7}{6}$	$-\frac{1}{6}$
0	2	0	$\frac{7}{12}$	$\frac{5}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{1}{4} - \left(\frac{1}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{6\,(3+x)} + \frac{5}{12x} + (-)\,(0) \\
&= -\frac{1}{6\,(3+x)} + \frac{5}{12x} \\
&= \frac{x+5}{4x\,(3+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{6\,(3+x)} + \frac{5}{12x}\right)(0) + \left(\left(\frac{1}{6\,(3+x)^2} - \frac{5}{12x^2}\right) + \left(-\frac{1}{6\,(3+x)} + \frac{5}{12x}\right)^2 - \left(\frac{-3x^2 - 30x - 35}{16\,(x^2+3x)^2}\right)\right.$$
$$0 =$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{6(3+x)} + \frac{5}{12x}\right) dx} \\ &= \frac{x^{5/12}}{(3+x)^{1/6}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{5x^2+x}{2x^3+6x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{12} - \frac{7\ln(3+x)}{6}} \\ &= z_1 \left(\frac{1}{x^{1/12}\,(3+x)^{7/6}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/3}}{(3+x)^{4/3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{5x^2+x}{2x^3+6x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{\ln(x)}{6} - \frac{7\ln(3+x)}{3}}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{-\frac{\ln(x)}{6} - \frac{7\ln(3+x)}{3}}\,(3+x)^{8/3}}{x^{2/3}} dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{x^{1/3}}{(3+x)^{4/3}}\right) + c_2 \left(\frac{x^{1/3}}{(3+x)^{4/3}} \left(\int \frac{\mathrm{e}^{-\frac{\ln(x)}{6} - \frac{7\ln(3+x)}{3}}\,(3+x)^{8/3}}{x^{2/3}} dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2(x+3)\left(\frac{d^2}{dx^2}y(x)\right)+x(5x+1)\left(\frac{d}{dx}y(x)\right)+(x+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(x+1)y(x)}{2x^2(x+3)}-\frac{(5x+1)\left(\frac{d}{dx}y(x)\right)}{2x(x+3)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{(5x+1)\left(\frac{d}{dx}y(x)\right)}{2x(x+3)}+\frac{(x+1)y(x)}{2x^2(x+3)}=0$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x)=\frac{5x+1}{2x(x+3)},P_3(x)=\frac{x+1}{2x^2(x+3)}\right]$
  - $(x+3)\cdot P_2(x)$ is analytic at $x=-3$

    $\left((x+3)\cdot P_2(x)\right)\Big|_{x=-3}=\frac{7}{3}$
  - $(x+3)^2\cdot P_3(x)$ is analytic at $x=-3$

    $\left((x+3)^2\cdot P_3(x)\right)\Big|_{x=-3}=0$
  - $x=-3$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-3$

- Multiply by denominators

  $2x^2(x+3)\left(\frac{d^2}{dx^2}y(x)\right)+x(5x+1)\left(\frac{d}{dx}y(x)\right)+(x+1)\,y(x)=0$

- Change variables using $x=u-3$ so that the regular singular point is at $u=0$

  $(2u^3-12u^2+18u)\left(\frac{d^2}{du^2}y(u)\right)+(5u^2-29u+42)\left(\frac{d}{du}y(u)\right)+(u-2)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

    $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$
  - Shift index using $k->k-m$

    $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$
  - Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$6a_0r(4+3r)\,u^{-1+r} + (6a_1(1+r)\,(7+3r) - a_0(12r^2+17r+2))\,u^r + \left(\sum_{k=1}^{\infty} (6a_{k+1}(k+r+1)\,(3k+\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $6r(4+3r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, -\frac{4}{3}\}$
- Each term must be 0
  $6a_1(1+r)\,(7+3r) - a_0(12r^2+17r+2) = 0$
- Each term in the series must be 0, giving the recursion relation
  $2(-6a_k + a_{k-1} + 9a_{k+1})\,k^2 + (4(-6a_k + a_{k-1} + 9a_{k+1})\,r - 17a_k - a_{k-1} + 60a_{k+1})\,k + 2(-6a_k + a_{k-}$
- Shift index using $k- >k+1$
  $2(-6a_{k+1} + a_k + 9a_{k+2})\,(k+1)^2 + (4(-6a_{k+1} + a_k + 9a_{k+2})\,r - 17a_{k+1} - a_k + 60a_{k+2})\,(k+1) + 2$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} + 4kra_k - 24kra_{k+1} + 2r^2a_k - 12r^2a_{k+1} + 3ka_k - 41ka_{k+1} + 3ra_k - 41ra_{k+1} + a_k - 31a_{k+1}}{6(3k^2+6kr+3r^2+16k+16r+20)}$
- Recursion relation for $r = 0$
  $a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} + 3ka_k - 41ka_{k+1} + a_k - 31a_{k+1}}{6(3k^2+16k+20)}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} + 3ka_k - 41ka_{k+1} + a_k - 31a_{k+1}}{6(3k^2+16k+20)}, 42a_1 - 2a_0 = 0\right]$
- Revert the change of variables $u = x+3$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+3)^k, a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} + 3ka_k - 41ka_{k+1} + a_k - 31a_{k+1}}{6(3k^2+16k+20)}, 42a_1 - 2a_0 = 0\right]$
- Recursion relation for $r = -\frac{4}{3}$
  $a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} - \frac{7}{3}ka_k - 9ka_{k+1} + \frac{5}{9}a_k + \frac{7}{3}a_{k+1}}{6(3k^2+8k+4)}$
- Solution for $r = -\frac{4}{3}$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{4}{3}}, a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} - \frac{7}{3}ka_k - 9ka_{k+1} + \frac{5}{9}a_k + \frac{7}{3}a_{k+1}}{6(3k^2+8k+4)}, -6a_1 - \frac{2a_0}{3} = 0\right]$
- Revert the change of variables $u = x+3$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+3)^{k-\frac{4}{3}}, a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} - \frac{7}{3}ka_k - 9ka_{k+1} + \frac{5}{9}a_k + \frac{7}{3}a_{k+1}}{6(3k^2+8k+4)}, -6a_1 - \frac{2a_0}{3} = 0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k(x+3)^k\right) + \left(\sum_{k=0}^{\infty} b_k(x+3)^{k-\frac{4}{3}}\right), a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} + 3ka_k - 41ka_{k+1} + a_k - 31a_{k+1}}{6(3k^2+16k+20)}, 42\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
```

```
Group is reducible, not completely reducible
Solution has integrals. Trying a special function solution free of integrals...
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful
   -> Trying to convert hypergeometric functions to elementary form...
   <- elementary form for at least one hypergeometric solution is achieved - return
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.091 (sec)
Leaf size : 36

```
dsolve(2*x^2*(x+3)*diff(diff(y(x),x),x)+x*(5*x+1)*diff(y(x),x)+(x+1)*y(x) = 0,y(x),si
```

$$y = c_1\sqrt{x}\,\mathrm{hypergeom}\left(\left[1, \frac{3}{2}\right], \left[\frac{7}{6}\right], -\frac{x}{3}\right) + \frac{c_2 x^{1/3}}{(x+3)\left(1+\frac{x}{3}\right)^{1/3}}$$

Mathematica DSolve solution

Solving time : 0.236 (sec)
Leaf size : 108

```
DSolve[{2*x^2*(3+x)*D[y[x],{x,2}]+x*(1+5*x)*D[y[x],x]+(1+x)*y[x]==0,{}},y[x],x,IncludeSingul
```

$$y(x) \to \exp\left(\int_1^x \frac{K[1]+5}{4K[1]^2+12K[1]}dK[1] - \frac{1}{2}\int_1^x \frac{5K[2]+1}{2K[2]^2+6K[2]}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{K[1]+5}{4K[1]^2+12K[1]}dK[1]\right)dK[3] + c_1\right)$$

2.1.513 Problem 529

Internal problem ID [9685]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 529
Date solved : `Monday, January 27, 2025 at 06:13:04 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(4+x)\,y'' - x(1-3x)\,y' + y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.487 (sec)

Writing the ode as

$$x^2(4+x)\,y'' + \left(3x^2 - x\right)y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(4+x) \\ B &= 3x^2 - x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^2 - 6x - 7}{4\left(x^2 + 4x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3x^2 - 6x - 7 \\ t &= 4\left(x^2 + 4x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^2 - 6x - 7}{4\left(x^2 + 4x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.971: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + 4x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -4$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{65}{64\,(4+x)^2} + \frac{5}{128\,(4+x)} - \frac{7}{64x^2} - \frac{5}{128x}$$

For the pole at $x = -4$ let b be the coefficient of $\frac{1}{(4+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{65}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{13}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{8} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x^2 - 6x - 7}{4(x^2 + 4x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3x^2 - 6x - 7}{4(x^2 + 4x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-4	2	0	$\frac{13}{8}$	$-\frac{5}{8}$
0	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{5}{8(4+x)} + \frac{1}{8x} + (-)(0) \\
&= -\frac{5}{8(4+x)} + \frac{1}{8x} \\
&= -\frac{x-1}{2x(4+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{5}{8\,(4+x)} + \frac{1}{8x}\right)(0) + \left(\left(\frac{5}{8\,(4+x)^2} - \frac{1}{8x^2}\right) + \left(-\frac{5}{8\,(4+x)} + \frac{1}{8x}\right)^2 - \left(\frac{3x^2-6x-7}{4\,(x^2+4x)^2}\right)\right) =$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(-\frac{5}{8(4+x)} + \frac{1}{8x}\right) dx} \\ &= \frac{x^{1/8}}{(4+x)^{5/8}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3x^2-x}{x^2(4+x)}\, dx} \\ &= z_1 e^{-\frac{13\ln(4+x)}{8} + \frac{\ln(x)}{8}} \\ &= z_1 \left(\frac{x^{1/8}}{(4+x)^{13/8}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/4}}{(4+x)^{9/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3x^2-x}{x^2(4+x)}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{13\ln(4+x)}{4} + \frac{\ln(x)}{4}}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{-\frac{13\ln(4+x)}{4} + \frac{\ln(x)}{4}}(4+x)^{9/2}}{\sqrt{x}}\, dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{x^{1/4}}{(4+x)^{9/4}}\right) + c_2\left(\frac{x^{1/4}}{(4+x)^{9/4}}\left(\int \frac{\mathrm{e}^{-\frac{13\ln(4+x)}{4} + \frac{\ln(x)}{4}}(4+x)^{9/2}}{\sqrt{x}}\, dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x+4)\left(\frac{d^2}{dx^2}y(x)\right)-x(1-3x)\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{y(x)}{x^2(x+4)}-\frac{(3x-1)\left(\frac{d}{dx}y(x)\right)}{x(x+4)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x)+\frac{(3x-1)\left(\frac{d}{dx}y(x)\right)}{x(x+4)}+\frac{y(x)}{x^2(x+4)}=0$

□ Check to see if x_0 is a regular singular point

- Define functions

$\left[P_2(x)=\frac{3x-1}{x(x+4)},P_3(x)=\frac{1}{x^2(x+4)}\right]$

- $(x+4)\cdot P_2(x)$ is analytic at $x=-4$

$\left((x+4)\cdot P_2(x)\right)\Big|_{x=-4}=\frac{13}{4}$

- $(x+4)^2\cdot P_3(x)$ is analytic at $x=-4$

$\left((x+4)^2\cdot P_3(x)\right)\Big|_{x=-4}=0$

- $x=-4$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-4$

- Multiply by denominators

$x^2(x+4)\left(\frac{d^2}{dx^2}y(x)\right)+x(3x-1)\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Change variables using $x=u-4$ so that the regular singular point is at $u=0$

$(u^3-8u^2+16u)\left(\frac{d^2}{du^2}y(u)\right)+(3u^2-25u+52)\left(\frac{d}{du}y(u)\right)+y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

Rewrite ODE with series expansions

$4a_0r(9+4r)\,u^{-1+r}+\left(4a_1(1+r)(13+4r)-a_0(8r^2+17r-1)\right)u^r+\left(\sum_{k=1}^{\infty}(4a_{k+1}(k+1+r)(4k+\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$4r(9+4r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0, -\frac{9}{4}\right\}$$

- Each term must be 0

$$4a_1(1+r)(13+4r)-a_0(8r^2+17r-1)=0$$

- Each term in the series must be 0, giving the recursion relation

$$(-8a_k+a_{k-1}+16a_{k+1})k^2+(2(-8a_k+a_{k-1}+16a_{k+1})r-17a_k+68a_{k+1})k+(-8a_k+a_{k-1}+$$

- Shift index using $k->k+1$

$$(-8a_{k+1}+a_k+16a_{k+2})(k+1)^2+(2(-8a_{k+1}+a_k+16a_{k+2})r-17a_{k+1}+68a_{k+2})(k+1)+(-$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{k^2a_k-8k^2a_{k+1}+2kra_k-16kra_{k+1}+r^2a_k-8r^2a_{k+1}+2ka_k-33ka_{k+1}+2ra_k-33ra_{k+1}-24a_{k+1}}{4(4k^2+8kr+4r^2+25k+25r+34)}$$

- Recursion relation for $r=0$

$$a_{k+2}=-\frac{k^2a_k-8k^2a_{k+1}+2ka_k-33ka_{k+1}-24a_{k+1}}{4(4k^2+25k+34)}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty}a_ku^k, a_{k+2}=-\frac{k^2a_k-8k^2a_{k+1}+2ka_k-33ka_{k+1}-24a_{k+1}}{4(4k^2+25k+34)}, 52a_1+a_0=0\right]$$

- Revert the change of variables $u=x+4$

$$\left[y(x)=\sum_{k=0}^{\infty}a_k(x+4)^k, a_{k+2}=-\frac{k^2a_k-8k^2a_{k+1}+2ka_k-33ka_{k+1}-24a_{k+1}}{4(4k^2+25k+34)}, 52a_1+a_0=0\right]$$

- Recursion relation for $r=-\frac{9}{4}$

$$a_{k+2}=-\frac{k^2a_k-8k^2a_{k+1}-\frac{5}{2}ka_k+3ka_{k+1}+\frac{9}{16}a_k+\frac{39}{4}a_{k+1}}{4(4k^2+7k-2)}$$

- Solution for $r=-\frac{9}{4}$

$$\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k-\frac{9}{4}}, a_{k+2}=-\frac{k^2a_k-8k^2a_{k+1}-\frac{5}{2}ka_k+3ka_{k+1}+\frac{9}{16}a_k+\frac{39}{4}a_{k+1}}{4(4k^2+7k-2)}, -20a_1-\frac{5a_0}{4}=0\right]$$

- Revert the change of variables $u=x+4$

$$\left[y(x)=\sum_{k=0}^{\infty}a_k(x+4)^{k-\frac{9}{4}}, a_{k+2}=-\frac{k^2a_k-8k^2a_{k+1}-\frac{5}{2}ka_k+3ka_{k+1}+\frac{9}{16}a_k+\frac{39}{4}a_{k+1}}{4(4k^2+7k-2)}, -20a_1-\frac{5a_0}{4}=0\right]$$

- Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty}a_k(x+4)^k\right)+\left(\sum_{k=0}^{\infty}b_k(x+4)^{k-\frac{9}{4}}\right), a_{k+2}=-\frac{k^2a_k-8k^2a_{k+1}+2ka_k-33ka_{k+1}-24a_{k+1}}{4(4k^2+25k+34)}, 52a_1\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
```

```
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
   <- hypergeometric successful
<- special function solution successful
   -> Trying to convert hypergeometric functions to elementary form...
   <- elementary form for at least one hypergeometric solution is achieved - returnin
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.122 (sec)
Leaf size : 27

```
dsolve(x^2*(x+4)*diff(diff(y(x),x),x)-x*(-3*x+1)*diff(y(x),x)+y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_1 x^{1/4}}{(x+4)^{9/4}} + c_2 \,\mathrm{hypergeom}\left([1,3], \left[\frac{7}{4}\right], -\frac{x}{4}\right) x$$

Mathematica DSolve solution

Solving time : 0.208 (sec)
Leaf size : 109

```
DSolve[{x^2*(4+x)*D[y[x],{x,2}]-x*(1-3*x)*D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \exp\left(\int_1^x \frac{1-K[1]}{2K[1]^2+8K[1]}dK[1] - \frac{1}{2}\int_1^x \frac{3K[2]-1}{K[2](K[2]+4)}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{1-K[1]}{2K[1]^2+8K[1]}dK[1]\right)dK[3] + c_1\right)$$

2.1.514 Problem 530

Internal problem ID [9686]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 530
Date solved : `Monday, January 27, 2025 at 06:13:05 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2y'' + 5xy' + (1+x)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.216 (sec)

Writing the ode as

$$2x^2y'' + 5xy' + (1+x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= 5x \\ C &= 1+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3-8x}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3-8x \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3-8x}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.973: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{2x} - \frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 1, \; e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(1 - (1)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{1}{(x-(0))}\right) \\ &= \frac{1}{2x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2x} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2x} + \frac{1+8x}{16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{1 + 2\sqrt{2}\,\sqrt{-x}}{4x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \frac{1+2\sqrt{2}\,\sqrt{-x}}{4x} dx} \\ &= x^{1/4}\mathrm{e}^{\sqrt{2}\,\sqrt{-x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{5x}{2x^2}\,dx} \\
&= z_1 e^{-\frac{5\ln(x)}{4}} \\
&= z_1 \left(\frac{1}{x^{5/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{e^{\sqrt{2}\,\sqrt{-x}}}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5x}{2x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{5\ln(x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\sqrt{2}\,\sqrt{-x}\left(1-e^{-2\sqrt{2}\,\sqrt{-x}}\right)}{2\sqrt{x}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{e^{\sqrt{2}\,\sqrt{-x}}}{x}\right) + c_2\left(\frac{e^{\sqrt{2}\,\sqrt{-x}}}{x}\left(-\frac{\sqrt{2}\,\sqrt{-x}\left(1-e^{-2\sqrt{2}\,\sqrt{-x}}\right)}{2\sqrt{x}}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$2x^2\left(\frac{d^2}{dx^2}y(x)\right) + 5x\left(\frac{d}{dx}y(x)\right) + (x+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{(x+1)y(x)}{2x^2} - \frac{5\left(\frac{d}{dx}y(x)\right)}{2x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) + \frac{5\left(\frac{d}{dx}y(x)\right)}{2x} + \frac{(x+1)y(x)}{2x^2} = 0$
- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

$\left[P_2(x) = \frac{5}{2x}, P_3(x) = \frac{x+1}{2x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{5}{2}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{1}{2}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x^2\left(\frac{d^2}{dx^2}y(x)\right) + 5x\left(\frac{d}{dx}y(x)\right) + (x+1)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(1+2r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(2k+2r+1) + a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-1, -\frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2(k+r+1)\left(k+r+\frac{1}{2}\right) a_k + a_{k-1} = 0$

- Shift index using $k- >k + 1$

  $2(k+2+r)\left(k+\frac{3}{2}+r\right) a_{k+1} + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{(k+2+r)(2k+3+2r)}$

- Recursion relation for $r = -1$

  $a_{k+1} = -\frac{a_k}{(k+1)(2k+1)}$

- Solution for $r = -1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = -\frac{a_k}{(k+1)(2k+1)}\right]$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+1} = -\frac{a_k}{\left(k+\frac{3}{2}\right)(2k+2)}$

- Solution for $r = -\frac{1}{2}$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = -\frac{a_k}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{2}}\right), a_{k+1} = -\frac{a_k}{(k+1)(2k+1)}, b_{k+1} = -\frac{b_k}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 29

```
dsolve(2*x^2*diff(diff(y(x),x),x)+5*diff(y(x),x)*x+(x+1)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_1 \sin\left(\sqrt{x}\,\sqrt{2}\right) + c_2 \cos\left(\sqrt{x}\,\sqrt{2}\right)}{x}$$

Mathematica DSolve solution

Solving time : 0.066 (sec)
Leaf size : 60

```
DSolve[{2*x^2*D[y[x],{x,2}]+5*x*D[y[x],x]+(1+x)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to \frac{2c_1 e^{i\sqrt{2}\sqrt{x}} + i\sqrt{2}c_2 e^{-i\sqrt{2}\sqrt{x}}}{2x}$$

2.1.515 Problem 531

Internal problem ID [9687]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 531
Date solved : `Monday, January 27, 2025 at 06:13:05 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$6x^2y'' + x(10 - x)\,y' - (2 + x)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.318 (sec)

Writing the ode as

$$6x^2y'' + \left(-x^2 + 10x\right) y' + (-x - 2)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 6x^2 \\ B &= -x^2 + 10x \\ C &= -x - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x + 28}{144x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x + 28 \\ t &= 144x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 4x + 28}{144x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.975: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 144x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{144} + \frac{7}{36x^2} + \frac{1}{36x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{12} + \frac{1}{6x} + \frac{1}{x^2} - \frac{2}{x^3} - \frac{2}{x^4} + \frac{28}{x^5} - \frac{56}{x^6} - \frac{272}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{12}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{12}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{144}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 + 4x + 28}{144x^2} \\ &= Q + \frac{R}{144x^2} \\ &= \left(\frac{1}{144}\right) + \left(\frac{4x + 28}{144x^2}\right) \\ &= \frac{1}{144} + \frac{4x + 28}{144x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 144 gives $\frac{1}{36}$. Now b can be found.

$$\begin{aligned}b &= \left(\frac{1}{36}\right) - (0) \\ &= \frac{1}{36}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{12} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{36}}{\frac{1}{12}} - 0\right) &&= \frac{1}{6} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{36}}{\frac{1}{12}} - 0\right) &&= -\frac{1}{6}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 4x + 28}{144x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{12}$	$\frac{1}{6}$	$-\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{6}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= -\frac{1}{6} - \left(-\frac{1}{6}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{6x} + (-)\left(\frac{1}{12}\right) \\
&= -\frac{1}{6x} - \frac{1}{12} \\
&= -\frac{2+x}{12x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{6x} - \frac{1}{12}\right)(0) + \left(\left(\frac{1}{6x^2}\right) + \left(-\frac{1}{6x} - \frac{1}{12}\right)^2 - \left(\frac{x^2+4x+28}{144x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(-\frac{1}{6x} - \frac{1}{12}\right) dx} \\
&= \frac{e^{-\frac{x}{12}}}{x^{1/6}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2+10x}{6x^2}\,dx} \\
&= z_1 e^{\frac{x}{12}-\frac{5\ln(x)}{6}} \\
&= z_1\left(\frac{e^{\frac{x}{12}}}{x^{5/6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+10x}{6x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{x}{6}-\frac{5\ln(x)}{3}}}{(y_1)^2}\,dx \\
&= y_1\left(\int e^{\frac{x}{6}-\frac{5\ln(x)}{3}}x^2dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x}\left(\int e^{\frac{x}{6}-\frac{5\ln(x)}{3}}x^2dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$6x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(10-x)\left(\frac{d}{dx}y(x)\right) - (x+2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = \frac{(x+2)y(x)}{6x^2} + \frac{(-10+x)\left(\frac{d}{dx}y(x)\right)}{6x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x) - \frac{(-10+x)\left(\frac{d}{dx}y(x)\right)}{6x} - \frac{(x+2)y(x)}{6x^2} = 0$
- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{-10+x}{6x}, P_3(x) = -\frac{x+2}{6x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

$(x \cdot P_2(x)) \bigg|_{x=0} = \frac{5}{3}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x)) \bigg|_{x=0} = -\frac{1}{3}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $6x^2\left(\frac{d^2}{dx^2}y(x)\right) - x(-10+x)\left(\frac{d}{dx}y(x)\right) + (-x-2)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- > k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k- > k + 1 - m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $2a_0(1+r)(-1+3r)\,x^r + \left(\sum_{k=1}^{\infty} (2a_k(k+r+1)(3k+3r-1) - a_{k-1}(k+r))\,x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $2(1+r)(-1+3r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-1, \frac{1}{3}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $6(k+r+1)\left(k - \frac{1}{3} + r\right)a_k - a_{k-1}(k+r) = 0$

- Shift index using $k- > k + 1$

  $6(k+2+r)\left(k + \frac{2}{3} + r\right)a_{k+1} - a_k(k+r+1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r+1)}{2(k+2+r)(3k+2+3r)}$

- Recursion relation for $r = -1$

  $a_{k+1} = \frac{a_k k}{2(k+1)(3k-1)}$

- Solution for $r = -1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = \frac{a_k k}{2(k+1)(3k-1)}\right]$

- Recursion relation for $r = \frac{1}{3}$

$$a_{k+1} = \frac{a_k \left(k+\frac{4}{3}\right)}{2\left(k+\frac{7}{3}\right)(3k+3)}$$

- Solution for $r = \frac{1}{3}$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+1} = \frac{a_k \left(k+\frac{4}{3}\right)}{2\left(k+\frac{7}{3}\right)(3k+3)}\right]$$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+1} = \frac{a_k k}{2(k+1)(3k-1)}, b_{k+1} = \frac{b_k \left(k+\frac{4}{3}\right)}{2\left(k+\frac{7}{3}\right)(3k+3)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Whittaker successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - return
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.071 (sec)
Leaf size : 27

```
dsolve(6*x^2*diff(diff(y(x),x),x)+x*(10-x)*diff(y(x),x)-(x+2)*y(x) = 0,y(x),singsol=al
```

$$y = \frac{c_2 x^{5/6} + c_1 \operatorname{WhittakerM}\left(-\frac{1}{6}, \frac{2}{3}, \frac{x}{6}\right) e^{\frac{x}{12}} x}{x^{11/6}}$$

Mathematica DSolve solution

Solving time : 0.034 (sec)
Leaf size : 38

```
DSolve[{6*x^2*D[y[x],{x,2}]+x*(10-x)*D[y[x],x]-(2+x)*y[x]==0,{}},y[x],x,IncludeSingularSolut
```

$$y(x) \to c_2 \sqrt[3]{x} L_{-\frac{4}{3}}^{\frac{4}{3}}\left(\frac{x}{6}\right) + \frac{6\sqrt[3]{6} c_1}{x}$$

2.1.516 Problem 532

Internal problem ID [9688]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 532
Date solved : `Monday, January 27, 2025 at 06:13:06 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(3+4x)\,y''+x(11+4x)\,y'-(3+4x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.302 (sec)

Writing the ode as

$$\left(4x^3+3x^2\right)y''+\left(4x^2+11x\right)y'+(-3-4x)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^3+3x^2 \\ B &= 4x^2+11x \\ C &= -3-4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{48x^2+8x+91}{4\left(4x^2+3x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 48x^2+8x+91 \\ t &= 4\left(4x^2+3x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{48x^2+8x+91}{4\left(4x^2+3x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.977: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(4x^2 + 3x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{3}{4}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{176}{27x} + \frac{91}{36x^2} + \frac{28}{9\left(x + \frac{3}{4}\right)^2} + \frac{176}{27\left(x + \frac{3}{4}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{91}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{13}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{7}{6} \end{aligned}$$

For the pole at $x = -\frac{3}{4}$ let b be the coefficient of $\frac{1}{\left(x+\frac{3}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{28}{9}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{3} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{4}{3} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{48x^2 + 8x + 91}{4\left(4x^2 + 3x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{48x^2 + 8x + 91}{4\left(4x^2 + 3x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{13}{6}$	$-\frac{7}{6}$
$-\frac{3}{4}$	2	0	$\frac{7}{3}$	$-\frac{4}{3}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{5}{2}\right) \\
&= 2
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{7}{6x} - \frac{4}{3\left(x + \frac{3}{4}\right)} + (-)(0) \\
&= -\frac{7}{6x} - \frac{4}{3\left(x + \frac{3}{4}\right)} \\
&= \frac{-7 - 20x}{8x^2 + 6x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(-\frac{7}{6x} - \frac{4}{3\left(x+\frac{3}{4}\right)}\right)(2x + a_1) + \left(\left(\frac{7}{6x^2} + \frac{4}{3\left(x+\frac{3}{4}\right)^2}\right) + \left(-\frac{7}{6x} - \frac{4}{3\left(x+\frac{3}{4}\right)}\right)^2 - \left(\frac{48x^2 + 8x}{4(4x^2 + 3}\right.\right.$$

$$\frac{12a_1 x - 8x + 32a_0 -}{x(3+4x)}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{7}{48}, a_1 = \frac{2}{3}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + \frac{2}{3}x + \frac{7}{48}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 + \frac{2}{3}x + \frac{7}{48}\right) e^{\int \left(-\frac{7}{6x} - \frac{4}{3\left(x+\frac{3}{4}\right)}\right) dx} \\
&= \left(x^2 + \frac{2}{3}x + \frac{7}{48}\right) e^{-\frac{7\ln(x)}{6} - \frac{4\ln(3+4x)}{3}} \\
&= \frac{x^2 + \frac{2}{3}x + \frac{7}{48}}{x^{7/6}(3+4x)^{4/3}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x^2+11x}{4x^3+3x^2} \, dx} \\
&= z_1 e^{-\frac{11\ln(x)}{6} + \frac{4\ln(3+4x)}{3}} \\
&= z_1 \left(\frac{(3+4x)^{4/3}}{x^{11/6}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2 + \frac{2}{3}x + \frac{7}{48}}{x^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x^2+11x}{4x^3+3x^2} dx}}{(y_1)^2} dx \\
&= y_1 \int \frac{e^{-\frac{11\ln(x)}{3}+\frac{8\ln(3+4x)}{3}}}{(y_1)^2} dx \\
&= y_1 \left(\int \frac{e^{-\frac{11\ln(x)}{3}+\frac{8\ln(3+4x)}{3}} x^6}{\left(x^2+\frac{2}{3}x+\frac{7}{48}\right)^2} dx \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^2+\frac{2}{3}x+\frac{7}{48}}{x^3} \right) + c_2 \left(\frac{x^2+\frac{2}{3}x+\frac{7}{48}}{x^3} \left(\int \frac{e^{-\frac{11\ln(x)}{3}+\frac{8\ln(3+4x)}{3}} x^6}{\left(x^2+\frac{2}{3}x+\frac{7}{48}\right)^2} dx \right) \right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(3+4x)\left(\frac{d^2}{dx^2}y(x)\right)+x(11+4x)\left(\frac{d}{dx}y(x)\right)-(3+4x)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{y(x)}{x^2}-\frac{(11+4x)\left(\frac{d}{dx}y(x)\right)}{x(3+4x)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{(11+4x)\left(\frac{d}{dx}y(x)\right)}{x(3+4x)}-\frac{y(x)}{x^2}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{11+4x}{x(3+4x)}, P_3(x)=-\frac{1}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x=0$

    $\left.(x \cdot P_2(x))\right|_{x=0}=\frac{11}{3}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0}=-1$

  ◦ $x=0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=0$

- Multiply by denominators

  $x^2(3+4x)\left(\frac{d^2}{dx^2}y(x)\right)+x(11+4x)\left(\frac{d}{dx}y(x)\right)+(-3-4x)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y(x)$ to series expansion for $m=0..1$

$x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..3$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(3+r)\,(-1+3r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+3)\,(3k+3r-1) + 4a_{k-1}(k+r)\,(k-2+r)\right) x^{k+r}\right) =$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(3+r)\,(-1+3r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-3, \frac{1}{3}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $3(k+r+3)\left(k-\frac{1}{3}+r\right) a_k + 4a_{k-1}(k+r)\,(k-2+r) = 0$
- Shift index using $k->k+1$

  $3(k+4+r)\left(k+\frac{2}{3}+r\right) a_{k+1} + 4a_k(k+r+1)\,(k+r-1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{4a_k(k+r+1)(k+r-1)}{(k+4+r)(3k+2+3r)}$
- Recursion relation for $r = -3$; series terminates at $k = 2$

  $a_{k+1} = -\frac{4a_k(k-2)(k-4)}{(k+1)(3k-7)}$
- Apply recursion relation for $k = 0$

  $a_1 = \frac{32a_0}{7}$
- Apply recursion relation for $k = 1$

  $a_2 = \frac{3a_1}{2}$
- Express in terms of a_0

  $a_2 = \frac{48a_0}{7}$
- Terminating series solution of the ODE for $r = -3$. Use reduction of order to find the second

  $y(x) = a_0 \cdot \left(\frac{48}{7}x^2 + \frac{32}{7}x + 1\right)$
- Recursion relation for $r = \frac{1}{3}$

  $a_{k+1} = -\frac{4a_k\left(k+\frac{4}{3}\right)\left(k-\frac{2}{3}\right)}{\left(k+\frac{13}{3}\right)(3k+3)}$
- Solution for $r = \frac{1}{3}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+1} = -\frac{4a_k\left(k+\frac{4}{3}\right)\left(k-\frac{2}{3}\right)}{\left(k+\frac{13}{3}\right)(3k+3)}\right]$
- Combine solutions and rename parameters

$$\left[y(x)=a_0\cdot\left(\frac{48}{7}x^2+\frac{32}{7}x+1\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{3}}\right), b_{k+1}=-\frac{4b_k\left(k+\frac{4}{3}\right)\left(k-\frac{2}{3}\right)}{\left(k+\frac{13}{3}\right)(3k+3)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returnin
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.127 (sec)
Leaf size : 41

```
dsolve(x^2*(4*x+3)*diff(diff(y(x),x),x)+x*(11+4*x)*diff(y(x),x)-(4*x+3)*y(x) = 0,y(x),si
```

$$y=\frac{c_1(48x^2+32x+7)}{x^3}+c_2\,\text{hypergeom}\left([3,5],\left[\frac{13}{3}\right],-\frac{4x}{3}\right)(4x+3)^{11/3}x^{1/3}$$

Mathematica DSolve solution

Solving time : 0.521 (sec)
Leaf size : 143

```
DSolve[{x^2*(3+4*x)*D[y[x],{x,2}]+x*(11+4*x)*D[y[x],x]-(3+4*x)*y[x]==0,{}},y[x],x,IncludeSin
```

$$y(x) \to \frac{1}{48}(48x^2+32x+7)\exp\left(\int_1^x -\frac{20K[1]+7}{8K[1]^2+6K[1]}dK[1] - \frac{1}{2}\int_1^x \frac{4K[2]+11}{4K[2]^2+3K[2]}dK[2]\right)\left(c_2\int_1^x \frac{2304\exp\left(-2\int_1^{K[3]} -\frac{20K[1]+7}{8K[1]^2+6K[1]}dK[1]\right)}{(48K[3]^2+32K[3]+7)^2}dK[3] + c_1\right)$$

2.1.517 Problem 533

Internal problem ID [9689]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 533
Date solved : `Monday, January 27, 2025 at 06:13:07 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(2+3x)\,y''+x(4+11x)\,y'-(1-x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.161 (sec)

Writing the ode as

$$\left(6x^3+4x^2\right)y''+\left(11x^2+4x\right)y'+(x-1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=6x^3+4x^2\\ B&=11x^2+4x\\ C&=x-1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-35}{16\,(2+3x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=-35\\ t&=16(2+3x)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{35}{16\,(2+3x)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.979: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(2+3x)^2$. There is a pole at $x = -\frac{2}{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{35}{144 \left(x + \frac{2}{3}\right)^2}$$

For the pole at $x = -\frac{2}{3}$ let b be the coefficient of $\frac{1}{\left(x+\frac{2}{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{35}{16 (2+3x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{35}{144}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{7}{12}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{5}{12}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=-\frac{35}{16\,(2+3x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$-\frac{2}{3}$	2	0	$\frac{7}{12}$	$\frac{5}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{7}{12}$	$\frac{5}{12}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{5}{12}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}\right)\\
&= \frac{5}{12}-\left(\frac{5}{12}\right)\\
&= 0
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^{-}}{x-c_1}\right)+(-)[\sqrt{r}]_\infty\\
&= \frac{5}{12\left(x+\frac{2}{3}\right)}+(-)\,(0)\\
&= \frac{5}{12\left(x+\frac{2}{3}\right)}\\
&= \frac{5}{8+12x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{5}{12\left(x+\frac{2}{3}\right)}\right)(0)+\left(\left(-\frac{5}{12\left(x+\frac{2}{3}\right)^{2}}\right)+\left(\frac{5}{12\left(x+\frac{2}{3}\right)}\right)^{2}-\left(-\frac{35}{16\left(2+3x\right)^{2}}\right)\right)=0$$

$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \frac{5}{12\left(x+\frac{2}{3}\right)}dx} \\ &= (2+3x)^{5/12} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{11x^2+4x}{6x^3+4x^2}\, dx} \\ &= z_1 e^{-\frac{5\ln(2+3x)}{12}-\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{1}{(2+3x)^{5/12}\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1=\frac{1}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1\int \frac{e^{\int -\frac{11x^2+4x}{6x^3+4x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1\int \frac{e^{-\frac{5\ln(2+3x)}{6}-\ln(x)}}{(y_1)^2}\, dx \\ &= y_1\left(2\,\mathrm{e}^{-\frac{5\ln(2+3x)}{6}-\ln(x)}x(2+3x)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{1}{\sqrt{x}}\right)+c_2\left(\frac{1}{\sqrt{x}}\left(2\,\mathrm{e}^{-\frac{5\ln(2+3x)}{6}-\ln(x)}x(2+3x)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2(3x+2)\left(\frac{d^2}{dx^2}y(x)\right)+x(4+11x)\left(\frac{d}{dx}y(x)\right)-(1-x)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{(x-1)y(x)}{2(3x+2)x^2}-\frac{(4+11x)\left(\frac{d}{dx}y(x)\right)}{2x(3x+2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x)+\frac{(4+11x)\left(\frac{d}{dx}y(x)\right)}{2x(3x+2)}+\frac{(x-1)y(x)}{2(3x+2)x^2}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

$\left[P_2(x)=\frac{4+11x}{2x(3x+2)},P_3(x)=\frac{x-1}{2(3x+2)x^2}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

$\left(x\cdot P_2(x)\right)\Big|_{x=0}=1$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=-\frac{1}{4}$

- ○ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$2x^2(3x+2)\left(\frac{d^2}{dx^2}y(x)\right)+x(4+11x)\left(\frac{d}{dx}y(x)\right)+(x-1)\,y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

$x^m\cdot y(x)=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

- ○ Shift index using $k->k-m$

$x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ○ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ○ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..3$

$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- ○ Shift index using $k->k+2-m$

$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

Rewrite ODE with series expansions

$a_0(1+2r)(-1+2r)x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)(2k+2r-1) + a_{k-1}(2k+2r-1)(3k-2+3r\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(1+2r)(-1+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $4\left(\left(\frac{3k}{2} + \frac{3r}{2} - 1\right)a_{k-1} + a_k\left(k + r + \frac{1}{2}\right)\right)\left(k + r - \frac{1}{2}\right) = 0$
- Shift index using $k- >k+1$
  $4\left(\left(\frac{3k}{2} + \frac{1}{2} + \frac{3r}{2}\right)a_k + a_{k+1}\left(k + \frac{3}{2} + r\right)\right)\left(k + r + \frac{1}{2}\right) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = -\frac{(3k+3r+1)a_k}{2k+3+2r}$
- Recursion relation for $r = -\frac{1}{2}$
  $a_{k+1} = -\frac{\left(3k-\frac{1}{2}\right)a_k}{2k+2}$
- Solution for $r = -\frac{1}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = -\frac{\left(3k-\frac{1}{2}\right)a_k}{2k+2}\right]$
- Recursion relation for $r = \frac{1}{2}$
  $a_{k+1} = -\frac{\left(3k+\frac{5}{2}\right)a_k}{2k+4}$
- Solution for $r = \frac{1}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{\left(3k+\frac{5}{2}\right)a_k}{2k+4}\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = -\frac{\left(3k-\frac{1}{2}\right)a_k}{2k+2}, b_{k+1} = -\frac{\left(3k+\frac{5}{2}\right)b_k}{2k+4}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.025 (sec)
Leaf size : 19

```
dsolve(2*x^2*(2+3*x)*diff(diff(y(x),x),x)+x*(4+11*x)*diff(y(x),x)-(1-x)*y(x) = 0,y(x),
```

$$y = \frac{c_2(2+3x)^{1/6} + c_1}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.321 (sec)
Leaf size : 69

```
DSolve[{2*x^2*(2+3*x)*D[y[x],{x,2}]+x*(4+11*x)*D[y[x],x]-(1-x)*y[x]==0,{}},y[x],x,IncludeSingul
```

$$y(x)\to \sqrt[6]{2}(3x+2)^{5/12}\left(c_2\sqrt[6]{3x+2}+2^{2/3}c_1\right)\exp\left(-\frac{1}{2}\int_1^x\left(\frac{5}{6K[1]+4}+\frac{1}{K[1]}\right)dK[1]\right)$$

2.1.518 Problem 534

Internal problem ID [9690]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 534
Date solved : `Monday, January 27, 2025 at 06:13:07 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(2+x)\,y''+5x(1-x)\,y'-(2-8x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.822 (sec)

Writing the ode as

$$x^2(2+x)\,y''+\left(-5x^2+5x\right)y'+(8x-2)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(2+x) \\ B &= -5x^2+5x \\ C &= 8x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{3x^2-126x+21}{4\left(x^2+2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3x^2-126x+21 \\ t &= 4\left(x^2+2x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{3x^2-126x+21}{4\left(x^2+2x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.981: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{21}{16x^2} + \frac{285}{16\,(2+x)^2} + \frac{147}{16\,(2+x)} - \frac{147}{16x}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(2+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{285}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{19}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{15}{4} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x^2 - 126x + 21}{4(x^2 + 2x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3x^2 - 126x + 21}{4(x^2 + 2x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{19}{4}$	$-\frac{15}{4}$
0	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^- + \alpha_{c_2}^-) \\
&= -\frac{1}{2} - \left(-\frac{9}{2}\right) \\
&= 4
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{15}{4(2+x)} - \frac{3}{4x} + (-)(0) \\
&= -\frac{15}{4(2+x)} - \frac{3}{4x} \\
&= -\frac{3(3x+1)}{2x(2+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(12x^2 + 6xa_3 + 2a_2) + 2\left(-\frac{15}{4\,(2+x)} - \frac{3}{4x}\right)(4x^3 + 3x^2a_3 + 2a_2x + a_1) + \left(\left(\frac{15}{4\,(2+x)^2} + \frac{3}{4x^2}\right) + \left(-\frac{1}{4\,(2\cdots}\right.\right.$$

$$\frac{3(4+a_3)\,x^3 + (8a_2 + 3a_3\cdots}{}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{1}{40}, a_1 = \frac{1}{5}, a_2 = \frac{3}{2}, a_3 = -4\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 - 4x^3 + \frac{3}{2}x^2 + \frac{1}{5}x + \frac{1}{40}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\,dx} \\
&= \left(x^4 - 4x^3 + \frac{3}{2}x^2 + \frac{1}{5}x + \frac{1}{40}\right) e^{\int \left(-\frac{15}{4(2+x)} - \frac{3}{4x}\right)dx} \\
&= \left(x^4 - 4x^3 + \frac{3}{2}x^2 + \frac{1}{5}x + \frac{1}{40}\right) e^{-\frac{3\ln(x)}{4} - \frac{15\ln(2+x)}{4}} \\
&= \frac{40x^4 - 160x^3 + 60x^2 + 8x + 1}{40x^{3/4}\,(2+x)^{15/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-5x^2+5x}{x^2(2+x)}\,dx} \\
&= z_1 e^{-\frac{5\ln(x)}{4} + \frac{15\ln(2+x)}{4}} \\
&= z_1\left(\frac{(2+x)^{15/4}}{x^{5/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{40x^4 - 160x^3 + 60x^2 + 8x + 1}{40x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-5x^2+5x}{x^2(2+x)}\,dx}}{(y_1)^2}\,dx$$

$$= y_1 \int \frac{e^{-\frac{5\ln(x)}{2}+\frac{15\ln(2+x)}{2}}}{(y_1)^2}\,dx$$

$$= y_1 \left(\frac{10\sqrt{2+x}\,x^{5/2}\left(8x^5\sqrt{x(2+x)} + 4200\ln\left(\frac{x+\sqrt{x(2+x)}}{x}\right)x^4 - 4200\ln\left(\frac{\sqrt{x(2+x)}-x}{x}\right)x^4 + 328x^4\sqrt{x(2\right.}{}\right.$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\frac{40x^4 - 160x^3 + 60x^2 + 8x + 1}{40x^2}\right)$$

$$+ c_2 \left(\frac{40x^4 - 160x^3 + 60x^2 + 8x + 1}{40x^2} \left(\frac{10\sqrt{2+x}\,x^{5/2}\left(8x^5\sqrt{x(2+x)} + 4200\ln\left(\frac{x+\sqrt{x(2+x)}}{x}\right)x^4 - 420\right.}{}\right.\right.$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x+2)\left(\frac{d^2}{dx^2}y(x)\right) + 5x(1-x)\left(\frac{d}{dx}y(x)\right) - (2-8x)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{2(4x-1)y(x)}{(x+2)x^2} + \frac{5(x-1)\left(\frac{d}{dx}y(x)\right)}{x(x+2)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) - \frac{5(x-1)\left(\frac{d}{dx}y(x)\right)}{x(x+2)} + \frac{2(4x-1)y(x)}{(x+2)x^2} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{5(x-1)}{x(x+2)}, P_3(x) = \frac{2(4x-1)}{(x+2)x^2}\right]$
  - $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

    $\left((x+2)\cdot P_2(x)\right)\bigg|_{x=-2} = -\frac{15}{2}$
  - $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

    $\left((x+2)^2\cdot P_3(x)\right)\bigg|_{x=-2} = 0$
  - $x=-2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -2$

- Multiply by denominators

  $x^2(x+2)\left(\frac{d^2}{dx^2}y(x)\right) - 5x(x-1)\left(\frac{d}{dx}y(x)\right) + (8x-2)\,y(x) = 0$

- Change variables using $x = u - 2$ so that the regular singular point is at $u = 0$

$(u^3 - 4u^2 + 4u)\left(\frac{d^2}{du^2}y(u)\right) + (-5u^2 + 25u - 30)\left(\frac{d}{du}y(u)\right) + (8u - 18)\,y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

  - Shift index using $k- > k - m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k- > k + 1 - m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k- > k + 2 - m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

Rewrite ODE with series expansions

$2a_0 r(-17+2r)\,u^{-1+r} + (2a_1(1+r)(-15+2r) - a_0(4r^2 - 29r + 18))\,u^r + \left(\sum_{k=1}^{\infty} (2a_{k+1}(k+1+r)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$2r(-17+2r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{0, \frac{17}{2}\right\}$

- Each term must be 0

$2a_1(1+r)(-15+2r) - a_0(4r^2 - 29r + 18) = 0$

- Each term in the series must be 0, giving the recursion relation

$(-4a_k + a_{k-1} + 4a_{k+1})\,k^2 + ((-8a_k + 2a_{k-1} + 8a_{k+1})\,r + 29a_k - 8a_{k-1} - 26a_{k+1})\,k + (-4a_k + a_{k-1}$

- Shift index using $k- > k + 1$

$(-4a_{k+1} + a_k + 4a_{k+2})\,(k+1)^2 + ((-8a_{k+1} + 2a_k + 8a_{k+2})\,r + 29a_{k+1} - 8a_k - 26a_{k+2})\,(k+1) + ($

- Recursion relation that defines series solution to ODE

$a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} + 2kra_k - 8kra_{k+1} + r^2 a_k - 4r^2 a_{k+1} - 6ka_k + 21ka_{k+1} - 6ra_k + 21ra_{k+1} + 8a_k + 7a_{k+1}}{2(2k^2 + 4kr + 2r^2 - 9k - 9r - 26)}$

- Recursion relation for $r = 0$

$a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} - 6ka_k + 21ka_{k+1} + 8a_k + 7a_{k+1}}{2(2k^2 - 9k - 26)}$

- Solution for $r = 0$

$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} - 6ka_k + 21ka_{k+1} + 8a_k + 7a_{k+1}}{2(2k^2 - 9k - 26)}, -30a_1 - 18a_0 = 0\right]$

- Revert the change of variables $u = x + 2$

$\left[y(x) = \sum_{k=0}^{\infty} a_k (x+2)^k, a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} - 6ka_k + 21ka_{k+1} + 8a_k + 7a_{k+1}}{2(2k^2 - 9k - 26)}, -30a_1 - 18a_0 = 0\right]$

- Recursion relation for $r = \frac{17}{2}$

$$a_{k+2} = -\frac{k^2a_k-4k^2a_{k+1}+11ka_k-47ka_{k+1}+\frac{117}{4}a_k-\frac{207}{2}a_{k+1}}{2(2k^2+25k+42)}$$

- Solution for $r = \frac{17}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{17}{2}}, a_{k+2} = -\frac{k^2a_k-4k^2a_{k+1}+11ka_k-47ka_{k+1}+\frac{117}{4}a_k-\frac{207}{2}a_{k+1}}{2(2k^2+25k+42)}, 38a_1 - \frac{121a_0}{2} = 0\right]$$

- Revert the change of variables $u = x + 2$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k(x+2)^{k+\frac{17}{2}}, a_{k+2} = -\frac{k^2a_k-4k^2a_{k+1}+11ka_k-47ka_{k+1}+\frac{117}{4}a_k-\frac{207}{2}a_{k+1}}{2(2k^2+25k+42)}, 38a_1 - \frac{121a_0}{2} = 0\right]$$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k(x+2)^k\right) + \left(\sum_{k=0}^{\infty} b_k(x+2)^{k+\frac{17}{2}}\right), a_{k+2} = -\frac{k^2a_k-4k^2a_{k+1}-6ka_k+21ka_{k+1}+8a_k+7a_{k+1}}{2(2k^2-9k-26)}, \cdot\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - return
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.138 (sec)
Leaf size : 113

```
dsolve(x^2*(x+2)*diff(diff(y(x),x),x)+5*x*(1-x)*diff(y(x),x)-(2-8*x)*y(x) = 0,y(x),si
```

$$y = \frac{c_1(40x^4-160x^3+60x^2+8x+1)}{x^2} + \frac{4(-x-2)^{3/4} c_2\left(1050\left(x^4-4x^3+\frac{3}{2}x^2+\frac{1}{5}x+\frac{1}{40}\right)x^{3/2}\operatorname{arcsinh}\left(\frac{\sqrt{x}\sqrt{2}}{2}\right)+\sqrt{x+2}\,x^2\left(x^5+41x^4-\frac{6987}{4}\right.\right.}{105x^{7/2}(x+2)^{3/4}}$$

Mathematica DSolve solution

Solving time : `0.714` (sec)
Leaf size : 163

```
DSolve[{x^2*(2+x)*D[y[x],{x,2}]+5*x*(1-x)*D[y[x],x]-(2-8*x)*y[x]==0,{}},y[x],x,IncludeSingular
```

$$y(x) \to \frac{1}{40}\left(40x^4-160x^3+60x^2+8x+1\right)\exp\left(\int_1^x -\frac{9K[1]+3}{2K[1]^2+4K[1]}dK[1] - \frac{1}{2}\int_1^x \frac{5-5K[2]}{K[2]^2+2K[2]}dK[2]\right)\left(c_2\int_1^x \frac{1600\exp\left(-2\int_1^{K[3]} -\frac{9K[1]+3}{2K[1]^2+4K[1]}dK[1]\right)}{\left(40K[3]^4-160K[3]^3+60K[3]^2+8K[3]+1\right)^2}dK[3] + c_1\right)$$

2.1.519 Problem 535

Internal problem ID [9691]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 535
Date solved : `Monday, January 27, 2025 at 06:13:09 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$8x^2\left(-x^2+1\right)y''+2x\left(-13x^2+1\right)y'+\left(-9x^2+1\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.325 (sec)

Writing the ode as

$$\left(-8x^4+8x^2\right)y''+\left(-26x^3+2x\right)y'+\left(-9x^2+1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=-8x^4+8x^2\\ B&=-26x^3+2x\\ C&=-9x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-7x^4-26x^2-15}{64\left(x^3-x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=-7x^4-26x^2-15\\ t&=64\left(x^3-x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-7x^4-26x^2-15}{64\left(x^3-x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.983: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64(x^3 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16\,(x+1)^2} - \frac{1}{4\,(x+1)} + \frac{1}{4x-4} - \frac{3}{16\,(x-1)^2} - \frac{15}{64x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{15}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{8} \end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-7x^4 - 26x^2 - 15}{64\,(x^3 - x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{7}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-7x^4 - 26x^2 - 15}{64\,(x^3 - x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{8}$	$\frac{3}{8}$
1	2	0	$\frac{3}{4}$	$\frac{1}{4}$
-1	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{8}$	$\frac{1}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{8}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{7}{8} - \left(\frac{7}{8}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{3}{8x} + \frac{1}{4x-4} + \frac{1}{4x+4} + (0) \\
&= \frac{3}{8x} + \frac{1}{4x-4} + \frac{1}{4x+4} \\
&= \frac{7x^2-3}{8x^3-8x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{3}{8x} + \frac{1}{4x-4} + \frac{1}{4x+4}\right)(0) + \left(\left(-\frac{3}{8x^2} - \frac{1}{4(x-1)^2} - \frac{1}{4(x+1)^2}\right) + \left(\frac{3}{8x} + \frac{1}{4x-4} + \frac{1}{4x+4}\right)^2\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{3}{8x} + \frac{1}{4x-4} + \frac{1}{4x+4}\right) dx} \\
&= x^{3/8}(x+1)^{1/4}(x-1)^{1/4}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-26x^3+2x}{-8x^4+8x^2}\, dx} \\
&= z_1 e^{-\frac{\ln(x)}{8} - \frac{3\ln(x+1)}{4} - \frac{3\ln(x-1)}{4}} \\
&= z_1 \left(\frac{1}{x^{1/8}(x+1)^{3/4}(x-1)^{3/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^{1/4}(x^2-1)^{1/4}}{(x+1)^{3/4}(x-1)^{3/4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-26x^3+2x}{-8x^4+8x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x)}{4} - \frac{3\ln(x+1)}{2} - \frac{3\ln(x-1)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{e^{-\frac{\ln(x)}{4} - \frac{3\ln(x+1)}{2} - \frac{3\ln(x-1)}{2}}(x+1)^{3/2}(x-1)^{3/2}}{\sqrt{x}\,\sqrt{x^2-1}}\, dx\right)
\end{aligned}
$$

Therefore the solution is

$$y = c_1y_1 + c_2y_2$$
$$= c_1\left(\frac{x^{1/4}(x^2-1)^{1/4}}{(x+1)^{3/4}(x-1)^{3/4}}\right) + c_2\left(\frac{x^{1/4}(x^2-1)^{1/4}}{(x+1)^{3/4}(x-1)^{3/4}}\left(\int \frac{e^{-\frac{\ln(x)}{4}-\frac{3\ln(x+1)}{2}-\frac{3\ln(x-1)}{2}}(x+1)^{3/2}(x-1)^{3/2}}{\sqrt{x}\sqrt{x^2-1}}dx\right.\right.$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$8x^2(-x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + 2x(-13x^2+1)\left(\frac{d}{dx}y(x)\right) + (-9x^2+1)y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(9x^2-1)y(x)}{8(x^2-1)x^2} - \frac{(13x^2-1)\left(\frac{d}{dx}y(x)\right)}{4x(x^2-1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(13x^2-1)\left(\frac{d}{dx}y(x)\right)}{4x(x^2-1)} + \frac{(9x^2-1)y(x)}{8(x^2-1)x^2} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{13x^2-1}{4x(x^2-1)}, P_3(x) = \frac{9x^2-1}{8(x^2-1)x^2}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1} = \frac{3}{2}$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $8(x^2-1)x^2\left(\frac{d^2}{dx^2}y(x)\right) + 2x(13x^2-1)\left(\frac{d}{dx}y(x)\right) + (9x^2-1)y(x) = 0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(8u^4-32u^3+40u^2-16u)\left(\frac{d^2}{du^2}y(u)\right) + (26u^3-78u^2+76u-24)\left(\frac{d}{du}y(u)\right) + (9u^2-18u+8)y$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot y(u)$ to series expansion for $m=0..2$

    $u^m\cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

  ◦ Shift index using $k->k-m$

    $u^m\cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..3$

$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..4$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-8a_0r(1+2r)\, u^{-1+r} + (-8a_1(1+r)\,(3+2r) + 4a_0(1+2r)\,(2+5r))\, u^r + (-8a_2(2+r)\,(5+2r) +$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-8r(1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{0, -\frac{1}{2}\right\}$
- The coefficients of each power of u must be 0

  $[-8a_1(1+r)\,(3+2r) + 4a_0(1+2r)\,(2+5r) = 0, -8a_2(2+r)\,(5+2r) + 4a_1(3+2r)\,(7+5r) - 2a$
- Solve for the dependent coefficient(s)

  $\left\{a_1 = \frac{a_0(10r^2+9r+2)}{2(2r^2+5r+3)}, a_2 = \frac{a_0(34r^3+76r^2+41r+5)}{4(2r^3+11r^2+19r+10)}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $8(5a_k + a_{k-2} - 4a_{k-1} - 2a_{k+1})\, k^2 + 2(8(5a_k + a_{k-2} - 4a_{k-1} - 2a_{k+1})\, r + 18a_k - 7a_{k-2} + 9a_{k-1} - 2$
- Shift index using $k->k+2$

  $8(5a_{k+2} + a_k - 4a_{k+1} - 2a_{k+3})\,(k+2)^2 + 2(8(5a_{k+2} + a_k - 4a_{k+1} - 2a_{k+3})\, r + 18a_{k+2} - 7a_k + 9a_k$
- Recursion relation that defines series solution to ODE

  $a_{k+3} = \frac{8k^2a_k - 32k^2a_{k+1} + 40k^2a_{k+2} + 16kra_k - 64kra_{k+1} + 80kra_{k+2} + 8r^2a_k - 32r^2a_{k+1} + 40r^2a_{k+2} + 18ka_k - 110ka_{k+1} + 196ka_{k+2} +}{8(2k^2+4kr+2r^2+13k+13r+21)}$
- Recursion relation for $r = 0$

  $a_{k+3} = \frac{8k^2a_k - 32k^2a_{k+1} + 40k^2a_{k+2} + 18ka_k - 110ka_{k+1} + 196ka_{k+2} + 9a_k - 96a_{k+1} + 240a_{k+2}}{8(2k^2+13k+21)}$
- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+3} = \frac{8k^2a_k - 32k^2a_{k+1} + 40k^2a_{k+2} + 18ka_k - 110ka_{k+1} + 196ka_{k+2} + 9a_k - 96a_{k+1} + 240a_{k+2}}{8(2k^2+13k+21)}, a_1 = \frac{a_0}{3}, a\right.$
- Revert the change of variables $u = x+1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+3} = \frac{8k^2a_k - 32k^2a_{k+1} + 40k^2a_{k+2} + 18ka_k - 110ka_{k+1} + 196ka_{k+2} + 9a_k - 96a_{k+1} + 240a_{k+2}}{8(2k^2+13k+21)}, a_1 =\right.$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+3} = \frac{8k^2a_k - 32k^2a_{k+1} + 40k^2a_{k+2} + 10ka_k - 78ka_{k+1} + 156ka_{k+2} + 2a_k - 49a_{k+1} + 152a_{k+2}}{8(2k^2+11k+15)}$
- Solution for $r = -\frac{1}{2}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{1}{2}}, a_{k+3} = \frac{8k^2a_k - 32k^2a_{k+1} + 40k^2a_{k+2} + 10ka_k - 78ka_{k+1} + 156ka_{k+2} + 2a_k - 49a_{k+1} + 152a_{k+2}}{8(2k^2+11k+15)}, a_1 = 0, a\right.$
- Revert the change of variables $u = x+1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^{k-\frac{1}{2}}, a_{k+3} = \frac{8k^2a_k - 32k^2a_{k+1} + 40k^2a_{k+2} + 10ka_k - 78ka_{k+1} + 156ka_{k+2} + 2a_k - 49a_{k+1} + 152a_{k+2}}{8(2k^2+11k+15)}, a_1\right.$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k(x+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k(x+1)^{k-\frac{1}{2}}\right), a_{k+3} = \frac{8k^2a_k - 32k^2a_{k+1} + 40k^2a_{k+2} + 18ka_k - 110ka_{k+1} + 196ka}{8(2k^2+13k+21)}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special functio
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.106 (sec)
Leaf size : 34

```
dsolve(8*x^2*(-x^2+1)*diff(diff(y(x),x),x)+2*x*(-13*x^2+1)*diff(y(x),x)+(-9*x^2+1)*y(x
```

$$y = \frac{x^{1/4}\left(\operatorname{LegendreQ}\left(-\frac{1}{8}, \frac{1}{8}, \sqrt{-x^2+1}\right) c_2 x^{1/8} + c_1\right)}{\sqrt{x^2-1}}$$

Mathematica DSolve solution

Solving time : 0.276 (sec)
Leaf size : 118

```
DSolve[{8*x^2*(1-x^2)*D[y[x],{x,2}]+2*x*(1-13*x^2)*D[y[x],x]+(1-9*x^2)*y[x]==0,{}},y[x],x,In
```

$$y(x) \to \exp\left(\int_1^x \frac{3-7K[1]^2}{8K[1]-8K[1]^3}dK[1] - \frac{1}{2}\int_1^x \frac{1-13K[2]^2}{4K[2]-4K[2]^3}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{3-7K[1]^2}{8K[1]-8K[1]^3}dK[1]\right)dK[3] + c_1\right)$$

2.1.520 Problem 536

Internal problem ID [9692]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 536
Date solved : `Monday, January 27, 2025 at 06:13:09 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(x^2+1)\,y''-2x(-x^2+2)\,y'+4y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.327 (sec)

Writing the ode as

$$(x^4+x^2)\,y''+(2x^3-4x)\,y'+4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^4+x^2\\B&=2x^3-4x\\C&=4\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{-x^2+2}{(x^3+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=-x^2+2\\t&=(x^3+x)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-x^2+2}{(x^3+x)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.985: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{7i}{4(x-i)} - \frac{7i}{4(x+i)} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-x^2+2}{(x^3+x)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{2}{x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)} + (-)(0) \\
&= \frac{2}{x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)} \\
&= \frac{x^2+2}{x^3+x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{2}{x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)}\right)(0) + \left(\left(-\frac{2}{x^2} + \frac{1}{2(x-i)^2} + \frac{1}{2(x+i)^2}\right) + \left(\frac{2}{x} - \frac{1}{2(x-i)} - \frac{1}{2(x+\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(\frac{2}{x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)}\right) dx} \\ &= \frac{x^2}{\sqrt{x^2+1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x^3-4x}{x^4+x^2} \, dx} \\ &= z_1 e^{2\ln(x) - \frac{3\ln(x^2+1)}{2}} \\ &= z_1 \left(\frac{x^2}{(x^2+1)^{3/2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^4}{(x^2+1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3-4x}{x^4+x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{4\ln(x) - 3\ln(x^2+1)}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{(3x^2+1)(x^2+1)^3 \, \mathrm{e}^{4\ln(x) - 3\ln(x^2+1)}}{3x^7}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{x^4}{(x^2+1)^2}\right) + c_2 \left(\frac{x^4}{(x^2+1)^2} \left(-\frac{(3x^2+1)(x^2+1)^3 \, \mathrm{e}^{4\ln(x) - 3\ln(x^2+1)}}{3x^7}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)-2x(-x^2+2)\left(\frac{d}{dx}y(x)\right)+4y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{4y(x)}{x^2(x^2+1)}-\frac{2(x^2-2)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{2(x^2-2)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}+\frac{4y(x)}{x^2(x^2+1)}=0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{2(x^2-2)}{x(x^2+1)},P_3(x)=\frac{4}{x^2(x^2+1)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\bigg|_{x=0}=-4$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\bigg|_{x=0}=4$

  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)+2x(x^2-2)\left(\frac{d}{dx}y(x)\right)+4y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..4$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)\,(-4+r)\,x^r+a_1r(-3+r)\,x^{1+r}+\left(\sum_{k=2}^{\infty}(a_k(k+r-1)\,(k+r-4)+a_{k-2}(k-2+r)\,(k\cdot$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-4+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{1,4\}$

- Each term must be 0
  $a_1 r(-3+r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-1)\left(a_k(k+r-4)+a_{k-2}(k-2+r)\right) = 0$
- Shift index using $k->k+2$
  $(k+r+1)\left(a_{k+2}(k-2+r)+a_k(k+r)\right) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{a_k(k+r)}{k-2+r}$
- Recursion relation for $r = 1$
  $a_{k+2} = -\frac{a_k(k+1)}{k-1}$
- Solution for $r = 1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k(k+1)}{k-1}, a_1 = 0\right]$
- Recursion relation for $r = 4$
  $a_{k+2} = -\frac{a_k(k+4)}{k+2}$
- Solution for $r = 4$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+4}, a_{k+2} = -\frac{a_k(k+4)}{k+2}, a_1 = 0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{4+k}\right), a_{k+2} = -\frac{a_k(k+1)}{k-1}, a_1 = 0, b_{k+2} = -\frac{b_k(4+k)}{k+2}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 26

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)-2*x*(-x^2+2)*diff(y(x),x)+4*y(x) = 0,y(x),sing
```

$$y = \frac{x(c_1 x^3 + 3c_2 x^2 + c_2)}{(x^2+1)^2}$$

Mathematica DSolve solution

Solving time : 0.197 (sec)
Leaf size : 101

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]-2*x*(2-x^2)*D[y[x],x]+4*y[x]==0,{}},y[x],x,IncludeSingularSol
```

$$y(x)\to \exp\left(\int_1^x \frac{K[1]^2+2}{K[1]^3+K[1]}dK[1]-\frac{1}{2}\int_1^x \frac{2(K[2]^2-2)}{K[2]^3+K[2]}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]}\frac{K[1]^2+2}{K[1]^3+K[1]}dK[1]\right)dK[3]+c_1\right)$$

2.1.521 Problem 537

Internal problem ID [9693]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 537
Date solved : `Monday, January 27, 2025 at 06:13:10 PM`
CAS classification : `[[_2nd_order, _exact, _linear, _homogeneous]]`

Solve

$$x\left(x^2+3\right)y''+\left(-x^2+2\right)y'-8xy=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.280 (sec)

Writing the ode as

$$\left(x^3+3x\right)y''+\left(-x^2+2\right)y'-8xy=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^3+3x \\ B &= -x^2+2 \\ C &= -8x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{35x^4+74x^2-8}{4\left(x^3+3x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 35x^4+74x^2-8 \\ t &= 4\left(x^3+3x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{35x^4+74x^2-8}{4\left(x^3+3x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.987: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + 3x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i\sqrt{3}$ of order 2. There is a pole at $x = -i\sqrt{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{85}{144\left(x - i\sqrt{3}\right)^2} + \frac{85}{144\left(x + i\sqrt{3}\right)^2} - \frac{187i\sqrt{3}}{144\left(x - i\sqrt{3}\right)} + \frac{187i\sqrt{3}}{144\left(x + i\sqrt{3}\right)} - \frac{2}{9x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{2}{9}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{2}{3} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{3} \end{aligned}$$

For the pole at $x = i\sqrt{3}$ let b be the coefficient of $\frac{1}{\left(x-i\sqrt{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{85}{144}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{17}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{12}\end{aligned}$$

For the pole at $x = -i\sqrt{3}$ let b be the coefficient of $\frac{1}{\left(x+i\sqrt{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{85}{144}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{17}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{12}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{35x^4 + 74x^2 - 8}{4\left(x^3 + 3x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{35}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{35x^4 + 74x^2 - 8}{4\left(x^3 + 3x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{2}{3}$	$\frac{1}{3}$
$i\sqrt{3}$	2	0	$\frac{17}{12}$	$-\frac{5}{12}$
$-i\sqrt{3}$	2	0	$\frac{17}{12}$	$-\frac{5}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{2}$ then

$$\begin{aligned}d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\ &= \frac{7}{2} - \left(\frac{7}{2}\right) \\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^+}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{2}{3x} + \frac{17}{12\left(x-i\sqrt{3}\right)} + \frac{17}{12\left(x+i\sqrt{3}\right)} + (0) \\
&= \frac{2}{3x} + \frac{17}{12\left(x-i\sqrt{3}\right)} + \frac{17}{12\left(x+i\sqrt{3}\right)} \\
&= \frac{2}{3x} + \frac{17x}{6x^2+18}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{2}{3x} + \frac{17}{12\left(x-i\sqrt{3}\right)} + \frac{17}{12\left(x+i\sqrt{3}\right)}\right)(0) + \left(\left(-\frac{2}{3x^2} - \frac{17}{12\left(x-i\sqrt{3}\right)^2} - \frac{17}{12\left(x+i\sqrt{3}\right)^2}\right) + \left(\frac{2}{3x}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(\frac{2}{3x} + \frac{17}{12\left(x-i\sqrt{3}\right)} + \frac{17}{12\left(x+i\sqrt{3}\right)}\right) dx} \\
&= \left(x^2+3\right)^{17/12} x^{2/3}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2+2}{x^3+3x}\, dx} \\
&= z_1 e^{-\frac{\ln(x)}{3} + \frac{5\ln\left(x^2+3\right)}{12}} \\
&= z_1 \left(\frac{\left(x^2+3\right)^{5/12}}{x^{1/3}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^{1/3}\left(x^2+3\right)^{11/6}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+2}{x^3+3x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{2\ln(x)}{3}+\frac{5\ln\left(x^2+3\right)}{6}}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{x^{1/3}(8x^4+44x^2+55)\,\mathrm{e}^{-\frac{2\ln(x)}{3}+\frac{5\ln\left(x^2+3\right)}{6}}}{55\,(x^2+3)^{8/3}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^{1/3}\left(x^2+3\right)^{11/6}\right) + c_2\left(x^{1/3}\left(x^2+3\right)^{11/6}\left(-\frac{x^{1/3}(8x^4+44x^2+55)\,\mathrm{e}^{-\frac{2\ln(x)}{3}+\frac{5\ln\left(x^2+3\right)}{6}}}{55\,(x^2+3)^{8/3}}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x(x^2+3)\left(\frac{d^2}{dx^2}y(x)\right)+(-x^2+2)\left(\frac{d}{dx}y(x)\right)-8xy(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{8y(x)}{x^2+3}+\frac{(x^2-2)\left(\frac{d}{dx}y(x)\right)}{x(x^2+3)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)-\frac{(x^2-2)\left(\frac{d}{dx}y(x)\right)}{x(x^2+3)}-\frac{8y(x)}{x^2+3}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=-\frac{x^2-2}{x(x^2+3)}, P_3(x)=-\frac{8}{x^2+3}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{2}{3}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x(x^2+3)\left(\frac{d^2}{dx^2}y(x)\right)+(-x^2+2)\left(\frac{d}{dx}y(x)\right)-8xy(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - ∘ Convert $x \cdot y(x)$ to series expansion

    $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+1}$
  - ∘ Shift index using $k->k-1$

    $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$
  - ∘ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..2$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
  - ∘ Shift index using $k->k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - ∘ Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 1..3$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$
  - ∘ Shift index using $k->k+2-m$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-1+3r)\, x^{-1+r} + a_1(1+r)\,(2+3r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(3k+2+3r) + a_{k-1}(k+r+1\right.\right.$
- • $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+3r) = 0$
- • Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1}{3}\right\}$
- • Each term must be 0

  $a_1(1+r)\,(2+3r) = 0$
- • Each term in the series must be 0, giving the recursion relation

  $(k+r+1)\left(a_{k-1}(k-5+r) + 3\left(k+r+\frac{2}{3}\right) a_{k+1}\right) = 0$
- • Shift index using $k->k+1$

  $(k+r+2)\left(a_k(k+r-4) + 3\left(k+\frac{5}{3}+r\right) a_{k+2}\right) = 0$
- • Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r-4)}{3k+5+3r}$
- • Recursion relation for $r = 0$; series terminates at $k = 4$

  $a_{k+2} = -\frac{a_k(k-4)}{3k+5}$
- • Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(k-4)}{3k+5}, 2a_1 = 0\right]$
- • Recursion relation for $r = \frac{1}{3}$

  $a_{k+2} = -\frac{a_k\left(k-\frac{11}{3}\right)}{3k+6}$
- • Solution for $r = \frac{1}{3}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -\frac{a_k\left(k-\frac{11}{3}\right)}{3k+6}, 4a_1 = 0\right]$
- • Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = -\frac{a_k(-4+k)}{3k+5}, 2a_1 = 0, b_{k+2} = -\frac{b_k\left(k-\frac{11}{3}\right)}{3k+6}, 4b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 32

```
dsolve(x*(x^2+3)*diff(diff(y(x),x),x)+(-x^2+2)*diff(y(x),x)-8*x*y(x) = 0,y(x),singsol=
```

$$y = c_1 \left(x^2+3\right)^{11/6} x^{1/3} + \frac{c_2 (8x^4 + 44x^2 + 55)}{8}$$

Mathematica DSolve solution

Solving time : 0.219 (sec)
Leaf size : 116

```
DSolve[{x*(3+x^2)*D[y[x],{x,2}]+(2-x^2)*D[y[x],x]-8*x*y[x]==0,{}},y[x],x,IncludeSingularSolu
```

$$y(x) \to \exp\left(\int_1^x \frac{7K[1]^2+4}{2K[1]^3+6K[1]}dK[1] - \frac{1}{2}\int_1^x \frac{2-K[2]^2}{K[2]^3+3K[2]}dK[2]\right)\left(c_2 \int_1^x \exp\left(-2\int_1^{K[3]} \frac{7K[1]^2+4}{2K[1]^3+6K[1]}dK[1]\right)dK[3] + c_1\right)$$

2.1.522 Problem 538

Internal problem ID [9694]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 538
Date solved : `Monday, January 27, 2025 at 06:13:11 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2\left(-x^2+1\right)y''+x\left(-19x^2+7\right)y'-\left(14x^2+1\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.313 (sec)

Writing the ode as

$$\left(-4x^4+4x^2\right)y''+\left(-19x^3+7x\right)y'+\left(-14x^2-1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-4x^4+4x^2\\B&=-19x^3+7x\\C&=-14x^2-1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-15x^4-42x^2+9}{64\left(x^3-x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=-15x^4-42x^2+9\\t&=64\left(x^3-x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-15x^4-42x^2+9}{64\left(x^3-x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.989: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64(x^3 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16\,(x+1)^2} + \frac{9}{64x^2} - \frac{3}{16\,(x-1)^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{9}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{8} \end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{-15x^4 - 42x^2 + 9}{64\left(x^3 - x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{15}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{8} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{8}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-15x^4 - 42x^2 + 9}{64\left(x^3 - x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{9}{8}$	$-\frac{1}{8}$
1	2	0	$\frac{3}{4}$	$\frac{1}{4}$
-1	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{5}{8}$	$\frac{3}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{3}{8}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-} + \alpha_{c_3}^{-}\right) \\
&= \frac{3}{8} - \left(\frac{3}{8}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x - c_3}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{8x} + \frac{1}{4x - 4} + \frac{1}{4x + 4} + (-)(0) \\
&= -\frac{1}{8x} + \frac{1}{4x - 4} + \frac{1}{4x + 4} \\
&= \frac{3x^2 + 1}{8x^3 - 8x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{8x} + \frac{1}{4x - 4} + \frac{1}{4x + 4}\right)(0) + \left(\left(\frac{1}{8x^2} - \frac{1}{4(x-1)^2} - \frac{1}{4(x+1)^2}\right) + \left(-\frac{1}{8x} + \frac{1}{4x - 4} + \frac{1}{4x +
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(-\frac{1}{8x} + \frac{1}{4x-4} + \frac{1}{4x+4}\right) dx} \\
&= \frac{(x-1)^{1/4} (x+1)^{1/4}}{x^{1/8}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-19x^3 + 7x}{-4x^4 + 4x^2}\, dx} \\
&= z_1 e^{-\frac{7\ln(x)}{8} - \frac{3\ln(x-1)}{4} - \frac{3\ln(x+1)}{4}} \\
&= z_1 \left(\frac{1}{x^{7/8} (x-1)^{3/4} (x+1)^{3/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(x^2 - 1)^{1/4}}{x (x-1)^{3/4} (x+1)^{3/4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-19x^3 + 7x}{-4x^4 + 4x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{7\ln(x)}{4} - \frac{3\ln(x-1)}{2} - \frac{3\ln(x+1)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{e^{-\frac{7\ln(x)}{4} - \frac{3\ln(x-1)}{2} - \frac{3\ln(x+1)}{2}} x^2 (x-1)^{3/2} (x+1)^{3/2}}{\sqrt{x^2 - 1}}\, dx\right)
\end{aligned}
$$

Therefore the solution is

$$y = c_1y_1 + c_2y_2$$

$$= c_1\left(\frac{(x^2-1)^{1/4}}{x(x-1)^{3/4}(x+1)^{3/4}}\right) + c_2\left(\frac{(x^2-1)^{1/4}}{x(x-1)^{3/4}(x+1)^{3/4}}\left(\int \frac{e^{-\frac{7\ln(x)}{4}-\frac{3\ln(x-1)}{2}-\frac{3\ln(x+1)}{2}}x^2(x-1)^{3/2}(x+1)^{3}}{\sqrt{x^2-1}}\right.\right.$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2(-x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + x(-19x^2+7)\left(\frac{d}{dx}y(x)\right) - (14x^2+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(14x^2+1)y(x)}{4(x^2-1)x^2} - \frac{(19x^2-7)\left(\frac{d}{dx}y(x)\right)}{4x(x^2-1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(19x^2-7)\left(\frac{d}{dx}y(x)\right)}{4x(x^2-1)} + \frac{(14x^2+1)y(x)}{4(x^2-1)x^2} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{19x^2-7}{4x(x^2-1)}, P_3(x) = \frac{14x^2+1}{4(x^2-1)x^2}\right]$

  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left.((x+1)\cdot P_2(x))\right|_{x=-1} = \frac{3}{2}$

  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left.((x+1)^2\cdot P_3(x))\right|_{x=-1} = 0$

  - $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $4(x^2-1)\,x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(19x^2-7)\left(\frac{d}{dx}y(x)\right) + (14x^2+1)\,y(x) = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(4u^4-16u^3+20u^2-8u)\left(\frac{d^2}{du^2}y(u)\right) + (19u^3-57u^2+50u-12)\left(\frac{d}{du}y(u)\right) + (14u^2-28u+15)\,y($

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m\cdot y(u)$ to series expansion for $m = 0..2$

    $u^m\cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

  - Shift index using $k->k-m$

    $u^m\cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..3$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..4$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-4a_0r(1+2r)\, u^{-1+r} + \left(-4a_1(1+r)\,(3+2r) + 5a_0(4r^2+6r+3)\right) u^r + \left(-4a_2(2+r)\,(5+2r) + \right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-4r(1+2r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{0, -\frac{1}{2}\right\}$

- The coefficients of each power of u must be 0

$[-4a_1(1+r)\,(3+2r) + 5a_0(4r^2+6r+3) = 0, -4a_2(2+r)\,(5+2r) + 5a_1(4r^2+14r+13) - a_0$

- Solve for the dependent coefficient(s)

$\left\{a_1 = \frac{5a_0(4r^2+6r+3)}{4(2r^2+5r+3)}, a_2 = \frac{a_0(272r^4+1352r^3+2464r^2+1948r+639)}{16(4r^4+28r^3+71r^2+77r+30)}\right\}$

- Each term in the series must be 0, giving the recursion relation

$4(5a_k + a_{k-2} - 4a_{k-1} - 2a_{k+1})\, k^2 + (8(5a_k + a_{k-2} - 4a_{k-1} - 2a_{k+1})\, r + 30a_k - a_{k-2} - 9a_{k-1} -$

- Shift index using $k->k+2$

$4(5a_{k+2} + a_k - 4a_{k+1} - 2a_{k+3})\,(k+2)^2 + (8(5a_{k+2} + a_k - 4a_{k+1} - 2a_{k+3})\, r + 30a_{k+2} - a_k - 9a_k$

- Recursion relation that defines series solution to ODE

$a_{k+3} = \frac{4k^2a_k-16k^2a_{k+1}+20k^2a_{k+2}+8kra_k-32kra_{k+1}+40kra_{k+2}+4r^2a_k-16r^2a_{k+1}+20r^2a_{k+2}+15ka_k-73ka_{k+1}+110ka_{k+2}}{4(2k^2+4kr+2r^2+13k+13r+21)}$

- Recursion relation for $r = 0$

$a_{k+3} = \frac{4k^2a_k-16k^2a_{k+1}+20k^2a_{k+2}+15ka_k-73ka_{k+1}+110ka_{k+2}+14a_k-85a_{k+1}+155a_{k+2}}{4(2k^2+13k+21)}$

- Solution for $r = 0$

$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+3} = \frac{4k^2a_k-16k^2a_{k+1}+20k^2a_{k+2}+15ka_k-73ka_{k+1}+110ka_{k+2}+14a_k-85a_{k+1}+155a_{k+2}}{4(2k^2+13k+21)}, a_1 = \right.$

- Revert the change of variables $u = x+1$

$\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+3} = \frac{4k^2a_k-16k^2a_{k+1}+20k^2a_{k+2}+15ka_k-73ka_{k+1}+110ka_{k+2}+14a_k-85a_{k+1}+155a_{k+2}}{4(2k^2+13k+21)}, \right.$

- Recursion relation for $r = -\frac{1}{2}$

$a_{k+3} = \frac{4k^2a_k-16k^2a_{k+1}+20k^2a_{k+2}+11ka_k-57ka_{k+1}+90ka_{k+2}+\frac{15}{2}a_k-\frac{105}{2}a_{k+1}+105a_{k+2}}{4(2k^2+11k+15)}$

- Solution for $r = -\frac{1}{2}$

$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{1}{2}}, a_{k+3} = \frac{4k^2a_k-16k^2a_{k+1}+20k^2a_{k+2}+11ka_k-57ka_{k+1}+90ka_{k+2}+\frac{15}{2}a_k-\frac{105}{2}a_{k+1}+105a_{k+2}}{4(2k^2+11k+15)}, a_1 = \right.$

- Revert the change of variables $u = x+1$

$\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^{k-\frac{1}{2}}, a_{k+3} = \frac{4k^2a_k-16k^2a_{k+1}+20k^2a_{k+2}+11ka_k-57ka_{k+1}+90ka_{k+2}+\frac{15}{2}a_k-\frac{105}{2}a_{k+1}+105a_{k+}}{4(2k^2+11k+15)}\right.$

- Combine solutions and rename parameters

$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k(x+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k(x+1)^{k-\frac{1}{2}}\right), a_{k+3} = \frac{4k^2a_k-16k^2a_{k+1}+20k^2a_{k+2}+15ka_k-73ka_{k+1}+11}{4(2k^2+13k+21)}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.107` (sec)
Leaf size : 44

```
dsolve(4*x^2*(-x^2+1)*diff(diff(y(x),x),x)+x*(-19*x^2+7)*diff(y(x),x)-(14*x^2+1)*y(x) =
```

$$y = \frac{c_1 \operatorname{LegendreP}\left(-\frac{3}{8}, \frac{5}{8}, \sqrt{-x^2+1}\right) + c_2 \operatorname{LegendreQ}\left(-\frac{3}{8}, \frac{5}{8}, \sqrt{-x^2+1}\right)}{x^{3/8}\sqrt{x^2-1}}$$

Mathematica DSolve solution

Solving time : `0.291` (sec)
Leaf size : 120

```
DSolve[{4*x^2*(1-x^2)*D[y[x],{x,2}]+x*(7-19*x^2)*D[y[x],x]-(1+14*x^2)*y[x]==0,{}},y[x],x,Inclu
```

$$y(x) \to \exp\left(\int_1^x -\frac{3K[1]^2+1}{8K[1]-8K[1]^3}dK[1] - \frac{1}{2}\int_1^x \frac{7-19K[2]^2}{4K[2]-4K[2]^3}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} -\frac{3K[1]^2+1}{8K[1]-8K[1]^3}dK[1]\right)dK[3]+c_1\right)$$

2.1.523 Problem 539

Internal problem ID [9695]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 539
Date solved : `Monday, January 27, 2025 at 06:13:11 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$3x^2\left(-x^2+2\right)y''+x\left(-11x^2+1\right)y'+\left(-5x^2+1\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.411 (sec)

Writing the ode as

$$\left(-3x^4+6x^2\right)y''+\left(-11x^3+x\right)y'+\left(-5x^2+1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=-3x^4+6x^2\\ B&=-11x^3+x\\ C&=-5x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{-5x^4-4x^2-35}{36\left(x^3-2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=-5x^4-4x^2-35\\ t&=36\left(x^3-2x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-5x^4-4x^2-35}{36\left(x^3-2x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.991: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^3 - 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \sqrt{2}$ of order 2. There is a pole at $x = -\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{64\left(x-\sqrt{2}\right)^2} - \frac{7}{64\left(x+\sqrt{2}\right)^2} + \frac{31\sqrt{2}}{384\left(x-\sqrt{2}\right)} - \frac{31\sqrt{2}}{384\left(x+\sqrt{2}\right)} - \frac{35}{144x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12} \end{aligned}$$

For the pole at $x=\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x-\sqrt{2}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{7}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{7}{8} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{8}
\end{aligned}
$$

For the pole at $x=-\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x+\sqrt{2}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{7}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{7}{8} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{8}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty=0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r=\frac{s}{t}=\frac{-5x^{4}-4x^{2}-35}{36\left(x^{3}-2x\right)^{2}}
$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{5}{36}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{6} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{6}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-5x^{4}-4x^{2}-35}{36\left(x^{3}-2x\right)^{2}}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{12}$	$\frac{5}{12}$
$\sqrt{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$
$-\sqrt{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{5}{6}$	$\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=\frac{5}{6}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{+}+\alpha_{c_2}^{-}+\alpha_{c_3}^{-}\right) \\
&= \frac{5}{6}-\left(\frac{5}{6}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{7}{12x} + \frac{1}{8x-8\sqrt{2}} + \frac{1}{8x+8\sqrt{2}} + (0) \\
&= \frac{7}{12x} + \frac{1}{8x-8\sqrt{2}} + \frac{1}{8x+8\sqrt{2}} \\
&= \frac{5x^2-7}{6x^3-12x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{7}{12x} + \frac{1}{8x-8\sqrt{2}} + \frac{1}{8x+8\sqrt{2}}\right)(0) + \left(\left(-\frac{7}{12x^2} - \frac{1}{8\left(x-\sqrt{2}\right)^2} - \frac{1}{8\left(x+\sqrt{2}\right)^2}\right) + \left(\frac{7}{12x} + \frac{1}{8x-\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(\frac{7}{12x} + \frac{1}{8x-8\sqrt{2}} + \frac{1}{8x+8\sqrt{2}}\right)dx} \\
&= \left(x+\sqrt{2}\right)^{1/8} x^{7/12} \left(x-\sqrt{2}\right)^{1/8}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-11x^3+x}{-3x^4+6x^2}\, dx} \\
&= z_1 e^{-\frac{\ln(x)}{12} - \frac{7\ln\left(x^2-2\right)}{8}} \\
&= z_1 \left(\frac{1}{x^{1/12}\left(x^2-2\right)^{7/8}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{\left(x^2-2\right)^{3/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-11x^3+x}{-3x^4+6x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x)}{6}-\frac{7\ln(x^2-2)}{4}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{\ln(x)}{6}-\frac{7\ln(x^2-2)}{4}}(x^2-2)^{3/2}}{x}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\sqrt{x}}{(x^2-2)^{3/4}}\right) + c_2\left(\frac{\sqrt{x}}{(x^2-2)^{3/4}}\left(\int \frac{\mathrm{e}^{-\frac{\ln(x)}{6}-\frac{7\ln(x^2-2)}{4}}(x^2-2)^{3/2}}{x}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$3x^2(-x^2+2)\left(\frac{d^2}{dx^2}y(x)\right)+x(-11x^2+1)\left(\frac{d}{dx}y(x)\right)+(-5x^2+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
$\frac{d^2}{dx^2}y(x)=-\frac{(5x^2-1)y(x)}{3x^2(x^2-2)}-\frac{(11x^2-1)\left(\frac{d}{dx}y(x)\right)}{3x(x^2-2)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
$\frac{d^2}{dx^2}y(x)+\frac{(11x^2-1)\left(\frac{d}{dx}y(x)\right)}{3x(x^2-2)}+\frac{(5x^2-1)y(x)}{3x^2(x^2-2)}=0$

☐ Check to see if x_0 is a regular singular point
- ○ Define functions
$\left[P_2(x)=\frac{11x^2-1}{3x(x^2-2)}, P_3(x)=\frac{5x^2-1}{3x^2(x^2-2)}\right]$
- ○ $x\cdot P_2(x)$ is analytic at $x=0$
$\left.(x\cdot P_2(x))\right|_{x=0}=\frac{1}{6}$
- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$
$\left.(x^2\cdot P_3(x))\right|_{x=0}=\frac{1}{6}$
- ○ $x=0$is a regular singular point
Check to see if x_0 is a regular singular point
$x_0=0$
- Multiply by denominators
$3x^2(x^2-2)\left(\frac{d^2}{dx^2}y(x)\right)+x(11x^2-1)\left(\frac{d}{dx}y(x)\right)+(5x^2-1)\,y(x)=0$
- Assume series solution for $y(x)$
$y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
- Shift index using $k- >k + 1 - m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$
- Shift index using $k- >k + 2 - m$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0(-1+3r)\,(-1+2r)\, x^r - a_1(2+3r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (-a_k(3k+3r-1)\,(2k+2r-1) + a_k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-(-1+3r)\,(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2}, \frac{1}{3}\right\}$
- Each term must be 0

  $-a_1(2+3r)\,(1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $-6\left(\frac{(-k-r+1)a_{k-2}}{2} + \left(k+r-\frac{1}{2}\right) a_k\right)\left(k-\frac{1}{3}+r\right) = 0$
- Shift index using $k- >k + 2$

  $-6\left(\frac{(-k-1-r)a_k}{2} + \left(k+\frac{3}{2}+r\right) a_{k+2}\right)\left(k+\frac{5}{3}+r\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{(k+r+1)a_k}{2k+3+2r}$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = \frac{\left(k+\frac{3}{2}\right)a_k}{2k+4}$
- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{\left(k+\frac{3}{2}\right)a_k}{2k+4}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{3}$

  $a_{k+2} = \frac{\left(k+\frac{4}{3}\right)a_k}{2k+\frac{11}{3}}$
- Solution for $r = \frac{1}{3}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = \frac{\left(k+\frac{4}{3}\right)a_k}{2k+\frac{11}{3}}, a_1 = 0\right]$
- Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{3}}\right), a_{k+2}=\frac{\left(k+\frac{3}{2}\right)a_k}{2k+4}, a_1=0, b_{k+2}=\frac{\left(k+\frac{4}{3}\right)b_k}{2k+\frac{11}{3}}, b_1=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - return
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.089 (sec)
Leaf size : 35

```
dsolve(3*x^2*(-x^2+2)*diff(diff(y(x),x),x)+x*(-11*x^2+1)*diff(y(x),x)+(-5*x^2+1)*y(x)
```

$$y=\frac{c_1\sqrt{x}}{\left(-2x^2+4\right)^{3/4}}+c_2 x^{1/3}\,\mathrm{hypergeom}\left(\left[\frac{2}{3},1\right],\left[\frac{11}{12}\right],\frac{x^2}{2}\right)$$

Mathematica DSolve solution

Solving time : 0.286 (sec)
Leaf size : 118

```
DSolve[{3*x^2*(2-x^2)*D[y[x],{x,2}]+x*(1-11*x^2)*D[y[x],x]+(1-5*x^2)*y[x]==0,{}},y[x],x,Includ
```

$$y(x) \to \exp\left(\int_1^x \frac{7-5K[1]^2}{12K[1]-6K[1]^3}dK[1] - \frac{1}{2}\int_1^x \frac{1-11K[2]^2}{6K[2]-3K[2]^3}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{7-5K[1]^2}{12K[1]-6K[1]^3}dK[1]\right)dK[3] + c_1\right)$$

2.1.524 Problem 540

Internal problem ID [9696]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 540
Date solved : Monday, January 27, 2025 at 06:13:12 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$2x^2\left(x^2+2\right)y''-x\left(-7x^2+12\right)y'+\left(3x^2+7\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.332 (sec)

Writing the ode as

$$\left(2x^4+4x^2\right)y''+\left(7x^3-12x\right)y'+\left(3x^2+7\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^4+4x^2 \\ B &= 7x^3-12x \\ C &= 3x^2+7 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-3x^4-72x^2+128}{16\left(x^3+2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3x^4-72x^2+128 \\ t &= 16\left(x^3+2x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-3x^4-72x^2+128}{16\left(x^3+2x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.993: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^3 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i\sqrt{2}$ of order 2. There is a pole at $x = -i\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{65}{64\left(x - i\sqrt{2}\right)^2} + \frac{65}{64\left(x + i\sqrt{2}\right)^2} + \frac{135i\sqrt{2}}{128\left(x - i\sqrt{2}\right)} - \frac{135i\sqrt{2}}{128\left(x + i\sqrt{2}\right)} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

For the pole at $x = i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x-i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{65}{64}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{13}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{8}\end{aligned}$$

For the pole at $x = -i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x+i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{65}{64}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{13}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{8}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^4 - 72x^2 + 128}{16\left(x^3+2x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x^4 - 72x^2 + 128}{16\left(x^3+2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1
$i\sqrt{2}$	2	0	$\frac{13}{8}$	$-\frac{5}{8}$
$-i\sqrt{2}$	2	0	$\frac{13}{8}$	$-\frac{5}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{4}$ then

$$\begin{aligned}d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\ &= \frac{3}{4} - \left(\frac{3}{4}\right) \\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\ &= \frac{2}{x} - \frac{5}{8\left(x-i\sqrt{2}\right)} - \frac{5}{8\left(x+i\sqrt{2}\right)} + (0) \\ &= \frac{2}{x} - \frac{5}{8\left(x-i\sqrt{2}\right)} - \frac{5}{8\left(x+i\sqrt{2}\right)} \\ &= \frac{2}{x} - \frac{5x}{4x^2+8}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{2}{x} - \frac{5}{8\left(x-i\sqrt{2}\right)} - \frac{5}{8\left(x+i\sqrt{2}\right)}\right)(0) + \left(\left(-\frac{2}{x^2} + \frac{5}{8\left(x-i\sqrt{2}\right)^2} + \frac{5}{8\left(x+i\sqrt{2}\right)^2}\right) + \left(\frac{2}{x} - \frac{5}{8\left(x-\right.}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\,dx} \\ &= e^{\int \left(\frac{2}{x} - \frac{5}{8\left(x-i\sqrt{2}\right)} - \frac{5}{8\left(x+i\sqrt{2}\right)}\right)dx} \\ &= \frac{x^2}{\left(x^2+2\right)^{5/8}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{7x^3-12x}{2x^4+4x^2}\,dx} \\ &= z_1 e^{-\frac{13\ln\left(x^2+2\right)}{8} + \frac{3\ln(x)}{2}} \\ &= z_1\left(\frac{x^{3/2}}{\left(x^2+2\right)^{13/8}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{7/2}}{\left(x^2+2\right)^{9/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x^3-12x}{2x^4+4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{13\ln(x^2+2)}{4}+3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{e^{-\frac{13\ln(x^2+2)}{4}+3\ln(x)}\left(x^2+2\right)^{9/2}}{x^7}\,dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^{7/2}}{(x^2+2)^{9/4}}\right) + c_2\left(\frac{x^{7/2}}{(x^2+2)^{9/4}}\left(\int \frac{e^{-\frac{13\ln(x^2+2)}{4}+3\ln(x)}\left(x^2+2\right)^{9/2}}{x^7}\,dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2(x^2+2)\left(\frac{d^2}{dx^2}y(x)\right) - x(-7x^2+12)\left(\frac{d}{dx}y(x)\right) + (3x^2+7)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(3x^2+7)y(x)}{2(x^2+2)x^2} - \frac{(7x^2-12)\left(\frac{d}{dx}y(x)\right)}{2x(x^2+2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(7x^2-12)\left(\frac{d}{dx}y(x)\right)}{2x(x^2+2)} + \frac{(3x^2+7)y(x)}{2(x^2+2)x^2} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{7x^2-12}{2x(x^2+2)}, P_3(x) = \frac{3x^2+7}{2(x^2+2)x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -3$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{7}{4}$

  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $2x^2(x^2+2)\left(\frac{d^2}{dx^2}y(x)\right) + x(7x^2-12)\left(\frac{d}{dx}y(x)\right) + (3x^2+7)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$
    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k-m$
    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$
    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
  - Shift index using $k- >k+1-m$
    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$
    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$
  - Shift index using $k- >k+2-m$
    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions
  $a_0(-1+2r)\,(-7+2r)\, x^r + a_1(1+2r)\,(-5+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r-1)\,(2k+2r-7) + a_{k-}\right.$
- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+2r)\,(-7+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{\frac{1}{2}, \frac{7}{2}\right\}$
- Each term must be 0
  $a_1(1+2r)\,(-5+2r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $4\left(\frac{a_{k-2}(k+r-1)}{2} + a_k\left(k+r-\frac{7}{2}\right)\right)\left(k+r-\frac{1}{2}\right) = 0$
- Shift index using $k- >k+2$
  $4\left(\frac{a_k(k+r+1)}{2} + a_{k+2}\left(k-\frac{3}{2}+r\right)\right)\left(k+\frac{3}{2}+r\right) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{a_k(k+r+1)}{2k-3+2r}$
- Recursion relation for $r = \frac{1}{2}$
  $a_{k+2} = -\frac{a_k\left(k+\frac{3}{2}\right)}{2k-2}$
- Solution for $r = \frac{1}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{a_k\left(k+\frac{3}{2}\right)}{2k-2}, a_1 = 0\right]$
- Recursion relation for $r = \frac{7}{2}$
  $a_{k+2} = -\frac{a_k\left(k+\frac{9}{2}\right)}{2k+4}$
- Solution for $r = \frac{7}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{7}{2}}, a_{k+2} = -\frac{a_k\left(k+\frac{9}{2}\right)}{2k+4}, a_1 = 0\right]$

- Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{7}{2}}\right), a_{k+2}=-\frac{a_k\left(k+\frac{3}{2}\right)}{2k-2}, a_1=0, b_{k+2}=-\frac{b_k\left(k+\frac{9}{2}\right)}{2k+4}, b_1=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - return
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `1.849` (sec)
Leaf size : 35

```
dsolve(2*x^2*(x^2+2)*diff(diff(y(x),x),x)-x*(-7*x^2+12)*diff(y(x),x)+(3*x^2+7)*y(x) =
```

$$y=\frac{c_1 x^{7/2}}{(2x^2+4)^{9/4}}+c_2\sqrt{x}\,\mathrm{hypergeom}\left(\left[\frac{3}{4},1\right],\left[-\frac{1}{2}\right],-\frac{x^2}{2}\right)$$

Mathematica DSolve solution

Solving time : `0.305` (sec)
Leaf size : 117

```
DSolve[{2*x^2*(2+x^2)*D[y[x],{x,2}]-x*(12-7*x^2)*D[y[x],x]+(7+3*x^2)*y[x]==0,{}},y[x],x,Incl
```

$$y(x)\to\exp\left(\int_1^x\frac{3K[1]^2+16}{4K[1]^3+8K[1]}dK[1]-\frac{1}{2}\int_1^x\left(\frac{13K[2]}{2\left(K[2]^2+2\right)}-\frac{3}{K[2]}\right)dK[2]\right)\left(c_2\int_1^x\exp\left(-2\int_1^{K[3]}\frac{3K[1]^2+16}{4K[1]^3+8K[1]}dK[1]\right)dK[3]+c_1\right)$$

2.1.525 Problem 541

Internal problem ID [9697]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 541
Date solved : `Monday, January 27, 2025 at 06:13:13 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2\left(x^2+2\right)y''+x\left(7x^2+4\right)y'-\left(-3x^2+1\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.305 (sec)

Writing the ode as

$$\left(2x^4+4x^2\right)y''+\left(7x^3+4x\right)y'+\left(3x^2-1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=2x^4+4x^2\\ B&=7x^3+4x\\ C&=3x^2-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-3x^2+24}{16\left(x^2+2\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=-3x^2+24\\ t&=16\left(x^2+2\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-3x^2+24}{16\left(x^2+2\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.995: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2+2)^2$. There is a pole at $x = i\sqrt{2}$ of order 2. There is a pole at $x = -i\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{15}{64\left(x-i\sqrt{2}\right)^2} - \frac{15}{64\left(x+i\sqrt{2}\right)^2} - \frac{9i\sqrt{2}}{128\left(x-i\sqrt{2}\right)} + \frac{9i\sqrt{2}}{128\left(x+i\sqrt{2}\right)}$$

For the pole at $x = i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x-i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{15}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{8} \end{aligned}$$

For the pole at $x = -i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x+i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{15}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{8} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 + 24}{16\left(x^2+2\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x^2 + 24}{16\left(x^2+2\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$i\sqrt{2}$	2	0	$\frac{5}{8}$	$\frac{3}{8}$
$-i\sqrt{2}$	2	0	$\frac{5}{8}$	$\frac{3}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{3}{4} - \left(\frac{3}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{3}{8\left(x-i\sqrt{2}\right)} + \frac{3}{8\left(x+i\sqrt{2}\right)} + (0) \\
&= \frac{3}{8\left(x-i\sqrt{2}\right)} + \frac{3}{8\left(x+i\sqrt{2}\right)} \\
&= \frac{3x}{4x^2+8}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{8\,(x - i\sqrt{2})} + \frac{3}{8\,(x + i\sqrt{2})}\right)(0) + \left(\left(-\frac{3}{8\,(x - i\sqrt{2})^2} - \frac{3}{8\,(x + i\sqrt{2})^2}\right) + \left(\frac{3}{8\,(x - i\sqrt{2})} + \frac{}{8\,(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(\frac{3}{8\left(x - i\sqrt{2}\right)} + \frac{3}{8\left(x + i\sqrt{2}\right)}\right) dx} \\
&= \left(-x^2 - 2\right)^{3/8}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7x^3+4x}{2x^4+4x^2}\, dx} \\
&= z_1 e^{-\frac{\ln(x)}{2} - \frac{5\ln\left(x^2+2\right)}{8}} \\
&= z_1 \left(\frac{1}{\sqrt{x}\,\left(x^2+2\right)^{5/8}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(-1)^{3/8}}{\sqrt{x}\,\left(x^2+2\right)^{1/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x^3+4x}{2x^4+4x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\ln(x) - \frac{5\ln\left(x^2+2\right)}{4}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int -\mathrm{e}^{-\ln(x) - \frac{5\ln\left(x^2+2\right)}{4}} x\sqrt{x^2+2}\,(-1)^{1/4}\, dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(-1)^{3/8}}{\sqrt{x}\,\left(x^2+2\right)^{1/4}}\right) + c_2\left(\frac{(-1)^{3/8}}{\sqrt{x}\,\left(x^2+2\right)^{1/4}}\left(\int -\mathrm{e}^{-\ln(x) - \frac{5\ln\left(x^2+2\right)}{4}} x\sqrt{x^2+2}\,(-1)^{1/4}\, dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2(x^2+2)\left(\frac{d^2}{dx^2}y(x)\right)+x(7x^2+4)\left(\frac{d}{dx}y(x)\right)-(-3x^2+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{(3x^2-1)y(x)}{2(x^2+2)x^2}-\frac{(7x^2+4)\left(\frac{d}{dx}y(x)\right)}{2x(x^2+2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x)+\frac{(7x^2+4)\left(\frac{d}{dx}y(x)\right)}{2x(x^2+2)}+\frac{(3x^2-1)y(x)}{2(x^2+2)x^2}=0$

- □ Check to see if x_0 is a regular singular point
  - ◦ Define functions

    $\left[P_2(x)=\frac{7x^2+4}{2x(x^2+2)},P_3(x)=\frac{3x^2-1}{2(x^2+2)x^2}\right]$
  - ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\bigg|_{x=0}=1$
  - ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\bigg|_{x=0}=-\frac{1}{4}$
  - ◦ $x=0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=0$
- Multiply by denominators

$2x^2(x^2+2)\left(\frac{d^2}{dx^2}y(x)\right)+x(7x^2+4)\left(\frac{d}{dx}y(x)\right)+(3x^2-1)\,y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

- □ Rewrite ODE with series expansions
  - ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

    $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$
  - ◦ Shift index using $k->k-m$

    $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$
  - ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$
  - ◦ Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - ◦ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..4$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$
  - ◦ Shift index using $k->k+2-m$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$a_0(1+2r)(-1+2r)x^r + a_1(3+2r)(1+2r)x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+1)(2k+2r-1) + a_{k-2}\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(1+2r)(-1+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0
  $a_1(3+2r)(1+2r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $4\left(\frac{a_{k-2}(k+r-1)}{2} + a_k\left(k+r+\frac{1}{2}\right)\right)\left(k+r-\frac{1}{2}\right) = 0$
- Shift index using $k->k+2$
  $4\left(\frac{a_k(k+r+1)}{2} + a_{k+2}\left(k+\frac{5}{2}+r\right)\right)\left(k+\frac{3}{2}+r\right) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{a_k(k+r+1)}{2k+5+2r}$
- Recursion relation for $r = -\frac{1}{2}$
  $a_{k+2} = -\frac{a_k\left(k+\frac{1}{2}\right)}{2k+4}$
- Solution for $r = -\frac{1}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{a_k\left(k+\frac{1}{2}\right)}{2k+4}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{2}$
  $a_{k+2} = -\frac{a_k\left(k+\frac{3}{2}\right)}{2k+6}$
- Solution for $r = \frac{1}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{a_k\left(k+\frac{3}{2}\right)}{2k+6}, a_1 = 0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{a_k\left(k+\frac{1}{2}\right)}{2k+4}, a_1 = 0, b_{k+2} = -\frac{b_k\left(k+\frac{3}{2}\right)}{2k+6}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Legendre successful
```

```
<- special function solution successful
   -> Trying to convert hypergeometric functions to elementary form...
   <- elementary form is not straightforward to achieve - returning special function
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.057 (sec)
Leaf size : 35

```
dsolve(2*x^2*(x^2+2)*diff(diff(y(x),x),x)+x*(7*x^2+4)*diff(y(x),x)-(-3*x^2+1)*y(x) = 0,y
```

$$y = \frac{c_2 \operatorname{LegendreQ}\left(-\frac{1}{4}, \frac{1}{4}, \frac{i\sqrt{2}\, x}{2}\right) \left(x^2+2\right)^{1/8} + c_1}{\left(x^2+2\right)^{1/4} \sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.257 (sec)
Leaf size : 95

```
DSolve[{2*x^2*(2+x^2)*D[y[x],{x,2}]+x*(4+7*x^2)*D[y[x],x]-(1-3*x^2)*y[x]==0,{}},y[x],x,IncludeS
```

$$y(x) \to \frac{\left(c_2 \sqrt[8]{x^2+2}\, \text{Gamma}\left(\frac{3}{4}\right) Q_{-\frac{1}{4}}^{\frac{1}{4}}\left(\frac{ix}{\sqrt{2}}\right) + 2^{3/8} c_1\right) \exp\left(\int_1^x -\frac{3K[1]^2+4}{4K[1]^3+8K[1]} dK[1]\right)}{\sqrt[8]{x^2+2}\, \text{Gamma}\left(\frac{3}{4}\right)}$$

2.1.526 Problem 542

Internal problem ID [9698]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 542
Date solved : Monday, January 27, 2025 at 06:13:14 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$2x^2\left(2x^2+1\right)y''+5x\left(6x^2+1\right)y'-\left(-40x^2+2\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.500 (sec)

Writing the ode as

$$\left(4x^4+2x^2\right)y''+\left(30x^3+5x\right)y'+\left(40x^2-2\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=4x^4+2x^2\\B&=30x^3+5x\\C&=40x^2-2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{20x^4+12x^2+21}{16\left(2x^3+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=20x^4+12x^2+21\\t&=16\left(2x^3+x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{20x^4+12x^2+21}{16\left(2x^3+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.997: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(2x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{i\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{i\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16\left(x - \frac{i\sqrt{2}}{2}\right)^2} + \frac{5}{16\left(x + \frac{i\sqrt{2}}{2}\right)^2} + \frac{13i\sqrt{2}}{16\left(x - \frac{i\sqrt{2}}{2}\right)} - \frac{13i\sqrt{2}}{16\left(x + \frac{i\sqrt{2}}{2}\right)} + \frac{21}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4} \end{aligned}$$

For the pole at $x = \frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

For the pole at $x = -\frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{20x^4 + 12x^2 + 21}{16\left(2x^3 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{20x^4 + 12x^2 + 21}{16\left(2x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
$\frac{i\sqrt{2}}{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-\frac{i\sqrt{2}}{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{5}{4} - \left(\frac{5}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x - c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{7}{4x} - \frac{1}{4\left(x - \frac{i\sqrt{2}}{2}\right)} - \frac{1}{4\left(x + \frac{i\sqrt{2}}{2}\right)} + (0) \\
&= \frac{7}{4x} - \frac{1}{4\left(x - \frac{i\sqrt{2}}{2}\right)} - \frac{1}{4\left(x + \frac{i\sqrt{2}}{2}\right)} \\
&= \frac{10x^2 + 7}{8x^3 + 4x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{7}{4x} - \frac{1}{4\left(x - \frac{i\sqrt{2}}{2}\right)} - \frac{1}{4\left(x + \frac{i\sqrt{2}}{2}\right)} \right)(0) + \left(\left(-\frac{7}{4x^2} + \frac{1}{4\left(x - \frac{i\sqrt{2}}{2}\right)^2} + \frac{1}{4\left(x + \frac{i\sqrt{2}}{2}\right)^2} \right) + \left(\frac{7}{4x} - \frac{}{4} \right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{7}{4x} - \frac{1}{4\left(x - \frac{i\sqrt{2}}{2}\right)} - \frac{1}{4\left(x + \frac{i\sqrt{2}}{2}\right)} \right) dx} \\
&= \frac{2^{3/4} x^{7/4}}{2\left(2x^2 + 1\right)^{1/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{30x^3 + 5x}{4x^4 + 2x^2} \, dx} \\
&= z_1 e^{-\frac{5 \ln\left(x\left(2x^2 + 1\right)\right)}{4}} \\
&= z_1 \left(\frac{1}{\left(2x^3 + x\right)^{5/4}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{2^{3/4} x^{3/4}}{2\left(2x^2 + 1\right)^{5/4} \left(2x^3 + x\right)^{1/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{30x^3+5x}{4x^4+2x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{5\ln(2x^3+x)}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{(2x^2+1)^{5/2}\sqrt{2}}{(2x^3+x)^2\, x^{3/2}} dx \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{2^{3/4} x^{3/4}}{2\,(2x^2+1)^{5/4}\,(2x^3+x)^{1/4}} \right) + c_2 \left(\frac{2^{3/4} x^{3/4}}{2\,(2x^2+1)^{5/4}\,(2x^3+x)^{1/4}} \left(\int \frac{(2x^2+1)^{5/2}\sqrt{2}}{(2x^3+x)^2\, x^{3/2}} dx \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$2x^2(2x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + 5x(6x^2+1)\left(\frac{d}{dx}y(x)\right) - (-40x^2+2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{(20x^2-1)y(x)}{x^2(2x^2+1)} - \frac{5(6x^2+1)\left(\frac{d}{dx}y(x)\right)}{2x(2x^2+1)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lir
  $\frac{d^2}{dx^2}y(x) + \frac{5(6x^2+1)\left(\frac{d}{dx}y(x)\right)}{2x(2x^2+1)} + \frac{(20x^2-1)y(x)}{x^2(2x^2+1)} = 0$
- □ Check to see if x_0 is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{5(6x^2+1)}{2x(2x^2+1)}, P_3(x) = \frac{20x^2-1}{x^2(2x^2+1)}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{5}{2}$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -1$
  - $x = 0$is a regular singular point
    Check to see if x_0 is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $2x^2(2x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + 5x(6x^2+1)\left(\frac{d}{dx}y(x)\right) + (40x^2-2)\,y(x) = 0$
- Assume series solution for $y(x)$

$y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- >k + 2 - m$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0(2+r)\,(-1+2r)\, x^r + a_1(3+r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+2)\,(2k+2r-1) + 2a_{k-2}(k+r$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-2, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3+r)\,(1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $2(k+r+2)\left(a_{k-2}(2k+1+2r) + \left(k+r-\frac{1}{2}\right) a_k\right) = 0$

- Shift index using $k- >k + 2$

  $2(k+r+4)\left(a_k(2k+2r+5) + \left(k+\frac{3}{2}+r\right) a_{k+2}\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2a_k(2k+2r+5)}{2k+3+2r}$

- Recursion relation for $r = -2$

  $a_{k+2} = -\frac{2a_k(2k+1)}{2k-1}$

- Solution for $r = -2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{2a_k(2k+1)}{2k-1}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{2a_k(2k+6)}{2k+4}$

- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{2a_k(2k+6)}{2k+4}, a_1 = 0\right]$

- Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^{k-2}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2}=-\frac{2a_k(2k+1)}{2k-1}, a_1=0, b_{k+2}=-\frac{2b_k(2k+6)}{2k+4}, b_1=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - return
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.090 (sec)
Leaf size : 35

```
dsolve(2*x^2*(2*x^2+1)*diff(diff(y(x),x),x)+5*x*(6*x^2+1)*diff(y(x),x)-(-40*x^2+2)*y(x
```

$$y=\frac{c_1\sqrt{x}}{(2x^2+1)^{3/2}}+\frac{c_2 \operatorname{hypergeom}\left(\left[\frac{1}{4},1\right],\left[-\frac{1}{4}\right],-2x^2\right)}{x^2}$$

Mathematica DSolve solution

Solving time : 0.283 (sec)
Leaf size : 118

```
DSolve[{2*x^2*(1+2*x^2)*D[y[x],{x,2}]+5*x*(1+6*x^2)*D[y[x],x]-(2-40*x^2)*y[x]==0,{}},y[x],x,Inc
```

$$y(x) \to \exp\left(\int_1^x \frac{10K[1]^2+7}{8K[1]^3+4K[1]}dK[1] - \frac{1}{2}\int_1^x \frac{30K[2]^2+5}{4K[2]^3+2K[2]}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{10K[1]^2+7}{8K[1]^3+4K[1]}dK[1]\right)dK[3] + c_1\right)$$

2.1.527 Problem 543

Internal problem ID [9699]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 543
Date solved : `Monday, January 27, 2025 at 06:13:15 PM`
CAS classification : `[[_2nd_order, _exact, _linear, _homogeneous]]`

Solve

$$x(x^2+1)\,y''+(7x^2+4)\,y'+8xy=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.268 (sec)

Writing the ode as

$$(x^3+x)\,y''+(7x^2+4)\,y'+8xy=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=x^3+x\\ B&=7x^2+4\\ C&=8x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{3x^4+14x^2+8}{4\,(x^3+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=3x^4+14x^2+8\\ t&=4(x^3+x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{3x^4+14x^2+8}{4\,(x^3+x)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.999: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16 (x - i)^2} - \frac{3}{16 (x + i)^2} + \frac{7i}{16 (x - i)} - \frac{7i}{16 (x + i)} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x^4 + 14x^2 + 8}{4\left(x^3 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3x^4 + 14x^2 + 8}{4\left(x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1
i	2	0	$\frac{3}{4}$	$\frac{1}{4}$
$-i$	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + \frac{1}{4x-4i} + \frac{1}{4x+4i} + (-)(0) \\
&= -\frac{1}{x} + \frac{1}{4x-4i} + \frac{1}{4x+4i} \\
&= -\frac{1}{x} + \frac{x}{2x^2+2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x} + \frac{1}{4x-4i} + \frac{1}{4x+4i}\right)(0) + \left(\left(\frac{1}{x^2} - \frac{1}{4(x-i)^2} - \frac{1}{4(x+i)^2}\right) + \left(-\frac{1}{x} + \frac{1}{4x-4i} + \frac{1}{4x+4i}\right)^2\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(-\frac{1}{x} + \frac{1}{4x-4i} + \frac{1}{4x+4i}\right) dx} \\
&= \frac{(x^2+1)^{1/4}}{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7x^2+4}{x^3+x}\, dx} \\
&= z_1 e^{-2\ln(x) - \frac{3\ln(x^2+1)}{4}} \\
&= z_1 \left(\frac{1}{x^2 (x^2+1)^{3/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^3\sqrt{x^2+1}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x^2+4}{x^3+x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-4\ln(x)-\frac{3\ln(x^2+1)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{x^5}{\sqrt{x^2+1}} - \frac{x^3}{3\sqrt{x^2+1}} + \frac{4x^7}{\sqrt{x^2+1}} + \frac{8x^9}{3\sqrt{x^2+1}} + \frac{x\sqrt{x^2+1}}{2} - \frac{\operatorname{arcsinh}(x)}{2} \right. \\
&\qquad \left. + \frac{x^3\sqrt{x^2+1}}{3} - \frac{4x^5\sqrt{x^2+1}}{3} - \frac{8x^7\sqrt{x^2+1}}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x^3\sqrt{x^2+1}}\right) + c_2 \left(\frac{1}{x^3\sqrt{x^2+1}} \left(\frac{x^5}{\sqrt{x^2+1}} - \frac{x^3}{3\sqrt{x^2+1}} + \frac{4x^7}{\sqrt{x^2+1}} + \frac{8x^9}{3\sqrt{x^2+1}} \right.\right. \\
&\qquad \left.\left. + \frac{x\sqrt{x^2+1}}{2} - \frac{\operatorname{arcsinh}(x)}{2} + \frac{x^3\sqrt{x^2+1}}{3} - \frac{4x^5\sqrt{x^2+1}}{3} - \frac{8x^7\sqrt{x^2+1}}{3}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + (7x^2+4)\left(\frac{d}{dx}y(x)\right) + 8xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{8y(x)}{x^2+1} - \frac{(7x^2+4)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(7x^2+4)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)} + \frac{8y(x)}{x^2+1} = 0$

☐ Check to see if x_0 is a regular singular point

  ○ Define functions

    $\left[P_2(x) = \frac{7x^2+4}{x(x^2+1)}, P_3(x) = \frac{8}{x^2+1}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 4$

  ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  ○ $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

$x(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)+(7x^2+4)\left(\frac{d}{dx}y(x)\right)+8xy(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty} a_k x^{k+r}$

- Rewrite ODE with series expansions

  - Convert $x\cdot y(x)$ to series expansion

    $x\cdot y(x)=\sum_{k=0}^{\infty} a_k x^{k+r+1}$

  - Shift index using $k->k-1$

    $x\cdot y(x)=\sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..2$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=1..3$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,x^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(3+r)\,x^{-1+r}+a_1(1+r)(4+r)\,x^r+\left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+r+1)(k+r+4)+a_{k-1}(k+r+3)(k+\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(3+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{-3,0\}$

- Each term must be 0

  $a_1(1+r)(4+r)=0$

- Each term in the series must be 0, giving the recursion relation

  $(k+r+1)\left(a_{k+1}(k+r+4)+a_{k-1}(k+r+3)\right)=0$

- Shift index using $k->k+1$

  $(k+r+2)\left(a_{k+2}(k+5+r)+a_k(k+r+4)\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_k(k+r+4)}{k+5+r}$

- Recursion relation for $r=-3$

  $a_{k+2}=-\frac{a_k(k+1)}{k+2}$

- Solution for $r=-3$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2}=-\frac{a_k(k+1)}{k+2}, -2a_1=0\right]$

- Recursion relation for $r=0$

  $a_{k+2}=-\frac{a_k(k+4)}{k+5}$

- Solution for $r=0$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=-\frac{a_k(k+4)}{k+5}, 4a_1=0\right]$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-3}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = -\frac{a_k(k+1)}{k+2}, -2a_1 = 0, b_{k+2} = -\frac{b_k(4+k)}{5+k}, 4b_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.039 (sec)
Leaf size : 32

```
dsolve(x*(x^2+1)*diff(diff(y(x),x),x)+(7*x^2+4)*diff(y(x),x)+8*x*y(x) = 0,y(x),singsol
```

$$y = \frac{-\sqrt{x^2+1}\, c_2 x + c_2 \operatorname{arcsinh}(x) + c_1}{\sqrt{x^2+1}\, x^3}$$

Mathematica DSolve solution

Solving time : 0.206 (sec)
Leaf size : 108

```
DSolve[{x*(1+x^2)*D[y[x],{x,2}]+(4+7*x^2)*D[y[x],x]+8*x*y[x]==0,{}},y[x],x,IncludeSingularSo
```

$$y(x) \to \exp\left(\int_1^x -\frac{K[1]^2+2}{2\left(K[1]^3+K[1]\right)}dK[1] - \frac{1}{2}\int_1^x \frac{7K[2]^2+4}{K[2]^3+K[2]}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} -\frac{K[1]^2+2}{2\left(K[1]^3+K[1]\right)}dK[1]\right)dK[3] + c_1\right)$$

2.1.528 Problem 544

Internal problem ID [9700]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 544
Date solved : `Monday, January 27, 2025 at 06:13:15 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2\left(x^2+1\right)y''+x\left(8x^2+3\right)y'-\left(-4x^2+3\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.305 (sec)

Writing the ode as

$$\left(2x^4+2x^2\right)y''+\left(8x^3+3x\right)y'+\left(4x^2-3\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=2x^4+2x^2\\ B&=8x^3+3x\\ C&=4x^2-3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{36x^2+21}{16\left(x^3+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=36x^2+21\\ t&=16\left(x^3+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{36x^2+21}{16\left(x^3+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1001: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{21}{16x^2} - \frac{15}{64\,(x-i)^2} - \frac{15}{64\,(x+i)^2} + \frac{27i}{64\,(x-i)} - \frac{27i}{64\,(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{15}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{8} \end{aligned}$$

For the pole at $x=-i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{15}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b} = \frac{5}{8} \\
\alpha_c^- &= \frac{1}{2}-\sqrt{1+4b} = \frac{3}{8}
\end{aligned}
$$

Since the order of r at ∞ is $4>2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{36x^2+21}{16\left(x^3+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
i	2	0	$\frac{5}{8}$	$\frac{3}{8}$
$-i$	2	0	$\frac{5}{8}$	$\frac{3}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=0$ then

$$
\begin{aligned}
d &= \alpha_\infty^+-\left(\alpha_{c_1}^-+\alpha_{c_2}^-+\alpha_{c_3}^-\right) \\
&= 0-(0) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^-}{x-c_1}\right)+\left((-)[\sqrt{r}]_{c_2}+\frac{\alpha_{c_2}^-}{x-c_2}\right)+\left((-)[\sqrt{r}]_{c_3}+\frac{\alpha_{c_3}^-}{x-c_3}\right)+(+)[\sqrt{r}]_\infty \\
&= -\frac{3}{4x}+\frac{3}{8(x-i)}+\frac{3}{8(x+i)}+(0) \\
&= -\frac{3}{4x}+\frac{3}{8(x-i)}+\frac{3}{8(x+i)} \\
&= -\frac{3}{4x\left(x^2+1\right)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{4x} + \frac{3}{8(x-i)} + \frac{3}{8(x+i)}\right)(0) + \left(\left(\frac{3}{4x^2} - \frac{3}{8(x-i)^2} - \frac{3}{8(x+i)^2}\right) + \left(-\frac{3}{4x} + \frac{3}{8(x-i)} + \right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{3}{4x} + \frac{3}{8(x-i)} + \frac{3}{8(x+i)}\right) dx} \\
&= \frac{(x^2+1)^{3/8}}{x^{3/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{8x^3+3x}{2x^4+2x^2}\, dx} \\
&= z_1 e^{-\frac{3\ln(x)}{4} - \frac{5\ln(x^2+1)}{8}} \\
&= z_1 \left(\frac{1}{x^{3/4}(x^2+1)^{5/8}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^{3/2}(x^2+1)^{1/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{8x^3+3x}{2x^4+2x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{3\ln(x)}{2} - \frac{5\ln(x^2+1)}{4}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \mathrm{e}^{-\frac{3\ln(x)}{2} - \frac{5\ln(x^2+1)}{4}} x^3\sqrt{x^2+1}\, dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x^{3/2}(x^2+1)^{1/4}}\right) + c_2\left(\frac{1}{x^{3/2}(x^2+1)^{1/4}}\left(\int \mathrm{e}^{-\frac{3\ln(x)}{2} - \frac{5\ln(x^2+1)}{4}} x^3\sqrt{x^2+1}\, dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(8x^2+3)\left(\frac{d}{dx}y(x)\right)-(-4x^2+3)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(4x^2-3)y(x)}{2x^2(x^2+1)}-\frac{(8x^2+3)\left(\frac{d}{dx}y(x)\right)}{2x(x^2+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{(8x^2+3)\left(\frac{d}{dx}y(x)\right)}{2x(x^2+1)}+\frac{(4x^2-3)y(x)}{2x^2(x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{8x^2+3}{2x(x^2+1)},P_3(x)=\frac{4x^2-3}{2x^2(x^2+1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\bigg|_{x=0}=\frac{3}{2}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\bigg|_{x=0}=-\frac{3}{2}$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(8x^2+3)\left(\frac{d}{dx}y(x)\right)+(4x^2-3)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

    $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  ◦ Shift index using $k->k-m$

    $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..4$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  ◦ Shift index using $k->k+2-m$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$a_0(2r+3)(-1+r)x^r + a_1(5+2r)rx^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+3)(k+r-1) + 2a_{k-2}(k+r)(k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(2r+3)(-1+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{1, -\frac{3}{2}\}$
- Each term must be 0
  $a_1(5+2r)r = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $2\left(\left(k+r+\frac{3}{2}\right)a_k + a_{k-2}(k+r)\right)(k+r-1) = 0$
- Shift index using $k->k+2$
  $2\left(\left(k+\frac{7}{2}+r\right)a_{k+2} + a_k(k+r+2)\right)(k+r+1) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{2a_k(k+r+2)}{2k+7+2r}$
- Recursion relation for $r = 1$
  $a_{k+2} = -\frac{2a_k(k+3)}{2k+9}$
- Solution for $r = 1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{2a_k(k+3)}{2k+9}, a_1 = 0\right]$
- Recursion relation for $r = -\frac{3}{2}$
  $a_{k+2} = -\frac{2a_k\left(k+\frac{1}{2}\right)}{2k+4}$
- Solution for $r = -\frac{3}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}, a_{k+2} = -\frac{2a_k\left(k+\frac{1}{2}\right)}{2k+4}, a_1 = 0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{3}{2}}\right), a_{k+2} = -\frac{2a_k(k+3)}{2k+9}, a_1 = 0, b_{k+2} = -\frac{2b_k\left(k+\frac{1}{2}\right)}{2k+4}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
```

```
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returnin
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.112 (sec)
Leaf size : 31

```
dsolve(2*x^2*(x^2+1)*diff(diff(y(x),x),x)+x*(8*x^2+3)*diff(y(x),x)-(-4*x^2+3)*y(x) = 0,y
```

$$y = c_1 x \,\text{hypergeom}\left(\left[1, \frac{3}{2}\right], \left[\frac{9}{4}\right], -x^2\right) + \frac{c_2}{(x^2+1)^{1/4}\, x^{3/2}}$$

Mathematica DSolve solution

Solving time : 0.175 (sec)
Leaf size : 99

```
DSolve[{2*x^2*(1+x^2)*D[y[x],{x,2}]+x*(3+8*x^2)*D[y[x],x]-(3-4*x^2)*y[x]==0,{}},y[x],x,IncludeS
```

$$y(x) \to \exp\left(\int_1^x -\frac{3}{4\left(K[1]^3+K[1]\right)}dK[1] - \frac{1}{2}\int_1^x \frac{8K[2]^2+3}{2\left(K[2]^3+K[2]\right)}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} -\frac{3}{4\left(K[1]^3+K[1]\right)}dK[1]\right)dK[3] + c_1\right)$$

2.1.529 Problem 545

Internal problem ID [9701]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 545
Date solved : `Monday, January 27, 2025 at 06:13:16 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$9x^2y'' + 3x\left(x^2+3\right)y' - \left(-5x^2+1\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.280 (sec)

Writing the ode as

$$9x^2y'' + \left(3x^3+9x\right)y' + \left(5x^2-1\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^2 \\ B &= 3x^3+9x \\ C &= 5x^2-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4-8x^2-5}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4-8x^2-5 \\ t &= 36x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4-8x^2-5}{36x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1003: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 4\\ &= -2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{36} - \frac{2}{9} - \frac{5}{36x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{1} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{6} - \frac{2}{3x} - \frac{7}{4x^3} - \frac{7}{x^5} - \frac{595}{16x^7} - \frac{889}{4x^9} - \frac{45647}{32x^{11}} - \frac{76811}{8x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{6}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{36}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^4 - 8x^2 - 5}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{x^2}{36} - \frac{2}{9}\right) + \left(-\frac{5}{36x^2}\right) \\ &= \frac{x^2}{36} - \frac{2}{9} - \frac{5}{36x^2}\end{aligned}$$

We see that the coefficient of the term x in the quotient is $-\frac{2}{9}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{2}{9}\right) - (0) \\ &= -\frac{2}{9}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{x}{6} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{2}{9}}{\frac{1}{6}} - 1\right) &&= -\frac{7}{6} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{2}{9}}{\frac{1}{6}} - 1\right) &&= \frac{1}{6}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 8x^2 - 5}{36x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{6}$	$-\frac{7}{6}$	$\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{6}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{6} - \left(\frac{1}{6}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{6x} + (-)\left(\frac{x}{6}\right) \\ &= \frac{1}{6x} - \frac{x}{6} \\ &= \frac{1}{6x} - \frac{x}{6} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{6x} - \frac{x}{6}\right)(0) + \left(\left(-\frac{1}{6x^2} - \frac{1}{6}\right) + \left(\frac{1}{6x} - \frac{x}{6}\right)^2 - \left(\frac{x^4 - 8x^2 - 5}{36x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{1}{6x} - \frac{x}{6}\right) dx} \\ &= x^{1/6} e^{-\frac{x^2}{12}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3x^3+9x}{9x^2}\,dx} \\ &= z_1 e^{-\frac{x^2}{12}-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{e^{-\frac{x^2}{12}}}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{-\frac{x^2}{6}}}{x^{1/3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3x^3+9x}{9x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{6}-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1 \left(\int e^{-\frac{x^2}{6}-\ln(x)} x^{2/3} e^{\frac{x^2}{3}}\,dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{e^{-\frac{x^2}{6}}}{x^{1/3}}\right) + c_2 \left(\frac{e^{-\frac{x^2}{6}}}{x^{1/3}}\left(\int e^{-\frac{x^2}{6}-\ln(x)} x^{2/3} e^{\frac{x^2}{3}}\,dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$9x^2\left(\frac{d^2}{dx^2}y(x)\right) + 3x(x^2+3)\left(\frac{d}{dx}y(x)\right) - (-5x^2+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(5x^2-1)y(x)}{9x^2} - \frac{(x^2+3)\left(\frac{d}{dx}y(x)\right)}{3x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(x^2+3)\left(\frac{d}{dx}y(x)\right)}{3x} + \frac{(5x^2-1)y(x)}{9x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = \frac{x^2+3}{3x}, P_3(x) = \frac{5x^2-1}{9x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{1}{9}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $9x^2\left(\frac{d^2}{dx^2}y(x)\right) + 3x(x^2+3)\left(\frac{d}{dx}y(x)\right) + (5x^2-1)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+3r)\,(-1+3r)\,x^r + a_1(4+3r)\,(2+3r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(3k+3r+1)\,(3k+3r-1) + a_{k-2}(3k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+3r)\,(-1+3r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{3}, \frac{1}{3}\right\}$

- Each term must be 0

  $a_1(4+3r)\,(2+3r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(3k+3r-1)\,(3a_k k + 3a_k r + a_k + a_{k-2}) = 0$

- Shift index using $k- >k + 2$

  $(3k+3r+5)\,(3a_{k+2}(k+2) + 3a_{k+2}r + a_{k+2} + a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{3k+7+3r}$

- Recursion relation for $r=-\frac{1}{3}$

  $a_{k+2}=-\frac{a_k}{3k+6}$
- Solution for $r=-\frac{1}{3}$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}, a_{k+2}=-\frac{a_k}{3k+6}, a_1=0\right]$
- Recursion relation for $r=\frac{1}{3}$

  $a_{k+2}=-\frac{a_k}{3k+8}$
- Solution for $r=\frac{1}{3}$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2}=-\frac{a_k}{3k+8}, a_1=0\right]$
- Combine solutions and rename parameters

  $\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2}=-\frac{a_k}{3k+6}, a_1=0, b_{k+2}=-\frac{b_k}{3k+8}, b_1=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Whittaker successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - return
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.044 (sec)
Leaf size : 37

```
dsolve(9*x^2*diff(diff(y(x),x),x)+3*x*(x^2+3)*diff(y(x),x)-(-5*x^2+1)*y(x) = 0,y(x),si
```

$$y=\frac{\mathrm{e}^{-\frac{x^2}{12}}\left(\mathrm{e}^{-\frac{x^2}{12}} c_2 x+\mathrm{WhittakerM}\left(\frac{1}{3}, \frac{1}{6}, \frac{x^2}{6}\right) x^{1/3} c_1\right)}{x^{4/3}}$$

Mathematica DSolve solution

Solving time : 0.115 (sec)
Leaf size : 70

```
DSolve[{9*x^2*D[y[x],{x,2}]+3*x*(3+x^2)*D[y[x],x]-(1-5*x^2)*y[x]==0,{}},y[x],x,IncludeSingular
```

$$y(x) \to \frac{e^{-\frac{x^2}{6}-\frac{1}{6}}\left(2\sqrt[3]{e}c_1x^{4/3}+\sqrt[3]{6}c_2(-x^2)^{2/3}\,\Gamma\left(\frac{1}{3},-\frac{x^2}{6}\right)\right)}{2x^{5/3}}$$

2.1.530 Problem 546

Internal problem ID [9702]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 546
Date solved : `Monday, January 27, 2025 at 06:13:17 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$6x^2y'' + x\left(6x^2+1\right)y' + \left(9x^2+1\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.339 (sec)

Writing the ode as

$$6x^2y'' + \left(6x^3+x\right)y' + \left(9x^2+1\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 6x^2 \\ B &= 6x^3+x \\ C &= 9x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{36x^4 - 132x^2 - 35}{144x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 36x^4 - 132x^2 - 35 \\ t &= 144x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{36x^4 - 132x^2 - 35}{144x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1005: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 144x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{4} - \frac{11}{12} - \frac{35}{144x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{11}{12x} - \frac{13}{12x^3} - \frac{143}{72x^5} - \frac{130}{27x^7} - \frac{17017}{1296x^9} - \frac{597961}{15552x^{11}} - \frac{11016863}{93312x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{36x^4 - 132x^2 - 35}{144x^2} \\ &= Q + \frac{R}{144x^2} \\ &= \left(\frac{x^2}{4} - \frac{11}{12}\right) + \left(-\frac{35}{144x^2}\right) \\ &= \frac{x^2}{4} - \frac{11}{12} - \frac{35}{144x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is $-\frac{11}{12}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{11}{12}\right) - (0) \\ &= -\frac{11}{12} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{11}{12}}{\frac{1}{2}} - 1\right) &&= -\frac{17}{12} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{11}{12}}{\frac{1}{2}} - 1\right) &&= \frac{5}{12} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{36x^4 - 132x^2 - 35}{144x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{12}$	$\frac{5}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	$-\frac{17}{12}$	$\frac{5}{12}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{5}{12}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{5}{12} - \left(\frac{5}{12}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{5}{12x} + (-)\left(\frac{x}{2}\right) \\ &= \frac{5}{12x} - \frac{x}{2} \\ &= \frac{5}{12x} - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{5}{12x} - \frac{x}{2}\right)(0) + \left(\left(-\frac{5}{12x^2} - \frac{1}{2}\right) + \left(\frac{5}{12x} - \frac{x}{2}\right)^2 - \left(\frac{36x^4 - 132x^2 - 35}{144x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{5}{12x} - \frac{x}{2}\right) dx} \\ &= x^{5/12} e^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{6x^3+x}{6x^2}\,dx} \\
&= z_1 e^{-\frac{x^2}{4}-\frac{\ln(x)}{12}} \\
&= z_1 \left(\frac{e^{-\frac{x^2}{4}}}{x^{1/12}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^{1/3} e^{-\frac{x^2}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{6x^3+x}{6x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}-\frac{\ln(x)}{6}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{e^{-\frac{x^2}{2}-\frac{\ln(x)}{6}} e^{x^2}}{x^{2/3}}\,dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{1/3} e^{-\frac{x^2}{2}}\right) + c_2 \left(x^{1/3} e^{-\frac{x^2}{2}} \left(\int \frac{e^{-\frac{x^2}{2}-\frac{\ln(x)}{6}} e^{x^2}}{x^{2/3}}\,dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$6x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(6x^2+1)\left(\frac{d}{dx}y(x)\right) + (9x^2+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(9x^2+1)y(x)}{6x^2} - \frac{(6x^2+1)\left(\frac{d}{dx}y(x)\right)}{6x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(6x^2+1)\left(\frac{d}{dx}y(x)\right)}{6x} + \frac{(9x^2+1)y(x)}{6x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

$\left[P_2(x) = \frac{6x^2+1}{6x}, P_3(x) = \frac{9x^2+1}{6x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = \frac{1}{6}$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{1}{6}$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $6x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(6x^2+1)\left(\frac{d}{dx}y(x)\right) + (9x^2+1)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions

  - ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - ◦ Shift index using $k- >k - m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - ◦ Shift index using $k- >k + 1 - m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - ◦ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+3r)\,(-1+2r)\,x^r + a_1(2+3r)\,(1+2r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(3k+3r-1)\,(2k+2r-1) + 3a_{k-2}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+3r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2}, \frac{1}{3}\right\}$

- Each term must be 0

  $a_1(2+3r)\,(1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $6\left(\left(k - \frac{1}{3} + r\right)a_k + a_{k-2}\right)\left(k + r - \frac{1}{2}\right) = 0$

- Shift index using $k- >k + 2$

  $6\left(\left(k + \frac{5}{3} + r\right)a_{k+2} + a_k\right)\left(k + \frac{3}{2} + r\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{3a_k}{3k+5+3r}$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{3a_k}{3k+\frac{13}{2}}$
- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{3a_k}{3k+\frac{13}{2}}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{3}$

  $a_{k+2} = -\frac{3a_k}{3k+6}$
- Solution for $r = \frac{1}{3}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -\frac{3a_k}{3k+6}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = -\frac{3a_k}{3k+\frac{13}{2}}, a_1 = 0, b_{k+2} = -\frac{3b_k}{3k+6}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Whittaker successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - return
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.087 (sec)
Leaf size : 36

```
dsolve(6*x^2*diff(diff(y(x),x),x)+x*(6*x^2+1)*diff(y(x),x)+(9*x^2+1)*y(x) = 0,y(x),si
```

$$y = \frac{e^{-\frac{x^2}{4}}\left(e^{-\frac{x^2}{4}} x^{11/12} c_2 + \text{WhittakerM}\left(\frac{11}{24}, \frac{1}{24}, \frac{x^2}{2}\right) c_1\right)}{x^{7/12}}$$

Mathematica DSolve solution

Solving time : 0.084 (sec)
Leaf size : 61

```
DSolve[{6*x^2*D[y[x],{x,2}]+x*(1+6*x^2)*D[y[x],x]+(1+9*x^2)*y[x]==0,{}},y[x],x,IncludeSingularS
```

$$y(x) \to \frac{e^{-\frac{x^2}{2}} \left(2c_1 x^{11/6} + \sqrt[12]{2} c_2 (-x^2)^{11/12} \, \Gamma\left(\frac{1}{12}, -\frac{x^2}{2}\right)\right)}{2x^{3/2}}$$

2.1.531 Problem 547

Internal problem ID [9703]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 547
Date solved : `Monday, January 27, 2025 at 06:13:18 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$9x^2\left(x^2+1\right)y''+3x\left(13x^2+3\right)y'-\left(-25x^2+1\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.301 (sec)

Writing the ode as

$$\left(9x^4+9x^2\right)y''+\left(39x^3+9x\right)y'+\left(25x^2-1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^4+9x^2 \\ B &= 39x^3+9x \\ C &= 25x^2-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-9x^4+6x^2-5}{36\left(x^3+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9x^4+6x^2-5 \\ t &= 36\left(x^3+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-9x^4+6x^2-5}{36\left(x^3+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1007: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{36\,(x-i)^2} - \frac{5}{36\,(x+i)^2} - \frac{i}{12\,(x-i)} + \frac{i}{12x+12i} - \frac{5}{36x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-9x^4 + 6x^2 - 5}{36\left(x^3 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-9x^4 + 6x^2 - 5}{36\left(x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{6}$	$\frac{1}{6}$
i	2	0	$\frac{5}{6}$	$\frac{1}{6}$
$-i$	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{6x} + \frac{1}{6x-6i} + \frac{1}{6x+6i} + (-)(0) \\
&= \frac{1}{6x} + \frac{1}{6x-6i} + \frac{1}{6x+6i} \\
&= \frac{1}{6x} + \frac{x}{3x^2+3}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{6x} + \frac{1}{6x-6i} + \frac{1}{6x+6i}\right)(0) + \left(\left(-\frac{1}{6x^2} - \frac{1}{6(x-i)^2} - \frac{1}{6(x+i)^2}\right) + \left(\frac{1}{6x} + \frac{1}{6x-6i} + \frac{1}{6x+6i}\right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(\frac{1}{6x} + \frac{1}{6x-6i} + \frac{1}{6x+6i}\right) dx} \\
&= (x^2+1)^{1/6} (-x)^{1/6}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{39x^3+9x}{9x^4+9x^2}\, dx} \\
&= z_1 e^{-\frac{\ln(x)}{2} - \frac{5\ln(x^2+1)}{6}} \\
&= z_1 \left(\frac{1}{\sqrt{x}\,(x^2+1)^{5/6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(-x)^{1/6}}{\sqrt{x}\,(x^2+1)^{2/3}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{39x^3+9x}{9x^4+9x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\ln(x) - \frac{5\ln(x^2+1)}{3}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{e^{-\ln(x) - \frac{5\ln(x^2+1)}{3}}\, x(x^2+1)^{4/3}}{(-x)^{1/3}}\, dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(-x)^{1/6}}{\sqrt{x}\,(x^2+1)^{2/3}} \right) + c_2 \left(\frac{(-x)^{1/6}}{\sqrt{x}\,(x^2+1)^{2/3}} \left(\int \frac{\mathrm{e}^{-\ln(x)-\frac{5\ln(x^2+1)}{3}} x(x^2+1)^{4/3}}{(-x)^{1/3}} dx \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$9x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + 3x(13x^2+3)\left(\frac{d}{dx}y(x)\right) - (-25x^2+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(25x^2-1)y(x)}{9x^2(x^2+1)} - \frac{(13x^2+3)\left(\frac{d}{dx}y(x)\right)}{3x(x^2+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(13x^2+3)\left(\frac{d}{dx}y(x)\right)}{3x(x^2+1)} + \frac{(25x^2-1)y(x)}{9x^2(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{13x^2+3}{3x(x^2+1)}, P_3(x) = \frac{25x^2-1}{9x^2(x^2+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{1}{9}$

  ◦ $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $9x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + 3x(13x^2+3)\left(\frac{d}{dx}y(x)\right) + (25x^2-1)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k-m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$
- Shift index using $k- >k+2-m$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+3r)\,(-1+3r)\, x^r + a_1(4+3r)\,(2+3r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(3k+3r+1)\,(3k+3r-1) + a_{k-2}(3k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+3r)\,(-1+3r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{3}, \frac{1}{3}\right\}$
- Each term must be 0

  $a_1(4+3r)\,(2+3r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $9\left(\left(k-\frac{1}{3}+r\right)a_{k-2} + \left(k+r+\frac{1}{3}\right)a_k\right)\left(k-\frac{1}{3}+r\right) = 0$
- Shift index using $k- >k+2$

  $9\left(\left(k+\frac{5}{3}+r\right)a_k + \left(k+\frac{7}{3}+r\right)a_{k+2}\right)\left(k+\frac{5}{3}+r\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{(3k+3r+5)a_k}{3k+7+3r}$
- Recursion relation for $r = -\frac{1}{3}$

  $a_{k+2} = -\frac{(3k+4)a_k}{3k+6}$
- Solution for $r = -\frac{1}{3}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}, a_{k+2} = -\frac{(3k+4)a_k}{3k+6}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{3}$

  $a_{k+2} = -\frac{(3k+6)a_k}{3k+8}$
- Solution for $r = \frac{1}{3}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -\frac{(3k+6)a_k}{3k+8}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = -\frac{(3k+4)a_k}{3k+6}, a_1 = 0, b_{k+2} = -\frac{(3k+6)b_k}{3k+8}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - return
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.160 (sec)
Leaf size : 33

```
dsolve(9*x^2*(x^2+1)*diff(diff(y(x),x),x)+3*x*(13*x^2+3)*diff(y(x),x)-(-25*x^2+1)*y(x)
```

$$y = \frac{c_1}{\left(x^2+1\right)^{2/3} x^{1/3}} + c_2 x^{1/3} \operatorname{hypergeom}\left(\left[1,1\right], \left[\frac{4}{3}\right], -x^2\right)$$

Mathematica DSolve solution

Solving time : 0.235 (sec)
Leaf size : 113

```
DSolve[{9*x^2*(1+x^2)*D[y[x],{x,2}]+3*x*(3+13*x^2)*D[y[x],x]-(1-25*x^2)*y[x]==0,{}},y[x],x,I
```

$$y(x) \to \exp\left(\int_1^x \frac{3K[1]^2+1}{6\left(K[1]^3+K[1]\right)} dK[1] - \frac{1}{2}\int_1^x \left(\frac{10K[2]}{3\left(K[2]^2+1\right)} + \frac{1}{K[2]}\right) dK[2]\right)\left(c_2 \int_1^x \exp\left(-2\int_1^{K[3]} \frac{3K[1]^2+1}{6\left(K[1]^3+K[1]\right)} dK[1]\right) dK[3] + c_1\right)$$

2.1.532 Problem 548

Internal problem ID [9704]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 548
Date solved : Monday, January 27, 2025 at 06:13:18 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$4x^2\left(x^2+1\right)y''+4x\left(6x^2+1\right)y'-\left(-25x^2+1\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.270 (sec)

Writing the ode as

$$\left(4x^4+4x^2\right)y''+\left(24x^3+4x\right)y'+\left(25x^2-1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^4+4x^2 \\ B &= 24x^3+4x \\ C &= 25x^2-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-x^2-6}{4\left(x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s=-x^2-6$$

$$t=4\left(x^2+1\right)^2$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-x^2-6}{4\left(x^2+1\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1009: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16\,(x-i)^2} + \frac{5}{16\,(x+i)^2} + \frac{7i}{16\,(x-i)} - \frac{7i}{16\,(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 6}{4\left(x^2+1\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 - 6}{4\left(x^2+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-i$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{4\left(x-i\right)} - \frac{1}{4\left(x+i\right)} + (-)\left(0\right) \\
&= -\frac{1}{4\left(x-i\right)} - \frac{1}{4\left(x+i\right)} \\
&= -\frac{x}{2x^2+2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{4(x-i)} - \frac{1}{4(x+i)}\right)(1) + \left(\left(\frac{1}{4(x-i)^2} + \frac{1}{4(x+i)^2}\right) + \left(-\frac{1}{4(x-i)} - \frac{1}{4(x+i)}\right)^2 - \left(\frac{-}{4(\ldots}\right.\right. \quad \frac{(x^2+1\ldots}{(-x+i)^2(\ldots}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, e^{\int \left(-\frac{1}{4(x-i)} - \frac{1}{4(x+i)}\right) dx} \\
&= (x) \frac{1}{((-x+i)(x+i))^{1/4}} \\
&= \frac{x}{(-x^2-1)^{1/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{24x^3+4x}{4x^4+4x^2} \, dx} \\
&= z_1 e^{-\frac{\ln(x)}{2} - \frac{5\ln(x^2+1)}{4}} \\
&= z_1 \left(\frac{1}{\sqrt{x}\,(x^2+1)^{5/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(\frac{1}{2} - \frac{i}{2}\right)\sqrt{x}\,\sqrt{2}}{(x^2+1)^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{24x^3+4x}{4x^4+4x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\ln(x) - \frac{5\ln(x^2+1)}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(i\left(-\frac{(x^2+1)^{3/2}}{x} + x\sqrt{x^2+1} + \operatorname{arcsinh}(x)\right)\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\left(\frac{1}{2}-\frac{i}{2}\right)\sqrt{x}\,\sqrt{2}}{\left(x^2+1\right)^{3/2}}\right) \\
&\quad + c_2 \left(\frac{\left(\frac{1}{2}-\frac{i}{2}\right)\sqrt{x}\,\sqrt{2}}{\left(x^2+1\right)^{3/2}}\left(i\left(-\frac{\left(x^2+1\right)^{3/2}}{x}+x\sqrt{x^2+1}+\operatorname{arcsinh}(x)\right)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)+4x(6x^2+1)\left(\frac{d}{dx}y(x)\right)-(-25x^2+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(25x^2-1)y(x)}{4x^2(x^2+1)}-\frac{(6x^2+1)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{(6x^2+1)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}+\frac{(25x^2-1)y(x)}{4x^2(x^2+1)}=0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=\frac{6x^2+1}{x(x^2+1)},P_3(x)=\frac{25x^2-1}{4x^2(x^2+1)}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=1$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=-\frac{1}{4}$

- ○ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)+4x(6x^2+1)\left(\frac{d}{dx}y(x)\right)+(25x^2-1)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

- ○ Shift index using $k->k-m$

  $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ○ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-1) + a_{k-2}\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $4\left(\left(k+r+\frac{1}{2}\right)a_{k-2} + \left(k+r-\frac{1}{2}\right)a_k\right)\left(k+r+\frac{1}{2}\right) = 0$

- Shift index using $k->k+2$

  $4\left(\left(k+\frac{5}{2}+r\right)a_k + \left(k+\frac{3}{2}+r\right)a_{k+2}\right)\left(k+\frac{5}{2}+r\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{(2k+2r+5)a_k}{2k+3+2r}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{(2k+4)a_k}{2k+2}$

- Solution for $r = -\frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{(2k+4)a_k}{2k+2}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{(2k+6)a_k}{2k+4}$

- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{(2k+6)a_k}{2k+4}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{(2k+4)a_k}{2k+2}, a_1 = 0, b_{k+2} = -\frac{b_k(2k+6)}{2k+4}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.047 (sec)
Leaf size : 34

```
dsolve(4*x^2*(x^2+1)*diff(diff(y(x),x),x)+4*x*(6*x^2+1)*diff(y(x),x)-(-25*x^2+1)*y(x) =
```

$$y = \frac{-\sqrt{x^2+1}\, c_2 + x(c_2 \operatorname{arcsinh}(x) + c_1)}{\sqrt{x}\, (x^2+1)^{3/2}}$$

Mathematica DSolve solution

Solving time : 0.392 (sec)
Leaf size : 70

```
DSolve[{4*x^2*(1+x^2)*D[y[x],{x,2}]+4*x*(1+6*x^2)*D[y[x],x]-(1-25*x^2)*y[x]==0,{}},y[x],x,Inclu
```

$$y(x) \to \frac{\left(c_2 x \operatorname{arcsinh}(x) - c_2\sqrt{x^2+1} + c_1 x\right) \exp\left(-\frac{1}{2}\int_1^x \frac{6K[1]^2+1}{K[1]^3+K[1]} dK[1]\right)}{\sqrt[4]{x^2+1}}$$

2.1.533 Problem 549

Internal problem ID [9705]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 549
Date solved : Monday, January 27, 2025 at 06:13:19 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$8x^2(2x^2+1)\,y''+2x(34x^2+5)\,y'-(-30x^2+1)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.462 (sec)

Writing the ode as

$$(16x^4+8x^2)\,y''+(68x^3+10x)\,y'+(30x^2-1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 16x^4+8x^2\\ B &= 68x^3+10x\\ C &= 30x^2-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t}\\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{132x^4+148x^2-7}{64\,(2x^3+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 132x^4+148x^2-7\\ t &= 64(2x^3+x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{132x^4+148x^2-7}{64\,(2x^3+x)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1011: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64(2x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{i\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{i\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{64x^2} - \frac{3}{16\left(x - \frac{i\sqrt{2}}{2}\right)^2} - \frac{3}{16\left(x + \frac{i\sqrt{2}}{2}\right)^2} - \frac{i\sqrt{2}}{2\left(x - \frac{i\sqrt{2}}{2}\right)} + \frac{i\sqrt{2}}{2x + i\sqrt{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8} \end{aligned}$$

For the pole at $x = \frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4}\\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

For the pole at $x = -\frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4}\\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{132x^4 + 148x^2 - 7}{64\left(2x^3 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{33}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{8}\\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{8}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{132x^4 + 148x^2 - 7}{64\left(2x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{8}$	$\frac{1}{8}$
$\frac{i\sqrt{2}}{2}$	2	0	$\frac{3}{4}$	$\frac{1}{4}$
$-\frac{i\sqrt{2}}{2}$	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{11}{8}$	$-\frac{3}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{11}{8}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right)\\
&= \frac{11}{8} - \left(\frac{11}{8}\right)\\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{7}{8x} + \frac{1}{4x-2i\sqrt{2}} + \frac{1}{4x+2i\sqrt{2}} + (0) \\
&= \frac{7}{8x} + \frac{1}{4x-2i\sqrt{2}} + \frac{1}{4x+2i\sqrt{2}} \\
&= \frac{22x^2+7}{16x^3+8x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{7}{8x} + \frac{1}{4x-2i\sqrt{2}} + \frac{1}{4x+2i\sqrt{2}}\right)(0) + \left(\left(-\frac{7}{8x^2} - \frac{1}{4\left(x-\frac{i\sqrt{2}}{2}\right)^2} - \frac{1}{4\left(x+\frac{i\sqrt{2}}{2}\right)^2}\right) + \left(\frac{7}{8x} + \frac{}{4x -}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(\frac{7}{8x} + \frac{1}{4x-2i\sqrt{2}} + \frac{1}{4x+2i\sqrt{2}}\right) dx} \\
&= x^{7/8} 2^{1/4} (2x^2+1)^{1/4}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{68x^3+10x}{16x^4+8x^2}\, dx} \\
&= z_1 e^{-\frac{5\ln(x)}{8} - \frac{3\ln(2x^2+1)}{4}} \\
&= z_1 \left(\frac{1}{x^{5/8}(2x^2+1)^{3/4}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/4} 2^{1/4}}{\sqrt{2x^2+1}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{68x^3+10x}{16x^4+8x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{5\ln(x)}{4}-\frac{3\ln(2x^2+1)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{e^{-\frac{5\ln(x)}{4}-\frac{3\ln(2x^2+1)}{2}}(2x^2+1)\sqrt{2}}{2\sqrt{x}} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^{1/4}2^{1/4}}{\sqrt{2x^2+1}}\right) + c_2\left(\frac{x^{1/4}2^{1/4}}{\sqrt{2x^2+1}}\left(\int \frac{e^{-\frac{5\ln(x)}{4}-\frac{3\ln(2x^2+1)}{2}}(2x^2+1)\sqrt{2}}{2\sqrt{x}} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$8x^2(2x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + 2x(34x^2+5)\left(\frac{d}{dx}y(x)\right) - (-30x^2+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(30x^2-1)y(x)}{8x^2(2x^2+1)} - \frac{(34x^2+5)\left(\frac{d}{dx}y(x)\right)}{4x(2x^2+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(34x^2+5)\left(\frac{d}{dx}y(x)\right)}{4x(2x^2+1)} + \frac{(30x^2-1)y(x)}{8x^2(2x^2+1)} = 0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{34x^2+5}{4x(2x^2+1)}, P_3(x) = \frac{30x^2-1}{8x^2(2x^2+1)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{5}{4}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\frac{1}{8}$

  - $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $8x^2(2x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + 2x(34x^2+5)\left(\frac{d}{dx}y(x)\right) + (30x^2-1)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- > k-m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$
  - Shift index using $k- > k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$
  - Shift index using $k- > k+2-m$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+4r)\, x^r + a_1(3+2r)\,(3+4r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+1)\,(4k+4r-1) + 2a_{k-2}(2\right.\right.$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+4r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{4}\right\}$
- Each term must be 0

  $a_1(3+2r)\,(3+4r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $8\left(k+r+\frac{1}{2}\right)\left(\left(2k+2r-\frac{5}{2}\right) a_{k-2} + a_k\left(k+r-\frac{1}{4}\right)\right) = 0$
- Shift index using $k- > k+2$

  $8\left(k+\frac{5}{2}+r\right)\left(\left(2k+\frac{3}{2}+2r\right) a_k + a_{k+2}\left(k+\frac{7}{4}+r\right)\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2(4k+4r+3)a_k}{4k+7+4r}$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{2(4k+1)a_k}{4k+5}$
- Solution for $r = -\frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{2(4k+1)a_k}{4k+5}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{4}$

  $a_{k+2} = -\frac{2(4k+4)a_k}{4k+8}$
- Solution for $r = \frac{1}{4}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+2} = -\frac{2(4k+4)a_k}{4k+8}, a_1 = 0\right]$
- Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{4}}\right), a_{k+2}=-\frac{2(4k+1)a_k}{4k+5}, a_1=0, b_{k+2}=-\frac{2(4k+4)b_k}{4k+8}, b_1=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special functio
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.113 (sec)
Leaf size : 46

```
dsolve(8*x^2*(2*x^2+1)*diff(diff(y(x),x),x)+2*x*(34*x^2+5)*diff(y(x),x)-(-30*x^2+1)*y(
```

$$y=\frac{c_2 \operatorname{LegendreQ}\left(\frac{3}{8}, \frac{3}{8}, \sqrt{2x^2+1}\right)+c_1 \operatorname{LegendreP}\left(\frac{3}{8}, \frac{3}{8}, \sqrt{2x^2+1}\right)}{\sqrt{2x^2+1}\, x^{1/8}}$$

Mathematica DSolve solution

Solving time : `0.342` (sec)
Leaf size : 118

```
DSolve[{8*x^2*(1+2*x^2)*D[y[x],{x,2}]+2*x*(5+34*x^2)*D[y[x],x]-(1-30*x^2)*y[x]==0,{}},y[x],x,In
```

$$y(x) \to \exp\left(\int_1^x \left(\frac{K[1]}{2K[1]^2+1}+\frac{7}{8K[1]}\right)dK[1] - \frac{1}{2}\int_1^x \frac{34K[2]^2+5}{8K[2]^3+4K[2]}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]}\left(\frac{K[1]}{2K[1]^2+1}+\frac{7}{8K[1]}\right)dK[1]\right)dK[3] + c_1\right)$$

2.1.534 Problem 550

Internal problem ID [9706]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 550
Date solved : `Monday, January 27, 2025 at 06:13:20 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(1+x)\,y''-x(1-3x)\,y'+y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.158 (sec)

Writing the ode as

$$\left(2x^3+2x^2\right)y''+\left(3x^2-x\right)y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x^3+2x^2\\B&=3x^2-x\\C&=1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-3}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=-3\\t&=16x^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{3}{16x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1013: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{3}{16x^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{4}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=-\frac{3}{16x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}\right)\\
&= \frac{1}{4}-\left(\frac{1}{4}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^{-}}{x-c_1}\right)+(-)[\sqrt{r}]_\infty\\
&= \frac{1}{4x}+(-)\,(0)\\
&= \frac{1}{4x}\\
&= \frac{1}{4x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}
$$

Let

$$
p(x)=1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{1}{4x}\right)(0)+\left(\left(-\frac{1}{4x^2}\right)+\left(\frac{1}{4x}\right)^2-\left(-\frac{3}{16x^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\,dx} \\ &= \mathrm{e}^{\int \frac{1}{4x}dx} \\ &= x^{1/4} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3x^2-x}{2x^3+2x^2}\,dx} \\ &= z_1 e^{-\ln(1+x)+\frac{\ln(x)}{4}} \\ &= z_1\left(\frac{x^{1/4}}{1+x}\right) \end{aligned}$$

Which simplifies to

$$y_1=\frac{\sqrt{x}}{1+x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1\int \frac{e^{\int -\frac{3x^2-x}{2x^3+2x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1\int \frac{e^{-2\ln(1+x)+\frac{\ln(x)}{2}}}{(y_1)^2}\,dx \\ &= y_1\left(2\,\mathrm{e}^{-2\ln(1+x)+\frac{\ln(x)}{2}}(1+x)^2\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1y_1+c_2y_2 \\ &= c_1\left(\frac{\sqrt{x}}{1+x}\right)+c_2\left(\frac{\sqrt{x}}{1+x}\left(2\,\mathrm{e}^{-2\ln(1+x)+\frac{\ln(x)}{2}}(1+x)^2\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)-x(1-3x)\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{y(x)}{2(x+1)x^2}-\frac{(3x-1)\left(\frac{d}{dx}y(x)\right)}{2x(x+1)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{(3x-1)\left(\frac{d}{dx}y(x)\right)}{2x(x+1)}+\frac{y(x)}{2(x+1)x^2}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=\frac{3x-1}{2x(x+1)},P_3(x)=\frac{1}{2(x+1)x^2}\right]$
- ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=2$
- ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$
- ◦ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1$

- Multiply by denominators

  $2x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(3x-1)\left(\frac{d}{dx}y(x)\right)+y(x)=0$
- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(2u^3-4u^2+2u)\left(\frac{d^2}{du^2}y(u)\right)+(3u^2-7u+4)\left(\frac{d}{du}y(u)\right)+y(u)=0$
- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$
- ◦ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$
- ◦ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$
- ◦ Shift index using $k->k+2-m$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

Rewrite ODE with series expansions

$2a_0r(1+r)\,u^{-1+r}+(2a_1(1+r)(2+r)-a_0(1+r)(-1+4r))\,u^r+\left(\sum_{k=1}^{\infty}(2a_{k+1}(k+r+1)(k+\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$2r(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $2a_1(1+r)(2+r) - a_0(1+r)(-1+4r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $(-4a_k + 2a_{k-1} + 2a_{k+1})k^2 + ((-8a_k + 4a_{k-1} + 4a_{k+1})r - 3a_k - 3a_{k-1} + 6a_{k+1})k + (-4a_k + 2a_{k-}$

- Shift index using $k- >k+1$

  $(-4a_{k+1} + 2a_k + 2a_{k+2})(k+1)^2 + ((-8a_{k+1} + 4a_k + 4a_{k+2})r - 3a_{k+1} - 3a_k + 6a_{k+2})(k+1) + (-$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} + 4kra_k - 8kra_{k+1} + 2r^2a_k - 4r^2a_{k+1} + ka_k - 11ka_{k+1} + ra_k - 11ra_{k+1} - 6a_{k+1}}{2(k^2+2kr+r^2+5k+5r+6)}$

- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - 3ka_k - 3ka_{k+1} + a_k + a_{k+1}}{2(k^2+3k+2)}$

- Solution for $r = -1$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-1}, a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - 3ka_k - 3ka_{k+1} + a_k + a_{k+1}}{2(k^2+3k+2)}, 0 = 0\right]$

- Revert the change of variables $u = x + 1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x+1)^{k-1}, a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - 3ka_k - 3ka_{k+1} + a_k + a_{k+1}}{2(k^2+3k+2)}, 0 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} + ka_k - 11ka_{k+1} - 6a_{k+1}}{2(k^2+5k+6)}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k}, a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} + ka_k - 11ka_{k+1} - 6a_{k+1}}{2(k^2+5k+6)}, 4a_1 + a_0 = 0\right]$

- Revert the change of variables $u = x + 1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x+1)^{k}, a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} + ka_k - 11ka_{k+1} - 6a_{k+1}}{2(k^2+5k+6)}, 4a_1 + a_0 = 0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k (x+1)^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k}\right), a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - 3ka_k - 3ka_{k+1} + a_k + a_{k+1}}{2(k^2+3k+2)}, 0 = 0,\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.034 (sec)
Leaf size : 19

```
dsolve(2*x^2*(x+1)*diff(diff(y(x),x),x)-x*(-3*x+1)*diff(y(x),x)+y(x) = 0,y(x),singsol=
```

$$y = \frac{c_2\sqrt{x} + c_1 x}{x+1}$$

Mathematica DSolve solution

Solving time : 0.244 (sec)
Leaf size : 53

```
DSolve[{2*x^2*(1+x)*D[y[x],{x,2}]-x*(1-3*x)*D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolu
```

$$y(x) \to \sqrt[4]{x}\left(2c_2\sqrt{x} + c_1\right) \exp\left(-\frac{1}{2}\int_1^x \left(\frac{2}{K[1]+1} - \frac{1}{2K[1]}\right) dK[1]\right)$$

2.1.535 Problem 551

Internal problem ID [9707]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 551
Date solved : `Monday, January 27, 2025 at 06:13:20 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$6x^2(2x^2+1)\,y''+x(50x^2+1)\,y'+(30x^2+1)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.164 (sec)

Writing the ode as

$$(12x^4+6x^2)\,y''+(50x^3+x)\,y'+(30x^2+1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 12x^4+6x^2 \\ B &= 50x^3+x \\ C &= 30x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-35}{144x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -35 \\ t &= 144x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{35}{144x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1015: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 144x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{35}{144x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{5}{12} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{35}{144x^2}$$

Since the gcd$(s,t) = 1$. This gives $b = -\frac{35}{144}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{35}{144x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{12}$	$\frac{5}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{12}$	$\frac{5}{12}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{5}{12}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{5}{12} - \left(\frac{5}{12}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{5}{12x} + (-)(0) \\
&= \frac{5}{12x} \\
&= \frac{5}{12x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{gathered}
(0)+2\left(\frac{5}{12x}\right)(0)+\left(\left(-\frac{5}{12x^2}\right)+\left(\frac{5}{12x}\right)^2-\left(-\frac{35}{144x^2}\right)\right)=0 \\
0=0
\end{gathered}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \frac{5}{12x} dx} \\
&= x^{5/12}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{50x^3+x}{12x^4+6x^2}\, dx} \\
&= z_1 e^{-\ln(2x^2+1)-\frac{\ln(x)}{12}} \\
&= z_1\left(\frac{1}{(2x^2+1)\, x^{1/12}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1=\frac{x^{1/3}}{2x^2+1}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2=y_1\int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1\int \frac{e^{\int -\frac{50x^3+x}{12x^4+6x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1\int \frac{e^{-2\ln(2x^2+1)-\frac{\ln(x)}{6}}}{(y_1)^2}\, dx \\
&= y_1\left(6x^{1/3}\mathrm{e}^{-2\ln(2x^2+1)-\frac{\ln(x)}{6}}\left(2x^2+1\right)^2\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^{1/3}}{2x^2+1}\right)+c_2\left(\frac{x^{1/3}}{2x^2+1}\left(6x^{1/3}\mathrm{e}^{-2\ln(2x^2+1)-\frac{\ln(x)}{6}}\left(2x^2+1\right)^2\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$6x^2(2x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(50x^2+1)\left(\frac{d}{dx}y(x)\right)+(30x^2+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(30x^2+1)y(x)}{6x^2(2x^2+1)}-\frac{(50x^2+1)\left(\frac{d}{dx}y(x)\right)}{6x(2x^2+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{(50x^2+1)\left(\frac{d}{dx}y(x)\right)}{6x(2x^2+1)}+\frac{(30x^2+1)y(x)}{6x^2(2x^2+1)}=0$

- ☐ Check to see if x_0 is a regular singular point
  - ∘ Define functions

    $\left[P_2(x)=\frac{50x^2+1}{6x(2x^2+1)}, P_3(x)=\frac{30x^2+1}{6x^2(2x^2+1)}\right]$

  - ∘ $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\bigg|_{x=0}=\frac{1}{6}$

  - ∘ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\bigg|_{x=0}=\frac{1}{6}$

  - ∘ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $6x^2(2x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(50x^2+1)\left(\frac{d}{dx}y(x)\right)+(30x^2+1)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - ∘ Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

    $x^m\cdot y(x)=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - ∘ Shift index using $k->k-m$

    $x^m\cdot y(x)=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - ∘ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - ∘ Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - ∘ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..4$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - ∘ Shift index using $k->k+2-m$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$a_0(-1+3r)(-1+2r)x^r + a_1(2+3r)(1+2r)x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(3k+3r-1)(2k+2r-1)+2a$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+3r)(-1+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{\frac{1}{2}, \frac{1}{3}\right\}$
- Each term must be 0
  $a_1(2+3r)(1+2r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $(3k+3r-1)(2k+2r-1)(a_k+2a_{k-2}) = 0$
- Shift index using $k- > k+2$
  $(3k+3r+5)(2k+2r+3)(a_{k+2}+2a_k) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -2a_k$
- Recursion relation for $r = \frac{1}{2}$
  $a_{k+2} = -2a_k$
- Solution for $r = \frac{1}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -2a_k, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{3}$
  $a_{k+2} = -2a_k$
- Solution for $r = \frac{1}{3}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -2a_k, a_1 = 0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = -2a_k, a_1 = 0, b_{k+2} = -2b_k, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.037 (sec)
Leaf size : 24

```
dsolve(6*x^2*(2*x^2+1)*diff(diff(y(x),x),x)+x*(50*x^2+1)*diff(y(x),x)+(30*x^2+1)*y(x) =
```

$$y = \frac{x^{1/3}\left(c_1 x^{1/6} + c_2\right)}{2x^2 + 1}$$

Mathematica DSolve solution

Solving time : 0.264 (sec)
Leaf size : 58

```
DSolve[{6*x^2*(1+2*x^2)*D[y[x],{x,2}]+x*(1+50*x^2)*D[y[x],x]+(1+30*x^2)*y[x]==0,{}},y[x],x,Inc
```

$$y(x) \to x^{5/12}\left(6c_2\sqrt[6]{x} + c_1\right)\exp\left(-\frac{1}{2}\int_1^x \frac{50K[1]^2 + 1}{12K[1]^3 + 6K[1]} dK[1]\right)$$

2.1.536 Problem 552

Internal problem ID [9708]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 552
Date solved : `Monday, January 27, 2025 at 06:13:21 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$28x^2(1-3x)\, y'' - 7x(5+9x)\, y' + 7(2+9x)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.153 (sec)

Writing the ode as

$$\left(-84x^3+28x^2\right) y'' + \left(-63x^2-35x\right) y' + (63x+14)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -84x^3+28x^2 \\ B &= -63x^2-35x \\ C &= 63x+14 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{33}{64x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 33 \\ t &= 64x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{33}{64x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1017: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{33}{64x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{33}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{11}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{8} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{33}{64x^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{33}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{11}{8}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{3}{8}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{33}{64x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{11}{8}$	$-\frac{3}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{11}{8}$	$-\frac{3}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=-\frac{3}{8}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}\right)\\
&= -\frac{3}{8}-\left(-\frac{3}{8}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^{-}}{x-c_1}\right)+(-)[\sqrt{r}]_\infty\\
&= -\frac{3}{8x}+(-)(0)\\
&= -\frac{3}{8x}\\
&= -\frac{3}{8x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}
$$

Let

$$
p(x)=1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{3}{8x}\right)(0)+\left(\left(\frac{3}{8x^2}\right)+\left(-\frac{3}{8x}\right)^2-\left(\frac{33}{64x^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\,dx}\\
&= e^{\int -\frac{3}{8x}dx}\\
&= \frac{1}{x^{3/8}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\
&= z_1 e^{-\int \frac{1}{2}\frac{-63x^2-35x}{-84x^3+28x^2}\,dx}\\
&= z_1 e^{-\ln(-1+3x)+\frac{5\ln(x)}{8}}\\
&= z_1\left(\frac{x^{5/8}}{-1+3x}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\frac{x^{1/4}}{-1+3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int\frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1\int\frac{e^{\int -\frac{-63x^2-35x}{-84x^3+28x^2}\,dx}}{(y_1)^2}\,dx\\
&= y_1\int\frac{e^{-2\ln(-1+3x)+\frac{5\ln(x)}{4}}}{(y_1)^2}\,dx\\
&= y_1\left(\frac{4\sqrt{x}\,e^{-2\ln(-1+3x)+\frac{5\ln(x)}{4}}(-1+3x)^2}{7}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1y_1+c_2y_2\\
&= c_1\left(\frac{x^{1/4}}{-1+3x}\right)+c_2\left(\frac{x^{1/4}}{-1+3x}\left(\frac{4\sqrt{x}\,e^{-2\ln(-1+3x)+\frac{5\ln(x)}{4}}(-1+3x)^2}{7}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$28x^2(1-3x)\left(\frac{d^2}{dx^2}y(x)\right)-7x(5+9x)\left(\frac{d}{dx}y(x)\right)+7(2+9x)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=\frac{(2+9x)y(x)}{4(3x-1)x^2}-\frac{(5+9x)\left(\frac{d}{dx}y(x)\right)}{4x(3x-1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)+\frac{(5+9x)\left(\frac{d}{dx}y(x)\right)}{4x(3x-1)}-\frac{(2+9x)y(x)}{4(3x-1)x^2}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=\frac{5+9x}{4x(3x-1)},P_3(x)=-\frac{2+9x}{4(3x-1)x^2}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

$\left(x\cdot P_2(x)\right)\bigg|_{x=0}=-\frac{5}{4}$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=\frac{1}{2}$

- ◦ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$4(3x-1)\,x^2\left(\frac{d^2}{dx^2}y(x)\right)+x(5+9x)\left(\frac{d}{dx}y(x)\right)+(-9x-2)\,y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

$x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ◦ Shift index using $k->k-m$

$x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ◦ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..3$

$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- ◦ Shift index using $k->k+2-m$

$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

Rewrite ODE with series expansions

$$-a_0(-1+4r)(-2+r)x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(4k+4r-1)(k+r-2) + 3a_{k-1}(4k+4r-1)(k+r-2)\right)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-(-1+4r)(-2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{2, \frac{1}{4}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $-4\left(k+r-\frac{1}{4}\right)(a_k - 3a_{k-1})(k+r-2) = 0$
- Shift index using $k->k+1$
  $-4\left(k+\frac{3}{4}+r\right)(a_{k+1} - 3a_k)(k+r-1) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = 3a_k$
- Recursion relation for $r = 2$
  $a_{k+1} = 3a_k$
- Solution for $r = 2$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = 3a_k\right]$
- Recursion relation for $r = \frac{1}{4}$
  $a_{k+1} = 3a_k$
- Solution for $r = \frac{1}{4}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+1} = 3a_k\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{4}}\right), a_{k+1} = 3a_k, b_{k+1} = 3b_k\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.024 (sec)
Leaf size : 23

```
dsolve(28*x^2*(-3*x+1)*diff(diff(y(x),x),x)-7*x*(5+9*x)*diff(y(x),x)+7*(2+9*x)*y(x) = 0,
```

$$y = \frac{c_1 x^2 + c_2 x^{1/4}}{3x - 1}$$

Mathematica DSolve solution

Solving time : `0.278` (sec)
Leaf size : 60

```
DSolve[{28*x^2*(1-3*x)*D[y[x],{x,2}]-7*x*(5+9*x)*D[y[x],x]+7*(2+9*x)*y[x]==0,{}},y[x],x,Incl
```

$$y(x) \to \frac{\left(4c_2x^{7/4} + 7c_1\right) \exp\left(-\frac{1}{2}\int_1^x \left(\frac{6}{3K[1]-1} - \frac{5}{4K[1]}\right) dK[1]\right)}{7x^{3/8}}$$

2.1.537 Problem 553

Internal problem ID [9709]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 553
Date solved : Monday, January 27, 2025 at 06:13:22 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$8x^2\left(-x^2+2\right)y''+2x\left(-21x^2+10\right)y'-\left(35x^2+2\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.161 (sec)

Writing the ode as

$$\left(-8x^4+16x^2\right)y''+\left(-42x^3+20x\right)y'+\left(-35x^2-2\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-8x^4+16x^2\\B&=-42x^3+20x\\C&=-35x^2-2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-7}{64x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=-7\\t&=64x^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{7}{64x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1019: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{64x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{7}{64x^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{7}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{7}{8}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{8}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=-\frac{7}{64x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{7}{8}$	$\frac{1}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{8}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}\right)\\
&= \frac{1}{8}-\left(\frac{1}{8}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^{-}}{x-c_1}\right)+(-)[\sqrt{r}]_\infty\\
&= \frac{1}{8x}+(-)\,(0)\\
&= \frac{1}{8x}\\
&= \frac{1}{8x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}
$$

Let

$$
p(x)=1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{1}{8x}\right)(0)+\left(\left(-\frac{1}{8x^2}\right)+\left(\frac{1}{8x}\right)^2-\left(-\frac{7}{64x^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}z_1(x)&=pe^{\int \omega\,dx}\\&=\mathrm{e}^{\int \frac{1}{8x}dx}\\&=x^{1/8}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1&=z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\&=z_1e^{-\int \frac{1}{2}\frac{-42x^3+20x}{-8x^4+16x^2}\,dx}\\&=z_1e^{-\frac{5\ln(x)}{8}-\ln(x^2-2)}\\&=z_1\left(\frac{1}{x^{5/8}\,(x^2-2)}\right)\end{aligned}$$

Which simplifies to

$$y_1=\frac{1}{\sqrt{x}\,(x^2-2)}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2&=y_1\int \frac{e^{\int -\frac{-42x^3+20x}{-8x^4+16x^2}\,dx}}{(y_1)^2}\,dx\\&=y_1\int \frac{e^{-\frac{5\ln(x)}{4}-2\ln(x^2-2)}}{(y_1)^2}\,dx\\&=y_1\left(\frac{4x^2\mathrm{e}^{-\frac{5\ln(x)}{4}-2\ln(x^2-2)}\,(x^2-2)^2}{3}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y&=c_1y_1+c_2y_2\\&=c_1\left(\frac{1}{\sqrt{x}\,(x^2-2)}\right)+c_2\left(\frac{1}{\sqrt{x}\,(x^2-2)}\left(\frac{4x^2\mathrm{e}^{-\frac{5\ln(x)}{4}-2\ln(x^2-2)}\,(x^2-2)^2}{3}\right)\right)\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$8x^2(-x^2+2)\left(\frac{d^2}{dx^2}y(x)\right)+2x(-21x^2+10)\left(\frac{d}{dx}y(x)\right)-(35x^2+2)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(35x^2+2)y(x)}{8x^2(x^2-2)}-\frac{(21x^2-10)\left(\frac{d}{dx}y(x)\right)}{4x(x^2-2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{(21x^2-10)\left(\frac{d}{dx}y(x)\right)}{4x(x^2-2)}+\frac{(35x^2+2)y(x)}{8x^2(x^2-2)}=0$

- ☐ Check to see if x_0 is a regular singular point
  - ◦ Define functions

    $\left[P_2(x)=\frac{21x^2-10}{4x(x^2-2)},P_3(x)=\frac{35x^2+2}{8x^2(x^2-2)}\right]$

  - ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{5}{4}$

  - ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=-\frac{1}{8}$

  - ◦ $x=0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=0$

- Multiply by denominators

  $8x^2(x^2-2)\left(\frac{d^2}{dx^2}y(x)\right)+2x(21x^2-10)\left(\frac{d}{dx}y(x)\right)+(35x^2+2)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

- ☐ Rewrite ODE with series expansions
  - ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

    $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  - ◦ Shift index using $k->k-m$

    $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  - ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  - ◦ Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - ◦ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..4$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - ◦ Shift index using $k->k+2-m$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

    Rewrite ODE with series expansions

$$-2a_0(1+2r)(-1+4r)x^r - 2a_1(3+2r)(3+4r)x^{1+r} + \left(\sum_{k=2}^{\infty} (-2a_k(2k+2r+1)(4k+4r-1)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-2(1+2r)(-1+4r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{-\frac{1}{2}, \frac{1}{4}\right\}$
- Each term must be 0
  $-2a_1(3+2r)(3+4r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $-(2k+2r+1)(4k+4r-1)(2a_k - a_{k-2}) = 0$
- Shift index using $k- >k+2$
  $-(2k+2r+5)(4k+4r+7)(2a_{k+2} - a_k) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{a_k}{2}$
- Recursion relation for $r = -\frac{1}{2}$
  $a_{k+2} = \frac{a_k}{2}$
- Solution for $r = -\frac{1}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = \frac{a_k}{2}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{4}$
  $a_{k+2} = \frac{a_k}{2}$
- Solution for $r = \frac{1}{4}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+2} = \frac{a_k}{2}, a_1 = 0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{4}}\right), a_{k+2} = \frac{a_k}{2}, a_1 = 0, b_{k+2} = \frac{b_k}{2}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.043 (sec)
Leaf size : 22

```
dsolve(8*x^2*(-x^2+2)*diff(diff(y(x),x),x)+2*x*(-21*x^2+10)*diff(y(x),x)-(35*x^2+2)*y(x)
```

$$y = \frac{c_2 x^{3/4} + c_1}{(x^2 - 2)\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.277 (sec)
Leaf size : 62

```
DSolve[{8*x^2*(2-x^2)*D[y[x],{x,2}]+2*x*(10-21*x^2)*D[y[x],x]-(2+35*x^2)*y[x]==0,{}},y[x],x,In
```

$$y(x) \to \frac{1}{3}\sqrt[8]{x}(4c_2 x^{3/4} + 3c_1) \exp\left(-\frac{1}{2}\int_1^x \left(\frac{4K[1]}{K[1]^2 - 2} + \frac{5}{4K[1]}\right) dK[1]\right)$$

2.1.538 Problem 554

Internal problem ID [9710]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 554
Date solved : `Monday, January 27, 2025 at 06:13:22 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2\left(x^2+3x+1\right)y''-4x\left(-3x^2-3x+1\right)y'+3\left(x^2-x+1\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.097 (sec)

Writing the ode as

$$\left(4x^4+12x^3+4x^2\right)y''+\left(12x^3+12x^2-4x\right)y'+\left(3x^2-3x+3\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^4+12x^3+4x^2 \\ B &= 12x^3+12x^2-4x \\ C &= 3x^2-3x+3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1021: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{12x^3+12x^2-4x}{4x^4+12x^3+4x^2}\, dx} \\ &= z_1 e^{-\ln(x^2+3x+1)+\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{\sqrt{x}}{x^2+3x+1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{x^2+3x+1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{12x^3+12x^2-4x}{4x^4+12x^3+4x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2\ln(x^2+3x+1)+\ln(x)}}{(y_1)^2}\, dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\sqrt{x}}{x^2+3x+1}\right) + c_2\left(\frac{\sqrt{x}}{x^2+3x+1}(x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2(x^2+3x+1)\left(\frac{d^2}{dx^2}y(x)\right) - 4x(-3x^2-3x+1)\left(\frac{d}{dx}y(x)\right) + 3(x^2-x+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{3(x^2-x+1)y(x)}{4x^2(x^2+3x+1)} - \frac{(3x^2+3x-1)\left(\frac{d}{dx}y(x)\right)}{x(x^2+3x+1)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(3x^2+3x-1)\left(\frac{d}{dx}y(x)\right)}{x(x^2+3x+1)} + \frac{3(x^2-x+1)y(x)}{4x^2(x^2+3x+1)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{3x^2+3x-1}{x(x^2+3x+1)}, P_3(x) = \frac{3(x^2-x+1)}{4x^2(x^2+3x+1)}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{3}{4}$
  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4x^2(x^2+3x+1)\left(\frac{d^2}{dx^2}y(x)\right) + 4x(3x^2+3x-1)\left(\frac{d}{dx}y(x)\right) + (3x^2-3x+3)\,y(x) = 0$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- > k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x^m \cdot \left(\frac{d}{dx} y(x)\right)$ to series expansion for $m = 1..3$

  $x^m \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$
- Shift index using $k- > k + 1 - m$

  $x^m \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$
- Convert $x^m \cdot \left(\frac{d^2}{dx^2} y(x)\right)$ to series expansion for $m = 2..4$

  $x^m \cdot \left(\frac{d^2}{dx^2} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$
- Shift index using $k- > k + 2 - m$

  $x^m \cdot \left(\frac{d^2}{dx^2} y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)\,(-3+2r)\, x^r + (a_1(1+2r)\,(-1+2r) + 3a_0(1+2r)\,(-1+2r))\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)\,(-3+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$
- Each term must be 0

  $a_1(1+2r)\,(-1+2r) + 3a_0(1+2r)\,(-1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = -3a_0$
- Each term in the series must be 0, giving the recursion relation

  $(2k+2r-1)\,(2k+2r-3)\,(a_k + 3a_{k-1} + a_{k-2}) = 0$
- Shift index using $k- > k + 2$

  $(2k+2r+3)\,(2k+2r+1)\,(a_{k+2} + 3a_{k+1} + a_k) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -3a_{k+1} - a_k$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -3a_{k+1} - a_k$
- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -3a_{k+1} - a_k, a_1 = -3a_0\right]$
- Recursion relation for $r = \frac{3}{2}$

  $a_{k+2} = -3a_{k+1} - a_k$
- Solution for $r = \frac{3}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2} = -3a_{k+1} - a_k, a_1 = -3a_0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+2} = -3a_{k+1} - a_k, a_1 = -3a_0, b_{k+2} = -3b_{k+1} - b_k, b_1 =\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.041 (sec)
Leaf size : 23

```
dsolve(4*x^2*(x^2+3*x+1)*diff(diff(y(x),x),x)-4*x*(-3*x^2-3*x+1)*diff(y(x),x)+3*(x^2-x
```

$$y = \frac{\sqrt{x}\,(c_2 x + c_1)}{x^2 + 3x + 1}$$

Mathematica DSolve solution

Solving time : 0.184 (sec)
Leaf size : 52

```
DSolve[{4*x^2*(1+3*x+x^2)*D[y[x],{x,2}]-4*x*(1-3*x-3*x^2)*D[y[x],x]+3*(1-x+x^2)*y[x]==0,{}},
```

$$y(x) \to (c_2 x + c_1) \exp\left(-\frac{1}{2}\int_1^x \frac{3K[1](K[1]+1)-1}{K[1](K[1](K[1]+3)+1)}dK[1]\right)$$

2.1.539 Problem 555

Internal problem ID [9711]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 555
Date solved : Monday, January 27, 2025 at 06:13:23 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$3x^2(1+x)^2\, y'' - x\left(-11x^2 - 10x + 1\right) y' + \left(5x^2 + 1\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.161 (sec)

Writing the ode as

$$3x^2(1+x)^2\, y'' + \left(11x^3 + 10x^2 - x\right) y' + \left(5x^2 + 1\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^2(1+x)^2 \\ B &= 11x^3 + 10x^2 - x \\ C &= 5x^2 + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -5 \\ t &= 36x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{5}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1023: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{36x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{5}{36x^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{5}{36}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{6}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{6}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=-\frac{5}{36x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{5}{6}$	$\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{6}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}\right)\\
&= \frac{1}{6}-\left(\frac{1}{6}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^{-}}{x-c_1}\right)+(-)[\sqrt{r}]_\infty\\
&= \frac{1}{6x}+(-)\,(0)\\
&= \frac{1}{6x}\\
&= \frac{1}{6x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}
$$

Let

$$
p(x)=1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0)+2\left(\frac{1}{6x}\right)(0)+\left(\left(-\frac{1}{6x^2}\right)+\left(\frac{1}{6x}\right)^2-\left(-\frac{5}{36x^2}\right)\right)=0
$$
$$
0=0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \,dx} \\
&= \mathrm{e}^{\int \frac{1}{6x} dx} \\
&= x^{1/6}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{11x^3+10x^2-x}{3x^2(1+x)^2} \,dx} \\
&= z_1 e^{\frac{\ln(x)}{6}-2\ln(1+x)} \\
&= z_1\left(\frac{x^{1/6}}{(1+x)^2}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1=\frac{x^{1/3}}{(1+x)^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2=y_1\int \frac{e^{\int -\frac{B}{A} \,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1\int \frac{e^{\int -\frac{11x^3+10x^2-x}{3x^2(1+x)^2} \,dx}}{(y_1)^2}\,dx \\
&= y_1\int \frac{e^{\frac{\ln(x)}{3}-4\ln(1+x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{3x^{1/3}\mathrm{e}^{\frac{\ln(x)}{3}-4\ln(1+x)}(1+x)^4}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1+c_2y_2 \\
&= c_1\left(\frac{x^{1/3}}{(1+x)^2}\right)+c_2\left(\frac{x^{1/3}}{(1+x)^2}\left(\frac{3x^{1/3}\mathrm{e}^{\frac{\ln(x)}{3}-4\ln(1+x)}(1+x)^4}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$3x^2(x+1)^2\left(\frac{d^2}{dx^2}y(x)\right)-x(-11x^2-10x+1)\left(\frac{d}{dx}y(x)\right)+(5x^2+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(5x^2+1)y(x)}{3x^2(x+1)^2}-\frac{\left(\frac{d}{dx}y(x)\right)(11x-1)}{3x(x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{\left(\frac{d}{dx}y(x)\right)(11x-1)}{3x(x+1)}+\frac{(5x^2+1)y(x)}{3x^2(x+1)^2}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{11x-1}{3x(x+1)},P_3(x)=\frac{5x^2+1}{3x^2(x+1)^2}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1}=4$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1}=2$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $3x^2(x+1)^2\left(\frac{d^2}{dx^2}y(x)\right)+x(x+1)(11x-1)\left(\frac{d}{dx}y(x)\right)+(5x^2+1)\,y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(3u^4-6u^3+3u^2)\left(\frac{d^2}{du^2}y(u)\right)+(11u^3-23u^2+12u)\left(\frac{d}{du}y(u)\right)+(5u^2-10u+6)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot y(u)$ to series expansion for $m=0..2$

    $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

  ◦ Shift index using $k->k-m$

    $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=1..3$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  ◦ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=2..4$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

  ◦ Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$3a_0(2+r)(1+r)\,u^r + (3a_1(3+r)(2+r) - a_0(2+r)(5+6r))\,u^{1+r} + \left(\sum_{k=2}^{\infty} (3a_k(k+r+2)(k+\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $3(2+r)(1+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-2, -1\}$
- Each term must be 0
  $3a_1(3+r)(2+r) - a_0(2+r)(5+6r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = \frac{a_0(5+6r)}{3(3+r)}$
- Each term in the series must be 0, giving the recursion relation
  $3(a_k + a_{k-2} - 2a_{k-1})\,k^2 + (6(a_k + a_{k-2} - 2a_{k-1})\,r + 9a_k - 4a_{k-2} - 5a_{k-1})\,k + 3(a_k + a_{k-2} - 2a_{}$
- Shift index using $k- >k+2$
  $3(a_{k+2} + a_k - 2a_{k+1})\,(k+2)^2 + (6(a_{k+2} + a_k - 2a_{k+1})\,r + 9a_{k+2} - 4a_k - 5a_{k+1})\,(k+2) + 3(a_{k+}$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{3k^2a_k - 6k^2a_{k+1} + 6kra_k - 12kra_{k+1} + 3r^2a_k - 6r^2a_{k+1} + 8ka_k - 29ka_{k+1} + 8ra_k - 29ra_{k+1} + 5a_k - 33a_{k+1}}{3(k^2 + 2kr + r^2 + 7k + 7r + 12)}$
- Recursion relation for $r = -2$
  $a_{k+2} = -\frac{3k^2a_k - 6k^2a_{k+1} - 4ka_k - 5ka_{k+1} + a_k + a_{k+1}}{3(k^2+3k+2)}$
- Solution for $r = -2$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-2}, a_{k+2} = -\frac{3k^2a_k - 6k^2a_{k+1} - 4ka_k - 5ka_{k+1} + a_k + a_{k+1}}{3(k^2+3k+2)}, a_1 = -\frac{7a_0}{3}\right]$
- Revert the change of variables $u = x + 1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x+1)^{k-2}, a_{k+2} = -\frac{3k^2a_k - 6k^2a_{k+1} - 4ka_k - 5ka_{k+1} + a_k + a_{k+1}}{3(k^2+3k+2)}, a_1 = -\frac{7a_0}{3}\right]$
- Recursion relation for $r = -1$
  $a_{k+2} = -\frac{3k^2a_k - 6k^2a_{k+1} + 2ka_k - 17ka_{k+1} - 10a_{k+1}}{3(k^2+5k+6)}$
- Solution for $r = -1$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-1}, a_{k+2} = -\frac{3k^2a_k - 6k^2a_{k+1} + 2ka_k - 17ka_{k+1} - 10a_{k+1}}{3(k^2+5k+6)}, a_1 = -\frac{a_0}{6}\right]$
- Revert the change of variables $u = x + 1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x+1)^{k-1}, a_{k+2} = -\frac{3k^2a_k - 6k^2a_{k+1} + 2ka_k - 17ka_{k+1} - 10a_{k+1}}{3(k^2+5k+6)}, a_1 = -\frac{a_0}{6}\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k (x+1)^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k-1}\right), a_{k+2} = -\frac{3k^2a_k - 6k^2a_{k+1} - 4ka_k - 5ka_{k+1} + a_k + a_{k+1}}{3(k^2+3k+2)}, a\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
```

```
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.019 (sec)
Leaf size : 19

```
dsolve(3*x^2*(x+1)^2*diff(diff(y(x),x),x)-x*(-11*x^2-10*x+1)*diff(y(x),x)+(5*x^2+1)*y(x)
```

$$y = \frac{x^{1/3} c_2 + c_1 x}{(x+1)^2}$$

Mathematica DSolve solution

Solving time : 0.243 (sec)
Leaf size : 58

```
DSolve[{3*x^2*(1+x)^2*D[y[x],{x,2}]-x*(1-10*x-11*x^2)*D[y[x],x]+(1+5*x^2)*y[x]==0,{}},y[x],x,I
```

$$y(x) \to \frac{1}{2}\sqrt[6]{x}\left(3c_2 x^{2/3} + 2c_1\right)\exp\left(-\frac{1}{2}\int_1^x \left(\frac{4}{K[1]+1} - \frac{1}{3K[1]}\right) dK[1]\right)$$

2.1.540 Problem 556

Internal problem ID [9712]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 556
Date solved : `Monday, January 27, 2025 at 06:13:23 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2\left(x^2+2x+3\right)y''-x\left(-15x^2-14x+3\right)y'+\left(7x^2+3\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.162 (sec)

Writing the ode as

$$\left(4x^4+8x^3+12x^2\right)y''+\left(15x^3+14x^2-3x\right)y'+\left(7x^2+3\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^4+8x^3+12x^2 \\ B &= 15x^3+14x^2-3x \\ C &= 7x^2+3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-7}{64x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -7 \\ t &= 64x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{7}{64x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1025: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{64x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{7}{64x^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{7}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b} = \frac{7}{8}\\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{8}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{7}{64x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{8}$	$\frac{1}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=\frac{1}{8}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right)\\
&= \frac{1}{8} - \left(\frac{1}{8}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty\\
&= \frac{1}{8x} + (-)\,(0)\\
&= \frac{1}{8x}\\
&= \frac{1}{8x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{gathered}
(0)+2\left(\frac{1}{8x}\right)(0)+\left(\left(-\frac{1}{8x^2}\right)+\left(\frac{1}{8x}\right)^2-\left(-\frac{7}{64x^2}\right)\right)=0 \\
0=0
\end{gathered}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \frac{1}{8x} dx} \\
&= x^{1/8}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{15x^3+14x^2-3x}{4x^4+8x^3+12x^2} dx} \\
&= z_1 e^{-\ln(x^2+2x+3)+\frac{\ln(x)}{8}} \\
&= z_1\left(\frac{x^{1/8}}{x^2+2x+3}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1=\frac{x^{1/4}}{x^2+2x+3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2=y_1\int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1\int \frac{e^{\int -\frac{15x^3+14x^2-3x}{4x^4+8x^3+12x^2} dx}}{(y_1)^2}\, dx \\
&= y_1\int \frac{e^{-2\ln(x^2+2x+3)+\frac{\ln(x)}{4}}}{(y_1)^2}\, dx \\
&= y_1\left(\frac{4\sqrt{x}\,\mathrm{e}^{-2\ln(x^2+2x+3)+\frac{\ln(x)}{4}}(x^2+2x+3)^2}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1+c_2y_2 \\
&= c_1\left(\frac{x^{1/4}}{x^2+2x+3}\right)+c_2\left(\frac{x^{1/4}}{x^2+2x+3}\left(\frac{4\sqrt{x}\,\mathrm{e}^{-2\ln(x^2+2x+3)+\frac{\ln(x)}{4}}(x^2+2x+3)^2}{3}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2(x^2+2x+3)\left(\frac{d^2}{dx^2}y(x)\right)-x(-15x^2-14x+3)\left(\frac{d}{dx}y(x)\right)+(7x^2+3)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(7x^2+3)y(x)}{4x^2(x^2+2x+3)}-\frac{(15x^2+14x-3)\left(\frac{d}{dx}y(x)\right)}{4x(x^2+2x+3)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{(15x^2+14x-3)\left(\frac{d}{dx}y(x)\right)}{4x(x^2+2x+3)}+\frac{(7x^2+3)y(x)}{4x^2(x^2+2x+3)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{15x^2+14x-3}{4x(x^2+2x+3)},P_3(x)=\frac{7x^2+3}{4x^2(x^2+2x+3)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\bigg|_{x=0}=-\frac{1}{4}$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\bigg|_{x=0}=\frac{1}{4}$

  - $x=0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=0$

- Multiply by denominators

  $4x^2(x^2+2x+3)\left(\frac{d^2}{dx^2}y(x)\right)+x(15x^2+14x-3)\left(\frac{d}{dx}y(x)\right)+(7x^2+3)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

    $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..4$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$3a_0(-1+4r)(-1+r)x^r + (3a_1(3+4r)r + 2a_0r(3+4r))x^{1+r} + \left(\sum_{k=2}^{\infty}(3a_k(4k+4r-1)(k+r-\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $3(-1+4r)(-1+r)=0$
- Values of r that satisfy the indicial equation
  $r \in \{1, \frac{1}{4}\}$
- Each term must be 0
  $3a_1(3+4r)r + 2a_0r(3+4r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = -\frac{2a_0}{3}$
- Each term in the series must be 0, giving the recursion relation
  $(4k+4r-1)(k+r-1)(3a_k + 2a_{k-1} + a_{k-2}) = 0$
- Shift index using $k->k+2$
  $(4k+4r+7)(k+r+1)(3a_{k+2} + 2a_{k+1} + a_k) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{2a_{k+1}}{3} - \frac{a_k}{3}$
- Recursion relation for $r=1$
  $a_{k+2} = -\frac{2a_{k+1}}{3} - \frac{a_k}{3}$
- Solution for $r=1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{2a_{k+1}}{3} - \frac{a_k}{3}, a_1 = -\frac{2a_0}{3}\right]$
- Recursion relation for $r=\frac{1}{4}$
  $a_{k+2} = -\frac{2a_{k+1}}{3} - \frac{a_k}{3}$
- Solution for $r=\frac{1}{4}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+2} = -\frac{2a_{k+1}}{3} - \frac{a_k}{3}, a_1 = -\frac{2a_0}{3}\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{4}}\right), a_{k+2} = -\frac{2a_{k+1}}{3} - \frac{a_k}{3}, a_1 = -\frac{2a_0}{3}, b_{k+2} = -\frac{2b_{k+1}}{3} - \frac{b_k}{3}, b_1 = -\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.028 (sec)
Leaf size : 24

```
dsolve(4*x^2*(x^2+2*x+3)*diff(diff(y(x),x),x)-x*(-15*x^2-14*x+3)*diff(y(x),x)+(7*x^2+3
```

$$y = \frac{c_2 x^{1/4} + c_1 x}{x^2 + 2x + 3}$$

Mathematica DSolve solution

Solving time : 0.272 (sec)
Leaf size : 67

```
DSolve[{4*x^2*(3+2*x+x^2)*D[y[x],{x,2}]-x*(3-14*x-15*x^2)*D[y[x],x]+(3+7*x^2)*y[x]==0,{}},y[
```

$$y(x) \to \frac{1}{3}\sqrt[8]{x}\left(4c_2 x^{3/4} + 3c_1\right)\exp\left(-\frac{1}{2}\int_1^x \left(\frac{4(K[1]+1)}{K[1](K[1]+2)+3} - \frac{1}{4K[1]}\right) dK[1]\right)$$

2.1.541 Problem 557

Internal problem ID [9713]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 557
Date solved : `Monday, January 27, 2025 at 06:13:24 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(x^2-2x+1\right)y''-x(3+x)\,y'+(4+x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.303 (sec)

Writing the ode as

$$x^2(x-1)^2\,y''+\left(-x^2-3x\right)y'+(4+x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=x^2(x-1)^2\\ B&=-x^2-3x\\ C&=4+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{7x^2+10x-1}{4x^2\left(x-1\right)^4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=7x^2+10x-1\\ t&=4x^2(x-1)^4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{7x^2+10x-1}{4x^2\left(x-1\right)^4}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1027: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2(x-1)^4$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{3}{2x} - \frac{2}{(x-1)^3} + \frac{7}{4(x-1)^2} - \frac{3}{2(x-1)} + \frac{4}{(x-1)^4}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned}\alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a}+v\right)\\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a}+v\right)\end{aligned}$$

The partial fraction decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{3}{2x} - \frac{2}{(x-1)^3} + \frac{7}{4(x-1)^2} - \frac{3}{2(x-1)} + \frac{4}{(x-1)^4}$$

There is pole in r at $x=1$ of order 4, hence $v=2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c=1$ gives

$$[\sqrt{r}]_c \approx \frac{2}{(x-1)^2} - \frac{1}{2(x-1)} + \frac{21}{32} - \frac{9x}{32} + \frac{53(x-1)^2}{256} - \frac{149(x-1)^3}{1024} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v=2$ the above becomes

$$[\sqrt{r}]_c = \frac{2}{(x-1)^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-1)^2}$ is

$$a = 2$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c=1$. This term becomes $\frac{1}{(x-1)^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be -2. Therefore

$$\begin{aligned}b &= (-2)-(0)\\ &= -2\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_c &= \frac{2}{(x-1)^2}\\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a}+v\right) = \frac{1}{2}\left(\frac{-2}{2}+2\right) = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a}+v\right) = \frac{1}{2}\left(-\frac{-2}{2}+2\right) = \frac{3}{2}\end{aligned}$$

Since the order of r at ∞ is $4>2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= 0\\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{7x^2+10x-1}{4x^2(x-1)^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
1	4	$\frac{2}{(x-1)^2}$	$\frac{1}{2}$	$\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{+}\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2} + (-)(0) \\ &= \frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2} \\ &= \frac{2x^2 + x + 1}{2x(x-1)^2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{4}{(x-1)^3} - \frac{1}{2(x-1)^2}\right) + \left(\frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x -$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2}\right) dx} \\ &= \sqrt{x}\,\sqrt{x-1}\,e^{-\frac{2}{x-1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2-3x}{x^2(x-1)^2}\,dx} \\
&= z_1 e^{\frac{3\ln(x)}{2}-\frac{2}{x-1}-\frac{3\ln(x-1)}{2}} \\
&= z_1\left(\frac{x^{3/2}\mathrm{e}^{-\frac{2}{x-1}}}{(x-1)^{3/2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^{3/2}\mathrm{e}^{-\frac{4}{x-1}}\sqrt{x\,(x-1)}}{(x-1)^{3/2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-3x}{x^2(x-1)^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3\ln(x)-\frac{4}{x-1}-3\ln(x-1)}}{(y_1)^2}\,dx \\
&= y_1\left(\mathrm{e}^{-4}\,\mathrm{Ei}_1\left(-\frac{4}{x-1}-4\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{x^{3/2}\mathrm{e}^{-\frac{4}{x-1}}\sqrt{x\,(x-1)}}{(x-1)^{3/2}}\right) + c_2\left(\frac{x^{3/2}\mathrm{e}^{-\frac{4}{x-1}}\sqrt{x\,(x-1)}}{(x-1)^{3/2}}\left(\mathrm{e}^{-4}\,\mathrm{Ei}_1\left(-\frac{4}{x-1}-4\right)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2(x^2-2x+1)\left(\frac{d^2}{dx^2}y(x)\right) - x(x+3)\left(\frac{d}{dx}y(x)\right) + (x+4)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{(x+4)y(x)}{x^2(x^2-2x+1)} + \frac{(x+3)\left(\frac{d}{dx}y(x)\right)}{x(x^2-2x+1)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) - \frac{(x+3)\left(\frac{d}{dx}y(x)\right)}{x(x^2-2x+1)} + \frac{(x+4)y(x)}{x^2(x^2-2x+1)} = 0$

☐ Check to see if x_0 is a regular singular point
  - Define functions

$\left[P_2(x) = -\frac{x+3}{x(x^2-2x+1)}, P_3(x) = \frac{x+4}{x^2(x^2-2x+1)}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = -3$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 4$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2 - 2x + 1)\left(\frac{d^2}{dx^2}y(x)\right) - x(x+3)\left(\frac{d}{dx}y(x)\right) + (x+4)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- Rewrite ODE with series expansions

  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k- >k + 1 - m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - Shift index using $k- >k + 2 - m$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)^2\,x^r + \left(a_1(-1+r)^2 - a_0(1+2r)\,(-1+r)\right)x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-2)^2 - a_{k-1}(2k -\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 2$

- Each term must be 0

  $a_1(-1+r)^2 - a_0(1+2r)\,(-1+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = \frac{a_0(1+2r)}{-1+r}$

- Each term in the series must be 0, giving the recursion relation

  $((a_k + a_{k-2} - 2a_{k-1})\,k + (a_k + a_{k-2} - 2a_{k-1})\,r - 2a_k - 3a_{k-2} + a_{k-1})\,(k+r-2) = 0$

- Shift index using $k- >k + 2$

$$((a_{k+2}+a_k-2a_{k+1})(k+2)+(a_{k+2}+a_k-2a_{k+1})r-2a_{k+2}-3a_k+a_{k+1})(k+r)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{ka_k-2ka_{k+1}+ra_k-2ra_{k+1}-a_k-3a_{k+1}}{k+r}$$

- Recursion relation for $r=2$

$$a_{k+2}=-\frac{ka_k-2ka_{k+1}+a_k-7a_{k+1}}{k+2}$$

- Solution for $r=2$

$$\left[y(x)=\sum_{k=0}^{\infty}a_k x^{k+2}, a_{k+2}=-\frac{ka_k-2ka_{k+1}+a_k-7a_{k+1}}{k+2}, a_1=5a_0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 45

```
dsolve(x^2*(x^2-2*x+1)*diff(diff(y(x),x),x)-x*(x+3)*diff(y(x),x)+(x+4)*y(x) = 0,y(x),sin
```

$$y=\frac{x^2\left(\mathrm{Ei}_1\left(-\frac{4x}{x-1}\right)\mathrm{e}^{-\frac{4x}{x-1}}c_2+\mathrm{e}^{-\frac{4}{x-1}}c_1\right)}{x-1}$$

Mathematica DSolve solution

Solving time : 0.285 (sec)
Leaf size : 116

```
DSolve[{x^2*(1-2*x+x^2)*D[y[x],{x,2}]-x*(3+x)*D[y[x],x]+(4+x)*y[x]==0,{}},y[x],x,IncludeSingula
```

$$y(x)\to\exp\left(\int_1^x\frac{2K[1]^2+K[1]+1}{2(K[1]-1)^2K[1]}dK[1]-\frac{1}{2}\int_1^x-\frac{K[2]+3}{(K[2]-1)^2K[2]}dK[2]\right)\left(c_2\int_1^x\exp\left(-2\int_1^{K[3]}\frac{2K[1]^2+K[1]+1}{2(K[1]-1)^2K[1]}dK[1]\right)dK[3]+c_1\right)$$

2.1.542 Problem 558

Internal problem ID [9714]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 558
Date solved : `Monday, January 27, 2025 at 06:13:25 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(2+x)\,y''+5x^2y'+(1+x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.200 (sec)

Writing the ode as

$$\left(2x^3+4x^2\right)y''+5x^2y'+(1+x)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^3+4x^2 \\ B &= 5x^2 \\ C &= 1+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-3x^2-24x-16}{16\left(x^2+2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3x^2-24x-16 \\ t &= 16\left(x^2+2x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-3x^2-24x-16}{16\left(x^2+2x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1029: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2+2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{8x} - \frac{1}{4x^2} + \frac{5}{16\,(2+x)^2} + \frac{1}{16+8x}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(2+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 - 24x - 16}{16\left(x^2 + 2x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x^2 - 24x - 16}{16\left(x^2 + 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= \frac{1}{4} - \left(\frac{1}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{4\left(2+x\right)} + \frac{1}{2x} + (-)\left(0\right) \\
&= -\frac{1}{4\left(2+x\right)} + \frac{1}{2x} \\
&= \frac{x+4}{4x\left(2+x\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{4(2+x)} + \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{4(2+x)^2} - \frac{1}{2x^2}\right) + \left(-\frac{1}{4(2+x)} + \frac{1}{2x}\right)^2 - \left(\frac{-3x^2 - 24x - 16}{16(x^2+2x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{4(2+x)} + \frac{1}{2x}\right) dx} \\ &= \frac{\sqrt{x}}{(2+x)^{1/4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{5x^2}{2x^3+4x^2} \, dx} \\ &= z_1 e^{-\frac{5\ln(2+x)}{4}} \\ &= z_1 \left(\frac{1}{(2+x)^{5/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{(2+x)^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{5x^2}{2x^3+4x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\frac{5\ln(2+x)}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(2\sqrt{2+x} - 2\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\sqrt{x}}{(2+x)^{3/2}}\right) + c_2 \left(\frac{\sqrt{x}}{(2+x)^{3/2}} \left(2\sqrt{2+x} - 2\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2(x+2)\left(\frac{d^2}{dx^2}y(x)\right)+5x^2\left(\frac{d}{dx}y(x)\right)+(x+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(x+1)y(x)}{2(x+2)x^2}-\frac{5\left(\frac{d}{dx}y(x)\right)}{2(x+2)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{5\left(\frac{d}{dx}y(x)\right)}{2(x+2)}+\frac{(x+1)y(x)}{2(x+2)x^2}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=\frac{5}{2(x+2)}, P_3(x)=\frac{x+1}{2(x+2)x^2}\right]$
- ○ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

  $\left((x+2)\cdot P_2(x)\right)\Big|_{x=-2}=\frac{5}{2}$
- ○ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

  $\left((x+2)^2\cdot P_3(x)\right)\Big|_{x=-2}=0$
- ○ $x=-2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-2$

- Multiply by denominators

  $2x^2(x+2)\left(\frac{d^2}{dx^2}y(x)\right)+5x^2\left(\frac{d}{dx}y(x)\right)+(x+1)\,y(x)=0$
- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

  $(2u^3-8u^2+8u)\left(\frac{d^2}{du^2}y(u)\right)+(5u^2-20u+20)\left(\frac{d}{du}y(u)\right)+(u-1)\,y(u)=0$
- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty} a_k u^{k+r+m}$
- ○ Shift index using $k->k-m$

  $u^m\cdot y(u)=\sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
- ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
- ○ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$
- ○ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$
- ○ Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$4a_0r(3+2r)\,u^{-1+r} + (4a_1(1+r)\,(5+2r) - a_0(8r^2+12r+1))\,u^r + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+r+1)\,(2k+5\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$4r(3+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, -\frac{3}{2}\right\}$$

- Each term must be 0

  $$4a_1(1+r)\,(5+2r) - a_0(8r^2+12r+1) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$2(-4a_k + a_{k-1} + 4a_{k+1})\,k^2 + (4(-4a_k + a_{k-1} + 4a_{k+1})\,r - 12a_k - a_{k-1} + 28a_{k+1})\,k + 2(-4a_k + a_{k-}$$

- Shift index using $k- >k+1$

  $$2(-4a_{k+1} + a_k + 4a_{k+2})\,(k+1)^2 + (4(-4a_{k+1} + a_k + 4a_{k+2})\,r - 12a_{k+1} - a_k + 28a_{k+2})\,(k+1) + 2$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 4kra_k - 16kra_{k+1} + 2r^2a_k - 8r^2a_{k+1} + 3ka_k - 28ka_{k+1} + 3ra_k - 28ra_{k+1} + a_k - 21a_{k+1}}{4(2k^2+4kr+2r^2+11k+11r+14)}$$

- Recursion relation for $r = 0$

  $$a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 3ka_k - 28ka_{k+1} + a_k - 21a_{k+1}}{4(2k^2+11k+14)}$$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 3ka_k - 28ka_{k+1} + a_k - 21a_{k+1}}{4(2k^2+11k+14)}, 20a_1 - a_0 = 0\right]$$

- Revert the change of variables $u = x+2$

  $$\left[y(x) = \sum_{k=0}^{\infty} a_k(x+2)^k, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 3ka_k - 28ka_{k+1} + a_k - 21a_{k+1}}{4(2k^2+11k+14)}, 20a_1 - a_0 = 0\right]$$

- Recursion relation for $r = -\frac{3}{2}$

  $$a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} - 3ka_k - 4ka_{k+1} + a_k + 3a_{k+1}}{4(2k^2+5k+2)}$$

- Solution for $r = -\frac{3}{2}$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{3}{2}}, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} - 3ka_k - 4ka_{k+1} + a_k + 3a_{k+1}}{4(2k^2+5k+2)}, -4a_1 - a_0 = 0\right]$$

- Revert the change of variables $u = x+2$

  $$\left[y(x) = \sum_{k=0}^{\infty} a_k(x+2)^{k-\frac{3}{2}}, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} - 3ka_k - 4ka_{k+1} + a_k + 3a_{k+1}}{4(2k^2+5k+2)}, -4a_1 - a_0 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k(x+2)^k\right) + \left(\sum_{k=0}^{\infty} b_k(x+2)^{k-\frac{3}{2}}\right), a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 3ka_k - 28ka_{k+1} + a_k - 21a_{k+1}}{4(2k^2+11k+14)}, 20a\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
```

```
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.043 (sec)
Leaf size : 39

```
dsolve(2*x^2*(x+2)*diff(diff(y(x),x),x)+5*diff(y(x),x)*x^2+(x+1)*y(x) = 0,y(x),singsol
```

$$y = \frac{\sqrt{x}\left(\sqrt{2}\,\sqrt{x+2}\,c_2 - 2\,\operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)c_2 + c_1\right)}{(x+2)^{3/2}}$$

Mathematica DSolve solution

Solving time : 0.392 (sec)
Leaf size : 83

```
DSolve[{2*x^2*(2+x)*D[y[x],{x,2}]+5*x^2*D[y[x],x]+(1+x)*y[x]==0,{}},y[x],x,IncludeSingularSo
```

$$y(x) \to \frac{\exp\left(\int_1^x \frac{K[1]+4}{4K[1]^2+8K[1]}dK[1]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[2]}\frac{K[1]+4}{4K[1]^2+8K[1]}dK[1]\right)dK[2] + c_1\right)}{(x+2)^{5/4}}$$

2.1.543 Problem 559

Internal problem ID [9715]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 559
Date solved : `Monday, January 27, 2025 at 06:13:25 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(-x^2+2\right) y''-2x\left(2x^2+1\right) y'+\left(-2x^2+2\right) y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.362 (sec)

Writing the ode as

$$\left(-x^4+2x^2\right) y''+\left(-4x^3-2x\right) y'+\left(-2x^2+2\right) y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^4+2x^2 \\ B &= -4x^3-2x \\ C &= -2x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{3x^2-1}{\left(x^3-2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3x^2-1 \\ t &= \left(x^3-2x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{3x^2-1}{\left(x^3-2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1031: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 6 - 2 \\
&= 4
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^3 - 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \sqrt{2}$ of order 2. There is a pole at $x = -\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16\,(x-\sqrt{2})^2} + \frac{5}{16\,(x+\sqrt{2})^2} - \frac{3\sqrt{2}}{32\,(x-\sqrt{2})} + \frac{3\sqrt{2}}{32\,(x+\sqrt{2})} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x=\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x-\sqrt{2}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{4} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{4}
\end{aligned}$$

For the pole at $x=-\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x+\sqrt{2}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{4} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is $4>2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{3x^{2}-1}{\left(x^{3}-2x\right)^{2}}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$\sqrt{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-\sqrt{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=0$ then

$$\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{+}+\alpha_{c_2}^{-}+\alpha_{c_3}^{-}\right) \\
&= 0-(0) \\
&= 0
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + (+)[\sqrt{r}]_{\infty} \\ &= \frac{1}{2x} - \frac{1}{4\left(x-\sqrt{2}\right)} - \frac{1}{4\left(x+\sqrt{2}\right)} + (0) \\ &= \frac{1}{2x} - \frac{1}{4\left(x-\sqrt{2}\right)} - \frac{1}{4\left(x+\sqrt{2}\right)} \\ &= -\frac{1}{x^3-2x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{4\left(x-\sqrt{2}\right)} - \frac{1}{4\left(x+\sqrt{2}\right)}\right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{1}{4\left(x-\sqrt{2}\right)^2} + \frac{1}{4\left(x+\sqrt{2}\right)^2}\right) + \left(\frac{1}{2x} - \frac{}{4\left(\right.}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{1}{4\left(x-\sqrt{2}\right)} - \frac{1}{4\left(x+\sqrt{2}\right)}\right) dx} \\ &= \frac{\sqrt{x}}{\left(x-\sqrt{2}\right)^{1/4}\left(x+\sqrt{2}\right)^{1/4}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x^3-2x}{-x^4+2x^2}\, dx} \\ &= z_1 e^{-\frac{5\ln\left(x^2-2\right)}{4} + \frac{\ln(x)}{2}} \\ &= z_1\left(\frac{\sqrt{x}}{\left(x^2-2\right)^{5/4}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x}{\left(x^2-2\right)^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^3-2x}{-x^4+2x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\frac{5\ln(x^2-2)}{2}+\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(\sqrt{x^2-2}+\sqrt{2}\arctan\left(\frac{\sqrt{2}}{\sqrt{x^2-2}}\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{x}{(x^2-2)^{3/2}}\right) + c_2\left(\frac{x}{(x^2-2)^{3/2}}\left(\sqrt{x^2-2}+\sqrt{2}\arctan\left(\frac{\sqrt{2}}{\sqrt{x^2-2}}\right)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(-x^2+2)\left(\frac{d^2}{dx^2}y(x)\right) - 2x(2x^2+1)\left(\frac{d}{dx}y(x)\right) + (-2x^2+2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{2(x^2-1)y(x)}{x^2(x^2-2)} - \frac{2(2x^2+1)\left(\frac{d}{dx}y(x)\right)}{x(x^2-2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{2(2x^2+1)\left(\frac{d}{dx}y(x)\right)}{x(x^2-2)} + \frac{2(x^2-1)y(x)}{x^2(x^2-2)} = 0$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{2(2x^2+1)}{x(x^2-2)}, P_3(x) = \frac{2(x^2-1)}{x^2(x^2-2)}\right]$
  - $x \cdot P_2(x)$ is analytic at $x=0$

    $(x \cdot P_2(x))\Big|_{x=0} = -1$
  - $x^2 \cdot P_3(x)$ is analytic at $x=0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 1$
  - $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2-2)\left(\frac{d^2}{dx^2}y(x)\right) + 2x(2x^2+1)\left(\frac{d}{dx}y(x)\right) + (2x^2-2)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
- Shift index using $k- >k + 1 - m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$
- Shift index using $k- >k + 2 - m$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0(-1+r)^2 x^r - 2a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(-2a_k(k+r-1)^2 + a_{k-2}(k+r)\,(k+r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2(-1+r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 1$
- Each term must be 0

  $-2a_1 r^2 = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $-2a_k(k+r-1)^2 + a_{k-2}(k+r)\,(k+r-1) = 0$
- Shift index using $k- >k + 2$

  $-2a_{k+2}(k+r+1)^2 + a_k(k+r+2)\,(k+r+1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k(k+r+2)}{2(k+r+1)}$
- Recursion relation for $r = 1$

  $a_{k+2} = \frac{a_k(k+3)}{2(k+2)}$
- Solution for $r = 1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = \frac{a_k(k+3)}{2(k+2)}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.035 (sec)
Leaf size : 42

```
dsolve(x^2*(-x^2+2)*diff(diff(y(x),x),x)-2*x*(2*x^2+1)*diff(y(x),x)+(-2*x^2+2)*y(x) = 0,
```

$$y = \frac{x\left(\sqrt{2}\, c_2\sqrt{x^2-2}+2\arctan\left(\frac{\sqrt{2}}{\sqrt{x^2-2}}\right) c_2+c_1\right)}{\left(x^2-2\right)^{3/2}}$$

Mathematica DSolve solution

Solving time : 0.256 (sec)
Leaf size : 97

```
DSolve[{x^2*(2-x^2)*D[y[x],{x,2}]-2*x*(1+2*x^2)*D[y[x],x]+(2-2*x^2)*y[x]==0,{}},y[x],x,IncludeS
```

$$y(x) \to \exp\left(\int_1^x \frac{1}{2K[1]-K[1]^3}dK[1]-\frac{1}{2}\int_1^x\left(\frac{5K[2]}{K[2]^2-2}-\frac{1}{K[2]}\right)dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]}\frac{1}{2K[1]-K[1]^3}dK[1]\right)dK[3]+c_1\right)$$

2.1.544 Problem 560

Internal problem ID [9716]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 560
Date solved : `Monday, January 27, 2025 at 06:13:26 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x(5-x)\,y' + (9-4x)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.424 (sec)

Writing the ode as

$$x^2y'' + \left(x^2-5x\right)y' + (9-4x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2-5x \\ C &= 9-4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2+6x-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2+6x-1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2+6x-1}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1033: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{3}{2x} - \frac{5}{2x^2} + \frac{15}{2x^3} - \frac{115}{4x^4} + \frac{495}{4x^5} - \frac{2285}{4x^6} + \frac{11055}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 6x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{6x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{6x - 1}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 6. Dividing this by leading coefficient in t which is 4 gives $\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{3}{2}\right) - (0) \\ &= \frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{3}{2}}{\frac{1}{2}} - 0\right) &&= \frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{3}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{3}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 6x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{3}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+\right) \\ &= \frac{3}{2} - \left(\frac{1}{2}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2x} + \frac{1}{2} \\ &= \frac{1+x}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2x} + \frac{1}{2}\right)(1) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} + \frac{1}{2}\right)^2 - \left(\frac{x^2+6x-1}{4x^2}\right)\right) &= 0 \\ \frac{1-a_0}{x} &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 1 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (1+x)\, \mathrm{e}^{\int \left(\frac{1}{2x}+\frac{1}{2}\right) dx} \\ &= (1+x)\, \mathrm{e}^{\frac{x}{2}+\frac{\ln(x)}{2}} \\ &= (1+x)\sqrt{x}\, \mathrm{e}^{\frac{x}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^2-5x}{x^2}\, dx} \\ &= z_1 e^{-\frac{x}{2}+\frac{5\ln(x)}{2}} \\ &= z_1\left(x^{5/2} \mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^3(1+x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2-5x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-x+5\ln(x)}}{(y_1)^2}\, dx \\ &= y_1\left(-\operatorname{Ei}_1(x) - \frac{\mathrm{e}^{-x}}{-1-x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(x^3(1+x)\right) + c_2\left(x^3(1+x)\left(-\operatorname{Ei}_1(x) - \frac{\mathrm{e}^{-x}}{-1-x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right) - x(5-x)\left(\frac{d}{dx}y(x)\right) + (9-4x)\, y(x) = 0$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
$\frac{d^2}{dx^2}y(x) = \frac{(-9+4x)y(x)}{x^2} - \frac{(x-5)\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)+\frac{(x-5)\left(\frac{d}{dx}y(x)\right)}{x}-\frac{(-9+4x)y(x)}{x^2}=0$

- □ Check to see if $x_0=0$ is a regular singular point
  - ○ Define functions
    $\left[P_2(x)=\frac{x-5}{x},P_3(x)=-\frac{-9+4x}{x^2}\right]$
  - ○ $x\cdot P_2(x)$ is analytic at $x=0$
    $\left.(x\cdot P_2(x))\right|_{x=0}=-5$
  - ○ $x^2\cdot P_3(x)$ is analytic at $x=0$
    $\left.(x^2\cdot P_3(x))\right|_{x=0}=9$
  - ○ $x=0$is a regular singular point
    Check to see if $x_0=0$ is a regular singular point
    $x_0=0$
- • Multiply by denominators
  $x^2\left(\frac{d^2}{dx^2}y(x)\right)+x(x-5)\left(\frac{d}{dx}y(x)\right)+(9-4x)\,y(x)=0$
- • Assume series solution for $y(x)$
  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$
- □ Rewrite ODE with series expansions
  - ○ Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$
    $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$
  - ○ Shift index using $k->k-m$
    $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$
  - ○ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$
    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$
  - ○ Shift index using $k->k+1-m$
    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - ○ Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion
    $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$
    Rewrite ODE with series expansions
    $a_0(-3+r)^2\,x^r+\left(\sum_{k=1}^{\infty}\left(a_k(k+r-3)^2+a_{k-1}(k-5+r)\right)x^{k+r}\right)=0$
- • $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-3+r)^2=0$
- • Values of r that satisfy the indicial equation
  $r=3$
- • Each term in the series must be 0, giving the recursion relation
  $a_k(k+r-3)^2+a_{k-1}(k-5+r)=0$
- • Shift index using $k->k+1$
  $a_{k+1}(k-2+r)^2+a_k(k+r-4)=0$
- • Recursion relation that defines series solution to ODE
  $a_{k+1}=-\frac{a_k(k+r-4)}{(k-2+r)^2}$
- • Recursion relation for $r=3$; series terminates at $k=1$

$a_{k+1} = -\frac{a_k(k-1)}{(k+1)^2}$

- Apply recursion relation for $k = 0$

  $a_1 = a_0$
- Terminating series solution of the ODE for $r = 3$. Use reduction of order to find the second li

  $y(x) = a_0 \cdot (x + 1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 27

```
dsolve(x^2*diff(diff(y(x),x),x)-x*(-x+5)*diff(y(x),x)+(9-4*x)*y(x) = 0,y(x),singsol=al
```

$$y = \left(-c_2 \mathrm{e}^{-x} + \left(\mathrm{Ei}_1\left(x\right) c_2 + c_1\right)\left(x+1\right)\right) x^3$$

Mathematica DSolve solution

Solving time : 0.501 (sec)
Leaf size : 72

```
DSolve[{x^2*D[y[x],{x,2}]-x*(5-x)*D[y[x],x]+(9-4*x)*y[x]==0,{}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to \sqrt{x}(x+1)\left(c_2 \int_1^x \frac{e^{-K[2]-1}}{K[2](K[2]+1)^2} dK[2] + c_1\right) \exp\left(\frac{1}{2}\left(-\int_1^x \left(1 - \frac{5}{K[1]}\right) dK[1] + x + 1\right)\right)$$

2.1.545 Problem 561

Internal problem ID [9717]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 561
Date solved : Monday, January 27, 2025 at 06:13:27 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$4x^2\left(x^2+x+1\right)y''+12x^2(1+x)\,y'+\left(3x^2+3x+1\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.750 (sec)

Writing the ode as

$$\left(4x^4+4x^3+4x^2\right)y''+\left(12x^3+12x^2\right)y'+\left(3x^2+3x+1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^4+4x^3+4x^2 \\ B &= 12x^3+12x^2 \\ C &= 3x^2+3x+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{2x^2-4x-1}{4\left(x^3+x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2x^2-4x-1 \\ t &= 4\left(x^3+x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{2x^2-4x-1}{4\left(x^3+x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1035: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ of order 2. There is a pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{-\frac{3}{8} - \frac{i\sqrt{3}}{8}}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{3}{8} + \frac{i\sqrt{3}}{8}}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2} + \frac{\frac{1}{4} - \frac{5i\sqrt{3}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{4} + \frac{5i\sqrt{3}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} - \frac{1}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x=-\frac{1}{2}+\frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{3}{8}-\frac{i\sqrt{3}}{8}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b} = \frac{1}{2}+\frac{\sqrt{-2-2i\sqrt{3}}}{4} \\
\alpha_c^- &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{2}-\frac{\sqrt{-2-2i\sqrt{3}}}{4}
\end{aligned}$$

For the pole at $x=-\frac{1}{2}-\frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{3}{8}+\frac{i\sqrt{3}}{8}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b} = \frac{1}{2}+\frac{\sqrt{-2+2i\sqrt{3}}}{4} \\
\alpha_c^- &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{2}-\frac{\sqrt{-2+2i\sqrt{3}}}{4}
\end{aligned}$$

Since the order of r at ∞ is $4>2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{2x^2-4x-1}{4\left(x^3+x^2+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-\frac{1}{2}+\frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2}+\frac{\sqrt{-2-2i\sqrt{3}}}{4}$	$\frac{1}{2}-\frac{\sqrt{-2-2i\sqrt{3}}}{4}$
$-\frac{1}{2}-\frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2}+\frac{\sqrt{-2+2i\sqrt{3}}}{4}$	$\frac{1}{2}-\frac{\sqrt{-2+2i\sqrt{3}}}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= 1-(1) \\
&= 0
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + \frac{\frac{1}{2} - \frac{\sqrt{-2-2i\sqrt{3}}}{4}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} - \frac{\sqrt{-2+2i\sqrt{3}}}{4}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}} + (-)(0) \\
&= \frac{1}{2x} + \frac{\frac{1}{2} - \frac{\sqrt{-2-2i\sqrt{3}}}{4}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} - \frac{\sqrt{-2+2i\sqrt{3}}}{4}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}} \\
&= \frac{2x^2+1}{2x\left(x^2+x+1\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{\frac{1}{2} - \frac{\sqrt{-2-2i\sqrt{3}}}{4}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} - \frac{\sqrt{-2+2i\sqrt{3}}}{4}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{\frac{1}{2} - \frac{\sqrt{-2-2i\sqrt{3}}}{4}}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^2} - \frac{\frac{1}{2} - \frac{\sqrt{-2+2i\sqrt{3}}}{4}}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2}\right) -$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} + \frac{\frac{1}{2} - \frac{\sqrt{-2-2i\sqrt{3}}}{4}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} - \frac{\sqrt{-2+2i\sqrt{3}}}{4}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}}\right) dx} \\
&= \sqrt{2}\,\sqrt{x}\,\left(x^2+x+1\right)^{1/4} \mathrm{e}^{-\frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{12x^3+12x^2}{4x^4+4x^3+4x^2}\, dx} \\
&= z_1 e^{-\frac{3\ln\left(x^2+x+1\right)}{4} - \frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{2}}}{\left(x^2+x+1\right)^{3/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}\sqrt{2}\,\sqrt{x}}{\sqrt{x^2+x+1}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{12x^3+12x^2}{4x^4+4x^3+4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\frac{3\ln(x^2+x+1)}{2}-\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}}{(y_1)^2}\,dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{-\frac{3\ln(x^2+x+1)}{2}-\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}(x^2+x+1)\,\mathrm{e}^{2\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}}{2x}dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\mathrm{e}^{-\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}\sqrt{2}\,\sqrt{x}}{\sqrt{x^2+x+1}}\right) \\ &\quad + c_2 \left(\frac{\mathrm{e}^{-\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}\sqrt{2}\,\sqrt{x}}{\sqrt{x^2+x+1}}\left(\int \frac{\mathrm{e}^{-\frac{3\ln(x^2+x+1)}{2}-\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}(x^2+x+1)\,\mathrm{e}^{2\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}}{2x}dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$4x^2(x^2+x+1)\left(\frac{d^2}{dx^2}y(x)\right)+12x^2(x+1)\left(\frac{d}{dx}y(x)\right)+(3x^2+3x+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x)=-\frac{(3x^2+3x+1)y(x)}{4x^2(x^2+x+1)}-\frac{3(x+1)\left(\frac{d}{dx}y(x)\right)}{x^2+x+1}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d^2}{dx^2}y(x)+\frac{3(x+1)\left(\frac{d}{dx}y(x)\right)}{x^2+x+1}+\frac{(3x^2+3x+1)y(x)}{4x^2(x^2+x+1)}=0$
- □ Check to see if x_0 is a regular singular point
  - Define functions
    $\left[P_2(x)=\frac{3(x+1)}{x^2+x+1}, P_3(x)=\frac{3x^2+3x+1}{4x^2(x^2+x+1)}\right]$
  - $x\cdot P_2(x)$ is analytic at $x=0$
    $\left.(x\cdot P_2(x))\right|_{x=0}=0$
  - $x^2\cdot P_3(x)$ is analytic at $x=0$
    $\left.(x^2\cdot P_3(x))\right|_{x=0}=\frac{1}{4}$
  - $x=0$is a regular singular point
    Check to see if x_0 is a regular singular point
    $x_0=0$
- Multiply by denominators

$$4x^2(x^2+x+1)\left(\frac{d^2}{dx^2}y(x)\right)+12x^2(x+1)\left(\frac{d}{dx}y(x)\right)+(3x^2+3x+1)\,y(x)=0$$

- Assume series solution for $y(x)$

$$y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

$$x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$$

- Shift index using $k->k-m$

$$x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$$

- Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=2..3$

$$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..4$

$$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+2r)^2\,x^r+\left(a_1(1+2r)^2+a_0(3+2r)\,(1+2r)\right)x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_k(2k+2r-1)^2+a_{k-1}(2k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+2r)^2=0$$

- Values of r that satisfy the indicial equation

$$r=\frac{1}{2}$$

- Each term must be 0

$$a_1(1+2r)^2+a_0(3+2r)\,(1+2r)=0$$

- Solve for the dependent coefficient(s)

$$a_1=-\frac{(3+2r)a_0}{1+2r}$$

- Each term in the series must be 0, giving the recursion relation

$$4\left(k+r-\frac{1}{2}\right)\left((a_k+a_{k-2}+a_{k-1})\,k+(a_k+a_{k-2}+a_{k-1})\,r-\frac{a_k}{2}-\frac{3a_{k-2}}{2}+\frac{a_{k-1}}{2}\right)=0$$

- Shift index using $k->k+2$

$$4\left(k+\frac{3}{2}+r\right)\left((a_{k+2}+a_k+a_{k+1})\,(k+2)+(a_{k+2}+a_k+a_{k+1})\,r-\frac{a_{k+2}}{2}-\frac{3a_k}{2}+\frac{a_{k+1}}{2}\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{2ka_k+2ka_{k+1}+2ra_k+2ra_{k+1}+a_k+5a_{k+1}}{2k+2r+3}$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+2}=-\frac{2ka_k+2ka_{k+1}+2a_k+6a_{k+1}}{2k+4}$$

- Solution for $r=\frac{1}{2}$

$$\left[y(x)=\sum_{k=0}^{\infty}a_kx^{k+\frac{1}{2}},a_{k+2}=-\frac{2ka_k+2ka_{k+1}+2a_k+6a_{k+1}}{2k+4},a_1=-2a_0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.689 (sec)
Leaf size : 143

```
dsolve(4*x^2*(x^2+x+1)*diff(diff(y(x),x),x)+12*x^2*(x+1)*diff(y(x),x)+(3*x^2+3*x+1)*y(x)
```

$$y = \frac{e^{-\frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{2}}\sqrt{-2x+i\sqrt{3}-1}\,\sqrt{x}\left(c_1\left(\frac{-2ix+\sqrt{3}-i}{\sqrt{3}+2ix+i}\right)^{\frac{1}{4}-\frac{i\sqrt{3}}{4}}+c_2\left(\frac{-2ix+\sqrt{3}-i}{\sqrt{3}+2ix+i}\right)^{\frac{3}{4}+\frac{i\sqrt{3}}{4}}\text{hypergeom}\left(\left[1,\frac{1}{2}+\right.\right.\right.}{(x^2+x+1)^{3/4}}$$

Mathematica DSolve solution

Solving time : 0.467 (sec)
Leaf size : 120

```
DSolve[{4*x^2*(1+x+x^2)*D[y[x],{x,2}]+12*x^2*(1+x)*D[y[x],x]+(1+3*x+3*x^2)*y[x]==0,{}},y[x],
```

$$y(x)\to \exp\left(\int_1^x \frac{2K[1]^2+1}{2K[1]\left(K[1]^2+K[1]+1\right)}dK[1] -\frac{1}{2}\int_1^x \frac{3(K[2]+1)}{K[2]^2+K[2]+1}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{2K[1]^2+1}{2K[1]\left(K[1]^2+K[1]+1\right)}dK[1]\right)dK[3] + c_1\right)$$

2.1.546 Problem 562

Internal problem ID [9718]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 562
Date solved : `Monday, January 27, 2025 at 06:13:28 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(x^2+x+1\right)y''-x\left(-2x^2-4x+1\right)y'+y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.868 (sec)

Writing the ode as

$$x^2\left(x^2+x+1\right)y''+\left(2x^3+4x^2-x\right)y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=x^2\left(x^2+x+1\right)\\ B&=2x^3+4x^2-x \\ C&=1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{10x^2-8x-1}{4\left(x^3+x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=10x^2-8x-1\\ t&=4\left(x^3+x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{10x^2-8x-1}{4\left(x^3+x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1037: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ of order 2. There is a pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{2x} - \frac{1}{4x^2} + \frac{-\frac{29}{24} - \frac{7i\sqrt{3}}{24}}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{29}{24} + \frac{7i\sqrt{3}}{24}}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2} + \frac{\frac{3}{4} - \frac{41i\sqrt{3}}{36}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{3}{4} + \frac{41i\sqrt{3}}{36}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{29}{24} - \frac{7i\sqrt{3}}{24}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{-138-42i\sqrt{3}}}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{-138-42i\sqrt{3}}}{12}
\end{aligned}
$$

For the pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{29}{24} + \frac{7i\sqrt{3}}{24}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{-138+42i\sqrt{3}}}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{-138+42i\sqrt{3}}}{12}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{10x^2 - 8x - 1}{4\left(x^3 + x^2 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-\frac{1}{2} + \frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2} + \frac{\sqrt{-138-42i\sqrt{3}}}{12}$	$\frac{1}{2} - \frac{\sqrt{-138-42i\sqrt{3}}}{12}$
$-\frac{1}{2} - \frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2} + \frac{\sqrt{-138+42i\sqrt{3}}}{12}$	$\frac{1}{2} - \frac{\sqrt{-138+42i\sqrt{3}}}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \frac{\frac{1}{2}-\frac{\sqrt{-138-42i\sqrt{3}}}{12}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2}-\frac{\sqrt{-138+42i\sqrt{3}}}{12}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}} + (-)(0) \\
&= \frac{1}{2x} + \frac{\frac{1}{2}-\frac{\sqrt{-138-42i\sqrt{3}}}{12}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2}-\frac{\sqrt{-138+42i\sqrt{3}}}{12}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}} \\
&= \frac{2x^2-2x+1}{2x\left(x^2+x+1\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{1}{2x}+\frac{\frac{1}{2}-\frac{\sqrt{-138-42i\sqrt{3}}}{12}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}}+\frac{\frac{1}{2}-\frac{\sqrt{-138+42i\sqrt{3}}}{12}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}}\right)(0)+\left(\left(-\frac{1}{2x^2}-\frac{\frac{1}{2}-\frac{\sqrt{-138-42i\sqrt{3}}}{12}}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^2}-\frac{\frac{1}{2}-\frac{\sqrt{-138+42i}}{12}}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right.}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\,dx} \\
&= e^{\int \left(\frac{1}{2x}+\frac{\frac{1}{2}-\frac{\sqrt{-138-42i\sqrt{3}}}{12}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}}+\frac{\frac{1}{2}-\frac{\sqrt{-138+42i\sqrt{3}}}{12}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}}\right)dx} \\
&= \left(x^2+x+1\right)^{1/4}\sqrt{x}\,\sqrt{2}\,e^{-\frac{7\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^3+4x^2-x}{x^2\left(x^2+x+1\right)}\,dx} \\
&= z_1 e^{-\frac{3\ln\left(x^2+x+1\right)}{4}-\frac{7\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}+\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{\sqrt{x}\,e^{-\frac{7\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}}{\left(x^2+x+1\right)^{3/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\frac{x\,e^{-\frac{7\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}\sqrt{2}}{\sqrt{x^2+x+1}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int\frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{2x^3+4x^2-x}{x^2(x^2+x+1)} dx}}{(y_1)^2} dx$$

$$= y_1 \int \frac{e^{-\frac{3\ln(x^2+x+1)}{2} - \frac{7\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3} + \ln(x)}}{(y_1)^2} dx$$

$$= y_1 \left(\int \frac{e^{-\frac{3\ln(x^2+x+1)}{2} - \frac{7\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3} + \ln(x)} (x^2+x+1) e^{\frac{14\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{2x^2} dx \right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\frac{x e^{-\frac{7\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}} \sqrt{2}}{\sqrt{x^2+x+1}} \right)$$

$$+ c_2 \left(\frac{x e^{-\frac{7\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}} \sqrt{2}}{\sqrt{x^2+x+1}} \left(\int \frac{e^{-\frac{3\ln(x^2+x+1)}{2} - \frac{7\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3} + \ln(x)} (x^2+x+1) e^{\frac{14\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{2x^2} dx \right.\right.$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x^2+x+1)\left(\frac{d^2}{dx^2}y(x)\right) - x(-2x^2-4x+1)\left(\frac{d}{dx}y(x)\right) + y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{y(x)}{x^2(x^2+x+1)} - \frac{(2x^2+4x-1)\left(\frac{d}{dx}y(x)\right)}{x(x^2+x+1)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(2x^2+4x-1)\left(\frac{d}{dx}y(x)\right)}{x(x^2+x+1)} + \frac{y(x)}{x^2(x^2+x+1)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{2x^2+4x-1}{x(x^2+x+1)}, P_3(x) = \frac{1}{x^2(x^2+x+1)}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 1$
  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

$$x^2(x^2+x+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(2x^2+4x-1)\left(\frac{d}{dx}y(x)\right)+y(x)=0$$

- Assume series solution for $y(x)$

$$y(x)=\sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

$$x^m \cdot \left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot \left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..4$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+r)^2 x^r+\left(a_1 r^2+a_0 r(3+r)\right)x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_k(k+r-1)^2+a_{k-1}(k+r-1)(k+2+r)\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+r)^2=0$$

- Values of r that satisfy the indicial equation

$$r=1$$

- Each term must be 0

$$a_1 r^2+a_0 r(3+r)=0$$

- Solve for the dependent coefficient(s)

$$a_1=-\frac{(3+r)a_0}{r}$$

- Each term in the series must be 0, giving the recursion relation

$$((a_k+a_{k-2}+a_{k-1})\,k+(a_k+a_{k-2}+a_{k-1})\,r-a_k-2a_{k-2}+2a_{k-1})(k+r-1)=0$$

- Shift index using $k->k+2$

$$((a_{k+2}+a_k+a_{k+1})(k+2)+(a_{k+2}+a_k+a_{k+1})\,r-a_{k+2}-2a_k+2a_{k+1})(k+r+1)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{ka_k+ka_{k+1}+ra_k+ra_{k+1}+4a_{k+1}}{k+r+1}$$

- Recursion relation for $r=1$

$$a_{k+2}=-\frac{ka_k+ka_{k+1}+a_k+5a_{k+1}}{k+2}$$

- Solution for $r=1$

$$\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2}=-\frac{ka_k+ka_{k+1}+a_k+5a_{k+1}}{k+2}, a_1=-4a_0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.248 (sec)
Leaf size : 147

```
dsolve(x^2*(x^2+x+1)*diff(diff(y(x),x),x)-x*(-2*x^2-4*x+1)*diff(y(x),x)+y(x) = 0,y(x),si
```

$$y = \frac{e^{-\frac{7\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}} x \left(c_2 \left(-2x+i\sqrt{3}-1\right)^{-\frac{1}{4}-\frac{7i\sqrt{3}}{12}} \left(2x+i\sqrt{3}+1\right)^{\frac{3}{4}+\frac{7i\sqrt{3}}{12}} \operatorname{hypergeom}\left(\left[1, \frac{1}{2}+\frac{7i\sqrt{3}}{6}\right], \left[\frac{3}{2}+ \right.\right.\right.}{\left(x^2+x+1\right)^{3/4}}$$

Mathematica DSolve solution

Solving time : 0.552 (sec)
Leaf size : 130

```
DSolve[{x^2*(1+x+x^2)*D[y[x],{x,2}]-x*(1-4*x-2*x^2)*D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSing
```

$$y(x) \to \exp\left(\int_1^x \frac{1}{2}\left(\frac{K[1]-3}{K[1]^2+K[1]+1}+\frac{1}{K[1]}\right)dK[1]-\frac{1}{2}\int_1^x\left(\frac{3K[2]+5}{K[2]^2+K[2]+1}-\frac{1}{K[2]}\right)dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]}\frac{2K[1]^2-2K[1]+1}{2K[1]\left(K[1]^2+K[1]+1\right)}dK[1]\right)dK[3]+c_1\right)$$

2.1.547 Problem 563

Internal problem ID [9719]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 563
Date solved : `Monday, January 27, 2025 at 06:13:29 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$9x^2y'' + 3x\left(-2x^2+3x+5\right)y' + \left(-14x^2+12x+1\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.484 (sec)

Writing the ode as

$$9x^2y'' + \left(-6x^3+9x^2+15x\right)y' + \left(-14x^2+12x+1\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^2 \\ B &= -6x^3+9x^2+15x \\ C &= -14x^2+12x+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^4 - 12x^3 + 33x^2 - 18x - 9}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^4 - 12x^3 + 33x^2 - 18x - 9 \\ t &= 36x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4 - 12x^3 + 33x^2 - 18x - 9}{36x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1039: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{9} - \frac{x}{3} + \frac{11}{12} - \frac{1}{4x^2} - \frac{1}{2x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{3} - \frac{1}{2} + \frac{1}{x} + \frac{3}{4x^2} - \frac{3}{4x^3} - \frac{27}{8x^4} - \frac{117}{32x^5} + \frac{405}{64x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{3}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= -\frac{1}{2} + \frac{x}{3} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4} - \frac{1}{3}x + \frac{1}{9}x^2$$

This shows that the coefficient of 1 in the above is $\frac{1}{4}$. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^4 - 12x^3 + 33x^2 - 18x - 9}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{1}{9}x^2 - \frac{1}{3}x + \frac{11}{12}\right) + \left(\frac{-18x - 9}{36x^2}\right) \\ &= \frac{x^2}{9} - \frac{x}{3} + \frac{11}{12} + \frac{-18x - 9}{36x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is $\frac{11}{12}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{11}{12}\right) - \left(\frac{1}{4}\right) \\ &= \frac{2}{3} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= -\frac{1}{2} + \frac{x}{3} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{2}{3}}{\frac{1}{3}} - 1\right) &= \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{2}{3}}{\frac{1}{3}} - 1\right) &= -\frac{3}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^4 - 12x^3 + 33x^2 - 18x - 9}{36x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$-\frac{1}{2}+\frac{x}{3}$	$\frac{1}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^+) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + \left(-\frac{1}{2} + \frac{x}{3}\right) \\ &= \frac{1}{2x} - \frac{1}{2} + \frac{x}{3} \\ &= \frac{1}{2x} - \frac{1}{2} + \frac{x}{3} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{2} + \frac{x}{3}\right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{1}{3}\right) + \left(\frac{1}{2x} - \frac{1}{2} + \frac{x}{3}\right)^2 - \left(\frac{4x^4 - 12x^3 + 33x^2 - 18x - 9}{36x^2}\right)\right) = 0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{1}{2x} - \frac{1}{2} + \frac{x}{3}\right) dx} \\ &= \sqrt{x}\, e^{\frac{x(x-3)}{6}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-6x^3+9x^2+15x}{9x^2}\,dx} \\
&= z_1 e^{\frac{x^2}{6}-\frac{x}{2}-\frac{5\ln(x)}{6}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{x(x-3)}{6}}}{x^{5/6}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{\frac{x(x-3)}{3}}}{x^{1/3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-6x^3+9x^2+15x}{9x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{x^2}{3}-x-\frac{5\ln(x)}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \mathrm{e}^{\frac{x^2}{3}-x-\frac{5\ln(x)}{3}} x^{2/3} \mathrm{e}^{-\frac{2x(x-3)}{3}}\,dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{\frac{x(x-3)}{3}}}{x^{1/3}}\right) + c_2 \left(\frac{\mathrm{e}^{\frac{x(x-3)}{3}}}{x^{1/3}} \left(\int \mathrm{e}^{\frac{x^2}{3}-x-\frac{5\ln(x)}{3}} x^{2/3} \mathrm{e}^{-\frac{2x(x-3)}{3}}\,dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$9x^2\left(\frac{d^2}{dx^2}y(x)\right) + 3x(-2x^2+3x+5)\left(\frac{d}{dx}y(x)\right) + (-14x^2+12x+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{(14x^2-12x-1)y(x)}{9x^2} + \frac{(2x^2-3x-5)\left(\frac{d}{dx}y(x)\right)}{3x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) - \frac{(2x^2-3x-5)\left(\frac{d}{dx}y(x)\right)}{3x} - \frac{(14x^2-12x-1)y(x)}{9x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - ○ Define functions

$\left[P_2(x) = -\frac{2x^2-3x-5}{3x}, P_3(x) = -\frac{14x^2-12x-1}{9x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = \frac{5}{3}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{1}{9}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $9x^2\left(\frac{d^2}{dx^2}y(x)\right) - 3x(2x^2 - 3x - 5)\left(\frac{d}{dx}y(x)\right) + (-14x^2 + 12x + 1)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+3r)^2\,x^r + \left(a_1(4+3r)^2 + 3a_0(4+3r)\right)x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r+1)^2 + 3a_{k-1}(3k+3r+1\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+3r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = -\frac{1}{3}$

- Each term must be 0

  $a_1(4+3r)^2 + 3a_0(4+3r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = -\frac{3a_0}{4+3r}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(3k+3r+1)^2 + (3k+3r+1)\,(-2a_{k-2} + 3a_{k-1}) = 0$

- Shift index using $k->k+2$

  $a_{k+2}(3k+3r+7)^2 + (3k+3r+7)\,(-2a_k + 3a_{k+1}) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+2} = \frac{2a_k - 3a_{k+1}}{3k+3r+7}$

- Recursion relation for $r = -\frac{1}{3}$

$a_{k+2} = \frac{2a_k - 3a_{k+1}}{3k+6}$

- Solution for $r = -\frac{1}{3}$

$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}, a_{k+2} = \frac{2a_k - 3a_{k+1}}{3k+6}, a_1 = -a_0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Mo
      <- Heun successful: received ODE is equivalent to the  HeunB  ODE, case  c = 0
   Special function solution also has integrals. Returning default Liouvillian solution.
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 2.159 (sec)
Leaf size : 32

```
dsolve(9*x^2*diff(diff(y(x),x),x)+3*x*(-2*x^2+3*x+5)*diff(y(x),x)+(-14*x^2+12*x+1)*y(x)
```

$$y = \frac{e^{\frac{x(x-3)}{3}} \left(\left(\int \frac{e^{-\frac{x(x-3)}{3}}}{x} dx\right) c_2 + c_1\right)}{x^{1/3}}$$

Mathematica DSolve solution

Solving time : 0.32 (sec)
Leaf size : 52

```
DSolve[{9*x^2*D[y[x],{x,2}]+3*x*(5+3*x-2*x^2)*D[y[x],x]+(1+12*x-14*x^2)*y[x]==0,{}},y[x],x,I
```

$$y(x) \to \frac{e^{\frac{1}{3}(x-3)x}\left(c_2\int_1^x \frac{e^{K[1]-\frac{K[1]^2}{3}}}{K[1]}dK[1]+c_1\right)}{\sqrt[3]{x}}$$

2.1.548 Problem 564

Internal problem ID [9720]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 564
Date solved : `Monday, January 27, 2025 at 06:13:30 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+2x)\,y''+x\left(3x^2+14x+5\right)y'+\left(12x^2+18x+4\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.539 (sec)

Writing the ode as

$$\left(2x^3+x^2\right)y''+\left(3x^3+14x^2+5x\right)y'+\left(12x^2+18x+4\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^3+x^2\\ B &= 3x^3+14x^2+5x \\ C &= 12x^2+18x+4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{9x^4-12x^3-16x^2-4x-1}{4\left(2x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9x^4-12x^3-16x^2-4x-1 \\ t &= 4\left(2x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{9x^4-12x^3-16x^2-4x-1}{4\left(2x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1041: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{16} - \frac{15}{64\left(x + \frac{1}{2}\right)^2} - \frac{21}{16\left(x + \frac{1}{2}\right)} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x + \frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{15}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{3}{8} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3}{4} - \frac{7}{8x} - \frac{19}{48x^2} - \frac{151}{288x^3} - \frac{139}{192x^4} - \frac{11383}{10368x^5} - \frac{38729}{20736x^6} - \frac{1212655}{373248x^7} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{3}{4} \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9}{16}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{9x^4 - 12x^3 - 16x^2 - 4x - 1}{16x^4 + 16x^3 + 4x^2} \\
&= Q + \frac{R}{16x^4 + 16x^3 + 4x^2} \\
&= \left(\frac{9}{16}\right) + \left(\frac{-21x^3 - \frac{73}{4}x^2 - 4x - 1}{16x^4 + 16x^3 + 4x^2}\right) \\
&= \frac{9}{16} + \frac{-21x^3 - \frac{73}{4}x^2 - 4x - 1}{16x^4 + 16x^3 + 4x^2}
\end{aligned}$$

Since the degree of t is 4, then we see that the coefficient of the term x^3 in the remainder R is -21. Dividing this by leading coefficient in t which is 16 gives $-\frac{21}{16}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{21}{16}\right) - (0) \\
&= -\frac{21}{16}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{3}{4} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{21}{16}}{\frac{3}{4}} - 0\right) &&= -\frac{7}{8} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{21}{16}}{\frac{3}{4}} - 0\right) &&= \frac{7}{8}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{9x^4 - 12x^3 - 16x^2 - 4x - 1}{4\left(2x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-\frac{1}{2}$	2	0	$\frac{5}{8}$	$\frac{3}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{3}{4}$	$-\frac{7}{8}$	$\frac{7}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{7}{8}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{-}\right) \\
&= \frac{7}{8} - \left(\frac{7}{8}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \frac{3}{8\left(x + \frac{1}{2}\right)} + (-)\left(\frac{3}{4}\right) \\
&= \frac{1}{2x} + \frac{3}{8\left(x + \frac{1}{2}\right)} - \frac{3}{4} \\
&= \frac{-3x^2 + 2x + 1}{4x^2 + 2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{1}{2x}+\frac{3}{8\left(x+\frac{1}{2}\right)}-\frac{3}{4}\right)(0)+\left(\left(-\frac{1}{2x^2}-\frac{3}{8\left(x+\frac{1}{2}\right)^2}\right)+\left(\frac{1}{2x}+\frac{3}{8\left(x+\frac{1}{2}\right)}-\frac{3}{4}\right)^2-\left(\frac{9x^4-12x^3-}{4(2x}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\,dx} \\
&= \mathrm{e}^{\int\left(\frac{1}{2x}+\frac{3}{8\left(x+\frac{1}{2}\right)}-\frac{3}{4}\right)dx} \\
&= \sqrt{x}\,(1+2x)^{3/8}\,\mathrm{e}^{-\frac{3x}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x^3+14x^2+5x}{2x^3+x^2}\,dx} \\
&= z_1 e^{-\frac{3x}{4}-\frac{5\ln(x)}{2}-\frac{5\ln(1+2x)}{8}} \\
&= z_1\left(\frac{\mathrm{e}^{-\frac{3x}{4}}}{x^{5/2}\,(1+2x)^{5/8}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-\frac{3x}{2}}}{x^2\,(1+2x)^{1/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^3+14x^2+5x}{2x^3+x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{3x}{2}-5\ln(x)-\frac{5\ln(1+2x)}{4}}}{(y_1)^2}\,dx \\
&= y_1\left(\int \mathrm{e}^{-\frac{3x}{2}-5\ln(x)-\frac{5\ln(1+2x)}{4}}\,x^4\sqrt{1+2x}\,\mathrm{e}^{3x}\,dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\mathrm{e}^{-\frac{3x}{2}}}{x^2\,(1+2x)^{1/4}}\right)+c_2\left(\frac{\mathrm{e}^{-\frac{3x}{2}}}{x^2\,(1+2x)^{1/4}}\left(\int \mathrm{e}^{-\frac{3x}{2}-5\ln(x)-\frac{5\ln(1+2x)}{4}}\,x^4\sqrt{1+2x}\,\mathrm{e}^{3x}\,dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(2x+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(3x^2+14x+5)\left(\frac{d}{dx}y(x)\right)+(12x^2+18x+4)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{2(6x^2+9x+2)y(x)}{x^2(2x+1)}-\frac{(3x^2+14x+5)\left(\frac{d}{dx}y(x)\right)}{x(2x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)+\frac{(3x^2+14x+5)\left(\frac{d}{dx}y(x)\right)}{x(2x+1)}+\frac{2(6x^2+9x+2)y(x)}{x^2(2x+1)}=0$

☐ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=\frac{3x^2+14x+5}{x(2x+1)},P_3(x)=\frac{2(6x^2+9x+2)}{x^2(2x+1)}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

$\left.(x\cdot P_2(x))\right|_{x=0}=5$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left.(x^2\cdot P_3(x))\right|_{x=0}=4$

- ◦ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$x^2(2x+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(3x^2+14x+5)\left(\frac{d}{dx}y(x)\right)+(12x^2+18x+4)\,y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

$x^m\cdot y(x)=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

- ◦ Shift index using $k->k-m$

$x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ◦ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..3$

$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- ◦ Shift index using $k->k+2-m$

$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

Rewrite ODE with series expansions

$$a_0(2+r)^2 x^r + \left(a_1(3+r)^2 + 2a_0(3+r)^2\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+2)^2 + 2a_{k-1}(k+r+2)^2 + 3a_{k-2}\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
$$(2+r)^2 = 0$$
- Values of r that satisfy the indicial equation
$$r = -2$$
- Each term must be 0
$$a_1(3+r)^2 + 2a_0(3+r)^2 = 0$$
- Solve for the dependent coefficient(s)
$$a_1 = -2a_0$$
- Each term in the series must be 0, giving the recursion relation
$$((2k+2r+4)\, a_{k-1} + a_k(k+r+2) + 3a_{k-2})\,(k+r+2) = 0$$
- Shift index using $k->k+2$
$$((2k+8+2r)\, a_{k+1} + a_{k+2}(k+r+4) + 3a_k)\,(k+r+4) = 0$$
- Recursion relation that defines series solution to ODE
$$a_{k+2} = -\frac{2ka_{k+1} + 2ra_{k+1} + 3a_k + 8a_{k+1}}{k+r+4}$$
- Recursion relation for $r = -2$
$$a_{k+2} = -\frac{2ka_{k+1} + 3a_k + 4a_{k+1}}{k+2}$$
- Solution for $r = -2$
$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{2ka_{k+1} + 3a_k + 4a_{k+1}}{k+2}, a_1 = -2a_0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Mo
      <- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.407 (sec)
Leaf size : 53

```
dsolve(x^2*(2*x+1)*diff(diff(y(x),x),x)+x*(3*x^2+14*x+5)*diff(y(x),x)+(12*x^2+18*x+4)*
```

$$y = \frac{e^{-\frac{3x}{2}}\left((2x+1)^{1/4}\operatorname{HeunC}\left(-\frac{3}{4},\frac{1}{4},0,\frac{21}{32},-\frac{5}{32},2x+1\right)c_2+\operatorname{HeunC}\left(-\frac{3}{4},-\frac{1}{4},0,\frac{21}{32},-\frac{5}{32},2x+1\right)c_1\right)}{(2x+1)^{1/4}\,x^2}$$

Mathematica DSolve solution

Solving time : 0.328 (sec)
Leaf size : 120

```
DSolve[{x^2*(1+2*x)*D[y[x],{x,2}]+x*(5+14*x+3*x^2)*D[y[x],x]+(4+18*x+12*x^2)*y[x]==0,{}},y[x
```

$$y(x) \to \exp\left(\int_1^x \frac{1}{4}\left(\frac{3}{2K[1]+1}-3+\frac{2}{K[1]}\right)dK[1]-\frac{1}{2}\int_1^x\left(\frac{5}{4K[2]+2}+\frac{3}{2}+\frac{5}{K[2]}\right)dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]}\frac{-3K[1]^2+2K[1]+1}{4K[1]^2+2K[1]}dK[1]\right)dK[3]+c_1\right)$$

2.1.549 Problem 565

Internal problem ID [9721]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 565
Date solved : `Monday, January 27, 2025 at 06:13:31 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$16x^2y'' + 4x\left(2x^2+x+6\right)y' + \left(18x^2+5x+1\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.460 (sec)

Writing the ode as

$$16x^2y'' + \left(8x^3+4x^2+24x\right)y' + \left(18x^2+5x+1\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 16x^2 \\ B &= 8x^3+4x^2+24x \\ C &= 18x^2+5x+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^4+4x^3-31x^2-8x-16}{64x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^4+4x^3-31x^2-8x-16 \\ t &= 64x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4+4x^3-31x^2-8x-16}{64x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1043: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{16} + \frac{x}{16} - \frac{31}{64} - \frac{1}{4x^2} - \frac{1}{8x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{4} + \frac{1}{8} - \frac{1}{x} + \frac{1}{4x^2} - \frac{21}{8x^3} + \frac{37}{16x^4} - \frac{377}{32x^5} + \frac{1137}{64x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{1}{8} + \frac{x}{4} \end{aligned}\tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{64} + \frac{1}{16}x + \frac{1}{16}x^2$$

This shows that the coefficient of 1 in the above is $\frac{1}{64}$. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^4 + 4x^3 - 31x^2 - 8x - 16}{64x^2} \\ &= Q + \frac{R}{64x^2} \\ &= \left(\frac{1}{16}x^2 + \frac{1}{16}x - \frac{31}{64}\right) + \left(\frac{-8x - 16}{64x^2}\right) \\ &= \frac{x^2}{16} + \frac{x}{16} - \frac{31}{64} + \frac{-8x - 16}{64x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is $-\frac{31}{64}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{31}{64}\right) - \left(\frac{1}{64}\right) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{8} + \frac{x}{4} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{4}} - 1\right) &&= -\frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{4}} - 1\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^4 + 4x^3 - 31x^2 - 8x - 16}{64x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{1}{8}+\frac{x}{4}$	$-\frac{3}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\left(\frac{1}{8} + \frac{x}{4}\right) \\ &= \frac{1}{2x} - \frac{1}{8} - \frac{x}{4} \\ &= \frac{1}{2x} - \frac{1}{8} - \frac{x}{4} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{8} - \frac{x}{4}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{4}\right) + \left(\frac{1}{2x} - \frac{1}{8} - \frac{x}{4}\right)^2 - \left(\frac{4x^4 + 4x^3 - 31x^2 - 8x - 16}{64x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{1}{2x} - \frac{1}{8} - \frac{x}{4}\right) dx} \\ &= \sqrt{x}\, e^{-\frac{x(x+1)}{8}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{8x^3+4x^2+24x}{16x^2}\,dx} \\
&= z_1 e^{-\frac{x^2}{8}-\frac{x}{8}-\frac{3\ln(x)}{4}} \\
&= z_1\left(\frac{e^{-\frac{x(x+1)}{8}}}{x^{3/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{e^{-\frac{x(x+1)}{4}}}{x^{1/4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{8x^3+4x^2+24x}{16x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{4}-\frac{x}{4}-\frac{3\ln(x)}{2}}}{(y_1)^2}\,dx \\
&= y_1\left(\int e^{-\frac{x^2}{4}-\frac{x}{4}-\frac{3\ln(x)}{2}}\sqrt{x}\,e^{\frac{x(x+1)}{2}}\,dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{e^{-\frac{x(x+1)}{4}}}{x^{1/4}}\right) + c_2\left(\frac{e^{-\frac{x(x+1)}{4}}}{x^{1/4}}\left(\int e^{-\frac{x^2}{4}-\frac{x}{4}-\frac{3\ln(x)}{2}}\sqrt{x}\,e^{\frac{x(x+1)}{2}}\,dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$16x^2\left(\frac{d^2}{dx^2}y(x)\right)+4x(2x^2+x+6)\left(\frac{d}{dx}y(x)\right)+(18x^2+5x+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(18x^2+5x+1)y(x)}{16x^2}-\frac{(2x^2+x+6)\left(\frac{d}{dx}y(x)\right)}{4x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{(2x^2+x+6)\left(\frac{d}{dx}y(x)\right)}{4x}+\frac{(18x^2+5x+1)y(x)}{16x^2}=0$

- ☐ Check to see if $x_0=0$ is a regular singular point
  - ◦ Define functions

$\left[P_2(x) = \frac{2x^2+x+6}{4x}, P_3(x) = \frac{18x^2+5x+1}{16x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{3}{2}$
- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{1}{16}$
- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $16x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x(2x^2 + x + 6)\left(\frac{d}{dx}y(x)\right) + (18x^2 + 5x + 1)\, y(x) = 0$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r-1+m}$
- Shift index using $k- >k + 1 - m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, x^{k+r}$
- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1 + 4r)^2\, x^r + \left(a_1(5 + 4r)^2 + a_0(5 + 4r)\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(4k + 4r + 1)^2 + a_{k-1}(4k + 4r + 1) \right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1 + 4r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = -\frac{1}{4}$
- Each term must be 0

  $a_1(5 + 4r)^2 + a_0(5 + 4r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = -\frac{a_0}{5+4r}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k + 4r + 1)^2 + (4k + 4r + 1)\,(2a_{k-2} + a_{k-1}) = 0$
- Shift index using $k- >k + 2$

  $a_{k+2}(4k + 4r + 9)^2 + (4k + 4r + 9)\,(2a_k + a_{k+1}) = 0$
- Recursion relation that defines series solution to ODE

$a_{k+2} = -\frac{2a_k+a_{k+1}}{4k+4r+9}$

- Recursion relation for $r = -\frac{1}{4}$

$a_{k+2} = -\frac{2a_k+a_{k+1}}{4k+8}$

- Solution for $r = -\frac{1}{4}$

$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{4}}, a_{k+2} = -\frac{2a_k+a_{k+1}}{4k+8}, a_1 = -\frac{a_0}{4}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Mo
      <- Heun successful: received ODE is equivalent to the  HeunB  ODE, case  c = 0
   Special function solution also has integrals. Returning default Liouvillian solution.
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.421 (sec)
Leaf size : 32

```
dsolve(16*x^2*diff(diff(y(x),x),x)+4*x*(2*x^2+x+6)*diff(y(x),x)+(18*x^2+5*x+1)*y(x) = 0,
```

$$y = \frac{e^{-\frac{x(x+1)}{4}}\left(\left(\int \frac{e^{\frac{x(x+1)}{4}}}{x}dx\right)c_2 + c_1\right)}{x^{1/4}}$$

Mathematica DSolve solution

Solving time : 0.413 (sec)
Leaf size : 57

```
DSolve[{16*x^2*D[y[x],{x,2}]+4*x*(6+x+2*x^2)*D[y[x],x]+(1+5*x+18*x^2)*y[x]==0,{}},y[x],x,Inc
```

$$y(x) \to \frac{e^{\frac{1}{4}(-x^2-x-3)}\left(c_2 \int_1^x \frac{e^{\frac{1}{4}K[1](K[1]+1)}}{K[1]} dK[1] + c_1\right)}{\sqrt[4]{x}}$$

2.1.550 Problem 566

Internal problem ID [9722]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 566
Date solved : `Monday, January 27, 2025 at 06:13:32 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$9x^2(1+x)\,y''+3x\left(-x^2+11x+5\right)y'+\left(-7x^2+16x+1\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.342 (sec)

Writing the ode as

$$\left(9x^3+9x^2\right)y''+\left(-3x^3+33x^2+15x\right)y'+\left(-7x^2+16x+1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^3+9x^2 \\ B &= -3x^3+33x^2+15x \\ C &= -7x^2+16x+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{x^4+6x^3+3x^2-18x-9}{36\left(x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4+6x^3+3x^2-18x-9 \\ t &= 36\left(x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x^4+6x^3+3x^2-18x-9}{36\left(x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1045: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{36} - \frac{1}{4x^2} + \frac{1}{9 + 9x} + \frac{7}{36\,(1 + x)^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{6} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{6} + \frac{1}{3x} - \frac{5}{6x^2} + \frac{5}{6x^3} - \frac{7}{3x^4} + \frac{41}{6x^5} - \frac{149}{6x^6} + \frac{277}{3x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{6} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{36}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^4 + 6x^3 + 3x^2 - 18x - 9}{36x^4 + 72x^3 + 36x^2} \\ &= Q + \frac{R}{36x^4 + 72x^3 + 36x^2} \\ &= \left(\frac{1}{36}\right) + \left(\frac{4x^3 + 2x^2 - 18x - 9}{36x^4 + 72x^3 + 36x^2}\right) \\ &= \frac{1}{36} + \frac{4x^3 + 2x^2 - 18x - 9}{36x^4 + 72x^3 + 36x^2} \end{aligned}$$

Since the degree of t is 4, then we see that the coefficient of the term x^3 in the remainder R is 4. Dividing this by leading coefficient in t which is 36 gives $\frac{1}{9}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{9}\right) - (0) \\ &= \frac{1}{9} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{6}\\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a}-v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{9}}{\frac{1}{6}}-0\right) &&= \frac{1}{3}\\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a}-v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{9}}{\frac{1}{6}}-0\right) &&= -\frac{1}{3}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^4+6x^3+3x^2-18x-9}{36\left(x^2+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-1	2	0	$\frac{7}{6}$	$-\frac{1}{6}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{1}{3}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right)\\
&= \frac{1}{3} - \left(\frac{1}{3}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (+)[\sqrt{r}]_\infty\\
&= -\frac{1}{6\left(1+x\right)} + \frac{1}{2x} + \left(\frac{1}{6}\right)\\
&= -\frac{1}{6\left(1+x\right)} + \frac{1}{2x} + \frac{1}{6}\\
&= -\frac{1}{6+6x} + \frac{1}{2x} + \frac{1}{6}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{6\left(1+x\right)}+\frac{1}{2x}+\frac{1}{6}\right)(0)+\left(\left(\frac{1}{6\left(1+x\right)^{2}}-\frac{1}{2x^{2}}\right)+\left(-\frac{1}{6\left(1+x\right)}+\frac{1}{2x}+\frac{1}{6}\right)^{2}-\left(\frac{x^{4}+6x^{3}+3x}{36\left(x^{2}\right.}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\,dx}\\
&= \mathrm{e}^{\int \left(-\frac{1}{6(1+x)}+\frac{1}{2x}+\frac{1}{6}\right)dx}\\
&= \frac{\sqrt{x}\,\mathrm{e}^{\frac{x}{6}}}{\left(1+x\right)^{1/6}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\
&= z_1 e^{-\int \frac{1}{2}\frac{-3x^{3}+33x^{2}+15x}{9x^{3}+9x^{2}}\,dx}\\
&= z_1 e^{\frac{x}{6}-\frac{5\ln(x)}{6}-\frac{7\ln(1+x)}{6}}\\
&= z_1\left(\frac{\mathrm{e}^{\frac{x}{6}}}{x^{5/6}\left(1+x\right)^{7/6}}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\frac{\mathrm{e}^{\frac{x}{3}}}{x^{1/3}\left(1+x\right)^{4/3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^{2}}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1\int \frac{e^{\int -\frac{-3x^{3}+33x^{2}+15x}{9x^{3}+9x^{2}}\,dx}}{\left(y_1\right)^{2}}\,dx\\
&= y_1\int \frac{e^{\frac{x}{3}-\frac{5\ln(x)}{3}-\frac{7\ln(1+x)}{3}}}{\left(y_1\right)^{2}}\,dx\\
&= y_1\left(\int \mathrm{e}^{\frac{x}{3}-\frac{5\ln(x)}{3}-\frac{7\ln(1+x)}{3}}x^{2/3}\left(1+x\right)^{8/3}\mathrm{e}^{-\frac{2x}{3}}\,dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1+c_2 y_2\\
&= c_1\left(\frac{\mathrm{e}^{\frac{x}{3}}}{x^{1/3}\left(1+x\right)^{4/3}}\right)+c_2\left(\frac{\mathrm{e}^{\frac{x}{3}}}{x^{1/3}\left(1+x\right)^{4/3}}\left(\int \mathrm{e}^{\frac{x}{3}-\frac{5\ln(x)}{3}-\frac{7\ln(1+x)}{3}}x^{2/3}\left(1+x\right)^{8/3}\mathrm{e}^{-\frac{2x}{3}}\,dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$9x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)+3x(-x^2+11x+5)\left(\frac{d}{dx}y(x)\right)+(-7x^2+16x+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=\frac{(7x^2-16x-1)y(x)}{9x^2(x+1)}+\frac{(x^2-11x-5)\left(\frac{d}{dx}y(x)\right)}{3x(x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)-\frac{(x^2-11x-5)\left(\frac{d}{dx}y(x)\right)}{3x(x+1)}-\frac{(7x^2-16x-1)y(x)}{9x^2(x+1)}=0$

□ Check to see if x_0 is a regular singular point

- Define functions

$\left[P_2(x)=-\frac{x^2-11x-5}{3x(x+1)}, P_3(x)=-\frac{7x^2-16x-1}{9x^2(x+1)}\right]$

- $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

$\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=\frac{7}{3}$

- $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

$\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

- $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1$

- Multiply by denominators

$9x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)-3x(x^2-11x-5)\left(\frac{d}{dx}y(x)\right)+(-7x^2+16x+1)\,y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(9u^3-18u^2+9u)\left(\frac{d^2}{du^2}y(u)\right)+(-3u^3+42u^2-60u+21)\left(\frac{d}{du}y(u)\right)+(-7u^2+30u-22)\,y(u)=$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m\cdot y(u)$ to series expansion for $m=0..2$

$u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

- Shift index using $k->k-m$

$u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..3$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$3a_0r(4+3r)\,u^{-1+r} + (3a_1(1+r)(7+3r) - 2a_0(9r^2+21r+11))\,u^r + (3a_2(2+r)(10+3r) - 2a_1($$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $3r(4+3r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, -\frac{4}{3}\}$
- The coefficients of each power of u must be 0
  $[3a_1(1+r)(7+3r) - 2a_0(9r^2+21r+11) = 0, 3a_2(2+r)(10+3r) - 2a_1(9r^2+39r+41) + 3a_0(2$
- Solve for the dependent coefficient(s)
  $\left\{a_1 = \frac{2a_0(9r^2+21r+11)}{3(3r^2+10r+7)}, a_2 = \frac{a_0(243r^4+1593r^3+3699r^2+3567r+1174)}{9(9r^4+78r^3+241r^2+312r+140)}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $9(-2a_k + a_{k-1} + a_{k+1})\,k^2 + 3(6(-2a_k + a_{k-1} + a_{k+1})\,r - 14a_k - a_{k-2} + 5a_{k-1} + 10a_{k+1})\,k + 9(-2a$
- Shift index using $k- >k+2$
  $9(-2a_{k+2} + a_{k+1} + a_{k+3})\,(k+2)^2 + 3(6(-2a_{k+2} + a_{k+1} + a_{k+3})\,r - 14a_{k+2} - a_k + 5a_{k+1} + 10a_{k+3})$
- Recursion relation that defines series solution to ODE
  $a_{k+3} = -\frac{9k^2a_{k+1}-18k^2a_{k+2}+18kra_{k+1}-36kra_{k+2}+9r^2a_{k+1}-18r^2a_{k+2}-3ka_k+51ka_{k+1}-114ka_{k+2}-3ra_k+51ra_{k+1}-114ra_{k+2}}{3(3k^2+6kr+3r^2+22k+22r+39)}$
- Recursion relation for $r = 0$
  $a_{k+3} = -\frac{9k^2a_{k+1}-18k^2a_{k+2}-3ka_k+51ka_{k+1}-114ka_{k+2}-7a_k+72a_{k+1}-178a_{k+2}}{3(3k^2+22k+39)}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+3} = -\frac{9k^2a_{k+1}-18k^2a_{k+2}-3ka_k+51ka_{k+1}-114ka_{k+2}-7a_k+72a_{k+1}-178a_{k+2}}{3(3k^2+22k+39)}, a_1 = \frac{22a_0}{21}, a_2 = \right.$
- Revert the change of variables $u = x+1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+3} = -\frac{9k^2a_{k+1}-18k^2a_{k+2}-3ka_k+51ka_{k+1}-114ka_{k+2}-7a_k+72a_{k+1}-178a_{k+2}}{3(3k^2+22k+39)}, a_1 = \frac{22a_0}{21},\right.$
- Recursion relation for $r = -\frac{4}{3}$
  $a_{k+3} = -\frac{9k^2a_{k+1}-18k^2a_{k+2}-3ka_k+27ka_{k+1}-66ka_{k+2}-3a_k+20a_{k+1}-58a_{k+2}}{3(3k^2+14k+15)}$
- Solution for $r = -\frac{4}{3}$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{4}{3}}, a_{k+3} = -\frac{9k^2a_{k+1}-18k^2a_{k+2}-3ka_k+27ka_{k+1}-66ka_{k+2}-3a_k+20a_{k+1}-58a_{k+2}}{3(3k^2+14k+15)}, a_1 = \frac{2a_0}{3}, a_2 = \right.$
- Revert the change of variables $u = x+1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^{k-\frac{4}{3}}, a_{k+3} = -\frac{9k^2a_{k+1}-18k^2a_{k+2}-3ka_k+27ka_{k+1}-66ka_{k+2}-3a_k+20a_{k+1}-58a_{k+2}}{3(3k^2+14k+15)}, a_1 = \frac{2a_0}{3},\right.$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k(x+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k(x+1)^{k-\frac{4}{3}}\right), a_{k+3} = -\frac{9k^2a_{k+1}-18k^2a_{k+2}-3ka_k+51ka_{k+1}-114ka_{k+2}-7a}{3(3k^2+22k+39)}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @
      <- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0,
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 2.119 (sec)
Leaf size : 36

```
dsolve(9*x^2*(x+1)*diff(diff(y(x),x),x)+3*x*(-x^2+11*x+5)*diff(y(x),x)+(-7*x^2+16*x+1)
```

$$y = \frac{\frac{c_1 \operatorname{HeunC}\left(-\frac{1}{3},-\frac{4}{3},0,-\frac{1}{9},\frac{11}{18},x+1\right)}{(x+1)^{4/3}} + c_2 \operatorname{HeunC}\left(-\frac{1}{3},\frac{4}{3},0,-\frac{1}{9},\frac{11}{18},x+1\right)}{x^{1/3}}$$

Mathematica DSolve solution

Solving time : 0.33 (sec)
Leaf size : 120

```
DSolve[{9*x^2*(1+x)*D[y[x],{x,2}]+3*x*(5+11*x-x^2)*D[y[x],x]+(1+16*x-7*x^2)*y[x]==0,{}},y[x]
```

$$y(x) \to \exp\left(\int_1^x \frac{K[1]^2+3K[1]+3}{6K[1]^2+6K[1]}dK[1] - \frac{1}{2}\int_1^x \frac{1}{3}\left(\frac{7}{K[2]+1} - 1 + \frac{5}{K[2]}\right)dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{K[1]^2+3K[1]+3}{6K[1]^2+6K[1]}dK[1]\right)dK[3] + c_1\right)$$

2.1.551 Problem 567

Internal problem ID [9723]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 567
Date solved : `Monday, January 27, 2025 at 06:13:33 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$36x^2(1-2x)\,y''+24x(1-9x)\,y'+(1-70x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.320 (sec)

Writing the ode as

$$\left(-72x^3+36x^2\right)y''+\left(-216x^2+24x\right)y'+(1-70x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-72x^3+36x^2\\B&=-216x^2+24x\\C&=1-70x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-32x^2+48x-9}{36\left(2x^2-x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=-32x^2+48x-9\\t&=36\left(2x^2-x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-32x^2+48x-9}{36\left(2x^2-x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1047: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(2x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{1}{3x} + \frac{7}{36\left(x - \frac{1}{2}\right)^2} - \frac{1}{3\left(x - \frac{1}{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{6} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-32x^2 + 48x - 9}{36\left(2x^2 - x\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{2}{9}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{2}{3} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{3}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-32x^2 + 48x - 9}{36\left(2x^2 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{2}{3}$	$\frac{1}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{3}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= \frac{1}{3} - \left(\frac{1}{3}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} - \frac{1}{6\left(x - \frac{1}{2}\right)} + (-)(0) \\
&= \frac{1}{2x} - \frac{1}{6\left(x - \frac{1}{2}\right)} \\
&= \frac{-3 + 4x}{12x^2 - 6x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{6\left(x - \frac{1}{2}\right)}\right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{1}{6\left(x - \frac{1}{2}\right)^2}\right) + \left(\frac{1}{2x} - \frac{1}{6\left(x - \frac{1}{2}\right)}\right)^2 - \left(\frac{-32x^2 + 48x - 9}{36\left(2x^2 - x\right)^2}\right)\right.$$

$$0 =$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{1}{2x} - \frac{1}{6\left(x - \frac{1}{2}\right)}\right) dx} \\ &= \frac{\sqrt{x}}{(-1 + 2x)^{1/6}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-216x^2 + 24x}{-72x^3 + 36x^2} \, dx} \\ &= z_1 e^{-\frac{\ln(x)}{3} - \frac{7 \ln(-1+2x)}{6}} \\ &= z_1 \left(\frac{1}{x^{1/3} (-1 + 2x)^{7/6}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/6}}{(-1 + 2x)^{4/3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-216x^2 + 24x}{-72x^3 + 36x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\frac{2 \ln(x)}{3} - \frac{7 \ln(-1+2x)}{3}}}{(y_1)^2} \, dx \\ &= y_1 \left(3(-1 + 2x)^{1/3} + \frac{\ln\left((-1 + 2x)^{2/3} - (-1 + 2x)^{1/3} + 1\right)}{2} \right. \\ &\quad \left. - \sqrt{3} \arctan\left(\frac{\left(-1 + 2(-1 + 2x)^{1/3}\right)\sqrt{3}}{3}\right) - \ln\left((-1 + 2x)^{1/3} + 1\right)\right) \end{aligned}$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{x^{1/6}}{(-1+2x)^{4/3}}\right)$$
$$+c_2 \left(\frac{x^{1/6}}{(-1+2x)^{4/3}} \left(3(-1+2x)^{1/3} + \frac{\ln\left((-1+2x)^{2/3} - (-1+2x)^{1/3} + 1\right)}{2} - \sqrt{3}\arctan\left(\frac{(-1+2(-1-}{3}\right.\right.\right.$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$36x^2(-2x+1)\left(\frac{d^2}{dx^2}y(x)\right) + 24x(1-9x)\left(\frac{d}{dx}y(x)\right) + (1-70x)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(-1+70x)y(x)}{36x^2(2x-1)} - \frac{2(-1+9x)\left(\frac{d}{dx}y(x)\right)}{3x(2x-1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{2(-1+9x)\left(\frac{d}{dx}y(x)\right)}{3x(2x-1)} + \frac{(-1+70x)y(x)}{36x^2(2x-1)} = 0$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{2(-1+9x)}{3x(2x-1)}, P_3(x) = \frac{-1+70x}{36x^2(2x-1)}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{2}{3}$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{1}{36}$
  - $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $36x^2(2x-1)\left(\frac{d^2}{dx^2}y(x)\right) + 24x(-1+9x)\left(\frac{d}{dx}y(x)\right) + (-1+70x)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k - m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

$$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..3$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0(-1+6r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(6k+6r-1)^2 + 2a_{k-1}(6k+1+6r)\,(6k+6r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-(-1+6r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = \frac{1}{6}$
- Each term in the series must be 0, giving the recursion relation

  $-36\left(\left(-2k-2r-\frac{1}{3}\right)a_{k-1} + a_k\left(k+r-\frac{1}{6}\right)\right)\left(k+r-\frac{1}{6}\right) = 0$
- Shift index using $k->k+1$

  $-36\left(\left(-2k-\frac{7}{3}-2r\right)a_k + a_{k+1}\left(k+\frac{5}{6}+r\right)\right)\left(k+\frac{5}{6}+r\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2(6k+6r+7)a_k}{6k+6r+5}$
- Recursion relation for $r = \frac{1}{6}$

  $a_{k+1} = \frac{2(6k+8)a_k}{6k+6}$
- Solution for $r = \frac{1}{6}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{6}}, a_{k+1} = \frac{2(6k+8)a_k}{6k+6}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
```

```
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
         -> solution has integrals; searching for one without integrals...
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
            <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
         <- hypergeometric solution without integrals succesful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returnin
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.153 (sec)
Leaf size : 93

```
dsolve(36*x^2*(1-2*x)*diff(diff(y(x),x),x)+24*x*(1-9*x)*diff(y(x),x)+(1-70*x)*y(x) = 0,y
```

$$y = \frac{x^{1/6}\left(2\sqrt{3}\arctan\left(\frac{\sqrt{3}\,(-1+2x)^{1/3}}{-2+(-1+2x)^{1/3}}\right)c_2 - 2\ln\left(1+(-1+2x)^{1/3}\right)c_2 + \ln\left(1-(-1+2x)^{1/3}+(-1+2x)^{2/3}\right)\right.}{3\,(-1+2x)^{4/3}}$$

Mathematica DSolve solution

Solving time : 0.27 (sec)
Leaf size : 112

```
DSolve[{36*x^2*(1-2*x)*D[y[x],{x,2}]+24*x*(1-9*x)*D[y[x],x]+(1-70*x)*y[x]==0,{}},y[x],x,Includ
```

$$y(x) \to \exp\left(\int_1^x \frac{3-4K[1]}{6K[1]-12K[1]^2}dK[1] - \frac{1}{2}\int_1^x \frac{2-18K[2]}{3K[2]-6K[2]^2}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{3-4K[1]}{6K[1]-12K[1]^2}dK[1]\right)dK[3] + c_1\right)$$

2.1.552 Problem 568

Internal problem ID [9724]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 568
Date solved : `Monday, January 27, 2025 at 06:13:33 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+x)\,y''-x(3-x)\,y'+4y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.231 (sec)

Writing the ode as

$$x^2(1+x)\,y''+\left(x^2-3x\right)y'+4y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=x^2(1+x)\\ B&=x^2-3x \\ C&=4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-x^2-10x-1}{4\left(x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=-x^2-10x-1\\ t&=4\left(x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-x^2-10x-1}{4\left(x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1049: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{1+x} - \frac{1}{4x^2} - \frac{2}{x} + \frac{2}{(1+x)^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 10x - 1}{4\left(x^2 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 - 10x - 1}{4\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	2	-1
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right)\\ &= \frac{1}{2} - \left(-\frac{1}{2}\right)\\ &= 1\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (-)[\sqrt{r}]_\infty\\ &= -\frac{1}{1+x} + \frac{1}{2x} + (-)\,(0)\\ &= -\frac{1}{1+x} + \frac{1}{2x}\\ &= -\frac{x-1}{2x\,(1+x)}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{1+x} + \frac{1}{2x}\right)(1) + \left(\left(\frac{1}{(1+x)^2} - \frac{1}{2x^2}\right) + \left(-\frac{1}{1+x} + \frac{1}{2x}\right)^2 - \left(\frac{-x^2-10x-1}{4\left(x^2+x\right)^2}\right)\right) = 0$$

$$\frac{1+a_0}{x\left(1+x\right)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x-1)\, \mathrm{e}^{\int \left(-\frac{1}{1+x} + \frac{1}{2x}\right) dx} \\
&= (x-1)\, \mathrm{e}^{\frac{\ln(x)}{2} - \ln(1+x)} \\
&= \frac{(x-1)\sqrt{x}}{1+x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2-3x}{x^2(1+x)} \, dx} \\
&= z_1 e^{\frac{3\ln(x)}{2} - 2\ln(1+x)} \\
&= z_1 \left(\frac{x^{3/2}}{(1+x)^2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2(x-1)}{(1+x)^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2-3x}{x^2(1+x)} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{3\ln(x) - 4\ln(1+x)}}{(y_1)^2} \, dx \\
&= y_1 \left(\ln(x) - \frac{4}{x-1}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^2(x-1)}{(1+x)^3} \right) + c_2 \left(\frac{x^2(x-1)}{(1+x)^3} \left(\ln(x) - \frac{4}{x-1} \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right) - x(-x+3)\left(\frac{d}{dx}y(x)\right) + 4y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{4y(x)}{(x+1)x^2} - \frac{(x-3)\left(\frac{d}{dx}y(x)\right)}{x(x+1)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x) + \frac{(x-3)\left(\frac{d}{dx}y(x)\right)}{x(x+1)} + \frac{4y(x)}{(x+1)x^2} = 0$
- □ Check to see if x_0 is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{x-3}{x(x+1)}, P_3(x) = \frac{4}{(x+1)x^2}\right]$
  - $(x+1) \cdot P_2(x)$ is analytic at $x = -1$
    $\left((x+1) \cdot P_2(x)\right)\Big|_{x=-1} = 4$
  - $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$
    $\left((x+1)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$
  - $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point
  $x_0 = -1$
- Multiply by denominators
  $x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right) + x(x-3)\left(\frac{d}{dx}y(x)\right) + 4y(x) = 0$
- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$
  $(u^3 - 2u^2 + u)\left(\frac{d^2}{du^2}y(u)\right) + (u^2 - 5u + 4)\left(\frac{d}{du}y(u)\right) + 4y(u) = 0$
- Assume series solution for $y(u)$
  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$
- □ Rewrite ODE with series expansions
  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$
    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$
  - Shift index using $k- >k+1-m$
    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$
  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(3+r)\, u^{-1+r} + \left(a_1(1+r)(4+r) - a_0(2r^2+3r-4)\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)(k+4+r) -\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(3+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-3, 0\}$
- Each term must be 0
  $a_1(1+r)(4+r) - a_0(2r^2+3r-4) = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)(k+4+r) - a_k(2k^2+4kr+2r^2+3k+3r-4) + a_{k-1}(k+r-1)^2 = 0$
- Shift index using $k->k+1$
  $a_{k+2}(k+2+r)(k+5+r) - a_{k+1}\left(2(k+1)^2+4(k+1)r+2r^2+3k-1+3r\right) + a_k(k+r)^2 = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{k^2a_k-2k^2a_{k+1}+2kra_k-4kra_{k+1}+r^2a_k-2r^2a_{k+1}-7ka_{k+1}-7ra_{k+1}-a_{k+1}}{(k+2+r)(k+5+r)}$
- Recursion relation for $r = -3$
  $a_{k+2} = -\frac{k^2a_k-2k^2a_{k+1}-6ka_k+5ka_{k+1}+9a_k+2a_{k+1}}{(k-1)(k+2)}$
- Series not valid for $r = -3$, division by 0 in the recursion relation at $k = 1$
  $a_{k+2} = -\frac{k^2a_k-2k^2a_{k+1}-6ka_k+5ka_{k+1}+9a_k+2a_{k+1}}{(k-1)(k+2)}$
- Recursion relation for $r = 0$
  $a_{k+2} = -\frac{k^2a_k-2k^2a_{k+1}-7ka_{k+1}-a_{k+1}}{(k+2)(k+5)}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2a_k-2k^2a_{k+1}-7ka_{k+1}-a_{k+1}}{(k+2)(k+5)}, 4a_1+4a_0 = 0\right]$
- Revert the change of variables $u = x+1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+2} = -\frac{k^2a_k-2k^2a_{k+1}-7ka_{k+1}-a_{k+1}}{(k+2)(k+5)}, 4a_1+4a_0 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 30

```
dsolve(x^2*(x+1)*diff(diff(y(x),x),x)-x*(-x+3)*diff(y(x),x)+4*y(x) = 0,y(x),singsol=al
```

$$y = \frac{(c_2(x-1)\ln(x) + c_1x - c_1 - 4c_2)\, x^2}{(x+1)^3}$$

Mathematica DSolve solution

Solving time : 0.444 (sec)
Leaf size : 111

```
DSolve[{x^2*(1+x)*D[y[x],{x,2}]-x*(3-x)*D[y[x],x]+4*y[x]==0,{}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to (x-1)\exp\left(\int_1^x \left(\frac{1}{2K[1]} - \frac{1}{K[1]+1}\right)dK[1] - \frac{1}{2}\int_1^x \frac{K[2]-3}{K[2](K[2]+1)}dK[2]\right)\left(c_2\int_1^x \frac{\exp\left(-2\int_1^{K[3]}\left(\frac{1}{2K[1]} - \frac{1}{K[1]+1}\right)dK[1]\right)}{(K[3]-1)^2}dK[3] + c_1\right)$$

2.1.553 Problem 569

Internal problem ID [9725]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 569
Date solved : Monday, January 27, 2025 at 06:13:34 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$x^2(1-2x)\,y''-x(5-4x)\,y'+(9-4x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.210 (sec)

Writing the ode as

$$\left(-2x^3+x^2\right)y''+\left(4x^2-5x\right)y'+(9-4x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -2x^3+x^2 \\ B &= 4x^2-5x \\ C &= 9-4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{8x-1}{4\left(2x^2-x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 8x-1 \\ t &= 4\left(2x^2-x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{8x-1}{4\left(2x^2-x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1051: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 3 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 3 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4\left(x - \frac{1}{2}\right)^2} - \frac{1}{x - \frac{1}{2}} + \frac{1}{x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $3 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{8x - 1}{4\left(2x^2 - x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
3	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= 0 - (0) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} - \frac{1}{2\left(x - \frac{1}{2}\right)} + (0) \\
&= \frac{1}{2x} - \frac{1}{2\left(x - \frac{1}{2}\right)} \\
&= -\frac{1}{2x\left(-1 + 2x\right)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{1}{2x}-\frac{1}{2\left(x-\frac{1}{2}\right)}\right)(0)+\left(\left(-\frac{1}{2x^2}+\frac{1}{2\left(x-\frac{1}{2}\right)^2}\right)+\left(\frac{1}{2x}-\frac{1}{2\left(x-\frac{1}{2}\right)}\right)^2-\left(\frac{8x-1}{4\left(2x^2-x\right)^2}\right)\right)=$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\,dx}\\
&= \mathrm{e}^{\int \left(\frac{1}{2x}-\frac{1}{2\left(x-\frac{1}{2}\right)}\right)dx}\\
&= \frac{\sqrt{x}}{\sqrt{-1+2x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\
&= z_1 e^{-\int \frac{1}{2}\frac{4x^2-5x}{-2x^3+x^2}\,dx}\\
&= z_1 e^{-\frac{3\ln(-1+2x)}{2}+\frac{5\ln(x)}{2}}\\
&= z_1\left(\frac{x^{5/2}}{(-1+2x)^{3/2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\frac{x^3}{(-1+2x)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1\int \frac{e^{\int -\frac{4x^2-5x}{-2x^3+x^2}\,dx}}{(y_1)^2}\,dx\\
&= y_1\int \frac{e^{-3\ln(-1+2x)+5\ln(x)}}{(y_1)^2}\,dx\\
&= y_1(2x-\ln(x))
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1y_1+c_2y_2\\
&= c_1\left(\frac{x^3}{(-1+2x)^2}\right)+c_2\left(\frac{x^3}{(-1+2x)^2}(2x-\ln(x))\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(-2x+1)\left(\frac{d^2}{dx^2}y(x)\right)-x(5-4x)\left(\frac{d}{dx}y(x)\right)+(9-4x)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(-9+4x)y(x)}{x^2(2x-1)}+\frac{(-5+4x)\left(\frac{d}{dx}y(x)\right)}{x(2x-1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)-\frac{(-5+4x)\left(\frac{d}{dx}y(x)\right)}{x(2x-1)}+\frac{(-9+4x)y(x)}{x^2(2x-1)}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=-\frac{-5+4x}{x(2x-1)},P_3(x)=\frac{-9+4x}{x^2(2x-1)}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=-5$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=9$

- ○ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

  $x^2(2x-1)\left(\frac{d^2}{dx^2}y(x)\right)-x(-5+4x)\left(\frac{d}{dx}y(x)\right)+(-9+4x)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ○ Shift index using $k->k-m$

  $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ○ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ○ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..3$

  $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- ○ Shift index using $k->k+2-m$

  $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

Rewrite ODE with series expansions

$$-a_0(-3+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r-3)^2 + 2a_{k-1}(k+r-2)(k+r-3)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-(-3+r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 3$
- Each term in the series must be 0, giving the recursion relation
  $-a_k(k+r-3)^2 + 2a_{k-1}(k+r-2)(k+r-3) = 0$
- Shift index using $k->k+1$
  $-a_{k+1}(k+r-2)^2 + 2a_k(k+r-1)(k+r-2) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{2a_k(k+r-1)}{k+r-2}$
- Recursion relation for $r = 3$
  $a_{k+1} = \frac{2a_k(k+2)}{k+1}$
- Solution for $r = 3$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = \frac{2a_k(k+2)}{k+1}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.007` (sec)
Leaf size : 26

```
dsolve(x^2*(1-2*x)*diff(diff(y(x),x),x)-x*(5-4*x)*diff(y(x),x)+(9-4*x)*y(x) = 0,y(x),s
```

$$y = \frac{x^3(2c_2x - c_2\ln(x) + c_1)}{(-1+2x)^2}$$

Mathematica DSolve solution

Solving time : 0.201 (sec)
Leaf size : 95

```
DSolve[{x^2*(1-2*x)*D[y[x],{x,2}]-x*(5-4*x)*D[y[x],x]+(9-4*x)*y[x]==0,{}},y[x],x,IncludeSingula
```

$$y(x) \to \exp\left(\int_1^x \frac{1}{2K[1]-4K[1]^2}dK[1]-\frac{1}{2}\int_1^x\left(\frac{6}{2K[2]-1}-\frac{5}{K[2]}\right)dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]}\frac{1}{2K[1]-4K[1]^2}dK[1]\right)dK[3]+c_1\right)$$

2.1.554 Problem 570

Internal problem ID [9726]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 570
Date solved : `Monday, January 27, 2025 at 06:13:35 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(2+x)\,y''+x^2y'+(1-x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.201 (sec)

Writing the ode as

$$(2x^3+4x^2)\,y''+x^2y'+(1-x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x^3+4x^2\\B&=x^2\\C&=1-x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{5x^2+8x-16}{16\,(x^2+2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=5x^2+8x-16\\t&=16(x^2+2x)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{5x^2+8x-16}{16\,(x^2+2x)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1053: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{8x} - \frac{1}{4x^2} - \frac{3}{16\,(2 + x)^2} - \frac{3}{8\,(2 + x)}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(2+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{5x^2+8x-16}{16\left(x^2+2x\right)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=\frac{5}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b} = \frac{5}{4}\\ \alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b} = -\frac{1}{4}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{5x^2+8x-16}{16\left(x^2+2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{3}{4}$	$\frac{1}{4}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{4}$ then

$$\begin{aligned}d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right)\\ &= \frac{5}{4} - \left(\frac{5}{4}\right)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (+)[\sqrt{r}]_\infty\\ &= \frac{3}{4(2+x)} + \frac{1}{2x} + (0)\\ &= \frac{3}{4(2+x)} + \frac{1}{2x}\\ &= \frac{5x+4}{4x(2+x)}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{3}{4\,(2+x)}+\frac{1}{2x}\right)(0)+\left(\left(-\frac{3}{4\,(2+x)^2}-\frac{1}{2x^2}\right)+\left(\frac{3}{4\,(2+x)}+\frac{1}{2x}\right)^2-\left(\frac{5x^2+8x-16}{16\,(x^2+2x)^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x)&=pe^{\int \omega\,dx}\\
&=\mathrm{e}^{\int\left(\frac{3}{4(2+x)}+\frac{1}{2x}\right)dx}\\
&=\sqrt{x}\,(2+x)^{3/4}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1&=z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\
&=z_1e^{-\int \frac{1}{2}\frac{x^2}{2x^3+4x^2}\,dx}\\
&=z_1e^{-\frac{\ln(2+x)}{4}}\\
&=z_1\left(\frac{1}{(2+x)^{1/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\sqrt{2+x}\,\sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2&=y_1\int \frac{e^{\int -\frac{x^2}{2x^3+4x^2}\,dx}}{(y_1)^2}\,dx\\
&=y_1\int \frac{e^{-\frac{\ln(2+x)}{2}}}{(y_1)^2}\,dx\\
&=y_1\left(-\frac{\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right)}{2}+\frac{1}{\sqrt{2+x}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y&=c_1y_1+c_2y_2\\
&=c_1\left(\sqrt{2+x}\,\sqrt{x}\right)+c_2\left(\sqrt{2+x}\,\sqrt{x}\left(-\frac{\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right)}{2}+\frac{1}{\sqrt{2+x}}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2(x+2)\left(\frac{d^2}{dx^2}y(x)\right)+x^2\left(\frac{d}{dx}y(x)\right)+(1-x)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{(x-1)y(x)}{2x^2(x+2)}-\frac{\frac{d}{dx}y(x)}{2(x+2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{\frac{d}{dx}y(x)}{2(x+2)}-\frac{(x-1)y(x)}{2x^2(x+2)}=0$

□ Check to see if x_0 is a regular singular point

  ○ Define functions

  $\left[P_2(x)=\frac{1}{2(x+2)},P_3(x)=-\frac{x-1}{2x^2(x+2)}\right]$

  ○ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

  $\left((x+2)\cdot P_2(x)\right)\bigg|_{x=-2}=\frac{1}{2}$

  ○ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

  $\left((x+2)^2\cdot P_3(x)\right)\bigg|_{x=-2}=0$

  ○ $x=-2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-2$

- Multiply by denominators

  $2x^2(x+2)\left(\frac{d^2}{dx^2}y(x)\right)+x^2\left(\frac{d}{dx}y(x)\right)+(1-x)\,y(x)=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

  $(2u^3-8u^2+8u)\left(\frac{d^2}{du^2}y(u)\right)+(u^2-4u+4)\left(\frac{d}{du}y(u)\right)+(3-u)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ○ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

  ○ Shift index using $k->k-m$

  $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

  ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  ○ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  ○ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

  ○ Shift index using $k->k+2-m$

$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

Rewrite ODE with series expansions

$4a_0r(-1+2r)\, u^{-1+r} + (4a_1(1+r)(1+2r) - a_0(8r^2-4r-3))\, u^r + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+1+r)(2k+\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $4r(-1+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, \frac{1}{2}\}$
- Each term must be 0
  $4a_1(1+r)(1+2r) - a_0(8r^2-4r-3) = 0$
- Each term in the series must be 0, giving the recursion relation
  $2(-4a_k + a_{k-1} + 4a_{k+1})\, k^2 + (4(-4a_k + a_{k-1} + 4a_{k+1})\, r + 4a_k - 5a_{k-1} + 12a_{k+1})\, k + 2(-4a_k + a_{k-}$
- Shift index using $k->k+1$
  $2(-4a_{k+1} + a_k + 4a_{k+2})(k+1)^2 + (4(-4a_{k+1} + a_k + 4a_{k+2})\, r + 4a_{k+1} - 5a_k + 12a_{k+2})(k+1) + 2$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 4kra_k - 16kra_{k+1} + 2r^2a_k - 8r^2a_{k+1} - ka_k - 12ka_{k+1} - ra_k - 12ra_{k+1} - a_k - a_{k+1}}{4(2k^2+4kr+2r^2+7k+7r+6)}$
- Recursion relation for $r = 0$
  $a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} - ka_k - 12ka_{k+1} - a_k - a_{k+1}}{4(2k^2+7k+6)}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} - ka_k - 12ka_{k+1} - a_k - a_{k+1}}{4(2k^2+7k+6)}, 4a_1 + 3a_0 = 0\right]$
- Revert the change of variables $u = x+2$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+2)^k, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} - ka_k - 12ka_{k+1} - a_k - a_{k+1}}{4(2k^2+7k+6)}, 4a_1 + 3a_0 = 0\right]$
- Recursion relation for $r = \frac{1}{2}$
  $a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + ka_k - 20ka_{k+1} - a_k - 9a_{k+1}}{4(2k^2+9k+10)}$
- Solution for $r = \frac{1}{2}$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{2}}, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + ka_k - 20ka_{k+1} - a_k - 9a_{k+1}}{4(2k^2+9k+10)}, 12a_1 + 3a_0 = 0\right]$
- Revert the change of variables $u = x+2$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+2)^{k+\frac{1}{2}}, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + ka_k - 20ka_{k+1} - a_k - 9a_{k+1}}{4(2k^2+9k+10)}, 12a_1 + 3a_0 = 0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k(x+2)^k\right) + \left(\sum_{k=0}^{\infty} b_k(x+2)^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} - ka_k - 12ka_{k+1} - a_k - a_{k+1}}{4(2k^2+7k+6)}, 4a_1 +\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
```

```
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.026 (sec)
Leaf size : 50

```
dsolve(2*x^2*(x+2)*diff(diff(y(x),x),x)+diff(y(x),x)*x^2+(1-x)*y(x) = 0,y(x),singsol=a
```

$$y = c_1\sqrt{x\left(x+2\right)}+\frac{c_2\left(\left(x+2\right)\operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)-\sqrt{2}\,\sqrt{x+2}\right)\sqrt{x}}{\sqrt{x+2}}$$

Mathematica DSolve solution

Solving time : 0.51 (sec)
Leaf size : 92

```
DSolve[{2*x^2*(2+x)*D[y[x],{x,2}]+x^2*D[y[x],x]+(1-x)*y[x]==0,{}},y[x],x,IncludeSingularSolu
```

$$y(x)\to \frac{\exp\left(\int_1^x \frac{5K[1]+4}{4K[1]^2+8K[1]}dK[1]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[2]}\frac{5K[1]+4}{4K[1]^2+8K[1]}dK[1]\right)dK[2]+c_1\right)}{\sqrt[4]{2}\sqrt[4]{x+2}}$$

2.1.555 Problem 571

Internal problem ID [9727]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 571
Date solved : `Monday, January 27, 2025 at 06:13:35 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(1+x)\,y''-x(6-x)\,y'+(8-x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.201 (sec)

Writing the ode as

$$\left(2x^3+2x^2\right)y''+\left(x^2-6x\right)y'+(8-x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^3+2x^2 \\ B &= x^2-6x \\ C &= 8-x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{5x^2-20x-4}{16\left(x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 5x^2-20x-4 \\ t &= 16\left(x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{5x^2-20x-4}{16\left(x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1055: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2+x)^2$. There is a pole at $x=0$ of order 2. There is a pole at $x=-1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4\,(1+x)} - \frac{3}{4x} - \frac{1}{4x^2} + \frac{21}{16\,(1+x)^2}$$

For the pole at $x=-1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4} \end{aligned}$$

For the pole at $x=0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{5x^2 - 20x - 4}{16\left(x^2+x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{5x^2 - 20x - 4}{16\left(x^2+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= -\frac{1}{4} - \left(-\frac{1}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{4\left(1+x\right)} + \frac{1}{2x} + (-)\left(0\right) \\
&= -\frac{3}{4\left(1+x\right)} + \frac{1}{2x} \\
&= -\frac{x-2}{4x\left(1+x\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{3}{4\,(1+x)}+\frac{1}{2x}\right)(0)+\left(\left(\frac{3}{4\,(1+x)^2}-\frac{1}{2x^2}\right)+\left(-\frac{3}{4\,(1+x)}+\frac{1}{2x}\right)^2-\left(\frac{5x^2-20x-4}{16\,(x^2+x)^2}\right)\right)=0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x)&=pe^{\int\omega\,dx}\\
&=\mathrm{e}^{\int\left(-\frac{3}{4(1+x)}+\frac{1}{2x}\right)dx}\\
&=\frac{\sqrt{x}}{(1+x)^{3/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1&=z_1e^{\int-\frac{1}{2}\frac{B}{A}\,dx}\\
&=z_1e^{-\int\frac{1}{2}\frac{x^2-6x}{2x^3+2x^2}\,dx}\\
&=z_1e^{-\frac{7\ln(1+x)}{4}+\frac{3\ln(x)}{2}}\\
&=z_1\left(\frac{x^{3/2}}{(1+x)^{7/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\frac{x^2}{(1+x)^{5/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int\frac{e^{\int-\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2&=y_1\int\frac{e^{\int-\frac{x^2-6x}{2x^3+2x^2}\,dx}}{(y_1)^2}\,dx\\
&=y_1\int\frac{e^{-\frac{7\ln(1+x)}{2}+3\ln(x)}}{(y_1)^2}\,dx\\
&=y_1\left(\frac{2(1+x)^{3/2}}{3}+2\sqrt{1+x}+\ln\left(\sqrt{1+x}-1\right)-\ln\left(1+\sqrt{1+x}\right)\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^2}{(1+x)^{5/2}}\right) \\
&\quad + c_2 \left(\frac{x^2}{(1+x)^{5/2}} \left(\frac{2(1+x)^{3/2}}{3} + 2\sqrt{1+x} + \ln\left(\sqrt{1+x}-1\right) - \ln\left(1+\sqrt{1+x}\right)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right) - x(-x+6)\left(\frac{d}{dx}y(x)\right) + (8-x)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{(-8+x)y(x)}{2x^2(x+1)} - \frac{(-6+x)\left(\frac{d}{dx}y(x)\right)}{2x(x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(-6+x)\left(\frac{d}{dx}y(x)\right)}{2x(x+1)} - \frac{(-8+x)y(x)}{2x^2(x+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{-6+x}{2x(x+1)}, P_3(x) = -\frac{-8+x}{2x^2(x+1)}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1} = \frac{7}{2}$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $2x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right) + x(-6+x)\left(\frac{d}{dx}y(x)\right) + (8-x)\,y(x) = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(2u^3-4u^2+2u)\left(\frac{d^2}{du^2}y(u)\right) + (u^2-8u+7)\left(\frac{d}{du}y(u)\right) + (9-u)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

  ◦ Shift index using $k->k-m$

  $u^m\cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(5+2r)\, u^{-1+r} + \left(a_1(1+r)(7+2r) - a_0(4r^2+4r-9)\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)(2k+7+\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(5+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{0, -\frac{5}{2}\right\}$
- Each term must be 0
  $a_1(1+r)(7+2r) - a_0(4r^2+4r-9) = 0$
- Each term in the series must be 0, giving the recursion relation
  $(-4a_k + 2a_{k-1} + 2a_{k+1})\, k^2 + \left((-8a_k + 4a_{k-1} + 4a_{k+1})\, r - 4a_k - 5a_{k-1} + 9a_{k+1}\right) k + (-4a_k + 2a$
- Shift index using $k->k+1$
  $(-4a_{k+1} + 2a_k + 2a_{k+2})(k+1)^2 + \left((-8a_{k+1} + 4a_k + 4a_{k+2})\, r - 4a_{k+1} - 5a_k + 9a_{k+2}\right)(k+1) +$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} + 4kra_k - 8kra_{k+1} + 2r^2a_k - 4r^2a_{k+1} - ka_k - 12ka_{k+1} - ra_k - 12ra_{k+1} - a_k + a_{k+1}}{2k^2 + 4kr + 2r^2 + 13k + 13r + 18}$
- Recursion relation for $r = 0$
  $a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - ka_k - 12ka_{k+1} - a_k + a_{k+1}}{2k^2 + 13k + 18}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - ka_k - 12ka_{k+1} - a_k + a_{k+1}}{2k^2 + 13k + 18}, 7a_1 + 9a_0 = 0\right]$
- Revert the change of variables $u = x+1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x+1)^k, a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - ka_k - 12ka_{k+1} - a_k + a_{k+1}}{2k^2 + 13k + 18}, 7a_1 + 9a_0 = 0\right]$
- Recursion relation for $r = -\frac{5}{2}$
  $a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - 11ka_k + 8ka_{k+1} + 14a_k + 6a_{k+1}}{2k^2 + 3k - 2}$
- Solution for $r = -\frac{5}{2}$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{5}{2}}, a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - 11ka_k + 8ka_{k+1} + 14a_k + 6a_{k+1}}{2k^2 + 3k - 2}, -3a_1 - 6a_0 = 0\right]$
- Revert the change of variables $u = x+1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x+1)^{k-\frac{5}{2}}, a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - 11ka_k + 8ka_{k+1} + 14a_k + 6a_{k+1}}{2k^2 + 3k - 2}, -3a_1 - 6a_0 = 0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k (x+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k-\frac{5}{2}}\right), a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - ka_k - 12ka_{k+1} - a_k + a_{k+1}}{2k^2 + 13k + 18}, 7a_1\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.033 (sec)
Leaf size : 52

```
dsolve(2*x^2*(x+1)*diff(diff(y(x),x),x)-x*(-x+6)*diff(y(x),x)+(8-x)*y(x) = 0,y(x),singso
```

$$y = \frac{x^2 \left(\frac{2\sqrt{x+1}\, c_2 x}{3} + \left(\ln\left(\sqrt{x+1}-1\right) - \ln\left(\sqrt{x+1}+1\right) + \frac{8\sqrt{x+1}}{3}\right) c_2 + c_1\right)}{\left(x+1\right)^{5/2}}$$

Mathematica DSolve solution

Solving time : 0.225 (sec)
Leaf size : 109

```
DSolve[{2*x^2*(1+x)*D[y[x],{x,2}]-x*(6-x)*D[y[x],x]+(8-x)*y[x]==0,{}},y[x],x,IncludeSingularSo
```

$$y(x) \to \exp\left(\int_1^x \frac{2-K[1]}{4K[1]^2+4K[1]}dK[1] - \frac{1}{2}\int_1^x \left(\frac{7}{2(K[2]+1)} - \frac{3}{K[2]}\right) dK[2]\right) \left(c_2 \int_1^x \exp\left(-2\int_1^{K[3]} \frac{2-K[1]}{4K[1]^2+4K[1]}dK[1]\right) dK[3] + c_1\right)$$

2.1.556 Problem 572

Internal problem ID [9728]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 572
Date solved : Monday, January 27, 2025 at 06:13:36 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$x^2(1+2x)\,y''+x(5+9x)\,y'+(4+3x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.223 (sec)

Writing the ode as

$$\left(2x^3+x^2\right)y''+\left(9x^2+5x\right)y'+(4+3x)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=2x^3+x^2\\ B&=9x^2+5x \\ C&=4+3x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{21x^2+6x-1}{4\left(2x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=21x^2+6x-1\\ t&=4\left(2x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{21x^2+6x-1}{4\left(2x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1057: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{5}{2x} + \frac{5}{16\left(x + \frac{1}{2}\right)^2} - \frac{5}{2\left(x + \frac{1}{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{21x^2 + 6x - 1}{4\left(2x^2 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{21x^2 + 6x - 1}{4\left(2x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-\frac{1}{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\
&= \frac{7}{4} - \left(\frac{7}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \frac{5}{4\left(x+\frac{1}{2}\right)} + (0) \\
&= \frac{1}{2x} + \frac{5}{4\left(x+\frac{1}{2}\right)} \\
&= \frac{1+7x}{4x^2+2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{5}{4\left(x+\frac{1}{2}\right)}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{5}{4\left(x+\frac{1}{2}\right)^2}\right) + \left(\frac{1}{2x} + \frac{5}{4\left(x+\frac{1}{2}\right)}\right)^2 - \left(\frac{21x^2+6x-1}{4\,(2x^2+x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{2x} + \frac{5}{4\left(x+\frac{1}{2}\right)}\right) dx} \\ &= \sqrt{x}\,(1+2x)^{5/4} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{9x^2+5x}{2x^3+x^2}\, dx} \\ &= z_1 e^{-\frac{5\ln(x)}{2} + \frac{\ln(1+2x)}{4}} \\ &= z_1 \left(\frac{(1+2x)^{1/4}}{x^{5/2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(1+2x)^{3/2}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{9x^2+5x}{2x^3+x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-5\ln(x) + \frac{\ln(1+2x)}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(\ln\left(\sqrt{1+2x} - 1\right) + \frac{2}{3\,(1+2x)^{3/2}} + \frac{2}{\sqrt{1+2x}} - \ln\left(\sqrt{1+2x} + 1\right)\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(1+2x)^{3/2}}{x^2}\right) \\
&+ c_2\left(\frac{(1+2x)^{3/2}}{x^2}\left(\ln\left(\sqrt{1+2x}-1\right)+\frac{2}{3(1+2x)^{3/2}}+\frac{2}{\sqrt{1+2x}}-\ln\left(\sqrt{1+2x}+1\right)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(2x+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(5+9x)\left(\frac{d}{dx}y(x)\right)+(3x+4)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(3x+4)y(x)}{x^2(2x+1)}-\frac{(5+9x)\left(\frac{d}{dx}y(x)\right)}{x(2x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{(5+9x)\left(\frac{d}{dx}y(x)\right)}{x(2x+1)}+\frac{(3x+4)y(x)}{x^2(2x+1)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{5+9x}{x(2x+1)}, P_3(x)=\frac{3x+4}{x^2(2x+1)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=5$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=4$

  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(2x+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(5+9x)\left(\frac{d}{dx}y(x)\right)+(3x+4)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

    $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

$$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..3$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(2+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+2)^2 + a_{k-1}(k+r+2)\,(2k-1+2r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(2+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = -2$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r+2)\,(a_k(k+r+2) + a_{k-1}(2k-1+2r)) = 0$$

- Shift index using $k- >k+1$

$$(k+r+3)\,(a_{k+1}(k+r+3) + a_k(2k+2r+1)) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k(2k+2r+1)}{k+r+3}$$

- Recursion relation for $r = -2$

$$a_{k+1} = -\frac{a_k(2k-3)}{k+1}$$

- Solution for $r = -2$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+1} = -\frac{a_k(2k-3)}{k+1}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.033 (sec)
Leaf size : 73

```
dsolve(x^2*(2*x+1)*diff(diff(y(x),x),x)+x*(5+9*x)*diff(y(x),x)+(3*x+4)*y(x) = 0,y(x),sin
```

$$y = \frac{c_2\left(x+\frac{1}{2}\right)^2 \ln\left(\sqrt{2x+1}-1\right) - c_2\left(x+\frac{1}{2}\right)^2 \ln\left(\sqrt{2x+1}+1\right) + c_2\left(x+\frac{2}{3}\right)\sqrt{2x+1} + 4\left(x+\frac{1}{2}\right)^2 c_1}{x^2\sqrt{2x+1}}$$

Mathematica DSolve solution

Solving time : 0.2 (sec)
Leaf size : 110

```
DSolve[{x^2*(1+2*x)*D[y[x],{x,2}]+x*(5+9*x)*D[y[x],x]+(4+3*x)*y[x]==0,{}},y[x],x,IncludeSing
```

$$y(x) \to \exp\left(\int_1^x \frac{7K[1]+1}{4K[1]^2+2K[1]}dK[1] - \frac{1}{2}\int_1^x \frac{9K[2]+5}{2K[2]^2+K[2]}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{7K[1]+1}{4K[1]^2+2K[1]}dK[1]\right)dK[3] + c_1\right)$$

2.1.557 Problem 573

Internal problem ID [9729]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 573
Date solved : `Monday, January 27, 2025 at 06:13:37 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1-2x)\,y''-x(5+4x)\,y'+(9+4x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.254 (sec)

Writing the ode as

$$\left(-2x^3+x^2\right)y''+\left(-4x^2-5x\right)y'+(9+4x)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-2x^3+x^2\\B&=-4x^2-5x\\C&=9+4x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{32x^2+56x-1}{4\left(2x^2-x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=32x^2+56x-1\\t&=4\left(2x^2-x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{32x^2+56x-1}{4\left(2x^2-x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1059: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{35}{4\left(x - \frac{1}{2}\right)^2} - \frac{13}{x - \frac{1}{2}} + \frac{13}{x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{5}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{32x^2 + 56x - 1}{4\left(2x^2 - x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{32x^2 + 56x - 1}{4\left(2x^2 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= -1 - (-2) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} - \frac{5}{2\left(x-\frac{1}{2}\right)} + (-)\,(0) \\
&= \frac{1}{2x} - \frac{5}{2\left(x-\frac{1}{2}\right)} \\
&= \frac{-1-8x}{4x^2-2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{5}{2\left(x-\frac{1}{2}\right)}\right)(1) + \left(\left(-\frac{1}{2x^2} + \frac{5}{2\left(x-\frac{1}{2}\right)^2}\right) + \left(\frac{1}{2x} - \frac{5}{2\left(x-\frac{1}{2}\right)}\right)^2 - \left(\frac{32x^2+56x-1}{4\left(2x^2-x\right)^2}\right)\right) \frac{-1+8a_0}{x(-1+2x)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{1}{8}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{1}{8}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{1}{8}\right) e^{\int \left(\frac{1}{2x} - \frac{5}{2\left(x-\frac{1}{2}\right)}\right) dx} \\
&= \left(x + \frac{1}{8}\right) e^{-\frac{5\ln(-1+2x)}{2} + \frac{\ln(x)}{2}} \\
&= \frac{\left(x+\frac{1}{8}\right)\sqrt{x}}{(-1+2x)^{5/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x^2-5x}{-2x^3+x^2} \, dx} \\
&= z_1 e^{-\frac{7\ln(-1+2x)}{2} + \frac{5\ln(x)}{2}} \\
&= z_1 \left(\frac{x^{5/2}}{(-1+2x)^{7/2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^3\left(x+\frac{1}{8}\right)}{(-1+2x)^6}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^2-5x}{-2x^3+x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-7\ln(-1+2x)+5\ln(x)}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{32x^3}{3} - 44x^2 + \frac{203x}{2} - \frac{3125}{16(1+8x)} - 64\ln(x)\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^3 \left(x + \frac{1}{8}\right)}{\left(-1 + 2x\right)^6} \right) + c_2 \left(\frac{x^3 \left(x + \frac{1}{8}\right)}{\left(-1 + 2x\right)^6} \left(\frac{32x^3}{3} - 44x^2 + \frac{203x}{2} - \frac{3125}{16\left(1 + 8x\right)} - 64 \ln\left(x\right) \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(-2x+1)\left(\frac{d^2}{dx^2}y(x)\right) - x(5+4x)\left(\frac{d}{dx}y(x)\right) + (4x+9)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{(4x+9)y(x)}{x^2(2x-1)} - \frac{(5+4x)\left(\frac{d}{dx}y(x)\right)}{x(2x-1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(5+4x)\left(\frac{d}{dx}y(x)\right)}{x(2x-1)} - \frac{(4x+9)y(x)}{x^2(2x-1)} = 0$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{5+4x}{x(2x-1)}, P_3(x) = -\frac{4x+9}{x^2(2x-1)}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -5$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 9$
  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2(2x-1)\left(\frac{d^2}{dx^2}y(x)\right) + x(5+4x)\left(\frac{d}{dx}y(x)\right) + (-4x-9)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k-m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$
  - Shift index using $k- >k+1-m$

$$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..3$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0(-3+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r-3)^2 + 2a_{k-1}(k+1+r)\,(k-2+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-(-3+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 3$$

- Each term in the series must be 0, giving the recursion relation

$$-a_k(k+r-3)^2 + 2a_{k-1}(k+1+r)\,(k-2+r) = 0$$

- Shift index using $k->k+1$

$$-a_{k+1}(k-2+r)^2 + 2a_k(k+r+2)\,(k+r-1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{2a_k(k+r+2)(k+r-1)}{(k-2+r)^2}$$

- Recursion relation for $r = 3$

$$a_{k+1} = \frac{2a_k(k+5)(k+2)}{(k+1)^2}$$

- Solution for $r = 3$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = \frac{2a_k(k+5)(k+2)}{(k+1)^2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 54

```
dsolve(x^2*(1-2*x)*diff(diff(y(x),x),x)-x*(5+4*x)*diff(y(x),x)+(4*x+9)*y(x) = 0,y(x),s
```

$$y = \frac{\left(-6c_2\left(x+\frac{1}{8}\right)\ln(x) + c_2x^4 - 4c_2x^3 + 9c_2x^2 + \left(8c_1 + \frac{609c_2}{512}\right)x + c_1 - \frac{9375c_2}{4096}\right)x^3}{(-1+2x)^6}$$

Mathematica DSolve solution

Solving time : 0.458 (sec)
Leaf size : 129

```
DSolve[{x^2*(1-2*x)*D[y[x],{x,2}]-x*(5+4*x)*D[y[x],x]+(9+4*x)*y[x]==0,{}},y[x],x,IncludeSingula
```

$$y(x) \to \frac{1}{8}(8x+1)\exp\left(\int_1^x \frac{8K[1]+1}{2K[1]-4K[1]^2}dK[1] - \frac{1}{2}\int_1^x -\frac{4K[2]+5}{K[2]-2K[2]^2}dK[2]\right)\left(c_2\int_1^x \frac{64\exp\left(-2\int_1^{K[3]}\frac{8K[1]+1}{2K[1]-4K[1]^2}dK[1]\right)}{(8K[3]+1)^2}dK[3]+c_1\right)$$

2.1.558 Problem 574

Internal problem ID [9730]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 574
Date solved : Monday, January 27, 2025 at 06:13:37 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$x^2(1-x)\,y''+x(7+x)\,y'+(9-x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.244 (sec)

Writing the ode as

$$\left(-x^3+x^2\right)y''+\left(x^2+7x\right)y'+(9-x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-x^3+x^2\\B&=x^2+7x\\C&=9-x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-x^2+82x-1}{4\left(x^2-x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=-x^2+82x-1\\t&=4\left(x^2-x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-x^2+82x-1}{4\left(x^2-x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1061: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} - \frac{20}{-1 + x} + \frac{20}{x} + \frac{20}{(-1 + x)^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 20$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 5 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -4 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 + 82x - 1}{4(x^2 - x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 + 82x - 1}{4(x^2 - x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
1	2	0	5	-4

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= \frac{1}{2} - \left(-\frac{7}{2}\right) \\
&= 4
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} - \frac{4}{-1+x} + (-)(0) \\
&= \frac{1}{2x} - \frac{4}{-1+x} \\
&= -\frac{1+7x}{2x(-1+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(12x^2 + 6xa_3 + 2a_2\right) + 2\left(\frac{1}{2x} - \frac{4}{-1+x}\right)\left(4x^3 + 3x^2a_3 + 2a_2x + a_1\right) + \left(\left(-\frac{1}{2x^2} + \frac{4}{(-1+x)^2}\right) + \left(\frac{1}{2x} - \right.\right.$$

$$\frac{(a_3 - 16)\,x^3 + (4a_2 - 9a_3)\,x^2}{x\,(-1}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1, a_1 = 16, a_2 = 36, a_3 = 16\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 + 16x^3 + 36x^2 + 16x + 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\,dx} \\
&= \left(x^4 + 16x^3 + 36x^2 + 16x + 1\right) \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{4}{-1+x}\right)dx} \\
&= \left(x^4 + 16x^3 + 36x^2 + 16x + 1\right) \mathrm{e}^{\frac{\ln(x)}{2} - 4\ln(-1+x)} \\
&= \frac{\left(x^4 + 16x^3 + 36x^2 + 16x + 1\right)\sqrt{x}}{(-1+x)^4}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2+7x}{-x^3+x^2}\,dx} \\
&= z_1 e^{-\frac{7\ln(x)}{2} + 4\ln(-1+x)} \\
&= z_1 \left(\frac{(-1+x)^4}{x^{7/2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^4 + 16x^3 + 36x^2 + 16x + 1}{x^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+7x}{-x^3+x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-7\ln(x) + 8\ln(-1+x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{20\left(-2x^3 - \frac{15}{2}x^2 - \frac{14}{3}x - \frac{5}{12}\right)}{x^4 + 16x^3 + 36x^2 + 16x + 1} + \ln(x)\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^4+16x^3+36x^2+16x+1}{x^3}\right) \\
&\quad + c_2 \left(\frac{x^4+16x^3+36x^2+16x+1}{x^3}\left(-\frac{20\left(-2x^3-\frac{15}{2}x^2-\frac{14}{3}x-\frac{5}{12}\right)}{x^4+16x^3+36x^2+16x+1}+\ln(x)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(1-x)\left(\frac{d^2}{dx^2}y(x)\right)+x(7+x)\left(\frac{d}{dx}y(x)\right)+(9-x)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(x-9)y(x)}{x^2(x-1)}+\frac{(7+x)\left(\frac{d}{dx}y(x)\right)}{x(x-1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)-\frac{(7+x)\left(\frac{d}{dx}y(x)\right)}{x(x-1)}+\frac{(x-9)y(x)}{x^2(x-1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=-\frac{7+x}{x(x-1)},P_3(x)=\frac{x-9}{x^2(x-1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=7$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=9$

  ◦ $x=0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=0$

- Multiply by denominators

  $x^2(x-1)\left(\frac{d^2}{dx^2}y(x)\right)-x(7+x)\left(\frac{d}{dx}y(x)\right)+(x-9)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

    $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

  ◦ Shift index using $k->k-m$

    $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..3$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0(3+r)^2\,x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r+3)^2 + a_{k-1}(k-2+r)^2\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-(3+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = -3$

- Each term in the series must be 0, giving the recursion relation

  $-a_k(k+r+3)^2 + a_{k-1}(k-2+r)^2 = 0$

- Shift index using $k->k+1$

  $-a_{k+1}(k+4+r)^2 + a_k(k+r-1)^2 = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r-1)^2}{(k+4+r)^2}$

- Recursion relation for $r = -3$; series terminates at $k = 4$

  $a_{k+1} = \frac{a_k(k-4)^2}{(k+1)^2}$

- Apply recursion relation for $k = 0$

  $a_1 = 16a_0$

- Apply recursion relation for $k = 1$

  $a_2 = \frac{9a_1}{4}$

- Express in terms of a_0

  $a_2 = 36a_0$

- Apply recursion relation for $k = 2$

  $a_3 = \frac{4a_2}{9}$

- Express in terms of a_0

  $a_3 = 16a_0$

- Apply recursion relation for $k = 3$

  $a_4 = \frac{a_3}{16}$

- Express in terms of a_0

  $a_4 = a_0$

- Terminating series solution of the ODE for $r = -3$. Use reduction of order to find the second lin

  $y(x) = a_0 \cdot (x^4 + 16x^3 + 36x^2 + 16x + 1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 72

```
dsolve((1-x)*x^2*diff(diff(y(x),x),x)+x*(7+x)*diff(y(x),x)+(9-x)*y(x) = 0,y(x),singsol
```

$$y = \frac{3c_2(x^4+16x^3+36x^2+16x+1)\ln(x)+c_1x^4+(16c_1+120c_2)x^3+(36c_1+450c_2)x^2+(16c_1+280c_2)}{x^3}$$

Mathematica DSolve solution

Solving time : 0.722 (sec)
Leaf size : 145

```
DSolve[{x^2*(1-x)*D[y[x],{x,2}]+x*(7+x)*D[y[x],x]+(9-x)*y[x]==0,{}},y[x],x,IncludeSingularSo
```

$$y(x) \to (x^4+16x^3+36x^2+16x+1)\exp\left(\int_1^x\left(\frac{1}{2K[1]}-\frac{4}{K[1]-1}\right)dK[1] -\frac{1}{2}\int_1^x\frac{K[2]+7}{K[2]-K[2]^2}dK[2]\right)\left(c_2\int_1^x\frac{\exp\left(-2\int_1^{K[3]}\left(\frac{1}{2K[1]}-\frac{4}{K[1]-1}\right)dK[1]\right)}{(K[3]^4+16K[3]^3+36K[3]^2+16K[3]+1)^2}dK[3] + c_1\right)$$

2.1.559 Problem 575

Internal problem ID [9731]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 575
Date solved : `Monday, January 27, 2025 at 06:13:38 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x\left(-x^2+1\right)y' + \left(x^2+1\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.240 (sec)

Writing the ode as

$$x^2y'' + \left(x^3-x\right)y' + \left(x^2+1\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^3 - x \\ C &= x^2 + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 4x^2 - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4 - 4x^2 - 1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 4x^2 - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1063: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{4} - 1 - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{1}{x} - \frac{5}{4x^3} - \frac{5}{2x^5} - \frac{105}{16x^7} - \frac{155}{8x^9} - \frac{1965}{32x^{11}} - \frac{3265}{16x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^4 - 4x^2 - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{x^2}{4} - 1\right) + \left(-\frac{1}{4x^2}\right) \\ &= \frac{x^2}{4} - 1 - \frac{1}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is -1. Now b can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 1\right) &&= -\frac{3}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 1\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 4x^2 - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\left(\frac{x}{2}\right) \\ &= \frac{1}{2x} - \frac{x}{2} \\ &= \frac{1}{2x} - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2x} - \frac{x}{2}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{2}\right) + \left(\frac{1}{2x} - \frac{x}{2}\right)^2 - \left(\frac{x^4 - 4x^2 - 1}{4x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{1}{2x} - \frac{x}{2}\right) dx} \\ &= \sqrt{x}\, e^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^3-x}{x^2}\,dx} \\
&= z_1 e^{-\frac{x^2}{4}+\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\,\mathrm{e}^{-\frac{x^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{-\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^3-x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}+\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{\mathrm{Ei}_1\left(-\frac{x^2}{2}\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x\,\mathrm{e}^{-\frac{x^2}{2}}\right) + c_2\left(x\,\mathrm{e}^{-\frac{x^2}{2}}\left(-\frac{\mathrm{Ei}_1\left(-\frac{x^2}{2}\right)}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right) - x(-x^2+1)\left(\frac{d}{dx}y(x)\right) + (x^2+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{(x^2+1)y(x)}{x^2} - \frac{(x^2-1)\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) + \frac{(x^2-1)\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(x^2+1)y(x)}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

$\left[P_2(x)=\frac{x^2-1}{x}, P_3(x)=\frac{x^2+1}{x^2}\right]$

- ∘ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\Big|_{x=0}=-1$

- ∘ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\Big|_{x=0}=1$

- ∘ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right)+x(x^2-1)\left(\frac{d}{dx}y(x)\right)+(x^2+1)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ∘ Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

- ∘ Shift index using $k->k-m$

  $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ∘ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ∘ Shift index using $k->k+1-m$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ∘ Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)^2\,x^r+a_1r^2x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_k(k+r-1)^2+a_{k-2}(k+r-1)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)^2=0$

- Values of r that satisfy the indicial equation

  $r=1$

- Each term must be 0

  $a_1r^2=0$

- Solve for the dependent coefficient(s)

  $a_1=0$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\,(a_k(k+r-1)+a_{k-2})=0$

- Shift index using $k->k+2$

  $(k+r+1)\,(a_{k+2}(k+r+1)+a_k)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_k}{k+r+1}$

- Recursion relation for $r = 1$
  $a_{k+2} = -\frac{a_k}{k+2}$
- Solution for $r = 1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{k+2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 23

```
dsolve(x^2*diff(diff(y(x),x),x)-x*(-x^2+1)*diff(y(x),x)+(x^2+1)*y(x) = 0,y(x),singsol=al
```

$$y = x\,\mathrm{e}^{-\frac{x^2}{2}}\left(c_1 + c_2\,\mathrm{Ei}_1\left(-\frac{x^2}{2}\right)\right)$$

Mathematica DSolve solution

Solving time : 0.026 (sec)
Leaf size : 35

```
DSolve[{x^2*D[y[x],{x,2}]-x*(1-x^2)*D[y[x],x]+(1+x^2)*y[x]==0,{}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to \frac{1}{2}e^{-\frac{x^2}{2}}x\left(c_1\,\mathrm{ExpIntegralEi}\left(\frac{x^2}{2}\right) + 2c_2\right)$$

2.1.560 Problem 576

Internal problem ID [9732]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 576
Date solved : `Monday, January 27, 2025 at 06:13:39 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(x^2+1\right)y''-3x\left(-x^2+1\right)y'+4y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.332 (sec)

Writing the ode as

$$\left(x^4+x^2\right)y''+\left(3x^3-3x\right)y'+4y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2 \\ B &= 3x^3-3x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{3x^4-10x^2-1}{4\left(x^3+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3x^4-10x^2-1 \\ t &= 4\left(x^3+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{3x^4-10x^2-1}{4\left(x^3+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1065: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{i}{4x - 4i} - \frac{i}{4(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x^4 - 10x^2 - 1}{4\left(x^3 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3x^4 - 10x^2 - 1}{4\left(x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} - \frac{1}{2\left(x-i\right)} - \frac{1}{2\left(x+i\right)} + (-)\left(0\right) \\
&= \frac{1}{2x} - \frac{1}{2\left(x-i\right)} - \frac{1}{2\left(x+i\right)} \\
&= \frac{1}{2x} - \frac{x}{x^2+1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x} - \frac{1}{2\left(x-i\right)} - \frac{1}{2\left(x+i\right)}\right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{1}{2\left(x-i\right)^2} + \frac{1}{2\left(x+i\right)^2}\right) + \left(\frac{1}{2x} - \frac{1}{2\left(x-i\right)} - \frac{1}{2\left(x+\right.}\right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)}\right) dx} \\
&= \frac{\sqrt{x}}{\sqrt{x^2+1}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x^3-3x}{x^4+x^2} \, dx} \\
&= z_1 e^{-\frac{3\ln\left(x^2+1\right)}{2} + \frac{3\ln(x)}{2}} \\
&= z_1 \left(\frac{x^{3/2}}{\left(x^2+1\right)^{3/2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^2}{\left(x^2+1\right)^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^3-3x}{x^4+x^2} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{-3\ln\left(x^2+1\right)+3\ln(x)}}{\left(y_1\right)^2} \, dx \\
&= y_1 \left(\frac{x^2}{2} + \ln\left(x\right)\right)
\end{aligned}
$$

Therefore the solution is

$$\begin{aligned}y &= c_1y_1 + c_2y_2 \\ &= c_1\left(\frac{x^2}{(x^2+1)^2}\right) + c_2\left(\frac{x^2}{(x^2+1)^2}\left(\frac{x^2}{2}+\ln(x)\right)\right)\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) - 3x(-x^2+1)\left(\frac{d}{dx}y(x)\right) + 4y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{4y(x)}{x^2(x^2+1)} - \frac{3(x^2-1)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x) + \frac{3(x^2-1)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)} + \frac{4y(x)}{x^2(x^2+1)} = 0$
- □ Check to see if x_0 is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{3(x^2-1)}{x(x^2+1)}, P_3(x) = \frac{4}{x^2(x^2+1)}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = -3$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 4$
  - $x = 0$is a regular singular point
    Check to see if x_0 is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + 3x(x^2-1)\left(\frac{d}{dx}y(x)\right) + 4y(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$
    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k->k+1-m$
    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$
    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,x^{k+r-2+m}$
  - Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-2+r)^2\,x^r + a_1(-1+r)^2\,x^{1+r} + \left(\sum_{k=2}^{\infty}\left(a_k(k+r-2)^2 + a_{k-2}(k+r-2)\,(k+r)\right)x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 2$
- Each term must be 0

  $a_1(-1+r)^2 = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)\,(a_k(k+r-2) + a_{k-2}(k+r)) = 0$
- Shift index using $k- >k+2$

  $(k+r)\,(a_{k+2}(k+r) + a_k(k+r+2)) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r+2)}{k+r}$
- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{a_k(k+4)}{k+2}$
- Solution for $r = 2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{a_k(k+4)}{k+2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 27

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)-3*x*(-x^2+1)*diff(y(x),x)+4*y(x) = 0,y(x),singso
```

$$y = \frac{x^2\left(c_1 + c_2\left(\frac{x^2}{2} + \ln(x)\right)\right)}{(x^2+1)^2}$$

Mathematica DSolve solution

Solving time : 0.22 (sec)
Leaf size : 107

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]-3*x*(1-x^2)*D[y[x],x]+4*y[x]==0,{}},y[x],x,IncludeSingular
```

$$y(x) \to \exp\left(\int_1^x -\frac{K[1]^2-1}{2\left(K[1]^3+K[1]\right)}dK[1] - \frac{1}{2}\int_1^x \frac{3(K[2]^2-1)}{K[2]^3+K[2]}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} -\frac{K[1]^2-1}{2\left(K[1]^3+K[1]\right)}dK[1]\right)dK[3]+c_1\right)$$

2.1.561 Problem 577

Internal problem ID [9733]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 577
Date solved : `Monday, January 27, 2025 at 06:13:39 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' + 2x^3y' + (3x^2 + 1)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.238 (sec)

Writing the ode as

$$4x^2y'' + 2x^3y' + (3x^2 + 1)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= 2x^3 \\ C &= 3x^2 + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 8x^2 - 4}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4 - 8x^2 - 4 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 8x^2 - 4}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1067: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 4\\ &= -2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1,2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{16} - \frac{1}{2} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{4} - \frac{1}{x} - \frac{5}{2x^3} - \frac{10}{x^5} - \frac{105}{2x^7} - \frac{310}{x^9} - \frac{1965}{x^{11}} - \frac{13060}{x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{4} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{16}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^4 - 8x^2 - 4}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{x^2}{16} - \frac{1}{2}\right) + \left(-\frac{1}{4x^2}\right) \\ &= \frac{x^2}{16} - \frac{1}{2} - \frac{1}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{4} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{4}} - 1\right) &&= -\frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{4}} - 1\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 8x^2 - 4}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{4}$	$-\frac{3}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\left(\frac{x}{4}\right) \\ &= \frac{1}{2x} - \frac{x}{4} \\ &= \frac{1}{2x} - \frac{x}{4} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered} (0) + 2\left(\frac{1}{2x} - \frac{x}{4}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{4}\right) + \left(\frac{1}{2x} - \frac{x}{4}\right)^2 - \left(\frac{x^4 - 8x^2 - 4}{16x^2}\right)\right) = 0 \\ 0 = 0 \end{gathered}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{1}{2x} - \frac{x}{4}\right) dx} \\ &= \sqrt{x}\, e^{-\frac{x^2}{8}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x^3}{4x^2}\,dx} \\ &= z_1 e^{-\frac{x^2}{8}} \\ &= z_1\left(\mathrm{e}^{-\frac{x^2}{8}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^2}{4}}\sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3}{4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{4}}}{(y_1)^2}\,dx \\ &= y_1\left(-\frac{\mathrm{Ei}_1\left(-\frac{x^2}{4}\right)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-\frac{x^2}{4}}\sqrt{x}\right) + c_2\left(\mathrm{e}^{-\frac{x^2}{4}}\sqrt{x}\left(-\frac{\mathrm{Ei}_1\left(-\frac{x^2}{4}\right)}{2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2\left(\frac{d^2}{dx^2}y(x)\right) + 2\left(\frac{d}{dx}y(x)\right)x^3 + (3x^2+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(3x^2+1)y(x)}{4x^2} - \frac{x\left(\frac{d}{dx}y(x)\right)}{2}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{x\left(\frac{d}{dx}y(x)\right)}{2} + \frac{(3x^2+1)y(x)}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

$\left[P_2(x) = \frac{x}{2}, P_3(x) = \frac{3x^2+1}{4x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 0$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{1}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) + 2\left(\frac{d}{dx}y(x)\right)x^3 + (3x^2+1)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^3 \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x^3 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r+2}$

- Shift index using $k->k-2$

  $x^3 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=2}^{\infty} a_{k-2}(k-2+r)\,x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)^2\,x^r + a_1(1+2r)^2\,x^{1+r} + \left(\sum_{k=2}^{\infty}\left(a_k(2k+2r-1)^2 + a_{k-2}(2k+2r-1)\right)x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = \frac{1}{2}$

- Each term must be 0

  $a_1(1+2r)^2 = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(2k+2r-1)^2 + a_{k-2}(2k+2r-1) = 0$

- Shift index using $k->k+2$

  $a_{k+2}(2k+2r+3)^2 + a_k(2k+2r+3) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{2k+2r+3}$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{a_k}{2k+4}$
- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{a_k}{2k+4}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 25

```
dsolve(4*x^2*diff(diff(y(x),x),x)+2*x^3*diff(y(x),x)+(3*x^2+1)*y(x) = 0,y(x),singsol=all
```

$$y = \sqrt{x}\, \mathrm{e}^{-\frac{x^2}{4}} \left(c_1 + c_2 \operatorname{Ei}_1\left(-\frac{x^2}{4}\right)\right)$$

Mathematica DSolve solution

Solving time : 0.102 (sec)
Leaf size : 44

```
DSolve[{4*x^2*D[y[x],{x,2}]+2*x^3*D[y[x],x]+(1+3*x^2)*y[x]==0,{}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to \frac{1}{2} e^{-\frac{x^2}{4}-\frac{1}{2}} \sqrt{x} \left(c_2 \operatorname{ExpIntegralEi}\left(\frac{x^2}{4}\right) + 2ec_1\right)$$

2.1.562 Problem 578

Internal problem ID [9734]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 578
Date solved : `Monday, January 27, 2025 at 06:13:40 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(x^2+1)\,y''-x(-2x^2+1)\,y'+y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.242 (sec)

Writing the ode as

$$(x^4+x^2)\,y''+(2x^3-x)\,y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2 \\ B &= 2x^3-x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{2x^2-1}{4\,(x^3+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2x^2-1 \\ t &= 4(x^3+x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{2x^2-1}{4\,(x^3+x)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1069: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16\,(x-i)^2} - \frac{3}{16\,(x+i)^2} - \frac{5i}{16\,(x-i)} + \frac{5i}{16\,(x+i)} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2x^2 - 1}{4\left(x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
i	2	0	$\frac{3}{4}$	$\frac{1}{4}$
$-i$	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x - c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \frac{1}{4x - 4i} + \frac{1}{4x + 4i} + (-)\,(0) \\
&= \frac{1}{2x} + \frac{1}{4x - 4i} + \frac{1}{4x + 4i} \\
&= \frac{1}{2x} + \frac{x}{2x^2 + 2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{1}{4x-4i} + \frac{1}{4x+4i}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{4\left(x-i\right)^2} - \frac{1}{4\left(x+i\right)^2}\right) + \left(\frac{1}{2x} + \frac{1}{4x-4i} + \frac{1}{4x+4i}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{2x} + \frac{1}{4x-4i} + \frac{1}{4x+4i}\right) dx} \\ &= \left(x^2+1\right)^{1/4} \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{2x^3-x}{x^4+x^2} \, dx} \\ &= z_1 e^{-\frac{3\ln\left(x^2+1\right)}{4} + \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{\sqrt{x}}{\left(x^2+1\right)^{3/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x}{\sqrt{x^2+1}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3-x}{x^4+x^2} \, dx}}{\left(y_1\right)^2} \, dx \\ &= y_1 \int \frac{e^{-\frac{3\ln\left(x^2+1\right)}{2} + \ln(x)}}{\left(y_1\right)^2} \, dx \\ &= y_1 \left(-\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{x}{\sqrt{x^2+1}}\right) + c_2 \left(\frac{x}{\sqrt{x^2+1}} \left(-\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)-x(-2x^2+1)\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{y(x)}{x^2(x^2+1)}-\frac{(2x^2-1)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{(2x^2-1)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}+\frac{y(x)}{x^2(x^2+1)}=0$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x)=\frac{2x^2-1}{x(x^2+1)}, P_3(x)=\frac{1}{x^2(x^2+1)}\right]$
  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=-1$
  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=1$
  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(2x^2-1)\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..4$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,x^{k+r-2+m}$
  - Shift index using $k->k+2-m$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)^2x^r+a_1r^2x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_k(k+r-1)^2+a_{k-2}(k-2+r)(k+r-1)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)^2=0$

- Values of r that satisfy the indicial equation

$r = 1$

- Each term must be 0

  $a_1 r^2 = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\,(a_k(k+r-1)+a_{k-2}(k-2+r)) = 0$

- Shift index using $k- >k+2$

  $(k+r+1)\,(a_{k+2}(k+r+1)+a_k(k+r)) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r)}{k+r+1}$

- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{a_k(k+1)}{k+2}$

- Solution for $r = 1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k(k+1)}{k+2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.022 (sec)
Leaf size : 25

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)-x*(-2*x^2+1)*diff(y(x),x)+y(x) = 0,y(x),singsol=
```

$$y = \frac{x\left(\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) c_2 + c_1\right)}{\sqrt{x^2+1}}$$

Mathematica DSolve solution

Solving time : 0.207 (sec)
Leaf size : 112

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]-x*(1-2*x^2)*D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSo
```

$$y(x) \to \exp\left(\int_1^x \frac{2K[1]^2+1}{2\left(K[1]^3+K[1]\right)}dK[1] - \frac{1}{2}\int_1^x \frac{2K[2]^2-1}{K[2]^3+K[2]}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{2K[1]^2+1}{2\left(K[1]^3+K[1]\right)}dK[1]\right)dK[3] + c_1\right)$$

2.1.563 Problem 579

Internal problem ID [9735]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 579
Date solved : `Monday, January 27, 2025 at 06:13:41 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(x^2+2)\,y''+7x^3y'+(3x^2+1)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.312 (sec)

Writing the ode as

$$(2x^4+4x^2)\,y''+7x^3y'+(3x^2+1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x^4+4x^2\\B&=7x^3\\C&=3x^2+1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-3x^4-16}{16\,(x^3+2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=-3x^4-16\\t&=16(x^3+2x)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-3x^4-16}{16\,(x^3+2x)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1071: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^3 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i\sqrt{2}$ of order 2. There is a pole at $x = -i\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{64\,(x - i\sqrt{2})^2} - \frac{7}{64\,(x + i\sqrt{2})^2} - \frac{9i\sqrt{2}}{128\,(x - i\sqrt{2})} + \frac{9i\sqrt{2}}{128\,(x + i\sqrt{2})} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x-i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}\end{aligned}$$

For the pole at $x = -i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x+i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^4 - 16}{16\left(x^3 + 2x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4}\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x^4 - 16}{16\left(x^3 + 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$i\sqrt{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$
$-i\sqrt{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{4}$ then

$$\begin{aligned}d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right)\\ &= \frac{3}{4} - \left(\frac{3}{4}\right)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + \frac{1}{8x-8i\sqrt{2}} + \frac{1}{8x+8i\sqrt{2}} + (0) \\ &= \frac{1}{2x} + \frac{1}{8x-8i\sqrt{2}} + \frac{1}{8x+8i\sqrt{2}} \\ &= \frac{1}{2x} + \frac{x}{4x^2+8}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{1}{8x-8i\sqrt{2}} + \frac{1}{8x+8i\sqrt{2}}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{8\left(x-i\sqrt{2}\right)^2} - \frac{1}{8\left(x+i\sqrt{2}\right)^2}\right) + \left(\frac{1}{2x} + \frac{}{8x}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= pe^{\int \omega\,dx} \\ &= \mathrm{e}^{\int\left(\frac{1}{2x}+\frac{1}{8x-8i\sqrt{2}}+\frac{1}{8x+8i\sqrt{2}}\right)dx} \\ &= \sqrt{x}\,(x^2+2)^{1/8}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{7x^3}{2x^4+4x^2}\,dx} \\ &= z_1 e^{-\frac{7\ln\left(x^2+2\right)}{8}} \\ &= z_1\left(\frac{1}{(x^2+2)^{7/8}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{(x^2+2)^{3/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x^3}{2x^4+4x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{7\ln(x^2+2)}{4}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{1}{(x^2+2)^{1/4}\, x} dx \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x}}{(x^2+2)^{3/4}} \right) + c_2 \left(\frac{\sqrt{x}}{(x^2+2)^{3/4}} \left(\int \frac{1}{(x^2+2)^{1/4}\, x} dx \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2(x^2+2)\left(\frac{d^2}{dx^2}y(x)\right) + 7\left(\frac{d}{dx}y(x)\right)x^3 + (3x^2+1)\, y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(3x^2+1)y(x)}{2(x^2+2)x^2} - \frac{7x\left(\frac{d}{dx}y(x)\right)}{2(x^2+2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{7x\left(\frac{d}{dx}y(x)\right)}{2(x^2+2)} + \frac{(3x^2+1)y(x)}{2(x^2+2)x^2} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{7x}{2(x^2+2)}, P_3(x) = \frac{3x^2+1}{2(x^2+2)x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{1}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x^2(x^2+2)\left(\frac{d^2}{dx^2}y(x)\right) + 7\left(\frac{d}{dx}y(x)\right)x^3 + (3x^2+1)\, y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

$$x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^3 \cdot \left(\frac{d}{dx} y(x)\right)$ to series expansion

$$x^3 \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r+2}$$

- Shift index using $k- >k - 2$

$$x^3 \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=2}^{\infty} a_{k-2} (k-2+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2} y(x)\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d^2}{dx^2} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

$$x^m \cdot \left(\frac{d^2}{dx^2} y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+2r)^2 x^r + a_1(1+2r)^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r-1)^2 + a_{k-2}(2k+2r-1)\,(k+r-1)\right) x$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = \frac{1}{2}$

- Each term must be 0

  $a_1(1+2r)^2 = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $4\left(\frac{a_{k-2}(k+r-1)}{2} + \left(k+r-\frac{1}{2}\right) a_k\right) \left(k+r-\frac{1}{2}\right) = 0$

- Shift index using $k- >k + 2$

  $4\left(\frac{a_k(k+r+1)}{2} + \left(k+\frac{3}{2}+r\right) a_{k+2}\right) \left(k+\frac{3}{2}+r\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r+1)}{2k+2r+3}$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{a_k\left(k+\frac{3}{2}\right)}{2k+4}$

- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{a_k\left(k+\frac{3}{2}\right)}{2k+4}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
         -> solution has integrals; searching for one without integrals...
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
            <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
         <- hypergeometric solution without integrals succesful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returnin
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.234 (sec)
Leaf size : 81

```
dsolve(2*x^2*(x^2+2)*diff(diff(y(x),x),x)+7*x^3*diff(y(x),x)+(3*x^2+1)*y(x) = 0,y(x),sin
```

$$y = \frac{\sqrt{x}\left(2^{3/4}c_1 + 2\arctan\left(\frac{\sqrt{2}\,(2x^2+4)^{1/4}}{2}\right)c_2 + \ln\left(-\sqrt{2}\,(2x^2+4)^{1/4}+2\right)c_2 - \ln\left(\sqrt{2}\,(2x^2+4)^{1/4}+2\right)c_2\right)2}{2\,(x^2+2)^{3/4}}$$

Mathematica DSolve solution

Solving time : 0.429 (sec)
Leaf size : 93

```
DSolve[{2*x^2*(2+x^2)*D[y[x],{x,2}]+7*x^3*D[y[x],x]+(1+3*x^2)*y[x]==0,{}},y[x],x,IncludeSingula
```

$$y(x) \to \frac{\exp\left(\int_1^x \frac{3K[1]^2+4}{4K[1]^3+8K[1]}dK[1]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[2]} \frac{3K[1]^2+4}{4K[1]^3+8K[1]}dK[1]\right)dK[2]+c_1\right)}{(x^2+2)^{7/8}}$$

2.1.564 Problem 580

Internal problem ID [9736]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 580
Date solved : Monday, January 27, 2025 at 06:13:41 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$x^2(x^2+1)\,y''-x(-4x^2+1)\,y'+(2x^2+1)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.270 (sec)

Writing the ode as

$$(x^4+x^2)\,y''+(4x^3-x)\,y'+(2x^2+1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^4+x^2\\ B&=4x^3-x\\ C&=2x^2+1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-6x^2-1}{4\,(x^3+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=-6x^2-1\\ t&=4(x^3+x)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-6x^2-1}{4\,(x^3+x)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1073: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3+x)^2$. There is a pole at $x=0$ of order 2. There is a pole at $x=i$ of order 2. There is a pole at $x=-i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16\,(x-i)^2} + \frac{5}{16\,(x+i)^2} + \frac{3i}{16\,(x-i)} - \frac{3i}{16\,(x+i)} - \frac{1}{4x^2}$$

For the pole at $x=0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x=i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-6x^2 - 1}{4\left(x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
i	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-i$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= 0 - (0) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} - \frac{1}{4(x-i)} - \frac{1}{4(x+i)} + (0) \\
&= \frac{1}{2x} - \frac{1}{4(x-i)} - \frac{1}{4(x+i)} \\
&= \frac{1}{2x^3 + 2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{4\,(x-i)} - \frac{1}{4\,(x+i)}\right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{1}{4\,(x-i)^2} + \frac{1}{4\,(x+i)^2}\right) + \left(\frac{1}{2x} - \frac{1}{4\,(x-i)} - \frac{1}{4\,(x+\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{1}{2x} - \frac{1}{4(x-i)} - \frac{1}{4(x+i)}\right) dx} \\ &= \frac{\sqrt{x}}{\left(x^2+1\right)^{1/4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x^3-x}{x^4+x^2}\, dx} \\ &= z_1 e^{-\frac{5\ln\left(x^2+1\right)}{4} + \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{\sqrt{x}}{\left(x^2+1\right)^{5/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x}{\left(x^2+1\right)^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x^3-x}{x^4+x^2}\, dx}}{\left(y_1\right)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{5\ln\left(x^2+1\right)}{2} + \ln(x)}}{\left(y_1\right)^2}\, dx \\ &= y_1 \left(\sqrt{x^2+1} - \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{x}{\left(x^2+1\right)^{3/2}}\right) + c_2 \left(\frac{x}{\left(x^2+1\right)^{3/2}} \left(\sqrt{x^2+1} - \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)-x(-4x^2+1)\left(\frac{d}{dx}y(x)\right)+(2x^2+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(2x^2+1)y(x)}{x^2(x^2+1)}-\frac{(4x^2-1)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{(4x^2-1)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}+\frac{(2x^2+1)y(x)}{x^2(x^2+1)}=0$

- □ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{4x^2-1}{x(x^2+1)},P_3(x)=\frac{2x^2+1}{x^2(x^2+1)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=-1$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=1$

  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(4x^2-1)\left(\frac{d}{dx}y(x)\right)+(2x^2+1)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

    $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..4$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$a_0(-1+r)^2 x^r + a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)^2 + a_{k-2}(k+r)(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 1$
- Each term must be 0
  $a_1 r^2 = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-1)\left(a_k(k+r-1) + a_{k-2}(k+r)\right) = 0$
- Shift index using $k- >k+2$
  $(k+r+1)\left(a_{k+2}(k+r+1) + a_k(k+r+2)\right) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{a_k(k+r+2)}{k+r+1}$
- Recursion relation for $r = 1$
  $a_{k+2} = -\frac{a_k(k+3)}{k+2}$
- Solution for $r = 1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k(k+3)}{k+2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.038` (sec)
Leaf size : 35

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)-x*(-4*x^2+1)*diff(y(x),x)+(2*x^2+1)*y(x) = 0,y(x
```

$$y = \frac{x\left(\sqrt{x^2+1}\, c_2 - \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) c_2 + c_1\right)}{(x^2+1)^{3/2}}$$

Mathematica DSolve solution

Solving time : `0.148` (sec)
Leaf size : 96

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]-x*(1-4*x^2)*D[y[x],x]+(1+2*x^2)*y[x]==0,{}},y[x],x,Include
```

$$y(x) \to \exp\left(\int_1^x \frac{1}{2K[1]^3+2K[1]}dK[1] - \frac{1}{2}\int_1^x \frac{4K[2]^2-1}{K[2]^3+K[2]}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{1}{2K[1]^3+2K[1]}dK[1]\right)dK[3] + c_1\right)$$

2.1.565 Problem 581

Internal problem ID [9737]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 581
Date solved : `Monday, January 27, 2025 at 06:13:42 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2\left(x^2+4\right)y''+3x\left(3x^2+8\right)y'+\left(-9x^2+1\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.338 (sec)

Writing the ode as

$$\left(4x^4+16x^2\right)y''+\left(9x^3+24x\right)y'+\left(-9x^2+1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=4x^4+16x^2\\ B&=9x^3+24x\\ C&=-9x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{153x^4+704x^2-256}{64\left(x^3+4x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=153x^4+704x^2-256\\ t&=64\left(x^3+4x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{153x^4+704x^2-256}{64\left(x^3+4x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1075: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64(x^3+4x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 2i$ of order 2. There is a pole at $x = -2i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} - \frac{39}{256(x-2i)^2} - \frac{39}{256(x+2i)^2} - \frac{377i}{512(x-2i)} + \frac{377i}{512(x+2i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = 2i$ let b be the coefficient of $\frac{1}{(x-2i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{39}{256}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{13}{16} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{16} \end{aligned}$$

For the pole at $x = -2i$ let b be the coefficient of $\frac{1}{(x+2i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{39}{256}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{13}{16} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{16}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{153x^4 + 704x^2 - 256}{64\left(x^3 + 4x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{153}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{17}{8} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{9}{8}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{153x^4 + 704x^2 - 256}{64\left(x^3 + 4x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$2i$	2	0	$\frac{13}{16}$	$\frac{3}{16}$
$-2i$	2	0	$\frac{13}{16}$	$\frac{3}{16}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{17}{8}$	$-\frac{9}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{17}{8}$ then

$$\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{+} + \alpha_{c_3}^{+}\right) \\
&= \frac{17}{8} - \left(\frac{17}{8}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{+}}{x-c_3}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + \frac{13}{16\,(x-2i)} + \frac{13}{16\,(x+2i)} + (0) \\
&= \frac{1}{2x} + \frac{13}{16\,(x-2i)} + \frac{13}{16\,(x+2i)} \\
&= \frac{1}{2x} + \frac{13x}{8x^2+32}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{13}{16\,(x-2i)} + \frac{13}{16\,(x+2i)}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{13}{16\,(x-2i)^2} - \frac{13}{16\,(x+2i)^2}\right) + \left(\frac{1}{2x} + \frac{1}{16\,(x}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} + \frac{13}{16(x-2i)} + \frac{13}{16(x+2i)}\right) dx} \\
&= \left(x^2+4\right)^{13/16} \sqrt{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{9x^3+24x}{4x^4+16x^2}\, dx} \\
&= z_1 e^{-\frac{3\ln\left(x^2+4\right)}{16} - \frac{3\ln(x)}{4}} \\
&= z_1 \left(\frac{1}{\left(x^2+4\right)^{3/16} x^{3/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(x^2+4\right)^{5/8}}{x^{1/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{\mathrm{e}^{\int -\frac{9x^3+24x}{4x^4+16x^2}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{\mathrm{e}^{-\frac{3\ln\left(x^2+4\right)}{8} - \frac{3\ln(x)}{2}}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{3\ln\left(x^2+4\right)}{8} - \frac{3\ln(x)}{2}} \sqrt{x}}{\left(x^2+4\right)^{5/4}}\, dx\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x^2+4)^{5/8}}{x^{1/4}} \right) + c_2 \left(\frac{(x^2+4)^{5/8}}{x^{1/4}} \left(\int \frac{e^{-\frac{3\ln(x^2+4)}{8} - \frac{3\ln(x)}{2}} \sqrt{x}}{(x^2+4)^{5/4}} dx \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2(x^2+4)\left(\frac{d^2}{dx^2}y(x)\right) + 3x(3x^2+8)\left(\frac{d}{dx}y(x)\right) + (-9x^2+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{(9x^2-1)y(x)}{4x^2(x^2+4)} - \frac{3(3x^2+8)\left(\frac{d}{dx}y(x)\right)}{4x(x^2+4)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{3(3x^2+8)\left(\frac{d}{dx}y(x)\right)}{4x(x^2+4)} - \frac{(9x^2-1)y(x)}{4x^2(x^2+4)} = 0$

- Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{3(3x^2+8)}{4x(x^2+4)}, P_3(x) = -\frac{9x^2-1}{4x^2(x^2+4)}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{3}{2}$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{1}{16}$
  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4x^2(x^2+4)\left(\frac{d^2}{dx^2}y(x)\right) + 3x(3x^2+8)\left(\frac{d}{dx}y(x)\right) + (-9x^2+1)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k-m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k- >k+1-m$

$$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+4r)^2\, x^r + a_1(5+4r)^2\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(4k+4r+1)^2 + a_{k-2}(4k+4r+1)\,(k-3+r)\right) x^{k+}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+4r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = -\frac{1}{4}$$

- Each term must be 0

$$a_1(5+4r)^2 = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$16\left(\frac{a_{k-2}(k-3+r)}{4} + \left(k+r+\frac{1}{4}\right)a_k\right)\left(k+r+\frac{1}{4}\right) = 0$$

- Shift index using $k- >k+2$

$$16\left(\frac{a_k(k+r-1)}{4} + \left(k+\frac{9}{4}+r\right)a_{k+2}\right)\left(k+\frac{9}{4}+r\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k(k+r-1)}{4k+4r+9}$$

- Recursion relation for $r = -\frac{1}{4}$

$$a_{k+2} = -\frac{a_k\left(k-\frac{5}{4}\right)}{4k+8}$$

- Solution for $r = -\frac{1}{4}$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{4}}, a_{k+2} = -\frac{a_k\left(k-\frac{5}{4}\right)}{4k+8}, a_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
```

```
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.115 (sec)
Leaf size : 66

```
dsolve(4*x^2*(x^2+4)*diff(diff(y(x),x),x)+3*x*(3*x^2+8)*diff(y(x),x)+(-9*x^2+1)*y(x) = 0
```

$$y = \frac{c_2(x^2+4)^{5/8}\left(x^2 \operatorname{hypergeom}\left(\left[1,1,\frac{13}{8}\right],[2,2],-\frac{x^2}{4}\right)-\frac{32\gamma}{5}+\frac{64\ln(2)}{5}-\frac{64\ln(x)}{5}-\frac{32\Psi\left(\frac{5}{8}\right)}{5}\right)2^{3/4}+c_1(x^2+4)^{5/}}{x^{1/4}}$$

Mathematica DSolve solution

Solving time : 0.264 (sec)
Leaf size : 118

```
DSolve[{4*x^2*(4+x^2)*D[y[x],{x,2}]+3*x*(8+3*x^2)*D[y[x],x]+(1-9*x^2)*y[x]==0,{}},y[x],x,Inclu
```

$$y(x) \to \exp\left(\int_1^x \frac{17K[1]^2+16}{8K[1]^3+32K[1]}dK[1] - \frac{1}{2}\int_1^x \frac{9K[2]^2+24}{4K[2]^3+16K[2]}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{17K[1]^2+16}{8K[1]^3+32K[1]}dK[1]\right)dK[3] + c_1\right)$$

2.1.566 Problem 582

Internal problem ID [9738]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 582
Date solved : `Monday, January 27, 2025 at 06:13:43 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$3x^2(x^2+3)\,y''+x(11x^2+3)\,y'+(5x^2+1)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.319 (sec)

Writing the ode as

$$\left(3x^4+9x^2\right)y''+\left(11x^3+3x\right)y'+\left(5x^2+1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^4+9x^2\\ B &= 11x^3+3x\\ C &= 5x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-5x^4+18x^2-81}{36\left(x^3+3x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -5x^4+18x^2-81\\ t &= 36\left(x^3+3x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-5x^4+18x^2-81}{36\left(x^3+3x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1077: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^3 + 3x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i\sqrt{3}$ of order 2. There is a pole at $x = -i\sqrt{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} - \frac{5}{36\left(x - i\sqrt{3}\right)^2} - \frac{5}{36\left(x + i\sqrt{3}\right)^2} - \frac{7i\sqrt{3}}{108\left(x - i\sqrt{3}\right)} + \frac{7i\sqrt{3}}{108\left(x + i\sqrt{3}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = i\sqrt{3}$ let b be the coefficient of $\frac{1}{\left(x-i\sqrt{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}
\end{aligned}$$

For the pole at $x = -i\sqrt{3}$ let b be the coefficient of $\frac{1}{\left(x+i\sqrt{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-5x^4 + 18x^2 - 81}{36\left(x^3 + 3x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{5}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-5x^4 + 18x^2 - 81}{36\left(x^3 + 3x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$i\sqrt{3}$	2	0	$\frac{5}{6}$	$\frac{1}{6}$
$-i\sqrt{3}$	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{6}$	$\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{6}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{5}{6} - \left(\frac{5}{6}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \frac{1}{6x-6i\sqrt{3}} + \frac{1}{6x+6i\sqrt{3}} + (0) \\
&= \frac{1}{2x} + \frac{1}{6x-6i\sqrt{3}} + \frac{1}{6x+6i\sqrt{3}} \\
&= \frac{1}{2x} + \frac{x}{3x^2+9}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{1}{6x-6i\sqrt{3}} + \frac{1}{6x+6i\sqrt{3}}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{6\,(x-i\sqrt{3})^2} - \frac{1}{6\,(x+i\sqrt{3})^2}\right) + \left(\frac{1}{2x} + \frac{1}{6x-\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(\frac{1}{2x} + \frac{1}{6x-6i\sqrt{3}} + \frac{1}{6x+6i\sqrt{3}}\right) dx} \\
&= \sqrt{x}\,(x^2+3)^{1/6}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{11x^3+3x}{3x^4+9x^2}\, dx} \\
&= z_1 e^{-\frac{5\ln(x^2+3)}{6} - \frac{\ln(x)}{6}} \\
&= z_1 \left(\frac{1}{(x^2+3)^{5/6}\, x^{1/6}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/3}}{(x^2+3)^{2/3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{11x^3+3x}{3x^4+9x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\frac{5\ln(x^2+3)}{3}-\frac{\ln(x)}{3}}}{(y_1)^2}\,dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{-\frac{5\ln(x^2+3)}{3}-\frac{\ln(x)}{3}}(x^2+3)^{4/3}}{x^{2/3}}dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{x^{1/3}}{(x^2+3)^{2/3}}\right) + c_2 \left(\frac{x^{1/3}}{(x^2+3)^{2/3}}\left(\int \frac{\mathrm{e}^{-\frac{5\ln(x^2+3)}{3}-\frac{\ln(x)}{3}}(x^2+3)^{4/3}}{x^{2/3}}dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$3x^2(x^2+3)\left(\frac{d^2}{dx^2}y(x)\right)+x(11x^2+3)\left(\frac{d}{dx}y(x)\right)+(5x^2+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x)=-\frac{(5x^2+1)y(x)}{3x^2(x^2+3)}-\frac{(11x^2+3)\left(\frac{d}{dx}y(x)\right)}{3x(x^2+3)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x)+\frac{(11x^2+3)\left(\frac{d}{dx}y(x)\right)}{3x(x^2+3)}+\frac{(5x^2+1)y(x)}{3x^2(x^2+3)}=0$

□ Check to see if x_0 is a regular singular point
  - Define functions
    $\left[P_2(x)=\frac{11x^2+3}{3x(x^2+3)}, P_3(x)=\frac{5x^2+1}{3x^2(x^2+3)}\right]$
  - $x\cdot P_2(x)$ is analytic at $x=0$
    $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{1}{3}$
  - $x^2\cdot P_3(x)$ is analytic at $x=0$
    $\left.(x^2\cdot P_3(x))\right|_{x=0}=\frac{1}{9}$
  - $x=0$is a regular singular point
    Check to see if x_0 is a regular singular point
    $x_0=0$
- Multiply by denominators
  $3x^2(x^2+3)\left(\frac{d^2}{dx^2}y(x)\right)+x(11x^2+3)\left(\frac{d}{dx}y(x)\right)+(5x^2+1)\,y(x)=0$
- Assume series solution for $y(x)$
  $y(x)=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- >k + 2 - m$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+3r)^2 x^r + a_1(2+3r)^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r-1)^2 + a_{k-2}(3k+3r-1)(k+r-1)\right) x^{k+r}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+3r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = \frac{1}{3}$

- Each term must be 0

  $a_1(2+3r)^2 = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $9\left(\frac{a_{k-2}(k+r-1)}{3} + \left(k - \frac{1}{3} + r\right) a_k\right)\left(k - \frac{1}{3} + r\right) = 0$

- Shift index using $k- >k + 2$

  $9\left(\frac{a_k(k+r+1)}{3} + \left(k + \frac{5}{3} + r\right) a_{k+2}\right)\left(k + \frac{5}{3} + r\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r+1)}{3k+3r+5}$

- Recursion relation for $r = \frac{1}{3}$

  $a_{k+2} = -\frac{a_k\left(k+\frac{4}{3}\right)}{3k+6}$

- Solution for $r = \frac{1}{3}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -\frac{a_k\left(k+\frac{4}{3}\right)}{3k+6}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
         -> solution has integrals; searching for one without integrals...
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
            <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
         <- hypergeometric solution without integrals succesful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returr
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.243 (sec)
Leaf size : 102

```
dsolve(3*x^2*(x^2+3)*diff(diff(y(x),x),x)+x*(11*x^2+3)*diff(y(x),x)+(5*x^2+1)*y(x) = (
```

$$y = \frac{x^{1/3}\left(2\sqrt{3}\arctan\left(\frac{(9x^2+27)^{1/3}\sqrt{3}}{6+(9x^2+27)^{1/3}}\right)c_2+3\,3^{1/3}c_1+2\ln\left(3-(9x^2+27)^{1/3}\right)c_2-\ln\left(9+3(9x^2+27)^{1/3}+\right.\right.}{9\,(x^2+3)^{2/3}}$$

Mathematica DSolve solution

Solving time : 0.104 (sec)
Leaf size : 57

```
DSolve[{3*x^2*(3+x^2)*D[y[x],x]+x*(3+11*x^2)*D[y[x],x]+(1+5*x^2)*y[x]==0,{}},y[x],x,IncludeS
```

$$y(x) \to c_1 \exp\left(\int_1^x -\frac{5K[1]^2+1}{3K[1]^4+11K[1]^3+9K[1]^2+3K[1]}dK[1]\right)$$
$$y(x) \to 0$$

2.1.567 Problem 583

Internal problem ID [9739]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 583
Date solved : `Monday, January 27, 2025 at 06:13:43 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$9x^2y'' - 3x\left(-2x^2+7\right)y' + \left(2x^2+25\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.278 (sec)

Writing the ode as

$$9x^2y'' + \left(6x^3-21x\right)y' + \left(2x^2+25\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^2 \\ B &= 6x^3-21x \\ C &= 2x^2+25 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^4-24x^2-9}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^4-24x^2-9 \\ t &= 36x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4-24x^2-9}{36x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1079: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{9} - \frac{2}{3} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{3} - \frac{1}{x} - \frac{15}{8x^3} - \frac{45}{8x^5} - \frac{2835}{128x^7} - \frac{12555}{128x^9} - \frac{477495}{1024x^{11}} - \frac{2380185}{1024x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{3}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{3}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{9}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{4x^4 - 24x^2 - 9}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{x^2}{9} - \frac{2}{3}\right) + \left(-\frac{1}{4x^2}\right) \\ &= \frac{x^2}{9} - \frac{2}{3} - \frac{1}{4x^2}\end{aligned}$$

We see that the coefficient of the term x in the quotient is $-\frac{2}{3}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{2}{3}\right) - (0) \\ &= -\frac{2}{3}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{x}{3} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{2}{3}}{\frac{1}{3}} - 1\right) &&= -\frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{2}{3}}{\frac{1}{3}} - 1\right) &&= \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^4 - 24x^2 - 9}{36x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{3}$	$-\frac{3}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\left(\frac{x}{3}\right) \\ &= \frac{1}{2x} - \frac{x}{3} \\ &= \frac{1}{2x} - \frac{x}{3} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2x} - \frac{x}{3}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{3}\right) + \left(\frac{1}{2x} - \frac{x}{3}\right)^2 - \left(\frac{4x^4 - 24x^2 - 9}{36x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \,dx} \\ &= e^{\int \left(\frac{1}{2x} - \frac{x}{3}\right)dx} \\ &= \sqrt{x}\, e^{-\frac{x^2}{6}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{6x^3-21x}{9x^2}\,dx} \\
&= z_1 e^{-\frac{x^2}{6}+\frac{7\ln(x)}{6}} \\
&= z_1\left(x^{7/6}\mathrm{e}^{-\frac{x^2}{6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^{5/3}\mathrm{e}^{-\frac{x^2}{3}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{6x^3-21x}{9x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{3}+\frac{7\ln(x)}{3}}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{\mathrm{Ei}_1\left(-\frac{x^2}{3}\right)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^{5/3}\mathrm{e}^{-\frac{x^2}{3}}\right) + c_2\left(x^{5/3}\mathrm{e}^{-\frac{x^2}{3}}\left(-\frac{\mathrm{Ei}_1\left(-\frac{x^2}{3}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$9x^2\left(\frac{d^2}{dx^2}y(x)\right) - 3x(-2x^2+7)\left(\frac{d}{dx}y(x)\right) + (2x^2+25)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(2x^2+25)y(x)}{9x^2} - \frac{(2x^2-7)\left(\frac{d}{dx}y(x)\right)}{3x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(2x^2-7)\left(\frac{d}{dx}y(x)\right)}{3x} + \frac{(2x^2+25)y(x)}{9x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = \frac{2x^2-7}{3x}, P_3(x) = \frac{2x^2+25}{9x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = -\frac{7}{3}$
- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{25}{9}$
- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $9x^2\left(\frac{d^2}{dx^2}y(x)\right) + 3x(2x^2 - 7)\left(\frac{d}{dx}y(x)\right) + (2x^2 + 25)\, y(x) = 0$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r-1+m}$
- Shift index using $k- >k + 1 - m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, x^{k+r}$
- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-5 + 3r)^2\, x^r + a_1(-2 + 3r)^2\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k + 3r - 5)^2 + 2a_{k-2}(3k + 3r - 5)\right) x^{k+r}\right) = ($

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-5 + 3r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = \frac{5}{3}$
- Each term must be 0

  $a_1(-2 + 3r)^2 = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(3k + 3r - 5)^2 + 2a_{k-2}(3k + 3r - 5) = 0$
- Shift index using $k- >k + 2$

  $a_{k+2}(3k + 3r + 1)^2 + 2a_k(3k + 3r + 1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2a_k}{3k+3r+1}$

- Recursion relation for $r = \frac{5}{3}$

  $a_{k+2} = -\frac{2a_k}{3k+6}$

- Solution for $r = \frac{5}{3}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{3}}, a_{k+2} = -\frac{2a_k}{3k+6}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 25

```
dsolve(9*x^2*diff(diff(y(x),x),x)-3*x*(-2*x^2+7)*diff(y(x),x)+(2*x^2+25)*y(x) = 0,y(x),s
```

$$y = x^{5/3} \mathrm{e}^{-\frac{x^2}{3}} \left(c_1 + c_2 \operatorname{Ei}_1\left(-\frac{x^2}{3}\right)\right)$$

Mathematica DSolve solution

Solving time : `0.065` (sec)
Leaf size : 39

```
DSolve[{9*x^2*D[y[x],{x,2}]-3*x*(7-2*x^2)*D[y[x],x]+(25+2*x^2)*y[x]==0,{}},y[x],x,IncludeSingul
```

$$y(x) \to \frac{1}{2} e^{-\frac{x^2}{3}} x^{5/3} \left(c_2 \operatorname{ExpIntegralEi}\left(\frac{x^2}{3}\right) + 2c_1\right)$$

2.1.568 Problem 584

Internal problem ID [9740]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 584
Date solved : `Monday, January 27, 2025 at 06:13:44 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x(-x^2+1)\,y' + (x^2+1)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.245 (sec)

Writing the ode as

$$x^2y'' + (x^3-x)\,y' + (x^2+1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^3 - x \\ C &= x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4-4x^2-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4-4x^2-1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4-4x^2-1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1081: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{4} - 1 - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{1}{x} - \frac{5}{4x^3} - \frac{5}{2x^5} - \frac{105}{16x^7} - \frac{155}{8x^9} - \frac{1965}{32x^{11}} - \frac{3265}{16x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^4 - 4x^2 - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{x^2}{4} - 1\right) + \left(-\frac{1}{4x^2}\right) \\ &= \frac{x^2}{4} - 1 - \frac{1}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is -1. Now b can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 1\right) &= -\frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 1\right) &= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 4x^2 - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\left(\frac{x}{2}\right) \\ &= \frac{1}{2x} - \frac{x}{2} \\ &= \frac{1}{2x} - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered} (0) + 2\left(\frac{1}{2x} - \frac{x}{2}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{2}\right) + \left(\frac{1}{2x} - \frac{x}{2}\right)^2 - \left(\frac{x^4 - 4x^2 - 1}{4x^2}\right)\right) = 0 \\ 0 = 0 \end{gathered}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{1}{2x} - \frac{x}{2}\right) dx} \\ &= \sqrt{x}\, e^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^3-x}{x^2}\,dx} \\ &= z_1 e^{-\frac{x^2}{4}+\frac{\ln(x)}{2}} \\ &= z_1\left(\sqrt{x}\,\mathrm{e}^{-\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{-\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^3-x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{2}+\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(-\frac{\mathrm{Ei}_1\left(-\frac{x^2}{2}\right)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(x\,\mathrm{e}^{-\frac{x^2}{2}}\right) + c_2\left(x\,\mathrm{e}^{-\frac{x^2}{2}}\left(-\frac{\mathrm{Ei}_1\left(-\frac{x^2}{2}\right)}{2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right) - x(-x^2+1)\left(\frac{d}{dx}y(x)\right) + (x^2+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{(x^2+1)y(x)}{x^2} - \frac{(x^2-1)\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x) + \frac{(x^2-1)\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(x^2+1)y(x)}{x^2} = 0$
- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - ○ Define functions

$\left[P_2(x) = \frac{x^2-1}{x}, P_3(x) = \frac{x^2+1}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 1$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(x^2-1)\left(\frac{d}{dx}y(x)\right) + (x^2+1)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)^2 x^r + a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)^2 + a_{k-2}(k+r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 1$

- Each term must be 0

  $a_1 r^2 = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\,(a_k(k+r-1) + a_{k-2}) = 0$

- Shift index using $k- >k+2$

  $(k+r+1)\,(a_{k+2}(k+r+1) + a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{k+r+1}$

- Recursion relation for $r = 1$
  $a_{k+2} = -\frac{a_k}{k+2}$
- Solution for $r = 1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{k+2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 23

```
dsolve(x^2*diff(diff(y(x),x),x)-x*(-x^2+1)*diff(y(x),x)+(x^2+1)*y(x) = 0,y(x),singsol=
```

$$y = x\,\mathrm{e}^{-\frac{x^2}{2}}\left(c_1 + c_2\,\mathrm{Ei}_1\left(-\frac{x^2}{2}\right)\right)$$

Mathematica DSolve solution

Solving time : 0.026 (sec)
Leaf size : 35

```
DSolve[{x^2*D[y[x],{x,2}]-x*(1-x^2)*D[y[x],x]+(1+x^2)*y[x]==0,{}},y[x],x,IncludeSingularSolu
```

$$y(x) \to \frac{1}{2}e^{-\frac{x^2}{2}}x\left(c_1\,\text{ExpIntegralEi}\left(\frac{x^2}{2}\right) + 2c_2\right)$$

2.1.569 Problem 585

Internal problem ID [9741]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 585
Date solved : `Monday, January 27, 2025 at 06:13:45 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1-2x)\, y''+3xy'+(1+4x)\, y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.222 (sec)

Writing the ode as

$$\left(-2x^3+x^2\right) y''+3xy'+(1+4x)\, y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -2x^3+x^2 \\ B &= 3x \\ C &= 1+4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{32x^2+16x-1}{4\left(2x^2-x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 32x^2+16x-1 \\ t &= 4\left(2x^2-x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{32x^2+16x-1}{4\left(2x^2-x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1083: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{x} - \frac{1}{4x^2} + \frac{15}{4\left(x - \frac{1}{2}\right)^2} - \frac{3}{x - \frac{1}{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{32x^2 + 16x - 1}{4\left(2x^2 - x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{32x^2 + 16x - 1}{4\left(2x^2 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} - \frac{3}{2\left(x-\frac{1}{2}\right)} + (-)\,(0) \\
&= \frac{1}{2x} - \frac{3}{2\left(x-\frac{1}{2}\right)} \\
&= \frac{-1-4x}{4x^2-2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{3}{2\left(x - \frac{1}{2}\right)}\right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{3}{2\left(x - \frac{1}{2}\right)^2}\right) + \left(\frac{1}{2x} - \frac{3}{2\left(x - \frac{1}{2}\right)}\right)^2 - \left(\frac{32x^2 + 16x - 1}{4\left(2x^2 - x\right)^2}\right)\right)$$

$$0 = ($$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{2x} - \frac{3}{2\left(x - \frac{1}{2}\right)}\right) dx} \\
&= \frac{\sqrt{x}}{\left(-1 + 2x\right)^{3/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{3x}{-2x^3 + x^2} \, dx} \\
&= z_1 e^{-\frac{3\ln(x)}{2} + \frac{3\ln(-1+2x)}{2}} \\
&= z_1 \left(\frac{\left(-1 + 2x\right)^{3/2}}{x^{3/2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{-2x^3 + x^2} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{-3\ln(x) + 3\ln(-1+2x)}}{\left(y_1\right)^2} \, dx \\
&= y_1 \left(\frac{8x^3}{3} + \frac{1}{2} + 6x - 6x^2 - \ln\left(x\right)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x}\left(\frac{8x^3}{3} + \frac{1}{2} + 6x - 6x^2 - \ln\left(x\right)\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(-2x+1)\left(\frac{d^2}{dx^2}y(x)\right)+3x\left(\frac{d}{dx}y(x)\right)+(4x+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{(4x+1)y(x)}{x^2(2x-1)}+\frac{3\left(\frac{d}{dx}y(x)\right)}{x(2x-1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)-\frac{3\left(\frac{d}{dx}y(x)\right)}{x(2x-1)}-\frac{(4x+1)y(x)}{x^2(2x-1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{3}{x(2x-1)},P_3(x)=-\frac{4x+1}{x^2(2x-1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\Big|_{x=0}=3$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=1$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(2x-1)\left(\frac{d^2}{dx^2}y(x)\right)-3x\left(\frac{d}{dx}y(x)\right)+(-4x-1)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

  ◦ Shift index using $k->k-m$

  $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r}$

  ◦ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..3$

  $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  ◦ Shift index using $k->k+2-m$

  $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0(1+r)^2x^r+\left(\sum_{k=1}^{\infty}\left(-a_k(k+r+1)^2+2a_{k-1}(k+r)\,(k-3+r)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-(1+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = -1$

- Each term in the series must be 0, giving the recursion relation

  $-a_k(k+r+1)^2 + 2a_{k-1}(k+r)(k-3+r) = 0$

- Shift index using $k->k+1$

  $-a_{k+1}(k+2+r)^2 + 2a_k(k+r+1)(k+r-2) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k(k+r+1)(k+r-2)}{(k+2+r)^2}$

- Recursion relation for $r = -1$; series terminates at $k = 3$

  $a_{k+1} = \frac{2a_k k(k-3)}{(k+1)^2}$

- Apply recursion relation for $k = 0$

  $a_1 = 0$

- Apply recursion relation for $k = 1$

  $a_2 = -a_1$

- Express in terms of a_0

  $a_2 = 0$

- Apply recursion relation for $k = 2$

  $a_3 = -\frac{4a_2}{9}$

- Express in terms of a_0

  $a_3 = 0$

- Terminating series solution of the ODE for $r = -1$. Use reduction of order to find the second

  $y(x) = a_0 \cdot 0$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 31

```
dsolve(x^2*(1-2*x)*diff(diff(y(x),x),x)+3*diff(y(x),x)*x+(4*x+1)*y(x) = 0,y(x),singsol
```

$$y = \frac{3c_2 \ln(x) + (-8x^3 + 18x^2 - 18x)c_2 + c_1}{x}$$

Mathematica DSolve solution

Solving time : 0.221 (sec)
Leaf size : 105

```
DSolve[{x^2*(1-2*x)*D[y[x],{x,2}]+3*x*D[y[x],x]+(1+4*x)*y[x]==0,{}},y[x],x,IncludeSingularSolu
```

$$y(x) \to \exp\left(\int_1^x \frac{4K[1]+1}{2K[1]-4K[1]^2}dK[1] - \frac{1}{2}\int_1^x \frac{3}{K[2]-2K[2]^2}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{4K[1]+1}{2K[1]-4K[1]^2}dK[1]\right)dK[3] + c_1\right)$$

2.1.570 Problem 586

Internal problem ID [9742]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 586
Date solved : `Monday, January 27, 2025 at 06:13:45 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x(1+x)\,y''+(1-x)\,y'+y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.218 (sec)

Writing the ode as

$$\left(x^2+x\right)y''+(1-x)\,y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+x \\ B &= 1-x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{-x^2-10x-1}{4\left(x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2-10x-1 \\ t &= 4\left(x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-x^2-10x-1}{4\left(x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1085: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} - \frac{2}{x} + \frac{2}{1 + x} + \frac{2}{(1 + x)^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 10x - 1}{4\left(x^2 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 - 10x - 1}{4\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	2	-1
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{1+x} + \frac{1}{2x} + (-)(0) \\
&= -\frac{1}{1+x} + \frac{1}{2x} \\
&= -\frac{x-1}{2x(1+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{1+x} + \frac{1}{2x}\right)(1) + \left(\left(\frac{1}{(1+x)^2} - \frac{1}{2x^2}\right) + \left(-\frac{1}{1+x} + \frac{1}{2x}\right)^2 - \left(\frac{-x^2-10x-1}{4\left(x^2+x\right)^2}\right)\right) = 0$$

$$\frac{1+a_0}{x\left(1+x\right)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x-1)\, \mathrm{e}^{\int \left(-\frac{1}{1+x} + \frac{1}{2x}\right) dx} \\ &= (x-1)\, \mathrm{e}^{-\ln(1+x) + \frac{\ln(x)}{2}} \\ &= \frac{(x-1)\sqrt{x}}{1+x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1-x}{x^2+x} \, dx} \\ &= z_1 e^{\ln(1+x) - \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1+x}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x - 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1-x}{x^2+x} \, dx}}{\left(y_1\right)^2} \, dx \\ &= y_1 \int \frac{e^{2\ln(1+x) - \ln(x)}}{\left(y_1\right)^2} \, dx \\ &= y_1 \left(-\frac{4}{x-1} + \ln\left(x\right)\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x-1) + c_2\left(x - 1\left(-\frac{4}{x-1} + \ln(x)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x(x+1)\left(\frac{d^2}{dx^2}y(x)\right) + (1-x)\left(\frac{d}{dx}y(x)\right) + y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{y(x)}{x(x+1)} + \frac{(x-1)\left(\frac{d}{dx}y(x)\right)}{x(x+1)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x) - \frac{(x-1)\left(\frac{d}{dx}y(x)\right)}{x(x+1)} + \frac{y(x)}{x(x+1)} = 0$

□ Check to see if x_0 is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{x-1}{x(x+1)}, P_3(x) = \frac{1}{x(x+1)}\right]$
  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$
    $\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1} = -2$
  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$
    $\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1} = 0$
  - $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point
  $x_0 = -1$
- Multiply by denominators
  $x(x+1)\left(\frac{d^2}{dx^2}y(x)\right) + (1-x)\left(\frac{d}{dx}y(x)\right) + y(x) = 0$
- Change variables using $x = u-1$ so that the regular singular point is at $u=0$
  $(u^2-u)\left(\frac{d^2}{du^2}y(u)\right) + (2-u)\left(\frac{d}{du}y(u)\right) + y(u) = 0$
- Assume series solution for $y(u)$
  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$
    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$
    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-a_0 r(-3+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(k-2+r) + a_k(k+r-1)^2\right) u^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 3\}$

- Each term in the series must be 0, giving the recursion relation

  $-a_{k+1}(k+1+r)\,(k-2+r) + a_k(k+r-1)^2 = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r-1)^2}{(k+1+r)(k-2+r)}$

- Recursion relation for $r=0$; series terminates at $k=1$

  $a_{k+1} = \frac{a_k(k-1)^2}{(k+1)(k-2)}$

- Apply recursion relation for $k=0$

  $a_1 = -\frac{a_0}{2}$

- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second linea

  $y(u) = a_0 \cdot \left(1-\frac{u}{2}\right)$

- Revert the change of variables $u=x+1$

  $\left[y(x) = a_0\left(-\frac{x}{2}+\frac{1}{2}\right)\right]$

- Recursion relation for $r=3$

  $a_{k+1} = \frac{a_k(k+2)^2}{(k+4)(k+1)}$

- Solution for $r=3$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+3}, a_{k+1} = \frac{a_k(k+2)^2}{(k+4)(k+1)}\right]$

- Revert the change of variables $u=x+1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^{k+3}, a_{k+1} = \frac{a_k(k+2)^2}{(k+4)(k+1)}\right]$

- Combine solutions and rename parameters

  $\left[y(x) = a_0\left(-\frac{x}{2}+\frac{1}{2}\right) + \left(\sum_{k=0}^{\infty} b_k(x+1)^{k+3}\right), b_{k+1} = \frac{b_k(k+2)^2}{(4+k)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 20

```
dsolve(x*(x+1)*diff(diff(y(x),x),x)+(1-x)*diff(y(x),x)+y(x) = 0,y(x),singsol=all)
```

$$y = c_2(x-1)\ln(x) - 4c_2 + c_1(x-1)$$

Mathematica DSolve solution

Solving time : 0.435 (sec)
Leaf size : 112

```
DSolve[{x*(1+x)*D[y[x],{x,2}]+(1-x)*D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to (x-1)\exp\left(\int_1^x \left(\frac{1}{2K[1]} - \frac{1}{K[1]+1}\right) dK[1] - \frac{1}{2}\int_1^x \left(\frac{1}{K[2]} - \frac{2}{K[2]+1}\right) dK[2]\right)\left(c_2\int_1^x \frac{\exp\left(-2\int_1^{K[3]} \frac{1-K[1]}{2K[1]^2+2K[1]} dK[1]\right)}{(K[3]-1)^2} dK[3] + c_1\right)$$

2.1.571 Problem 587

Internal problem ID [9743]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 587
Date solved : `Monday, January 27, 2025 at 06:13:46 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1-x)\,y''-x(3-5x)\,y'+(4-5x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.220 (sec)

Writing the ode as

$$\left(-x^3+x^2\right)y''+\left(5x^2-3x\right)y'+(4-5x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=-x^3+x^2\\ B&=5x^2-3x\\ C&=4-5x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{15x^2-6x-1}{4\left(x^2-x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=15x^2-6x-1\\ t&=4\left(x^2-x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{15x^2-6x-1}{4\left(x^2-x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1087: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{(-1 + x)^2} + \frac{2}{-1 + x} - \frac{1}{4x^2} - \frac{2}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15x^2 - 6x - 1}{4\left(x^2 - x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15x^2 - 6x - 1}{4\left(x^2 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
1	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\
&= \frac{5}{2} - \left(\frac{5}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \frac{2}{-1+x} + (0) \\
&= \frac{1}{2x} + \frac{2}{-1+x} \\
&= \frac{-1+5x}{2x\left(-1+x\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{2}{-1+x}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{2}{(-1+x)^2}\right) + \left(\frac{1}{2x} + \frac{2}{-1+x}\right)^2 - \left(\frac{15x^2 - 6x - 1}{4(x^2 - x)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{1}{2x} + \frac{2}{-1+x}\right) dx} \\ &= \sqrt{x}\,(-1+x)^2 \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{5x^2 - 3x}{-x^3 + x^2} \, dx} \\ &= z_1 e^{\frac{3\ln(x)}{2} + \ln(-1+x)} \\ &= z_1 \left(x^{3/2}(-1+x)\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^2(-1+x)^3$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{5x^2 - 3x}{-x^3 + x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{3\ln(x) + 2\ln(-1+x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\ln(x) - \frac{1}{3(-1+x)^3} - \frac{1}{-1+x} + \frac{1}{2(-1+x)^2} - \ln(-1+x)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(x^2(-1+x)^3\right) \\ &\quad + c_2 \left(x^2(-1+x)^3 \left(\ln(x) - \frac{1}{3(-1+x)^3} - \frac{1}{-1+x} + \frac{1}{2(-1+x)^2} - \ln(-1+x)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(1-x)\left(\frac{d^2}{dx^2}y(x)\right)-x(3-5x)\left(\frac{d}{dx}y(x)\right)+(4-5x)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(-4+5x)y(x)}{x^2(x-1)}+\frac{(-3+5x)\left(\frac{d}{dx}y(x)\right)}{x(x-1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)-\frac{(-3+5x)\left(\frac{d}{dx}y(x)\right)}{x(x-1)}+\frac{(-4+5x)y(x)}{x^2(x-1)}=0$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x)=-\frac{-3+5x}{x(x-1)},P_3(x)=\frac{-4+5x}{x^2(x-1)}\right]$
  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=-3$
  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=4$
  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(x-1)\left(\frac{d^2}{dx^2}y(x)\right)-x(-3+5x)\left(\frac{d}{dx}y(x)\right)+(-4+5x)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

    $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$
  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..3$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$
  - Shift index using $k->k+2-m$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$-a_0(-2+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r-2)^2 + a_{k-1}(k+r-2)(k-6+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-(-2+r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 2$
- Each term in the series must be 0, giving the recursion relation
  $-a_k(k+r-2)^2 + a_{k-1}(k+r-2)(k-6+r) = 0$
- Shift index using $k- >k+1$
  $-a_{k+1}(k+r-1)^2 + a_k(k+r-1)(k+r-5) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_k(k+r-5)}{k+r-1}$
- Recursion relation for $r = 2$; series terminates at $k = 3$
  $a_{k+1} = \frac{a_k(k-3)}{k+1}$
- Apply recursion relation for $k = 0$
  $a_1 = -3a_0$
- Apply recursion relation for $k = 1$
  $a_2 = -a_1$
- Express in terms of a_0
  $a_2 = 3a_0$
- Apply recursion relation for $k = 2$
  $a_3 = -\frac{a_2}{3}$
- Express in terms of a_0
  $a_3 = -a_0$
- Terminating series solution of the ODE for $r = 2$. Use reduction of order to find the second li
  $y(x) = a_0 \cdot (-x^3 + 3x^2 - 3x + 1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 47

```
dsolve((1-x)*x^2*diff(diff(y(x),x),x)-x*(3-5*x)*diff(y(x),x)+(4-5*x)*y(x) = 0,y(x),sin
```

$$y = x^2\left(c_1(x-1)^3 + c_2\left(-(x-1)^3 \ln(x-1) + (x-1)^3 \ln(x) - x^2 + \frac{5x}{2} - \frac{11}{6}\right)\right)$$

Mathematica DSolve solution

Solving time : `0.244` (sec)
Leaf size : 104

```
DSolve[{x^2*(1-x)*D[y[x],{x,2}]-x*(3-5*x)*D[y[x],x]+(4-5*x)*y[x]==0,{}},y[x],x,IncludeSingularS
```

$$y(x) \to \exp\left(\int_1^x \left(\frac{1}{2K[1]} + \frac{2}{K[1]-1}\right) dK[1] - \frac{1}{2}\int_1^x \left(-\frac{3}{K[2]} - \frac{2}{K[2]-1}\right) dK[2]\right) \left(c_2 \int_1^x \exp\left(-2\int_1^{K[3]} \frac{1-5K[1]}{2K[1]-2K[1]^2} dK[1]\right) dK[3] + c_1\right)$$

2.1.572 Problem 588

Internal problem ID [9744]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 588
Date solved : Monday, January 27, 2025 at 06:13:47 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$x^2(x^2+1)\,y''-x(9x^2+1)\,y'+(25x^2+1)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.309 (sec)

Writing the ode as

$$(x^4+x^2)\,y''+(-9x^3-x)\,y'+(25x^2+1)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2 \\ B &= -9x^3-x \\ C &= 25x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-x^4-98x^2-1}{4\,(x^3+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^4-98x^2-1 \\ t &= 4(x^3+x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-x^4-98x^2-1}{4\,(x^3+x)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1089: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{(x - i)^2} + \frac{6}{(x + i)^2} + \frac{6i}{x - i} - \frac{6i}{x + i} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2 \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$[\sqrt{r}]_c = 0$$
$$\alpha_c^+ = \frac{1}{2} + \sqrt{1+4b} = 3$$
$$\alpha_c^- = \frac{1}{2} - \sqrt{1+4b} = -2$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^4 - 98x^2 - 1}{4\left(x^3 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$[\sqrt{r}]_\infty = 0$$
$$\alpha_\infty^+ = \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}$$
$$\alpha_\infty^- = \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^4 - 98x^2 - 1}{4\left(x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
i	2	0	3	-2
$-i$	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\ &= \frac{1}{2} - \left(-\frac{7}{2}\right) \\ &= 4 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x - c_3}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} - \frac{2}{x-i} - \frac{2}{x+i} + (-)\,(0) \\
&= \frac{1}{2x} - \frac{2}{x-i} - \frac{2}{x+i} \\
&= \frac{1}{2x} - \frac{4x}{x^2+1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\left(12x^2 + 6xa_3 + 2a_2\right) + 2\left(\frac{1}{2x} - \frac{2}{x-i} - \frac{2}{x+i}\right)\left(4x^3 + 3x^2 a_3 + 2a_2 x + a_1\right) + \Bigg(\bigg(-\frac{1}{2x^2} + \frac{2}{(x-i)^2} + \frac{2}{(x+i}
$$

$$
\left(x^2+1\right)\left(x^4 a_3 +\right.
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 1, a_1 = 0, a_2 = -4, a_3 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^4 - 4x^2 + 1
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^4 - 4x^2 + 1\right) \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{2}{x-i} - \frac{2}{x+i}\right) dx} \\
&= \left(x^4 - 4x^2 + 1\right) \mathrm{e}^{-2\ln\left(x^2+1\right) + \frac{\ln(x)}{2}} \\
&= \frac{\left(x^4 - 4x^2 + 1\right)\sqrt{x}}{\left(x^2+1\right)^2}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-9x^3 - x}{x^4 + x^2}\, dx} \\
&= z_1 e^{2\ln\left(x^2+1\right) + \frac{\ln(x)}{2}} \\
&= z_1 \left(\left(x^2+1\right)^2 \sqrt{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^5 - 4x^3 + x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-9x^3-x}{x^4+x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{4\ln(x^2+1)+\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{-6x^2+3}{x^4-4x^2+1} + \ln(x)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 (x^5 - 4x^3 + x) + c_2 \left(x^5 - 4x^3 + x\left(\frac{-6x^2+3}{x^4-4x^2+1} + \ln(x)\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) - x(9x^2+1)\left(\frac{d}{dx}y(x)\right) + (25x^2+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(25x^2+1)y(x)}{x^2(x^2+1)} + \frac{(9x^2+1)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) - \frac{(9x^2+1)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)} + \frac{(25x^2+1)y(x)}{x^2(x^2+1)} = 0$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{9x^2+1}{x(x^2+1)}, P_3(x) = \frac{25x^2+1}{x^2(x^2+1)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 1$

  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) - x(9x^2+1)\left(\frac{d}{dx}y(x)\right) + (25x^2+1)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

$$x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k-m$

$$x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

$$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+r)^2 x^r + a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)^2 + a_{k-2}(k-7+r)^2\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 1$
- Each term must be 0
  $a_1 r^2 = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_k(k+r-1)^2 + a_{k-2}(k-7+r)^2 = 0$
- Shift index using $k- >k+2$
  $a_{k+2}(k+1+r)^2 + a_k(k+r-5)^2 = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{a_k(k+r-5)^2}{(k+1+r)^2}$
- Recursion relation for $r = 1$; series terminates at $k = 4$
  $a_{k+2} = -\frac{a_k(k-4)^2}{(k+2)^2}$
- Solution for $r = 1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k(k-4)^2}{(k+2)^2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 41

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)-x*(9*x^2+1)*diff(y(x),x)+(25*x^2+1)*y(x) = 0,y
```

$$y = x\left(c_2\left(x^4-4x^2+1\right)\ln\left(x\right)+c_1x^4+\left(-4c_1-6c_2\right)x^2+c_1+3c_2\right)$$

Mathematica DSolve solution

Solving time : 0.609 (sec)
Leaf size : 138

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]-x*(1+9*x^2)*D[y[x],x]+(1+25*x^2)*y[x]==0,{}},y[x],x,Includ
```

$$y(x)\to\left(x^4-4x^2+1\right)\exp\left(\int_1^x\frac{1-7K[1]^2}{2\left(K[1]^3+K[1]\right)}dK[1]-\frac{1}{2}\int_1^x-\frac{9K[2]^2+1}{K[2]^3+K[2]}dK[2]\right)\left(c_2\int_1^x\frac{\exp\left(-2\int_1^{K[3]}\frac{1-7K[1]^2}{2(K[1]^3+K[1])}dK[1]\right)}{\left(K[3]^4-4K[3]^2+1\right)^2}dK[3]+c_1\right)$$

2.1.573 Problem 589

Internal problem ID [9745]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 589
Date solved : `Monday, January 27, 2025 at 06:13:47 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$9x^2y'' + 3x(-x^2+1)\,y' + (7x^2+1)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 1.120 (sec)

Writing the ode as

$$9x^2y'' + (-3x^3+3x)\,y' + (7x^2+1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^2 \\ B &= -3x^3+3x \\ C &= 7x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 36x^2 - 9}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4 - 36x^2 - 9 \\ t &= 36x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 36x^2 - 9}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1091: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{36} - 1 - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{6} - \frac{3}{x} - \frac{111}{4x^3} - \frac{999}{2x^5} - \frac{180819}{16x^7} - \frac{2292705}{8x^9} - \frac{249239511}{32x^{11}} - \frac{3548540907}{16x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{6} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{36}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^4 - 36x^2 - 9}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{x^2}{36} - 1\right) + \left(-\frac{1}{4x^2}\right) \\ &= \frac{x^2}{36} - 1 - \frac{1}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is -1. Now b can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{6} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{\frac{1}{6}} - 1\right) &&= -\frac{7}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{\frac{1}{6}} - 1\right) &&= \frac{5}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 36x^2 - 9}{36x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{6}$	$-\frac{7}{2}$	$\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{5}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^+) \\ &= \frac{5}{2} - \left(\frac{1}{2}\right) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\left(\frac{x}{6}\right) \\ &= \frac{1}{2x} - \frac{x}{6} \\ &= \frac{1}{2x} - \frac{x}{6} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{1}{2x} - \frac{x}{6}\right)(2x + a_1) + \left(\left(-\frac{1}{2x^2} - \frac{1}{6}\right) + \left(\frac{1}{2x} - \frac{x}{6}\right)^2 - \left(\frac{x^4 - 36x^2 - 9}{36x^2}\right)\right) = 0$$

$$\frac{x^2 a_1 + 2(6 + a_0)\, x + 3a_1}{3x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -6, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 6$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2-6\right) \mathrm{e}^{\int \left(\frac{1}{2x}-\frac{x}{6}\right) dx} \\
&= \left(x^2-6\right) \mathrm{e}^{-\frac{x^2}{12}+\frac{\ln(x)}{2}} \\
&= \left(x^2-6\right) \sqrt{x}\, \mathrm{e}^{-\frac{x^2}{12}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3x^3+3x}{9x^2} \, dx} \\
&= z_1 e^{\frac{x^2}{12}-\frac{\ln(x)}{6}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{x^2}{12}}}{x^{1/6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^{1/3}\left(x^2-6\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x^3+3x}{9x^2} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{6}-\frac{\ln(x)}{3}}}{\left(y_1\right)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{x^2}{6}-\frac{\ln(x)}{3}}}{x^{2/3}\left(x^2-6\right)^2} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^{1/3}\left(x^2-6\right)\right) + c_2\left(x^{1/3}\left(x^2-6\right)\left(\int \frac{\mathrm{e}^{\frac{x^2}{6}-\frac{\ln(x)}{3}}}{x^{2/3}\left(x^2-6\right)^2} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$9x^2\left(\frac{d^2}{dx^2}y(x)\right)+3x(-x^2+1)\left(\frac{d}{dx}y(x)\right)+(7x^2+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{(7x^2+1)y(x)}{9x^2}+\frac{(x^2-1)\left(\frac{d}{dx}y(x)\right)}{3x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)-\frac{(x^2-1)\left(\frac{d}{dx}y(x)\right)}{3x}+\frac{(7x^2+1)y(x)}{9x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x)=-\frac{x^2-1}{3x}, P_3(x)=\frac{7x^2+1}{9x^2}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{1}{3}$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=\frac{1}{9}$

- ○ $x=0$is a regular singular point

Check to see if $x_0=0$ is a regular singular point

$x_0=0$

- Multiply by denominators

$9x^2\left(\frac{d^2}{dx^2}y(x)\right)-3x(x^2-1)\left(\frac{d}{dx}y(x)\right)+(7x^2+1)\,y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

- ○ Shift index using $k->k-m$

  $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ○ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ○ Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,x^{k+r}$

Rewrite ODE with series expansions

$a_0(-1+3r)^2x^r+a_1(2+3r)^2x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_k(3k+3r-1)^2-a_{k-2}(3k-13+3r)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(-1+3r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = \frac{1}{3}$
- Each term must be 0

  $a_1(2+3r)^2 = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(3k+3r-1)^2 + (-3k+13-3r)\, a_{k-2} = 0$
- Shift index using $k- > k+2$

  $a_{k+2}(3k+5+3r)^2 + a_k(-3k-3r+7) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k(3k+3r-7)}{(3k+5+3r)^2}$
- Recursion relation for $r = \frac{1}{3}$; series terminates at $k = 2$

  $a_{k+2} = \frac{a_k(3k-6)}{(3k+6)^2}$
- Solution for $r = \frac{1}{3}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = \frac{a_k(3k-6)}{(3k+6)^2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solut
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special fu
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.033 (sec)
Leaf size : 19

```
dsolve(9*x^2*diff(diff(y(x),x),x)+3*x*(-x^2+1)*diff(y(x),x)+(7*x^2+1)*y(x) = 0,y(x),si
```

$$y = -\frac{x^{1/3}(x^2-6)(c_1-c_2)}{6}$$

Mathematica DSolve solution

Solving time : 0.364 (sec)
Leaf size : 59

```
DSolve[{9*x^2*D[y[x],{x,2}]+3*x*(1-x^2)*D[y[x],x]+(1+7*x^2)*y[x]==0,{}},y[x],x,IncludeSingul
```

$$y(x) \to \sqrt[3]{e}\sqrt[3]{x}(x^2-6)\left(c_2\int_1^x \frac{e^{\frac{K[1]^2}{6}-1}}{K[1](K[1]^2-6)^2}dK[1]+c_1\right)$$

2.1.574 Problem 590

Internal problem ID [9746]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 590
Date solved : `Monday, January 27, 2025 at 06:13:49 PM`
CAS classification : `[[_2nd_order, _exact, _linear, _homogeneous]]`

Solve

$$x\left(x^2+1\right)y''+\left(-x^2+1\right)y'-8xy=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.339 (sec)

Writing the ode as

$$\left(x^3+x\right)y''+\left(-x^2+1\right)y'-8xy=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^3+x \\ B &= -x^2+1 \\ C &= -8x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{35x^4+22x^2-1}{4\left(x^3+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 35x^4+22x^2-1 \\ t &= 4\left(x^3+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{35x^4+22x^2-1}{4\left(x^3+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1093: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} - \frac{15i}{4(x-i)} + \frac{15i}{4(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{35x^4 + 22x^2 - 1}{4\left(x^3 + x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{35}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{35x^4 + 22x^2 - 1}{4\left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\
&= \frac{7}{2} - \left(\frac{7}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2}\right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{+}}{x - c_3}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)} + (0) \\
&= \frac{1}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)} \\
&= \frac{1}{2x} + \frac{3x}{x^2+1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{3}{2(x-i)^2} - \frac{3}{2(x+i)^2}\right) + \left(\frac{1}{2x} + \frac{3}{2(x-i)} + \frac{}{2(}\right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)}\right) dx} \\
&= (x^2+1)^{3/2} \sqrt{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2+1}{x^3+x}\, dx} \\
&= z_1 e^{\frac{\ln(x^2+1)}{2} - \frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\sqrt{x^2+1}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (x^2+1)^2
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+1}{x^3+x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\ln(x^2+1) - \ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{1}{2x^2+2} - \frac{\ln(x^2+1)}{2} + \frac{1}{4(x^2+1)^2} + \ln(x)\right)
\end{aligned}
$$

Therefore the solution is

$$\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\left(x^2+1\right)^2\right) + c_2\left(\left(x^2+1\right)^2\left(\frac{1}{2x^2+2} - \frac{\ln\left(x^2+1\right)}{2} + \frac{1}{4\left(x^2+1\right)^2} + \ln\left(x\right)\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + (-x^2+1)\left(\frac{d}{dx}y(x)\right) - 8xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{8y(x)}{x^2+1} + \frac{(x^2-1)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) - \frac{(x^2-1)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)} - \frac{8y(x)}{x^2+1} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{x^2-1}{x(x^2+1)}, P_3(x) = -\frac{8}{x^2+1}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + (-x^2+1)\left(\frac{d}{dx}y(x)\right) - 8xy(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x \cdot y(x)$ to series expansion

    $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

  - Shift index using $k- >k-1$

    $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..2$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

$$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 1..3$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 x^{-1+r} + a_1(1+r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)^2 + a_{k-1}(k+r+1)\,(k-5+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 0$
- Each term must be 0
  $a_1(1+r)^2 = 0$
- Each term in the series must be 0, giving the recursion relation
  $((a_{k-1} + a_{k+1})\, k - 5a_{k-1} + a_{k+1})\,(k+1) = 0$
- Shift index using $k- >k+1$
  $((a_k + a_{k+2})\,(k+1) - 5a_k + a_{k+2})\,(k+2) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{a_k(k-4)}{k+2}$
- Recursion relation for $r = 0$; series terminates at $k = 4$
  $a_{k+2} = -\frac{a_k(k-4)}{k+2}$
- Solution for $r = 0$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(k-4)}{k+2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 48

```
dsolve(x*(x^2+1)*diff(diff(y(x),x),x)+(-x^2+1)*diff(y(x),x)-8*x*y(x) = 0,y(x),singsol=al
```

$$y = c_1 (x^2+1)^2 + c_2 \left(-\frac{(x^2+1)^2 \ln(x^2+1)}{2} + (x^2+1)^2 \ln(x) + \frac{x^2}{2} + \frac{3}{4}\right)$$

Mathematica DSolve solution

Solving time : 0.201 (sec)
Leaf size : 112

```
DSolve[{x*(1+x^2)*D[y[x],{x,2}]+(1-x^2)*D[y[x],x]-8*x*y[x]==0,{}},y[x],x,IncludeSingularSolutio
```

$$y(x) \to \exp\left(\int_1^x \frac{7K[1]^2+1}{2(K[1]^3+K[1])} dK[1] - \frac{1}{2}\int_1^x \frac{1-K[2]^2}{K[2]^3+K[2]} dK[2]\right) \left(c_2 \int_1^x \exp\left(-2\int_1^{K[3]} \frac{7K[1]^2+1}{2(K[1]^3+K[1])} dK[1]\right) dK[3] + c_1\right)$$

2.1.575 Problem 591

Internal problem ID [9747]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 591
Date solved : `Monday, January 27, 2025 at 06:13:50 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' + 2x\left(-x^2+4\right)y' + \left(7x^2+1\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.678 (sec)

Writing the ode as

$$4x^2y'' + \left(-2x^3+8x\right)y' + \left(7x^2+1\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -2x^3+8x \\ C &= 7x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 40x^2 - 4}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4 - 40x^2 - 4 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 40x^2 - 4}{16x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1095: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{16} - \frac{5}{2} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{4} - \frac{5}{x} - \frac{101}{2x^3} - \frac{1010}{x^5} - \frac{50601}{2x^7} - \frac{710030}{x^9} - \frac{21351501}{x^{11}} - \frac{672670100}{x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{4}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{16}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^4 - 40x^2 - 4}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{x^2}{16} - \frac{5}{2}\right) + \left(-\frac{1}{4x^2}\right) \\ &= \frac{x^2}{16} - \frac{5}{2} - \frac{1}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is $-\frac{5}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{5}{2}\right) - (0) \\ &= -\frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{4} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{4}} - 1\right) &&= -\frac{11}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{4}} - 1\right) &&= \frac{9}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 40x^2 - 4}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{4}$	$-\frac{11}{2}$	$\frac{9}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{9}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c1}^+\right) \\ &= \frac{9}{2} - \left(\frac{1}{2}\right) \\ &= 4 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\left(\frac{x}{4}\right) \\ &= \frac{1}{2x} - \frac{x}{4} \\ &= \frac{1}{2x} - \frac{x}{4} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(12x^2 + 6xa_3 + 2a_2\right) + 2\left(\frac{1}{2x} - \frac{x}{4}\right)\left(4x^3 + 3a_3x^2 + 2a_2x + a_1\right) + \left(\left(-\frac{1}{2x^2} - \frac{1}{4}\right) + \left(\frac{1}{2x} - \frac{x}{4}\right)^2 - \left(\frac{x^4 - 4}{16}\right.\right.$$

$$\frac{x^4a_3 + 2(16 + a_2)x^3 + 3(a_1 + 6a_3)x^2 + 4(a_0 + 2a_2)x}{2x}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 32, a_1 = 0, a_2 = -16, a_3 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 - 16x^2 + 32$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega \,dx} \\ &= \left(x^4 - 16x^2 + 32\right) \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{x}{4}\right)dx} \\ &= \left(x^4 - 16x^2 + 32\right) \mathrm{e}^{-\frac{x^2}{8} + \frac{\ln(x)}{2}} \\ &= \left(x^4 - 16x^2 + 32\right) \sqrt{x}\, \mathrm{e}^{-\frac{x^2}{8}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x^3+8x}{4x^2}\,dx} \\ &= z_1 e^{\frac{x^2}{8} - \ln(x)} \\ &= z_1 \left(\frac{\mathrm{e}^{\frac{x^2}{8}}}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^4 - 16x^2 + 32}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^3+8x}{4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\frac{x^2}{4} - 2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{\frac{x^2}{4} - 2\ln(x)}x}{\left(x^4 - 16x^2 + 32\right)^2}dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{x^4 - 16x^2 + 32}{\sqrt{x}}\right) + c_2 \left(\frac{x^4 - 16x^2 + 32}{\sqrt{x}} \left(\int \frac{\mathrm{e}^{\frac{x^2}{4} - 2\ln(x)}x}{\left(x^4 - 16x^2 + 32\right)^2}dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2\left(\frac{d^2}{dx^2}y(x)\right)+2x(-x^2+4)\left(\frac{d}{dx}y(x)\right)+(7x^2+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(7x^2+1)y(x)}{4x^2}+\frac{(x^2-4)\left(\frac{d}{dx}y(x)\right)}{2x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)-\frac{(x^2-4)\left(\frac{d}{dx}y(x)\right)}{2x}+\frac{(7x^2+1)y(x)}{4x^2}=0$

☐ Check to see if $x_0=0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x)=-\frac{x^2-4}{2x},P_3(x)=\frac{7x^2+1}{4x^2}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=2$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=\frac{1}{4}$

- ○ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right)-2x(x^2-4)\left(\frac{d}{dx}y(x)\right)+(7x^2+1)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ○ Shift index using $k->k-m$

  $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ○ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ○ Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)^2\,x^r+a_1(3+2r)^2\,x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_k(2k+2r+1)^2-a_{k-2}(2k-11+2r)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(1+2r)^2 = 0$

- Values of r that satisfy the indicial equation
  $r = -\frac{1}{2}$
- Each term must be 0
  $a_1(3+2r)^2 = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_k(2k+2r+1)^2 + (-2k+11-2r)\, a_{k-2} = 0$
- Shift index using $k->k+2$
  $a_{k+2}(2k+5+2r)^2 + a_k(-2k-2r+7) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{a_k(2k+2r-7)}{(2k+5+2r)^2}$
- Recursion relation for $r = -\frac{1}{2}$; series terminates at $k = 4$
  $a_{k+2} = \frac{a_k(2k-8)}{(2k+4)^2}$
- Solution for $r = -\frac{1}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = \frac{a_k(2k-8)}{(2k+4)^2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric sol
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.103` (sec)
Leaf size : 24

```
dsolve(4*x^2*diff(diff(y(x),x),x)+2*x*(-x^2+4)*diff(y(x),x)+(7*x^2+1)*y(x) = 0,y(x),sing
```

$$y = \frac{(x^4 - 16x^2 + 32)(c_1 + 2c_2)}{32\sqrt{x}}$$

Mathematica DSolve solution

Solving time : `0.4` (sec)
Leaf size : 70

```
DSolve[{4*x^2*D[y[x],{x,2}]+2*x*(4-x^2)*D[y[x],x]+(1+7*x^2)*y[x]==0,{}},y[x],x,IncludeSingular
```

$$y(x) \to \frac{\sqrt{e}(x^4 - 16x^2 + 32)\left(c_2 \int_1^x \frac{e^{\frac{K[1]^2}{4}-1}}{K[1](K[1]^4 - 16K[1]^2 + 32)^2} dK[1] + c_1\right)}{\sqrt{x}}$$

2.1.576 Problem 592

Internal problem ID [9748]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 592
Date solved : `Monday, January 27, 2025 at 06:13:51 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2(1+x)\,y''+8x^2y'+(1+x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.149 (sec)

Writing the ode as

$$\left(4x^3+4x^2\right)y''+8x^2y'+(1+x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^3+4x^2 \\ B &= 8x^2 \\ C &= 1+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{1}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1097: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the gcd$(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\,(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{1}{2x}\right)(0)+\left(\left(-\frac{1}{2x^2}\right)+\left(\frac{1}{2x}\right)^2-\left(-\frac{1}{4x^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{8x^2}{4x^3+4x^2} dx} \\ &= z_1 e^{-\ln(1+x)} \\ &= z_1\left(\frac{1}{1+x}\right) \end{aligned}$$

Which simplifies to

$$y_1=\frac{\sqrt{x}}{1+x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1\int \frac{e^{\int -\frac{8x^2}{4x^3+4x^2} dx}}{(y_1)^2} dx \\ &= y_1\int \frac{e^{-2\ln(1+x)}}{(y_1)^2} dx \\ &= y_1(\ln(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1y_1+c_2y_2 \\ &= c_1\left(\frac{\sqrt{x}}{1+x}\right)+c_2\left(\frac{\sqrt{x}}{1+x}(\ln(x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)+8x^2\left(\frac{d}{dx}y(x)\right)+(x+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{y(x)}{4x^2}-\frac{2\left(\frac{d}{dx}y(x)\right)}{x+1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)+\frac{2\left(\frac{d}{dx}y(x)\right)}{x+1}+\frac{y(x)}{4x^2}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

  $\left[P_2(x)=\frac{2}{x+1},P_3(x)=\frac{1}{4x^2}\right]$

  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left.((x+1)\cdot P_2(x))\right|_{x=-1}=2$

  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left.\left((x+1)^2\cdot P_3(x)\right)\right|_{x=-1}=0$

  - $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

$4x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)+8x^2\left(\frac{d}{dx}y(x)\right)+(x+1)\,y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(4u^3-8u^2+4u)\left(\frac{d^2}{du^2}y(u)\right)+(8u^2-16u+8)\left(\frac{d}{du}y(u)\right)+uy(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u\cdot y(u)$ to series expansion

  $u\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+1}$

  - Shift index using $k->k-1$

  $u\cdot y(u)=\sum_{k=1}^{\infty}a_{k-1}u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)u^{k+r}$$

Rewrite ODE with series expansions

$$4a_0r(1+r)u^{-1+r} + (4a_1(1+r)(2+r) - 8a_0r(1+r))u^r + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+r+1)(k+2+r) - 8$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $4r(1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$
- Each term must be 0

  $4a_1(1+r)(2+r) - 8a_0r(1+r) = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k-1}(2k-1+2r)^2 - 8(k+r+1)\left(\left(-\frac{k}{2}-\frac{r}{2}-1\right)a_{k+1} + a_k(k+r)\right) = 0$
- Shift index using $k->k+1$

  $a_k(2k+2r+1)^2 - 8(k+2+r)\left(\left(-\frac{k}{2}-\frac{3}{2}-\frac{r}{2}\right)a_{k+2} + a_{k+1}(k+r+1)\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 8kra_k - 16kra_{k+1} + 4r^2a_k - 8r^2a_{k+1} + 4ka_k - 24ka_{k+1} + 4ra_k - 24ra_{k+1} + a_k - 16a_{k+1}}{4(k+2+r)(k+3+r)}$
- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} - 4ka_k - 8ka_{k+1} + a_k}{4(k+1)(k+2)}$
- Solution for $r = -1$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-1}, a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} - 4ka_k - 8ka_{k+1} + a_k}{4(k+1)(k+2)}, 0 = 0\right]$
- Revert the change of variables $u = x+1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^{k-1}, a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} - 4ka_k - 8ka_{k+1} + a_k}{4(k+1)(k+2)}, 0 = 0\right]$
- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 4ka_k - 24ka_{k+1} + a_k - 16a_{k+1}}{4(k+2)(k+3)}$
- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 4ka_k - 24ka_{k+1} + a_k - 16a_{k+1}}{4(k+2)(k+3)}, 8a_1 = 0\right]$
- Revert the change of variables $u = x+1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 4ka_k - 24ka_{k+1} + a_k - 16a_{k+1}}{4(k+2)(k+3)}, 8a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k(x+1)^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k(x+1)^k\right), a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} - 4ka_k - 8ka_{k+1} + a_k}{4(k+1)(k+2)}, 0 = 0, b_{k+2}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
```

```
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.042` (sec)
Leaf size : 19

```
dsolve(4*x^2*(x+1)*diff(diff(y(x),x),x)+8*diff(y(x),x)*x^2+(x+1)*y(x) = 0,y(x),singsol
```

$$y = \frac{\sqrt{x}\,(c_2 \ln(x) + c_1)}{x + 1}$$

Mathematica DSolve solution

Solving time : `0.031` (sec)
Leaf size : 24

```
DSolve[{4*x^2*(1+x)*D[y[x],{x,2}]+8*x^2*D[y[x],x]+(1+x)*y[x]==0,{}},y[x],x,IncludeSingularSo
```

$$y(x) \to \frac{\sqrt{x}(c_2 \log(x) + c_1)}{x + 1}$$

2.1.577 Problem 593

Internal problem ID [9749]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 593
Date solved : Monday, January 27, 2025 at 06:13:51 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$9x^2(3+x)\,y''+3x(3+7x)\,y'+(3+4x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.213 (sec)

Writing the ode as

$$\left(9x^3+27x^2\right)y''+\left(21x^2+9x\right)y'+(3+4x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^3+27x^2 \\ B &= 21x^2+9x \\ C &= 3+4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{1}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1099: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=-\frac{1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{2}-\left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^{-}}{x-c_1}\right)+(-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x}+(-)(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}
$$

Let

$$
p(x)=1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{21x^2+9x}{9x^3+27x^2}\, dx} \\ &= z_1 e^{-\ln(3+x) - \frac{\ln(x)}{6}} \\ &= z_1 \left(\frac{1}{(3+x)\, x^{1/6}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/3}}{3+x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{21x^2+9x}{9x^3+27x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-2\ln(3+x) - \frac{\ln(x)}{3}}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{2\ln(3+x)\, x}{3} - 2\ln(3+x) + \frac{x^2}{9+3x} + \frac{2x}{3+x} + \frac{3}{3+x} + \ln(x) \right. \\ &\qquad \left. + \frac{2\ln(3+x)(3+x)}{3} - \frac{x}{3} - 2\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{x^{1/3}}{3+x}\right) \\ &\quad + c_2 \left(\frac{x^{1/3}}{3+x}\left(-\frac{2\ln(3+x)\, x}{3} - 2\ln(3+x) + \frac{x^2}{9+3x} + \frac{2x}{3+x} + \frac{3}{3+x} + \ln(x) + \frac{2\ln(3+x)(3+x)}{3} - \frac{x}{3} - 2 \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$9x^2(x+3)\left(\frac{d^2}{dx^2}y(x)\right)+3x(7x+3)\left(\frac{d}{dx}y(x)\right)+(3+4x)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(3+4x)y(x)}{9x^2(x+3)}-\frac{(7x+3)\left(\frac{d}{dx}y(x)\right)}{3x(x+3)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{(7x+3)\left(\frac{d}{dx}y(x)\right)}{3x(x+3)}+\frac{(3+4x)y(x)}{9x^2(x+3)}=0$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x)=\frac{7x+3}{3x(x+3)},P_3(x)=\frac{3+4x}{9x^2(x+3)}\right]$

  - $(x+3)\cdot P_2(x)$ is analytic at $x=-3$

    $\left((x+3)\cdot P_2(x)\right)\Big|_{x=-3}=2$

  - $(x+3)^2\cdot P_3(x)$ is analytic at $x=-3$

    $\left((x+3)^2\cdot P_3(x)\right)\Big|_{x=-3}=0$

  - $x=-3$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=-3$

- Multiply by denominators

  $9x^2(x+3)\left(\frac{d^2}{dx^2}y(x)\right)+3x(7x+3)\left(\frac{d}{dx}y(x)\right)+(3+4x)\,y(x)=0$

- Change variables using $x=u-3$ so that the regular singular point is at $u=0$

  $(9u^3-54u^2+81u)\left(\frac{d^2}{du^2}y(u)\right)+(21u^2-117u+162)\left(\frac{d}{du}y(u)\right)+(-9+4u)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

    $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

  - Shift index using $k->k-m$

    $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$81a_0r(1+r)\,u^{-1+r} + (81a_1(1+r)(2+r) - 9a_0(1+r)(1+6r))\,u^r + \left(\sum_{k=1}^{\infty} (81a_{k+1}(k+r+1)(k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $81r(1+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-1, 0\}$
- Each term must be 0
  $81a_1(1+r)(2+r) - 9a_0(1+r)(1+6r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $81a_{k+1}(k+r+1)(k+2+r) - 54\left(k+r+\frac{1}{6}\right)(k+r+1)\,a_k + a_{k-1}(3k-1+3r)^2 = 0$
- Shift index using $k->k+1$
  $81a_{k+2}(k+2+r)(k+3+r) - 54\left(k+\frac{7}{6}+r\right)(k+2+r)\,a_{k+1} + a_k(3k+3r+2)^2 = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{9k^2a_k - 54k^2a_{k+1} + 18kra_k - 108kra_{k+1} + 9r^2a_k - 54r^2a_{k+1} + 12ka_k - 171ka_{k+1} + 12ra_k - 171ra_{k+1} + 4a_k - 126a_{k+1}}{81(k+2+r)(k+3+r)}$
- Recursion relation for $r = -1$
  $a_{k+2} = -\frac{9k^2a_k - 54k^2a_{k+1} - 6ka_k - 63ka_{k+1} + a_k - 9a_{k+1}}{81(k+1)(k+2)}$
- Solution for $r = -1$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-1}, a_{k+2} = -\frac{9k^2a_k - 54k^2a_{k+1} - 6ka_k - 63ka_{k+1} + a_k - 9a_{k+1}}{81(k+1)(k+2)}, 0 = 0\right]$
- Revert the change of variables $u = x+3$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x+3)^{k-1}, a_{k+2} = -\frac{9k^2a_k - 54k^2a_{k+1} - 6ka_k - 63ka_{k+1} + a_k - 9a_{k+1}}{81(k+1)(k+2)}, 0 = 0\right]$
- Recursion relation for $r = 0$
  $a_{k+2} = -\frac{9k^2a_k - 54k^2a_{k+1} + 12ka_k - 171ka_{k+1} + 4a_k - 126a_{k+1}}{81(k+2)(k+3)}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{9k^2a_k - 54k^2a_{k+1} + 12ka_k - 171ka_{k+1} + 4a_k - 126a_{k+1}}{81(k+2)(k+3)}, 162a_1 - 9a_0 = 0\right]$
- Revert the change of variables $u = x+3$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x+3)^k, a_{k+2} = -\frac{9k^2a_k - 54k^2a_{k+1} + 12ka_k - 171ka_{k+1} + 4a_k - 126a_{k+1}}{81(k+2)(k+3)}, 162a_1 - 9a_0 = 0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k (x+3)^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k (x+3)^k\right), a_{k+2} = -\frac{9k^2a_k - 54k^2a_{k+1} - 6ka_k - 63ka_{k+1} + a_k - 9a_{k+1}}{81(k+1)(k+2)}, 0\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
```

```
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.031 (sec)
Leaf size : 19

```
dsolve(9*x^2*(x+3)*diff(diff(y(x),x),x)+3*x*(3+7*x)*diff(y(x),x)+(4*x+3)*y(x) = 0,y(x),s
```

$$y = \frac{x^{1/3}(c_2 \ln(x) + c_1)}{x + 3}$$

Mathematica DSolve solution

Solving time : 0.286 (sec)
Leaf size : 49

```
DSolve[{9*x^2*(3+x)*D[y[x],{x,2}]+3*x*(3+7*x)*D[y[x],x]+(3+4*x)*y[x]==0,{}},y[x],x,IncludeSing
```

$$y(x) \to \sqrt{x}(c_2 \log(x) + c_1) \exp\left(-\frac{1}{2}\int_1^x \left(\frac{2}{K[1]+3} + \frac{1}{3K[1]}\right) dK[1]\right)$$

2.1.578 Problem 594

Internal problem ID [9750]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 594
Date solved : `Monday, January 27, 2025 at 06:13:52 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(-x^2+2\right) y''-x\left(3x^2+2\right) y'+\left(-x^2+2\right) y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.165 (sec)

Writing the ode as

$$\left(-x^4+2x^2\right) y''+\left(-3x^3-2x\right) y'+\left(-x^2+2\right) y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^4+2x^2 \\ B &= -3x^3-2x \\ C &= -x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1101: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2}\\ \alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=-\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$\begin{aligned}d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}\right)\\ &= \frac{1}{2}-\left(\frac{1}{2}\right)\\ &= 0\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^{-}}{x-c_1}\right)+(-)[\sqrt{r}]_\infty\\ &= \frac{1}{2x}+(-)(0)\\ &= \frac{1}{2x}\\ &= \frac{1}{2x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-3x^3-2x}{-x^4+2x^2} \, dx} \\ &= z_1 e^{-\ln(x^2-2) + \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{\sqrt{x}}{x^2-2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x}{x^2-2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3x^3-2x}{-x^4+2x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-2\ln(x^2-2)+\ln(x)}}{(y_1)^2} \, dx \\ &= y_1(\ln(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{x}{x^2-2}\right) + c_2 \left(\frac{x}{x^2-2}(\ln(x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(-x^2+2)\left(\frac{d^2}{dx^2}y(x)\right)-x(3x^2+2)\left(\frac{d}{dx}y(x)\right)+(-x^2+2)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{y(x)}{x^2}-\frac{(3x^2+2)\left(\frac{d}{dx}y(x)\right)}{x(x^2-2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)+\frac{(3x^2+2)\left(\frac{d}{dx}y(x)\right)}{x(x^2-2)}+\frac{y(x)}{x^2}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=\frac{3x^2+2}{x(x^2-2)},P_3(x)=\frac{1}{x^2}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

$\left.(x\cdot P_2(x))\right|_{x=0}=-1$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left.(x^2\cdot P_3(x))\right|_{x=0}=1$

- ◦ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$x^2(x^2-2)\left(\frac{d^2}{dx^2}y(x)\right)+x(3x^2+2)\left(\frac{d}{dx}y(x)\right)+(x^2-2)\,y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

$x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ◦ Shift index using $k->k-m$

$x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ◦ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..4$

$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- ◦ Shift index using $k->k+2-m$

$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

Rewrite ODE with series expansions

$$-2a_0(-1+r)^2 x^r - 2a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(-2a_k(k+r-1)^2 + a_{k-2}(k+r-1)^2\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-2(-1+r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 1$
- Each term must be 0
  $-2a_1 r^2 = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $-2\left(a_k - \frac{a_{k-2}}{2}\right)(k+r-1)^2 = 0$
- Shift index using $k- > k+2$
  $-2\left(a_{k+2} - \frac{a_k}{2}\right)(k+r+1)^2 = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{a_k}{2}$
- Recursion relation for $r = 1$
  $a_{k+2} = \frac{a_k}{2}$
- Solution for $r = 1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = \frac{a_k}{2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 19

```
dsolve(x^2*(-x^2+2)*diff(diff(y(x),x),x)-x*(3*x^2+2)*diff(y(x),x)+(-x^2+2)*y(x) = 0,y(x)
```

$$y = \frac{x(c_2 \ln(x) + c_1)}{x^2 - 2}$$

Mathematica DSolve solution

Solving time : 0.265 (sec)
Leaf size : 51

```
DSolve[{x^2*(2-x^2)*D[y[x],{x,2}]-x*(2+3*x^2)*D[y[x],x]+(2-x^2)*y[x]==0,{}},y[x],x,IncludeSi
```

$$y(x)\to \sqrt{x}(c_2\log(x)+c_1)\exp\left(-\frac{1}{2}\int_1^x\left(\frac{4K[1]}{K[1]^2-2}-\frac{1}{K[1]}\right)dK[1]\right)$$

2.1.579 Problem 595

Internal problem ID [9751]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 595
Date solved : `Monday, January 27, 2025 at 06:13:53 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$16x^2(x^2+1)\,y''+8x(9x^2+1)\,y'+(49x^2+1)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.207 (sec)

Writing the ode as

$$\left(16x^4+16x^2\right)y''+\left(72x^3+8x\right)y'+\left(49x^2+1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=16x^4+16x^2\\B&=72x^3+8x\\C&=49x^2+1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=-1\\t&=4x^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{1}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1103: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=-\frac{1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2}-\left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^-}{x-c_1}\right)+(-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x}+(-)\,(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}
$$

Let

$$
p(x)=1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{1}{2x}\right)(0)+\left(\left(-\frac{1}{2x^2}\right)+\left(\frac{1}{2x}\right)^2-\left(-\frac{1}{4x^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}z_1(x)&=pe^{\int \omega\,dx}\\&=e^{\int \frac{1}{2x}dx}\\&=\sqrt{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1&=z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\&=z_1e^{-\int \frac{1}{2}\frac{72x^3+8x}{16x^4+16x^2}\,dx}\\&=z_1e^{-\frac{\ln(x)}{4}-\ln(x^2+1)}\\&=z_1\left(\frac{1}{x^{1/4}\left(x^2+1\right)}\right)\end{aligned}$$

Which simplifies to

$$y_1=\frac{x^{1/4}}{x^2+1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2&=y_1\int \frac{e^{\int -\frac{72x^3+8x}{16x^4+16x^2}\,dx}}{\left(y_1\right)^2}\,dx\\&=y_1\int \frac{e^{-\frac{\ln(x)}{2}-2\ln(x^2+1)}}{\left(y_1\right)^2}\,dx\\&=y_1\left(\ln(x)-\ln\left(x^2+1\right)x^2-\ln\left(x^2+1\right)+\frac{x^4}{2x^2+2}+\frac{x^2}{x^2+1}+\frac{1}{2x^2+2}\right.\\&\qquad\left.+\ln\left(x^2+1\right)\left(x^2+1\right)-\frac{x^2}{2}-1\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y&=c_1y_1+c_2y_2\\&=c_1\left(\frac{x^{1/4}}{x^2+1}\right)\\&\quad+c_2\left(\frac{x^{1/4}}{x^2+1}\left(\ln(x)-\ln\left(x^2+1\right)x^2-\ln\left(x^2+1\right)+\frac{x^4}{2x^2+2}+\frac{x^2}{x^2+1}+\frac{1}{2x^2+2}+\ln\left(x^2+1\right)\left(x^2+1\right)-\frac{x^2}{2}\right.\right.\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$16x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)+8x(9x^2+1)\left(\frac{d}{dx}y(x)\right)+(49x^2+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{(49x^2+1)y(x)}{16x^2(x^2+1)}-\frac{(9x^2+1)\left(\frac{d}{dx}y(x)\right)}{2x(x^2+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x)+\frac{(9x^2+1)\left(\frac{d}{dx}y(x)\right)}{2x(x^2+1)}+\frac{(49x^2+1)y(x)}{16x^2(x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

- ∘ Define functions

$\left[P_2(x)=\frac{9x^2+1}{2x(x^2+1)},P_3(x)=\frac{49x^2+1}{16x^2(x^2+1)}\right]$

- ∘ $x\cdot P_2(x)$ is analytic at $x=0$

$\left.(x\cdot P_2(x))\right|_{x=0}=\frac{1}{2}$

- ∘ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left.(x^2\cdot P_3(x))\right|_{x=0}=\frac{1}{16}$

- ∘ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$16x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)+8x(9x^2+1)\left(\frac{d}{dx}y(x)\right)+(49x^2+1)\,y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ∘ Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

$x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ∘ Shift index using $k->k-m$

$x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ∘ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ∘ Shift index using $k->k+1-m$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ∘ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..4$

$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- ∘ Shift index using $k->k+2-m$

$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

Rewrite ODE with series expansions

$$a_0(-1+4r)^2 x^r + a_1(3+4r)^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(4k+4r-1)^2 + a_{k-2}(4k+4r-1)^2\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+4r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = \frac{1}{4}$
- Each term must be 0
  $a_1(3+4r)^2 = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $(4k+4r-1)^2 (a_k + a_{k-2}) = 0$
- Shift index using $k- > k+2$
  $(4k+4r+7)^2 (a_{k+2} + a_k) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -a_k$
- Recursion relation for $r = \frac{1}{4}$
  $a_{k+2} = -a_k$
- Solution for $r = \frac{1}{4}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+2} = -a_k, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.040 (sec)
Leaf size : 21

```
dsolve(16*x^2*(x^2+1)*diff(diff(y(x),x),x)+8*x*(9*x^2+1)*diff(y(x),x)+(49*x^2+1)*y(x)
```

$$y = \frac{x^{1/4}(c_2 \ln(x) + c_1)}{x^2+1}$$

Mathematica DSolve solution

Solving time : `0.241` (sec)
Leaf size : 53

```
DSolve[{16*x^2*(1+x^2)*D[y[x],{x,2}]+8*x*(1+9*x^2)*D[y[x],x]+(1+49*x^2)*y[x]==0,{}},y[x],x,Inc
```

$$y(x) \to \sqrt{x}(c_2 \log(x) + c_1) \exp\left(-\frac{1}{2} \int_1^x \frac{9K[1]^2 + 1}{2\left(K[1]^3 + K[1]\right)} dK[1]\right)$$

2.1.580 Problem 596

Internal problem ID [9752]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 596
Date solved : `Monday, January 27, 2025 at 06:13:53 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(4+3x)\,y''-x(4-3x)\,y'+4y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.164 (sec)

Writing the ode as

$$\left(3x^3+4x^2\right)y''+\left(3x^2-4x\right)y'+4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^3+4x^2 \\ B &= 3x^2-4x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{1}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1105: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=-\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}\right)\\
&= \frac{1}{2}-\left(\frac{1}{2}\right)\\
&= 0
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^{-}}{x-c_1}\right)+(-)[\sqrt{r}]_\infty\\
&= \frac{1}{2x}+(-)(0)\\
&= \frac{1}{2x}\\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3x^2-4x}{3x^3+4x^2}\, dx} \\ &= z_1 e^{\frac{\ln(x)}{2} - \ln(4+3x)} \\ &= z_1\left(\frac{\sqrt{x}}{4+3x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x}{4+3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3x^2-4x}{3x^3+4x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\ln(x) - 2\ln(4+3x)}}{(y_1)^2}\, dx \\ &= y_1(\ln(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{x}{4+3x}\right) + c_2\left(\frac{x}{4+3x}(\ln(x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(3x+4)\left(\frac{d^2}{dx^2}y(x)\right)-x(4-3x)\left(\frac{d}{dx}y(x)\right)+4y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{4y(x)}{x^2(3x+4)}-\frac{(3x-4)\left(\frac{d}{dx}y(x)\right)}{x(3x+4)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)+\frac{(3x-4)\left(\frac{d}{dx}y(x)\right)}{x(3x+4)}+\frac{4y(x)}{x^2(3x+4)}=0$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x)=\frac{3x-4}{x(3x+4)}, P_3(x)=\frac{4}{x^2(3x+4)}\right]$
  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\Big|_{x=0}=-1$
  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=1$
  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

$x^2(3x+4)\left(\frac{d^2}{dx^2}y(x)\right)+x(3x-4)\left(\frac{d}{dx}y(x)\right)+4y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..3$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$
  - Shift index using $k->k+2-m$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $4a_0(-1+r)^2\,x^r+\left(\sum_{k=1}^{\infty}\left(4a_k(k+r-1)^2+3a_{k-1}(k+r-1)^2\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$4(-1+r)^2=0$

- Values of r that satisfy the indicial equation

$r = 1$

- Each term in the series must be 0, giving the recursion relation
  $(k + r - 1)^2 (4a_k + 3a_{k-1}) = 0$
- Shift index using $k- > k+1$
  $(k + r)^2 (4a_{k+1} + 3a_k) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = -\frac{3a_k}{4}$
- Recursion relation for $r = 1$
  $a_{k+1} = -\frac{3a_k}{4}$
- Solution for $r = 1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{3a_k}{4}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 19

```
dsolve(x^2*(3*x+4)*diff(diff(y(x),x),x)-x*(4-3*x)*diff(y(x),x)+4*y(x) = 0,y(x),singsol=a
```

$$y = \frac{x(c_2 \ln(x) + c_1)}{3x + 4}$$

Mathematica DSolve solution

Solving time : 0.254 (sec)
Leaf size : 49

```
DSolve[{x^2*(4+3*x)*D[y[x],{x,2}]-x*(4-3*x)*D[y[x],x]+4*y[x]==0,{}},y[x],x,IncludeSingularSolu
```

$$y(x) \to \sqrt{x}(c_2 \log(x) + c_1) \exp\left(-\frac{1}{2}\int_1^x \left(\frac{6}{3K[1]+4} - \frac{1}{K[1]}\right) dK[1]\right)$$

2.1.581 Problem 597

Internal problem ID [9753]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 597
Date solved : `Monday, January 27, 2025 at 06:13:54 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2\left(x^2+3x+1\right)y''+8x^2(3+2x)\,y'+\left(9x^2+3x+1\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.155 (sec)

Writing the ode as

$$\left(4x^4+12x^3+4x^2\right)y''+\left(16x^3+24x^2\right)y'+\left(9x^2+3x+1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=4x^4+12x^3+4x^2\\B&=16x^3+24x^2\\C&=9x^2+3x+1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=-1\\t&=4x^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{1}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1107: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=-\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2}-\left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^-}{x-c_1}\right)+(-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x}+(-)\,(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{1}{2x}\right)(0)+\left(\left(-\frac{1}{2x^2}\right)+\left(\frac{1}{2x}\right)^2-\left(-\frac{1}{4x^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}z_1(x)&=pe^{\int\omega\,dx}\\&=e^{\int\frac{1}{2x}dx}\\&=\sqrt{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1&=z_1e^{\int-\frac{1}{2}\frac{B}{A}\,dx}\\&=z_1e^{-\int\frac{1}{2}\frac{16x^3+24x^2}{4x^4+12x^3+4x^2}\,dx}\\&=z_1e^{-\ln(x^2+3x+1)}\\&=z_1\left(\frac{1}{x^2+3x+1}\right)\end{aligned}$$

Which simplifies to

$$y_1=\frac{\sqrt{x}}{x^2+3x+1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int\frac{e^{\int-\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2&=y_1\int\frac{e^{\int-\frac{16x^3+24x^2}{4x^4+12x^3+4x^2}\,dx}}{(y_1)^2}\,dx\\&=y_1\int\frac{e^{-2\ln(x^2+3x+1)}}{(y_1)^2}\,dx\\&=y_1(\ln(x))\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y&=c_1y_1+c_2y_2\\&=c_1\left(\frac{\sqrt{x}}{x^2+3x+1}\right)+c_2\left(\frac{\sqrt{x}}{x^2+3x+1}(\ln(x))\right)\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2(x^2+3x+1)\left(\frac{d^2}{dx^2}y(x)\right)+8x^2(2x+3)\left(\frac{d}{dx}y(x)\right)+(9x^2+3x+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(9x^2+3x+1)y(x)}{4x^2(x^2+3x+1)}-\frac{2(2x+3)\left(\frac{d}{dx}y(x)\right)}{x^2+3x+1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{2(2x+3)\left(\frac{d}{dx}y(x)\right)}{x^2+3x+1}+\frac{(9x^2+3x+1)y(x)}{4x^2(x^2+3x+1)}=0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{2(2x+3)}{x^2+3x+1}, P_3(x)=\frac{9x^2+3x+1}{4x^2(x^2+3x+1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=0$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=\frac{1}{4}$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4x^2(x^2+3x+1)\left(\frac{d^2}{dx^2}y(x)\right)+8x^2(2x+3)\left(\frac{d}{dx}y(x)\right)+(9x^2+3x+1)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

  ◦ Shift index using $k->k-m$

  $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=2..3$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..4$

  $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  ◦ Shift index using $k->k+2-m$

  $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$a_0(-1+2r)^2 x^r + \left(a_1(1+2r)^2 + 3a_0(1+2r)^2\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r-1)^2 + 3a_{k-1}(2k+2r-1\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+2r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = \frac{1}{2}$
- Each term must be 0
  $a_1(1+2r)^2 + 3a_0(1+2r)^2 = 0$
- Solve for the dependent coefficient(s)
  $a_1 = -3a_0$
- Each term in the series must be 0, giving the recursion relation
  $(2k+2r-1)^2 (a_k + 3a_{k-1} + a_{k-2}) = 0$
- Shift index using $k- >k+2$
  $(2k+2r+3)^2 (a_{k+2} + 3a_{k+1} + a_k) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -3a_{k+1} - a_k$
- Recursion relation for $r = \frac{1}{2}$
  $a_{k+2} = -3a_{k+1} - a_k$
- Solution for $r = \frac{1}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -3a_{k+1} - a_k, a_1 = -3a_0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.042 (sec)
Leaf size : 24

```
dsolve(4*x^2*(x^2+3*x+1)*diff(diff(y(x),x),x)+8*x^2*(2*x+3)*diff(y(x),x)+(9*x^2+3*x+1)*y
```

$$y = \frac{\sqrt{x}\,(c_2 \ln(x) + c_1)}{x^2 + 3x + 1}$$

Mathematica DSolve solution

Solving time : 0.058 (sec)
Leaf size : 29

```
DSolve[{4*x^2*(1+3*x+x^2)*D[y[x],{x,2}]+8*x^2*(3+2*x)*D[y[x],x]+(1+3*x+9*x^2)*y[x]==0,{}},y[
```

$$y(x)\to \frac{\sqrt{x}(c_2\log(x)+c_1)}{x^2+3x+1}$$

2.1.582 Problem 598

Internal problem ID [9754]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 598
Date solved : `Monday, January 27, 2025 at 06:13:54 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1-x)^2\,y'' - x\left(-3x^2+2x+1\right)y' + \left(x^2+1\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.164 (sec)

Writing the ode as

$$x^2(-1+x)^2\,y'' + \left(3x^3-2x^2-x\right)y' + \left(x^2+1\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(-1+x)^2 \\ B &= 3x^3-2x^2-x \\ C &= x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1109: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=-\frac{1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{2}-\left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^{-}}{x-c_1}\right)+(-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x}+(-)\,(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}
$$

Let

$$
p(x)=1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{gathered}
(0)+2\left(\frac{1}{2x}\right)(0)+\left(\left(-\frac{1}{2x^2}\right)+\left(\frac{1}{2x}\right)^2-\left(-\frac{1}{4x^2}\right)\right)=0 \\
0=0
\end{gathered}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \frac{1}{2x} dx} \\
&= \sqrt{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x^3-2x^2-x}{x^2(-1+x)^2}\, dx} \\
&= z_1 e^{-2\ln(-1+x)+\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{\sqrt{x}}{(-1+x)^2}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1=\frac{x}{(-1+x)^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2=y_1\int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1\int \frac{e^{\int -\frac{3x^3-2x^2-x}{x^2(-1+x)^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1\int \frac{e^{-4\ln(-1+x)+\ln(x)}}{(y_1)^2}\, dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1+c_2y_2 \\
&= c_1\left(\frac{x}{(-1+x)^2}\right)+c_2\left(\frac{x}{(-1+x)^2}(\ln(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(1-x)^2 \left(\frac{d^2}{dx^2}y(x)\right) - x(-3x^2+2x+1)\left(\frac{d}{dx}y(x)\right) + (x^2+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(x^2+1)y(x)}{x^2(x-1)^2} - \frac{\left(\frac{d}{dx}y(x)\right)(3x+1)}{x(x-1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{\left(\frac{d}{dx}y(x)\right)(3x+1)}{x(x-1)} + \frac{(x^2+1)y(x)}{x^2(x-1)^2} = 0$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{3x+1}{x(x-1)}, P_3(x) = \frac{x^2+1}{x^2(x-1)^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x=0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -1$
  - $x^2 \cdot P_3(x)$ is analytic at $x=0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 1$
  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2(x-1)^2\left(\frac{d^2}{dx^2}y(x)\right) + x(x-1)(3x+1)\left(\frac{d}{dx}y(x)\right) + (x^2+1)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,x^{k+r-2+m}$
  - Shift index using $k->k+2-m$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,x^{k+r}$

Rewrite ODE with series expansions

$$a_0(-1+r)^2 x^r + (-2a_0r^2 + a_1r^2) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)^2 - 2a_{k-1}(k+r-1)^2 + a_{k-2}(k+r\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $$(-1+r)^2 = 0$$
- Values of r that satisfy the indicial equation
  $$r = 1$$
- Each term must be 0
  $$-2a_0r^2 + a_1r^2 = 0$$
- Solve for the dependent coefficient(s)
  $$a_1 = 2a_0$$
- Each term in the series must be 0, giving the recursion relation
  $$(k+r-1)^2 (a_k - 2a_{k-1} + a_{k-2}) = 0$$
- Shift index using $k- > k+2$
  $$(k+r+1)^2 (a_{k+2} - 2a_{k+1} + a_k) = 0$$
- Recursion relation that defines series solution to ODE
  $$a_{k+2} = 2a_{k+1} - a_k$$
- Recursion relation for $r = 1$
  $$a_{k+2} = 2a_{k+1} - a_k$$
- Solution for $r = 1$
  $$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = 2a_{k+1} - a_k, a_1 = 2a_0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 17

```
dsolve(x^2*(1-x)^2*diff(diff(y(x),x),x)-x*(-3*x^2+2*x+1)*diff(y(x),x)+(x^2+1)*y(x) =
```

$$y = \frac{x(c_2 \ln(x) + c_1)}{(x-1)^2}$$

Mathematica DSolve solution

Solving time : 0.273 (sec)
Leaf size : 47

```
DSolve[{x^2*(1-x)^2*D[y[x],{x,2}]-x*(1+2*x-3*x^2)*D[y[x],x]+(1+x^2)*y[x]==0,{}},y[x],x,IncludeS
```

$$y(x) \to \sqrt{x}(c_2 \log(x) + c_1) \exp\left(-\frac{1}{2}\int_1^x \left(\frac{4}{K[1]-1} - \frac{1}{K[1]}\right) dK[1]\right)$$

2.1.583 Problem 599

Internal problem ID [9755]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 599
Date solved : `Monday, January 27, 2025 at 06:13:55 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$9x^2\left(x^2+x+1\right)y''+3x\left(13x^2+7x+1\right)y'+\left(25x^2+4x+1\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.271 (sec)

Writing the ode as

$$\left(9x^4+9x^3+9x^2\right)y''+\left(39x^3+21x^2+3x\right)y'+\left(25x^2+4x+1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=9x^4+9x^3+9x^2\\ B&=39x^3+21x^2+3x\\ C&=25x^2+4x+1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=-1\\ t&=4x^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{1}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1111: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=-\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{2}-\left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^{-}}{x-c_1}\right)+(-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x}+(-)(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{39x^3+21x^2+3x}{9x^4+9x^3+9x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{6} - \ln(x^2+x+1)} \\ &= z_1 \left(\frac{1}{x^{1/6}(x^2+x+1)}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/3}}{x^2+x+1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{39x^3+21x^2+3x}{9x^4+9x^3+9x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{\ln(x)}{3} - 2\ln(x^2+x+1)}}{(y_1)^2}\, dx \\ &= y_1 \left(2x - \frac{19}{24} + (x-1)^2 - \frac{x^5}{3(x^2+x+1)} + \frac{x}{3x^2+3x+3} - \frac{x^4}{3(x^2+x+1)} \right. \\ &\quad \left. - \frac{x^3}{3(x^2+x+1)} + \frac{x^2}{3x^2+3x+3} + \frac{1}{3x^2+3x+3} + \frac{x^3}{3} - x^2 + \ln(x)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{x^{1/3}}{x^2+x+1}\right) \\ &\quad + c_2 \left(\frac{x^{1/3}}{x^2+x+1} \left(2x - \frac{19}{24} + (x-1)^2 - \frac{x^5}{3(x^2+x+1)} + \frac{x}{3x^2+3x+3} - \frac{x^4}{3(x^2+x+1)} - \frac{x^3}{3(x^2+x+1)} + \right.\right. \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$$9x^2(x^2+x+1)\left(\frac{d^2}{dx^2}y(x)\right)+3x(13x^2+7x+1)\left(\frac{d}{dx}y(x)\right)+(25x^2+4x+1)\,y(x)=0$$

- Highest derivative means the order of the ODE is 2

$$\frac{d^2}{dx^2}y(x)$$

- Isolate 2nd derivative

$$\frac{d^2}{dx^2}y(x)=-\frac{(25x^2+4x+1)\,y(x)}{9x^2(x^2+x+1)}-\frac{(13x^2+7x+1)\left(\frac{d}{dx}y(x)\right)}{3x(x^2+x+1)}$$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$$\frac{d^2}{dx^2}y(x)+\frac{(13x^2+7x+1)\left(\frac{d}{dx}y(x)\right)}{3x(x^2+x+1)}+\frac{(25x^2+4x+1)\,y(x)}{9x^2(x^2+x+1)}=0$$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$$\left[P_2(x)=\frac{13x^2+7x+1}{3x(x^2+x+1)},P_3(x)=\frac{25x^2+4x+1}{9x^2(x^2+x+1)}\right]$$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

$$\left.(x\cdot P_2(x))\right|_{x=0}=\frac{1}{3}$$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$$\left.(x^2\cdot P_3(x))\right|_{x=0}=\frac{1}{9}$$

- ◦ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$$x_0=0$$

- Multiply by denominators

$$9x^2(x^2+x+1)\left(\frac{d^2}{dx^2}y(x)\right)+3x(13x^2+7x+1)\left(\frac{d}{dx}y(x)\right)+(25x^2+4x+1)\,y(x)=0$$

- Assume series solution for $y(x)$

$$y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

$$x^m\cdot y(x)=\sum_{k=0}^{\infty}a_k x^{k+r+m}$$

- ◦ Shift index using $k->k-m$

$$x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$$

- ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

$$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$$

- ◦ Shift index using $k->k+1-m$

$$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- ◦ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..4$

$$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

- ◦ Shift index using $k->k+2-m$

$$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+3r)^2 x^r + \left(a_1(2+3r)^2 + a_0(2+3r)^2\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r-1)^2 + a_{k-1}(3k+3r-1)^2\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+3r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = \frac{1}{3}$
- Each term must be 0
  $a_1(2+3r)^2 + a_0(2+3r)^2 = 0$
- Solve for the dependent coefficient(s)
  $a_1 = -a_0$
- Each term in the series must be 0, giving the recursion relation
  $(3k+3r-1)^2 (a_k + a_{k-1} + a_{k-2}) = 0$
- Shift index using $k- >k+2$
  $(3k+3r+5)^2 (a_{k+2} + a_{k+1} + a_k) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -a_{k+1} - a_k$
- Recursion relation for $r = \frac{1}{3}$
  $a_{k+2} = -a_{k+1} - a_k$
- Solution for $r = \frac{1}{3}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -a_{k+1} - a_k, a_1 = -a_0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.035 (sec)
Leaf size : 22

```
dsolve(9*x^2*(x^2+x+1)*diff(diff(y(x),x),x)+3*x*(13*x^2+7*x+1)*diff(y(x),x)+(25*x^2+4*x+
```

$$y = \frac{x^{1/3}(c_2 \ln(x) + c_1)}{x^2 + x + 1}$$

Mathematica DSolve solution

Solving time : 0.303 (sec)
Leaf size : 58

```
DSolve[{9*x^2*(1+x+x^2)*D[y[x],{x,2}]+3*x*(1+7*x+13*x^2)*D[y[x],x]+(1+4*x+25*x^2)*y[x]==0,{}
```

$$y(x) \to \sqrt{x}(c_2 \log(x) + c_1) \exp\left(-\frac{1}{2}\int_1^x \left(\frac{4K[1]+2}{K[1]^2+K[1]+1} + \frac{1}{3K[1]}\right) dK[1]\right)$$

2.1.584 Problem 600

Internal problem ID [9756]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 600
Date solved : Monday, January 27, 2025 at 06:13:56 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$2x^2(2+x)\,y''-x(4-7x)\,y'-(5-3x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.216 (sec)

Writing the ode as

$$\left(2x^3+4x^2\right)y''+\left(7x^2-4x\right)y'+(3x-5)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^3+4x^2 \\ B &= 7x^2-4x \\ C &= 3x-5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{-3x^2-32x+128}{16\left(x^2+2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3x^2-32x+128 \\ t &= 16\left(x^2+2x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-3x^2-32x+128}{16\left(x^2+2x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1113: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{2x} + \frac{5}{2\,(2+x)} + \frac{2}{x^2} + \frac{45}{16\,(2+x)^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(2+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{45}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{4} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 - 32x + 128}{16\left(x^2 + 2x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x^2 - 32x + 128}{16\left(x^2 + 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{9}{4}$	$-\frac{5}{4}$
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= \frac{3}{4} - \left(\frac{3}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{5}{4\left(2+x\right)} + \frac{2}{x} + (0) \\
&= -\frac{5}{4\left(2+x\right)} + \frac{2}{x} \\
&= \frac{3x+16}{4x\left(2+x\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{5}{4(2+x)} + \frac{2}{x}\right)(0) + \left(\left(\frac{5}{4(2+x)^2} - \frac{2}{x^2}\right) + \left(-\frac{5}{4(2+x)} + \frac{2}{x}\right)^2 - \left(\frac{-3x^2 - 32x + 128}{16(x^2 + 2x)^2}\right)\right) =$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(-\frac{5}{4(2+x)} + \frac{2}{x}\right) dx} \\ &= \frac{x^2}{(2+x)^{5/4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{7x^2-4x}{2x^3+4x^2}\, dx} \\ &= z_1 e^{-\frac{9\ln(2+x)}{4} + \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{\sqrt{x}}{(2+x)^{9/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{5/2}}{(2+x)^{7/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{7x^2-4x}{2x^3+4x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{9\ln(2+x)}{2} + \ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{-\frac{11(2+x)^{5/2}}{8} + \frac{10(2+x)^{3/2}}{3} - \frac{5\sqrt{2+x}}{2}}{x^3} - \frac{5\sqrt{2}\, \operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right)}{16}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{5/2}}{(2+x)^{7/2}}\right) \\
&\quad + c_2 \left(\frac{x^{5/2}}{(2+x)^{7/2}} \left(\frac{-\frac{11(2+x)^{5/2}}{8} + \frac{10(2+x)^{3/2}}{3} - \frac{5\sqrt{2+x}}{2}}{x^3} - \frac{5\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right)}{16}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2(x+2)\left(\frac{d^2}{dx^2}y(x)\right) - x(4-7x)\left(\frac{d}{dx}y(x)\right) - (5-3x)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(3x-5)y(x)}{2(x+2)x^2} - \frac{(-4+7x)\left(\frac{d}{dx}y(x)\right)}{2x(x+2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(-4+7x)\left(\frac{d}{dx}y(x)\right)}{2x(x+2)} + \frac{(3x-5)y(x)}{2(x+2)x^2} = 0$

□ Check to see if x_0 is a regular singular point

  ○ Define functions

    $\left[P_2(x) = \frac{-4+7x}{2x(x+2)}, P_3(x) = \frac{3x-5}{2(x+2)x^2}\right]$

  ○ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

    $\left((x+2)\cdot P_2(x)\right)\Big|_{x=-2} = \frac{9}{2}$

  ○ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

    $\left((x+2)^2\cdot P_3(x)\right)\Big|_{x=-2} = 0$

  ○ $x=-2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -2$

- Multiply by denominators

  $2x^2(x+2)\left(\frac{d^2}{dx^2}y(x)\right) + x(-4+7x)\left(\frac{d}{dx}y(x)\right) + (3x-5)\,y(x) = 0$

- Change variables using $x = u-2$ so that the regular singular point is at $u=0$

  $(2u^3 - 8u^2 + 8u)\left(\frac{d^2}{du^2}y(u)\right) + (7u^2 - 32u + 36)\left(\frac{d}{du}y(u)\right) + (3u-11)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ○ Convert $u^m\cdot y(u)$ to series expansion for $m = 0..1$

    $u^m\cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

  ○ Shift index using $k- >k-m$

    $u^m\cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$4a_0r(7+2r)\,u^{-1+r} + (4a_1(1+r)\,(9+2r) - a_0(8r^2+24r+11))\,u^r + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+r+1)\,(2k\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $4r(7+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{0, -\frac{7}{2}\right\}$
- Each term must be 0
  $4a_1(1+r)\,(9+2r) - a_0(8r^2+24r+11) = 0$
- Each term in the series must be 0, giving the recursion relation
  $2(-4a_k + a_{k-1} + 4a_{k+1})\,k^2 + (4(-4a_k + a_{k-1} + 4a_{k+1})\,r - 24a_k + a_{k-1} + 44a_{k+1})\,k + 2(-4a_k +$
- Shift index using $k->k+1$
  $2(-4a_{k+1} + a_k + 4a_{k+2})\,(k+1)^2 + (4(-4a_{k+1} + a_k + 4a_{k+2})\,r - 24a_{k+1} + a_k + 44a_{k+2})\,(k+1)$ -
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 4kra_k - 16kra_{k+1} + 2r^2a_k - 8r^2a_{k+1} + 5ka_k - 40ka_{k+1} + 5ra_k - 40ra_{k+1} + 3a_k - 43a_{k+1}}{4(2k^2+4kr+2r^2+15k+15r+22)}$
- Recursion relation for $r = 0$
  $a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 5ka_k - 40ka_{k+1} + 3a_k - 43a_{k+1}}{4(2k^2+15k+22)}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 5ka_k - 40ka_{k+1} + 3a_k - 43a_{k+1}}{4(2k^2+15k+22)}, 36a_1 - 11a_0 = 0\right]$
- Revert the change of variables $u = x+2$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+2)^k, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 5ka_k - 40ka_{k+1} + 3a_k - 43a_{k+1}}{4(2k^2+15k+22)}, 36a_1 - 11a_0 = 0\right]$
- Recursion relation for $r = -\frac{7}{2}$
  $a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} - 9ka_k + 16ka_{k+1} + 10a_k - a_{k+1}}{4(2k^2+k-6)}$
- Solution for $r = -\frac{7}{2}$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{7}{2}}, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} - 9ka_k + 16ka_{k+1} + 10a_k - a_{k+1}}{4(2k^2+k-6)}, -20a_1 - 25a_0 = 0\right]$
- Revert the change of variables $u = x+2$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+2)^{k-\frac{7}{2}}, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} - 9ka_k + 16ka_{k+1} + 10a_k - a_{k+1}}{4(2k^2+k-6)}, -20a_1 - 25a_0 = 0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k(x+2)^k\right) + \left(\sum_{k=0}^{\infty} b_k(x+2)^{k-\frac{7}{2}}\right), a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 5ka_k - 40ka_{k+1} + 3a_k - 43a_{k+1}}{4(2k^2+15k+22)}, \right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.044 (sec)
Leaf size : 55

```
dsolve(2*x^2*(x+2)*diff(diff(y(x),x),x)-x*(4-7*x)*diff(y(x),x)-(5-3*x)*y(x) = 0,y(x),sin
```

$$y=\frac{15\,\mathrm{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)c_2x^3+33c_2\sqrt{2}\left(x^2+\frac{52}{33}x+\frac{32}{33}\right)\sqrt{x+2}+c_1x^3}{\sqrt{x}\,(x+2)^{7/2}}$$

Mathematica DSolve solution

Solving time : 0.257 (sec)
Leaf size : 106

```
DSolve[{2*x^2*(2+x)*D[y[x],{x,2}]-x*(4-7*x)*D[y[x],x]-(5-3*x)*y[x]==0,{}},y[x],x,IncludeSingul
```

$$y(x)\to\exp\left(\int_1^x\left(\frac{2}{K[1]}-\frac{5}{4(K[1]+2)}\right)dK[1]-\frac{1}{2}\int_1^x\left(\frac{9}{2(K[2]+2)}-\frac{1}{K[2]}\right)dK[2]\right)\left(c_2\int_1^x\exp\left(-2\int_1^{K[3]}\frac{3K[1]+16}{4K[1]^2+8K[1]}dK[1]\right)dK[3]+c_1\right)$$

2.1.585 Problem 601

Internal problem ID [9757]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 601
Date solved : Monday, January 27, 2025 at 06:13:56 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$x^2(1-2x)\,y''+x(8-9x)\,y'+(6-3x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.211 (sec)

Writing the ode as

$$\left(-2x^3+x^2\right)y''+\left(-9x^2+8x\right)y'+(6-3x)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -2x^3+x^2 \\ B &= -9x^2+8x \\ C &= 6-3x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{21x^2-20x+24}{4\left(2x^2-x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 21x^2-20x+24 \\ t &= 4\left(2x^2-x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{21x^2-20x+24}{4\left(2x^2-x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1115: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{x^2} + \frac{19}{x} + \frac{77}{16\left(x - \frac{1}{2}\right)^2} - \frac{19}{x - \frac{1}{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{77}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{7}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{21x^2 - 20x + 24}{4\left(2x^2 - x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{21}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{21x^2 - 20x + 24}{4\left(2x^2 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	3	-2
$\frac{1}{2}$	2	0	$\frac{11}{4}$	$-\frac{7}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{4}$ then

$$\begin{aligned}d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= \frac{7}{4} - \left(\frac{3}{4}\right) \\ &= 1\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{2}{x} + \frac{11}{4\left(x-\frac{1}{2}\right)} + (0) \\ &= -\frac{2}{x} + \frac{11}{4\left(x-\frac{1}{2}\right)} \\ &= \frac{4+3x}{4x^2-2x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{2}{x} + \frac{11}{4\left(x-\frac{1}{2}\right)}\right)(1) + \left(\left(\frac{2}{x^2} - \frac{11}{4\left(x-\frac{1}{2}\right)^2}\right) + \left(-\frac{2}{x} + \frac{11}{4\left(x-\frac{1}{2}\right)}\right)^2 - \left(\frac{21x^2-20x+24}{4\left(2x^2-x\right)^2}\right)\right) = \frac{4-3a_0}{x(-1+2x)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{4}{3}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{4}{3}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \left(x+\frac{4}{3}\right) e^{\int \left(-\frac{2}{x}+\frac{11}{4\left(x-\frac{1}{2}\right)}\right) dx} \\ &= \left(x+\frac{4}{3}\right) e^{-2\ln(x)+\frac{11\ln(-1+2x)}{4}} \\ &= \frac{\left(x+\frac{4}{3}\right)(-1+2x)^{11/4}}{x^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-9x^2+8x}{-2x^3+x^2}\, dx} \\ &= z_1 e^{-4\ln(x)+\frac{7\ln(-1+2x)}{4}} \\ &= z_1\left(\frac{(-1+2x)^{7/4}}{x^4}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(-1+2x)^{9/2}(4+3x)}{3x^6}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-9x^2+8x}{-2x^3+x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-8\ln(x)+\frac{7\ln(-1+2x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{(231x^3-198x^2+66x-8)\,x^8 \mathrm{e}^{-8\ln(x)+\frac{7\ln(-1+2x)}{2}}}{385\,(4+3x)\,(-1+2x)^8}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(-1+2x)^{9/2}\,(4+3x)}{3x^6}\right) \\
&\quad + c_2 \left(\frac{(-1+2x)^{9/2}\,(4+3x)}{3x^6}\left(-\frac{(231x^3-198x^2+66x-8)\,x^8 \mathrm{e}^{-8\ln(x)+\frac{7\ln(-1+2x)}{2}}}{385\,(4+3x)\,(-1+2x)^8}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(-2x+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(8-9x)\left(\frac{d}{dx}y(x)\right)+(6-3x)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{3(x-2)y(x)}{x^2(2x-1)}-\frac{(-8+9x)\left(\frac{d}{dx}y(x)\right)}{x(2x-1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{(-8+9x)\left(\frac{d}{dx}y(x)\right)}{x(2x-1)}+\frac{3(x-2)y(x)}{x^2(2x-1)}=0$

- ☐ Check to see if x_0 is a regular singular point
  - ○ Define functions

    $\left[P_2(x)=\frac{-8+9x}{x(2x-1)}, P_3(x)=\frac{3(x-2)}{x^2(2x-1)}\right]$
  - ○ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=8$
  - ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=6$
  - ○ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(2x-1)\left(\frac{d^2}{dx^2}y(x)\right)+x(-8+9x)\left(\frac{d}{dx}y(x)\right)+(3x-6)\,y(x)=0$

- Assume series solution for $y(x)$

$y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- > k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k- > k + 1 - m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..3$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- > k + 2 - m$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

Rewrite ODE with series expansions

$-a_0(6+r)\,(1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r+6)\,(k+r+1) + a_{k-1}(k+2+r)\,(2k-1+2r)\right) x^{k+r}\right) =$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-(6+r)\,(1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-6, -1\}$
- Each term in the series must be 0, giving the recursion relation

  $2(k+2+r)\left(k+r-\frac{1}{2}\right) a_{k-1} - a_k(k+r+6)\,(k+r+1) = 0$
- Shift index using $k- > k + 1$

  $2(k+r+3)\left(k+\frac{1}{2}+r\right) a_k - a_{k+1}(k+7+r)\,(k+2+r) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{(k+r+3)(2k+2r+1)a_k}{(k+7+r)(k+2+r)}$
- Recursion relation for $r = -6$; series terminates at $k = 3$

  $a_{k+1} = \frac{(k-3)(2k-11)a_k}{(k+1)(k-4)}$
- Apply recursion relation for $k = 0$

  $a_1 = -\frac{33a_0}{4}$
- Apply recursion relation for $k = 1$

  $a_2 = -3a_1$
- Express in terms of a_0

  $a_2 = \frac{99a_0}{4}$
- Apply recursion relation for $k = 2$

  $a_3 = -\frac{7a_2}{6}$
- Express in terms of a_0

  $a_3 = -\frac{231a_0}{8}$
- Terminating series solution of the ODE for $r = -6$. Use reduction of order to find the second lin

  $y(x) = a_0 \cdot \left(-\frac{231}{8}x^3 + \frac{99}{4}x^2 - \frac{33}{4}x + 1\right)$

- Recursion relation for $r = -1$

  $a_{k+1} = \frac{(k+2)(2k-1)a_k}{(k+6)(k+1)}$

- Solution for $r = -1$

  $$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = \frac{(k+2)(2k-1)a_k}{(k+6)(k+1)}\right]$$

- Combine solutions and rename parameters

  $$\left[y(x) = a_0 \cdot \left(-\frac{231}{8}x^3 + \frac{99}{4}x^2 - \frac{33}{4}x + 1\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), b_{k+1} = \frac{(k+2)(2k-1)b_k}{(k+6)(k+1)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.020 (sec)
Leaf size : 43

```
dsolve(x^2*(1-2*x)*diff(diff(y(x),x),x)+x*(8-9*x)*diff(y(x),x)+(6-3*x)*y(x) = 0,y(x),s
```

$$y = \frac{48\left(x+\frac{4}{3}\right)\left(-\frac{1}{2}+x\right)^4 c_1\sqrt{-1+2x} + 231c_2\left(x^3 - \frac{6}{7}x^2 + \frac{2}{7}x - \frac{8}{231}\right)}{x^6}$$

Mathematica DSolve solution

Solving time : 0.489 (sec)
Leaf size : 130

```
DSolve[{x^2*(1-2*x)*D[y[x],{x,2}]+x*(8-9*x)*D[y[x],x]+(6-3*x)*y[x]==0,{}},y[x],x,IncludeSing
```

$$y(x) \to \frac{1}{3}(3x+4)\exp\left(\int_1^x -\frac{3K[1]+4}{2K[1]-4K[1]^2}dK[1] - \frac{1}{2}\int_1^x \frac{8-9K[2]}{K[2]-2K[2]^2}dK[2]\right)\left(c_2\int_1^x \frac{9\exp\left(-2\int_1^{K[3]} -\frac{3K[1]+4}{2K[1]-4K[1]^2}dK[1]\right)}{(3K[3]+4)^2}dK[3] + c_1\right)$$

2.1.586 Problem 602

Internal problem ID [9758]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 602
Date solved : `Monday, January 27, 2025 at 06:13:57 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(x^2+1)\,y''+x(10x^2+3)\,y'-(-14x^2+15)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.331 (sec)

Writing the ode as

$$(x^4+x^2)\,y''+(10x^3+3x)\,y'+(14x^2-15)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2 \\ B &= 10x^3+3x \\ C &= 14x^2-15 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{24x^4+66x^2+63}{4\,(x^3+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 24x^4+66x^2+63 \\ t &= 4(x^3+x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{24x^4+66x^2+63}{4\,(x^3+x)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1117: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{63}{4x^2} + \frac{21}{16\,(x-i)^2} + \frac{21}{16\,(x+i)^2} + \frac{99i}{16\,(x-i)} - \frac{99i}{16\,(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{63}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{7}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{24x^4 + 66x^2 + 63}{4\,(x^3+x)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = 6$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{24x^4 + 66x^2 + 63}{4\,(x^3+x)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{9}{2}$	$-\frac{7}{2}$
i	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
$-i$	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	3	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 3$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= 3 - (3) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{9}{2x} - \frac{3}{4(x-i)} - \frac{3}{4(x+i)} + (0) \\
&= \frac{9}{2x} - \frac{3}{4(x-i)} - \frac{3}{4(x+i)} \\
&= \frac{9}{2x} - \frac{3x}{2x^2+2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{9}{2x} - \frac{3}{4(x-i)} - \frac{3}{4(x+i)}\right)(0) + \left(\left(-\frac{9}{2x^2} + \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2}\right) + \left(\frac{9}{2x} - \frac{3}{4(x-i)} - \frac{3}{4(
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{9}{2x} - \frac{3}{4(x-i)} - \frac{3}{4(x+i)}\right) dx} \\
&= \frac{x^{9/2}}{(x^2+1)^{3/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{10x^3+3x}{x^4+x^2} \, dx} \\
&= z_1 e^{-\frac{7\ln(x^2+1)}{4} - \frac{3\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{(x^2+1)^{7/4} x^{3/2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^3}{(x^2+1)^{5/2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{10x^3+3x}{x^4+x^2} dx}}{(y_1)^2} dx \\
&= y_1 \int \frac{e^{-\frac{7\ln(x^2+1)}{2}-3\ln(x)}}{(y_1)^2} dx \\
&= y_1 \left(-\frac{(x^2+1)^{5/2}}{8x^8} + \frac{(x^2+1)^{5/2}}{16x^6} - \frac{(x^2+1)^{5/2}}{64x^4} - \frac{(x^2+1)^{5/2}}{128x^2} + \frac{(x^2+1)^{3/2}}{128} + \frac{3\sqrt{x^2+1}}{128} \right. \\
&\quad \left. - \frac{3\,\text{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)}{128} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^3}{(x^2+1)^{5/2}} \right) \\
&\quad + c_2 \left(\frac{x^3}{(x^2+1)^{5/2}} \left(-\frac{(x^2+1)^{5/2}}{8x^8} + \frac{(x^2+1)^{5/2}}{16x^6} - \frac{(x^2+1)^{5/2}}{64x^4} - \frac{(x^2+1)^{5/2}}{128x^2} + \frac{(x^2+1)^{3/2}}{128} + \frac{3\sqrt{x^2+1}}{128} - \frac{3\,\text{ar}}{} \right. \right.
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + x(10x^2+3)\left(\frac{d}{dx}y(x)\right) - (-14x^2+15)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(14x^2-15)y(x)}{x^2(x^2+1)} - \frac{(10x^2+3)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(10x^2+3)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)} + \frac{(14x^2-15)y(x)}{x^2(x^2+1)} = 0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{10x^2+3}{x(x^2+1)}, P_3(x) = \frac{14x^2-15}{x^2(x^2+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 3$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -15$

  ◦ $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(10x^2+3)\left(\frac{d}{dx}y(x)\right)+(14x^2-15)\,y(x)=0$
- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty} a_k x^{k+r}$
- □ Rewrite ODE with series expansions
  - Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

    $x^m\cdot y(x)=\sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m\cdot y(x)=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..4$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$
  - Shift index using $k->k+2-m$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(5+r)\,(-3+r)\,x^r+a_1(6+r)\,(-2+r)\,x^{1+r}+\left(\sum_{k=2}^{\infty}(a_k(k+r+5)\,(k+r-3)+a_{k-2}(k+r-$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(5+r)\,(-3+r)=0$
- Values of r that satisfy the indicial equation

  $r\in\{-5,3\}$
- Each term must be 0

  $a_1(6+r)\,(-2+r)=0$
- Solve for the dependent coefficient(s)

  $a_1=0$
- Each term in the series must be 0, giving the recursion relation

  $(k+r+5)\,(a_k(k+r-3)+a_{k-2}(k+r))=0$
- Shift index using $k->k+2$

  $(k+r+7)\,(a_{k+2}(k+r-1)+a_k(k+r+2))=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_k(k+r+2)}{k+r-1}$
- Recursion relation for $r=-5$

  $a_{k+2}=-\frac{a_k(k-3)}{k-6}$
- Series not valid for $r=-5$, division by 0 in the recursion relation at $k=6$

  $a_{k+2}=-\frac{a_k(k-3)}{k-6}$
- Recursion relation for $r=3$

  $a_{k+2}=-\frac{a_k(k+5)}{k+2}$
- Solution for $r=3$

$$\left[y(x)=\sum_{k=0}^{\infty}a_k x^{k+3}, a_{k+2}=-\frac{a_k(k+5)}{k+2}, a_1=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.037 (sec)
Leaf size : 61

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)+x*(10*x^2+3)*diff(y(x),x)-(-14*x^2+15)*y(x) = 0,
```

$$y=\frac{3\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)c_2x^8+c_2(-3x^6+2x^4+24x^2+16)\sqrt{x^2+1}+c_1x^8}{(x^2+1)^{5/2}x^5}$$

Mathematica DSolve solution

Solving time : 0.247 (sec)
Leaf size : 112

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]+x*(3+10*x^2)*D[y[x],x]-(15-14*x^2)*y[x]==0,{}},y[x],x,Includ
```

$$y(x)\to\exp\left(\int_1^x\frac{6K[1]^2+9}{2(K[1]^3+K[1])}dK[1]-\frac{1}{2}\int_1^x\frac{10K[2]^2+3}{K[2]^3+K[2]}dK[2]\right)\left(c_2\int_1^x\exp\left(-2\int_1^{K[3]}\frac{6K[1]^2+9}{2(K[1]^3+K[1])}dK[1]\right)dK[3]+c_1\right)$$

2.1.587 Problem 603

Internal problem ID [9759]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 603
Date solved : `Monday, January 27, 2025 at 06:13:58 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(-2x^2+1\right)y''+x\left(-13x^2+7\right)y'-14x^2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.286 (sec)

Writing the ode as

$$\left(-2x^4+x^2\right)y''+\left(-13x^3+7x\right)y'-14x^2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -2x^4+x^2 \\ B &= -13x^3+7x \\ C &= -14x^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{5x^4-68x^2+35}{4\left(2x^3-x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 5x^4-68x^2+35 \\ t &= 4\left(2x^3-x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{5x^4-68x^2+35}{4\left(2x^3-x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1119: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^3 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{35}{4x^2} + \frac{9}{64\left(x - \frac{\sqrt{2}}{2}\right)^2} + \frac{9}{64\left(x + \frac{\sqrt{2}}{2}\right)^2} - \frac{279\sqrt{2}}{64\left(x - \frac{\sqrt{2}}{2}\right)} + \frac{279\sqrt{2}}{64\left(x + \frac{\sqrt{2}}{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2} \end{aligned}$$

For the pole at $x = \frac{\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{9}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{8}
\end{aligned}
$$

For the pole at $x = -\frac{\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{9}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{8}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{5x^4 - 68x^2 + 35}{4\left(2x^3 - x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{5x^4 - 68x^2 + 35}{4\left(2x^3 - x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$
$\frac{\sqrt{2}}{2}$	2	0	$\frac{9}{8}$	$-\frac{1}{8}$
$-\frac{\sqrt{2}}{2}$	2	0	$\frac{9}{8}$	$-\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\
&= -\frac{1}{4} - \left(-\frac{1}{4}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^+}{x-c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{5}{2x} + \frac{9}{8\left(x-\frac{\sqrt{2}}{2}\right)} + \frac{9}{8\left(x+\frac{\sqrt{2}}{2}\right)} + (-)(0) \\
&= -\frac{5}{2x} + \frac{9}{8\left(x-\frac{\sqrt{2}}{2}\right)} + \frac{9}{8\left(x+\frac{\sqrt{2}}{2}\right)} \\
&= \frac{-x^2+5}{4x^3-2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{5}{2x} + \frac{9}{8\left(x-\frac{\sqrt{2}}{2}\right)} + \frac{9}{8\left(x+\frac{\sqrt{2}}{2}\right)} \right)(0) + \left(\left(\frac{5}{2x^2} - \frac{9}{8\left(x-\frac{\sqrt{2}}{2}\right)^2} - \frac{9}{8\left(x+\frac{\sqrt{2}}{2}\right)^2} \right) + \left(-\frac{5}{2x} + \frac{}{8} \right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(-\frac{5}{2x} + \frac{9}{8\left(x-\frac{\sqrt{2}}{2}\right)} + \frac{9}{8\left(x+\frac{\sqrt{2}}{2}\right)} \right) dx} \\
&= \frac{\left(2x-\sqrt{2}\right)^{9/8} \left(2x+\sqrt{2}\right)^{9/8}}{x^{5/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-13x^3+7x}{-2x^4+x^2}\, dx} \\
&= z_1 e^{\frac{\ln\left(2x^2-1\right)}{8} - \frac{7\ln(x)}{2}} \\
&= z_1 \left(\frac{\left(2x^2-1\right)^{1/8}}{x^{7/2}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{2\left(2x^2-1\right)^{5/4} 2^{1/8}}{x^6}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-13x^3+7x}{-2x^4+x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\frac{\ln(2x^2-1)}{4}-7\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{(5x^4-20x^2+8)\, x^7 \mathrm{e}^{\frac{\ln(2x^2-1)}{4}-7\ln(x)} 2^{3/4}}{120\,(2x^2-1)^{3/2}}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{2(2x^2-1)^{5/4}\, 2^{1/8}}{x^6}\right) \\ &\quad + c_2 \left(\frac{2(2x^2-1)^{5/4}\, 2^{1/8}}{x^6} \left(\frac{(5x^4-20x^2+8)\, x^7 \mathrm{e}^{\frac{\ln(2x^2-1)}{4}-7\ln(x)} 2^{3/4}}{120\,(2x^2-1)^{3/2}}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(-2x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + x(-13x^2+7)\left(\frac{d}{dx}y(x)\right) - 14x^2 y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{14y(x)}{2x^2-1} - \frac{(13x^2-7)\left(\frac{d}{dx}y(x)\right)}{x(2x^2-1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(13x^2-7)\left(\frac{d}{dx}y(x)\right)}{x(2x^2-1)} + \frac{14y(x)}{2x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{13x^2-7}{x(2x^2-1)}, P_3(x) = \frac{14}{2x^2-1}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 7$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

  $x(2x^2-1)\left(\frac{d^2}{dx^2}y(x)\right)+(13x^2-7)\left(\frac{d}{dx}y(x)\right)+14xy(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - ∘ Convert $x\cdot y(x)$ to series expansion

    $x\cdot y(x)=\sum_{k=0}^{\infty} a_k x^{k+r+1}$
  - ∘ Shift index using $k->k-1$

    $x\cdot y(x)=\sum_{k=1}^{\infty} a_{k-1} x^{k+r}$
  - ∘ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..2$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - ∘ Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - ∘ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=1..3$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$
  - ∘ Shift index using $k->k+2-m$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0r(6+r)\,x^{-1+r}-a_1(1+r)\,(7+r)\,x^r+\left(\sum_{k=1}^{\infty}(-a_{k+1}(k+r+1)\,(k+7+r)+a_{k-1}(2k+5+2r)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(6+r)=0$
- Values of r that satisfy the indicial equation

  $r\in\{-6,0\}$
- Each term must be 0

  $-a_1(1+r)\,(7+r)=0$
- Each term in the series must be 0, giving the recursion relation

  $2(k+r+1)\left(\left(k+r+\frac{5}{2}\right)a_{k-1}-\frac{a_{k+1}(k+7+r)}{2}\right)=0$
- Shift index using $k->k+1$

  $2(k+r+2)\left(\left(k+\frac{7}{2}+r\right)a_k-\frac{a_{k+2}(k+8+r)}{2}\right)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{(2k+2r+7)a_k}{k+8+r}$
- Recursion relation for $r=-6$

  $a_{k+2}=\frac{(2k-5)a_k}{k+2}$
- Solution for $r=-6$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k-6}, a_{k+2}=\frac{(2k-5)a_k}{k+2}, 5a_1=0\right]$
- Recursion relation for $r=0$

  $a_{k+2}=\frac{(2k+7)a_k}{k+8}$

- Solution for $r = 0$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{(2k+7)a_k}{k+8}, -7a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-6}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = \frac{(2k-5)a_k}{k+2}, 5a_1 = 0, b_{k+2} = \frac{(2k+7)b_k}{k+8}, -7b_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.022` (sec)
Leaf size : 35

```
dsolve(x^2*(-2*x^2+1)*diff(diff(y(x),x),x)+x*(-13*x^2+7)*diff(y(x),x)-14*x^2*y(x) = 0,
```

$$y = \frac{c_1 (2x^2 - 1)^{5/4} + 5c_2 x^4 - 20c_2 x^2 + 8c_2}{x^6}$$

Mathematica DSolve solution

Solving time : `0.214` (sec)
Leaf size : 116

```
DSolve[{x^2*(1-2*x^2)*D[y[x],{x,2}]+x*(7-13*x^2)*D[y[x],x]-14*x^2*y[x]==0,{}},y[x],x,Include
```

$$y(x) \to \exp\left(\int_1^x \frac{5 - K[1]^2}{4K[1]^3 - 2K[1]} dK[1] - \frac{1}{2}\int_1^x \frac{7 - 13K[2]^2}{K[2] - 2K[2]^3} dK[2]\right) \left(c_2 \int_1^x \exp\left(-2\int_1^{K[3]} \frac{5 - K[1]^2}{4K[1]^3 - 2K[1]} dK[1]\right) dK[3] + c_1\right)$$

2.1.588 Problem 604

Internal problem ID [9760]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 604
Date solved : `Monday, January 27, 2025 at 06:13:58 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2(1+x)\,y''+4x(1+2x)\,y'-(1+3x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.213 (sec)

Writing the ode as

$$\left(4x^3+4x^2\right)y''+\left(8x^2+4x\right)y'+(-3x-1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^3+4x^2 \\ B &= 8x^2+4x \\ C &= -3x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{3x+4}{4x\left(1+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3x+4 \\ t &= 4x(1+x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{3x+4}{4x\left(1+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1121: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 3 - 1\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x(1+x)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= 1\\ \alpha_c^- &= 1\end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{1+x} + \frac{1}{x} - \frac{1}{4\,(1+x)^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x+4}{4x(1+x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3x+4}{4x(1+x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right)\\
&= \frac{3}{2} - \left(\frac{3}{2}\right)\\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (+)[\sqrt{r}]_\infty\\
&= \frac{1}{x} + \frac{1}{2+2x} + (0)\\
&= \frac{1}{x} + \frac{1}{2+2x}\\
&= \frac{1}{x} + \frac{1}{2+2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x} + \frac{1}{2+2x}\right)(0) + \left(\left(-\frac{1}{x^2} - \frac{1}{2(1+x)^2}\right) + \left(\frac{1}{x} + \frac{1}{2+2x}\right)^2 - \left(\frac{3x+4}{4x(1+x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{1}{x} + \frac{1}{2+2x}\right) dx} \\ &= x\sqrt{1+x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{8x^2+4x}{4x^3+4x^2} \, dx} \\ &= z_1 e^{-\frac{\ln(x(1+x))}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x(1+x)}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x\sqrt{1+x}}{\sqrt{x(1+x)}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{8x^2+4x}{4x^3+4x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\ln(x(1+x))}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{1}{x} - \ln(x) + \ln(1+x)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{x\sqrt{1+x}}{\sqrt{x(1+x)}}\right) + c_2 \left(\frac{x\sqrt{1+x}}{\sqrt{x(1+x)}}\left(-\frac{1}{x} - \ln(x) + \ln(1+x)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)+4x(2x+1)\left(\frac{d}{dx}y(x)\right)-(3x+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{(3x+1)y(x)}{4x^2(x+1)}-\frac{(2x+1)\left(\frac{d}{dx}y(x)\right)}{x(x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{(2x+1)\left(\frac{d}{dx}y(x)\right)}{x(x+1)}-\frac{(3x+1)y(x)}{4x^2(x+1)}=0$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x)=\frac{2x+1}{x(x+1)}, P_3(x)=-\frac{3x+1}{4x^2(x+1)}\right]$

  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=1$

  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

  - $x=-1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=-1$

- Multiply by denominators

  $4x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)+4x(2x+1)\left(\frac{d}{dx}y(x)\right)+(-3x-1)\,y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(4u^3-8u^2+4u)\left(\frac{d^2}{du^2}y(u)\right)+(8u^2-12u+4)\left(\frac{d}{du}y(u)\right)+(-3u+2)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

    $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

  - Shift index using $k->k-m$

    $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$4a_0 r^2 u^{-1+r} + \left(4a_1(1+r)^2 - 2a_0(4r^2+2r-1)\right)u^r + \left(\sum_{k=1}^{\infty} \left(4a_{k+1}(k+1+r)^2 - 2a_k(4k^2+8kr\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $4r^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 0$
- Each term must be 0
  $4a_1(1+r)^2 - 2a_0(4r^2+2r-1) = 0$
- Each term in the series must be 0, giving the recursion relation
  $(4k^2-4k-3)\,a_{k-1} + (-8k^2-4k+2)\,a_k + 4a_{k+1}(k+1)^2 = 0$
- Shift index using $k->k+1$
  $\left(4(k+1)^2-4k-7\right)a_k + \left(-8(k+1)^2-4k-2\right)a_{k+1} + 4a_{k+2}(k+2)^2 = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{4k^2a_k-8k^2a_{k+1}+4ka_k-20ka_{k+1}-3a_k-10a_{k+1}}{4(k+2)^2}$
- Recursion relation for $r = 0$
  $a_{k+2} = -\frac{4k^2a_k-8k^2a_{k+1}+4ka_k-20ka_{k+1}-3a_k-10a_{k+1}}{4(k+2)^2}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{4k^2a_k-8k^2a_{k+1}+4ka_k-20ka_{k+1}-3a_k-10a_{k+1}}{4(k+2)^2}, 4a_1+2a_0 = 0\right]$
- Revert the change of variables $u = x+1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+2} = -\frac{4k^2a_k-8k^2a_{k+1}+4ka_k-20ka_{k+1}-3a_k-10a_{k+1}}{4(k+2)^2}, 4a_1+2a_0 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.025 (sec)
Leaf size : 26

```
dsolve(4*x^2*(x+1)*diff(diff(y(x),x),x)+4*x*(2*x+1)*diff(y(x),x)-(3*x+1)*y(x) = 0,y(x)
```

$$y = \frac{\ln(x)\, c_2 x + c_1 x - \ln(x+1)\, c_2 x + c_2}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.236 (sec)
Leaf size : 96

```
DSolve[{4*x^2*(1+x)*D[y[x],{x,2}]+4*x*(1+2*x)*D[y[x],x]-(1+3*x)*y[x]==0,{}},y[x],x,IncludeSingu
```

$$y(x)\to \exp\left(\int_1^x\left(\frac{1}{2K[1]+2}+\frac{1}{K[1]}\right)dK[1]-\frac{1}{2}\int_1^x\left(\frac{1}{K[2]+1}+\frac{1}{K[2]}\right)dK[2]\right)\left(c_2\int_1^x\exp\left(-2\int_1^{K[3]}\frac{3K[1]+2}{2K[1]^2+2K[1]}dK[1]\right)dK[3]+c_1\right)$$

2.1.589 Problem 605

Internal problem ID [9761]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 605
Date solved : Monday, January 27, 2025 at 06:13:59 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$2x^2(2+3x)\,y''+x(4+21x)\,y'-(1-9x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.220 (sec)

Writing the ode as

$$\left(6x^3+4x^2\right)y''+\left(21x^2+4x\right)y'+(9x-1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 6x^3+4x^2 \\ B &= 21x^2+4x \\ C &= 9x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-27x-48}{16x\,(2+3x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -27x-48 \\ t &= 16x(2+3x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-27x-48}{16x\,(2+3x)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1123: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x(2+3x)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = -\frac{2}{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16\left(x+\frac{2}{3}\right)^2} + \frac{3}{4\left(x+\frac{2}{3}\right)} - \frac{3}{4x}$$

For the pole at $x = -\frac{2}{3}$ let b be the coefficient of $\frac{1}{\left(x+\frac{2}{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-27x - 48}{16x\,(2 + 3x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-27x - 48}{16x\,(2 + 3x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1
$-\frac{2}{3}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{3}{4} - \left(\frac{3}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{x} - \frac{1}{4\left(x + \frac{2}{3}\right)} + (0) \\
&= \frac{1}{x} - \frac{1}{4\left(x + \frac{2}{3}\right)} \\
&= \frac{8 + 9x}{12x^2 + 8x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x} - \frac{1}{4\left(x+\frac{2}{3}\right)}\right)(0) + \left(\left(-\frac{1}{x^2} + \frac{1}{4\left(x+\frac{2}{3}\right)^2}\right) + \left(\frac{1}{x} - \frac{1}{4\left(x+\frac{2}{3}\right)}\right)^2 - \left(\frac{-27x-48}{16x\left(2+3x\right)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{x} - \frac{1}{4\left(x+\frac{2}{3}\right)}\right) dx} \\ &= \frac{x}{\left(2+3x\right)^{1/4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{21x^2+4x}{6x^3+4x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2} - \frac{5\ln(2+3x)}{4}} \\ &= z_1 \left(\frac{1}{\sqrt{x}\left(2+3x\right)^{5/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{\left(2+3x\right)^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{21x^2+4x}{6x^3+4x^2}\, dx}}{\left(y_1\right)^2}\, dx \\ &= y_1 \int \frac{e^{-\ln(x) - \frac{5\ln(2+3x)}{2}}}{\left(y_1\right)^2}\, dx \\ &= y_1 \left(-\frac{\sqrt{2+3x}}{x} - \frac{3\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2+3x}\,\sqrt{2}}{2}\right)}{2}\right) \end{aligned}$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\frac{\sqrt{x}}{(2+3x)^{3/2}}\right) + c_2 \left(\frac{\sqrt{x}}{(2+3x)^{3/2}} \left(-\frac{\sqrt{2+3x}}{x} - \frac{3\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2+3x}\,\sqrt{2}}{2}\right)}{2}\right)\right)$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2(3x+2)\left(\frac{d^2}{dx^2}y(x)\right) + x(4+21x)\left(\frac{d}{dx}y(x)\right) - (1-9x)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(-1+9x)y(x)}{2(3x+2)x^2} - \frac{(4+21x)\left(\frac{d}{dx}y(x)\right)}{2x(3x+2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(4+21x)\left(\frac{d}{dx}y(x)\right)}{2x(3x+2)} + \frac{(-1+9x)y(x)}{2(3x+2)x^2} = 0$

□ Check to see if x_0 is a regular singular point

  ○ Define functions

    $\left[P_2(x) = \frac{4+21x}{2x(3x+2)}, P_3(x) = \frac{-1+9x}{2(3x+2)x^2}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x=0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ○ $x^2 \cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$

  ○ $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $2x^2(3x+2)\left(\frac{d^2}{dx^2}y(x)\right) + x(4+21x)\left(\frac{d}{dx}y(x)\right) + (-1+9x)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ○ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ○ Shift index using $k- >k-m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ○ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  ○ Shift index using $k- >k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- ∘ Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..3$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$
- ∘ Shift index using $k- >k+2-m$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)(-1+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)(2k+2r-1) + 3a_{k-1}(2k+2r+1)(k+r)\right) x^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $4\left(\left(k+r-\frac{1}{2}\right) a_k + \frac{3a_{k-1}(k+r)}{2}\right)\left(k+r+\frac{1}{2}\right) = 0$
- Shift index using $k- >k+1$

  $4\left(\left(k+r+\frac{1}{2}\right) a_{k+1} + \frac{3a_k(k+r+1)}{2}\right)\left(k+\frac{3}{2}+r\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{3a_k(k+r+1)}{2k+2r+1}$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+1} = -\frac{3a_k\left(k+\frac{1}{2}\right)}{2k}$
- Solution for $r = -\frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = -\frac{3a_k\left(k+\frac{1}{2}\right)}{2k}\right]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = -\frac{3a_k\left(k+\frac{3}{2}\right)}{2k+2}$
- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{3a_k\left(k+\frac{3}{2}\right)}{2k+2}\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = -\frac{3a_k\left(k+\frac{1}{2}\right)}{2k}, b_{k+1} = -\frac{3b_k\left(k+\frac{3}{2}\right)}{2k+2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.043 (sec)
Leaf size : 48

```
dsolve(2*x^2*(2+3*x)*diff(diff(y(x),x),x)+x*(4+21*x)*diff(y(x),x)-(1-9*x)*y(x) = 0,y(x
```

$$y = \frac{\sqrt{2+3x}\,\sqrt{2}\,c_2 + c_1 x + 3\,\text{arctanh}\left(\frac{\sqrt{2}\,\sqrt{2+3x}}{2}\right) c_2 x}{(2+3x)^{3/2}\,\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.297 (sec)
Leaf size : 102

```
DSolve[{2*x^2*(2+3*x)*D[y[x],{x,2}]+x*(4+21*x)*D[y[x],x]-(1-9*x)*y[x]==0,{}},y[x],x,IncludeS
```

$$y(x) \to \exp\left(\int_1^x \left(\frac{1}{K[1]} - \frac{3}{12K[1]+8}\right) dK[1] - \frac{1}{2}\int_1^x \left(\frac{15}{6K[2]+4} + \frac{1}{K[2]}\right) dK[2]\right) \left(c_2 \int_1^x \exp\left(-2\int_1^{K[3]} \frac{9K[1]+8}{12K[1]^2+8K[1]} dK[1]\right) dK[3] + c_1\right)$$

2.1.590 Problem 606

Internal problem ID [9762]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 606
Date solved : Monday, January 27, 2025 at 06:13:59 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$x^2y'' + x(2+x)\,y' - (2-3x)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.238 (sec)

Writing the ode as

$$x^2y'' + \left(x^2+2x\right)y' + (3x-2)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2+2x \\ C &= 3x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2-8x+8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2-8x+8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2-8x+8}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1125: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{x^2} - \frac{2}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{2}{x} - \frac{2}{x^2} - \frac{8}{x^3} - \frac{36}{x^4} - \frac{176}{x^5} - \frac{912}{x^6} - \frac{4928}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 8x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-8x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-8x + 8}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -8. Dividing this by leading coefficient in t which is 4 gives -2. Now b can be found.

$$\begin{aligned} b &= (-2) - (0) \\ &= -2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-2}{\frac{1}{2}} - 0\right) &&= -2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-2}{\frac{1}{2}} - 0\right) &&= 2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 8x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	-2	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\
&= 2 - (2) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{2}{x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{2}{x} - \frac{1}{2} \\
&= -\frac{x - 4}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered}
(0) + 2\left(\frac{2}{x} - \frac{1}{2}\right)(0) + \left(\left(-\frac{2}{x^2}\right) + \left(\frac{2}{x} - \frac{1}{2}\right)^2 - \left(\frac{x^2 - 8x + 8}{4x^2}\right)\right) = 0 \\
0 = 0
\end{gathered}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{2}{x} - \frac{1}{2}\right) dx} \\
&= x^2 e^{-\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2+2x}{x^2}\,dx} \\
&= z_1 e^{-\frac{x}{2}-\ln(x)} \\
&= z_1\left(\frac{\mathrm{e}^{-\frac{x}{2}}}{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+2x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{\mathrm{e}^x}{3x^3}-\frac{\mathrm{e}^x}{6x^2}-\frac{\mathrm{e}^x}{6x}-\frac{\mathrm{Ei}_1(-x)}{6}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x\,\mathrm{e}^{-x}\right) + c_2\left(x\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^x}{3x^3}-\frac{\mathrm{e}^x}{6x^2}-\frac{\mathrm{e}^x}{6x}-\frac{\mathrm{Ei}_1(-x)}{6}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right)+x(x+2)\left(\frac{d}{dx}y(x)\right)-(-3x+2)\,y(x)=0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x)=-\frac{(3x-2)y(x)}{x^2}-\frac{(x+2)\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x)+\frac{(x+2)\left(\frac{d}{dx}y(x)\right)}{x}+\frac{(3x-2)y(x)}{x^2}=0$
- ☐ Check to see if $x_0=0$ is a regular singular point
  - Define functions
    $\left[P_2(x)=\frac{x+2}{x}, P_3(x)=\frac{3x-2}{x^2}\right]$
  - $x\cdot P_2(x)$ is analytic at $x=0$

$$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 2$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -2$$

- $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(x+2)\left(\frac{d}{dx}y(x)\right) + (3x-2)\,y(x) = 0$$

- Assume series solution for $y(x)$

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

$$x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k->k-m$

$$x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

$$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

  - Shift index using $k->k+1-m$

$$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

  - Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

$$x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$$

  Rewrite ODE with series expansions

$$a_0(2+r)\,(-1+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+2)\,(k+r-1) + a_{k-1}(k+r+2)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(2+r)\,(-1+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{-2, 1\}$

- Each term in the series must be 0, giving the recursion relation

$(k+r+2)\,(a_k(k+r-1) + a_{k-1}) = 0$

- Shift index using $k->k+1$

$(k+r+3)\,(a_{k+1}(k+r) + a_k) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = -\frac{a_k}{k+r}$

- Recursion relation for $r = -2$

$a_{k+1} = -\frac{a_k}{k-2}$

- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 2$

$a_{k+1} = -\frac{a_k}{k-2}$

- Recursion relation for $r = 1$

$a_{k+1} = -\frac{a_k}{k+1}$

- Solution for $r = 1$

$$\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1}=-\frac{a_k}{k+1}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 40

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(x+2)*diff(y(x),x)-(2-3*x)*y(x) = 0,y(x),singsol=all)
```

$$y=\frac{\operatorname{Ei}_1(-x)\,\mathrm{e}^{-x} c_2 x^3+\mathrm{e}^{-x} c_1 x^3+c_2(x^2+x+2)}{x^2}$$

Mathematica DSolve solution

Solving time : 0.211 (sec)
Leaf size : 36

```
DSolve[{x^2*D[y[x],{x,2}]+x*(2+x)*D[y[x],x]-(2-3*x)*y[x]==0,{}},y[x],x,IncludeSingularSolutions
```

$$y(x)\to e^{-x-1}x\left(c_2\int_1^x \frac{e^{K[1]}}{K[1]^4}dK[1]+c_1\right)$$

2.1.591 Problem 607

Internal problem ID [9763]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 607
Date solved : `Monday, January 27, 2025 at 06:14:00 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2(1+x)\,y''+4x(3+8x)\,y'-(5-49x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.212 (sec)

Writing the ode as

$$\left(4x^3+4x^2\right)y''+\left(32x^2+12x\right)y'+(49x-5)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=4x^3+4x^2\\ B&=32x^2+12x\\ C&=49x-5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-x^2-8x+8}{4\left(x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=-x^2-8x+8\\ t&=4\left(x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-x^2-8x+8}{4\left(x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1127: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{1 + x} + \frac{2}{x^2} + \frac{15}{4\,(1 + x)^2} - \frac{6}{x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 8x + 8}{4\left(x^2 + x\right)^2}$$

Since the gcd$(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 - 8x + 8}{4\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{2(1+x)} + \frac{2}{x} + (-)(0) \\
&= -\frac{3}{2(1+x)} + \frac{2}{x} \\
&= \frac{x+4}{2x(1+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{2(1+x)} + \frac{2}{x}\right)(0) + \left(\left(\frac{3}{2(1+x)^2} - \frac{2}{x^2}\right) + \left(-\frac{3}{2(1+x)} + \frac{2}{x}\right)^2 - \left(\frac{-x^2-8x+8}{4(x^2+x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{3}{2(1+x)} + \frac{2}{x}\right) dx} \\ &= \frac{x^2}{(1+x)^{3/2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{32x^2+12x}{4x^3+4x^2} \, dx} \\ &= z_1 e^{-\frac{3\ln(x)}{2} - \frac{5\ln(1+x)}{2}} \\ &= z_1 \left(\frac{1}{x^{3/2}(1+x)^{5/2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{(1+x)^4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{32x^2+12x}{4x^3+4x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-3\ln(x) - 5\ln(1+x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\ln(x) - \frac{3}{x} - \frac{1}{3x^3} - \frac{3}{2x^2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\sqrt{x}}{(1+x)^4}\right) + c_2 \left(\frac{\sqrt{x}}{(1+x)^4} \left(\ln(x) - \frac{3}{x} - \frac{1}{3x^3} - \frac{3}{2x^2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)+4x(3+8x)\left(\frac{d}{dx}y(x)\right)-(5-49x)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{(-5+49x)y(x)}{4(x+1)x^2}-\frac{(3+8x)\left(\frac{d}{dx}y(x)\right)}{x(x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)+\frac{(3+8x)\left(\frac{d}{dx}y(x)\right)}{x(x+1)}+\frac{(-5+49x)y(x)}{4(x+1)x^2}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=\frac{3+8x}{x(x+1)},P_3(x)=\frac{-5+49x}{4(x+1)x^2}\right]$

- ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

$\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=5$

- ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

$\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

- ◦ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1$

- Multiply by denominators

$4x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)+4x(3+8x)\left(\frac{d}{dx}y(x)\right)+(-5+49x)\,y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(4u^3-8u^2+4u)\left(\frac{d^2}{du^2}y(u)\right)+(32u^2-52u+20)\left(\frac{d}{du}y(u)\right)+(-54+49u)\,y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

$u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

- ◦ Shift index using $k->k-m$

$u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ◦ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- ◦ Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$4a_0r(4+r)\,u^{-1+r} + (4a_1(1+r)\,(5+r) - 2a_0(4r^2+22r+27))\,u^r + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+1+r)\,(k+5\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $$4r(4+r) = 0$$
- Values of r that satisfy the indicial equation
  $$r \in \{-4, 0\}$$
- Each term must be 0
  $$4a_1(1+r)\,(5+r) - 2a_0(4r^2+22r+27) = 0$$
- Each term in the series must be 0, giving the recursion relation
  $$4a_{k+1}(k+1+r)\,(k+5+r) - 2a_k(4k^2+8kr+4r^2+22k+22r+27) + a_{k-1}(2k+5+2r)^2 = 0$$
- Shift index using $k- >k+1$
  $$4a_{k+2}(k+2+r)\,(k+6+r) - 2a_{k+1}\left(4(k+1)^2+8(k+1)\,r+4r^2+22k+49+22r\right) + a_k(2k+2r$$
- Recursion relation that defines series solution to ODE
  $$a_{k+2} = -\frac{4k^2a_k-8k^2a_{k+1}+8kra_k-16kra_{k+1}+4r^2a_k-8r^2a_{k+1}+28ka_k-60ka_{k+1}+28ra_k-60ra_{k+1}+49a_k-106a_{k+1}}{4(k+2+r)(k+6+r)}$$
- Recursion relation for $r = -4$
  $$a_{k+2} = -\frac{4k^2a_k-8k^2a_{k+1}-4ka_k+4ka_{k+1}+a_k+6a_{k+1}}{4(k-2)(k+2)}$$
- Series not valid for $r = -4$, division by 0 in the recursion relation at $k = 2$
  $$a_{k+2} = -\frac{4k^2a_k-8k^2a_{k+1}-4ka_k+4ka_{k+1}+a_k+6a_{k+1}}{4(k-2)(k+2)}$$
- Recursion relation for $r = 0$
  $$a_{k+2} = -\frac{4k^2a_k-8k^2a_{k+1}+28ka_k-60ka_{k+1}+49a_k-106a_{k+1}}{4(k+2)(k+6)}$$
- Solution for $r = 0$
  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{4k^2a_k-8k^2a_{k+1}+28ka_k-60ka_{k+1}+49a_k-106a_{k+1}}{4(k+2)(k+6)}, 20a_1-54a_0 = 0\right]$$
- Revert the change of variables $u = x+1$
  $$\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+2} = -\frac{4k^2a_k-8k^2a_{k+1}+28ka_k-60ka_{k+1}+49a_k-106a_{k+1}}{4(k+2)(k+6)}, 20a_1-54a_0 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.042 (sec)
Leaf size : 40

```
dsolve(4*x^2*(x+1)*diff(diff(y(x),x),x)+4*x*(3+8*x)*diff(y(x),x)-(5-49*x)*y(x) = 0,y(x
```

$$y = \frac{c_1 x^3 + 6c_2 \ln(x) x^3 - 18c_2 x^2 - 9c_2 x - 2c_2}{(x+1)^4 x^{5/2}}$$

Mathematica DSolve solution

Solving time : 0.194 (sec)
Leaf size : 104

```
DSolve[{4*x^2*(1+x)*D[y[x],{x,2}]+4*x*(3+8*x)*D[y[x],x]-(5-49*x)*y[x]==0,{}},y[x],x,IncludeS
```

$$y(x) \to \exp\left(\int_1^x \frac{K[1]+4}{2K[1]^2+2K[1]}dK[1] - \frac{1}{2}\int_1^x \frac{8K[2]+3}{K[2]^2+K[2]}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{K[1]+4}{2K[1]^2+2K[1]}dK[1]\right)dK[3] + c_1\right)$$

2.1.592 Problem 608

Internal problem ID [9764]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 608
Date solved : `Monday, January 27, 2025 at 06:14:01 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+x)\,y''-x(3+10x)\,y'+30xy=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.231 (sec)

Writing the ode as

$$x^2(1+x)\,y''+\left(-10x^2-3x\right)y'+30xy=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2(1+x)\\B&=-10x^2-3x\\C&=30x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-48x+15}{4\left(x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=-48x+15\\t&=4\left(x^2+x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-48x+15}{4\left(x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1129: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+x)^2$. There is a pole at $x=0$ of order 2. There is a pole at $x=-1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 3 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 3 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{63}{4(1+x)^2} + \frac{39}{2(1+x)} - \frac{39}{2x} + \frac{15}{4x^2}$$

For the pole at $x=-1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{63}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{7}{2} \end{aligned}$$

For the pole at $x=0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is $3 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-48x + 15}{4\left(x^2 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-1	2	0	$\frac{9}{2}$	$-\frac{7}{2}$
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
3	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= 0 - (-1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{7}{2(1+x)} + \frac{5}{2x} + (0) \\
&= -\frac{7}{2(1+x)} + \frac{5}{2x} \\
&= -\frac{2x - 5}{2x(1+x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{7}{2(1+x)}+\frac{5}{2x}\right)(1)+\left(\left(\frac{7}{2(1+x)^2}-\frac{5}{2x^2}\right)+\left(-\frac{7}{2(1+x)}+\frac{5}{2x}\right)^2-\left(\frac{-48x+15}{4(x^2+x)^2}\right)\right)=0$$

$$\frac{5+2a_0}{x(1+x)}=0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0=-\frac{5}{2}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x)=x-\frac{5}{2}$$

Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\,dx}\\
&= \left(x-\frac{5}{2}\right)\mathrm{e}^{\int\left(-\frac{7}{2(1+x)}+\frac{5}{2x}\right)dx}\\
&= \left(x-\frac{5}{2}\right)\mathrm{e}^{\frac{5\ln(x)}{2}-\frac{7\ln(1+x)}{2}}\\
&= \frac{\left(x-\frac{5}{2}\right)x^{5/2}}{(1+x)^{7/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\
&= z_1 e^{-\int \frac{1}{2}\frac{-10x^2-3x}{x^2(1+x)}\,dx}\\
&= z_1 e^{\frac{3\ln(x)}{2}+\frac{7\ln(1+x)}{2}}\\
&= z_1\left(x^{3/2}(1+x)^{7/2}\right)
\end{aligned}$$

Which simplifies to

$$y_1=x^5-\frac{5}{2}x^4$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int\frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1\int\frac{e^{\int -\frac{-10x^2-3x}{x^2(1+x)}\,dx}}{(y_1)^2}\,dx\\
&= y_1\int\frac{e^{3\ln(x)+7\ln(1+x)}}{(y_1)^2}\,dx\\
&= y_1\left(x-\frac{1}{25x^4}-\frac{52}{125x^3}-\frac{1354}{625x^2}-\frac{27708}{3125x}+12\ln(x)-\frac{823543}{6250(2x-5)}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^5 - \frac{5}{2}x^4\right) \\
&\quad + c_2\left(x^5 - \frac{5}{2}x^4\left(x - \frac{1}{25x^4} - \frac{52}{125x^3} - \frac{1354}{625x^2} - \frac{27708}{3125x} + 12\ln(x) - \frac{823543}{6250(2x-5)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right) - x(10x+3)\left(\frac{d}{dx}y(x)\right) + 30xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{30y(x)}{x(x+1)} + \frac{(10x+3)\left(\frac{d}{dx}y(x)\right)}{x(x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) - \frac{(10x+3)\left(\frac{d}{dx}y(x)\right)}{x(x+1)} + \frac{30y(x)}{x(x+1)} = 0$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{10x+3}{x(x+1)}, P_3(x) = \frac{30}{x(x+1)}\right]$
  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1} = -7$
  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1} = 0$
  - $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $x(x+1)\left(\frac{d^2}{dx^2}y(x)\right) + (-10x-3)\left(\frac{d}{dx}y(x)\right) + 30y(x) = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(u^2-u)\left(\frac{d^2}{du^2}y(u)\right) + (-10u+7)\left(\frac{d}{du}y(u)\right) + 30y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-a_0r(-8+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)(k-7+r) + a_k(k+r-5)(k+r-6)\right) u^{k+r}\right) =$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-8+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 8\}$
- Each term in the series must be 0, giving the recursion relation

  $-a_{k+1}(k+1+r)(k-7+r) + a_k(k+r-5)(k+r-6) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r-5)(k+r-6)}{(k+1+r)(k-7+r)}$
- Recursion relation for $r = 0$; series terminates at $k = 5$

  $a_{k+1} = \frac{a_k(k-5)(k-6)}{(k+1)(k-7)}$
- Apply recursion relation for $k = 0$

  $a_1 = -\frac{30a_0}{7}$
- Apply recursion relation for $k = 1$

  $a_2 = -\frac{5a_1}{3}$
- Express in terms of a_0

  $a_2 = \frac{50a_0}{7}$
- Apply recursion relation for $k = 2$

  $a_3 = -\frac{4a_2}{5}$
- Express in terms of a_0

  $a_3 = -\frac{40a_0}{7}$
- Apply recursion relation for $k = 3$

  $a_4 = -\frac{3a_3}{8}$
- Express in terms of a_0

  $a_4 = \frac{15a_0}{7}$
- Apply recursion relation for $k = 4$

  $a_5 = -\frac{2a_4}{15}$
- Express in terms of a_0

  $a_5 = -\frac{2a_0}{7}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - \frac{30}{7}u + \frac{50}{7}u^2 - \frac{40}{7}u^3 + \frac{15}{7}u^4 - \frac{2}{7}u^5\right)$
- Revert the change of variables $u = x + 1$

  $\left[y(x) = a_0\left(\frac{5}{7}x^4 - \frac{2}{7}x^5\right)\right]$
- Recursion relation for $r = 8$

  $a_{k+1} = \frac{a_k(k+3)(k+2)}{(k+9)(k+1)}$
- Solution for $r = 8$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+8}, a_{k+1} = \frac{a_k(k+3)(k+2)}{(k+9)(k+1)}\right]$
- Revert the change of variables $u = x + 1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^{k+8}, a_{k+1} = \frac{a_k(k+3)(k+2)}{(k+9)(k+1)}\right]$

- Combine solutions and rename parameters

$$\left[y(x) = a_0\left(\frac{5}{7}x^4 - \frac{2}{7}x^5\right) + \left(\sum_{k=0}^{\infty} b_k(x+1)^{k+8}\right), b_{k+1} = \frac{b_k(k+3)(k+2)}{(k+9)(k+1)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 65

```
dsolve(x^2*(x+1)*diff(diff(y(x),x),x)-x*(10*x+3)*diff(y(x),x)+30*x*y(x) = 0,y(x),singsol
```

$$y = 3c_2x^4\left(x-\frac{5}{2}\right)\ln(x) + \frac{c_2x^6}{4} + \frac{(16c_1 - 5c_2)x^5}{8} + \frac{(-80c_1 - 299c_2)x^4}{16} + 5c_2x^3 + \frac{5c_2x^2}{4} + \frac{c_2x}{4} + \frac{c_2}{40}$$

Mathematica DSolve solution

Solving time : 0.485 (sec)
Leaf size : 125

```
DSolve[{x^2*(1+x)*D[y[x],{x,2}]-x*(3+10*x)*D[y[x],x]+30*x*y[x]==0,{}},y[x],x,IncludeSingularSol
```

$$y(x) \to \frac{1}{2}(2x-5)\exp\left(\int_1^x \frac{5-2K[1]}{2K[1]^2+2K[1]}dK[1] - \frac{1}{2}\int_1^x\left(-\frac{7}{K[2]+1} - \frac{3}{K[2]}\right)dK[2]\right)\left(c_2\int_1^x \frac{4\exp\left(-2\int_1^{K[3]}\frac{5-2K[1]}{2K[1]^2+2K[1]}dK[1]\right)}{(5-2K[3])^2}dK[3] + c_1\right)$$

2.1.593 Problem 609

Internal problem ID [9765]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 609
Date solved : `Monday, January 27, 2025 at 06:14:01 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x(1+x)\,y' - 3(3+x)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.290 (sec)

Writing the ode as

$$x^2y'' + (x^2+x)\,y' + (-3x-9)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2+x \\ C &= -3x-9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2+14x+35}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2+14x+35 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2+14x+35}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1131: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{7}{2x} + \frac{35}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{5}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{7}{2x} - \frac{7}{2x^2} + \frac{49}{2x^3} - \frac{735}{4x^4} + \frac{5831}{4x^5} - \frac{48363}{4x^6} + \frac{415373}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 + 14x + 35}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{14x + 35}{4x^2}\right) \\ &= \frac{1}{4} + \frac{14x + 35}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 14. Dividing this by leading coefficient in t which is 4 gives $\frac{7}{2}$. Now b can be found.

$$\begin{aligned}b &= \left(\frac{7}{2}\right) - (0) \\ &= \frac{7}{2}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{7}{2}}{\frac{1}{2}} - 0\right) &&= \frac{7}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{7}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{7}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 14x + 35}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{7}{2}$	$-\frac{7}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+\right) \\
&= \frac{7}{2} - \left(\frac{7}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{7}{2x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} + \frac{7}{2x} \\
&= \frac{x+7}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2} + \frac{7}{2x}\right)(0) + \left(\left(-\frac{7}{2x^2}\right) + \left(\frac{1}{2} + \frac{7}{2x}\right)^2 - \left(\frac{x^2+14x+35}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(\frac{1}{2} + \frac{7}{2x}\right) dx} \\
&= x^{7/2} e^{\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^2+x}{x^2}\,dx} \\ &= z_1 e^{-\frac{x}{2}-\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{\mathrm{e}^{-\frac{x}{2}}}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^3$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-x-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(-\frac{\mathrm{e}^{-x}}{6x^6}+\frac{\mathrm{e}^{-x}}{30x^5}-\frac{\mathrm{e}^{-x}}{120x^4}+\frac{\mathrm{e}^{-x}}{360x^3}-\frac{\mathrm{e}^{-x}}{720x^2}+\frac{\mathrm{e}^{-x}}{720x}-\frac{\mathrm{Ei}_1(x)}{720}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(x^3\right) + c_2\left(x^3\left(-\frac{\mathrm{e}^{-x}}{6x^6}+\frac{\mathrm{e}^{-x}}{30x^5}-\frac{\mathrm{e}^{-x}}{120x^4}+\frac{\mathrm{e}^{-x}}{360x^3}-\frac{\mathrm{e}^{-x}}{720x^2}+\frac{\mathrm{e}^{-x}}{720x}-\frac{\mathrm{Ei}_1(x)}{720}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(x+1)\left(\frac{d}{dx}y(x)\right) - 3(x+3)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = \frac{3(x+3)y(x)}{x^2} - \frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x) + \frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{x} - \frac{3(x+3)y(x)}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{x+1}{x}, P_3(x) = -\frac{3(x+3)}{x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 1$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = -9$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(x+1)\left(\frac{d}{dx}y(x)\right) + (-3x-9)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ◦ Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- ◦ Shift index using $k- >k + 1 - m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ◦ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(3+r)\,(-3+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+3)\,(k+r-3) + a_{k-1}(k-4+r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(3+r)\,(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-3, 3\}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+3)\,(k+r-3) + a_{k-1}(k-4+r) = 0$

- Shift index using $k- >k + 1$

  $a_{k+1}(k+4+r)\,(k-2+r) + a_k(k+r-3) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r-3)}{(k+4+r)(k-2+r)}$

- Recursion relation for $r = -3$; series terminates at $k = 6$

  $a_{k+1} = -\frac{a_k(k-6)}{(k+1)(k-5)}$

- Series not valid for $r = -3$, division by 0 in the recursion relation at $k = 5$

  $a_{k+1} = -\frac{a_k(k-6)}{(k+1)(k-5)}$

- Recursion relation for $r = 3$

$$a_{k+1} = -\frac{a_k k}{(k+7)(k+1)}$$

- Solution for $r = 3$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = -\frac{a_k k}{(k+7)(k+1)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 50

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(x+1)*diff(y(x),x)-3*(x+3)*y(x) = 0,y(x),singsol=all
```

$$y = \frac{c_2(x^5 - x^4 + 2x^3 - 6x^2 + 24x - 120)\,e^{-x} + x^6(-\operatorname{Ei}_1(x)\,c_2 + c_1)}{x^3}$$

Mathematica DSolve solution

Solving time : 0.225 (sec)
Leaf size : 40

```
DSolve[{x^2*D[y[x],{x,2}]+x*(1+x)*D[y[x],x]-3*(3+x)*y[x]==0,{}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to e^{7/2} x^3 \left(c_2 \int_1^x \frac{e^{-K[1]-7}}{K[1]^7} dK[1] + c_1\right)$$

2.1.594 Problem 610

Internal problem ID [9766]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 610
Date solved : Monday, January 27, 2025 at 06:14:02 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$x^2(1+2x)\,y''+x(9+13x)\,y'+(7+5x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.234 (sec)

Writing the ode as

$$\left(2x^3+x^2\right)y''+\left(13x^2+9x\right)y'+(7+5x)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^3+x^2 \\ B &= 13x^2+9x \\ C &= 7+5x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{77x^2+86x+35}{4\left(2x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 77x^2+86x+35 \\ t &= 4\left(2x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{77x^2+86x+35}{4\left(2x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1133: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{45}{16\left(x + \frac{1}{2}\right)^2} + \frac{27}{2\left(x + \frac{1}{2}\right)} - \frac{27}{2x} + \frac{35}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{5}{2} \end{aligned}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{45}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{9}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{5}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{77x^2 + 86x + 35}{4\left(2x^2 + x\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{77}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{4} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{7}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{77x^2 + 86x + 35}{4\left(2x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$
$-\frac{1}{2}$	2	0	$\frac{9}{4}$	$-\frac{5}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{11}{4}$	$-\frac{7}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{7}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\ &= -\frac{7}{4} - \left(-\frac{15}{4}\right) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{5}{2x} - \frac{5}{4\left(x+\frac{1}{2}\right)} + (-)(0) \\ &= -\frac{5}{2x} - \frac{5}{4\left(x+\frac{1}{2}\right)} \\ &= \frac{-5-15x}{4x^2+2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(-\frac{5}{2x} - \frac{5}{4\left(x+\frac{1}{2}\right)}\right)(2x + a_1) + \left(\left(\frac{5}{2x^2} + \frac{5}{4\left(x+\frac{1}{2}\right)^2}\right) + \left(-\frac{5}{2x} - \frac{5}{4\left(x+\frac{1}{2}\right)}\right)^2 - \left(\frac{77x^2 + 86x}{4\,(2x^2 +}\right.\right.$$
$$\frac{(11a_1 - 8)\,x + 26a_0}{2x^2 + x}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{20}{143}, a_1 = \frac{8}{11}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + \frac{8}{11}x + \frac{20}{143}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \left(x^2 + \frac{8}{11}x + \frac{20}{143}\right) e^{\int \left(-\frac{5}{2x} - \frac{5}{4\left(x+\frac{1}{2}\right)}\right) dx} \\ &= \left(x^2 + \frac{8}{11}x + \frac{20}{143}\right) e^{-\frac{5\ln(x)}{2} - \frac{5\ln(1+2x)}{4}} \\ &= \frac{x^2 + \frac{8}{11}x + \frac{20}{143}}{x^{5/2}\,(1+2x)^{5/4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{13x^2+9x}{2x^3+x^2}\, dx} \\ &= z_1 e^{-\frac{9\ln(x)}{2} + \frac{5\ln(1+2x)}{4}} \\ &= z_1 \left(\frac{(1+2x)^{5/4}}{x^{9/2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2 + \frac{8}{11}x + \frac{20}{143}}{x^7}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{13x^2+9x}{2x^3+x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-9\ln(x)+\frac{5\ln(1+2x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{143(1+2x)\left(35x^3-45x^2+36x-20\right)x^9 \mathrm{e}^{-9\ln(x)+\frac{5\ln(1+2x)}{2}}}{315\left(143x^2+104x+20\right)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^2+\frac{8}{11}x+\frac{20}{143}}{x^7}\right) \\
&\quad + c_2 \left(\frac{x^2+\frac{8}{11}x+\frac{20}{143}}{x^7}\left(\frac{143(1+2x)\left(35x^3-45x^2+36x-20\right)x^9 \mathrm{e}^{-9\ln(x)+\frac{5\ln(1+2x)}{2}}}{315\left(143x^2+104x+20\right)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(2x+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(9+13x)\left(\frac{d}{dx}y(x)\right)+(7+5x)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(7+5x)y(x)}{x^2(2x+1)}-\frac{(9+13x)\left(\frac{d}{dx}y(x)\right)}{x(2x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{(9+13x)\left(\frac{d}{dx}y(x)\right)}{x(2x+1)}+\frac{(7+5x)y(x)}{x^2(2x+1)}=0$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x)=\frac{9+13x}{x(2x+1)}, P_3(x)=\frac{7+5x}{x^2(2x+1)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x=0$

    $\left.(x \cdot P_2(x))\right|_{x=0}=9$

  - $x^2 \cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0}=7$

  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(2x+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(9+13x)\left(\frac{d}{dx}y(x)\right)+(7+5x)\,y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m=0..1$

    $x^m \cdot y(x)=\sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m \cdot y(x)=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..3$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$
  - Shift index using $k->k+2-m$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(7+r)\,(1+r)\, x^r+\left(\sum_{k=1}^{\infty}\left(a_k(k+r+7)\,(k+r+1)+a_{k-1}(k+4+r)\,(2k-1+2r)\right) x^{k+r}\right)=$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(7+r)\,(1+r)=0$
- Values of r that satisfy the indicial equation

  $r\in\{-7,-1\}$
- Each term in the series must be 0, giving the recursion relation

  $2(k+4+r)\left(k+r-\frac{1}{2}\right)a_{k-1}+a_k(k+r+7)\,(k+r+1)=0$
- Shift index using $k->k+1$

  $2(k+r+5)\left(k+\frac{1}{2}+r\right)a_k+a_{k+1}(k+8+r)\,(k+2+r)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{(k+r+5)(2k+2r+1)a_k}{(k+8+r)(k+2+r)}$
- Recursion relation for $r=-7$; series terminates at $k=2$

  $a_{k+1}=-\frac{(k-2)(2k-13)a_k}{(k+1)(k-5)}$
- Apply recursion relation for $k=0$

  $a_1=\frac{26a_0}{5}$
- Apply recursion relation for $k=1$

  $a_2=\frac{11a_1}{8}$
- Express in terms of a_0

  $a_2=\frac{143a_0}{20}$
- Terminating series solution of the ODE for $r=-7$. Use reduction of order to find the second

  $y(x)=a_0\cdot\left(\frac{143}{20}x^2+\frac{26}{5}x+1\right)$
- Recursion relation for $r=-1$

  $a_{k+1}=-\frac{(k+4)(2k-1)a_k}{(k+7)(k+1)}$
- Solution for $r=-1$

$$\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1}=-\frac{(k+4)(2k-1)a_k}{(k+7)(k+1)}\right]$$

- Combine solutions and rename parameters

$$\left[y(x)=a_0\cdot\left(\frac{143}{20}x^2+\frac{26}{5}x+1\right)+\left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), b_{k+1}=-\frac{(4+k)(2k-1)b_k}{(k+7)(k+1)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.020 (sec)
Leaf size : 50

```
dsolve(x^2*(2*x+1)*diff(diff(y(x),x),x)+x*(13*x+9)*diff(y(x),x)+(7+5*x)*y(x) = 0,y(x),si
```

$$y=\frac{280c_2\left(x+\frac{1}{2}\right)^3\left(x^3-\frac{9}{7}x^2+\frac{36}{35}x-\frac{4}{7}\right)\sqrt{2x+1}+143c_1x^2+104c_1x+20c_1}{x^7}$$

Mathematica DSolve solution

Solving time : 0.54 (sec)
Leaf size : 141

```
DSolve[{x^2*(1+2*x)*D[y[x],{x,2}]+x*(9+13*x)*D[y[x],x]+(7+5*x)*y[x]==0,{}},y[x],x,IncludeSingul
```

$$y(x) \to \frac{1}{143}\left(143x^2+104x+20\right)\exp\left(\int_1^x -\frac{15K[1]+5}{4K[1]^2+2K[1]}dK[1]-\frac{1}{2}\int_1^x \frac{13K[2]+9}{2K[2]^2+K[2]}dK[2]\right)\left(c_2\int_1^x \frac{20449\exp\left(-2\int_1^{K[3]}-\frac{15K[1]+5}{4K[1]^2+2K[1]}dK[1]\right)}{\left(143K[3]^2+104K[3]+20\right)^2}dK[3]+c_1\right)$$

2.1.595 Problem 611

Internal problem ID [9767]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 611
Date solved : `Monday, January 27, 2025 at 06:14:03 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2(1+2x)\,y''-2x(4-x)\,y'-(7+5x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.196 (sec)

Writing the ode as

$$\left(8x^3+4x^2\right)y''+\left(2x^2-8x\right)y'+(-5x-7)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=8x^3+4x^2\\B&=2x^2-8x\\C&=-5x-7\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{33x^2+132x+60}{16\left(2x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=33x^2+132x+60\\t&=16\left(2x^2+x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{33x^2+132x+60}{16\left(2x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1135: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(2x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{64\left(x + \frac{1}{2}\right)^2} + \frac{27}{4\left(x + \frac{1}{2}\right)} - \frac{27}{4x} + \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{2} \end{aligned}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{9}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{9}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{8} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{33x^2 + 132x + 60}{16\left(2x^2 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{33}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{8} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{8}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{33x^2 + 132x + 60}{16\left(2x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
$-\frac{1}{2}$	2	0	$\frac{9}{8}$	$-\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{11}{8}$	$-\frac{3}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{3}{8}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= -\frac{3}{8} - \left(-\frac{3}{8}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{2x} + \frac{9}{8\left(x+\frac{1}{2}\right)} + (-)\,(0) \\
&= -\frac{3}{2x} + \frac{9}{8\left(x+\frac{1}{2}\right)} \\
&= -\frac{3(x+2)}{4x\,(1+2x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{2x} + \frac{9}{8\left(x+\frac{1}{2}\right)}\right)(0) + \left(\left(\frac{3}{2x^2} - \frac{9}{8\left(x+\frac{1}{2}\right)^2}\right) + \left(-\frac{3}{2x} + \frac{9}{8\left(x+\frac{1}{2}\right)}\right)^2 - \left(\frac{33x^2+132x+60}{16\left(2x^2+x\right)^2}\right)\right)$$

$$0 =$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(-\frac{3}{2x} + \frac{9}{8\left(x+\frac{1}{2}\right)}\right) dx} \\
&= \frac{(1+2x)^{9/8}}{x^{3/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^2-8x}{8x^3+4x^2}\, dx} \\
&= z_1 e^{-\frac{9\ln(1+2x)}{8} + \ln(x)} \\
&= z_1 \left(\frac{x}{(1+2x)^{9/8}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2-8x}{8x^3+4x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{9\ln(1+2x)}{4} + 2\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{2(1+2x)\left(5x^3 - 10x^2 - 40x - 16\right) e^{-\frac{9\ln(1+2x)}{4} + 2\ln(x)}}{35x^2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{\sqrt{x}}\right) + c_2 \left(\frac{1}{\sqrt{x}} \left(\frac{2(1+2x)\left(5x^3 - 10x^2 - 40x - 16\right) e^{-\frac{9\ln(1+2x)}{4} + 2\ln(x)}}{35x^2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2(2x+1)\left(\frac{d^2}{dx^2}y(x)\right)-2x(4-x)\left(\frac{d}{dx}y(x)\right)-(7+5x)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=\frac{(7+5x)y(x)}{4x^2(2x+1)}-\frac{(-4+x)\left(\frac{d}{dx}y(x)\right)}{2(2x+1)x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)+\frac{(-4+x)\left(\frac{d}{dx}y(x)\right)}{2(2x+1)x}-\frac{(7+5x)y(x)}{4x^2(2x+1)}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

$\left[P_2(x)=\frac{-4+x}{2(2x+1)x}, P_3(x)=-\frac{7+5x}{4x^2(2x+1)}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

$\left(x\cdot P_2(x)\right)\bigg|_{x=0}=-2$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=-\frac{7}{4}$

- ○ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$4x^2(2x+1)\left(\frac{d^2}{dx^2}y(x)\right)+2x(-4+x)\left(\frac{d}{dx}y(x)\right)+(-5x-7)\,y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

$x^m\cdot y(x)=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

- ○ Shift index using $k->k-m$

$x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ○ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ○ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..3$

$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,x^{k+r-2+m}$

- ○ Shift index using $k->k+2-m$

$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)\,x^{k+r}$

Rewrite ODE with series expansions

$$a_0(1+2r)(-7+2r)x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)(2k+2r-7) + a_{k-1}(2k-1+2r)(4k-9+4r)\right) x$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(1+2r)(-7+2r)=0$
- Values of r that satisfy the indicial equation
  $r \in \left\{-\frac{1}{2}, \frac{7}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $8\left(k-\frac{9}{4}+r\right)\left(k+r-\frac{1}{2}\right)a_{k-1} + 4\left(k+r-\frac{7}{2}\right)a_k\left(k+r+\frac{1}{2}\right)=0$
- Shift index using $k->k+1$
  $8\left(k-\frac{5}{4}+r\right)\left(k+r+\frac{1}{2}\right)a_k + 4\left(k-\frac{5}{2}+r\right)a_{k+1}\left(k+\frac{3}{2}+r\right)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = -\frac{(4k+4r-5)(2k+2r+1)a_k}{(2k-5+2r)(2k+3+2r)}$
- Recursion relation for $r=-\frac{1}{2}$
  $a_{k+1} = -\frac{2(4k-7)ka_k}{(2k-6)(2k+2)}$
- Series not valid for $r=-\frac{1}{2}$, division by 0 in the recursion relation at $k=3$
  $a_{k+1} = -\frac{2(4k-7)ka_k}{(2k-6)(2k+2)}$
- Recursion relation for $r=\frac{7}{2}$
  $a_{k+1} = -\frac{(4k+9)(2k+8)a_k}{(2k+2)(2k+10)}$
- Solution for $r=\frac{7}{2}$
  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+\frac{7}{2}}, a_{k+1} = -\frac{(4k+9)(2k+8)a_k}{(2k+2)(2k+10)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.026 (sec)
Leaf size : 34

```
dsolve(4*x^2*(2*x+1)*diff(diff(y(x),x),x)-2*x*(-x+4)*diff(y(x),x)-(7+5*x)*y(x) = 0,y(x),
```

$$y = \frac{c_1 + \frac{c_2(5x^3-10x^2-40x-16)}{(2x+1)^{5/4}}}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.262 (sec)
Leaf size : 111

```
DSolve[{4*x^2*(1+2*x)*D[y[x],{x,2}]-2*x*(4-x)*D[y[x],x]-(7+5*x)*y[x]==0,{}},y[x],x,IncludeSi
```

$$y(x) \to \exp\left(\int_1^x -\frac{3K[1]+6}{8K[1]^2+4K[1]}dK[1]-\frac{1}{2}\int_1^x\left(\frac{9}{4K[2]+2}-\frac{2}{K[2]}\right)dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]}-\frac{3K[1]+6}{8K[1]^2+4K[1]}dK[1]\right)dK[3]+c_1\right)$$

2.1.596 Problem 612

Internal problem ID [9768]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 612
Date solved : Monday, January 27, 2025 at 06:14:03 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$3x^2(3+x)\,y''-x(15+x)\,y'-20y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.224 (sec)

Writing the ode as

$$\left(3x^3+9x^2\right)y''+\left(-x^2-15x\right)y'-20y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^3+9x^2 \\ B &= -x^2-15x \\ C &= -20 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{7x^2+450x+1215}{36\left(x^2+3x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 7x^2+450x+1215 \\ t &= 36\left(x^2+3x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{7x^2+450x+1215}{36\left(x^2+3x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1137: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^2+3x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -3$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{9\,(3+x)^2} - \frac{10}{9x} + \frac{10}{9\,(3+x)} + \frac{15}{4x^2}$$

For the pole at $x = -3$ let b be the coefficient of $\frac{1}{(3+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{2}{9}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{2}{3} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{3} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{7x^2 + 450x + 1215}{36\left(x^2 + 3x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{7}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{7x^2 + 450x + 1215}{36\left(x^2 + 3x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-3	2	0	$\frac{2}{3}$	$\frac{1}{3}$
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{6}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= -\frac{1}{6} - \left(-\frac{7}{6}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{9+3x} - \frac{3}{2x} + (-)(0) \\
&= \frac{1}{9+3x} - \frac{3}{2x} \\
&= -\frac{7x+27}{6x(3+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{9+3x} - \frac{3}{2x}\right)(1) + \left(\left(-\frac{1}{3\left(3+x\right)^2} + \frac{3}{2x^2}\right) + \left(\frac{1}{9+3x} - \frac{3}{2x}\right)^2 - \left(\frac{7x^2+450x+1215}{36\left(x^2+3x\right)^2}\right)\right) = \frac{-27+7a_0}{3x\left(3+x\right)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{27}{7}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{27}{7}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{27}{7}\right) e^{\int \left(\frac{1}{9+3x} - \frac{3}{2x}\right) dx} \\
&= \left(x + \frac{27}{7}\right) e^{\frac{\ln(3+x)}{3} - \frac{3\ln(x)}{2}} \\
&= \frac{\left(x + \frac{27}{7}\right)\left(3+x\right)^{1/3}}{x^{3/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2-15x}{3x^3+9x^2} \, dx} \\
&= z_1 e^{-\frac{2\ln(3+x)}{3} + \frac{5\ln(x)}{6}} \\
&= z_1 \left(\frac{x^{5/6}}{\left(3+x\right)^{2/3}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{7x+27}{7\left(3+x\right)^{1/3} x^{2/3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-15x}{3x^3+9x^2} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{4\ln(3+x)}{3} + \frac{5\ln(x)}{3}}}{\left(y_1\right)^2} \, dx \\
&= y_1 \left(\frac{21\left(3+x\right)^{5/3}\left(x^2-36x-243\right) e^{-\frac{4\ln(3+x)}{3} + \frac{5\ln(x)}{3}}}{4\left(7x+27\right) x^{5/3}}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{7x+27}{7\left(3+x\right)^{1/3} x^{2/3}}\right) \\
&\quad + c_2\left(\frac{7x+27}{7\left(3+x\right)^{1/3} x^{2/3}}\left(\frac{21(3+x)^{5/3}\left(x^2-36x-243\right)\mathrm{e}^{-\frac{4\ln(3+x)}{3}+\frac{5\ln(x)}{3}}}{4\left(7x+27\right)x^{5/3}}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$3x^2(x+3)\left(\frac{d^2}{dx^2}y(x)\right) - x(15+x)\left(\frac{d}{dx}y(x)\right) - 20y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{20y(x)}{3x^2(x+3)} + \frac{(15+x)\left(\frac{d}{dx}y(x)\right)}{3x(x+3)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) - \frac{(15+x)\left(\frac{d}{dx}y(x)\right)}{3x(x+3)} - \frac{20y(x)}{3x^2(x+3)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{15+x}{3x(x+3)}, P_3(x) = -\frac{20}{3x^2(x+3)}\right]$

  - $(x+3)\cdot P_2(x)$ is analytic at $x=-3$

    $\left((x+3)\cdot P_2(x)\right)\Big|_{x=-3} = \frac{4}{3}$

  - $(x+3)^2\cdot P_3(x)$ is analytic at $x=-3$

    $\left((x+3)^2\cdot P_3(x)\right)\Big|_{x=-3} = 0$

  - $x=-3$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -3$

- Multiply by denominators

  $3x^2(x+3)\left(\frac{d^2}{dx^2}y(x)\right) - x(15+x)\left(\frac{d}{dx}y(x)\right) - 20y(x) = 0$

- Change variables using $x = u-3$ so that the regular singular point is at $u=0$

  $\left(3u^3-18u^2+27u\right)\left(\frac{d^2}{du^2}y(u)\right) + \left(-u^2-9u+36\right)\left(\frac{d}{du}y(u)\right) - 20y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- ◦ Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- ◦ Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$9a_0r(1+3r)\, u^{-1+r} + (9a_1(1+r)\,(4+3r) - a_0(18r^2-9r+20))\, u^r + \left(\sum_{k=1}^{\infty} (9a_{k+1}(k+1+r)\,(3k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$9r(1+3r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{1}{3}\right\}$$

- Each term must be 0

$$9a_1(1+r)\,(4+3r) - a_0(18r^2-9r+20) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$3(-6a_k + a_{k-1} + 9a_{k+1})\, k^2 + (6(-6a_k + a_{k-1} + 9a_{k+1})\, r + 9a_k - 10a_{k-1} + 63a_{k+1})\, k + 3(-6a_k +$$

- Shift index using $k- >k+1$

$$3(-6a_{k+1} + a_k + 9a_{k+2})\,(k+1)^2 + (6(-6a_{k+1} + a_k + 9a_{k+2})\, r + 9a_{k+1} - 10a_k + 63a_{k+2})\,(k+1)$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{3k^2a_k - 18k^2a_{k+1} + 6kra_k - 36kra_{k+1} + 3r^2a_k - 18r^2a_{k+1} - 4ka_k - 27ka_{k+1} - 4ra_k - 27ra_{k+1} - 29a_{k+1}}{9(3k^2+6kr+3r^2+13k+13r+14)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{3k^2a_k - 18k^2a_{k+1} - 4ka_k - 27ka_{k+1} - 29a_{k+1}}{9(3k^2+13k+14)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{3k^2a_k - 18k^2a_{k+1} - 4ka_k - 27ka_{k+1} - 29a_{k+1}}{9(3k^2+13k+14)}, 36a_1 - 20a_0 = 0\right]$$

- Revert the change of variables $u = x+3$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k(x+3)^k, a_{k+2} = -\frac{3k^2a_k - 18k^2a_{k+1} - 4ka_k - 27ka_{k+1} - 29a_{k+1}}{9(3k^2+13k+14)}, 36a_1 - 20a_0 = 0\right]$$

- Recursion relation for $r = -\frac{1}{3}$

$$a_{k+2} = -\frac{3k^2a_k - 18k^2a_{k+1} - 6ka_k - 15ka_{k+1} + \frac{5}{3}a_k - 22a_{k+1}}{9(3k^2+11k+10)}$$

- Solution for $r = -\frac{1}{3}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{1}{3}}, a_{k+2} = -\frac{3k^2a_k - 18k^2a_{k+1} - 6ka_k - 15ka_{k+1} + \frac{5}{3}a_k - 22a_{k+1}}{9(3k^2+11k+10)}, 18a_1 - 25a_0 = 0\right]$$

- Revert the change of variables $u = x+3$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k(x+3)^{k-\frac{1}{3}}, a_{k+2} = -\frac{3k^2a_k - 18k^2a_{k+1} - 6ka_k - 15ka_{k+1} + \frac{5}{3}a_k - 22a_{k+1}}{9(3k^2+11k+10)}, 18a_1 - 25a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k(x+3)^k\right) + \left(\sum_{k=0}^{\infty} b_k(x+3)^{k-\frac{1}{3}}\right), a_{k+2} = -\frac{3k^2a_k - 18k^2a_{k+1} - 4ka_k - 27ka_{k+1} - 29a_{k+1}}{9(3k^2+13k+14)}, 36a\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.037 (sec)
Leaf size : 31

```
dsolve(3*x^2*(x+3)*diff(diff(y(x),x),x)-x*(15+x)*diff(y(x),x)-20*y(x) = 0,y(x),singsol=a
```

$$y = \frac{c_1(x^2 - 36x - 243) + \frac{c_2(7x+27)}{(x+3)^{1/3}}}{x^{2/3}}$$

Mathematica DSolve solution

Solving time : 0.494 (sec)
Leaf size : 123

```
DSolve[{3*x^2*(3+x)*D[y[x],{x,2}]-x*(15+x)*D[y[x],x]-20*y[x]==0,{}},y[x],x,IncludeSingularSolu
```

$$y(x) \to \frac{1}{7}(7x+27)\exp\left(\int_1^x \left(\frac{1}{3K[1]+9} - \frac{3}{2K[1]}\right) dK[1] - \frac{1}{2}\int_1^x -\frac{K[2]+15}{3K[2]^2+9K[2]} dK[2]\right)\left(c_2\int_1^x \frac{49\exp\left(-2\int_1^{K[3]}\left(\frac{1}{3K[1]+9} - \frac{3}{2K[1]}\right) dK[1]\right)}{(7K[3]+27)^2} dK[3] + c_1\right)$$

2.1.597 Problem 613

Internal problem ID [9769]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 613
Date solved : `Monday, January 27, 2025 at 06:14:04 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+x)\,y''+x(1-10x)\,y'-(9-10x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.247 (sec)

Writing the ode as

$$x^2(1+x)\,y''+\left(-10x^2+x\right)y'+(10x-9)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2(1+x)\\B&=-10x^2+x\\C&=10x-9\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{80x^2-28x+35}{4\left(x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=80x^2-28x+35\\t&=4\left(x^2+x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{80x^2-28x+35}{4\left(x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1139: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{143}{4\,(1+x)^2} - \frac{49}{2x} + \frac{35}{4x^2} + \frac{49}{2\,(1+x)}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{143}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{13}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{11}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{80x^2 - 28x + 35}{4\left(x^2 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 20$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 5 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -4
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{80x^2 - 28x + 35}{4\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{13}{2}$	$-\frac{11}{2}$
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	5	-4

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 5$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= 5 - (4) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{13}{2\,(1+x)} - \frac{5}{2x} + (0) \\
&= \frac{13}{2\,(1+x)} - \frac{5}{2x} \\
&= \frac{8x - 5}{2x\,(1+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{13}{2(1+x)} - \frac{5}{2x}\right)(1) + \left(\left(-\frac{13}{2(1+x)^2} + \frac{5}{2x^2}\right) + \left(\frac{13}{2(1+x)} - \frac{5}{2x}\right)^2 - \left(\frac{80x^2 - 28x + 35}{4(x^2+x)^2}\right)\right) = 0$$

$$\frac{-5 - 8a_0}{x(1+x)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{5}{8}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - \frac{5}{8}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{5}{8}\right) e^{\int \left(\frac{13}{2(1+x)} - \frac{5}{2x}\right) dx} \\
&= \left(x - \frac{5}{8}\right) e^{\frac{13 \ln(1+x)}{2} - \frac{5 \ln(x)}{2}} \\
&= \frac{\left(x - \frac{5}{8}\right)(1+x)^{13/2}}{x^{5/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-10x^2+x}{x^2(1+x)} \, dx} \\
&= z_1 e^{\frac{11 \ln(1+x)}{2} - \frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{(1+x)^{11/2}}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(1+x)^{12}\left(x - \frac{5}{8}\right)}{x^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-10x^2+x}{x^2(1+x)} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{11 \ln(1+x) - \ln(x)}}{(y_1)^2} \, dx \\
&= y_1 \left(-\frac{8 \, e^{11 \ln(1+x) - \ln(x)} x(715x^4 + 572x^3 + 234x^2 + 52x + 5)}{6435(8x-5)(1+x)^{23}}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{(1+x)^{12}\left(x-\frac{5}{8}\right)}{x^3}\right) \\
&\quad + c_2\left(\frac{(1+x)^{12}\left(x-\frac{5}{8}\right)}{x^3}\left(-\frac{8\,\mathrm{e}^{11\ln(1+x)-\ln(x)}x(715x^4+572x^3+234x^2+52x+5)}{6435\,(8x-5)\,(1+x)^{23}}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(1-10x)\left(\frac{d}{dx}y(x)\right)-(9-10x)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(-9+10x)y(x)}{(x+1)x^2}+\frac{(-1+10x)\left(\frac{d}{dx}y(x)\right)}{x(x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)-\frac{(-1+10x)\left(\frac{d}{dx}y(x)\right)}{x(x+1)}+\frac{(-9+10x)y(x)}{(x+1)x^2}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=-\frac{-1+10x}{x(x+1)}, P_3(x)=\frac{-9+10x}{(x+1)x^2}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=-11$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=-1$

- Multiply by denominators

  $x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)-x(-1+10x)\left(\frac{d}{dx}y(x)\right)+(-9+10x)\,y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^3-2u^2+u)\left(\frac{d^2}{du^2}y(u)\right)+(-10u^2+21u-11)\left(\frac{d}{du}y(u)\right)+(-19+10u)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

    $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

  ◦ Shift index using $k->k-m$

    $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$
- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$
- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$
- Shift index using $k- >k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-12+r)\, u^{-1+r} + (a_1(1+r)\,(-11+r) - a_0(2r^2-23r+19))\, u^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(k-1\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-12+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 12\}$
- Each term must be 0

  $a_1(1+r)\,(-11+r) - a_0(2r^2-23r+19) = 0$
- Each term in the series must be 0, giving the recursion relation

  $(-2a_k + a_{k-1} + a_{k+1})\, k^2 + ((-4a_k + 2a_{k-1} + 2a_{k+1})\, r + 23a_k - 13a_{k-1} - 10a_{k+1})\, k + (-2a_k + a_{k-1}$
- Shift index using $k- >k+1$

  $(-2a_{k+1} + a_k + a_{k+2})\,(k+1)^2 + ((-4a_{k+1} + 2a_k + 2a_{k+2})\, r + 23a_{k+1} - 13a_k - 10a_{k+2})\,(k+1) + ($
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + 2kra_k - 4kra_{k+1} + r^2a_k - 2r^2a_{k+1} - 11ka_k + 19ka_{k+1} - 11ra_k + 19ra_{k+1} + 10a_k + 2a_{k+1}}{k^2+2kr+r^2-8k-8r-20}$
- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} - 11ka_k + 19ka_{k+1} + 10a_k + 2a_{k+1}}{k^2-8k-20}$
- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 10$

  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} - 11ka_k + 19ka_{k+1} + 10a_k + 2a_{k+1}}{k^2-8k-20}$
- Recursion relation for $r = 12$

  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + 13ka_k - 29ka_{k+1} + 22a_k - 58a_{k+1}}{k^2+16k+28}$
- Solution for $r = 12$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+12}, a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + 13ka_k - 29ka_{k+1} + 22a_k - 58a_{k+1}}{k^2+16k+28}, 13a_1 - 31a_0 = 0\right]$
- Revert the change of variables $u = x+1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^{k+12}, a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + 13ka_k - 29ka_{k+1} + 22a_k - 58a_{k+1}}{k^2+16k+28}, 13a_1 - 31a_0 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 82

```
dsolve(x^2*(x+1)*diff(diff(y(x),x),x)+x*(1-10*x)*diff(y(x),x)-(9-10*x)*y(x) = 0,y(x),s
```

$$y = \frac{8c_2x^{13} + 91c_2x^{12} + 468c_2x^{11} + 1430c_2x^{10} + 2860c_2x^9 + 3861c_2x^8 + 3432c_2x^7 + 1716c_2x^6 + 715c_1x^4 + 5}{x^3}$$

Mathematica DSolve solution

Solving time : 0.472 (sec)
Leaf size : 123

```
DSolve[{x^2*(1+x)*D[y[x],{x,2}]+x*(1-10*x)*D[y[x],x]-(9-10*x)*y[x]==0,{}},y[x],x,IncludeSing
```

$$y(x) \to \frac{1}{8}(8x-5)\exp\left(\int_1^x \frac{8K[1]-5}{2K[1](K[1]+1)}dK[1] - \frac{1}{2}\int_1^x \left(\frac{1}{K[2]} - \frac{11}{K[2]+1}\right)dK[2]\right)\left(c_2\int_1^x \frac{64\exp\left(-2\int_1^{K[3]} \frac{8K[1]-5}{2K[1](K[1]+1)}dK[1]\right)}{(5-8K[3])^2}dK[3] + c_1\right)$$

2.1.598 Problem 614

Internal problem ID [9770]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 614
Date solved : `Monday, January 27, 2025 at 06:14:05 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+x)\,y''+3x^2y'-(6-x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.198 (sec)

Writing the ode as

$$x^2(1+x)\,y''+3x^2y'+(x-6)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(1+x) \\ B &= 3x^2 \\ C &= x-6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-x^2+20x+24}{4\,(x^2+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2+20x+24 \\ t &= 4\left(x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-x^2+20x+24}{4\,(x^2+x)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1141: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4\,(1+x)^2} - \frac{7}{x} + \frac{7}{1+x} + \frac{6}{x^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 + 20x + 24}{4\left(x^2 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 + 20x + 24}{4\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{3}{2\,(1+x)} - \frac{2}{x} + (-)\,(0) \\
&= \frac{3}{2\,(1+x)} - \frac{2}{x} \\
&= -\frac{x+4}{2x\,(1+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{2(1+x)} - \frac{2}{x}\right)(1) + \left(\left(-\frac{3}{2(1+x)^2} + \frac{2}{x^2}\right) + \left(\frac{3}{2(1+x)} - \frac{2}{x}\right)^2 - \left(\frac{-x^2+20x+24}{4(x^2+x)^2}\right)\right) = 0$$

$$\frac{-4+a_0}{x(1+x)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 4\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 4$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (x+4)\, e^{\int \left(\frac{3}{2(1+x)} - \frac{2}{x}\right) dx} \\
&= (x+4)\, e^{-2\ln(x) + \frac{3\ln(1+x)}{2}} \\
&= \frac{(x+4)(1+x)^{3/2}}{x^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x^2}{x^2(1+x)}\, dx} \\
&= z_1 e^{-\frac{3\ln(1+x)}{2}} \\
&= z_1\left(\frac{1}{(1+x)^{3/2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x+4}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^2}{x^2(1+x)}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-3\ln(1+x)}}{(y_1)^2}\, dx \\
&= y_1\left(\frac{256}{27(x+4)} + \ln(1+x) - \frac{1}{18(1+x)^2} + \frac{14}{27(1+x)}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{x+4}{x^2}\right) + c_2\left(\frac{x+4}{x^2}\left(\frac{256}{27\,(x+4)} + \ln(1+x) - \frac{1}{18\,(1+x)^2} + \frac{14}{27\,(1+x)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right) + 3x^2\left(\frac{d}{dx}y(x)\right) - (-x+6)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(-6+x)y(x)}{(x+1)x^2} - \frac{3\left(\frac{d}{dx}y(x)\right)}{x+1}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{3\left(\frac{d}{dx}y(x)\right)}{x+1} + \frac{(-6+x)y(x)}{(x+1)x^2} = 0$
- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{3}{x+1}, P_3(x) = \frac{-6+x}{(x+1)x^2}\right]$
  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1} = 3$
  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$
  - $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$
- Multiply by denominators

  $x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right) + 3x^2\left(\frac{d}{dx}y(x)\right) + (-6+x)\,y(x) = 0$
- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(u^3 - 2u^2 + u)\left(\frac{d^2}{du^2}y(u)\right) + (3u^2 - 6u + 3)\left(\frac{d}{du}y(u)\right) + (-7+u)\,y(u) = 0$
- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

    $u^m\cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$
  - Shift index using $k->k-m$

    $u^m\cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(2+r)\, u^{-1+r} + \left(a_1(1+r)\,(3+r) - a_0(2r^2+4r+7)\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+3+r)\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(2+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{-2, 0\}$

- Each term must be 0

$a_1(1+r)\,(3+r) - a_0(2r^2+4r+7) = 0$

- Each term in the series must be 0, giving the recursion relation

$a_{k-1}(k+r)^2 + a_{k+1}(k+r+1)\,(k+3+r) - 2\left(k^2+(2r+2)\,k+r^2+2r+\frac{7}{2}\right) a_k = 0$

- Shift index using $k->k+1$

$a_k(k+r+1)^2 + a_{k+2}(k+r+2)\,(k+4+r) - 2\left((k+1)^2+(2r+2)\,(k+1)+r^2+2r+\frac{7}{2}\right) a_{k+1}$

- Recursion relation that defines series solution to ODE

$a_{k+2} = -\frac{k^2a_k-2k^2a_{k+1}+2kra_k-4kra_{k+1}+r^2a_k-2r^2a_{k+1}+2ka_k-8ka_{k+1}+2ra_k-8ra_{k+1}+a_k-13a_{k+1}}{(k+r+2)(k+4+r)}$

- Recursion relation for $r = -2$

$a_{k+2} = -\frac{k^2a_k-2k^2a_{k+1}-2ka_k+a_k-5a_{k+1}}{k(k+2)}$

- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 0$

$a_{k+2} = -\frac{k^2a_k-2k^2a_{k+1}-2ka_k+a_k-5a_{k+1}}{k(k+2)}$

- Recursion relation for $r = 0$

$a_{k+2} = -\frac{k^2a_k-2k^2a_{k+1}+2ka_k-8ka_{k+1}+a_k-13a_{k+1}}{(k+2)(k+4)}$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2a_k-2k^2a_{k+1}+2ka_k-8ka_{k+1}+a_k-13a_{k+1}}{(k+2)(k+4)}, 3a_1-7a_0 = 0\right]$$

- Revert the change of variables $u = x+1$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+2} = -\frac{k^2a_k-2k^2a_{k+1}+2ka_k-8ka_{k+1}+a_k-13a_{k+1}}{(k+2)(k+4)}, 3a_1-7a_0 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 45

```
dsolve(x^2*(x+1)*diff(diff(y(x),x),x)+3*diff(y(x),x)*x^2-(-x+6)*y(x) = 0,y(x),singsol=al
```

$$y = \frac{c_1(x+4) + \frac{c_2\left(6(x+4)(x+1)^2\ln(x+1)+60x^2+129x+68\right)}{(x+1)^2}}{x^2}$$

Mathematica DSolve solution

Solving time : 0.474 (sec)
Leaf size : 91

```
DSolve[{x^2*(1+x)*D[y[x],{x,2}]+3*x^2*D[y[x],x]-(6-x)*y[x]==0,{}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to \frac{(x+4)\exp\left(\int_1^x \left(\frac{3}{2(K[1]+1)} - \frac{2}{K[1]}\right) dK[1]\right)\left(c_2\int_1^x \frac{\exp\left(-2\int_1^{K[2]}\left(\frac{3}{2(K[1]+1)}-\frac{2}{K[1]}\right)dK[1]\right)}{(K[2]+4)^2}dK[2] + c_1\right)}{(x+1)^{3/2}}$$

2.1.599 Problem 615

Internal problem ID [9771]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 615
Date solved : `Monday, January 27, 2025 at 06:14:05 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+2x)\,y''-2x(3+14x)\,y'+(6+100x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.204 (sec)

Writing the ode as

$$\left(2x^3+x^2\right)y''+\left(-28x^2-6x\right)y'+(6+100x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x^3+x^2\\B&=-28x^2-6x\\C&=6+100x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{24x^2-16x+6}{\left(2x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=24x^2-16x+6\\t&=\left(2x^2+x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{24x^2-16x+6}{\left(2x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1143: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{40}{x} + \frac{6}{x^2} + \frac{20}{\left(x+\frac{1}{2}\right)^2} + \frac{40}{x+\frac{1}{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 20$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 5 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -4 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{24x^2 - 16x + 6}{(2x^2 + x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{24x^2 - 16x + 6}{(2x^2 + x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	3	-2
$-\frac{1}{2}$	2	0	5	-4

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	3	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 3$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 3 - (3) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{2}{x} + \frac{5}{x+\frac{1}{2}} + (0) \\ &= -\frac{2}{x} + \frac{5}{x+\frac{1}{2}} \\ &= \frac{-2+6x}{2x^2+x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{2}{x} + \frac{5}{x + \frac{1}{2}}\right)(0) + \left(\left(\frac{2}{x^2} - \frac{5}{\left(x + \frac{1}{2}\right)^2}\right) + \left(-\frac{2}{x} + \frac{5}{x + \frac{1}{2}}\right)^2 - \left(\frac{24x^2 - 16x + 6}{\left(2x^2 + x\right)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(-\frac{2}{x} + \frac{5}{x + \frac{1}{2}}\right) dx} \\ &= \frac{(1 + 2x)^5}{x^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-28x^2 - 6x}{2x^3 + x^2}\, dx} \\ &= z_1 e^{4\ln(1+2x) + 3\ln(x)} \\ &= z_1\left((1 + 2x)^4 x^3\right) \end{aligned}$$

Which simplifies to

$$y_1 = (1 + 2x)^9 x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-28x^2 - 6x}{2x^3 + x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{8\ln(1+2x) + 6\ln(x)}}{(y_1)^2}\, dx \\ &= y_1\left(-\frac{(2016x^4 + 672x^3 + 144x^2 + 18x + 1)\,\mathrm{e}^{8\ln(1+2x) + 6\ln(x)}}{20160\,(1 + 2x)^{17} x^6}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left((1+2x)^9 x\right) + c_2\left((1+2x)^9 x\left(-\frac{(2016x^4 + 672x^3 + 144x^2 + 18x + 1)\,\mathrm{e}^{8\ln(1+2x) + 6\ln(x)}}{20160\,(1 + 2x)^{17} x^6}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(2x+1)\left(\frac{d^2}{dx^2}y(x)\right)-2x(3+14x)\left(\frac{d}{dx}y(x)\right)+(6+100x)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{2(3+50x)y(x)}{x^2(2x+1)}+\frac{2(3+14x)\left(\frac{d}{dx}y(x)\right)}{x(2x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)-\frac{2(3+14x)\left(\frac{d}{dx}y(x)\right)}{x(2x+1)}+\frac{2(3+50x)y(x)}{x^2(2x+1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{2(3+14x)}{x(2x+1)},P_3(x)=\frac{2(3+50x)}{x^2(2x+1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=-6$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=6$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

$x^2(2x+1)\left(\frac{d^2}{dx^2}y(x)\right)-2x(3+14x)\left(\frac{d}{dx}y(x)\right)+(6+100x)\,y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

  ◦ Shift index using $k->k-m$

  $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..3$

  $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,x^{k+r-2+m}$

  ◦ Shift index using $k->k+2-m$

  $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$a_0(-1+r)(-6+r)x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)(k+r-6) + 2a_{k-1}(k+r-6)(k-11+r)\right)x^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+r)(-6+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{1, 6\}$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-6)\left((2k+2r-22)\,a_{k-1} + a_k(k+r-1)\right) = 0$
- Shift index using $k- >k+1$
  $(k+r-5)\left((2k+2r-20)\,a_k + a_{k+1}(k+r)\right) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = -\frac{2(k+r-10)a_k}{k+r}$
- Recursion relation for $r = 1$; series terminates at $k = 9$
  $a_{k+1} = -\frac{2(k-9)a_k}{k+1}$
- Recursion relation that defines the terminating series solution of the ODE for $r = 1$
  $\left[y(x) = \sum_{k=0}^{8} a_k x^{k+1}, a_{k+1} = -\frac{2(k-9)a_k}{k+1}\right]$
- Recursion relation for $r = 6$; series terminates at $k = 4$
  $a_{k+1} = -\frac{2(k-4)a_k}{k+6}$
- Apply recursion relation for $k = 0$
  $a_1 = \frac{4a_0}{3}$
- Apply recursion relation for $k = 1$
  $a_2 = \frac{6a_1}{7}$
- Express in terms of a_0
  $a_2 = \frac{8a_0}{7}$
- Apply recursion relation for $k = 2$
  $a_3 = \frac{a_2}{2}$
- Express in terms of a_0
  $a_3 = \frac{4a_0}{7}$
- Apply recursion relation for $k = 3$
  $a_4 = \frac{2a_3}{9}$
- Express in terms of a_0
  $a_4 = \frac{8a_0}{63}$
- Terminating series solution of the ODE for $r = 6$. Use reduction of order to find the second linea
  $y(x) = a_0 \cdot \left(1 + \frac{4}{3}x + \frac{8}{7}x^2 + \frac{4}{7}x^3 + \frac{8}{63}x^4\right)$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{8} a_k x^{k+1}\right) + b_0 \cdot \left(1 + \frac{4}{3}x + \frac{8}{7}x^2 + \frac{4}{7}x^3 + \frac{8}{63}x^4\right), a_{k+1} = -\frac{2(k-9)a_k}{k+1}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 62

```
dsolve(x^2*(2*x+1)*diff(diff(y(x),x),x)-2*x*(3+14*x)*diff(y(x),x)+(6+100*x)*y(x) = 0,y
```

$$y = 8c_2x^{10}+36c_2x^9+72c_2x^8+84c_2x^7+63c_2x^6+2016c_1x^5+672c_1x^4+144c_1x^3+18c_1x^2+c_1x$$

Mathematica DSolve solution

Solving time : 0.26 (sec)
Leaf size : 105

```
DSolve[{x^2*(1+2*x)*D[y[x],{x,2}]-2*x*(3+14*x)*D[y[x],x]+(6+100*x)*y[x]==0,{}},y[x],x,Includ
```

$$y(x) \to \exp\left(\int_1^x \left(\frac{10}{2K[1]+1}-\frac{2}{K[1]}\right)dK[1]-\frac{1}{2}\int_1^x -\frac{28K[2]+6}{2K[2]^2+K[2]}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]}\left(\frac{10}{2K[1]+1}-\frac{2}{K[1]}\right)dK[1]\right)dK[3]+c_1\right)$$

2.1.600 Problem 616

Internal problem ID [9772]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 616
Date solved : `Monday, January 27, 2025 at 06:14:06 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+x)\,y''-x(6+11x)\,y'+(6+32x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.225 (sec)

Writing the ode as

$$x^2(1+x)\,y''+\left(-11x^2-6x\right)y'+(6+32x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(1+x) \\ B &= -11x^2-6x \\ C &= 6+32x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{15x^2+4x+24}{4\left(x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15x^2+4x+24 \\ t &= 4\left(x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{15x^2+4x+24}{4\left(x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1145: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{x^2} + \frac{11}{1+x} + \frac{35}{4(1+x)^2} - \frac{11}{x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15x^2 + 4x + 24}{4\left(x^2 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15x^2 + 4x + 24}{4\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{7}{2}$	$-\frac{5}{2}$
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= \frac{5}{2} - \left(\frac{3}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{7}{2(1+x)} - \frac{2}{x} + (0) \\
&= \frac{7}{2(1+x)} - \frac{2}{x} \\
&= \frac{3x-4}{2x(1+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=x+a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{7}{2\left(1+x\right)}-\frac{2}{x}\right)(1)+\left(\left(-\frac{7}{2\left(1+x\right)^2}+\frac{2}{x^2}\right)+\left(\frac{7}{2\left(1+x\right)}-\frac{2}{x}\right)^2-\left(\frac{15x^2+4x+24}{4\left(x^2+x\right)^2}\right)\right)=0$$

$$\frac{-4-3a_0}{x\left(1+x\right)}=0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0=-\frac{4}{3}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x)=x-\frac{4}{3}$$

Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x)&=pe^{\int \omega\,dx}\\
&=\left(x-\frac{4}{3}\right)\mathrm{e}^{\int\left(\frac{7}{2(1+x)}-\frac{2}{x}\right)dx}\\
&=\left(x-\frac{4}{3}\right)\mathrm{e}^{-2\ln(x)+\frac{7\ln(1+x)}{2}}\\
&=\frac{\left(x-\frac{4}{3}\right)\left(1+x\right)^{7/2}}{x^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1&=z_1e^{\int-\frac{1}{2}\frac{B}{A}\,dx}\\
&=z_1e^{-\int\frac{1}{2}\frac{-11x^2-6x}{x^2(1+x)}\,dx}\\
&=z_1e^{3\ln(x)+\frac{5\ln(1+x)}{2}}\\
&=z_1\left(x^3(1+x)^{5/2}\right)
\end{aligned}$$

Which simplifies to

$$y_1=x(1+x)^6\left(x-\frac{4}{3}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int\frac{e^{\int-\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2&=y_1\int\frac{e^{\int-\frac{-11x^2-6x}{x^2(1+x)}\,dx}}{\left(y_1\right)^2}\,dx\\
&=y_1\int\frac{e^{6\ln(x)+5\ln(1+x)}}{\left(y_1\right)^2}\,dx\\
&=y_1\left(-\frac{3\,\mathrm{e}^{6\ln(x)+5\ln(1+x)}\left(35x^3+42x^2+21x+4\right)}{140\left(3x-4\right)x^6\left(1+x\right)^{11}}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x(1+x)^6\left(x-\frac{4}{3}\right)\right) \\
&\quad + c_2\left(x(1+x)^6\left(x-\frac{4}{3}\right)\left(-\frac{3\,\mathrm{e}^{6\ln(x)+5\ln(1+x)}(35x^3+42x^2+21x+4)}{140\,(3x-4)\,x^6\,(1+x)^{11}}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)-x(6+11x)\left(\frac{d}{dx}y(x)\right)+(6+32x)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{2(3+16x)y(x)}{(x+1)x^2}+\frac{(6+11x)\left(\frac{d}{dx}y(x)\right)}{x(x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)-\frac{(6+11x)\left(\frac{d}{dx}y(x)\right)}{x(x+1)}+\frac{2(3+16x)y(x)}{(x+1)x^2}=0$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x)=-\frac{6+11x}{x(x+1)},P_3(x)=\frac{2(3+16x)}{(x+1)x^2}\right]$
  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=-5$
  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$
  - $x=-1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=-1$

- Multiply by denominators

  $x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)-x(6+11x)\left(\frac{d}{dx}y(x)\right)+(6+32x)\,y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^3-2u^2+u)\left(\frac{d^2}{du^2}y(u)\right)+(-11u^2+16u-5)\left(\frac{d}{du}y(u)\right)+(-26+32u)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

    $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$
  - Shift index using $k->k-m$

    $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0r(-6+r)\,u^{-1+r} + (a_1(1+r)\,(-5+r) - 2a_0(r^2-9r+13))\,u^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(k-5$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-6+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 6\}$
- Each term must be 0

  $a_1(1+r)\,(-5+r) - 2a_0(r^2-9r+13) = 0$
- Each term in the series must be 0, giving the recursion relation

  $(-2a_k + a_{k-1} + a_{k+1})\,k^2 + 2((-2a_k + a_{k-1} + a_{k+1})\,r + 9a_k - 7a_{k-1} - 2a_{k+1})\,k + (-2a_k + a_{k-1}$
- Shift index using $k->k+1$

  $(-2a_{k+1} + a_k + a_{k+2})\,(k+1)^2 + 2((-2a_{k+1} + a_k + a_{k+2})\,r + 9a_{k+1} - 7a_k - 2a_{k+2})\,(k+1) + ($
- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + 2kra_k - 4kra_{k+1} + r^2a_k - 2r^2a_{k+1} - 12ka_k + 14ka_{k+1} - 12ra_k + 14ra_{k+1} + 32a_k - 10a_{k+1}}{k^2 + 2kr + r^2 - 2k - 2r - 8}$$
- Recursion relation for $r = 0$

  $$a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} - 12ka_k + 14ka_{k+1} + 32a_k - 10a_{k+1}}{k^2 - 2k - 8}$$
- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 4$

  $$a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} - 12ka_k + 14ka_{k+1} + 32a_k - 10a_{k+1}}{k^2 - 2k - 8}$$
- Recursion relation for $r = 6$

  $$a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} - 10ka_{k+1} - 4a_k + 2a_{k+1}}{k^2 + 10k + 16}$$
- Solution for $r = 6$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+6}, a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} - 10ka_{k+1} - 4a_k + 2a_{k+1}}{k^2 + 10k + 16}, 7a_1 + 10a_0 = 0\right]$$
- Revert the change of variables $u = x + 1$

  $$\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^{k+6}, a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} - 10ka_{k+1} - 4a_k + 2a_{k+1}}{k^2 + 10k + 16}, 7a_1 + 10a_0 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.011 (sec)
Leaf size : 45

```
dsolve(x^2*(x+1)*diff(diff(y(x),x),x)-x*(6+11*x)*diff(y(x),x)+(6+32*x)*y(x) = 0,y(x),sin
```

$$y = 3c_1x^8 + 14c_1x^7 + 21c_1x^6 + 35c_2x^4 + 42c_2x^3 + 21c_2x^2 + 4c_2x$$

Mathematica DSolve solution

Solving time : 0.48 (sec)
Leaf size : 122

```
DSolve[{x^2*(1+x)*D[y[x],{x,2}]-x*(6+11*x)*D[y[x],x]+(6+32*x)*y[x]==0,{}},y[x],x,IncludeSingula
```

$$y(x) \to \frac{1}{3}(3x-4)\exp\left(\int_1^x \left(\frac{7}{2(K[1]+1)} - \frac{2}{K[1]}\right) dK[1] - \frac{1}{2}\int_1^x \left(-\frac{5}{K[2]+1} - \frac{6}{K[2]}\right) dK[2]\right)\left(c_2\int_1^x \frac{9\exp\left(-2\int_1^{K[3]} \frac{3K[1]-4}{2K[1](K[1]+1)} dK[1]\right)}{(4-3K[3])^2} dK[3] + c_1\right)$$

2.1.601 Problem 617

Internal problem ID [9773]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 617
Date solved : `Monday, January 27, 2025 at 06:14:06 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2(1+x)\,y''+4x(1+4x)\,y'-(49+27x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.222 (sec)

Writing the ode as

$$\left(4x^3+4x^2\right)y''+\left(16x^2+4x\right)y'+(-27x-49)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^3+4x^2 \\ B &= 16x^2+4x \\ C &= -27x-49 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{35x^2+80x+48}{4\left(x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 35x^2+80x+48 \\ t &= 4\left(x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{35x^2+80x+48}{4\left(x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1147: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4\,(1+x)^2} + \frac{4}{1+x} + \frac{12}{x^2} - \frac{4}{x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 12$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 4 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -3 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{35x^2 + 80x + 48}{4\left(x^2 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{35}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{35x^2 + 80x + 48}{4\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
0	2	0	4	-3

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= \frac{7}{2} - \left(\frac{7}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(1+x\right)} + \frac{4}{x} + (0) \\
&= -\frac{1}{2\left(1+x\right)} + \frac{4}{x} \\
&= \frac{7x+8}{2x\left(1+x\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2(1+x)}+\frac{4}{x}\right)(0)+\left(\left(\frac{1}{2(1+x)^2}-\frac{4}{x^2}\right)+\left(-\frac{1}{2(1+x)}+\frac{4}{x}\right)^2-\left(\frac{35x^2+80x+48}{4(x^2+x)^2}\right)\right)=0$$

$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}z_1(x)&=pe^{\int \omega\,dx}\\&=\mathrm{e}^{\int\left(-\frac{1}{2(1+x)}+\frac{4}{x}\right)dx}\\&=\frac{x^4}{\sqrt{1+x}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1&=z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\&=z_1e^{-\int \frac{1}{2}\frac{16x^2+4x}{4x^3+4x^2}\,dx}\\&=z_1e^{-\frac{\ln(x)}{2}-\frac{3\ln(1+x)}{2}}\\&=z_1\left(\frac{1}{\sqrt{x}\,(1+x)^{3/2}}\right)\end{aligned}$$

Which simplifies to

$$y_1=\frac{x^{7/2}}{(1+x)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int\frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2&=y_1\int\frac{e^{\int -\frac{16x^2+4x}{4x^3+4x^2}\,dx}}{(y_1)^2}\,dx\\&=y_1\int\frac{e^{-\ln(x)-3\ln(1+x)}}{(y_1)^2}\,dx\\&=y_1\left(-\frac{(7x+6)(1+x)^3\,\mathrm{e}^{-\ln(x)-3\ln(1+x)}}{42x^6}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y&=c_1y_1+c_2y_2\\&=c_1\left(\frac{x^{7/2}}{(1+x)^2}\right)+c_2\left(\frac{x^{7/2}}{(1+x)^2}\left(-\frac{(7x+6)(1+x)^3\,\mathrm{e}^{-\ln(x)-3\ln(1+x)}}{42x^6}\right)\right)\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)+4x(4x+1)\left(\frac{d}{dx}y(x)\right)-(49+27x)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=\frac{(49+27x)y(x)}{4x^2(x+1)}-\frac{(4x+1)\left(\frac{d}{dx}y(x)\right)}{x(x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)+\frac{(4x+1)\left(\frac{d}{dx}y(x)\right)}{x(x+1)}-\frac{(49+27x)y(x)}{4x^2(x+1)}=0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

  $\left[P_2(x)=\frac{4x+1}{x(x+1)},P_3(x)=-\frac{49+27x}{4x^2(x+1)}\right]$

  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left.((x+1)\cdot P_2(x))\right|_{x=-1}=3$

  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left.((x+1)^2\cdot P_3(x))\right|_{x=-1}=0$

  - $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

$4x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)+4x(4x+1)\left(\frac{d}{dx}y(x)\right)+(-27x-49)\,y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(4u^3-8u^2+4u)\left(\frac{d^2}{du^2}y(u)\right)+(16u^2-28u+12)\left(\frac{d}{du}y(u)\right)+(-27u-22)\,y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

  - Shift index using $k->k-m$

  $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$4a_0r(2+r)\, u^{-1+r} + (4a_1(1+r)(3+r) - 2a_0(4r^2+10r+11))\, u^r + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+1+r)(k+3\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $4r(2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-2, 0\}$
- Each term must be 0
  $4a_1(1+r)(3+r) - 2a_0(4r^2+10r+11) = 0$
- Each term in the series must be 0, giving the recursion relation
  $4(-2a_k + a_{k-1} + a_{k+1})\, k^2 + 4(2(-2a_k + a_{k-1} + a_{k+1})\, r - 5a_k + a_{k-1} + 4a_{k+1})\, k + 4(-2a_k + a_{k-1} +$
- Shift index using $k- >k+1$
  $4(-2a_{k+1} + a_k + a_{k+2})\, (k+1)^2 + 4(2(-2a_{k+1} + a_k + a_{k+2})\, r - 5a_{k+1} + a_k + 4a_{k+2})\, (k+1) + 4(-2$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 8kra_k - 16kra_{k+1} + 4r^2a_k - 8r^2a_{k+1} + 12ka_k - 36ka_{k+1} + 12ra_k - 36ra_{k+1} - 27a_k - 50a_{k+1}}{4(k^2+2kr+r^2+6k+6r+8)}$
- Recursion relation for $r = -2$
  $a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} - 4ka_k - 4ka_{k+1} - 35a_k - 10a_{k+1}}{4(k^2+2k)}$
- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 0$
  $a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} - 4ka_k - 4ka_{k+1} - 35a_k - 10a_{k+1}}{4(k^2+2k)}$
- Recursion relation for $r = 0$
  $a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 12ka_k - 36ka_{k+1} - 27a_k - 50a_{k+1}}{4(k^2+6k+8)}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 12ka_k - 36ka_{k+1} - 27a_k - 50a_{k+1}}{4(k^2+6k+8)}, 12a_1 - 22a_0 = 0\right]$
- Revert the change of variables $u = x+1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x+1)^k, a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 12ka_k - 36ka_{k+1} - 27a_k - 50a_{k+1}}{4(k^2+6k+8)}, 12a_1 - 22a_0 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.042` (sec)
Leaf size : 26

```
dsolve(4*x^2*(x+1)*diff(diff(y(x),x),x)+4*x*(4*x+1)*diff(y(x),x)-(49+27*x)*y(x) = 0,y(
```

$$y = \frac{c_1 x^7 + 7c_2 x + 6c_2}{(x+1)^2 x^{7/2}}$$

Mathematica DSolve solution

Solving time : `0.235` (sec)
Leaf size : 102

```
DSolve[{4*x^2*(1+x)*D[y[x],{x,2}]+4*x*(1+4*x)*D[y[x],x]-(49+27*x)*y[x]==0,{}},y[x],x,Include
```

$$y(x) \to \exp\left(\int_1^x \left(\frac{4}{K[1]} - \frac{1}{2(K[1]+1)}\right) dK[1] - \frac{1}{2}\int_1^x \left(\frac{3}{K[2]+1} + \frac{1}{K[2]}\right) dK[2]\right) \left(c_2 \int_1^x \exp\left(-2\int_1^{K[3]} \frac{7K[1]+8}{2K[1]^2+2K[1]} dK[1]\right) dK[3] + c_1\right)$$

2.1.602 Problem 618

Internal problem ID [9774]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 618
Date solved : `Monday, January 27, 2025 at 06:14:07 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(x^2+1\right)y''-x\left(-2x^2+7\right)y'+12y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.296 (sec)

Writing the ode as

$$\left(x^4+x^2\right)y''+\left(2x^3-7x\right)y'+12y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^4+x^2\\B&=2x^3-7x\\C&=12\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-30x^2+15}{4\left(x^3+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=-30x^2+15\\t&=4\left(x^3+x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-30x^2+15}{4\left(x^3+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1149: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4x^2} + \frac{45}{16\,(x-i)^2} + \frac{45}{16\,(x+i)^2} + \frac{75i}{16\,(x-i)} - \frac{75i}{16\,(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{45}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{4} \end{aligned}$$

For the pole at $x=-i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{45}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{9}{4}\\ \alpha_c^- &= \frac{1}{2}-\sqrt{1+4b}=-\frac{5}{4}\end{aligned}$$

Since the order of r at ∞ is $4>2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= 0\\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{-30x^2+15}{4\left(x^3+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
i	2	0	$\frac{9}{4}$	$-\frac{5}{4}$
$-i$	2	0	$\frac{9}{4}$	$-\frac{5}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=0$ then

$$\begin{aligned}d &= \alpha_\infty^+-\left(\alpha_{c_1}^++\alpha_{c_2}^-+\alpha_{c_3}^-\right)\\ &= 0-(0)\\ &= 0\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^+}{x-c_1}\right)+\left((-)[\sqrt{r}]_{c_2}+\frac{\alpha_{c_2}^-}{x-c_2}\right)+\left((-)[\sqrt{r}]_{c_3}+\frac{\alpha_{c_3}^-}{x-c_3}\right)+(+)[\sqrt{r}]_\infty\\ &= \frac{5}{2x}-\frac{5}{4\left(x-i\right)}-\frac{5}{4\left(x+i\right)}+(0)\\ &= \frac{5}{2x}-\frac{5}{4\left(x-i\right)}-\frac{5}{4\left(x+i\right)}\\ &= \frac{5}{2x\left(x^2+1\right)}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{5}{2x} - \frac{5}{4(x-i)} - \frac{5}{4(x+i)}\right)(0) + \left(\left(-\frac{5}{2x^2} + \frac{5}{4(x-i)^2} + \frac{5}{4(x+i)^2}\right) + \left(\frac{5}{2x} - \frac{5}{4(x-i)} - \frac{5}{4(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{5}{2x} - \frac{5}{4(x-i)} - \frac{5}{4(x+i)}\right) dx} \\ &= \frac{x^{5/2}}{(x^2+1)^{5/4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x^3-7x}{x^4+x^2}\, dx} \\ &= z_1 e^{\frac{7\ln(x)}{2} - \frac{9\ln(x^2+1)}{4}} \\ &= z_1 \left(\frac{x^{7/2}}{(x^2+1)^{9/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^6}{(x^2+1)^{7/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3-7x}{x^4+x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{7\ln(x) - \frac{9\ln(x^2+1)}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{(x^2+1)^{7/2}}{4x^4} - \frac{3(x^2+1)^{7/2}}{8x^2} + \frac{3(x^2+1)^{5/2}}{8} + \frac{5(x^2+1)^{3/2}}{8} + \frac{15\sqrt{x^2+1}}{8} \right. \\ &\qquad \left. - \frac{15\,\mathrm{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)}{8}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{x^6}{(x^2+1)^{7/2}}\right) \\ &\quad + c_2\left(\frac{x^6}{(x^2+1)^{7/2}}\left(-\frac{(x^2+1)^{7/2}}{4x^4} - \frac{3(x^2+1)^{7/2}}{8x^2} + \frac{3(x^2+1)^{5/2}}{8} + \frac{5(x^2+1)^{3/2}}{8} + \frac{15\sqrt{x^2+1}}{8} - \frac{15\,\text{arctanh}\,(}{8}\right.\right. \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) - x(-2x^2+7)\left(\frac{d}{dx}y(x)\right) + 12y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{12y(x)}{x^2(x^2+1)} - \frac{(2x^2-7)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(2x^2-7)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)} + \frac{12y(x)}{x^2(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2x^2-7}{x(x^2+1)}, P_3(x) = \frac{12}{x^2(x^2+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -7$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 12$

  ◦ $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + x(2x^2-7)\left(\frac{d}{dx}y(x)\right) + 12y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k- > k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-2+r)(-6+r)\, x^r + a_1(-1+r)(-5+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-2)(k+r-6) + a_{k-2}(k +$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-2+r)(-6+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{2, 6\}$
- Each term must be 0
  $a_1(-1+r)(-5+r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-2)(a_k(k+r-6) + a_{k-2}(k+r-1)) = 0$
- Shift index using $k->k+2$
  $(k+r)(a_{k+2}(k-4+r) + a_k(k+r+1)) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{a_k(k+r+1)}{k-4+r}$
- Recursion relation for $r = 2$
  $a_{k+2} = -\frac{a_k(k+3)}{k-2}$
- Series not valid for $r = 2$, division by 0 in the recursion relation at $k = 2$
  $a_{k+2} = -\frac{a_k(k+3)}{k-2}$
- Recursion relation for $r = 6$
  $a_{k+2} = -\frac{a_k(k+7)}{k+2}$
- Solution for $r = 6$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+6}, a_{k+2} = -\frac{a_k(k+7)}{k+2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.037 (sec)
Leaf size : 56

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)-x*(-2*x^2+7)*diff(y(x),x)+12*y(x) = 0,y(x),sings
```

$$y = \frac{x^2 \left(-15 \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) c_2 x^4 + c_2 (8x^4 - 9x^2 - 2) \sqrt{x^2+1} + c_1 x^4\right)}{(x^2+1)^{7/2}}$$

Mathematica DSolve solution

Solving time : 0.17 (sec)
Leaf size : 96

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]-x*(7-2*x^2)*D[y[x],x]+12*y[x]==0,{}},y[x],x,IncludeSingularS
```

$$y(x) \to \exp\left(\int_1^x \frac{5}{2\left(K[1]^3+K[1]\right)} dK[1] - \frac{1}{2}\int_1^x \frac{2K[2]^2-7}{K[2]^3+K[2]} dK[2]\right)\left(c_2 \int_1^x \exp\left(-2\int_1^{K[3]} \frac{5}{2\left(K[1]^3+K[1]\right)} dK[1]\right) dK[3] + c_1\right)$$

2.1.603 Problem 619

Internal problem ID [9775]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 619
Date solved : `Monday, January 27, 2025 at 06:14:08 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x\left(-x^2+7\right)y' + 12y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.257 (sec)

Writing the ode as

$$x^2y'' + \left(x^3-7x\right)y' + 12y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^3 - 7x \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 12x^2 + 15}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4 - 12x^2 + 15 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 12x^2 + 15}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1151: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{4} - 3 + \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{x} - \frac{21}{4x^3} - \frac{63}{2x^5} - \frac{3465}{16x^7} - \frac{13041}{8x^9} - \frac{417501}{32x^{11}} - \frac{1744659}{16x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^4 - 12x^2 + 15}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{x^2}{4} - 3\right) + \left(\frac{15}{4x^2}\right) \\ &= \frac{x^2}{4} - 3 + \frac{15}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is -3. Now b can be found.

$$\begin{aligned} b &= (-3) - (0) \\ &= -3 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{-3}{\frac{1}{2}} - 1\right) = -\frac{7}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{-3}{\frac{1}{2}} - 1\right) = \frac{5}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 12x^2 + 15}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	$-\frac{7}{2}$	$\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{5}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{5}{2} - \left(\frac{5}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{5}{2x} + (-)\left(\frac{x}{2}\right) \\ &= \frac{5}{2x} - \frac{x}{2} \\ &= \frac{5}{2x} - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{5}{2x} - \frac{x}{2}\right)(0) + \left(\left(-\frac{5}{2x^2} - \frac{1}{2}\right) + \left(\frac{5}{2x} - \frac{x}{2}\right)^2 - \left(\frac{x^4 - 12x^2 + 15}{4x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{5}{2x} - \frac{x}{2}\right) dx} \\ &= x^{5/2} e^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^3-7x}{x^2}\,dx} \\
&= z_1 e^{-\frac{x^2}{4}+\frac{7\ln(x)}{2}} \\
&= z_1 \left(x^{7/2}\mathrm{e}^{-\frac{x^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^6 \mathrm{e}^{-\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^3-7x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}+7\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{4x^4} - \frac{\mathrm{e}^{\frac{x^2}{2}}}{8x^2} - \frac{\mathrm{Ei}_1\left(-\frac{x^2}{2}\right)}{16}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^6\mathrm{e}^{-\frac{x^2}{2}}\right) + c_2\left(x^6\mathrm{e}^{-\frac{x^2}{2}}\left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{4x^4} - \frac{\mathrm{e}^{\frac{x^2}{2}}}{8x^2} - \frac{\mathrm{Ei}_1\left(-\frac{x^2}{2}\right)}{16}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) - x(-x^2+7)\left(\frac{d}{dx}y(x)\right) + 12y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{12y(x)}{x^2} - \frac{(x^2-7)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(x^2-7)\left(\frac{d}{dx}y(x)\right)}{x} + \frac{12y(x)}{x^2} = 0$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - ◦ Define functions

$\left[P_2(x) = \frac{x^2-7}{x}, P_3(x) = \frac{12}{x^2}\right]$

- ∘ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = -7$

- ∘ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 12$

- ∘ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(x^2-7)\left(\frac{d}{dx}y(x)\right) + 12y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - ∘ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  - ∘ Shift index using $k- > k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - ∘ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)(-6+r)\,x^r + a_1(-1+r)(-5+r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-2)(k+r-6) + a_{k-2}(k+r\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)(-6+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{2, 6\}$

- Each term must be 0

  $a_1(-1+r)(-5+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)(a_k(k+r-6) + a_{k-2}) = 0$

- Shift index using $k- > k+2$

  $(k+r)(a_{k+2}(k-4+r) + a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{k-4+r}$

- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{a_k}{k-2}$

- Series not valid for $r = 2$, division by 0 in the recursion relation at $k = 2$

  $a_{k+2} = -\frac{a_k}{k-2}$

- Recursion relation for $r = 6$

  $a_{k+2} = -\frac{a_k}{k+2}$

- Solution for $r = 6$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+6}, a_{k+2} = -\frac{a_k}{k+2}, a_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 47

```
dsolve(x^2*diff(diff(y(x),x),x)-x*(-x^2+7)*diff(y(x),x)+12*y(x) = 0,y(x),singsol=all)
```

$$y = x^2 \left(\mathrm{Ei}_1\left(-\frac{x^2}{2}\right) \mathrm{e}^{-\frac{x^2}{2}} c_2 x^4 + \mathrm{e}^{-\frac{x^2}{2}} c_1 x^4 + 2c_2 x^2 + 4c_2\right)$$

Mathematica DSolve solution

Solving time : 0.106 (sec)
Leaf size : 68

```
DSolve[{x^2*D[y[x],{x,2}]-x*(7-x^2)*D[y[x],x]+12*y[x]==0,{}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{1}{16} e^{\frac{1}{2}(-x^2-5)} \left(c_2 x^6 \,\text{ExpIntegralEi}\left(\frac{x^2}{2}\right) + 16e^5 c_1 x^6 - 2c_2 e^{\frac{x^2}{2}} \left(x^2+2\right) x^2\right)$$

2.1.604 Problem 620

Internal problem ID [9776]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 620
Date solved : `Monday, January 27, 2025 at 06:14:08 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x(2x^2 + 1)\,y' - (-10x^2 + 1)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.340 (sec)

Writing the ode as

$$x^2y'' + (2x^3 + x)\,y' + (10x^2 - 1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 2x^3 + x \\ C &= 10x^2 - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^4 - 32x^2 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^4 - 32x^2 + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4 - 32x^2 + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1153: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = x^2 - 8 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x - \frac{4}{x} - \frac{61}{8x^3} - \frac{61}{2x^5} - \frac{19337}{128x^7} - \frac{26779}{32x^9} - \frac{5083557}{1024x^{11}} - \frac{7896633}{256x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^4 - 32x^2 + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (x^2 - 8) + \left(\frac{3}{4x^2}\right) \\ &= x^2 - 8 + \frac{3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is -8. Now b can be found.

$$\begin{aligned} b &= (-8) - (0) \\ &= -8 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= x \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-8}{1} - 1\right) &&= -\frac{9}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-8}{1} - 1\right) &&= \frac{7}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^4 - 32x^2 + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	x	$-\frac{9}{2}$	$\frac{7}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{7}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^+) \\ &= \frac{7}{2} - \left(\frac{3}{2}\right) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\ &= \frac{3}{2x} + (-)(x) \\ &= \frac{3}{2x} - x \\ &= \frac{3}{2x} - x \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{3}{2x} - x\right)(2x + a_1) + \left(\left(-\frac{3}{2x^2} - 1\right) + \left(\frac{3}{2x} - x\right)^2 - \left(\frac{4x^4 - 32x^2 + 3}{4x^2}\right)\right) = 0$$

$$\frac{2x^2a_1 + (4a_0 + 8)\,x + 3a_1}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -2, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= pe^{\int \omega \, dx} \\
&= \left(x^2-2\right) e^{\int \left(\frac{3}{2x}-x\right)dx} \\
&= \left(x^2-2\right) e^{-\frac{x^2}{2}+\frac{3\ln(x)}{2}} \\
&= \left(x^2-2\right) x^{3/2} e^{-\frac{x^2}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^3+x}{x^2}\, dx} \\
&= z_1 e^{-\frac{x^2}{2}-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{e^{-\frac{x^2}{2}}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x\, e^{-x^2} \left(x^2-2\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3+x}{x^2}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-x^2-\ln(x)}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\int \frac{e^{-x^2-\ln(x)} e^{2x^2}}{x^2 \left(x^2-2\right)^2} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x\, e^{-x^2} \left(x^2-2\right)\right) + c_2 \left(x\, e^{-x^2} \left(x^2-2\right) \left(\int \frac{e^{-x^2-\ln(x)} e^{2x^2}}{x^2 \left(x^2-2\right)^2} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right)+x(2x^2+1)\left(\frac{d}{dx}y(x)\right)-(-10x^2+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{(10x^2-1)y(x)}{x^2}-\frac{(2x^2+1)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)+\frac{(2x^2+1)\left(\frac{d}{dx}y(x)\right)}{x}+\frac{(10x^2-1)y(x)}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

- ◦ Define functions

$\left[P_2(x)=\frac{2x^2+1}{x},P_3(x)=\frac{10x^2-1}{x^2}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

$\left.(x\cdot P_2(x))\right|_{x=0}=1$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left.(x^2\cdot P_3(x))\right|_{x=0}=-1$

- ◦ $x=0$is a regular singular point

Check to see if $x_0=0$ is a regular singular point

$x_0=0$

- Multiply by denominators

$x^2\left(\frac{d^2}{dx^2}y(x)\right)+x(2x^2+1)\left(\frac{d}{dx}y(x)\right)+(10x^2-1)\,y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

$x^m\cdot y(x)=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

- ◦ Shift index using $k->k-m$

$x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ◦ Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

$x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

Rewrite ODE with series expansions

$a_0(1+r)\,(-1+r)\,x^r+a_1(2+r)\,r\,x^{1+r}+\left(\sum_{k=2}^{\infty}(a_k(k+r+1)\,(k+r-1)+2a_{k-2}(k+3+r))\,x^{k}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(1+r)(-1+r) = 0$

- Values of r that satisfy the indicial equation
  $r \in \{-1, 1\}$
- Each term must be 0
  $a_1(2+r)\,r = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_k(k+r+1)(k+r-1) + 2a_{k-2}(k+3+r) = 0$
- Shift index using $k- >k+2$
  $a_{k+2}(k+3+r)(k+r+1) + 2a_k(k+r+5) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{2a_k(k+r+5)}{(k+3+r)(k+r+1)}$
- Recursion relation for $r = -1$
  $a_{k+2} = -\frac{2a_k(k+4)}{(k+2)k}$
- Solution for $r = -1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{2a_k(k+4)}{(k+2)k}, a_1 = 0\right]$
- Recursion relation for $r = 1$
  $a_{k+2} = -\frac{2a_k(k+6)}{(k+4)(k+2)}$
- Solution for $r = 1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{2a_k(k+6)}{(k+4)(k+2)}, a_1 = 0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+2} = -\frac{2a_k(4+k)}{(k+2)k}, a_1 = 0, b_{k+2} = -\frac{2b_k(k+6)}{(4+k)(k+2)}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solut
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
```

```
    <- elementary form could result into a too large expression - returning special
  <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.032 (sec)
Leaf size : 23

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(2*x^2+1)*diff(y(x),x)-(-10*x^2+1)*y(x) = 0,y(x),sin
```

$$y = -\frac{x\,\mathrm{e}^{-x^2}(x^2-2)\,(c_1-2c_2)}{2}$$

Mathematica DSolve solution

Solving time : 0.225 (sec)
Leaf size : 51

```
DSolve[{x^2*D[y[x],{x,2}]+x*(1+2*x^2)*D[y[x],x]-(1-10*x^2)*y[x]==0,{}},y[x],x,IncludeSingula
```

$$y(x) \to e^{-x^2}x(x^2-2)\left(c_2\int_1^x \frac{e^{K[1]^2}}{K[1]^3\,(K[1]^2-2)^2}dK[1]+c_1\right)$$

2.1.605 Problem 621

Internal problem ID [9777]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 621
Date solved : Monday, January 27, 2025 at 06:14:09 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$x^2y'' + x\left(-2x^2+1\right)y' - 4\left(2x^2+1\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.263 (sec)

Writing the ode as

$$x^2y'' + \left(-2x^3+x\right)y' + \left(-8x^2-4\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x^3+x \\ C &= -8x^2-4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^4+24x^2+15}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^4+24x^2+15 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4+24x^2+15}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1155: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = x^2 + 6 + \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x + \frac{3}{x} - \frac{21}{8x^3} + \frac{63}{8x^5} - \frac{3465}{128x^7} + \frac{13041}{128x^9} - \frac{417501}{1024x^{11}} + \frac{1744659}{1024x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $([\sqrt{r}]_\infty)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$([\sqrt{r}]_\infty)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^4 + 24x^2 + 15}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (x^2 + 6) + \left(\frac{15}{4x^2}\right) \\ &= x^2 + 6 + \frac{15}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is 6. Now b can be found.

$$\begin{aligned} b &= (6) - (0) \\ &= 6 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= x \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{6}{1} - 1\right) &= \frac{5}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{6}{1} - 1\right) &= -\frac{7}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^4 + 24x^2 + 15}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	x	$\frac{5}{2}$	$-\frac{7}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+\right) \\ &= \frac{5}{2} - \left(\frac{5}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{5}{2x} + (x) \\ &= \frac{5}{2x} + x \\ &= \frac{5}{2x} + x \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{5}{2x} + x\right)(0) + \left(\left(-\frac{5}{2x^2} + 1\right) + \left(\frac{5}{2x} + x\right)^2 - \left(\frac{4x^4 + 24x^2 + 15}{4x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{5}{2x} + x\right) dx} \\ &= x^{5/2} e^{\frac{x^2}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x^3+x}{x^2}\,dx} \\
&= z_1 e^{\frac{x^2}{2}-\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{e^{\frac{x^2}{2}}}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^2 e^{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^3+x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x^2-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{e^{-x^2}}{4x^4}+\frac{e^{-x^2}}{4x^2}-\frac{\mathrm{Ei}_1(x^2)}{4}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^2 e^{x^2}\right) + c_2\left(x^2 e^{x^2}\left(-\frac{e^{-x^2}}{4x^4}+\frac{e^{-x^2}}{4x^2}-\frac{\mathrm{Ei}_1(x^2)}{4}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(-2x^2+1)\left(\frac{d}{dx}y(x)\right) - 4(2x^2+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = \frac{4(2x^2+1)y(x)}{x^2} + \frac{(2x^2-1)\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) - \frac{(2x^2-1)\left(\frac{d}{dx}y(x)\right)}{x} - \frac{4(2x^2+1)y(x)}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{2x^2-1}{x}, P_3(x) = -\frac{4(2x^2+1)}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -4$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$x^2\left(\frac{d^2}{dx^2}y(x)\right) - x(2x^2 - 1)\left(\frac{d}{dx}y(x)\right) + (-8x^2 - 4)\,y(x) = 0$$

- Assume series solution for $y(x)$

  $$y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $$x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k- >k - m$

    $$x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

    $$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

  - Shift index using $k- >k + 1 - m$

    $$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

  - Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $$x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(2+r)\,(-2+r)\,x^r + a_1(3+r)\,(-1+r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+2)\,(k+r-2) - 2a_{k-2}(k+r\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 2\}$

- Each term must be 0

  $a_1(3+r)\,(-1+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+r+2)\,(a_k(k+r-2) - 2a_{k-2}) = 0$

- Shift index using $k- >k + 2$

  $(k+r+4)\,(a_{k+2}(k+r) - 2a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2a_k}{k+r}$

- Recursion relation for $r = -2$

  $a_{k+2} = \frac{2a_k}{k-2}$

- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 2$
  $a_{k+2} = \frac{2a_k}{k-2}$
- Recursion relation for $r = 2$
  $a_{k+2} = \frac{2a_k}{k+2}$
- Solution for $r = 2$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{2a_k}{k+2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 39

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(-2*x^2+1)*diff(y(x),x)-4*(2*x^2+1)*y(x) = 0,y(x),sing
```

$$y = \frac{e^{x^2} \operatorname{Ei}_1\left(x^2\right) c_2 x^4 + c_1 x^4 e^{x^2} - c_2 x^2 + c_2}{x^2}$$

Mathematica DSolve solution

Solving time : 0.061 (sec)
Leaf size : 46

```
DSolve[{x^2*D[y[x],{x,2}]+x*(1-2*x^2)*D[y[x],x]-4*(1+2*x^2)*y[x]==0,{}},y[x],x,IncludeSingular
```

$$y(x) \to \frac{c_2\left(e^{x^2} x^4 \operatorname{ExpIntegralEi}\left(-x^2\right) + x^2 - 1\right)}{4x^2} + c_1 e^{x^2} x^2$$

2.1.606 Problem 622

Internal problem ID [9778]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 622
Date solved : `Monday, January 27, 2025 at 06:14:10 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x\left(-3x^2+1\right)y' - 4\left(-3x^2+1\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.582 (sec)

Writing the ode as

$$x^2y'' + \left(-3x^3+x\right)y' + \left(12x^2-4\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -3x^3+x \\ C &= 12x^2-4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9x^4 - 60x^2 + 15}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9x^4 - 60x^2 + 15 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{9x^4 - 60x^2 + 15}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1157: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2-4\\ &= -2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1,2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9x^2}{4} - 15 + \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3x}{2} - \frac{5}{x} - \frac{85}{12x^3} - \frac{425}{18x^5} - \frac{41225}{432x^7} - \frac{278375}{648x^9} - \frac{1787125}{864x^{11}} - \frac{40534375}{3888x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{3x}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{9x^4 - 60x^2 + 15}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{9x^2}{4} - 15\right) + \left(\frac{15}{4x^2}\right) \\ &= \frac{9x^2}{4} - 15 + \frac{15}{4x^2}\end{aligned}$$

We see that the coefficient of the term x in the quotient is -15. Now b can be found.

$$\begin{aligned}b &= (-15) - (0) \\ &= -15\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{3x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-15}{\frac{3}{2}} - 1\right) &&= -\frac{11}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-15}{\frac{3}{2}} - 1\right) &&= \frac{9}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{9x^4 - 60x^2 + 15}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{3x}{2}$	$-\frac{11}{2}$	$\frac{9}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{9}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^+) \\ &= \frac{9}{2} - \left(\frac{5}{2}\right) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{5}{2x} + (-)\left(\frac{3x}{2}\right) \\ &= \frac{5}{2x} - \frac{3x}{2} \\ &= \frac{5}{2x} - \frac{3x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{5}{2x} - \frac{3x}{2}\right)(2x + a_1) + \left(\left(-\frac{5}{2x^2} - \frac{3}{2}\right) + \left(\frac{5}{2x} - \frac{3x}{2}\right)^2 - \left(\frac{9x^4 - 60x^2 + 15}{4x^2}\right) \right) = 0$$

$$\frac{3x^2 a_1 + 6(2 + a_0)\, x + 5a_1}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -2, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x^2-2\right) e^{\int \left(\frac{5}{2x}-\frac{3x}{2}\right) dx} \\ &= \left(x^2-2\right) e^{-\frac{3x^2}{4}+\frac{5\ln(x)}{2}} \\ &= \left(x^2-2\right) x^{5/2} e^{-\frac{3x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-3x^3+x}{x^2} \, dx} \\ &= z_1 e^{\frac{3x^2}{4}-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{e^{\frac{3x^2}{4}}}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \left(x^2-2\right) x^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3x^3+x}{x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{3x^2}{2}-\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\int \frac{e^{\frac{3x^2}{2}-\ln(x)}}{\left(x^2-2\right)^2 x^4} dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\left(x^2-2\right) x^2\right) + c_2 \left(\left(x^2-2\right) x^2 \left(\int \frac{e^{\frac{3x^2}{2}-\ln(x)}}{\left(x^2-2\right)^2 x^4} dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right)+x(-3x^2+1)\left(\frac{d}{dx}y(x)\right)-4(-3x^2+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{4(3x^2-1)y(x)}{x^2}+\frac{(3x^2-1)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x)-\frac{(3x^2-1)\left(\frac{d}{dx}y(x)\right)}{x}+\frac{4(3x^2-1)y(x)}{x^2}=0$

- ☐ Check to see if $x_0=0$ is a regular singular point
  - Define functions

    $\left[P_2(x)=-\frac{3x^2-1}{x},P_3(x)=\frac{4(3x^2-1)}{x^2}\right]$
  - $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\Big|_{x=0}=1$
  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\Big|_{x=0}=-4$
  - $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$
- Multiply by denominators

$x^2\left(\frac{d^2}{dx^2}y(x)\right)-(3x^2-1)\,x\left(\frac{d}{dx}y(x)\right)+(12x^2-4)\,y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

    $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$
  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)\,(-2+r)\,x^r+a_1(3+r)\,(-1+r)\,x^{1+r}+\left(\sum_{k=2}^{\infty}(a_k(k+r+2)\,(k+r-2)-3a_{k-2}(k-6+\right.$
- $a_0$cannot be 0 by assumption, giving the indicial equation

$(2+r)(-2+r) = 0$

- Values of r that satisfy the indicial equation
  $r \in \{-2, 2\}$
- Each term must be 0
  $a_1(3+r)(-1+r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_k(k+r+2)(k+r-2) - 3a_{k-2}(k-6+r) = 0$
- Shift index using $k- >k+2$
  $a_{k+2}(k+4+r)(k+r) - 3a_k(k+r-4) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{3a_k(k+r-4)}{(k+4+r)(k+r)}$
- Recursion relation for $r = -2$; series terminates at $k = 6$
  $a_{k+2} = \frac{3a_k(k-6)}{(k+2)(k-2)}$
- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 2$
  $a_{k+2} = \frac{3a_k(k-6)}{(k+2)(k-2)}$
- Recursion relation for $r = 2$; series terminates at $k = 2$
  $a_{k+2} = \frac{3a_k(k-2)}{(k+6)(k+2)}$
- Solution for $r = 2$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{3a_k(k-2)}{(k+6)(k+2)}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric sol
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returr
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.034 (sec)
Leaf size : 19

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(-3*x^2+1)*diff(y(x),x)-4*(-3*x^2+1)*y(x) = 0,y(x),sin
```

$$y = -\frac{x^2(x^2-2)(c_1-c_2)}{2}$$

Mathematica DSolve solution

Solving time : 0.23 (sec)
Leaf size : 50

```
DSolve[{x^2*D[y[x],{x,2}]+x*(1-3*x^2)*D[y[x],x]-4*(1-3*x^2)*y[x]==0,{}},y[x],x,IncludeSingular
```

$$y(x) \to x^2(x^2-2)\left(c_2\int_1^x \frac{e^{\frac{3K[1]^2}{2}}}{K[1]^5(K[1]^2-2)^2}dK[1]+c_1\right)$$

2.1.607 Problem 623

Internal problem ID [9779]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 623
Date solved : Monday, January 27, 2025 at 06:14:11 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$x^2(x^2+1)y''+x(11x^2+5)y'+24x^2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.375 (sec)

Writing the ode as

$$(x^4+x^2)y''+(11x^3+5x)y'+24x^2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2 \\ B &= 11x^3+5x \\ C &= 24x^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{3x^4+6x^2+15}{4(x^3+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3x^4+6x^2+15 \\ t &= 4(x^3+x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{3x^4+6x^2+15}{4(x^3+x)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1159: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4x^2} + \frac{3}{4\,(x-i)^2} + \frac{3}{4\,(x+i)^2} + \frac{9i}{4\,(x-i)} - \frac{9i}{4\,(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x^4 + 6x^2 + 15}{4\left(x^3 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3x^4 + 6x^2 + 15}{4\left(x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\
&= \frac{3}{2} - \left(\frac{3}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{+}}{x-c_3}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{3}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)} + (0) \\
&= -\frac{3}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)} \\
&= -\frac{3}{2x} + \frac{3x}{x^2+1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{3}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)}\right)(0) + \left(\left(\frac{3}{2x^2} - \frac{3}{2(x-i)^2} - \frac{3}{2(x+i)^2}\right) + \left(-\frac{3}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{3}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)}\right) dx} \\
&= \frac{(x^2+1)^{3/2}}{x^{3/2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{11x^3+5x}{x^4+x^2}\, dx} \\
&= z_1 e^{-\frac{5\ln(x)}{2} - \frac{3\ln(x^2+1)}{2}} \\
&= z_1 \left(\frac{1}{x^{5/2}(x^2+1)^{3/2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x^4}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{11x^3+5x}{x^4+x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-5\ln(x) - 3\ln(x^2+1)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{(x^2+1)(2x^2+1)\, x^5 e^{-5\ln(x) - 3\ln(x^2+1)}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{1}{x^4}\right) + c_2\left(\frac{1}{x^4}\left(-\frac{(x^2+1)(2x^2+1)\,x^5 \mathrm{e}^{-5\ln(x)-3\ln(x^2+1)}}{4}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + x(11x^2+5)\left(\frac{d}{dx}y(x)\right) + 24x^2y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{24y(x)}{x^2+1} - \frac{(11x^2+5)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(11x^2+5)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)} + \frac{24y(x)}{x^2+1} = 0$

☐ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{11x^2+5}{x(x^2+1)}, P_3(x) = \frac{24}{x^2+1}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 5$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

- ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + (11x^2+5)\left(\frac{d}{dx}y(x)\right) + 24xy(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $x \cdot y(x)$ to series expansion

  $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

- ◦ Shift index using $k- >k-1$

  $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

- ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..2$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- ◦ Shift index using $k- >k+1-m$

$$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 1..3$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0r(4+r)\, x^{-1+r} + a_1(1+r)\,(5+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+5+r) + a_{k-1}(k+5+r)\,(k+3\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(4+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-4, 0\}$
- Each term must be 0
  $a_1(1+r)\,(5+r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $(k+5+r)\,(a_{k+1}(k+r+1) + a_{k-1}(k+3+r)) = 0$
- Shift index using $k- >k+1$
  $(k+r+6)\,(a_{k+2}(k+2+r) + a_k(k+r+4)) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{a_k(k+r+4)}{k+2+r}$
- Recursion relation for $r = -4$
  $a_{k+2} = -\frac{a_k k}{k-2}$
- Series not valid for $r = -4$, division by 0 in the recursion relation at $k = 2$
  $a_{k+2} = -\frac{a_k k}{k-2}$
- Recursion relation for $r = 0$
  $a_{k+2} = -\frac{a_k(k+4)}{k+2}$
- Solution for $r = 0$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(k+4)}{k+2}, 5a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 28

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)+x*(11*x^2+5)*diff(y(x),x)+24*x^2*y(x) = 0,y(x)
```

$$y = \frac{c_1 x^4 + 2c_2 x^2 + c_2}{(x^2+1)^2 x^4}$$

Mathematica DSolve solution

Solving time : 0.21 (sec)
Leaf size : 112

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]+x*(5+11*x^2)*D[y[x],x]+24*x^2*y[x]==0,{}},y[x],x,IncludeSi
```

$$y(x) \to \exp\left(\int_1^x \frac{3K[1]^2+5}{2(K[1]^3+K[1])}dK[1] - \frac{1}{2}\int_1^x \frac{11K[2]^2+5}{K[2]^3+K[2]}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{3K[1]^2+5}{2(K[1]^3+K[1])}dK[1]\right)dK[3] + c_1\right)$$

2.1.608 Problem 624

Internal problem ID [9780]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 624
Date solved : `Monday, January 27, 2025 at 06:14:12 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2\left(x^2+1\right)y''+8xy'-\left(-x^2+35\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.338 (sec)

Writing the ode as

$$\left(4x^4+4x^2\right)y''+8xy'+\left(x^2-35\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^4+4x^2 \\ B &= 8x \\ C &= x^2-35 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-x^4+22x^2+35}{4\left(x^3+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^4+22x^2+35 \\ t &= 4\left(x^3+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-x^4+22x^2+35}{4\left(x^3+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1161: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{35}{4x^2} + \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{21i}{4(x-i)} - \frac{21i}{4(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{-x^4 + 22x^2 + 35}{4\left(x^3 + x\right)^2}
$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-x^4 + 22x^2 + 35}{4\left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^+}{x-c_3}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{5}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)} + (-)(0) \\
&= -\frac{5}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)} \\
&= -\frac{5}{2x} + \frac{3x}{x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{5}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)}\right)(0) + \left(\left(\frac{5}{2x^2} - \frac{3}{2(x-i)^2} - \frac{3}{2(x+i)^2}\right) + \left(-\frac{5}{2x} + \frac{3}{2(x-i)} + \right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(-\frac{5}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)}\right) dx} \\
&= \frac{(x^2+1)^{3/2}}{x^{5/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{8x}{4x^4+4x^2}\, dx} \\
&= z_1 e^{-\ln(x) + \frac{\ln(x^2+1)}{2}} \\
&= z_1 \left(\frac{\sqrt{x^2+1}}{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2+1)^2}{x^{7/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{8x}{4x^4+4x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2\ln(x) + \ln(x^2+1)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{1}{4(x^2+1)^2} + \frac{1}{x^2+1} + \frac{\ln(x^2+1)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(x^2+1)^2}{x^{7/2}}\right) + c_2\left(\frac{(x^2+1)^2}{x^{7/2}}\left(-\frac{1}{4(x^2+1)^2} + \frac{1}{x^2+1} + \frac{\ln(x^2+1)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + 8x\left(\frac{d}{dx}y(x)\right) - (-x^2+35)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(x^2-35)y(x)}{4x^2(x^2+1)} - \frac{2\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{2\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)} + \frac{(x^2-35)y(x)}{4x^2(x^2+1)} = 0$
- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{2}{(x^2+1)x}, P_3(x) = \frac{x^2-35}{4x^2(x^2+1)}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{35}{4}$
  - $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$
- Multiply by denominators

  $4x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + 8x\left(\frac{d}{dx}y(x)\right) + (x^2-35)\,y(x) = 0$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k - m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r}$
  - Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(7+2r)(-5+2r)\, x^r + a_1(9+2r)(-3+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+7)(2k+2r-5) + a_k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(7+2r)(-5+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{-\frac{7}{2}, \frac{5}{2}\right\}$
- Each term must be 0
  $a_1(9+2r)(-3+2r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $4\left(k-\frac{5}{2}+r\right)\left(\left(k-\frac{5}{2}+r\right)a_{k-2} + a_k\left(k+r+\frac{7}{2}\right)\right) = 0$
- Shift index using $k->k+2$
  $4\left(k-\frac{1}{2}+r\right)\left(\left(k-\frac{1}{2}+r\right)a_k + a_{k+2}\left(k+\frac{11}{2}+r\right)\right) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{(2k+2r-1)a_k}{2k+11+2r}$
- Recursion relation for $r = -\frac{7}{2}$; series terminates at $k = 4$
  $a_{k+2} = -\frac{(2k-8)a_k}{2k+4}$
- Solution for $r = -\frac{7}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{7}{2}}, a_{k+2} = -\frac{(2k-8)a_k}{2k+4}, a_1 = 0\right]$
- Recursion relation for $r = \frac{5}{2}$
  $a_{k+2} = -\frac{(2k+4)a_k}{2k+16}$
- Solution for $r = \frac{5}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{2}}, a_{k+2} = -\frac{(2k+4)a_k}{2k+16}, a_1 = 0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{7}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{5}{2}}\right), a_{k+2} = -\frac{(2k-8)a_k}{2k+4}, a_1 = 0, b_{k+2} = -\frac{(2k+4)b_k}{2k+16}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
```

```
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.042 (sec)
Leaf size : 42

```
dsolve(4*x^2*(x^2+1)*diff(diff(y(x),x),x)+8*diff(y(x),x)*x-(-x^2+35)*y(x) = 0,y(x),sings
```

$$y = \frac{\left(x^2+1\right)^2 c_2 \ln\left(x^2+1\right) + \left(2x^2+\frac{3}{2}\right) c_2 + c_1\left(x^2+1\right)^2}{x^{7/2}}$$

Mathematica DSolve solution

Solving time : 0.156 (sec)
Leaf size : 101

```
DSolve[{4*x^2*(1+x^2)*D[y[x],{x,2}]+8*x*D[y[x],x]-(35-x^2)*y[x]==0,{}},y[x],x,IncludeSingularSo
```

$$y(x) \rightarrow \exp\left(\int_1^x \frac{K[1]^2-5}{2\left(K[1]^3+K[1]\right)} dK[1] - \frac{1}{2}\int_1^x \frac{2}{K[2]^3+K[2]} dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{K[1]^2-5}{2\left(K[1]^3+K[1]\right)} dK[1]\right) dK[3] + c_1\right)$$

2.1.609 Problem 625

Internal problem ID [9781]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 625
Date solved : `Monday, January 27, 2025 at 06:14:12 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(x^2+1\right)y''-x\left(-x^2+5\right)y'-\left(25x^2+7\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.345 (sec)

Writing the ode as

$$\left(x^4+x^2\right)y''+\left(x^3-5x\right)y'+\left(-25x^2-7\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^4+x^2\\B&=x^3-5x\\C&=-25x^2-7\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{99x^4+150x^2+63}{4\left(x^3+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=99x^4+150x^2+63\\t&=4\left(x^3+x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{99x^4+150x^2+63}{4\left(x^3+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1163: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 6 - 4\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{63}{4x^2} + \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} - \frac{15i}{4(x-i)} + \frac{15i}{4(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{63}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{7}{2}\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{99x^4 + 150x^2 + 63}{4\left(x^3 + x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{99}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{9}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{99x^4 + 150x^2 + 63}{4\left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{9}{2}$	$-\frac{7}{2}$
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{11}{2}$	$-\frac{9}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{9}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= -\frac{9}{2} - \left(-\frac{9}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\omega = \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x - c_3}\right) + (-)[\sqrt{r}]_{\infty}$$
$$= -\frac{7}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)} + (-)(0)$$
$$= -\frac{7}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)}$$
$$= -\frac{7}{2x} - \frac{x}{x^2+1}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{7}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)}\right)(0) + \left(\left(\frac{7}{2x^2} + \frac{1}{2(x-i)^2} + \frac{1}{2(x+i)^2}\right) + \left(-\frac{7}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$z_1(x) = p e^{\int \omega \, dx}$$
$$= e^{\int \left(-\frac{7}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)}\right) dx}$$
$$= \frac{1}{x^{7/2}\sqrt{x^2+1}}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$
$$= z_1 e^{-\int \frac{1}{2}\frac{x^3-5x}{x^4+x^2}\, dx}$$
$$= z_1 e^{\frac{5\ln(x)}{2} - \frac{3\ln(x^2+1)}{2}}$$
$$= z_1 \left(\frac{x^{5/2}}{(x^2+1)^{3/2}}\right)$$

Which simplifies to

$$y_1 = \frac{1}{x(x^2+1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{x^3-5x}{x^4+x^2}\, dx}}{(y_1)^2}\, dx$$
$$= y_1 \int \frac{e^{5\ln(x) - 3\ln(x^2+1)}}{(y_1)^2}\, dx$$
$$= y_1 \left(\frac{x^3(4x^2+5)(x^2+1)^3 e^{5\ln(x) - 3\ln(x^2+1)}}{40}\right)$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x\left(x^2+1\right)^2}\right) + c_2\left(\frac{1}{x\left(x^2+1\right)^2}\left(\frac{x^3(4x^2+5)\left(x^2+1\right)^3 \mathrm{e}^{5\ln(x)-3\ln(x^2+1)}}{40}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) - x(-x^2+5)\left(\frac{d}{dx}y(x)\right) - (25x^2+7)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{(25x^2+7)y(x)}{x^2(x^2+1)} - \frac{(x^2-5)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(x^2-5)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)} - \frac{(25x^2+7)y(x)}{x^2(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{x^2-5}{x(x^2+1)}, P_3(x) = -\frac{25x^2+7}{x^2(x^2+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -5$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -7$

  ◦ $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + x(x^2-5)\left(\frac{d}{dx}y(x)\right) + (-25x^2-7)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k-m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

$$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+r)\,(-7+r)\,x^r + a_1(2+r)\,(-6+r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+1)\,(k+r-7) + a_{k-2}(k+3+r$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
$(1+r)\,(-7+r) = 0$
- Values of r that satisfy the indicial equation
$r \in \{-1, 7\}$
- Each term must be 0
$a_1(2+r)\,(-6+r) = 0$
- Solve for the dependent coefficient(s)
$a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
$(k+r-7)\,(a_k(k+r+1) + a_{k-2}(k+3+r)) = 0$
- Shift index using $k->k+2$
$(k+r-5)\,(a_{k+2}(k+3+r) + a_k(k+r+5)) = 0$
- Recursion relation that defines series solution to ODE
$a_{k+2} = -\frac{a_k(k+r+5)}{k+3+r}$
- Recursion relation for $r = -1$
$a_{k+2} = -\frac{a_k(k+4)}{k+2}$
- Solution for $r = -1$
$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k(k+4)}{k+2}, a_1 = 0\right]$
- Recursion relation for $r = 7$
$a_{k+2} = -\frac{a_k(k+12)}{k+10}$
- Solution for $r = 7$
$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+7}, a_{k+2} = -\frac{a_k(k+12)}{k+10}, a_1 = 0\right]$
- Combine solutions and rename parameters
$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+7}\right), a_{k+2} = -\frac{a_k(4+k)}{k+2}, a_1 = 0, b_{k+2} = -\frac{b_k(k+12)}{k+10}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
```

```
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 29

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)-x*(-x^2+5)*diff(y(x),x)-(25*x^2+7)*y(x) = 0,y(
```

$$y = \frac{4c_2x^{10} + 5c_2x^8 + c_1}{x\left(x^2+1\right)^2}$$

Mathematica DSolve solution

Solving time : 0.2 (sec)
Leaf size : 110

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]-x*(5-x^2)*D[y[x],x]-(7+25*x^2)*y[x]==0,{}},y[x],x,IncludeS
```

$$y(x) \to \exp\left(\int_1^x -\frac{9K[1]^2+7}{2\left(K[1]^3+K[1]\right)}dK[1] - \frac{1}{2}\int_1^x \frac{K[2]^2-5}{K[2]^3+K[2]}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} -\frac{9K[1]^2+7}{2\left(K[1]^3+K[1]\right)}dK[1]\right)dK[3] + c_1\right)$$

2.1.610 Problem 626

Internal problem ID [9782]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 626
Date solved : `Monday, January 27, 2025 at 06:14:13 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(x^2+1)\,y''+x(2x^2+5)\,y'-21y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.329 (sec)

Writing the ode as

$$(x^4+x^2)\,y''+(2x^3+5x)\,y'-21y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2\\ B &= 2x^3+5x \\ C &= -21 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{78x^2+99}{4(x^3+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 78x^2+99\\ t &= 4(x^3+x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{78x^2+99}{4(x^3+x)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1165: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{99}{4x^2} + \frac{21}{16\,(x-i)^2} + \frac{21}{16\,(x+i)^2} + \frac{219i}{16\,(x-i)} - \frac{219i}{16\,(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{99}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{9}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4} \end{aligned}$$

For the pole at $x=-i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{21}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{7}{4}\\
\alpha_c^- &= \frac{1}{2}-\sqrt{1+4b}=-\frac{3}{4}
\end{aligned}
$$

Since the order of r at ∞ is $4>2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^+ &= 0\\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{78x^2+99}{4\left(x^3+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{11}{2}$	$-\frac{9}{2}$
i	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
$-i$	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=1$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+ + \alpha_{c_3}^+\right)\\
&= 1-(-1)\\
&= 2
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^-}{x-c_1}\right)+\left((+)[\sqrt{r}]_{c_2}+\frac{\alpha_{c_2}^+}{x-c_2}\right)+\left((+)[\sqrt{r}]_{c_3}+\frac{\alpha_{c_3}^+}{x-c_3}\right)+(-)[\sqrt{r}]_\infty\\
&= -\frac{9}{2x}+\frac{7}{4\left(x-i\right)}+\frac{7}{4\left(x+i\right)}+(-)\left(0\right)\\
&= -\frac{9}{2x}+\frac{7}{4\left(x-i\right)}+\frac{7}{4\left(x+i\right)}\\
&= -\frac{9}{2x}+\frac{7x}{2x^2+2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=x^2+a_1x+a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2)+2\left(-\frac{9}{2x}+\frac{7}{4(x-i)}+\frac{7}{4(x+i)}\right)(2x+a_1)+\left(\left(\frac{9}{2x^2}-\frac{7}{4(x-i)^2}-\frac{7}{4(x+i)^2}\right)+\left(-\frac{9}{2x}+\frac{7}{4(x-\right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0=8, a_1=0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x)=x^2+8$$

Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\,dx} \\
&= \left(x^2+8\right)e^{\int\left(-\frac{9}{2x}+\frac{7}{4(x-i)}+\frac{7}{4(x+i)}\right)dx} \\
&= \left(x^2+8\right)e^{\frac{7\ln\left(x^2+1\right)}{4}-\frac{9\ln(x)}{2}} \\
&= \frac{\left(x^2+8\right)\left(x^2+1\right)^{7/4}}{x^{9/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^3+5x}{x^4+x^2}\,dx} \\
&= z_1 e^{-\frac{5\ln(x)}{2}+\frac{3\ln\left(x^2+1\right)}{4}} \\
&= z_1\left(\frac{\left(x^2+1\right)^{3/4}}{x^{5/2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\frac{\left(x^2+1\right)^{5/2}\left(x^2+8\right)}{x^7}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int\frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1\int\frac{e^{\int -\frac{2x^3+5x}{x^4+x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1\int\frac{e^{-5\ln(x)+\frac{3\ln\left(x^2+1\right)}{2}}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(-\frac{\left(35x^6+140x^4+168x^2+64\right)x^5 e^{-5\ln(x)+\frac{3\ln\left(x^2+1\right)}{2}}}{35\left(x^2+1\right)^4\left(x^2+8\right)}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x^2+1)^{5/2}(x^2+8)}{x^7} \right) \\
&\quad + c_2 \left(\frac{(x^2+1)^{5/2}(x^2+8)}{x^7} \left(-\frac{(35x^6+140x^4+168x^2+64)\, x^5 \mathrm{e}^{-5\ln(x)+\frac{3\ln(x^2+1)}{2}}}{35(x^2+1)^4 (x^2+8)} \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + x(2x^2+5)\left(\frac{d}{dx}y(x)\right) - 21y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{21y(x)}{x^2(x^2+1)} - \frac{(2x^2+5)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(2x^2+5)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)} - \frac{21y(x)}{x^2(x^2+1)} = 0$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{2x^2+5}{x(x^2+1)}, P_3(x) = -\frac{21}{x^2(x^2+1)}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 5$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -21$
  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + x(2x^2+5)\left(\frac{d}{dx}y(x)\right) - 21y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(7+r)(-3+r)\, x^r + a_1(8+r)(-2+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+7)(k+r-3) + a_{k-2}(k-2$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(7+r)(-3+r)=0$
- Values of r that satisfy the indicial equation
  $r \in \{-7, 3\}$
- Each term must be 0
  $a_1(8+r)(-2+r)=0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_k(k+r+7)(k+r-3) + a_{k-2}(k-2+r)(k+r-1) = 0$
- Shift index using $k->k+2$
  $a_{k+2}(k+9+r)(k+r-1) + a_k(k+r)(k+r+1) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{a_k(k+r)(k+r+1)}{(k+9+r)(k+r-1)}$
- Recursion relation for $r=-7$; series terminates at $k=6$
  $a_{k+2} = -\frac{a_k(k-7)(k-6)}{(k+2)(k-8)}$
- Solution for $r=-7$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-7}, a_{k+2} = -\frac{a_k(k-7)(k-6)}{(k+2)(k-8)}, a_1 = 0\right]$
- Recursion relation for $r=3$
  $a_{k+2} = -\frac{a_k(k+3)(k+4)}{(k+12)(k+2)}$
- Solution for $r=3$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = -\frac{a_k(k+3)(k+4)}{(k+12)(k+2)}, a_1 = 0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-7}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), a_{k+2} = -\frac{a_k(k-7)(k-6)}{(k+2)(k-8)}, a_1 = 0, b_{k+2} = -\frac{b_k(k+3)(4+k)}{(k+12)(k+2)}, b_1 = \right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.023 (sec)
Leaf size : 41

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)+x*(2*x^2+5)*diff(y(x),x)-21*y(x) = 0,y(x),singso
```

$$y = \frac{c_1 (x^2+1)^{5/2} (x^2+8) + 35 c_2 \left(x^6 + 4x^4 + \frac{24}{5} x^2 + \frac{64}{35}\right)}{x^7}$$

Mathematica DSolve solution

Solving time : 0.431 (sec)
Leaf size : 126

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]+x*(5+2*x^2)*D[y[x],x]-21*y[x]==0,{}},y[x],x,IncludeSingularSo
```

$$y(x) \to (x^2+8) \exp\left(\int_1^x -\frac{2K[1]^2+9}{2(K[1]^3+K[1])} dK[1] - \frac{1}{2}\int_1^x \frac{2K[2]^2+5}{K[2]^3+K[2]} dK[2]\right) \left(c_2 \int_1^x \frac{\exp\left(-2\int_1^{K[3]} -\frac{2K[1]^2+9}{2(K[1]^3+K[1])} dK[1]\right)}{(K[3]^2+8)^2} dK[3] + c_1\right)$$

2.1.611 Problem 627

Internal problem ID [9783]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 627
Date solved : Monday, January 27, 2025 at 06:14:14 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$4x^2\left(x^2+1\right)y''+4x\left(x^2+2\right)y'-\left(x^2+15\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.261 (sec)

Writing the ode as

$$\left(4x^4+4x^2\right)y''+\left(4x^3+8x\right)y'+\left(-x^2-15\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^4+4x^2 \\ B &= 4x^3+8x \\ C &= -x^2-15 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{10x^2+15}{4\left(x^3+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 10x^2+15 \\ t &= 4\left(x^3+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{10x^2+15}{4\left(x^3+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1167: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4x^2} + \frac{5}{16\,(x-i)^2} + \frac{5}{16\,(x+i)^2} + \frac{35i}{16\,(x-i)} - \frac{35i}{16\,(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{10x^2 + 15}{4\left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
i	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-i$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^+}{x-c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{2x} + \frac{5}{4(x-i)} + \frac{5}{4(x+i)} + (-)(0) \\
&= -\frac{3}{2x} + \frac{5}{4(x-i)} + \frac{5}{4(x+i)} \\
&= -\frac{3}{2x} + \frac{5x}{2x^2+2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{2x} + \frac{5}{4(x-i)} + \frac{5}{4(x+i)}\right)(0) + \left(\left(\frac{3}{2x^2} - \frac{5}{4(x-i)^2} - \frac{5}{4(x+i)^2}\right) + \left(-\frac{3}{2x} + \frac{5}{4(x-i)} + \frac{5}{4(x$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{3}{2x} + \frac{5}{4(x-i)} + \frac{5}{4(x+i)}\right) dx} \\
&= \frac{(x^2+1)^{5/4}}{x^{3/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x^3+8x}{4x^4+4x^2}\, dx} \\
&= z_1 e^{-\ln(x) + \frac{\ln(x^2+1)}{4}} \\
&= z_1 \left(\frac{(x^2+1)^{1/4}}{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2+1)^{3/2}}{x^{5/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x^3+8x}{4x^4+4x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2\ln(x) + \frac{\ln(x^2+1)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{(3x^2+2)\, x^2 e^{-2\ln(x) + \frac{\ln(x^2+1)}{2}}}{3(x^2+1)^2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x^2+1)^{3/2}}{x^{5/2}}\right) + c_2 \left(\frac{(x^2+1)^{3/2}}{x^{5/2}} \left(-\frac{(3x^2+2)\, x^2 e^{-2\ln(x) + \frac{\ln(x^2+1)}{2}}}{3(x^2+1)^2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)+4x(x^2+2)\left(\frac{d}{dx}y(x)\right)-(x^2+15)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=\frac{(x^2+15)y(x)}{4x^2(x^2+1)}-\frac{(x^2+2)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)+\frac{(x^2+2)\left(\frac{d}{dx}y(x)\right)}{x(x^2+1)}-\frac{(x^2+15)y(x)}{4x^2(x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=\frac{x^2+2}{x(x^2+1)},P_3(x)=-\frac{x^2+15}{4x^2(x^2+1)}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

$(x\cdot P_2(x))\Big|_{x=0}=2$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$(x^2\cdot P_3(x))\Big|_{x=0}=-\frac{15}{4}$

- ◦ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$4x^2(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)+4x(x^2+2)\left(\frac{d}{dx}y(x)\right)+(-x^2-15)\,y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

$x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ◦ Shift index using $k->k-m$

$x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ◦ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..4$

$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- ◦ Shift index using $k->k+2-m$

$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

Rewrite ODE with series expansions

$$a_0(5+2r)(-3+2r)x^r + a_1(7+2r)(-1+2r)x^{1+r} + \left(\sum_{k=2}^{\infty}(a_k(2k+2r+5)(2k+2r-3)+a_{k-2}(\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(5+2r)(-3+2r)=0$
- Values of r that satisfy the indicial equation
  $r \in \left\{-\frac{5}{2}, \frac{3}{2}\right\}$
- Each term must be 0
  $a_1(7+2r)(-1+2r)=0$
- Solve for the dependent coefficient(s)
  $a_1=0$
- Each term in the series must be 0, giving the recursion relation
  $4\left(\left(k-\frac{5}{2}+r\right)a_{k-2}+\left(k+r+\frac{5}{2}\right)a_k\right)\left(k+r-\frac{3}{2}\right)=0$
- Shift index using $k->k+2$
  $4\left(\left(k-\frac{1}{2}+r\right)a_k+\left(k+\frac{9}{2}+r\right)a_{k+2}\right)\left(k+\frac{1}{2}+r\right)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+2}=-\frac{(2k+2r-1)a_k}{2k+9+2r}$
- Recursion relation for $r=-\frac{5}{2}$
  $a_{k+2}=-\frac{(2k-6)a_k}{2k+4}$
- Solution for $r=-\frac{5}{2}$
  $\left[y(x)=\sum_{k=0}^{\infty}a_k x^{k-\frac{5}{2}}, a_{k+2}=-\frac{(2k-6)a_k}{2k+4}, a_1=0\right]$
- Recursion relation for $r=\frac{3}{2}$
  $a_{k+2}=-\frac{(2k+2)a_k}{2k+12}$
- Solution for $r=\frac{3}{2}$
  $\left[y(x)=\sum_{k=0}^{\infty}a_k x^{k+\frac{3}{2}}, a_{k+2}=-\frac{(2k+2)a_k}{2k+12}, a_1=0\right]$
- Combine solutions and rename parameters
  $\left[y(x)=\left(\sum_{k=0}^{\infty}a_k x^{k-\frac{5}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{3}{2}}\right), a_{k+2}=-\frac{(2k-6)a_k}{2k+4}, a_1=0, b_{k+2}=-\frac{(2k+2)b_k}{2k+12}, b_1=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.046` (sec)
Leaf size : 27

```
dsolve(4*x^2*(x^2+1)*diff(diff(y(x),x),x)+4*x*(x^2+2)*diff(y(x),x)-(x^2+15)*y(x) = 0,y
```

$$y = \frac{c_2(x^2+1)^{3/2}+3c_1x^2+2c_1}{x^{5/2}}$$

Mathematica DSolve solution

Solving time : `0.195` (sec)
Leaf size : 110

```
DSolve[{4*x^2*(1+x^2)*D[y[x],{x,2}]+4*x*(2+x^2)*D[y[x],x]-(15+x^2)*y[x]==0,{}},y[x],x,Includ
```

$$y(x) \to \exp\left(\int_1^x \frac{2K[1]^2-3}{2(K[1]^3+K[1])}dK[1] - \frac{1}{2}\int_1^x \frac{K[2]^2+2}{K[2]^3+K[2]}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{2K[1]^2-3}{2(K[1]^3+K[1])}dK[1]\right)dK[3] + c_1\right)$$

2.1.612 Problem 628

Internal problem ID [9784]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 628
Date solved : `Monday, January 27, 2025 at 06:14:15 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - \frac{2(t+1)\,y'}{t^2+2t-1} + \frac{2y}{t^2+2t-1} = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.260 (sec)

Writing the ode as

$$y'' + \frac{(-2t-2)\,y'}{t^2+2t-1} + \frac{2y}{t^2+2t-1} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{-2t-2}{t^2+2t-1} \\ C &= \frac{2}{t^2+2t-1} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{6}{\left(t^2+2t-1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 6 \\ t &= \left(t^2+2t-1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{6}{\left(t^2+2t-1\right)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1169: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = \left(t^2+2t-1\right)^2$. There is a pole at $t = \sqrt{2}-1$ of order 2. There is a pole at $t = -1-\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4\left(t-\sqrt{2}+1\right)^2} + \frac{3}{4\left(t+1+\sqrt{2}\right)^2} - \frac{3\sqrt{2}}{8\left(t-\sqrt{2}+1\right)} + \frac{3\sqrt{2}}{8\left(t+1+\sqrt{2}\right)}$$

For the pole at $t = \sqrt{2}-1$ let b be the coefficient of $\frac{1}{\left(t-\sqrt{2}+1\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $t=-1-\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(t+1+\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{2} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $4>2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{6}{\left(t^2+2t-1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\sqrt{2}-1$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-1-\sqrt{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{+}\right) \\
&= 1-(1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{t-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^{-}}{t-c_1}\right)+\left((+)[\sqrt{r}]_{c_2}+\frac{\alpha_{c_2}^{+}}{t-c_2}\right)+(-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(t-\sqrt{2}+1\right)}+\frac{3}{2\left(t+1+\sqrt{2}\right)}+(-)\left(0\right) \\
&= -\frac{1}{2\left(t-\sqrt{2}+1\right)}+\frac{3}{2\left(t+1+\sqrt{2}\right)} \\
&= \frac{t+1-2\sqrt{2}}{t^2+2t-1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\left(t-\sqrt{2}+1\right)} + \frac{3}{2\left(t+1+\sqrt{2}\right)}\right)(0) + \left(\left(\frac{1}{2\left(t-\sqrt{2}+1\right)^2} - \frac{3}{2\left(t+1+\sqrt{2}\right)^2}\right) + \left(-\frac{}{2\,(t -}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2\left(t-\sqrt{2}+1\right)} + \frac{3}{2\left(t+1+\sqrt{2}\right)}\right) dt} \\ &= \frac{\left(t+1+\sqrt{2}\right)^{3/2}}{\sqrt{t-\sqrt{2}+1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{\frac{-2t-2}{t^2+2t-1}}{1}\, dt} \\ &= z_1 e^{\frac{\ln\left(t^2+2t-1\right)}{2}} \\ &= z_1 \left(\sqrt{t^2+2t-1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{t^2+2t-1}\,\left(t+1+\sqrt{2}\right)^{3/2}}{\sqrt{t-\sqrt{2}+1}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{\frac{-2t-2}{t^2+2t-1}}{1}\, dt}}{\left(y_1\right)^2}\, dt \\ &= y_1 \int \frac{e^{\ln\left(t^2+2t-1\right)}}{\left(y_1\right)^2}\, dt \\ &= y_1 \left(\frac{\sqrt{2}}{\left(t+1+\sqrt{2}\right)^2} - \frac{1}{t+1+\sqrt{2}}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{t^2+2t-1}\,\left(t+1+\sqrt{2}\right)^{3/2}}{\sqrt{t-\sqrt{2}+1}} \right) \\
&\quad + c_2 \left(\frac{\sqrt{t^2+2t-1}\,\left(t+1+\sqrt{2}\right)^{3/2}}{\sqrt{t-\sqrt{2}+1}} \left(\frac{\sqrt{2}}{\left(t+1+\sqrt{2}\right)^2} - \frac{1}{t+1+\sqrt{2}} \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dt^2}y(t) - \frac{2(t+1)\left(\frac{d}{dt}y(t)\right)}{t^2+2t-1} + \frac{2y(t)}{t^2+2t-1} = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dt^2}y(t)$

- □ Check to see if t_0 is a regular singular point
  - Define functions

    $\left[P_2(t) = -\frac{2(t+1)}{t^2+2t-1}, P_3(t) = \frac{2}{t^2+2t-1}\right]$
  - $\left(t+\sqrt{2}+1\right) \cdot P_2(t)$ is analytic at $t = -\sqrt{2}-1$

    $\left(\left(t+\sqrt{2}+1\right) \cdot P_2(t)\right)\Big|_{t=-\sqrt{2}-1} = 0$
  - $\left(t+\sqrt{2}+1\right)^2 \cdot P_3(t)$ is analytic at $t = -\sqrt{2}-1$

    $\left(\left(t+\sqrt{2}+1\right)^2 \cdot P_3(t)\right)\Big|_{t=-\sqrt{2}-1} = 0$
  - $t = -\sqrt{2}-1$is a regular singular point

    Check to see if t_0 is a regular singular point

    $t_0 = -\sqrt{2}-1$

- Multiply by denominators

  $(t^2+2t-1)\left(\frac{d^2}{dt^2}y(t)\right) + (-2t-2)\left(\frac{d}{dt}y(t)\right) + 2y(t) = 0$

- Change variables using $t = u-\sqrt{2}-1$ so that the regular singular point is at $u = 0$

  $\left(u^2-2u\sqrt{2}\right)\left(\frac{d^2}{du^2}y(u)\right) + \left(-2u+2\sqrt{2}\right)\left(\frac{d}{du}y(u)\right) + 2y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k- >k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$
  - Shift index using $k- >k+2-m$

$$u^{m} \cdot \left(\frac{d^{2}}{du^{2}}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$-2\sqrt{2}\,(r-2)\,r a_{0} u^{r-1} + \left(\sum_{k=0}^{\infty} \left(-2\sqrt{2}\,(k+r-1)\,(k+1+r)\,a_{k+1} + a_{k}(k+r-1)\,(k+r-2)\right) u$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-2\sqrt{2}\,(r-2)\,r = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 2\}$
- Each term in the series must be 0, giving the recursion relation
  $\left(-2a_{k+1}(k+1+r)\sqrt{2} + a_{k}(k+r-2)\right)(k+r-1) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_{k}(k+r-2)\sqrt{2}}{4(k+1+r)}$
- Recursion relation for $r = 0$; series terminates at $k = 2$
  $a_{k+1} = \frac{a_{k}(k-2)\sqrt{2}}{4(k+1)}$
- Apply recursion relation for $k = 0$
  $a_{1} = -\frac{a_{0}\sqrt{2}}{2}$
- Apply recursion relation for $k = 1$
  $a_{2} = -\frac{a_{1}\sqrt{2}}{8}$
- Express in terms of a_0
  $a_{2} = \frac{a_{0}}{8}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li
  $y(u) = a_{0} \cdot \left(1 - \frac{u\sqrt{2}}{2} + \frac{u^{2}}{8}\right)$
- Revert the change of variables $u = t + \sqrt{2} + 1$
  $\left[y(t) = a_{0}\left(\frac{(-2t-2)\sqrt{2}}{8} + \frac{t^{2}}{8} + \frac{t}{4} + \frac{3}{8}\right)\right]$
- Recursion relation for $r = 2$
  $a_{k+1} = \frac{a_{k}k\sqrt{2}}{4(k+3)}$
- Solution for $r = 2$
  $\left[y(u) = \sum_{k=0}^{\infty} a_{k}u^{k+2}, a_{k+1} = \frac{a_{k}k\sqrt{2}}{4(k+3)}\right]$
- Revert the change of variables $u = t + \sqrt{2} + 1$
  $\left[y(t) = \sum_{k=0}^{\infty} a_{k}\left(t + \sqrt{2} + 1\right)^{k+2}, a_{k+1} = \frac{a_{k}k\sqrt{2}}{4(k+3)}\right]$
- Combine solutions and rename parameters
  $\left[y(t) = a_{0}\left(\frac{(-2t-2)\sqrt{2}}{8} + \frac{t^{2}}{8} + \frac{t}{4} + \frac{3}{8}\right) + \left(\sum_{k=0}^{\infty} b_{k}\left(t + \sqrt{2} + 1\right)^{k+2}\right), b_{k+1} = \frac{b_{k}k\sqrt{2}}{4(k+3)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
```

```
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 15

```
dsolve(diff(diff(y(t),t),t)-2*(t+1)/(t^2+2*t-1)*diff(y(t),t)+2/(t^2+2*t-1)*y(t) = 0,y(t)
```

$$y = c_2 t^2 + c_1 t + c_1 + c_2$$

Mathematica DSolve solution

Solving time : 0.447 (sec)
Leaf size : 98

```
DSolve[{D[y[t],{t,2}]-2*(t+1)/(t^2+2*t-1)*D[y[t],t]+2/(t^2+2*t-1)*y[t]==0,{}},y[t],t,IncludeSi
```

$$y(t) \to \sqrt{t^2+2t-1}\exp\left(\int_1^t \frac{K[1]+2\sqrt{2}+1}{K[1](K[1]+2)-1}dK[1]\right)\left(c_2\int_1^t \exp\left(-2\int_1^{K[2]} \frac{K[1]+2\sqrt{2}+1}{K[1](K[1]+2)-1}dK[1]\right)dK[\right. + c_1\Bigg)$$

2.1.613 Problem 629

Internal problem ID [9785]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 629
Date solved : `Monday, January 27, 2025 at 06:14:15 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - 4ty' + \left(4t^2 - 2\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.058 (sec)

Writing the ode as

$$y'' - 4ty' + \left(4t^2 - 2\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4t \\ C &= 4t^2 - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1171: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4t}{1}\, dt} \\ &= z_1 e^{t^2} \\ &= z_1 \left(e^{t^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{t^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4t}{1} \, dt}}{(y_1)^2} \, dt \\ &= y_1 \int \frac{e^{2t^2}}{(y_1)^2} \, dt \\ &= y_1(t) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{t^2}\right) + c_2 \left(\mathrm{e}^{t^2}(t)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dt^2}y(t) - 4t\left(\frac{d}{dt}y(t)\right) + (4t^2 - 2)\,y(t) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dt^2}y(t)$

- Assume series solution for $y(t)$

  $y(t) = \sum_{k=0}^{\infty} a_k t^k$

- □ Rewrite ODE with series expansions
  - ○ Convert $t^m \cdot y(t)$ to series expansion for $m = 0..2$

    $t^m \cdot y(t) = \sum_{k=\max(0,-m)}^{\infty} a_k t^{k+m}$
  - ○ Shift index using $k->k-m$

    $t^m \cdot y(t) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} t^k$
  - ○ Convert $t \cdot \left(\frac{d}{dt}y(t)\right)$ to series expansion

    $t \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=0}^{\infty} a_k k\, t^k$
  - ○ Convert $\frac{d^2}{dt^2}y(t)$ to series expansion

    $\frac{d^2}{dt^2}y(t) = \sum_{k=2}^{\infty} a_k k(k-1)\, t^{k-2}$
  - ○ Shift index using $k->k+2$

    $\frac{d^2}{dt^2}y(t) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, t^k$

  Rewrite ODE with series expansions

  $2a_2 - 2a_0 + (6a_3 - 6a_1)\,t + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - 2a_k(2k+1) + 4a_{k-2}\right) t^k\right) = 0$

- The coefficients of each power of t must be 0

  $[2a_2 - 2a_0 = 0, 6a_3 - 6a_1 = 0]$

- Solve for the dependent coefficient(s)

  $\{a_2 = a_0, a_3 = a_1\}$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\,a_{k+2} - 4a_k k - 2a_k + 4a_{k-2} = 0$

- Shift index using $k->k+2$

  $$\left((k+2)^2+3k+8\right)a_{k+4}-4a_{k+2}(k+2)-2a_{k+2}+4a_k=0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y(t)=\sum_{k=0}^{\infty}a_k t^k, a_{k+4}=\frac{2(2ka_{k+2}-2a_k+5a_{k+2})}{k^2+7k+12}, a_2=a_0, a_3=a_1\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 14

```
dsolve(diff(diff(y(t),t),t)-4*t*diff(y(t),t)+(4*t^2-2)*y(t) = 0,y(t),singsol=all)
```

$$y=e^{t^2}(c_2t+c_1)$$

Mathematica DSolve solution

Solving time : 0.02 (sec)
Leaf size : 18

```
DSolve[{D[y[t],{t,2}]-4*t*D[y[t],t]+(4*t^2-2)*y[t]==0,{}},y[t],t,IncludeSingularSolutions->True
```

$$y(t)\to e^{t^2}(c_2t+c_1)$$

2.1.614 Problem 630

Internal problem ID [9786]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 630
Date solved : `Monday, January 27, 2025 at 06:14:16 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-t^2+1\right) y''-2ty'+2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.235 (sec)

Writing the ode as

$$\left(-t^2+1\right) y''-2ty'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -t^2+1 \\ B &= -2t \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{2t^2-3}{\left(t^2-1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2t^2-3 \\ t &= \left(t^2-1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t)=\left(\frac{2t^2-3}{\left(t^2-1\right)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1173: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (t^2 - 1)^2$. There is a pole at $t = 1$ of order 2. There is a pole at $t = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(t+1)^2} - \frac{1}{4(t-1)^2} - \frac{5}{4(t+1)} + \frac{5}{4(t-1)}$$

For the pole at $t = 1$ let b be the coefficient of $\frac{1}{(t-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $t = -1$ let b be the coefficient of $\frac{1}{(t+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2t^2 - 3}{(t^2 - 1)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2t^2 - 3}{(t^2 - 1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{t - c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2t - 2} + \frac{1}{2t + 2} + (0) \\
&= \frac{1}{2t - 2} + \frac{1}{2t + 2} \\
&= \frac{t}{t^2 - 1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 1$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2t-2} + \frac{1}{2t+2}\right)(1) + \left(\left(-\frac{1}{2(t-1)^2} - \frac{1}{2(t+1)^2}\right) + \left(\frac{1}{2t-2} + \frac{1}{2t+2}\right)^2 - \left(\frac{2t^2-3}{(t^2-1)^2}\right)\right) = 0$$

$$-\frac{2a_0}{t^2-1} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = t$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= pe^{\int \omega \, dt} \\
&= (t)\, e^{\int \left(\frac{1}{2t-2} + \frac{1}{2t+2}\right) dt} \\
&= (t)\sqrt{(t-1)(t+1)} \\
&= t\sqrt{t^2-1}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2t}{-t^2+1}\, dt} \\
&= z_1 e^{-\frac{\ln(t-1)}{2} - \frac{\ln(t+1)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{t-1}\,\sqrt{t+1}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{t\sqrt{t^2-1}}{\sqrt{t-1}\,\sqrt{t+1}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2t}{-t^2+1}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{-\ln(t-1) - \ln(t+1)}}{(y_1)^2}\, dt \\
&= y_1 \left(\frac{\ln(t-1)}{2} - \frac{\ln(t+1)}{2} + \frac{1}{t}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{t\sqrt{t^2-1}}{\sqrt{t-1}\,\sqrt{t+1}}\right) + c_2 \left(\frac{t\sqrt{t^2-1}}{\sqrt{t-1}\,\sqrt{t+1}} \left(\frac{\ln(t-1)}{2} - \frac{\ln(t+1)}{2} + \frac{1}{t}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(-t^2+1)\left(\frac{d^2}{dt^2}y(t)\right) - 2t\left(\frac{d}{dt}y(t)\right) + 2y(t) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dt^2}y(t)$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = \frac{2y(t)}{t^2-1} - \frac{2\left(\frac{d}{dt}y(t)\right)t}{t^2-1}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dt^2}y(t) + \frac{2\left(\frac{d}{dt}y(t)\right)t}{t^2-1} - \frac{2y(t)}{t^2-1} = 0$

□ Check to see if t_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(t) = \frac{2t}{t^2-1}, P_3(t) = -\frac{2}{t^2-1}\right]$

- ◦ $(t+1)\cdot P_2(t)$ is analytic at $t=-1$

  $\left((t+1)\cdot P_2(t)\right)\Big|_{t=-1} = 1$

- ◦ $(t+1)^2\cdot P_3(t)$ is analytic at $t=-1$

  $\left((t+1)^2\cdot P_3(t)\right)\Big|_{t=-1} = 0$

- ◦ $t=-1$is a regular singular point

  Check to see if t_0 is a regular singular point

  $t_0 = -1$

- Multiply by denominators

  $(t^2-1)\left(\frac{d^2}{dt^2}y(t)\right) + 2t\left(\frac{d}{dt}y(t)\right) - 2y(t) = 0$

- Change variables using $t = u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (2u-2)\left(\frac{d}{du}y(u)\right) - 2y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ◦ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- ◦ Shift index using $k->k+2-m$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r^2u^{-1+r} + \left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)^2 + a_k(k+r+2)\,(k+r-1)\right)u^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-2r^2 = 0$

- Values of r that satisfy the indicial equation
  $r = 0$
- Each term in the series must be 0, giving the recursion relation
  $-2a_{k+1}(k+1)^2 + a_k(k+2)(k-1) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$
- Recursion relation for $r = 0$; series terminates at $k = 1$
  $a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$
- Apply recursion relation for $k = 0$
  $a_1 = -a_0$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second linea
  $y(u) = a_0 \cdot (-u+1)$
- Revert the change of variables $u = t + 1$
  $[y(t) = -a_0 t]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 25

```
dsolve((-t^2+1)*diff(diff(y(t),t),t)-2*t*diff(y(t),t)+2*y(t) = 0,y(t),singsol=all)
```

$$y = -\frac{c_2 \ln(t+1) t}{2} + \frac{c_2 \ln(t-1) t}{2} + c_1 t + c_2$$

Mathematica DSolve solution

Solving time : 0.02 (sec)
Leaf size : 33

```
DSolve[{(1-t^2)*D[y[t],{t,2}]-2*t*D[y[t],t]+2*y[t]==0,{}},y[t],t,IncludeSingularSolutions->Tru
```

$$y(t) \to c_1 t - \frac{1}{2} c_2 (t \log(1-t) - t \log(t+1) + 2)$$

2.1.615 Problem 631

Internal problem ID [9787]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 631
Date solved : `Monday, January 27, 2025 at 06:14:16 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(t^2+1\right) y''-2ty'+2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.268 (sec)

Writing the ode as

$$\left(t^2+1\right) y''-2ty'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=t^2+1\\ B&=-2t\\ C&=2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-3}{\left(t^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=-3\\ t&=\left(t^2+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t)=\left(-\frac{3}{\left(t^2+1\right)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1175: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (t^2+1)^2$. There is a pole at $t = i$ of order 2. There is a pole at $t = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(t-i)^2} + \frac{3}{4(t+i)^2} + \frac{3i}{4(t-i)} - \frac{3i}{4(t+i)}$$

For the pole at $t = i$ let b be the coefficient of $\frac{1}{(t-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $t = -i$ let b be the coefficient of $\frac{1}{(t+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= 0\\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{(t^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right)\\ &= 1 - (1)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{t-c_2}\right) + (-)[\sqrt{r}]_\infty\\ &= -\frac{1}{2\,(t-i)} + \frac{3}{2\,(t+i)} + (-)\,(0)\\ &= -\frac{1}{2\,(t-i)} + \frac{3}{2\,(t+i)}\\ &= \frac{t-2i}{t^2+1}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d=0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(t-i)} + \frac{3}{2(t+i)}\right)(0) + \left(\left(\frac{1}{2(t-i)^2} - \frac{3}{2(t+i)^2}\right) + \left(-\frac{1}{2(t-i)} + \frac{3}{2(t+i)}\right)^2 - \left(-\frac{3}{(t^2+1}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= pe^{\int \omega \, dt} \\ &= e^{\int \left(-\frac{1}{2(t-i)} + \frac{3}{2(t+i)}\right) dt} \\ &= \frac{(t^2+1)^{3/2}}{(it+1)^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2t}{t^2+1} \, dt} \\ &= z_1 e^{\frac{\ln(t^2+1)}{2}} \\ &= z_1\left(\sqrt{t^2+1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(t^2+1)^2}{(it+1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2t}{t^2+1} \, dt}}{(y_1)^2} \, dt \\ &= y_1 \int \frac{e^{\ln(t^2+1)}}{(y_1)^2} \, dt \\ &= y_1\left(-\frac{t}{(t+i)^2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{(t^2+1)^2}{(it+1)^2}\right) + c_2\left(\frac{(t^2+1)^2}{(it+1)^2}\left(-\frac{t}{(t+i)^2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 16

```
dsolve((t^2+1)*diff(diff(y(t),t),t)-2*t*diff(y(t),t)+2*y(t) = 0,y(t),singsol=all)
```

$$y = c_2 t^2 + c_1 t - c_2$$

Mathematica DSolve solution

Solving time : 0.322 (sec)
Leaf size : 79

```
DSolve[{(1+t^2)*D[y[t],{t,2}]-2*t*D[y[t],t]+2*y[t]==0,{}},y[t],t,IncludeSingularSolutions->T
```

$$y(t) \to \sqrt{t^2+1}\exp\left(\int_1^t \frac{K[1]+2i}{K[1]^2+1}dK[1]\right)\left(c_2\int_1^t \exp\left(-2\int_1^{K[2]} \frac{K[1]+2i}{K[1]^2+1}dK[1]\right)dK[2] + c_1\right)$$

2.1.616 Problem 632

Internal problem ID [9788]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 632
Date solved : Monday, January 27, 2025 at 06:14:17 PM
CAS classification : [_Gegenbauer]

Solve

$$\left(-t^2+1\right) y''-2ty'+6y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.263 (sec)

Writing the ode as

$$\left(-t^2+1\right) y''-2ty'+6y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -t^2+1\\ B &= -2t \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{6t^2-7}{\left(t^2-1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 6t^2-7\\ t &= \left(t^2-1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t)=\left(\frac{6t^2-7}{\left(t^2-1\right)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1176: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (t^2 - 1)^2$. There is a pole at $t = 1$ of order 2. There is a pole at $t = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(t-1)^2} + \frac{13}{4(t-1)} - \frac{1}{4(t+1)^2} - \frac{13}{4(t+1)}$$

For the pole at $t = 1$ let b be the coefficient of $\frac{1}{(t-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $t = -1$ let b be the coefficient of $\frac{1}{(t+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{6t^2 - 7}{(t^2 - 1)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{6t^2 - 7}{(t^2 - 1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	3	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 3$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= 3 - (1) \\
&= 2
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{t - c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2t - 2} + \frac{1}{2t + 2} + (0) \\
&= \frac{1}{2t - 2} + \frac{1}{2t + 2} \\
&= \frac{t}{t^2 - 1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 2$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = t^2 + a_1 t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{1}{2t-2} + \frac{1}{2t+2}\right)(2t + a_1) + \left(\left(-\frac{1}{2(t-1)^2} - \frac{1}{2(t+1)^2}\right) + \left(\frac{1}{2t-2} + \frac{1}{2t+2}\right)^2 - \left(\frac{6t^2-7}{(t^2-1)^2}\right.\right.$$

$$\frac{-4a_1 t - 6a_0 - 2}{t^2-1}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{1}{3}, a_1 = 0\right\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = t^2 - \frac{1}{3}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \left(t^2 - \frac{1}{3}\right) e^{\int \left(\frac{1}{2t-2} + \frac{1}{2t+2}\right) dt} \\
&= \left(t^2 - \frac{1}{3}\right) \sqrt{(t-1)(t+1)} \\
&= \frac{(3t^2-1)\sqrt{t^2-1}}{3}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2t}{-t^2+1} \, dt} \\
&= z_1 e^{-\frac{\ln(t-1)}{2} - \frac{\ln(t+1)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{t-1}\sqrt{t+1}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(3t^2-1)\sqrt{t^2-1}}{3\sqrt{t-1}\sqrt{t+1}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2t}{-t^2+1} \, dt}}{(y_1)^2} \, dt \\
&= y_1 \int \frac{e^{-\ln(t-1) - \ln(t+1)}}{(y_1)^2} \, dt \\
&= y_1 \left(\frac{9t}{4\left(t^2 - \frac{1}{3}\right)} - \frac{9\ln(t+1)}{8} + \frac{9\ln(t-1)}{8}\right)
\end{aligned}$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{(3t^2-1)\sqrt{t^2-1}}{3\sqrt{t-1}\,\sqrt{t+1}}\right) + c_2 \left(\frac{(3t^2-1)\sqrt{t^2-1}}{3\sqrt{t-1}\,\sqrt{t+1}}\left(\frac{9t}{4\left(t^2-\frac{1}{3}\right)} - \frac{9\ln(t+1)}{8} + \frac{9\ln(t-1)}{8}\right)\right)$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$(-t^2+1)\left(\frac{d^2}{dt^2}y(t)\right) - 2t\left(\frac{d}{dt}y(t)\right) + 6y(t) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dt^2}y(t)$
- Isolate 2nd derivative
  $\frac{d^2}{dt^2}y(t) = \frac{6y(t)}{t^2-1} - \frac{2\left(\frac{d}{dt}y(t)\right)t}{t^2-1}$
- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d^2}{dt^2}y(t) + \frac{2\left(\frac{d}{dt}y(t)\right)t}{t^2-1} - \frac{6y(t)}{t^2-1} = 0$

☐ Check to see if t_0 is a regular singular point
  - Define functions
    $\left[P_2(t) = \frac{2t}{t^2-1}, P_3(t) = -\frac{6}{t^2-1}\right]$
  - $(t+1)\cdot P_2(t)$ is analytic at $t=-1$
    $\left((t+1)\cdot P_2(t)\right)\Big|_{t=-1} = 1$
  - $(t+1)^2\cdot P_3(t)$ is analytic at $t=-1$
    $\left((t+1)^2\cdot P_3(t)\right)\Big|_{t=-1} = 0$
  - $t=-1$is a regular singular point
    Check to see if t_0 is a regular singular point
    $t_0 = -1$
- Multiply by denominators
  $(t^2-1)\left(\frac{d^2}{dt^2}y(t)\right) + 2t\left(\frac{d}{dt}y(t)\right) - 6y(t) = 0$
- Change variables using $t = u-1$ so that the regular singular point is at $u=0$
  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (2u-2)\left(\frac{d}{du}y(u)\right) - 6y(u) = 0$
- Assume series solution for $y(u)$
  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$
    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$
    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0r^2u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)^2 + a_k(k+r+3)(k+r-2)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-2r^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 0$
- Each term in the series must be 0, giving the recursion relation
  $-2a_{k+1}(k+1)^2 + a_k(k+3)(k-2) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_k(k+3)(k-2)}{2(k+1)^2}$
- Recursion relation for $r = 0$; series terminates at $k = 2$
  $a_{k+1} = \frac{a_k(k+3)(k-2)}{2(k+1)^2}$
- Apply recursion relation for $k = 0$
  $a_1 = -3a_0$
- Apply recursion relation for $k = 1$
  $a_2 = -\frac{a_1}{2}$
- Express in terms of a_0
  $a_2 = \frac{3a_0}{2}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li
  $y(u) = a_0 \cdot \left(1 - 3u + \frac{3}{2}u^2\right)$
- Revert the change of variables $u = t + 1$
  $\left[y(t) = a_0\left(\frac{3t^2}{2} - \frac{1}{2}\right)\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 44

```
dsolve((-t^2+1)*diff(diff(y(t),t),t)-2*t*diff(y(t),t)+6*y(t) = 0,y(t),singsol=all)
```

$$y = \frac{c_2(3t^2-1)\ln(t-1)}{2} + \frac{(-3t^2+1)\,c_2\ln(t+1)}{2} - 3c_1t^2 + 3c_2t + c_1$$

Mathematica DSolve solution

Solving time : 0.023 (sec)
Leaf size : 55

```
DSolve[{(1-t^2)*D[y[t],{t,2}]-2*t*D[y[t],t]+6*y[t]==0,{}},y[t],t,IncludeSingularSolutions->Tru
```

$$y(t) \to \frac{1}{2}c_1\left(3t^2-1\right) - \frac{1}{4}c_2\left(\left(3t^2-1\right)\log(1-t) + \left(1-3t^2\right)\log(t+1) + 6t\right)$$

2.1.617 Problem 633

Internal problem ID [9789]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 633
Date solved : `Monday, January 27, 2025 at 06:14:18 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(2t+1)\,y''-4(t+1)\,y'+4y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.228 (sec)

Writing the ode as

$$(2t+1)\,y''+(-4t-4)\,y'+4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2t+1 \\ B &= -4t-4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{4t^2+2}{(2t+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4t^2+2 \\ t &= (2t+1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t)=\left(\frac{4t^2+2}{(2t+1)^2}\right)z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1178: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2t+1)^2$. There is a pole at $t = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{3}{4\left(t+\frac{1}{2}\right)^2} - \frac{1}{t+\frac{1}{2}}$$

For the pole at $t = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(t+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\
&= \sum_{i=0}^{0} a_i t^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r} \approx 1 - \frac{1}{2t} + \frac{1}{2t^2} - \frac{1}{4t^3} + \frac{3}{32t^4} - \frac{3}{64t^5} + \frac{1}{32t^6} - \frac{1}{64t^7} + \dots \tag{9}
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a = 1
$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\
&= 1
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = 1
$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{4t^2 + 2}{4t^2 + 4t + 1} \\
&= Q + \frac{R}{4t^2 + 4t + 1} \\
&= (1) + \left(\frac{-4t + 1}{4t^2 + 4t + 1}\right) \\
&= 1 + \frac{-4t + 1}{4t^2 + 4t + 1}
\end{aligned}
$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$
\begin{aligned}
b &= (-1) - (0) \\
&= -1
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 1 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{1} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{1} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4t^2+2}{(2t+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1}\right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{2\left(t+\frac{1}{2}\right)} + (1) \\ &= -\frac{1}{2\left(t+\frac{1}{2}\right)} + 1 \\ &= \frac{2t}{2t+1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\left(t+\frac{1}{2}\right)} + 1\right)(0) + \left(\left(\frac{1}{2\left(t+\frac{1}{2}\right)^2}\right) + \left(-\frac{1}{2\left(t+\frac{1}{2}\right)} + 1\right)^2 - \left(\frac{4t^2+2}{(2t+1)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2\left(t+\frac{1}{2}\right)}+1\right) dt} \\ &= \frac{\mathrm{e}^{t}}{\sqrt{2t+1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4t-4}{2t+1}\, dt} \\ &= z_1 e^{t+\frac{\ln(2t+1)}{2}} \\ &= z_1\left(\sqrt{2t+1}\,\mathrm{e}^{t}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{2t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4t-4}{2t+1}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{2t+\ln(2t+1)}}{(y_1)^2}\, dt \\ &= y_1\left(-\frac{(t+1)\,\mathrm{e}^{2t+\ln(2t+1)}\mathrm{e}^{-4t}}{2t+1}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{2t}\right) + c_2\left(\mathrm{e}^{2t}\left(-\frac{(t+1)\,\mathrm{e}^{2t+\ln(2t+1)}\mathrm{e}^{-4t}}{2t+1}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(2t+1)\left(\frac{d^2}{dt^2}y(t)\right)-4(t+1)\left(\frac{d}{dt}y(t)\right)+4y(t)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dt^2}y(t)$

- Isolate 2nd derivative

$\frac{d^2}{dt^2}y(t)=-\frac{4y(t)}{2t+1}+\frac{4(t+1)\left(\frac{d}{dt}y(t)\right)}{2t+1}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d^2}{dt^2}y(t)-\frac{4(t+1)\left(\frac{d}{dt}y(t)\right)}{2t+1}+\frac{4y(t)}{2t+1}=0$

□ Check to see if $t_0=-\frac{1}{2}$ is a regular singular point

- ◦ Define functions

  $\left[P_2(t)=-\frac{4(t+1)}{2t+1},P_3(t)=\frac{4}{2t+1}\right]$

- ◦ $\left(t+\frac{1}{2}\right)\cdot P_2(t)$ is analytic at $t=-\frac{1}{2}$

  $\left(\left(t+\frac{1}{2}\right)\cdot P_2(t)\right)\bigg|_{t=-\frac{1}{2}}=-1$

- ◦ $\left(t+\frac{1}{2}\right)^2\cdot P_3(t)$ is analytic at $t=-\frac{1}{2}$

  $\left(\left(t+\frac{1}{2}\right)^2\cdot P_3(t)\right)\bigg|_{t=-\frac{1}{2}}=0$

- ◦ $t=-\frac{1}{2}$is a regular singular point

Check to see if $t_0=-\frac{1}{2}$ is a regular singular point

$t_0=-\frac{1}{2}$

- Multiply by denominators

$(2t+1)\left(\frac{d^2}{dt^2}y(t)\right)+(-4t-4)\left(\frac{d}{dt}y(t)\right)+4y(t)=0$

- Change variables using $t=u-\frac{1}{2}$ so that the regular singular point is at $u=0$

$2u\left(\frac{d^2}{du^2}y(u)\right)+(-4u-2)\left(\frac{d}{du}y(u)\right)+4y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ◦ Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-1}$

- ◦ Shift index using $k->k+1$

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$

Rewrite ODE with series expansions

$2a_0r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(2a_{k+1}(k+1+r)\,(k+r-1)-4a_k(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $2r(-2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 2\}$
- Each term in the series must be 0, giving the recursion relation
  $2(a_{k+1}(k+1+r) - 2a_k)(k+r-1) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{2a_k}{k+1+r}$
- Recursion relation for $r = 0$
  $a_{k+1} = \frac{2a_k}{k+1}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{2a_k}{k+1}\right]$
- Revert the change of variables $u = t + \frac{1}{2}$
  $\left[y(t) = \sum_{k=0}^{\infty} a_k \left(t+\frac{1}{2}\right)^k, a_{k+1} = \frac{2a_k}{k+1}\right]$
- Recursion relation for $r = 2$
  $a_{k+1} = \frac{2a_k}{k+3}$
- Solution for $r = 2$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{2a_k}{k+3}\right]$
- Revert the change of variables $u = t + \frac{1}{2}$
  $\left[y(t) = \sum_{k=0}^{\infty} a_k \left(t+\frac{1}{2}\right)^{k+2}, a_{k+1} = \frac{2a_k}{k+3}\right]$
- Combine solutions and rename parameters
  $\left[y(t) = \left(\sum_{k=0}^{\infty} a_k \left(t+\frac{1}{2}\right)^k\right) + \left(\sum_{k=0}^{\infty} b_k \left(t+\frac{1}{2}\right)^{k+2}\right), a_{k+1} = \frac{2a_k}{k+1}, b_{k+1} = \frac{2b_k}{k+3}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 15

```
dsolve((2*t+1)*diff(diff(y(t),t),t)-4*(t+1)*diff(y(t),t)+4*y(t) = 0,y(t),singsol=all)
```

$$y = c_2 e^{2t} + c_1 t + c_1$$

Mathematica DSolve solution

Solving time : 0.169 (sec)
Leaf size : 88

```
DSolve[{(2*t+1)*D[y[t],{t,2}]-4*(t+1)*D[y[t],t]+4*y[t]==0,{}},y[t],t,IncludeSingularSolutions->
```

$$y(t) \to \exp\left(\int_1^t \frac{2K[1]}{2K[1]+1}dK[1] - \frac{1}{2}\int_1^t \left(-2-\frac{2}{2K[2]+1}\right)dK[2]\right)\left(c_2\int_1^t \exp\left(-2\int_1^{K[3]} \frac{2K[1]}{2K[1]+1}dK[1]\right)dK[3] + c_1\right)$$

2.1.618 Problem 634

Internal problem ID [9790]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 634
Date solved : `Monday, January 27, 2025 at 06:14:18 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$t^2y'' + ty' + \left(t^2 - \frac{1}{4}\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.141 (sec)

Writing the ode as

$$t^2y'' + ty' + \left(t^2 - \frac{1}{4}\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= t \\ C &= t^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1180: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2} \frac{t}{t^2}\, dt} \\ &= z_1 e^{-\frac{\ln(t)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{t}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(t)}{\sqrt{t}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{t}{t^2}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{-\ln(t)}}{(y_1)^2}\,dt \\ &= y_1(\tan(t)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\cos(t)}{\sqrt{t}}\right) + c_2\left(\frac{\cos(t)}{\sqrt{t}}(\tan(t))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dt^2}y(t)\right)t^2 + t\left(\frac{d}{dt}y(t)\right) + \left(t^2 - \frac{1}{4}\right)y(t) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dt^2}y(t)$
- Isolate 2nd derivative
  $\frac{d^2}{dt^2}y(t) = -\frac{(4t^2-1)y(t)}{4t^2} - \frac{\frac{d}{dt}y(t)}{t}$
- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dt^2}y(t) + \frac{\frac{d}{dt}y(t)}{t} + \frac{(4t^2-1)y(t)}{4t^2} = 0$
- □ Check to see if $t_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(t) = \frac{1}{t}, P_3(t) = \frac{4t^2-1}{4t^2}\right]$
  - $t \cdot P_2(t)$ is analytic at $t = 0$
    $\left.(t \cdot P_2(t))\right|_{t=0} = 1$
  - $t^2 \cdot P_3(t)$ is analytic at $t = 0$
    $\left.(t^2 \cdot P_3(t))\right|_{t=0} = -\frac{1}{4}$
  - $t = 0$is a regular singular point
    Check to see if $t_0 = 0$ is a regular singular point
    $t_0 = 0$
- Multiply by denominators
  $4\left(\frac{d^2}{dt^2}y(t)\right)t^2 + 4t\left(\frac{d}{dt}y(t)\right) + (4t^2-1)y(t) = 0$
- Assume series solution for $y(t)$
  $y(t) = \sum_{k=0}^{\infty} a_k t^{k+r}$
- □ Rewrite ODE with series expansions

- Convert $t^m \cdot y(t)$ to series expansion for $m = 0..2$

  $t^m \cdot y(t) = \sum_{k=0}^{\infty} a_k t^{k+r+m}$
- Shift index using $k->k-m$

  $t^m \cdot y(t) = \sum_{k=m}^{\infty} a_{k-m} t^{k+r}$
- Convert $t \cdot \left(\frac{d}{dt} y(t)\right)$ to series expansion

  $t \cdot \left(\frac{d}{dt} y(t)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, t^{k+r}$
- Convert $t^2 \cdot \left(\frac{d^2}{dt^2} y(t)\right)$ to series expansion

  $t^2 \cdot \left(\frac{d^2}{dt^2} y(t)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)(-1+2r)\, t^r + a_1(3+2r)(1+2r)\, t^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+1)(2k+2r-1) + 4a_{k-2}\right) t^k$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0

  $a_1(3+2r)(1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$
- Shift index using $k->k+2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$
- Solution for $r = -\frac{1}{2}$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$
- Solution for $r = \frac{1}{2}$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(t) = \left(\sum_{k=0}^{\infty} a_k t^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.040 (sec)
Leaf size : 17

```
dsolve(t^2*diff(diff(y(t),t),t)+t*diff(y(t),t)+(t^2-1/4)*y(t) = 0,y(t),singsol=all)
```

$$y = \frac{c_2 \cos(t) + c_1 \sin(t)}{\sqrt{t}}$$

Mathematica DSolve solution

Solving time : 0.034 (sec)
Leaf size : 39

```
DSolve[{t^2*D[y[t],{t,2}]+t*D[y[t],t]+(t^2-1/4)*y[t]==0,{}},y[t],t,IncludeSingularSolutions-
```

$$y(t) \to \frac{e^{-it}(2c_1 - ic_2e^{2it})}{2\sqrt{t}}$$

2.1.619 Problem 635

Internal problem ID [9791]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 635
Date solved : `Monday, January 27, 2025 at 06:14:19 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - \frac{2ty'}{t^2+1} + \frac{2y}{t^2+1} = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.270 (sec)

Writing the ode as

$$y'' - \frac{2ty'}{t^2+1} + \frac{2y}{t^2+1} = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -\frac{2t}{t^2+1} \\ C &= \frac{2}{t^2+1} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{(t^2+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= (t^2+1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(-\frac{3}{\left(t^2+1\right)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1182: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = \left(t^2+1\right)^2$. There is a pole at $t = i$ of order 2. There is a pole at $t = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4\,(t-i)^2} + \frac{3}{4\,(t+i)^2} + \frac{3i}{4\,(t-i)} - \frac{3i}{4\,(t+i)}$$

For the pole at $t = i$ let b be the coefficient of $\frac{1}{(t-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $t = -i$ let b be the coefficient of $\frac{1}{(t+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{3}{(t^2+1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{t-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(t-i)} + \frac{3}{2(t+i)} + (-)(0) \\
&= -\frac{1}{2(t-i)} + \frac{3}{2(t+i)} \\
&= \frac{t-2i}{t^2+1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(t-i)} + \frac{3}{2(t+i)}\right)(0) + \left(\left(\frac{1}{2(t-i)^2} - \frac{3}{2(t+i)^2}\right) + \left(-\frac{1}{2(t-i)} + \frac{3}{2(t+i)}\right)^2 - \left(-\frac{}{(t^2}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= pe^{\int \omega\, dt} \\ &= e^{\int \left(-\frac{1}{2(t-i)} + \frac{3}{2(t+i)}\right) dt} \\ &= \frac{(t^2+1)^{3/2}}{(it+1)^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-\frac{2t}{t^2+1}}{1}\, dt} \\ &= z_1 e^{\frac{\ln(t^2+1)}{2}} \\ &= z_1\left(\sqrt{t^2+1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(t^2+1)^2}{(it+1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-\frac{2t}{t^2+1}}{1}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{\ln(t^2+1)}}{(y_1)^2}\, dt \\ &= y_1\left(-\frac{t}{(t+i)^2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{(t^2+1)^2}{(it+1)^2}\right) + c_2\left(\frac{(t^2+1)^2}{(it+1)^2}\left(-\frac{t}{(t+i)^2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 16

```
dsolve(diff(diff(y(t),t),t)-2*t/(t^2+1)*diff(y(t),t)+2/(t^2+1)*y(t) = 0,y(t),singsol=all
```

$$y = c_2 t^2 + c_1 t - c_2$$

Mathematica DSolve solution

Solving time : 0.308 (sec)
Leaf size : 79

```
DSolve[{D[y[t],{t,2}]-2*t/(1+t^2)*D[y[t],t]+2/(1+t^2)*y[t]==0,{}},y[t],t,IncludeSingularSoluti
```

$$y(t) \to \sqrt{t^2+1} \exp\left(\int_1^t \frac{K[1]+2i}{K[1]^2+1} dK[1]\right) \left(c_2 \int_1^t \exp\left(-2\int_1^{K[2]} \frac{K[1]+2i}{K[1]^2+1} dK[1]\right) dK[2] + c_1\right)$$

2.1.620 Problem 636

Internal problem ID [9792]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 636
Date solved : `Monday, January 27, 2025 at 06:14:20 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + \left(t^2+2t+1\right) y' - (4+4t)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.260 (sec)

Writing the ode as

$$y'' + (1+t)^2 y' + (-4-4t)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= (1+t)^2 \\ C &= -4-4t \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^4+4t^3+6t^2+24t+21}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^4+4t^3+6t^2+24t+21 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{21}{4} + 6t + \frac{1}{4}t^4 + t^3 + \frac{3}{2}t^2\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1183: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 4 \\ &= -4 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{2} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{t^2}{2} + t + \frac{1}{2} + \frac{5}{t} - \frac{5}{t^2} + \frac{5}{t^3} - \frac{30}{t^4} + \frac{105}{t^5} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 2$ gives

$$[\sqrt{r}]_\infty = \sum_{i=0}^{2} a_i t^i = \frac{1}{2}t^2 + t + \frac{1}{2} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^1 = t$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}t^4 + t^3 + \frac{3}{2}t^2 + t + \frac{1}{4}$$

This shows that the coefficient of t in the above is 1. Now we need to find the coefficient of t in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of t in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^4 + 4t^3 + 6t^2 + 24t + 21}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{21}{4} + 6t + \frac{1}{4}t^4 + t^3 + \frac{3}{2}t^2\right) + (0) \\ &= \frac{21}{4} + 6t + \frac{1}{4}t^4 + t^3 + \frac{3}{2}t^2 \end{aligned}$$

We see that the coefficient of the term $\frac{1}{t}$ in the quotient is 6. Now b can be found.

$$\begin{aligned} b &= (6) - (1) \\ &= 5 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2}t^2 + t + \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{5}{\frac{1}{2}} - 2\right) = 4 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{5}{\frac{1}{2}} - 2\right) = -6 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{21}{4} + 6t + \frac{1}{4}t^4 + t^3 + \frac{3}{2}t^2$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-4	$\frac{1}{2}t^2 + t + \frac{1}{2}$	4	-6

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 4$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 4 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{1}{2}t^2 + t + \frac{1}{2}\right) \\ &= \frac{1}{2}t^2 + t + \frac{1}{2} \\ &= \frac{(1+t)^2}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 4$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(t) = t^4 + a_3t^3 + a_2t^2 + a_1t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(12t^2 + 6ta_3 + 2a_2) + 2\left(\frac{1}{2}t^2 + t + \frac{1}{2}\right)(4t^3 + 3t^2a_3 + 2ta_2 + a_1) + \left((1+t) + \left(\frac{1}{2}t^2 + t + \frac{1}{2}\right)^2 - \left(\frac{21}{4} + 6t\right.\right.$$

$$(-a_3 + 4)\,t^4 + 2(2 - a_2 + a_3)\,t^3 + 3(4 - a_1 + a_3)\,t^2 + 2(-2a_0 - a_1 + a_2 + 3a_3)\,t -$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 5, a_1 = 8, a_2 = 6, a_3 = 4\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = t^4 + 4t^3 + 6t^2 + 8t + 5$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= pe^{\int \omega\, dt} \\ &= (t^4 + 4t^3 + 6t^2 + 8t + 5)\, e^{\int (\frac{1}{2}t^2 + t + \frac{1}{2})dt} \\ &= (t^4 + 4t^3 + 6t^2 + 8t + 5)\, e^{\frac{(1+t)^3}{6}} \\ &= (1+t)\,(t^3 + 3t^2 + 3t + 5)\, e^{\frac{(1+t)^3}{6}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{(1+t)^2}{1}\, dt} \\ &= z_1 e^{-\frac{(1+t)^3}{6}} \\ &= z_1\left(e^{-\frac{(1+t)^3}{6}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (1+t)\left(t^3+3t^2+3t+5\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{(1+t)^2}{1}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-\frac{(1+t)^3}{3}}}{(y_1)^2}\,dt \\
&= y_1 \left(\int \frac{e^{-\frac{(1+t)^3}{3}}}{(1+t)^2\left(t^3+3t^2+3t+5\right)^2}dt\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((1+t)\left(t^3+3t^2+3t+5\right)\right) \\
&\quad + c_2\left((1+t)\left(t^3+3t^2+3t+5\right)\left(\int \frac{e^{-\frac{(1+t)^3}{3}}}{(1+t)^2\left(t^3+3t^2+3t+5\right)^2}dt\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dt^2}y(t) + (t^2+2t+1)\left(\frac{d}{dt}y(t)\right) - (4t+4)\,y(t) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dt^2}y(t)$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = -(t^2+2t+1)\left(\frac{d}{dt}y(t)\right) + (4t+4)\,y(t)$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dt^2}y(t) + (t^2+2t+1)\left(\frac{d}{dt}y(t)\right) + (-4t-4)\,y(t) = 0$

- Assume series solution for $y(t)$

  $y(t) = \sum_{k=0}^{\infty} a_k t^k$

□ Rewrite ODE with series expansions

  ◦ Convert $t^m \cdot y(t)$ to series expansion for $m = 0..1$

  $t^m \cdot y(t) = \sum_{k=\max(0,-m)}^{\infty} a_k t^{k+m}$

  ◦ Shift index using $k->k-m$

  $t^m \cdot y(t) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} t^k$

  ◦ Convert $t^m \cdot \left(\frac{d}{dt}y(t)\right)$ to series expansion for $m = 0..2$

$$t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, t^{k-1+m}$$

- Shift index using $k- >k+1-m$

$$t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, t^k$$

- Convert $\frac{d^2}{dt^2}y(t)$ to series expansion

$$\frac{d^2}{dt^2}y(t) = \sum_{k=2}^{\infty} a_k k(k-1)\, t^{k-2}$$

- Shift index using $k- >k+2$

$$\frac{d^2}{dt^2}y(t) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, t^k$$

Rewrite ODE with series expansions

$$2a_2 + a_1 - 4a_0 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_{k+1}(k+1) + 2a_k(k-2) + a_{k-1}(k-5)\right) t^k\right) = 0$$

- Each term must be 0

$$2a_2 + a_1 - 4a_0 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$k^2 a_{k+2} + (2a_k + a_{k-1} + a_{k+1} + 3a_{k+2})\, k - 4a_k - 5a_{k-1} + a_{k+1} + 2a_{k+2} = 0$$

- Shift index using $k- >k+1$

$$(k+1)^2\, a_{k+3} + (2a_{k+1} + a_k + a_{k+2} + 3a_{k+3})\,(k+1) - 4a_{k+1} - 5a_k + a_{k+2} + 2a_{k+3} = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y(t) = \sum_{k=0}^{\infty} a_k t^k, a_{k+3} = -\frac{a_k k + 2a_{k+1} k + k a_{k+2} - 4a_k - 2a_{k+1} + 2a_{k+2}}{k^2+5k+6}, 2a_2 + a_1 - 4a_0 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Mo
      <- Heun successful: received ODE is equivalent to the  HeunT  ODE, case  c = 0
   Special function solution also has integrals. Returning default Liouvillian solution.
```

```
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.291 (sec)
Leaf size : 60

```
dsolve(diff(diff(y(t),t),t)+(t^2+2*t+1)*diff(y(t),t)-(4*t+4)*y(t) = 0,y(t),singsol=all
```

$$y = (t+1)\left(t^3+3t^2+3t+5\right)\left(c_2\left(\int \frac{\mathrm{e}^{-\frac{t\left(t^2+3t+3\right)}{3}}}{(t+1)^2\left(t^3+3t^2+3t+5\right)^2}dt\right)+c_1\right)$$

Mathematica DSolve solution

Solving time : 0.373 (sec)
Leaf size : 78

```
DSolve[{D[y[t],{t,2}]+(t^2+2*t+1)*D[y[t],t]-(4+4*t)*y[t]==0,{}},y[t],t,IncludeSingularSoluti
```

$$y(t)\to (t+1)\left(t^3+3t^2+3t+5\right)\left(c_2\int_1^t \frac{e^{-\frac{1}{3}K[1]\left(K[1]^2+3K[1]+3\right)}}{(K[1]+1)^2\left(K[1]^3+3K[1]^2+3K[1]+5\right)^2}dK[1]+c_1\right)$$

2.1.621 Problem 638

Internal problem ID [9793]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 638
Date solved : `Monday, January 27, 2025 at 06:14:20 PM`
CAS classification : `[_Laguerre]`

Solve

$$2ty'' + (1 - 2t)\, y' - y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.215 (sec)

Writing the ode as

$$2ty'' + (1 - 2t)\, y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2t \\ B &= 1 - 2t \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4t^2 + 4t - 3}{16t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4t^2 + 4t - 3 \\ t &= 16t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{4t^2 + 4t - 3}{16t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1185: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{1}{4t} - \frac{3}{16t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{4t} - \frac{1}{4t^2} + \frac{1}{8t^3} - \frac{1}{8t^4} + \frac{1}{8t^5} - \frac{9}{64t^6} + \frac{21}{128t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{4t^2 + 4t - 3}{16t^2} \\ &= Q + \frac{R}{16t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{4t - 3}{16t^2}\right) \\ &= \frac{1}{4} + \frac{4t - 3}{16t^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is 4. Dividing this by leading coefficient in t which is 16 gives $\frac{1}{4}$. Now b can be found.

$$\begin{aligned}b &= \left(\frac{1}{4}\right) - (0) \\ &= \frac{1}{4}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{4}}{\frac{1}{2}} - 0\right) &&= \frac{1}{4} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{4}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{4}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4t^2 + 4t - 3}{16t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{4} - \left(\frac{1}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{4t} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} + \frac{1}{4t} \\
&= \frac{1}{2} + \frac{1}{4t}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2} + \frac{1}{4t}\right)(0) + \left(\left(-\frac{1}{4t^2}\right) + \left(\frac{1}{2} + \frac{1}{4t}\right)^2 - \left(\frac{4t^2+4t-3}{16t^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= e^{\int \left(\frac{1}{2} + \frac{1}{4t}\right) dt} \\
&= t^{1/4} e^{\frac{t}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1-2t}{2t}\,dt} \\ &= z_1 e^{\frac{t}{2}-\frac{\ln(t)}{4}} \\ &= z_1 \left(\frac{e^{\frac{t}{2}}}{t^{1/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1-2t}{2t}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{t-\frac{\ln(t)}{2}}}{(y_1)^2}\,dt \\ &= y_1\left(\sqrt{\pi}\,\operatorname{erf}\left(\sqrt{t}\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(e^t\right) + c_2\left(e^t\left(\sqrt{\pi}\,\operatorname{erf}\left(\sqrt{t}\right)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2\left(\frac{d^2}{dt^2}y(t)\right)t + (1-2t)\left(\frac{d}{dt}y(t)\right) - y(t) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dt^2}y(t)$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = \frac{y(t)}{2t} + \frac{(2t-1)\left(\frac{d}{dt}y(t)\right)}{2t}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d^2}{dt^2}y(t) - \frac{(2t-1)\left(\frac{d}{dt}y(t)\right)}{2t} - \frac{y(t)}{2t} = 0$

☐ Check to see if $t_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(t) = -\frac{2t-1}{2t}, P_3(t) = -\frac{1}{2t}\right]$

  ◦ $t \cdot P_2(t)$ is analytic at $t = 0$

$(t \cdot P_2(t))\Big|_{t=0} = \frac{1}{2}$

- ◦ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $(t^2 \cdot P_3(t))\Big|_{t=0} = 0$

- ◦ $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $2\left(\frac{d^2}{dt^2}y(t)\right)t + (1-2t)\left(\frac{d}{dt}y(t)\right) - y(t) = 0$

- Assume series solution for $y(t)$

  $y(t) = \sum_{k=0}^{\infty} a_k t^{k+r}$

- □ Rewrite ODE with series expansions

  - ◦ Convert $t^m \cdot \left(\frac{d}{dt}y(t)\right)$ to series expansion for $m = 0..1$

    $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$

  - ◦ Shift index using $k- >k+1-m$

    $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$

  - ◦ Convert $t \cdot \left(\frac{d^2}{dt^2}y(t)\right)$ to series expansion

    $t \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, t^{k+r-1}$

  - ◦ Shift index using $k- >k+1$

    $t \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-1+2r)\, t^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(2k+2r+1) - a_k(2k+2r+1)\right) t^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2(a_{k+1}(k+1+r) - a_k)\left(k+r+\frac{1}{2}\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1+r}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 0$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^k, a_{k+1} = \frac{a_k}{k+1}\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = \frac{a_k}{k+\frac{3}{2}}$

- Solution for $r = \frac{1}{2}$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k}{k+\frac{3}{2}}\right]$

- Combine solutions and rename parameters

$$\left[y(t)=\left(\sum_{k=0}^{\infty}a_k t^k\right)+\left(\sum_{k=0}^{\infty}b_k t^{k+\frac{1}{2}}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+\frac{3}{2}}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.032 (sec)
Leaf size : 15

```
dsolve(2*t*diff(diff(y(t),t),t)+(-2*t+1)*diff(y(t),t)-y(t) = 0,y(t),singsol=all)
```

$$y=\mathrm{e}^{t}\left(\operatorname{erf}\left(\sqrt{t}\right)c_1+c_2\right)$$

Mathematica DSolve solution

Solving time : 0.032 (sec)
Leaf size : 21

```
DSolve[{2*t*D[y[t],{t,2}]+(1-2*t)*D[y[t],t]-y[t]==0,{}},y[t],t,IncludeSingularSolutions->True]
```

$$y(t)\to e^t\left(c_1-c_2\Gamma\left(\frac{1}{2},t\right)\right)$$

2.1.622 Problem 639

Internal problem ID [9794]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 639
Date solved : `Monday, January 27, 2025 at 06:14:21 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2ty'' + (1+t)\,y' - 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.453 (sec)

Writing the ode as

$$2ty'' + (1+t)\,y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2t \\ B &= 1+t \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 + 18t - 3}{16t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^2 + 18t - 3 \\ t &= 16t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 + 18t - 3}{16t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1187: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16} - \frac{3}{16t^2} + \frac{9}{8t}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{4} + \frac{9}{4t} - \frac{21}{2t^2} + \frac{189}{2t^3} - \frac{1071}{t^4} + \frac{13608}{t^5} - \frac{370629}{2t^6} + \frac{5288409}{2t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{4} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{16}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 + 18t - 3}{16t^2} \\ &= Q + \frac{R}{16t^2} \\ &= \left(\frac{1}{16}\right) + \left(\frac{18t - 3}{16t^2}\right) \\ &= \frac{1}{16} + \frac{18t - 3}{16t^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is 18. Dividing this by leading coefficient in t which is 16 gives $\frac{9}{8}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{9}{8}\right) - (0) \\ &= \frac{9}{8} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{4} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{9}{8}}{\frac{1}{4}} - 0\right) &&= \frac{9}{4} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{9}{8}}{\frac{1}{4}} - 0\right) &&= -\frac{9}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 + 18t - 3}{16t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{4}$	$\frac{9}{4}$	$-\frac{9}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{9}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\ &= \frac{9}{4} - \left(\frac{1}{4}\right) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1}\right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{4t} + \left(\frac{1}{4}\right) \\ &= \frac{1}{4t} + \frac{1}{4} \\ &= \frac{1+t}{4t} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 2$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(t) = t^2 + a_1 t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{1}{4t} + \frac{1}{4}\right)(2t + a_1) + \left(\left(-\frac{1}{4t^2}\right) + \left(\frac{1}{4t} + \frac{1}{4}\right)^2 - \left(\frac{t^2 + 18t - 3}{16t^2}\right)\right) = 0$$

$$\frac{(-a_1 + 6)t - 2a_0 + a_1}{2t} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 3, a_1 = 6\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = t^2 + 6t + 3$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= pe^{\int \omega\, dt} \\
&= \left(t^2+6t+3\right) \mathrm{e}^{\int \left(\frac{1}{4t}+\frac{1}{4}\right)dt} \\
&= \left(t^2+6t+3\right) \mathrm{e}^{\frac{t}{4}+\frac{\ln(t)}{4}} \\
&= \left(t^2+6t+3\right) t^{1/4} \mathrm{e}^{\frac{t}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1+t}{2t}\, dt} \\
&= z_1 e^{-\frac{t}{4}-\frac{\ln(t)}{4}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{t}{4}}}{t^{1/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = t^2+6t+3$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1+t}{2t}\, dt}}{\left(y_1\right)^2}\, dt \\
&= y_1 \int \frac{e^{-\frac{t}{2}-\frac{\ln(t)}{2}}}{\left(y_1\right)^2}\, dt \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{t}{2}-\frac{\ln(t)}{2}}}{\left(t^2+6t+3\right)^2} dt\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(t^2+6t+3\right) + c_2 \left(t^2+6t+3 \left(\int \frac{\mathrm{e}^{-\frac{t}{2}-\frac{\ln(t)}{2}}}{\left(t^2+6t+3\right)^2} dt\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2\left(\frac{d^2}{dt^2}y(t)\right)t+(t+1)\left(\frac{d}{dt}y(t)\right)-2y(t)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dt^2}y(t)$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t)=\frac{y(t)}{t}-\frac{(t+1)\left(\frac{d}{dt}y(t)\right)}{2t}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d^2}{dt^2}y(t)+\frac{(t+1)\left(\frac{d}{dt}y(t)\right)}{2t}-\frac{y(t)}{t}=0$

- ☐ Check to see if $t_0=0$ is a regular singular point
  - Define functions

    $\left[P_2(t)=\frac{t+1}{2t}, P_3(t)=-\frac{1}{t}\right]$
  - $t\cdot P_2(t)$ is analytic at $t=0$

    $\left(t\cdot P_2(t)\right)\Big|_{t=0}=\frac{1}{2}$
  - $t^2\cdot P_3(t)$ is analytic at $t=0$

    $\left(t^2\cdot P_3(t)\right)\Big|_{t=0}=0$
  - $t=0$is a regular singular point

  Check to see if $t_0=0$ is a regular singular point

  $t_0=0$

- Multiply by denominators

  $2\left(\frac{d^2}{dt^2}y(t)\right)t+(t+1)\left(\frac{d}{dt}y(t)\right)-2y(t)=0$

- Assume series solution for $y(t)$

  $y(t)=\sum_{k=0}^{\infty}a_k t^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $t^m\cdot\left(\frac{d}{dt}y(t)\right)$ to series expansion for $m=0..1$

    $t^m\cdot\left(\frac{d}{dt}y(t)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,t^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $t^m\cdot\left(\frac{d}{dt}y(t)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,t^{k+r}$
  - Convert $t\cdot\left(\frac{d^2}{dt^2}y(t)\right)$ to series expansion

    $t\cdot\left(\frac{d^2}{dt^2}y(t)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,t^{k+r-1}$
  - Shift index using $k->k+1$

    $t\cdot\left(\frac{d^2}{dt^2}y(t)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,t^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-1+2r)\,t^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(2k+1+2r)+a_k(k+r-2)\right)t^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+2r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{0,\frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation
  $2(k+1+r)\left(k+r+\frac{1}{2}\right)a_{k+1}+a_k(k+r-2)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=-\frac{a_k(k+r-2)}{(k+1+r)(2k+1+2r)}$
- Recursion relation for $r=0$; series terminates at $k=2$
  $a_{k+1}=-\frac{a_k(k-2)}{(k+1)(2k+1)}$
- Apply recursion relation for $k=0$
  $a_1=2a_0$
- Apply recursion relation for $k=1$
  $a_2=\frac{a_1}{6}$
- Express in terms of a_0
  $a_2=\frac{a_0}{3}$
- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li
  $y(t)=a_0\cdot\left(1+2t+\frac{1}{3}t^2\right)$
- Recursion relation for $r=\frac{1}{2}$
  $a_{k+1}=-\frac{a_k\left(k-\frac{3}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}$
- Solution for $r=\frac{1}{2}$
  $\left[y(t)=\sum_{k=0}^{\infty}a_k t^{k+\frac{1}{2}}, a_{k+1}=-\frac{a_k\left(k-\frac{3}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$
- Combine solutions and rename parameters
  $\left[y(t)=a_0\cdot\left(1+2t+\frac{1}{3}t^2\right)+\left(\sum_{k=0}^{\infty}b_k t^{k+\frac{1}{2}}\right), b_{k+1}=-\frac{b_k\left(k-\frac{3}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special functio
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.040 (sec)
Leaf size : 56

```
dsolve(2*t*diff(diff(y(t),t),t)+(t+1)*diff(y(t),t)-2*y(t) = 0,y(t),singsol=all)
```

$$y = c_1\sqrt{\pi}\,(t^2+6t+3)\,\mathrm{erf}\left(\frac{\sqrt{2}\,\sqrt{t}}{2}\right)+5c_1\sqrt{2}\left(\sqrt{t}+\frac{t^{3/2}}{5}\right)\mathrm{e}^{-\frac{t}{2}}+c_2(t^2+6t+3)$$

Mathematica DSolve solution

Solving time : 0.308 (sec)
Leaf size : 58

```
DSolve[{2*t*D[y[t],{t,2}]+(1+t)*D[y[t],t]-2*y[t]==0,{}},y[t],t,IncludeSingularSolutions->True]
```

$$y(t)\to (t^2+6t+3)\left(c_2\int_1^t \frac{e^{-\frac{K[1]}{2}-\frac{1}{2}}}{\sqrt{K[1]}\,(K[1]^2+6K[1]+3)^2}dK[1]+c_1\right)$$

2.1.623 Problem 640

Internal problem ID [9795]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 640
Date solved : Monday, January 27, 2025 at 06:14:22 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$2t^2y'' - ty' + (1+t)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.214 (sec)

Writing the ode as

$$2t^2y'' - ty' + (1+t)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2t^2 \\ B &= -t \\ C &= 1+t \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3-8t}{16t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3-8t \\ t &= 16t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{-3-8t}{16t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1189: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16t^2$. There is a pole at $t = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16t^2} - \frac{1}{2t}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 1, \; e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(t)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(1 - (1)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{t-c} \\ &= \frac{1}{2}\left(\frac{1}{(t-(0))}\right) \\ &= \frac{1}{2t} \end{aligned}$$

Now we search for a monic polynomial $p(t)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(t)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2t} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2t} + \frac{1+8t}{16t^2} = 0$$

Solving for ω gives

$$\omega = \frac{1 + 2\sqrt{2}\,\sqrt{-t}}{4t}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= e^{\int \omega\, dt} \\ &= e^{\int \frac{1+2\sqrt{2}\,\sqrt{-t}}{4t} dt} \\ &= t^{1/4} e^{\sqrt{2}\,\sqrt{-t}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-t}{2t^2}\,dt} \\ &= z_1 e^{\frac{\ln(t)}{4}} \\ &= z_1\left(t^{1/4}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{t}\,\mathrm{e}^{\sqrt{2}\,\sqrt{-t}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-t}{2t^2}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{\frac{\ln(t)}{2}}}{(y_1)^2}\,dt \\ &= y_1\left(-\frac{\sqrt{2}\,\sqrt{-t}\left(1-\mathrm{e}^{-2\sqrt{2}\,\sqrt{-t}}\right)}{2\sqrt{t}}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\sqrt{t}\,\mathrm{e}^{\sqrt{2}\,\sqrt{-t}}\right) + c_2\left(\sqrt{t}\,\mathrm{e}^{\sqrt{2}\,\sqrt{-t}}\left(-\frac{\sqrt{2}\,\sqrt{-t}\left(1-\mathrm{e}^{-2\sqrt{2}\,\sqrt{-t}}\right)}{2\sqrt{t}}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$2\left(\frac{d^2}{dt^2}y(t)\right)t^2 - t\left(\frac{d}{dt}y(t)\right) + (t+1)\,y(t) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dt^2}y(t)$
- Isolate 2nd derivative
  $\frac{d^2}{dt^2}y(t) = -\frac{(t+1)y(t)}{2t^2} + \frac{\frac{d}{dt}y(t)}{2t}$
- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d^2}{dt^2}y(t) - \frac{\frac{d}{dt}y(t)}{2t} + \frac{(t+1)y(t)}{2t^2} = 0$

☐ Check to see if $t_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(t) = -\frac{1}{2t}, P_3(t) = \frac{t+1}{2t^2}\right]$
  - $t \cdot P_2(t)$ is analytic at $t = 0$

$(t \cdot P_2(t))\Big|_{t=0} = -\frac{1}{2}$

- $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $(t^2 \cdot P_3(t))\Big|_{t=0} = \frac{1}{2}$

- $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $2\left(\frac{d^2}{dt^2}y(t)\right)t^2 - t\left(\frac{d}{dt}y(t)\right) + (t+1)\,y(t) = 0$

- Assume series solution for $y(t)$

  $y(t) = \sum_{k=0}^{\infty} a_k t^{k+r}$

□ Rewrite ODE with series expansions

- Convert $t^m \cdot y(t)$ to series expansion for $m = 0..1$

  $t^m \cdot y(t) = \sum_{k=0}^{\infty} a_k t^{k+r+m}$

- Shift index using $k- > k - m$

  $t^m \cdot y(t) = \sum_{k=m}^{\infty} a_{k-m} t^{k+r}$

- Convert $t \cdot \left(\frac{d}{dt}y(t)\right)$ to series expansion

  $t \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, t^{k+r}$

- Convert $t^2 \cdot \left(\frac{d^2}{dt^2}y(t)\right)$ to series expansion

  $t^2 \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)\,(-1+r)\,t^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)\,(k+r-1) + a_{k-1}\right) t^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)\,(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{1, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k+r-\frac{1}{2}\right)(k+r-1)\,a_k + a_{k-1} = 0$

- Shift index using $k- > k + 1$

  $2\left(k+\frac{1}{2}+r\right)(k+r)\,a_{k+1} + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{(2k+1+2r)(k+r)}$

- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{a_k}{(2k+3)(k+1)}$

- Solution for $r = 1$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = -\frac{a_k}{(2k+3)(k+1)}\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = -\frac{a_k}{(2k+2)\left(k+\frac{1}{2}\right)}$

- Solution for $r = \frac{1}{2}$

$$\left[y(t)=\sum_{k=0}^{\infty} a_k t^{k+\frac{1}{2}}, a_{k+1}=-\frac{a_k}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y(t)=\left(\sum_{k=0}^{\infty} a_k t^{k+1}\right)+\left(\sum_{k=0}^{\infty} b_k t^{k+\frac{1}{2}}\right), a_{k+1}=-\frac{a_k}{(2k+3)(k+1)}, b_{k+1}=-\frac{b_k}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 29

```
dsolve(2*t^2*diff(diff(y(t),t),t)-t*diff(y(t),t)+(t+1)*y(t) = 0,y(t),singsol=all)
```

$$y=\sqrt{t}\left(c_1 \sin\left(\sqrt{2}\,\sqrt{t}\right)+c_2 \cos\left(\sqrt{2}\,\sqrt{t}\right)\right)$$

Mathematica DSolve solution

Solving time : 0.065 (sec)
Leaf size : 62

```
DSolve[{2*t^2*D[y[t],{t,2}]-t*D[y[t],t]+(1+t)*y[t]==0,{}},y[t],t,IncludeSingularSolutions->True
```

$$y(t)\to \frac{1}{2}e^{-i\sqrt{2}\sqrt{t}}\sqrt{t}\left(2c_1 e^{2i\sqrt{2}\sqrt{t}}+i\sqrt{2}c_2\right)$$

2.1.624 Problem 641

Internal problem ID [9796]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 641
Date solved : `Monday, January 27, 2025 at 06:14:22 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2t^2y'' + \left(t^2 - t\right)y' + y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.237 (sec)

Writing the ode as

$$2t^2y'' + \left(t^2 - t\right)y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2t^2 \\ B &= t^2 - t \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 - 2t - 3}{16t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^2 - 2t - 3 \\ t &= 16t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 - 2t - 3}{16t^2}\right)z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1191: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16} - \frac{3}{16t^2} - \frac{1}{8t}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{4} - \frac{1}{4t} - \frac{1}{2t^2} - \frac{1}{2t^3} - \frac{1}{t^4} - \frac{2}{t^5} - \frac{9}{2t^6} - \frac{21}{2t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{4} \end{aligned}\tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{16}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 - 2t - 3}{16t^2} \\ &= Q + \frac{R}{16t^2} \\ &= \left(\frac{1}{16}\right) + \left(\frac{-2t - 3}{16t^2}\right) \\ &= \frac{1}{16} + \frac{-2t - 3}{16t^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 16 gives $-\frac{1}{8}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{8}\right) - (0) \\ &= -\frac{1}{8} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{4} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{8}}{\frac{1}{4}} - 0\right) &&= -\frac{1}{4} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{8}}{\frac{1}{4}} - 0\right) &&= \frac{1}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 - 2t - 3}{16t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{4} - \left(\frac{1}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{4t} + (-)\left(\frac{1}{4}\right) \\ &= \frac{1}{4t} - \frac{1}{4} \\ &= -\frac{t-1}{4t} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{4t} - \frac{1}{4}\right)(0) + \left(\left(-\frac{1}{4t^2}\right) + \left(\frac{1}{4t} - \frac{1}{4}\right)^2 - \left(\frac{t^2-2t-3}{16t^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= \mathrm{e}^{\int \left(\frac{1}{4t} - \frac{1}{4}\right) dt} \\ &= t^{1/4} \mathrm{e}^{-\frac{t}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{t^2-t}{2t^2}\,dt} \\
&= z_1 e^{-\frac{t}{4}+\frac{\ln(t)}{4}} \\
&= z_1 \left(t^{1/4} \mathrm{e}^{-\frac{t}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \sqrt{t}\,\mathrm{e}^{-\frac{t}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{t^2-t}{2t^2}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-\frac{t}{2}+\frac{\ln(t)}{2}}}{(y_1)^2}\,dt \\
&= y_1 \left(-i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,\sqrt{t}}{2}\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{t}\,\mathrm{e}^{-\frac{t}{2}}\right) + c_2\left(\sqrt{t}\,\mathrm{e}^{-\frac{t}{2}}\left(-i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,\sqrt{t}}{2}\right)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2\left(\frac{d^2}{dt^2}y(t)\right)t^2 + (t^2-t)\left(\frac{d}{dt}y(t)\right) + y(t) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dt^2}y(t)$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = -\frac{y(t)}{2t^2} - \frac{(t-1)\left(\frac{d}{dt}y(t)\right)}{2t}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dt^2}y(t) + \frac{(t-1)\left(\frac{d}{dt}y(t)\right)}{2t} + \frac{y(t)}{2t^2} = 0$

- ☐ Check to see if $t_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(t) = \frac{t-1}{2t}, P_3(t) = \frac{1}{2t^2}\right]$

  - $t \cdot P_2(t)$ is analytic at $t = 0$

$(t \cdot P_2(t))\Big|_{t=0} = -\frac{1}{2}$

- $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $(t^2 \cdot P_3(t))\Big|_{t=0} = \frac{1}{2}$

- $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $2\left(\frac{d^2}{dt^2}y(t)\right)t^2 + t(t-1)\left(\frac{d}{dt}y(t)\right) + y(t) = 0$

- Assume series solution for $y(t)$

  $y(t) = \sum_{k=0}^{\infty} a_k t^{k+r}$

□ Rewrite ODE with series expansions

- Convert $t^m \cdot \left(\frac{d}{dt}y(t)\right)$ to series expansion for $m = 1..2$

  $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$

- Convert $t^2 \cdot \left(\frac{d^2}{dt^2}y(t)\right)$ to series expansion

  $t^2 \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)(-1+r)\, t^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)(k+r-1) + a_{k-1}(k+r-1)\right) t^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{1, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(\left(k+r-\frac{1}{2}\right)a_k + \frac{a_{k-1}}{2}\right)(k+r-1) = 0$

- Shift index using $k- >k+1$

  $2\left(\left(k+\frac{1}{2}+r\right)a_{k+1} + \frac{a_k}{2}\right)(k+r) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{2k+1+2r}$

- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{a_k}{2k+3}$

- Solution for $r = 1$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = -\frac{a_k}{2k+3}\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = -\frac{a_k}{2k+2}$

- Solution for $r = \frac{1}{2}$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k+\frac{1}{2}}, a_{k+1} = -\frac{a_k}{2k+2}\right]$

- Combine solutions and rename parameters

$$\left[y(t)=\left(\sum_{k=0}^{\infty}a_k t^{k+1}\right)+\left(\sum_{k=0}^{\infty}b_k t^{k+\frac{1}{2}}\right),a_{k+1}=-\frac{a_k}{2k+3},b_{k+1}=-\frac{b_k}{2k+2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Whittaker successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - return
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.021 (sec)
Leaf size : 47

```
dsolve(2*t^2*diff(diff(y(t),t),t)+(t^2-t)*diff(y(t),t)+y(t) = 0,y(t),singsol=all)
```

$$y=\frac{e^{-\frac{t}{2}}\left(\operatorname{erf}\left(\frac{\sqrt{2}\,\sqrt{-t}}{2}\right)2^{3/4}\sqrt{\pi}\,c_1 t+4\sqrt{t}\,\sqrt{-t}\,c_2\right)}{4\sqrt{-t}}$$

Mathematica DSolve solution

Solving time : 0.037 (sec)
Leaf size : 46

```
DSolve[{2*t^2*D[y[t],{t,2}]+(t^2-t)*D[y[t],t]+y[t]==0,{}},y[t],t,IncludeSingularSolutions->T
```

$$y(t)\to e^{-t/2}\left(c_2\sqrt{t}+\sqrt{2}c_1\sqrt{-t}\Gamma\left(\frac{1}{2},-\frac{t}{2}\right)\right)$$

2.1.625 Problem 642

Internal problem ID [9797]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 642
Date solved : `Monday, January 27, 2025 at 06:14:23 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$t^2y'' + (-t^2 + t)\, y' - y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.212 (sec)

Writing the ode as

$$t^2y'' + (-t^2 + t)\, y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= -t^2 + t \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 - 2t + 3}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^2 - 2t + 3 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 - 2t + 3}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1193: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2t} + \frac{3}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2t} + \frac{1}{2t^2} + \frac{1}{2t^3} + \frac{1}{4t^4} - \frac{1}{4t^5} - \frac{3}{4t^6} - \frac{3}{4t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 - 2t + 3}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2t + 3}{4t^2}\right) \\ &= \frac{1}{4} + \frac{-2t + 3}{4t^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 - 2t + 3}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{2t} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2} - \frac{1}{2t} \\ &= \frac{t-1}{2t} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2} - \frac{1}{2t}\right)(0) + \left(\left(\frac{1}{2t^2}\right) + \left(\frac{1}{2} - \frac{1}{2t}\right)^2 - \left(\frac{t^2-2t+3}{4t^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega\, dt} \\ &= e^{\int \left(\frac{1}{2} - \frac{1}{2t}\right) dt} \\ &= \frac{e^{\frac{t}{2}}}{\sqrt{t}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-t^2+t}{t^2}\,dt} \\
&= z_1 e^{\frac{t}{2}-\frac{\ln(t)}{2}} \\
&= z_1\left(\frac{e^{\frac{t}{2}}}{\sqrt{t}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{e^t}{t}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-t^2+t}{t^2}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{t-\ln(t)}}{(y_1)^2}\,dt \\
&= y_1\left(-(1+t)\,t\,e^{t-\ln(t)}e^{-2t}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{e^t}{t}\right) + c_2\left(\frac{e^t}{t}\left(-(1+t)\,t\,e^{t-\ln(t)}e^{-2t}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dt^2}y(t)\right)t^2 + (-t^2+t)\left(\frac{d}{dt}y(t)\right) - y(t) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dt^2}y(t)$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = \frac{y(t)}{t^2} + \frac{(t-1)\left(\frac{d}{dt}y(t)\right)}{t}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d^2}{dt^2}y(t) - \frac{(t-1)\left(\frac{d}{dt}y(t)\right)}{t} - \frac{y(t)}{t^2} = 0$

☐ Check to see if $t_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(t) = -\frac{t-1}{t}, P_3(t) = -\frac{1}{t^2}\right]$

  - $t \cdot P_2(t)$ is analytic at $t = 0$

$(t \cdot P_2(t))\bigg|_{t=0} = 1$

- $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $(t^2 \cdot P_3(t))\bigg|_{t=0} = -1$

- $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $\left(\frac{d^2}{dt^2}y(t)\right)t^2 - t(t-1)\left(\frac{d}{dt}y(t)\right) - y(t) = 0$

- Assume series solution for $y(t)$

  $y(t) = \sum_{k=0}^{\infty} a_k t^{k+r}$

□ Rewrite ODE with series expansions

- Convert $t^m \cdot \left(\frac{d}{dt}y(t)\right)$ to series expansion for $m = 1..2$

  $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$

- Shift index using $k- > k+1-m$

  $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$

- Convert $t^2 \cdot \left(\frac{d^2}{dt^2}y(t)\right)$ to series expansion

  $t^2 \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)(-1+r)\, t^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)(k+r-1) - a_{k-1}(k+r-1)\right) t^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 1\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_k(k+r+1) - a_{k-1}\right) = 0$

- Shift index using $k- > k+1$

  $(k+r)\left(a_{k+1}(k+2+r) - a_k\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+2+r}$

- Recursion relation for $r = -1$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = -1$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k-1}, a_{k+1} = \frac{a_k}{k+1}\right]$

- Recursion relation for $r = 1$

  $a_{k+1} = \frac{a_k}{k+3}$

- Solution for $r = 1$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = \frac{a_k}{k+3}\right]$

- Combine solutions and rename parameters

  $\left[y(t) = \left(\sum_{k=0}^{\infty} a_k t^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+1}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 17

```
dsolve(t^2*diff(diff(y(t),t),t)+(-t^2+t)*diff(y(t),t)-y(t) = 0,y(t),singsol=all)
```

$$y = \frac{c_2 \mathrm{e}^{t} + c_1 t + c_1}{t}$$

Mathematica DSolve solution

Solving time : 0.249 (sec)
Leaf size : 80

```
DSolve[{t^2*D[y[t],{t,2}]+(t-t^2)*D[y[t],t]-y[t]==0,{}},y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \exp\left(\int_1^t \left(1-\frac{1}{K[1]}\right) dK[1]\right) \left(\int_1^t \exp\left(-\int_1^{K[2]} \left(1-\frac{1}{K[1]}\right) dK[1]\right) c_1 dK[2]+c_2\right)$$
$$y(t) \to c_2 \exp\left(\int_1^t \left(1-\frac{1}{K[1]}\right) dK[1]\right)$$

2.1.626 Problem 643

Internal problem ID [9798]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 643
Date solved : `Monday, January 27, 2025 at 06:14:24 PM`
CAS classification : `[_Lienard]`

Solve

$$ty'' - \left(t^2+2\right)y' + ty = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.262 (sec)

Writing the ode as

$$ty'' + \left(-t^2-2\right)y' + ty = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= -t^2-2 \\ C &= t \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^4-2t^2+8}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^4-2t^2+8 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^4-2t^2+8}{4t^2}\right)z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1195: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{t^2}{4} - \frac{1}{2} + \frac{2}{t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{1} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{t}{2} - \frac{1}{2t} + \frac{7}{4t^3} + \frac{7}{4t^5} - \frac{21}{16t^7} - \frac{119}{16t^9} - \frac{189}{32t^{11}} + \frac{791}{32t^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i t^i \\ &= \frac{t}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{t^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{t^4 - 2t^2 + 8}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{t^2}{4} - \frac{1}{2}\right) + \left(\frac{2}{t^2}\right) \\ &= \frac{t^2}{4} - \frac{1}{2} + \frac{2}{t^2}\end{aligned}$$

We see that the coefficient of the term t in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{t}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 1\right) &&= -1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 1\right) &&= 0\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^4 - 2t^2 + 8}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{t}{2}$	-1	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -1$ then

$$\begin{aligned}d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right)\\ &= -1 - (-1)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1}\right) + (+)[\sqrt{r}]_\infty\\ &= -\frac{1}{t} + \left(\frac{t}{2}\right)\\ &= -\frac{1}{t} + \frac{t}{2}\\ &= -\frac{1}{t} + \frac{t}{2}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{t} + \frac{t}{2}\right)(0) + \left(\left(\frac{1}{t^2} + \frac{1}{2}\right) + \left(-\frac{1}{t} + \frac{t}{2}\right)^2 - \left(\frac{t^4 - 2t^2 + 8}{4t^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(t) &= p e^{\int \omega\, dt}\\ &= e^{\int \left(-\frac{1}{t} + \frac{t}{2}\right) dt}\\ &= \frac{e^{\frac{t^2}{4}}}{t}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-t^2-2}{t}\,dt} \\
&= z_1 e^{\frac{t^2}{4}+\ln(t)} \\
&= z_1\left(t\,\mathrm{e}^{\frac{t^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\frac{t^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-t^2-2}{t}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{\frac{t^2}{2}+2\ln(t)}}{(y_1)^2}\,dt \\
&= y_1\left(-t\,\mathrm{e}^{-\frac{t^2}{2}} + \frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,t}{2}\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{\frac{t^2}{2}}\right) + c_2\left(\mathrm{e}^{\frac{t^2}{2}}\left(-t\,\mathrm{e}^{-\frac{t^2}{2}} + \frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,t}{2}\right)}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dt^2}y(t)\right)t - (t^2+2)\left(\frac{d}{dt}y(t)\right) + ty(t) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dt^2}y(t)$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = -y(t) + \frac{(t^2+2)\left(\frac{d}{dt}y(t)\right)}{t}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dt^2}y(t) - \frac{(t^2+2)\left(\frac{d}{dt}y(t)\right)}{t} + y(t) = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(t) = -\frac{t^2+2}{t}, P_3(t) = 1\right]$

- $t \cdot P_2(t)$ is analytic at $t = 0$

  $(t \cdot P_2(t))\Big|_{t=0} = -2$

- $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $(t^2 \cdot P_3(t))\Big|_{t=0} = 0$

- $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $\left(\frac{d^2}{dt^2}y(t)\right)t + (-t^2 - 2)\left(\frac{d}{dt}y(t)\right) + ty(t) = 0$

- Assume series solution for $y(t)$

  $y(t) = \sum_{k=0}^{\infty} a_k t^{k+r}$

□ Rewrite ODE with series expansions

- Convert $t \cdot y(t)$ to series expansion

  $t \cdot y(t) = \sum_{k=0}^{\infty} a_k t^{k+r+1}$

- Shift index using $k- >k-1$

  $t \cdot y(t) = \sum_{k=1}^{\infty} a_{k-1} t^{k+r}$

- Convert $t^m \cdot \left(\frac{d}{dt}y(t)\right)$ to series expansion for $m = 0..2$

  $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$

- Convert $t \cdot \left(\frac{d^2}{dt^2}y(t)\right)$ to series expansion

  $t \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, t^{k+r-1}$

- Shift index using $k- >k+1$

  $t \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)(k+r)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-3+r)\, t^{-1+r} + a_1(1+r)(-2+r)\, t^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+r+1)(k-2+r) - a_{k-1}(k-2+r))\, t^k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 3\}$

- Each term must be 0

  $a_1(1+r)(-2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k-2+r)(a_{k+1}(k+r+1) - a_{k-1}) = 0$

- Shift index using $k- >k+1$

  $(k+r-1)(a_{k+2}(k+2+r) - a_k) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+2} = \frac{a_k}{k+2+r}$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{a_k}{k+2}$

- Solution for $r = 0$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^k, a_{k+2} = \frac{a_k}{k+2}, -2a_1 = 0\right]$

- Recursion relation for $r = 3$

  $a_{k+2} = \frac{a_k}{k+5}$

- Solution for $r = 3$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k+3}, a_{k+2} = \frac{a_k}{k+5}, 4a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y(t) = \left(\sum_{k=0}^{\infty} a_k t^k\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+3}\right), a_{k+2} = \frac{a_k}{k+2}, -2a_1 = 0, b_{k+2} = \frac{b_k}{5+k}, 4b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.007` (sec)
Leaf size : 34

```
dsolve(t*diff(diff(y(t),t),t)-(t^2+2)*diff(y(t),t)+y(t)*t = 0,y(t),singsol=all)
```

$$y = \left(-c_2\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,t}{2}\right) + c_1\right) e^{\frac{t^2}{2}} + 2c_2 t$$

Mathematica DSolve solution

Solving time : `0.105` (sec)
Leaf size : 56

```
DSolve[{t*D[y[t],{t,2}]-(t^2+2)*D[y[t],t]+t*y[t]==0,{}},y[t],t,IncludeSingularSolutions->Tru
```

$$y(t) \to \frac{1}{2}e\left(\sqrt{2\pi}c_2 e^{\frac{t^2}{2}}\operatorname{erf}\left(\frac{t}{\sqrt{2}}\right) + 2c_1 e^{\frac{t^2}{2}} - 2c_2 t\right)$$

2.1.627 Problem 644

Internal problem ID [9799]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 644
Date solved : `Monday, January 27, 2025 at 06:14:24 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$t^2y'' + t(t+1)\,y' - y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.204 (sec)

Writing the ode as

$$t^2y'' + \left(t^2+t\right)y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= t^2+t \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2+2t+3}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^2+2t+3 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2+2t+3}{4t^2}\right)z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1197: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{1}{2t} + \frac{3}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{2t} + \frac{1}{2t^2} - \frac{1}{2t^3} + \frac{1}{4t^4} + \frac{1}{4t^5} - \frac{3}{4t^6} + \frac{3}{4t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 + 2t + 3}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{2t + 3}{4t^2}\right) \\ &= \frac{1}{4} + \frac{2t + 3}{4t^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is 2. Dividing this by leading coefficient in t which is 4 gives $\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{2}\right) - (0) \\ &= \frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 + 2t + 3}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2t} + (-)\left(\frac{1}{2}\right) \\ &= -\frac{1}{2t} - \frac{1}{2} \\ &= -\frac{t+1}{2t} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{2t} - \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2t^2}\right) + \left(-\frac{1}{2t} - \frac{1}{2}\right)^2 - \left(\frac{t^2+2t+3}{4t^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= e^{\int \left(-\frac{1}{2t} - \frac{1}{2}\right) dt} \\ &= \frac{e^{-\frac{t}{2}}}{\sqrt{t}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{t^2+t}{t^2}\,dt} \\
&= z_1 e^{-\frac{t}{2}-\frac{\ln(t)}{2}} \\
&= z_1\left(\frac{\mathrm{e}^{-\frac{t}{2}}}{\sqrt{t}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{-t}}{t}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{t^2+t}{t^2}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-t-\ln(t)}}{(y_1)^2}\,dt \\
&= y_1\left((-1+t)\,t\,\mathrm{e}^{-t-\ln(t)}\mathrm{e}^{2t}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\mathrm{e}^{-t}}{t}\right) + c_2\left(\frac{\mathrm{e}^{-t}}{t}\left((-1+t)\,t\,\mathrm{e}^{-t-\ln(t)}\mathrm{e}^{2t}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dt^2}y(t)\right)t^2 + t(t+1)\left(\frac{d}{dt}y(t)\right) - y(t) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dt^2}y(t)$
- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = \frac{y(t)}{t^2} - \frac{(t+1)\left(\frac{d}{dt}y(t)\right)}{t}$
- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d^2}{dt^2}y(t) + \frac{(t+1)\left(\frac{d}{dt}y(t)\right)}{t} - \frac{y(t)}{t^2} = 0$

☐ Check to see if $t_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(t) = \frac{t+1}{t}, P_3(t) = -\frac{1}{t^2}\right]$

  ◦ $t \cdot P_2(t)$ is analytic at $t = 0$

$(t \cdot P_2(t))\Big|_{t=0} = 1$

- ∘ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $(t^2 \cdot P_3(t))\Big|_{t=0} = -1$

- ∘ $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $\left(\frac{d^2}{dt^2}y(t)\right)t^2 + t(t+1)\left(\frac{d}{dt}y(t)\right) - y(t) = 0$

- Assume series solution for $y(t)$

  $y(t) = \sum_{k=0}^{\infty} a_k t^{k+r}$

- □ Rewrite ODE with series expansions

  - ∘ Convert $t^m \cdot \left(\frac{d}{dt}y(t)\right)$ to series expansion for $m = 1..2$

    $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$

  - ∘ Shift index using $k- >k+1-m$

    $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$

  - ∘ Convert $t^2 \cdot \left(\frac{d^2}{dt^2}y(t)\right)$ to series expansion

    $t^2 \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)(-1+r)\, t^r + \left(\sum_{k=1}^{\infty} (a_k(k+r+1)(k+r-1) + a_{k-1}(k+r-1))\, t^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 1\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)(a_k(k+r+1) + a_{k-1}) = 0$

- Shift index using $k- >k+1$

  $(k+r)(a_{k+1}(k+2+r) + a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{k+2+r}$

- Recursion relation for $r = -1$

  $a_{k+1} = -\frac{a_k}{k+1}$

- Solution for $r = -1$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k-1}, a_{k+1} = -\frac{a_k}{k+1}\right]$

- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{a_k}{k+3}$

- Solution for $r = 1$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = -\frac{a_k}{k+3}\right]$

- Combine solutions and rename parameters

  $\left[y(t) = \left(\sum_{k=0}^{\infty} a_k t^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+1}\right), a_{k+1} = -\frac{a_k}{k+1}, b_{k+1} = -\frac{b_k}{k+3}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 20

```
dsolve(t^2*diff(diff(y(t),t),t)+t*(t+1)*diff(y(t),t)-y(t) = 0,y(t),singsol=all)
```

$$y = \frac{c_2 \mathrm{e}^{-t} + (t-1)\, c_1}{t}$$

Mathematica DSolve solution

Solving time : 0.594 (sec)
Leaf size : 54

```
DSolve[{t^2*D[y[t],{t,2}]+t*(t+1)*D[y[t],t]-y[t]==0,{}},y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{e^{-t-1}\left(\int_1^t e^{K[1]+1} c_1 K[1] dK[1] + c_2\right)}{t}$$
$$y(t) \to \frac{c_2 e^{-t-1}}{t}$$

2.1.628 Problem 645

Internal problem ID [9800]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 645
Date solved : Monday, January 27, 2025 at 06:14:25 PM
CAS classification : [_Laguerre]

Solve

$$ty'' - (4+t)\,y' + 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.256 (sec)

Writing the ode as

$$ty'' + (-4-t)\,y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= -4-t \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2+24}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^2 + 24 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2+24}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1199: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{6}{t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{6}{t^2} - \frac{36}{t^4} + \frac{432}{t^6} - \frac{6480}{t^8} + \frac{108864}{t^{10}} - \frac{1959552}{t^{12}} + \frac{36951552}{t^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{t^2 + 24}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{6}{t^2}\right) \\ &= \frac{1}{4} + \frac{6}{t^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is 0. Dividing this by leading coefficient in t which is 4 gives 0. Now b can be found.

$$\begin{aligned}b &= (0) - (0) \\ &= 0\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{\frac{1}{2}} - 0\right) &= 0 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{\frac{1}{2}} - 0\right) &= 0\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 + 24}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= 0 - (-2) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{2}{t} + (-)\left(\frac{1}{2}\right) \\ &= -\frac{2}{t} - \frac{1}{2} \\ &= -\frac{4+t}{2t} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 2$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = t^2 + a_1 t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (2) + 2\left(-\frac{2}{t} - \frac{1}{2}\right)(2t + a_1) + \left(\left(\frac{2}{t^2}\right) + \left(-\frac{2}{t} - \frac{1}{2}\right)^2 - \left(\frac{t^2+24}{4t^2}\right)\right) &= 0 \\ \frac{(a_1 - 6)\,t + 2a_0 - 4a_1}{t} &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 12, a_1 = 6\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = t^2 + 6t + 12$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= pe^{\int \omega\, dt} \\
&= \left(t^2+6t+12\right) e^{\int \left(-\frac{2}{t}-\frac{1}{2}\right)dt} \\
&= \left(t^2+6t+12\right) e^{-\frac{t}{2}-2\ln(t)} \\
&= \frac{\left(t^2+6t+12\right) e^{-\frac{t}{2}}}{t^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4-t}{t}\, dt} \\
&= z_1 e^{\frac{t}{2}+2\ln(t)} \\
&= z_1 \left(t^2 e^{\frac{t}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = t^2+6t+12$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4-t}{t}\, dt}}{\left(y_1\right)^2}\, dt \\
&= y_1 \int \frac{e^{t+4\ln(t)}}{\left(y_1\right)^2}\, dt \\
&= y_1 \left(\frac{\left(t^2-6t+12\right) e^{t+4\ln(t)}}{\left(t^2+6t+12\right) t^4}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(t^2+6t+12\right) + c_2 \left(t^2+6t+12\left(\frac{\left(t^2-6t+12\right) e^{t+4\ln(t)}}{\left(t^2+6t+12\right) t^4}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dt^2}y(t)\right) t - (t+4)\left(\frac{d}{dt}y(t)\right) + 2y(t) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dt^2}y(t)$
- Isolate 2nd derivative
  $\frac{d^2}{dt^2}y(t) = -\frac{2y(t)}{t} + \frac{(t+4)\left(\frac{d}{dt}y(t)\right)}{t}$
- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dt^2}y(t) - \frac{(t+4)\left(\frac{d}{dt}y(t)\right)}{t} + \frac{2y(t)}{t} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(t) = -\frac{t+4}{t}, P_3(t) = \frac{2}{t}\right]$

- $t \cdot P_2(t)$ is analytic at $t = 0$

  $\left(t \cdot P_2(t)\right)\Big|_{t=0} = -4$

- $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $\left(t^2 \cdot P_3(t)\right)\Big|_{t=0} = 0$

- $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $\left(\frac{d^2}{dt^2}y(t)\right) t + (-t-4)\left(\frac{d}{dt}y(t)\right) + 2y(t) = 0$

- Assume series solution for $y(t)$

  $y(t) = \sum_{k=0}^{\infty} a_k t^{k+r}$

□ Rewrite ODE with series expansions

- Convert $t^m \cdot \left(\frac{d}{dt}y(t)\right)$ to series expansion for $m = 0..1$

  $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$

- Convert $t \cdot \left(\frac{d^2}{dt^2}y(t)\right)$ to series expansion

  $t \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, t^{k+r-1}$

- Shift index using $k- >k+1$

  $t \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-5+r)\, t^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k-4+r) - a_k(k+r-2)\right) t^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-5+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 5\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k-4+r) - a_k(k+r-2) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r-2)}{(k+1+r)(k-4+r)}$

- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+1} = \frac{a_k(k-2)}{(k+1)(k-4)}$

- Apply recursion relation for $k = 0$

  $a_1 = \frac{a_0}{2}$

- Apply recursion relation for $k = 1$

$a_2 = \frac{a_1}{6}$

- Express in terms of a_0

  $a_2 = \frac{a_0}{12}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(t) = a_0 \cdot \left(1 + \frac{1}{2}t + \frac{1}{12}t^2\right)$

- Recursion relation for $r = 5$

  $a_{k+1} = \frac{a_k(k+3)}{(k+6)(k+1)}$

- Solution for $r = 5$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k+5}, a_{k+1} = \frac{a_k(k+3)}{(k+6)(k+1)}\right]$

- Combine solutions and rename parameters

  $\left[y(t) = a_0 \cdot \left(1 + \frac{1}{2}t + \frac{1}{12}t^2\right) + \left(\sum_{k=0}^{\infty} b_k t^{5+k}\right), b_{k+1} = \frac{b_k(k+3)}{(k+6)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 27

```
dsolve(t*diff(diff(y(t),t),t)-(4+t)*diff(y(t),t)+2*y(t) = 0,y(t),singsol=all)
```

$$y = c_1\left(t^2 + 6t + 12\right) + c_2 e^t \left(t^2 - 6t + 12\right)$$

Mathematica DSolve solution

Solving time : 0.078 (sec)
Leaf size : 87

```
DSolve[{t*D[y[t],{t,2}]-(4+t)*D[y[t],t]+2*y[t]==0,{}},y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{2e^{\frac{t+5}{2}}\sqrt{t}\left(\left(c_2t^2 - 6ic_1t + 12c_2\right)\cosh\left(\frac{t}{2}\right) + i\left(c_1\left(t^2 + 12\right) + 6ic_2t\right)\sinh\left(\frac{t}{2}\right)\right)}{\sqrt{\pi}\sqrt{-it}}$$

2.1.629 Problem 646

Internal problem ID [9801]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 646
Date solved : `Monday, January 27, 2025 at 06:14:26 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$t^2y'' + (t^2 - 3t)\, y' + 3y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.241 (sec)

Writing the ode as

$$t^2y'' + (t^2 - 3t)\, y' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= t^2 - 3t \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 - 6t + 3}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^2 - 6t + 3 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 - 6t + 3}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1201: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{3}{2t} + \frac{3}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{3}{2t} - \frac{3}{2t^2} - \frac{9}{2t^3} - \frac{63}{4t^4} - \frac{243}{4t^5} - \frac{999}{4t^6} - \frac{4293}{4t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{t^2 - 6t + 3}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-6t + 3}{4t^2}\right) \\ &= \frac{1}{4} + \frac{-6t + 3}{4t^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -6. Dividing this by leading coefficient in t which is 4 gives $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{3}{2}\right) - (0) \\ &= -\frac{3}{2}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &&= \frac{3}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 - 6t + 3}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{3}{2} - \left(\frac{3}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{3}{2t} + (-)\left(\frac{1}{2}\right) \\ &= \frac{3}{2t} - \frac{1}{2} \\ &= -\frac{t-3}{2t} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{3}{2t} - \frac{1}{2}\right)(0) + \left(\left(-\frac{3}{2t^2}\right) + \left(\frac{3}{2t} - \frac{1}{2}\right)^2 - \left(\frac{t^2 - 6t + 3}{4t^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= e^{\int \left(\frac{3}{2t} - \frac{1}{2}\right) dt} \\ &= t^{3/2} e^{-\frac{t}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{t^2-3t}{t^2}\,dt} \\
&= z_1 e^{-\frac{t}{2}+\frac{3\ln(t)}{2}} \\
&= z_1\left(t^{3/2}\mathrm{e}^{-\frac{t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = t^3\mathrm{e}^{-t}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{t^2-3t}{t^2}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-t+3\ln(t)}}{(y_1)^2}\,dt \\
&= y_1\left(-\frac{\mathrm{e}^t}{2t^2}-\frac{\mathrm{e}^t}{2t}-\frac{\mathrm{Ei}_1(-t)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(t^3\mathrm{e}^{-t}\right) + c_2\left(t^3\mathrm{e}^{-t}\left(-\frac{\mathrm{e}^t}{2t^2}-\frac{\mathrm{e}^t}{2t}-\frac{\mathrm{Ei}_1(-t)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dt^2}y(t)\right)t^2 + (t^2-3t)\left(\frac{d}{dt}y(t)\right) + 3y(t) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dt^2}y(t)$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = -\frac{3y(t)}{t^2} - \frac{(-3+t)\left(\frac{d}{dt}y(t)\right)}{t}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d^2}{dt^2}y(t) + \frac{(-3+t)\left(\frac{d}{dt}y(t)\right)}{t} + \frac{3y(t)}{t^2} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(t) = \frac{-3+t}{t}, P_3(t) = \frac{3}{t^2}\right]$

  ◦ $t \cdot P_2(t)$ is analytic at $t = 0$

$(t \cdot P_2(t))\bigg|_{t=0} = -3$

- ○ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $(t^2 \cdot P_3(t))\bigg|_{t=0} = 3$

- ○ $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $\left(\frac{d^2}{dt^2}y(t)\right)t^2 + (-3+t)\,t\left(\frac{d}{dt}y(t)\right) + 3y(t) = 0$

- Assume series solution for $y(t)$

  $y(t) = \sum_{k=0}^{\infty} a_k t^{k+r}$

- □ Rewrite ODE with series expansions

  - ○ Convert $t^m \cdot \left(\frac{d}{dt}y(t)\right)$ to series expansion for $m = 1..2$

    $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,t^{k+r-1+m}$

  - ○ Shift index using $k->k+1-m$

    $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,t^{k+r}$

  - ○ Convert $t^2 \cdot \left(\frac{d^2}{dt^2}y(t)\right)$ to series expansion

    $t^2 \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,t^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)\,(-3+r)\,t^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)\,(k+r-3) + a_{k-1}(k+r-1)\right) t^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{1, 3\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\,(a_k(k+r-3) + a_{k-1}) = 0$

- Shift index using $k->k+1$

  $(k+r)\,(a_{k+1}(k-2+r) + a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{k-2+r}$

- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{a_k}{k-1}$

- Series not valid for $r = 1$, division by 0 in the recursion relation at $k = 1$

  $a_{k+1} = -\frac{a_k}{k-1}$

- Recursion relation for $r = 3$

  $a_{k+1} = -\frac{a_k}{k+1}$

- Solution for $r = 3$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k+3}, a_{k+1} = -\frac{a_k}{k+1}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 34

```
dsolve(t^2*diff(diff(y(t),t),t)+(t^2-3*t)*diff(y(t),t)+3*y(t) = 0,y(t),singsol=all)
```

$$y = t\left(\mathrm{e}^{-t}\,\mathrm{Ei}_1\left(-t\right)c_2 t^2 + \mathrm{e}^{-t} c_1 t^2 + c_2 t + c_2\right)$$

Mathematica DSolve solution

Solving time : 46.85 (sec)
Leaf size : 50

```
DSolve[{t^2*D[y[t],{t,2}]+(t^2-3*t)*D[y[t],t]+3*y[t]==0,{}},y[t],t,IncludeSingularSolutions->T
```

$$y(t) \to e^{-t} t^3 \left(\int_1^t \frac{e^{K[1]} c_1}{K[1]^3} dK[1] + c_2\right)$$
$$y(t) \to c_2 e^{-t} t^3$$

2.1.630 Problem 647

Internal problem ID [9802]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 647
Date solved : `Monday, January 27, 2025 at 06:14:26 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$ty'' + ty' + 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.219 (sec)

Writing the ode as

$$ty'' + ty' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= t \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t-8}{4t} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t - 8 \\ t &= 4t \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t-8}{4t}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1203: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t$. There is a pole at $t = 0$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $t = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{2}{t} - \frac{4}{t^2} - \frac{16}{t^3} - \frac{80}{t^4} - \frac{448}{t^5} - \frac{2688}{t^6} - \frac{16896}{t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t-8}{4t} \\ &= Q + \frac{R}{4t} \\ &= \left(\frac{1}{4}\right) + \left(-\frac{2}{t}\right) \\ &= \frac{1}{4} - \frac{2}{t} \end{aligned}$$

Since the degree of t is 1, then we see that the coefficient of the term 1 in the remainder R is -8. Dividing this by leading coefficient in t which is 4 gives -2. Now b can be found.

$$\begin{aligned} b &= (-2) - (0) \\ &= -2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-2}{\frac{1}{2}} - 0\right) &&= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-2}{\frac{1}{2}} - 0\right) &&= 2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t-8}{4t}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	-2	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= 2 - (1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{t} + (-)\left(\frac{1}{2}\right) \\ &= \frac{1}{t} - \frac{1}{2} \\ &= \frac{1}{t} - \frac{1}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 1$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{t} - \frac{1}{2}\right)(1) + \left(\left(-\frac{1}{t^2}\right) + \left(\frac{1}{t} - \frac{1}{2}\right)^2 - \left(\frac{t-8}{4t}\right)\right) &= 0 \\ \frac{2+a_0}{t} &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -2\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = -2 + t$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= (-2+t)\, e^{\int \left(\frac{1}{t} - \frac{1}{2}\right) dt} \\ &= (-2+t)\, e^{-\frac{t}{2} + \ln(t)} \\ &= (-2+t)\, t\, e^{-\frac{t}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{t}{t}\,dt} \\
&= z_1 e^{-\frac{t}{2}} \\
&= z_1\left(\mathrm{e}^{-\frac{t}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-t}(-2+t)\,t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{t}{t}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-t}}{(y_1)^2}\,dt \\
&= y_1\left(-\frac{\mathrm{e}^{t}(-t+1)}{2\,(2-t)\,t} - \frac{\mathrm{Ei}_1(-t)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-t}(-2+t)\,t\right) + c_2\left(\mathrm{e}^{-t}(-2+t)\,t\left(-\frac{\mathrm{e}^{t}(-t+1)}{2\,(2-t)\,t} - \frac{\mathrm{Ei}_1(-t)}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dt^2}y(t)\right)t + t\left(\frac{d}{dt}y(t)\right) + 2y(t) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dt^2}y(t)$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = -\frac{2y(t)}{t} - \frac{d}{dt}y(t)$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dt^2}y(t) + \frac{d}{dt}y(t) + \frac{2y(t)}{t} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(t) = 1, P_3(t) = \frac{2}{t}\right]$

  ◦ $t \cdot P_2(t)$ is analytic at $t = 0$

    $\left(t \cdot P_2(t)\right)\Big|_{t=0} = 0$

- ◦ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $\left(t^2 \cdot P_3(t)\right)\Big|_{t=0} = 0$
- ◦ $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$
- Multiply by denominators

  $\left(\frac{d^2}{dt^2}y(t)\right)t + t\left(\frac{d}{dt}y(t)\right) + 2y(t) = 0$
- Assume series solution for $y(t)$

  $y(t) = \sum_{k=0}^{\infty} a_k t^{k+r}$
- □ Rewrite ODE with series expansions
  - ◦ Convert $t \cdot \left(\frac{d}{dt}y(t)\right)$ to series expansion

    $t \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r}$
  - ◦ Convert $t \cdot \left(\frac{d^2}{dt^2}y(t)\right)$ to series expansion

    $t \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, t^{k+r-1}$
  - ◦ Shift index using $k- >k+1$

    $t \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-1+r)\, t^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k+r) + a_k(k+r+2)\right) t^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 1\}$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k+r) + a_k(k+r+2) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r+2)}{(k+1+r)(k+r)}$
- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{a_k(k+2)}{(k+1)k}$
- Solution for $r = 0$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^k, a_{k+1} = -\frac{a_k(k+2)}{(k+1)k}\right]$
- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{a_k(k+3)}{(k+2)(k+1)}$
- Solution for $r = 1$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = -\frac{a_k(k+3)}{(k+2)(k+1)}\right]$
- Combine solutions and rename parameters

  $\left[y(t) = \left(\sum_{k=0}^{\infty} a_k t^k\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+1}\right), a_{k+1} = -\frac{a_k(k+2)}{(k+1)k}, b_{k+1} = -\frac{b_k(k+3)}{(k+2)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 35

```
dsolve(t*diff(diff(y(t),t),t)+t*diff(y(t),t)+2*y(t) = 0,y(t),singsol=all)
```

$$y = tc_2 \mathrm{e}^{-t}(t-2)\,\mathrm{Ei}_1(-t) + c_1 \mathrm{e}^{-t}(t-2)\,t + c_2(t-1)$$

Mathematica DSolve solution

Solving time : 0.203 (sec)
Leaf size : 43

```
DSolve[{t*D[y[t],{t,2}]+t*D[y[t],t]+2*y[t]==0,{}},y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to e^{-t}(t-2)t\left(c_2 \int_1^t \frac{e^{K[1]}}{(K[1]-2)^2 K[1]^2} dK[1] + c_1\right)$$

2.1.631 Problem 648

Internal problem ID [9803]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 648
Date solved : `Monday, January 27, 2025 at 06:14:27 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$ty'' + \left(-t^2+1\right)y' + 4ty = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.654 (sec)

Writing the ode as

$$ty'' + \left(-t^2+1\right)y' + 4ty = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= -t^2+1 \\ C &= 4t \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^4 - 20t^2 - 1}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^4 - 20t^2 - 1 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^4 - 20t^2 - 1}{4t^2}\right)z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1205: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{t^2}{4} - 5 - \frac{1}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{1} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{t}{2} - \frac{5}{t} - \frac{101}{4t^3} - \frac{505}{2t^5} - \frac{50601}{16t^7} - \frac{355015}{8t^9} - \frac{21351501}{32t^{11}} - \frac{168167525}{16t^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i t^i \\ &= \frac{t}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{t^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^4 - 20t^2 - 1}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{t^2}{4} - 5\right) + \left(-\frac{1}{4t^2}\right) \\ &= \frac{t^2}{4} - 5 - \frac{1}{4t^2} \end{aligned}$$

We see that the coefficient of the term t in the quotient is -5. Now b can be found.

$$\begin{aligned} b &= (-5) - (0) \\ &= -5 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{t}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-5}{\frac{1}{2}} - 1\right) &= -\frac{11}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-5}{\frac{1}{2}} - 1\right) &= \frac{9}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^4 - 20t^2 - 1}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{t}{2}$	$-\frac{11}{2}$	$\frac{9}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{9}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{9}{2} - \left(\frac{1}{2}\right) \\ &= 4 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2t} + (-)\left(\frac{t}{2}\right) \\ &= \frac{1}{2t} - \frac{t}{2} \\ &= \frac{1}{2t} - \frac{t}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 4$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(12t^2 + 6ta_3 + 2a_2\right) + 2\left(\frac{1}{2t} - \frac{t}{2}\right)\left(4t^3 + 3a_3t^2 + 2a_2t + a_1\right) + \left(\left(-\frac{1}{2t^2} - \frac{1}{2}\right) + \left(\frac{1}{2t} - \frac{t}{2}\right)^2 - \left(\frac{t^4 - 20t}{4t^2}\right.\right.$$

$$\frac{t^4 a_3 + 2(8 + a_2)\, t^3 + 3(a_1 + 3a_3)\, t^2 + 4(a_0 + a_2)\, t}{t}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 8, a_1 = 0, a_2 = -8, a_3 = 0\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = t^4 - 8t^2 + 8$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \left(t^4 - 8t^2 + 8\right) \mathrm{e}^{\int \left(\frac{1}{2t} - \frac{t}{2}\right) dt} \\
&= \left(t^4 - 8t^2 + 8\right) \mathrm{e}^{-\frac{t^2}{4} + \frac{\ln(t)}{2}} \\
&= \left(t^4 - 8t^2 + 8\right) \sqrt{t}\, \mathrm{e}^{-\frac{t^2}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-t^2+1}{t} \, dt} \\
&= z_1 e^{\frac{t^2}{4} - \frac{\ln(t)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{t^2}{4}}}{\sqrt{t}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = t^4 - 8t^2 + 8
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-t^2+1}{t} \, dt}}{(y_1)^2} \, dt \\
&= y_1 \int \frac{e^{\frac{t^2}{2} - \ln(t)}}{(y_1)^2} \, dt \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{t^2}{2} - \ln(t)}}{\left(t^4 - 8t^2 + 8\right)^2} dt \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(t^4 - 8t^2 + 8\right) + c_2 \left(t^4 - 8t^2 + 8 \left(\int \frac{\mathrm{e}^{\frac{t^2}{2} - \ln(t)}}{\left(t^4 - 8t^2 + 8\right)^2} dt \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dt^2}y(t)\right)t+(-t^2+1)\left(\frac{d}{dt}y(t)\right)+4ty(t)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dt^2}y(t)$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t)=-4y(t)+\frac{(t^2-1)\left(\frac{d}{dt}y(t)\right)}{t}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dt^2}y(t)-\frac{(t^2-1)\left(\frac{d}{dt}y(t)\right)}{t}+4y(t)=0$

□ Check to see if $t_0=0$ is a regular singular point

- ○ Define functions

  $\left[P_2(t)=-\frac{t^2-1}{t}, P_3(t)=4\right]$

- ○ $t\cdot P_2(t)$ is analytic at $t=0$

  $\left(t\cdot P_2(t)\right)\bigg|_{t=0}=1$

- ○ $t^2\cdot P_3(t)$ is analytic at $t=0$

  $\left(t^2\cdot P_3(t)\right)\bigg|_{t=0}=0$

- ○ $t=0$is a regular singular point

  Check to see if $t_0=0$ is a regular singular point

  $t_0=0$

- Multiply by denominators

  $\left(\frac{d^2}{dt^2}y(t)\right)t+(-t^2+1)\left(\frac{d}{dt}y(t)\right)+4ty(t)=0$

- Assume series solution for $y(t)$

  $y(t)=\sum_{k=0}^{\infty}a_k t^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $t\cdot y(t)$ to series expansion

  $t\cdot y(t)=\sum_{k=0}^{\infty}a_k t^{k+r+1}$

- ○ Shift index using $k->k-1$

  $t\cdot y(t)=\sum_{k=1}^{\infty}a_{k-1}t^{k+r}$

- ○ Convert $t^m\cdot\left(\frac{d}{dt}y(t)\right)$ to series expansion for $m=0..2$

  $t^m\cdot\left(\frac{d}{dt}y(t)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,t^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

  $t^m\cdot\left(\frac{d}{dt}y(t)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,t^{k+r}$

- ○ Convert $t\cdot\left(\frac{d^2}{dt^2}y(t)\right)$ to series expansion

  $t\cdot\left(\frac{d^2}{dt^2}y(t)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,t^{k+r-1}$

- ○ Shift index using $k->k+1$

  $t\cdot\left(\frac{d^2}{dt^2}y(t)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+r+1)\,(k+r)\,t^{k+r}$

  Rewrite ODE with series expansions

$$a_0 r^2 t^{-1+r} + a_1(1+r)^2 t^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)^2 - a_{k-1}(k-5+r)\right) t^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 0$
- Each term must be 0
  $a_1(1+r)^2 = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1)^2 - a_{k-1}(k-5) = 0$
- Shift index using $k->k+1$
  $a_{k+2}(k+2)^2 - a_k(k-4) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{a_k(k-4)}{(k+2)^2}$
- Recursion relation for $r = 0$; series terminates at $k = 4$
  $a_{k+2} = \frac{a_k(k-4)}{(k+2)^2}$
- Solution for $r = 0$
  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^k, a_{k+2} = \frac{a_k(k-4)}{(k+2)^2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solut
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returnin
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.034 (sec)
Leaf size : 21

```
dsolve(t*diff(diff(y(t),t),t)+(-t^2+1)*diff(y(t),t)+4*y(t)*t = 0,y(t),singsol=all)
```

$$y = \frac{(t^4 - 8t^2 + 8)(c_1 + 2c_2)}{8}$$

Mathematica DSolve solution

Solving time : 0.326 (sec)
Leaf size : 65

```
DSolve[{t*D[y[t],{t,2}]+(1-t^2)*D[y[t],t]+4*t*y[t]==0,{}},y[t],t,IncludeSingularSolutions->T
```

$$y(t) \to \sqrt{e}(t^4 - 8t^2 + 8)\left(c_2 \int_1^t \frac{e^{\frac{K[1]^2}{2}-1}}{K[1](K[1]^4 - 8K[1]^2 + 8)^2} dK[1] + c_1\right)$$

2.1.632 Problem 649

Internal problem ID [9804]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 649
Date solved : `Monday, January 27, 2025 at 06:14:28 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$t^2y'' - t(1+t)\,y' + y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.230 (sec)

Writing the ode as

$$t^2y'' + \left(-t^2 - t\right)y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= -t^2 - t \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 + 2t - 1}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^2 + 2t - 1 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 + 2t - 1}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1207: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{1}{2t} - \frac{1}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{2t} - \frac{1}{2t^2} + \frac{1}{2t^3} - \frac{3}{4t^4} + \frac{5}{4t^5} - \frac{9}{4t^6} + \frac{17}{4t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{t^2 + 2t - 1}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{2t - 1}{4t^2}\right) \\ &= \frac{1}{4} + \frac{2t - 1}{4t^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is 2. Dividing this by leading coefficient in t which is 4 gives $\frac{1}{2}$. Now b can be found.

$$\begin{aligned}b &= \left(\frac{1}{2}\right) - (0) \\ &= \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 + 2t - 1}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{t-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{2t} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2} + \frac{1}{2t} \\ &= \frac{1+t}{2t} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2} + \frac{1}{2t}\right)(0) + \left(\left(-\frac{1}{2t^2}\right) + \left(\frac{1}{2} + \frac{1}{2t}\right)^2 - \left(\frac{t^2+2t-1}{4t^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= e^{\int \left(\frac{1}{2} + \frac{1}{2t}\right) dt} \\ &= \sqrt{t}\, e^{\frac{t}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-t^2-t}{t^2}\,dt} \\
&= z_1 e^{\frac{t}{2}+\frac{\ln(t)}{2}} \\
&= z_1\left(\sqrt{t}\,\mathrm{e}^{\frac{t}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = t\,\mathrm{e}^t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-t^2-t}{t^2}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{t+\ln(t)}}{(y_1)^2}\,dt \\
&= y_1(-\operatorname{Ei}_1(t))
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(t\,\mathrm{e}^t\right) + c_2\left(t\,\mathrm{e}^t(-\operatorname{Ei}_1(t))\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dt^2}y(t)\right)t^2 - t(t+1)\left(\frac{d}{dt}y(t)\right) + y(t) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dt^2}y(t)$
- Isolate 2nd derivative
  $\frac{d^2}{dt^2}y(t) = \frac{(t+1)\left(\frac{d}{dt}y(t)\right)}{t} - \frac{y(t)}{t^2}$
- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d^2}{dt^2}y(t) + \frac{y(t)}{t^2} - \frac{(t+1)\left(\frac{d}{dt}y(t)\right)}{t} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(t) = -\frac{t+1}{t}, P_3(t) = \frac{1}{t^2}\right]$
  - $t \cdot P_2(t)$ is analytic at $t = 0$
    $\left.(t \cdot P_2(t))\right|_{t=0} = -1$
  - $t^2 \cdot P_3(t)$ is analytic at $t = 0$

$$\left(t^2 \cdot P_3(t)\right)\bigg|_{t=0} = 1$$

- ∘ $t = 0$is a regular singular point
  Check to see if $t_0 = 0$ is a regular singular point
  $t_0 = 0$
- Multiply by denominators
  $\left(\frac{d^2}{dt^2}y(t)\right)t^2 - t(t+1)\left(\frac{d}{dt}y(t)\right) + y(t) = 0$
- Assume series solution for $y(t)$
  $y(t) = \sum_{k=0}^{\infty} a_k t^{k+r}$
- □ Rewrite ODE with series expansions
  - ∘ Convert $t^m \cdot \left(\frac{d}{dt}y(t)\right)$ to series expansion for $m = 1..2$
    $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$
  - ∘ Shift index using $k->k+1-m$
    $t^m \cdot \left(\frac{d}{dt}y(t)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$
  - ∘ Convert $t^2 \cdot \left(\frac{d^2}{dt^2}y(t)\right)$ to series expansion
    $t^2 \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, t^{k+r}$
  Rewrite ODE with series expansions
  $a_0(-1+r)^2 t^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)^2 - a_{k-1}(k+r-1)\right) t^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 1$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-1)\left(a_k(k+r-1) - a_{k-1}\right) = 0$
- Shift index using $k->k+1$
  $(k+r)\left(a_{k+1}(k+r) - a_k\right) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_k}{k+r}$
- Recursion relation for $r = 1$
  $a_{k+1} = \frac{a_k}{k+1}$
- Solution for $r = 1$
  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = \frac{a_k}{k+1}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
```

```
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 15

```
dsolve(t^2*diff(diff(y(t),t),t)-t*(t+1)*diff(y(t),t)+y(t) = 0,y(t),singsol=all)
```

$$y = \mathrm{e}^{t} t (c_1 + c_2 \,\mathrm{Ei}_1(t))$$

Mathematica DSolve solution

Solving time : 35.564 (sec)
Leaf size : 44

```
DSolve[{t^2*D[y[t],{t,2}]-t*(1+t)*D[y[t],t]+y[t]==0,{}},y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to e^t t \left(\int_1^t \frac{e^{-K[1]} c_1}{K[1]} dK[1] + c_2 \right)$$
$$y(t) \to c_2 e^t t$$

2.1.633 Problem 650

Internal problem ID [9805]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 650
Date solved : `Monday, January 27, 2025 at 06:14:28 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + 4xy' + \left(4x^2 + 6\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.162 (sec)

Writing the ode as

$$y'' + 4xy' + \left(4x^2 + 6\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4x \\ C &= 4x^2 + 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1209: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x}{1}\, dx} \\ &= z_1 e^{-x^2} \\ &= z_1 \left(\mathrm{e}^{-x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x^2} \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-2x^2}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-x^2} \cos(2x)\right) + c_2 \left(\mathrm{e}^{-x^2} \cos(2x) \left(\frac{\tan(2x)}{2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + 4x\left(\frac{d}{dx}y(x)\right) + (4x^2+6)\, y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  - Shift index using $k- > k - m$

    $x^m \cdot y(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- > k+2$

    $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 + 6a_0 + (6a_3 + 10a_1)\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) + 2a_k(2k+3) + 4a_{k-2})\, x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 + 6a_0 = 0, 6a_3 + 10a_1 = 0]$

- Solve for the dependent coefficient(s)

  $\left\{a_2 = -3a_0, a_3 = -\frac{5a_1}{3}\right\}$

- Each term in the series must be 0, giving the recursion relation
  $(k^2+3k+2)\, a_{k+2}+4a_k k+6a_k+4a_{k-2}=0$
- Shift index using $k->k+2$
  $\left((k+2)^2+3k+8\right) a_{k+4}+4a_{k+2}(k+2)+6a_{k+2}+4a_k=0$
- Recursion relation that defines the series solution to the ODE
  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^k, a_{k+4}=-\frac{2(2k a_{k+2}+2a_k+7a_{k+2})}{k^2+7k+12}, a_2=-3a_0, a_3=-\frac{5a_1}{3}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.017 (sec)
Leaf size : 24

```
dsolve(diff(diff(y(x),x),x)+4*diff(y(x),x)*x+(4*x^2+6)*y(x) = 0,y(x),singsol=all)
```

$$y=\mathrm{e}^{-x^2}\left(\cos(2x)\, c_1+\sin(2x)\, c_2\right)$$

Mathematica DSolve solution

Solving time : 0.04 (sec)
Leaf size : 37

```
DSolve[{D[y[x],{x,2}]+4*x*D[y[x],x]+(4*x^2+6)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tru
```

$$y(x)\to\frac{1}{4}e^{-x(x+2i)}\left(4c_1-ic_2e^{4ix}\right)$$

2.1.634 Problem 651

Internal problem ID [9806]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 651
Date solved : `Monday, January 27, 2025 at 06:14:29 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-z^2+1\right) y''-3zy'+y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.349 (sec)

Writing the ode as

$$\left(-z^2+1\right) y''-3zy'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -z^2+1 \\ B &= -3z \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(z)=ye^{\int \frac{B}{2A}\,dz}$$

Then (2) becomes

$$z''(z)=rz(z) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{7z^2-10}{4\left(z^2-1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 7z^2-10 \\ t &= 4\left(z^2-1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(z)=\left(\frac{7z^2-10}{4\left(z^2-1\right)^2}\right) z(z) \tag{7}$$

Equation (7) is now solved. After finding $z(z)$ then y is found using the inverse transformation

$$y = z(z)\, e^{-\int \frac{B}{2A}\, dz}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1211: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(z^2-1)^2$. There is a pole at $z = 1$ of order 2. There is a pole at $z = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{17}{16(z-1)} - \frac{3}{16(z-1)^2} - \frac{17}{16(z+1)} - \frac{3}{16(z+1)^2}$$

For the pole at $z = 1$ let b be the coefficient of $\frac{1}{(z-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

For the pole at $z = -1$ let b be the coefficient of $\frac{1}{(z+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{z^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{7z^2 - 10}{4\left(z^2 - 1\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{7}{4}$. Hence

$$\begin{aligned} E_\infty &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{2\} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
1	2	$\{1,2,3\}$
-1	2	$\{1,2,3\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 1, \; e_2 = 1, \; e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(z)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(2 - (1 + (1))) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{z-c} \\ &= \frac{1}{2}\left(\frac{1}{(z-(1))} + \frac{1}{(z-(-1))}\right) \\ &= \frac{1}{2z-2} + \frac{1}{2z+2} \end{aligned}$$

Now we search for a monic polynomial $p(z)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(z)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2z-2} + \frac{1}{2z+2} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{2z-2} + \frac{1}{2z+2}\right)w + \frac{-7z^2+8}{4\left(z^2-1\right)^2} = 0$$

Solving for ω gives

$$\omega = \frac{z + 2\sqrt{2z^2-2}}{2\left(z-1\right)\left(z+1\right)}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(z) &= e^{\int \omega \, dz} \\ &= e^{\int \frac{z+2\sqrt{2z^2-2}}{2(z-1)(z+1)}dz} \\ &= \left(z^2-1\right)^{1/4} 2^{\frac{\sqrt{2}}{2}} \left(\sqrt{z^2-1}+z\right)^{\sqrt{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dz} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-3z}{-z^2+1}\, dz} \\ &= z_1 e^{-\frac{3\ln(z-1)}{4} - \frac{3\ln(z+1)}{4}} \\ &= z_1 \left(\frac{1}{\left(z-1\right)^{3/4}\left(z+1\right)^{3/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(z^2-1\right)^{1/4} 2^{\frac{\sqrt{2}}{2}} \left(\sqrt{z^2-1}+z\right)^{\sqrt{2}}}{\left(z-1\right)^{3/4}\left(z+1\right)^{3/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dz}}{y_1^2}\, dz$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3z}{-z^2+1}\, dz}}{\left(y_1\right)^2}\, dz \\ &= y_1 \int \frac{e^{-\frac{3\ln(z-1)}{2} - \frac{3\ln(z+1)}{2}}}{\left(y_1\right)^2}\, dz \\ &= y_1 \left(-\frac{2^{-\sqrt{2}}\sqrt{2}\left(\sqrt{z^2-1}+z\right)^{-2\sqrt{2}}}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\left(z^2-1\right)^{1/4} 2^{\frac{\sqrt{2}}{2}} \left(\sqrt{z^2-1}+z\right)^{\sqrt{2}}}{\left(z-1\right)^{3/4}\left(z+1\right)^{3/4}}\right) + c_2 \left(\frac{\left(z^2-1\right)^{1/4} 2^{\frac{\sqrt{2}}{2}} \left(\sqrt{z^2-1}+z\right)^{\sqrt{2}}}{\left(z-1\right)^{3/4}\left(z+1\right)^{3/4}} \left(-\frac{2^{-\sqrt{2}}\sqrt{2}\left(\sqrt{z^2-1}+z\right)^{-}}{4}\right.\right. \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(-z^2+1)\left(\frac{d^2}{dz^2}y(z)\right)-3z\left(\frac{d}{dz}y(z)\right)+y(z)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dz^2}y(z)$

- Isolate 2nd derivative

  $\frac{d^2}{dz^2}y(z)=\frac{y(z)}{z^2-1}-\frac{3z\left(\frac{d}{dz}y(z)\right)}{z^2-1}$

- Group terms with $y(z)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dz^2}y(z)+\frac{3z\left(\frac{d}{dz}y(z)\right)}{z^2-1}-\frac{y(z)}{z^2-1}=0$

□ Check to see if z_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(z)=\frac{3z}{z^2-1}, P_3(z)=-\frac{1}{z^2-1}\right]$

  ◦ $(z+1)\cdot P_2(z)$ is analytic at $z=-1$

    $\left((z+1)\cdot P_2(z)\right)\Big|_{z=-1}=\frac{3}{2}$

  ◦ $(z+1)^2\cdot P_3(z)$ is analytic at $z=-1$

    $\left((z+1)^2\cdot P_3(z)\right)\Big|_{z=-1}=0$

  ◦ $z=-1$is a regular singular point

  Check to see if z_0 is a regular singular point

  $z_0=-1$

- Multiply by denominators

  $(z^2-1)\left(\frac{d^2}{dz^2}y(z)\right)+3z\left(\frac{d}{dz}y(z)\right)-y(z)=0$

- Change variables using $z=u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(3u-3)\left(\frac{d}{du}y(u)\right)-y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  ◦ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

  ◦ Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-a_0r(1+2r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-a_{k+1}(k+1+r)(2k+3+2r)+a_k(k^2+2kr+r^2+2k+2r-1)\right)\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-r(1+2r)=0$

- Values of r that satisfy the indicial equation

$r\in\left\{0,-\frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

$-2\left(k+r+\frac{3}{2}\right)(k+1+r)\,a_{k+1}+\left(k^{2}+(2r+2)\,k+r^{2}+2r-1\right)a_{k}=0$

- Recursion relation that defines series solution to ODE

$a_{k+1}=\frac{\left(k^{2}+2kr+r^{2}+2k+2r-1\right)a_{k}}{(2k+3+2r)(k+1+r)}$

- Recursion relation for $r=0$

$a_{k+1}=\frac{\left(k^{2}+2k-1\right)a_{k}}{(2k+3)(k+1)}$

- Solution for $r=0$

$\left[y(u)=\sum_{k=0}^{\infty}a_{k}u^{k},a_{k+1}=\frac{\left(k^{2}+2k-1\right)a_{k}}{(2k+3)(k+1)}\right]$

- Revert the change of variables $u=z+1$

$\left[y(z)=\sum_{k=0}^{\infty}a_{k}(z+1)^{k},a_{k+1}=\frac{\left(k^{2}+2k-1\right)a_{k}}{(2k+3)(k+1)}\right]$

- Recursion relation for $r=-\frac{1}{2}$

$a_{k+1}=\frac{\left(k^{2}+k-\frac{7}{4}\right)a_{k}}{(2k+2)\left(k+\frac{1}{2}\right)}$

- Solution for $r=-\frac{1}{2}$

$\left[y(u)=\sum_{k=0}^{\infty}a_{k}u^{k-\frac{1}{2}},a_{k+1}=\frac{\left(k^{2}+k-\frac{7}{4}\right)a_{k}}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$

- Revert the change of variables $u=z+1$

$\left[y(z)=\sum_{k=0}^{\infty}a_{k}(z+1)^{k-\frac{1}{2}},a_{k+1}=\frac{\left(k^{2}+k-\frac{7}{4}\right)a_{k}}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$

- Combine solutions and rename parameters

$\left[y(z)=\left(\sum_{k=0}^{\infty}a_{k}(z+1)^{k}\right)+\left(\sum_{k=0}^{\infty}b_{k}(z+1)^{k-\frac{1}{2}}\right),a_{k+1}=\frac{\left(k^{2}+2k-1\right)a_{k}}{(2k+3)(k+1)},b_{k+1}=\frac{\left(k^{2}+k-\frac{7}{4}\right)b_{k}}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 1.803 (sec)
Leaf size : 45

```
dsolve((-z^2+1)*diff(diff(y(z),z),z)-3*z*diff(y(z),z)+y(z) = 0,y(z),singsol=all)
```

$$y(z)=\frac{c_{1}\left(z+\sqrt{z^{2}-1}\right)^{\sqrt{2}}+c_{2}\left(z+\sqrt{z^{2}-1}\right)^{-\sqrt{2}}}{\sqrt{z^{2}-1}}$$

Mathematica DSolve solution

Solving time : 0.061 (sec)
Leaf size : 90

```
DSolve[{(1-z^2)*D[y[z],{z,2}]-3*z*D[y[z],z]+y[z]==0,{}},y[z],z,IncludeSingularSolutions->Tru
```

$$y(z) \to \frac{\sqrt{2}c_1 \cos\left(2\sqrt{2}\arcsin\left(\frac{\sqrt{1-z}}{\sqrt{2}}\right)\right) + \sqrt{\pi}c_2\sqrt[4]{1-z^2}Q_{-\frac{1}{2}+\sqrt{2}}^{\frac{1}{2}}(z)}{\sqrt{\pi}\sqrt[4]{-\left(z^2-1\right)^2}}$$

2.1.635 Problem 652

Internal problem ID [9807]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 652
Date solved : `Monday, January 27, 2025 at 06:14:30 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4zy'' + 2(1 - z)\,y' - y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.232 (sec)

Writing the ode as

$$4zy'' + (-2z + 2)\,y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4z \\ B &= -2z + 2 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(z) = ye^{\int \frac{B}{2A}\,dz}$$

Then (2) becomes

$$z''(z) = rz(z) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{z^2 + 2z - 3}{16z^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= z^2 + 2z - 3 \\ t &= 16z^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(z) = \left(\frac{z^2 + 2z - 3}{16z^2}\right) z(z) \tag{7}$$

Equation (7) is now solved. After finding $z(z)$ then y is found using the inverse transformation

$$y = z(z)\, e^{-\int \frac{B}{2A}\, dz}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1213: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16z^2$. There is a pole at $z = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16} + \frac{1}{8z} - \frac{3}{16z^2}$$

For the pole at $z = 0$ let b be the coefficient of $\frac{1}{z^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving z^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i z^i \\ &= \sum_{i=0}^{0} a_i z^i \end{aligned} \tag{8}$$

Let a be the coefficient of $z^v = z^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{4} + \frac{1}{4z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{z^4} + \frac{2}{z^5} - \frac{9}{2z^6} + \frac{21}{2z^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i z^i \\ &= \frac{1}{4} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $z^{v-1} = z^{-1} = \frac{1}{z}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{16}$$

This shows that the coefficient of $\frac{1}{z}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{z}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{z}$ in r will be the coefficient in R of the term in z of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{z^2 + 2z - 3}{16z^2} \\ &= Q + \frac{R}{16z^2} \\ &= \left(\frac{1}{16}\right) + \left(\frac{2z - 3}{16z^2}\right) \\ &= \frac{1}{16} + \frac{2z - 3}{16z^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term z in the remainder R is 2. Dividing this by leading coefficient in t which is 16 gives $\frac{1}{8}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{8}\right) - (0) \\ &= \frac{1}{8} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{4} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{8}}{\frac{1}{4}} - 0\right) &&= \frac{1}{4} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{8}}{\frac{1}{4}} - 0\right) &&= -\frac{1}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{z^2 + 2z - 3}{16z^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{4} - \left(\frac{1}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{z-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{z-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{4z} + \left(\frac{1}{4}\right) \\ &= \frac{1}{4} + \frac{1}{4z} \\ &= \frac{z+1}{4z} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(z)$ of degree $d = 0$ to solve the ode. The polynomial $p(z)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(z) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{4} + \frac{1}{4z}\right)(0) + \left(\left(-\frac{1}{4z^2}\right) + \left(\frac{1}{4} + \frac{1}{4z}\right)^2 - \left(\frac{z^2+2z-3}{16z^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(z) &= p e^{\int \omega \, dz} \\ &= e^{\int \left(\frac{1}{4} + \frac{1}{4z}\right) dz} \\ &= z^{1/4} e^{\frac{z}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dz} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2z+2}{4z}\,dz} \\ &= z_1 e^{\frac{z}{4}-\frac{\ln(z)}{4}} \\ &= z_1\left(\frac{e^{\frac{z}{4}}}{z^{1/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{\frac{z}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dz}}{y_1^2}\,dz$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2z+2}{4z}\,dz}}{(y_1)^2}\,dz \\ &= y_1 \int \frac{e^{\frac{z}{2}-\frac{\ln(z)}{2}}}{(y_1)^2}\,dz \\ &= y_1\left(\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,\sqrt{z}}{2}\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(e^{\frac{z}{2}}\right) + c_2\left(e^{\frac{z}{2}}\left(\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,\sqrt{z}}{2}\right)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4z\left(\frac{d^2}{dz^2}y(z)\right) + 2(1-z)\left(\frac{d}{dz}y(z)\right) - y(z) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dz^2}y(z)$

- Isolate 2nd derivative

  $\frac{d^2}{dz^2}y(z) = \frac{y(z)}{4z} + \frac{(z-1)\left(\frac{d}{dz}y(z)\right)}{2z}$

- Group terms with $y(z)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d^2}{dz^2}y(z) - \frac{(z-1)\left(\frac{d}{dz}y(z)\right)}{2z} - \frac{y(z)}{4z} = 0$

- ☐ Check to see if $z_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(z) = -\frac{z-1}{2z}, P_3(z) = -\frac{1}{4z}\right]$
  - $z \cdot P_2(z)$ is analytic at $z = 0$

$(z \cdot P_2(z))\Big|_{z=0} = \frac{1}{2}$

- ◦ $z^2 \cdot P_3(z)$ is analytic at $z = 0$

  $(z^2 \cdot P_3(z))\Big|_{z=0} = 0$

- ◦ $z = 0$is a regular singular point

  Check to see if $z_0 = 0$ is a regular singular point

  $z_0 = 0$

- Multiply by denominators

  $4z\left(\frac{d^2}{dz^2}y(z)\right) + (-2z+2)\left(\frac{d}{dz}y(z)\right) - y(z) = 0$

- Assume series solution for $y(z)$

  $y(z) = \sum_{k=0}^{\infty} a_k z^{k+r}$

- □ Rewrite ODE with series expansions
  - ◦ Convert $z^m \cdot \left(\frac{d}{dz}y(z)\right)$ to series expansion for $m = 0..1$

    $z^m \cdot \left(\frac{d}{dz}y(z)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, z^{k+r-1+m}$

  - ◦ Shift index using $k->k+1-m$

    $z^m \cdot \left(\frac{d}{dz}y(z)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, z^{k+r}$

  - ◦ Convert $z \cdot \left(\frac{d^2}{dz^2}y(z)\right)$ to series expansion

    $z \cdot \left(\frac{d^2}{dz^2}y(z)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, z^{k+r-1}$

  - ◦ Shift index using $k->k+1$

    $z \cdot \left(\frac{d^2}{dz^2}y(z)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, z^{k+r}$

  Rewrite ODE with series expansions

  $2a_0r(-1+2r)\, z^{-1+r} + \left(\sum_{k=0}^{\infty} (2a_{k+1}(k+1+r)\,(2k+2r+1) - a_k(2k+2r+1))\, z^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $2r(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r+\frac{1}{2}\right)\left(a_{k+1}(k+1+r) - \frac{a_k}{2}\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{2(k+1+r)}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{2(k+1)}$

- Solution for $r = 0$

  $\left[y(z) = \sum_{k=0}^{\infty} a_k z^k, a_{k+1} = \frac{a_k}{2(k+1)}\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = \frac{a_k}{2\left(k+\frac{3}{2}\right)}$

- Solution for $r = \frac{1}{2}$

  $\left[y(z) = \sum_{k=0}^{\infty} a_k z^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k}{2\left(k+\frac{3}{2}\right)}\right]$

- Combine solutions and rename parameters

$$\left[y(z) = \left(\sum_{k=0}^{\infty} a_k z^k\right) + \left(\sum_{k=0}^{\infty} b_k z^{k+\frac{1}{2}}\right), a_{k+1} = \frac{a_k}{2(k+1)}, b_{k+1} = \frac{b_k}{2(k+\frac{3}{2})}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.031 (sec)
Leaf size : 22

```
dsolve(4*z*diff(diff(y(z),z),z)+2*(1-z)*diff(y(z),z)-y(z) = 0,y(z),singsol=all)
```

$$y(z) = e^{\frac{z}{2}}\left(\operatorname{erf}\left(\frac{\sqrt{2}\,\sqrt{z}}{2}\right) c_1 + c_2\right)$$

Mathematica DSolve solution

Solving time : 0.066 (sec)
Leaf size : 44

```
DSolve[{4*z*D[y[z],{z,2}]+2*(1-z)*D[y[z],z]-y[z]==0,{}},y[z],z,IncludeSingularSolutions->True]
```

$$y(z) \to e^{\frac{z}{2}-\frac{1}{4}}\left(\sqrt{e}c_1 - \sqrt{2}c_2\Gamma\left(\frac{1}{2}, \frac{z}{2}\right)\right)$$

2.1.636 Problem 653

Internal problem ID [9808]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 653
Date solved : `Monday, January 27, 2025 at 06:14:30 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$f'' + 2(z-1)\, f' + 4f = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.234 (sec)

Writing the ode as

$$f'' + (2z-2)\, f' + 4f = 0 \tag{1}$$
$$Af'' + Bf' + Cf = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2z-2 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(z) = f e^{\int \frac{B}{2A}\, dz}$$

Then (2) becomes

$$z''(z) = rz(z) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{z^2 - 2z - 2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= z^2 - 2z - 2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(z) = \left(z^2 - 2z - 2\right) z(z) \tag{7}$$

Equation (7) is now solved. After finding $z(z)$ then f is found using the inverse transformation

$$f = z(z)\, e^{-\int \frac{B}{2A}\, dz}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1215: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving z^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i z^i \\ &= \sum_{i=0}^{1} a_i z^i \end{aligned} \tag{8}$$

Let a be the coefficient of $z^v = z^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx z - 1 - \frac{3}{2z} - \frac{3}{2z^2} - \frac{21}{8z^3} - \frac{39}{8z^4} - \frac{159}{16z^5} - \frac{339}{16z^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i z^i \\ &= z - 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $z^{v-1} = z^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = z^2 - 2z + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{z^2 - 2z - 2}{1} \\ &= Q + \frac{R}{1} \\ &= (z^2 - 2z - 2) + (0) \\ &= z^2 - 2z - 2 \end{aligned}$$

We see that the coefficient of the term $\frac{1}{z}$ in the quotient is -2. Now b can be found.

$$\begin{aligned} b &= (-2) - (1) \\ &= -3 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= z - 1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-3}{1} - 1\right) &&= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-3}{1} - 1\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = z^2 - 2z - 2$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$z - 1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{z-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)(z-1) \\ &= 1-z \\ &= 1-z\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(z)$ of degree $d=1$ to solve the ode. The polynomial $p(z)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(z) = z + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2(1-z)(1) + \left((-1) + (1-z)^2 - (z^2 - 2z - 2)\right) &= 0 \\ 2 + 2a_0 &= 0\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1\}$$

Substituting these coefficients in $p(z)$ in eq. (2A) results in

$$p(z) = z - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(z) &= p e^{\int \omega \, dz} \\ &= (z-1)\, e^{\int (1-z)dz} \\ &= (z-1)\, e^{z - \frac{1}{2}z^2} \\ &= (z-1)\, e^{-\frac{z(-2+z)}{2}}\end{aligned}$$

The first solution to the original ode in f is found from

$$\begin{aligned}f_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dz} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2z-2}{1}\, dz} \\ &= z_1 e^{z - \frac{1}{2}z^2} \\ &= z_1 \left(e^{-\frac{z(-2+z)}{2}}\right)\end{aligned}$$

Which simplifies to

$$f_1 = e^{-z(-2+z)}(z-1)$$

The second solution f_2 to the original ode is found using reduction of order

$$f_2 = f_1 \int \frac{e^{\int -\frac{B}{A}\, dz}}{f_1^2}\, dz$$

Substituting gives

$$
\begin{aligned}
f_2 &= f_1 \int \frac{e^{\int -\frac{2z-2}{1}\, dz}}{(f_1)^2}\, dz \\
&= f_1 \int \frac{e^{-z^2+2z}}{(f_1)^2}\, dz \\
&= f_1 \left(-\frac{e^{(z-1)^2-1}}{z-1} - i\sqrt{\pi}\, e^{-1} \operatorname{erf}\left(i(z-1)\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
f &= c_1 f_1 + c_2 f_2 \\
&= c_1 \left(e^{-z(-2+z)}(z-1)\right) + c_2 \left(e^{-z(-2+z)}(z-1) \left(-\frac{e^{(z-1)^2-1}}{z-1} - i\sqrt{\pi}\, e^{-1} \operatorname{erf}\left(i(z-1)\right)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\frac{d^2}{dz^2}f(z) + 2(z-1)\left(\frac{d}{dz}f(z)\right) + 4f(z) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dz^2}f(z)$
- Isolate 2nd derivative
  $\frac{d^2}{dz^2}f(z) = -2(z-1)\left(\frac{d}{dz}f(z)\right) - 4f(z)$
- Group terms with $f(z)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dz^2}f(z) + (-2+2z)\left(\frac{d}{dz}f(z)\right) + 4f(z) = 0$
- Assume series solution for $f(z)$
  $f(z) = \sum_{k=0}^{\infty} a_k z^k$

□ Rewrite DE with series expansions
  - Convert $z^m \cdot \left(\frac{d}{dz}f(z)\right)$ to series expansion for $m = 0..1$
    $z^m \cdot \left(\frac{d}{dz}f(z)\right) = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, z^{k-1+m}$
  - Shift index using $k- >k+1-m$
    $z^m \cdot \left(\frac{d}{dz}f(z)\right) = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, z^k$
  - Convert $\frac{d^2}{dz^2}f(z)$ to series expansion
    $\frac{d^2}{dz^2}f(z) = \sum_{k=2}^{\infty} a_k k(k-1)\, z^{k-2}$
  - Shift index using $k- >k+2$
    $\frac{d^2}{dz^2}f(z) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, z^k$

  Rewrite DE with series expansions
  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)(k+1) - 2a_{k+1}(k+1) + 2a_k(k+2)\right) z^k = 0$

- Each term in the series must be 0, giving the recursion relation

$$k^2 a_{k+2} + (2a_k - 2a_{k+1} + 3a_{k+2})\, k + 4a_k - 2a_{k+1} + 2a_{k+2} = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[f(z) = \sum_{k=0}^{\infty} a_k z^k, a_{k+2} = -\frac{2(a_k k - a_{k+1} k + 2a_k - a_{k+1})}{k^2 + 3k + 2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 42

```
dsolve(diff(diff(f(z),z),z)+2*(z-1)*diff(f(z),z)+4*f(z) = 0,f(z),singsol=all)
```

$$f(z) = \sqrt{\pi}\, \operatorname{erf}(i(z-1))\, c_2 (z-1)\, \mathrm{e}^{-(z-1)^2} + c_1 \mathrm{e}^{-z(z-2)}(z-1) - i c_2$$

Mathematica DSolve solution

Solving time : 0.133 (sec)
Leaf size : 72

```
DSolve[{D[ f[z],{z,2}]+2*(z-a)*D[ f[z],z]+4*f[z]==0,{}},f[z],z,IncludeSingularSolutions->True]
```

$$f(z) \to e^{z(2a-z)}\left(-\sqrt{\pi} c_2 \sqrt{(a-z)^2} \operatorname{erfi}\left(\sqrt{(a-z)^2}\right) + c_2 e^{(a-z)^2} - 2a c_1 + 2c_1 z\right)$$

2.1.637 Problem 654

Internal problem ID [9809]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 654
Date solved : Monday, January 27, 2025 at 06:14:31 PM
CAS classification : [_Lienard]

Solve

$$zy'' - 2y' + zy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.263 (sec)

Writing the ode as

$$zy'' - 2y' + zy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= z \\ B &= -2 \\ C &= z \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(z) = ye^{\int \frac{B}{2A}\,dz}$$

Then (2) becomes

$$z''(z) = rz(z) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-z^2 + 2}{z^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -z^2 + 2 \\ t &= z^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(z) = \left(\frac{-z^2 + 2}{z^2}\right) z(z) \tag{7}$$

Equation (7) is now solved. After finding $z(z)$ then y is found using the inverse transformation

$$y = z(z)\, e^{-\int \frac{B}{2A}\, dz}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1217: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = z^2$. There is a pole at $z = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -1 + \frac{2}{z^2}$$

For the pole at $z = 0$ let b be the coefficient of $\frac{1}{z^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving z^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i z^i \\ &= \sum_{i=0}^{0} a_i z^i \end{aligned} \tag{8}$$

Let a be the coefficient of $z^v = z^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx i - \frac{i}{z^2} - \frac{i}{2z^4} - \frac{i}{2z^6} - \frac{5i}{8z^8} - \frac{7i}{8z^{10}} - \frac{21i}{16z^{12}} - \frac{33i}{16z^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i z^i \\ &= i \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $z^{v-1} = z^{-1} = \frac{1}{z}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -1$$

This shows that the coefficient of $\frac{1}{z}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{z}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{z}$ in r will be the coefficient in R of the term in z of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-z^2 + 2}{z^2} \\ &= Q + \frac{R}{z^2} \\ &= (-1) + \left(\frac{2}{z^2}\right) \\ &= -1 + \frac{2}{z^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term z in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= i \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{i} - 0\right) &= 0 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{i} - 0\right) &= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-z^2 + 2}{z^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	i	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{z-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{z-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{z} + (-)(i) \\ &= -\frac{1}{z} - i \\ &= -\frac{1}{z} - i \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(z)$ of degree $d = 1$ to solve the ode. The polynomial $p(z)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(z) = z + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{z} - i\right)(1) + \left(\left(\frac{1}{z^2}\right) + \left(-\frac{1}{z} - i\right)^2 - \left(\frac{-z^2+2}{z^2}\right) \right) = 0$$
$$\frac{2ia_0 - 2}{z} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -i\}$$

Substituting these coefficients in $p(z)$ in eq. (2A) results in

$$p(z) = z - i$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(z) &= p e^{\int \omega\, dz} \\ &= (z - i)\, e^{\int \left(-\frac{1}{z} - i\right) dz} \\ &= (z - i)\, e^{-\ln(z) - iz} \\ &= \frac{(z - i)\, e^{-iz}}{z} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dz} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{z}\,dz} \\ &= z_1 e^{\ln(z)} \\ &= z_1(z) \end{aligned}$$

Which simplifies to

$$y_1 = (z - i)\,\mathrm{e}^{-iz}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dz}}{y_1^2}\,dz$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{z}\,dz}}{(y_1)^2}\,dz \\ &= y_1 \int \frac{e^{2\ln(z)}}{(y_1)^2}\,dz \\ &= y_1 \left(\frac{(iz-1)\,\mathrm{e}^{2iz}}{-2z+2i}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left((z-i)\,\mathrm{e}^{-iz}\right) + c_2\left((z-i)\,\mathrm{e}^{-iz}\left(\frac{(iz-1)\,\mathrm{e}^{2iz}}{-2z+2i}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$z\left(\frac{d^2}{dz^2}y(z)\right) - 2\frac{d}{dz}y(z) + y(z)\,z = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dz^2}y(z)$

- Isolate 2nd derivative

  $\frac{d^2}{dz^2}y(z) = -y(z) + \frac{2\left(\frac{d}{dz}y(z)\right)}{z}$

- Group terms with $y(z)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dz^2}y(z) - \frac{2\left(\frac{d}{dz}y(z)\right)}{z} + y(z) = 0$

□ Check to see if $z_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(z) = -\frac{2}{z}, P_3(z) = 1\right]$

  ◦ $z \cdot P_2(z)$ is analytic at $z = 0$

    $\left(z \cdot P_2(z)\right)\Big|_{z=0} = -2$

- $z^2 \cdot P_3(z)$ is analytic at $z = 0$

  $\left(z^2 \cdot P_3(z)\right)\bigg|_{z=0} = 0$

- $z = 0$is a regular singular point

  Check to see if $z_0 = 0$ is a regular singular point

  $z_0 = 0$

- Multiply by denominators

  $z\left(\frac{d^2}{dz^2}y(z)\right) - 2\frac{d}{dz}y(z) + y(z)\,z = 0$

- Assume series solution for $y(z)$

  $y(z) = \sum_{k=0}^{\infty} a_k z^{k+r}$

- Rewrite ODE with series expansions

  - Convert $z \cdot y(z)$ to series expansion

    $z \cdot y(z) = \sum_{k=0}^{\infty} a_k z^{k+r+1}$

  - Shift index using $k- > k - 1$

    $z \cdot y(z) = \sum_{k=1}^{\infty} a_{k-1} z^{k+r}$

  - Convert $\frac{d}{dz}y(z)$ to series expansion

    $\frac{d}{dz}y(z) = \sum_{k=0}^{\infty} a_k(k+r)\, z^{k+r-1}$

  - Shift index using $k- > k + 1$

    $\frac{d}{dz}y(z) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, z^{k+r}$

  - Convert $z \cdot \left(\frac{d^2}{dz^2}y(z)\right)$ to series expansion

    $z \cdot \left(\frac{d^2}{dz^2}y(z)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, z^{k+r-1}$

  - Shift index using $k- > k + 1$

    $z \cdot \left(\frac{d^2}{dz^2}y(z)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, z^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-3+r)\, z^{-1+r} + a_1(1+r)\,(-2+r)\, z^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k-2+r) + a_{k-1}\right) z^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 3\}$

- Each term must be 0

  $a_1(1+r)\,(-2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)\,(k-2+r) + a_{k-1} = 0$

- Shift index using $k- > k + 1$

  $a_{k+2}(k+2+r)\,(k+r-1) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+2+r)(k+r-1)}$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{a_k}{(k+2)(k-1)}$

- Solution for $r = 0$

$$\left[y(z)=\sum_{k=0}^{\infty} a_k z^k, a_{k+2}=-\frac{a_k}{(k+2)(k-1)}, -2a_1=0\right]$$

- Recursion relation for $r=3$

$$a_{k+2}=-\frac{a_k}{(k+5)(k+2)}$$

- Solution for $r=3$

$$\left[y(z)=\sum_{k=0}^{\infty} a_k z^{k+3}, a_{k+2}=-\frac{a_k}{(k+5)(k+2)}, 4a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y(z)=\left(\sum_{k=0}^{\infty} a_k z^k\right)+\left(\sum_{k=0}^{\infty} b_k z^{k+3}\right), a_{k+2}=-\frac{a_k}{(k+2)(k-1)}, -2a_1=0, b_{k+2}=-\frac{b_k}{(5+k)(k+2)}, 4b_1=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.022 (sec)
Leaf size : 23

```
dsolve(z*diff(diff(y(z),z),z)-2*diff(y(z),z)+z*y(z) = 0,y(z),singsol=all)
```

$$y(z)=(c_1 z+c_2)\cos(z)+\sin(z)(c_2 z-c_1)$$

Mathematica DSolve solution

Solving time : 0.05 (sec)
Leaf size : 39

```
DSolve[{z*D[y[z],{z,2}]-2*D[y[z],z]+z*y[z]==0,{}},y[z],z,IncludeSingularSolutions->True]
```

$$y(z)\to -\sqrt{\frac{2}{\pi}}((c_1 z+c_2)\cos(z)+(c_2 z-c_1)\sin(z))$$

2.1.638 Problem 655

Internal problem ID [9810]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 655
Date solved : `Monday, January 27, 2025 at 06:14:32 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$zy'' + (2z-3)\,y' + \frac{4y}{z} = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.260 (sec)

Writing the ode as

$$zy'' + (2z-3)\,y' + \frac{4y}{z} = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= z \\ B &= 2z-3 \\ C &= \frac{4}{z} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(z) = ye^{\int \frac{B}{2A}\,dz}$$

Then (2) becomes

$$z''(z) = rz(z) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4z^2 - 12z - 1}{4z^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4z^2 - 12z - 1 \\ t &= 4z^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(z) = \left(\frac{4z^2 - 12z - 1}{4z^2}\right) z(z) \tag{7}$$

Equation (7) is now solved. After finding $z(z)$ then y is found using the inverse transformation

$$y = z(z)\, e^{-\int \frac{B}{2A}\, dz}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1219: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4z^2$. There is a pole at $z = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 - \frac{1}{4z^2} - \frac{3}{z}$$

For the pole at $z = 0$ let b be the coefficient of $\frac{1}{z^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving z^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i z^i \\
&= \sum_{i=0}^{0} a_i z^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $z^v = z^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r} \approx 1 - \frac{3}{2z} - \frac{5}{4z^2} - \frac{15}{8z^3} - \frac{115}{32z^4} - \frac{495}{64z^5} - \frac{2285}{128z^6} - \frac{11055}{256z^7} + \dots \tag{9}
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a = 1
$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i z^i \\
&= 1
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $z^{v-1} = z^{-1} = \frac{1}{z}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = 1
$$

This shows that the coefficient of $\frac{1}{z}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{z}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{z}$ in r will be the coefficient in R of the term in z of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{4z^2 - 12z - 1}{4z^2} \\
&= Q + \frac{R}{4z^2} \\
&= (1) + \left(\frac{-12z - 1}{4z^2}\right) \\
&= 1 + \frac{-12z - 1}{4z^2}
\end{aligned}
$$

Since the degree of t is 2, then we see that the coefficient of the term z in the remainder R is -12. Dividing this by leading coefficient in t which is 4 gives -3. Now b can be found.

$$
\begin{aligned}
b &= (-3) - (0) \\
&= -3
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 1 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{-3}{1} - 0\right) = -\frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{-3}{1} - 0\right) = \frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4z^2 - 12z - 1}{4z^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$-\frac{3}{2}$	$\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^+) \\ &= \frac{3}{2} - \left(\frac{1}{2}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{z-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{z-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2z} + (-)(1) \\ &= \frac{1}{2z} - 1 \\ &= \frac{1}{2z} - 1 \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(z)$ of degree $d = 1$ to solve the ode. The polynomial $p(z)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(z) = z + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2z} - 1\right)(1) + \left(\left(-\frac{1}{2z^2}\right) + \left(\frac{1}{2z} - 1\right)^2 - \left(\frac{4z^2 - 12z - 1}{4z^2}\right)\right) = 0$$

$$\frac{1 + 2a_0}{z} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{ a_0 = -\frac{1}{2} \right\}$$

Substituting these coefficients in $p(z)$ in eq. (2A) results in

$$p(z) = z - \frac{1}{2}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(z) &= p e^{\int \omega \, dz} \\ &= \left(z - \frac{1}{2}\right) e^{\int \left(\frac{1}{2z} - 1\right) dz} \\ &= \left(z - \frac{1}{2}\right) e^{-z + \frac{\ln(z)}{2}} \\ &= \frac{(-1 + 2z)\sqrt{z}\, e^{-z}}{2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dz} \\ &= z_1 e^{-\int \frac{1}{2} \frac{2z-3}{z} \, dz} \\ &= z_1 e^{-z + \frac{3\ln(z)}{2}} \\ &= z_1 \left(z^{3/2} e^{-z}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{z^2 e^{-2z}(-1 + 2z)}{2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dz}}{y_1^2} \, dz$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2z-3}{z} \, dz}}{(y_1)^2} \, dz \\ &= y_1 \int \frac{e^{-2z + 3\ln(z)}}{(y_1)^2} \, dz \\ &= y_1 \left(-4 \operatorname{Ei}_1(-2z) - \frac{4 e^{2z}}{-1 + 2z}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{z^2 e^{-2z}(-1 + 2z)}{2}\right) + c_2 \left(\frac{z^2 e^{-2z}(-1 + 2z)}{2} \left(-4 \operatorname{Ei}_1(-2z) - \frac{4 e^{2z}}{-1 + 2z}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$z\left(\frac{d^2}{dz^2}y(z)\right)+(-3+2z)\left(\frac{d}{dz}y(z)\right)+\frac{4y(z)}{z}=0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dz^2}y(z)$
- Isolate 2nd derivative
  $\frac{d^2}{dz^2}y(z)=-\frac{4y(z)}{z^2}-\frac{(-3+2z)\left(\frac{d}{dz}y(z)\right)}{z}$
- Group terms with $y(z)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dz^2}y(z)+\frac{(-3+2z)\left(\frac{d}{dz}y(z)\right)}{z}+\frac{4y(z)}{z^2}=0$

□ Check to see if $z_0=0$ is a regular singular point
  - Define functions
    $\left[P_2(z)=\frac{-3+2z}{z},P_3(z)=\frac{4}{z^2}\right]$
  - $z\cdot P_2(z)$ is analytic at $z=0$
    $(z\cdot P_2(z))\bigg|_{z=0}=-3$
  - $z^2\cdot P_3(z)$ is analytic at $z=0$
    $(z^2\cdot P_3(z))\bigg|_{z=0}=4$
  - $z=0$is a regular singular point
    Check to see if $z_0=0$ is a regular singular point
    $z_0=0$

- Multiply by denominators
  $\left(\frac{d^2}{dz^2}y(z)\right)z^2+z(-3+2z)\left(\frac{d}{dz}y(z)\right)+4y(z)=0$
- Assume series solution for $y(z)$
  $y(z)=\sum_{k=0}^{\infty}a_k z^{k+r}$

□ Rewrite ODE with series expansions
  - Convert $z^m\cdot\left(\frac{d}{dz}y(z)\right)$ to series expansion for $m=1..2$
    $z^m\cdot\left(\frac{d}{dz}y(z)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,z^{k+r-1+m}$
  - Shift index using $k->k+1-m$
    $z^m\cdot\left(\frac{d}{dz}y(z)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,z^{k+r}$
  - Convert $z^2\cdot\left(\frac{d^2}{dz^2}y(z)\right)$ to series expansion
    $z^2\cdot\left(\frac{d^2}{dz^2}y(z)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,z^{k+r}$

  Rewrite ODE with series expansions
  $a_0(-2+r)^2\,z^r+\left(\sum_{k=1}^{\infty}\left(a_k(k+r-2)^2+2a_{k-1}(k+r-1)\right)z^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-2+r)^2=0$
- Values of r that satisfy the indicial equation
  $r=2$
- Each term in the series must be 0, giving the recursion relation
  $a_k(k+r-2)^2+2a_{k-1}(k+r-1)=0$
- Shift index using $k->k+1$

$$a_{k+1}(k+r-1)^2+2a_k(k+r)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=-\frac{2a_k(k+r)}{(k+r-1)^2}$$

- Recursion relation for $r=2$

$$a_{k+1}=-\frac{2a_k(k+2)}{(k+1)^2}$$

- Solution for $r=2$

$$\left[y(z)=\sum_{k=0}^{\infty}a_k z^{k+2}, a_{k+1}=-\frac{2a_k(k+2)}{(k+1)^2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 36

```
dsolve(z*diff(diff(y(z),z),z)+(2*z-3)*diff(y(z),z)+4/z*y(z) = 0,y(z),singsol=all)
```

$$y(z)=2z^2\left(\mathrm{e}^{-2z}c_2\left(-\frac{1}{2}+z\right)\mathrm{Ei}_1\left(-2z\right)+c_1\left(-\frac{1}{2}+z\right)\mathrm{e}^{-2z}+\frac{c_2}{2}\right)$$

Mathematica DSolve solution

Solving time : 0.225 (sec)
Leaf size : 55

```
DSolve[{z*D[y[z],{z,2}]+(2*z-3)*D[y[z],z]+4/z*y[z]==0,{}},y[z],z,IncludeSingularSolutions->True
```

$$y(z)\to\frac{1}{2}e^{-2z}z^2(2z-1)\left(c_2\int_1^z\frac{4e^{2K[1]}}{(1-2K[1])^2K[1]}dK[1]+c_1\right)$$

2.1.639 Problem 656

Internal problem ID [9811]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 656
Date solved : `Monday, January 27, 2025 at 06:14:32 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + (1 - 2x)\, y' + (x - 1)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.151 (sec)

Writing the ode as

$$xy'' + (1 - 2x)\, y' + (x - 1)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 1 - 2x \\ C &= x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1221: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the gcd$(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\,(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0)+2\left(\frac{1}{2x}\right)(0)+\left(\left(-\frac{1}{2x^2}\right)+\left(\frac{1}{2x}\right)^2-\left(-\frac{1}{4x^2}\right)\right)&=0\\
0&=0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$
\begin{aligned}
z_1(x)&=pe^{\int \omega\,dx}\\
&=\mathrm{e}^{\int \frac{1}{2x}dx}\\
&=\sqrt{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1&=z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\
&=z_1e^{-\int \frac{1}{2}\frac{1-2x}{x}\,dx}\\
&=z_1e^{x-\frac{\ln(x)}{2}}\\
&=z_1\left(\frac{\mathrm{e}^x}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1=\mathrm{e}^x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2&=y_1\int \frac{e^{\int -\frac{1-2x}{x}\,dx}}{(y_1)^2}\,dx\\
&=y_1\int \frac{e^{2x-\ln(x)}}{(y_1)^2}\,dx\\
&=y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y&=c_1y_1+c_2y_2\\
&=c_1(\mathrm{e}^x)+c_2(\mathrm{e}^x(\ln(x)))
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dx^2}y(x)\right)x+(-2x+1)\left(\frac{d}{dx}y(x)\right)+(x-1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(x-1)y(x)}{x}+\frac{(2x-1)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)-\frac{(2x-1)\left(\frac{d}{dx}y(x)\right)}{x}+\frac{(x-1)y(x)}{x}=0$

- □ Check to see if $x_0=0$ is a regular singular point
  - Define functions

    $\left[P_2(x)=-\frac{2x-1}{x},P_3(x)=\frac{x-1}{x}\right]$
  - $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\Big|_{x=0}=1$
  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\Big|_{x=0}=0$
  - $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $\left(\frac{d^2}{dx^2}y(x)\right)x+(-2x+1)\left(\frac{d}{dx}y(x)\right)+(x-1)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

    $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$
  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..1$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-1}$
  - Shift index using $k->k+1$

    $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$a_0 r^2 x^{-1+r} + \left(a_1(1+r)^2 - a_0(1+2r)\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)^2 - a_k(2k+2r+1) + a_{k-1}\right) x^{k+r}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 0$
- Each term must be 0
  $a_1(1+r)^2 - a_0(1+2r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1)^2 + (-2k-1)\, a_k + a_{k-1} = 0$
- Shift index using $k- >k+1$
  $a_{k+2}(k+2)^2 + (-2k-3)\, a_{k+1} + a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{2k a_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}$
- Recursion relation for $r = 0$
  $a_{k+2} = \frac{2k a_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}$
- Solution for $r = 0$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2k a_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}, a_1 - a_0 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 13

```
dsolve(x*diff(diff(y(x),x),x)+(1-2*x)*diff(y(x),x)+(x-1)*y(x) = 0,y(x),singsol=all)
```

$$y = e^x (c_2 \ln(x) + c_1)$$

Mathematica DSolve solution

Solving time : 0.025 (sec)
Leaf size : 17

```
DSolve[{x*D[y[x],{x,2}]+(1-2*x)*D[y[x],x]+(x-1)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to e^x (c_2 \log(x) + c_1)$$

2.1.640 Problem 657

Internal problem ID [9812]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 657
Date solved : `Monday, January 27, 2025 at 06:14:33 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - 2xy' + \left(x^2 + 2\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.131 (sec)

Writing the ode as

$$x^2y'' - 2xy' + \left(x^2 + 2\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1223: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\, dx} \\ &= z_1 e^{\ln(x)} \\ &= z_1(x) \end{aligned}$$

Which simplifies to

$$y_1 = x\cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\, dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x\cos(x)) + c_2(x\cos(x)(\tan(x)))
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) - 2x\left(\frac{d}{dx}y(x)\right) + (x^2+2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(x^2+2)y(x)}{x^2} + \frac{2\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) - \frac{2\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(x^2+2)y(x)}{x^2} = 0$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{2}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 2$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) - 2x\left(\frac{d}{dx}y(x)\right) + (x^2+2)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

$x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)\,(-2+r)\,x^r + a_1 r(-1+r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-1)\,(k+r-2) + a_{k-2})\, x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$

- Each term must be 0

  $a_1 r(-1+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)\,(k+r-2) + a_{k-2} = 0$

- Shift index using $k- >k+2$

  $a_{k+2}(k+1+r)\,(k+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+1+r)(k+r)}$

- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$

- Solution for $r = 1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$

- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$

- Solution for $r = 2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+3)(k+2)}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
```

```
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 15

```
dsolve(x^2*diff(diff(y(x),x),x)-2*diff(y(x),x)*x+(x^2+2)*y(x) = 0,y(x),singsol=all)
```

$$y = x(\sin (x) c_1 + \cos (x) c_2)$$

Mathematica DSolve solution

Solving time : 0.04 (sec)
Leaf size : 33

```
DSolve[{x^2*D[y[x],{x,2}]-2*x*D[y[x],x]+(x^2+2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to c_1 e^{-ix} x - \frac{1}{2} i c_2 e^{ix} x$$

2.1.641 Problem 658

Internal problem ID [9813]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 658
Date solved : `Monday, January 27, 2025 at 06:14:33 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-x^2+1\right) y''-2xy'+2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.249 (sec)

Writing the ode as

$$\left(-x^2+1\right) y''-2xy'+2y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+1 \\ B &= -2x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{2x^2-3}{\left(x^2-1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2x^2-3 \\ t &= \left(x^2-1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{2x^2-3}{\left(x^2-1\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1225: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{4\,(x+1)} - \frac{1}{4\,(x-1)^2} - \frac{1}{4\,(x+1)^2} + \frac{5}{4\,(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^2 - 3}{(x^2 - 1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2x^2 - 3}{(x^2 - 1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x-2} + \frac{1}{2x+2} + (0) \\
&= \frac{1}{2x-2} + \frac{1}{2x+2} \\
&= \frac{x}{x^2 - 1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x-2} + \frac{1}{2x+2}\right)(1) + \left(\left(-\frac{1}{2(x-1)^2} - \frac{1}{2(x+1)^2}\right) + \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right)^2 - \left(\frac{2x^2-3}{(x^2-1)^2}\right.\right.$$
$$-\frac{2a_0}{x^2-1} =$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x) \, e^{\int \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right) dx} \\
&= (x) \sqrt{(x-1)(x+1)} \\
&= x\sqrt{x^2-1}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{-x^2+1} \, dx} \\
&= z_1 e^{-\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x-1}\,\sqrt{x+1}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{x+1}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{-x^2+1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\ln(x-1) - \ln(x+1)}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{1}{x} - \frac{\ln(x+1)}{2} + \frac{\ln(x-1)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{x+1}}\right) + c_2 \left(\frac{x\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{x+1}} \left(\frac{1}{x} - \frac{\ln(x+1)}{2} + \frac{\ln(x-1)}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(-x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)-2x\left(\frac{d}{dx}y(x)\right)+2y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{2y(x)}{x^2-1}-\frac{2\left(\frac{d}{dx}y(x)\right)x}{x^2-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{2\left(\frac{d}{dx}y(x)\right)x}{x^2-1}-\frac{2y(x)}{x^2-1}=0$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x)=\frac{2x}{x^2-1}, P_3(x)=-\frac{2}{x^2-1}\right]$
  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1}=1$
  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1}=0$
  - $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $(x^2-1)\left(\frac{d^2}{dx^2}y(x)\right)+2x\left(\frac{d}{dx}y(x)\right)-2y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(2u-2)\left(\frac{d}{du}y(u)\right)-2y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$
  - Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r^2u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)^2+a_k(k+r+2)(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-2r^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 0$
- Each term in the series must be 0, giving the recursion relation

  $-2a_{k+1}(k+1)^2 + a_k(k+2)(k-1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$
- Recursion relation for $r = 0$; series terminates at $k = 1$

  $a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$
- Apply recursion relation for $k = 0$

  $a_1 = -a_0$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot (-u+1)$
- Revert the change of variables $u = x+1$

  $[y(x) = -a_0 x]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 25

```
dsolve((-x^2+1)*diff(diff(y(x),x),x)-2*diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y = -\frac{\ln(x+1)\, c_2 x}{2} + \frac{c_2 \ln(x-1)\, x}{2} + c_1 x + c_2$$

Mathematica DSolve solution

Solving time : 0.02 (sec)
Leaf size : 33

```
DSolve[{(1-x^2)*D[y[x],{x,2}]-2*x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to c_1 x - \frac{1}{2} c_2 (x \log(1-x) - x \log(x+1) + 2)$$

2.1.642 Problem 659

Internal problem ID [9814]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 659
Date solved : `Monday, January 27, 2025 at 06:14:34 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' + 4xy' + \left(4x^2 - 1\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.138 (sec)

Writing the ode as

$$4x^2y'' + 4xy' + \left(4x^2 - 1\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= 4x \\ C &= 4x^2 - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1227: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x}{4x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{4x^2} dx}}{(y_1)^2} dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2} dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos(x)}{\sqrt{x}}(\tan(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x\left(\frac{d}{dx}y(x)\right) + (4x^2-1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(4x^2-1)y(x)}{4x^2} - \frac{\frac{d}{dx}y(x)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{\frac{d}{dx}y(x)}{x} + \frac{(4x^2-1)y(x)}{4x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x\left(\frac{d}{dx}y(x)\right) + (4x^2-1)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

$x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2+8kr+4r^2-1) + 4a_{k-2} = 0$
- Shift index using $k- >k+2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\,r + 4r^2 - 1\right) + 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$
- Solution for $r = -\frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$
- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.040 (sec)
Leaf size : 17

```
dsolve(4*x^2*diff(diff(y(x),x),x)+4*diff(y(x),x)*x+(4*x^2-1)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{\sin(x) c_1 + \cos(x) c_2}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.034 (sec)
Leaf size : 39

```
DSolve[{4*x^2*D[y[x],{x,2}]+4*x*D[y[x],x]+(4*x^2-1)*y[x]==0,{}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{e^{-ix}(2c_1 - ic_2 e^{2ix})}{2\sqrt{x}}$$

2.1.643 Problem 660

Internal problem ID [9815]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 660
Date solved : `Monday, January 27, 2025 at 06:14:35 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' - (2x + 1)\, y' + 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.211 (sec)

Writing the ode as

$$xy'' + (-2x - 1)\, y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -2x - 1 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 - 4x + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2 - 4x + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 - 4x + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1229: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 - \frac{1}{x} + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 - \frac{1}{2x} + \frac{1}{4x^2} + \frac{1}{8x^3} + \frac{1}{32x^4} - \frac{1}{64x^5} - \frac{3}{128x^6} - \frac{3}{256x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 - 4x + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (1) + \left(\frac{-4x + 3}{4x^2}\right) \\ &= 1 + \frac{-4x + 3}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{1} - 0\right) &&= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{1} - 0\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^2 - 4x + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (1) \\ &= 1 - \frac{1}{2x} \\ &= 1 - \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(1 - \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(1 - \frac{1}{2x}\right)^2 - \left(\frac{4x^2 - 4x + 3}{4x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(1 - \frac{1}{2x}\right) dx} \\ &= \frac{\mathrm{e}^x}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x-1}{x}\,dx} \\ &= z_1 e^{x+\frac{\ln(x)}{2}} \\ &= z_1\left(\sqrt{x}\,\mathrm{e}^{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-1}{x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x+\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(-\frac{(2x+1)\,\mathrm{e}^{2x+\ln(x)}\mathrm{e}^{-4x}}{4x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{2x}\right) + c_2\left(\mathrm{e}^{2x}\left(-\frac{(2x+1)\,\mathrm{e}^{2x+\ln(x)}\mathrm{e}^{-4x}}{4x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dx^2}y(x)\right)x - (2x+1)\left(\frac{d}{dx}y(x)\right) + 2y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{2y(x)}{x} + \frac{(2x+1)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) - \frac{(2x+1)\left(\frac{d}{dx}y(x)\right)}{x} + \frac{2y(x)}{x} = 0$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - ◦ Define functions

    $\left[P_2(x) = -\frac{2x+1}{x}, P_3(x) = \frac{2}{x}\right]$

  - ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -1$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- • Multiply by denominators

  $\left(\frac{d^2}{dx^2}y(x)\right)x + (-2x-1)\left(\frac{d}{dx}y(x)\right) + 2y(x) = 0$

- • Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..1$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  - ◦ Shift index using $k- >k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - ◦ Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

  - ◦ Shift index using $k- >k+1$

    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-2+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k+r-1) - 2a_k(k+r-1)\right) x^{k+r}\right) = 0$

- • $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$

- • Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- • Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_{k+1}(k+1+r) - 2a_k\right) = 0$

- • Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k}{k+1+r}$

- • Recursion relation for $r = 0$

  $a_{k+1} = \frac{2a_k}{k+1}$

- • Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{2a_k}{k+1}\right]$

- • Recursion relation for $r = 2$

  $a_{k+1} = \frac{2a_k}{k+3}$

- • Solution for $r = 2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{2a_k}{k+3}\right]$

- • Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+1} = \frac{2a_k}{k+1}, b_{k+1} = \frac{2b_k}{k+3}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 16

```
dsolve(x*diff(diff(y(x),x),x)-(2*x+1)*diff(y(x),x)+2*y(x) = 0,y(x),singsol=all)
```

$$y = e^{2x}c_2 + 2c_1x + c_1$$

Mathematica DSolve solution

Solving time : 0.199 (sec)
Leaf size : 33

```
DSolve[{x*D[y[x],{x,2}]-(2*x+1)*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tru
```

$$y(x) \to e^{2x}\left(c_2 \int_1^x e^{-2K[1]}K[1]dK[1] + c_1\right)$$

2.1.644 Problem 661

Internal problem ID [9816]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 661
Date solved : `Monday, January 27, 2025 at 06:14:35 PM`
CAS classification : `[_erf]`

Solve

$$y'' + 2xy' + 4y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.195 (sec)

Writing the ode as

$$y'' + 2xy' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 3}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 3 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(x^2 - 3\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1231: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x - \frac{3}{2x} - \frac{9}{8x^3} - \frac{27}{16x^5} - \frac{405}{128x^7} - \frac{1701}{256x^9} - \frac{15309}{1024x^{11}} - \frac{72171}{2048x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 3}{1} \\ &= Q + \frac{R}{1} \\ &= (x^2 - 3) + (0) \\ &= x^2 - 3 \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is -3. Now b can be found.

$$\begin{aligned} b &= (-3) - (0) \\ &= -3 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= x \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-3}{1} - 1\right) &= -2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-3}{1} - 1\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = x^2 - 3$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	x	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^{-} \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\,(x) \\ &= -x \\ &= -x\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2(-x)\,(1) + \left((-1) + (-x)^2 - (x^2-3)\right) &= 0 \\ 2a_0 &= 0\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= pe^{\int \omega\, dx} \\ &= (x)\,\mathrm{e}^{\int -x dx} \\ &= (x)\,\mathrm{e}^{-\frac{x^2}{2}} \\ &= x\,\mathrm{e}^{-\frac{x^2}{2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x}{1}\, dx} \\ &= z_1 e^{-\frac{x^2}{2}} \\ &= z_1\left(\mathrm{e}^{-\frac{x^2}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x^2} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-x^2}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{e^{x^2}}{x} + \sqrt{\pi}\, \operatorname{erfi}(x) \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(e^{-x^2} x \right) + c_2 \left(e^{-x^2} x \left(-\frac{e^{x^2}}{x} + \sqrt{\pi}\, \operatorname{erfi}(x) \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2} y(x) + 2x \left(\frac{d}{dx} y(x) \right) + 4y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2} y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

- ☐ Rewrite DE with series expansions
  - Convert $x \cdot \left(\frac{d}{dx} y(x) \right)$ to series expansion

    $x \cdot \left(\frac{d}{dx} y(x) \right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d^2}{dx^2} y(x)$ to series expansion

    $\frac{d^2}{dx^2} y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k+2$

    $\frac{d^2}{dx^2} y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 2a_k(k+2) \right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+2)\,(k a_{k+2} + 2a_k + a_{k+2}) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{2a_k}{k+1} \right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 25

```
dsolve(diff(diff(y(x),x),x)+2*diff(y(x),x)*x+4*y(x) = 0,y(x),singsol=all)
```

$$y = x\left(c_2\sqrt{\pi}\,\mathrm{erfi}\left(x\right)+c_1\right)\mathrm{e}^{-x^2}-c_2$$

Mathematica DSolve solution

Solving time : 0.036 (sec)
Leaf size : 51

```
DSolve[{D[y[x],{x,2}]+2*x*D[y[x],x]+4*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to e^{-x^2}\left(-\sqrt{\pi}c_2\sqrt{x^2}\mathrm{erfi}\left(\sqrt{x^2}\right)+c_2e^{x^2}+2c_1x\right)$$

2.1.645 Problem 662

Internal problem ID [9817]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 662
Date solved : `Monday, January 27, 2025 at 06:14:36 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + xy' + 3y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.246 (sec)

Writing the ode as

$$y'' + xy' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 10}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 10 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{5}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1233: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{5}{2x} - \frac{25}{4x^3} - \frac{125}{4x^5} - \frac{3125}{16x^7} - \frac{21875}{16x^9} - \frac{328125}{32x^{11}} - \frac{2578125}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{2}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}
$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 10}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} - \frac{5}{2}\right) + (0) \\
&= \frac{x^2}{4} - \frac{5}{2}
\end{aligned}
$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{5}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{5}{2}\right) - (0) \\
&= -\frac{5}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &= -3 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &= 2
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2}{4} - \frac{5}{2}
$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (2) + 2\left(-\frac{x}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{5}{2}\right)\right) &= 0 \\ a_1 x + 2a_0 + 2 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\,dx} \\ &= \left(x^2 - 1\right)e^{\int -\frac{x}{2}dx} \\ &= \left(x^2 - 1\right)e^{-\frac{x^2}{4}} \\ &= \left(x^2 - 1\right)e^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{1}\,dx} \\ &= z_1 e^{-\frac{x^2}{4}} \\ &= z_1\left(e^{-\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^2}{2}} \left(x^2 - 1\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2}\,dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{(x^2 - 1)^2} dx \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-\frac{x^2}{2}} \left(x^2 - 1\right) \right) + c_2 \left(\mathrm{e}^{-\frac{x^2}{2}} \left(x^2 - 1\right) \left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{(x^2 - 1)^2} dx \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2} y(x) + x\left(\frac{d}{dx} y(x)\right) + 3y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2} y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite DE with series expansions
  - ○ Convert $x \cdot \left(\frac{d}{dx} y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$
  - ○ Convert $\frac{d^2}{dx^2} y(x)$ to series expansion

    $\frac{d^2}{dx^2} y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$
  - ○ Shift index using $k- >k+2$

    $\frac{d^2}{dx^2} y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+3)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $\left(k^2 + 3k + 2\right) a_{k+2} + a_k(k+3) = 0$

- Recursion relation that defines the series solution to the ODE

$$\left[y(x)=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=-\frac{a_k(k+3)}{k^2+3k+2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.029 (sec)
Leaf size : 42

```
dsolve(diff(diff(y(x),x),x)+diff(y(x),x)*x+3*y(x) = 0,y(x),singsol=all)
```

$$y=-(x-1)(x+1)\left(c_1\sqrt{\pi}\,\mathrm{erfi}\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}-c_2\right)\mathrm{e}^{-\frac{x^2}{2}}+2c_1x$$

Mathematica DSolve solution

Solving time : 0.194 (sec)
Leaf size : 52

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]+3*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to e^{-\frac{x^2}{2}}(x^2-1)\left(c_2\int_1^x \frac{e^{\frac{K[1]^2}{2}}}{(K[1]^2-1)^2}dK[1]+c_1\right)$$

2.1.646 Problem 663

Internal problem ID [9818]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 663
Date solved : `Monday, January 27, 2025 at 06:14:37 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - x^2 y' - 3xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.253 (sec)

Writing the ode as

$$y'' - x^2 y' - 3xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x^2 \\ C &= -3x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x(x^3 + 8)}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x(x^3 + 8) \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x(x^3 + 8)}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1235: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 4 \\ &= -4 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{2} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x^2}{2} + \frac{2}{x} - \frac{4}{x^4} + \frac{16}{x^7} - \frac{80}{x^{10}} + \frac{448}{x^{13}} - \frac{2688}{x^{16}} + \frac{16896}{x^{19}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 2$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i x^i \\
&= \frac{x^2}{2}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^1 = x$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{x^4}{4}
$$

This shows that the coefficient of x in the above is 0. Now we need to find the coefficient of x in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of x in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x(x^3 + 8)}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^4 + 2x\right) + (0) \\
&= \frac{1}{4}x^4 + 2x
\end{aligned}
$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is 2. Now b can be found.

$$
\begin{aligned}
b &= (2) - (0) \\
&= 2
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x^2}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{2}{\frac{1}{2}} - 2\right) &&= 1 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{2}{\frac{1}{2}} - 2\right) &&= -3
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x(x^3 + 8)}{4}
$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-4	$\frac{x^2}{2}$	1	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 1$, and since there are no poles, then

$$
\begin{aligned}
d &= \alpha_\infty^+ \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= (+)[\sqrt{r}]_\infty \\
&= 0 + \left(\frac{x^2}{2}\right) \\
&= \frac{x^2}{2} \\
&= \frac{x^2}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{x^2}{2}\right)(1) + \left((x) + \left(\frac{x^2}{2}\right)^2 - \left(\frac{x(x^3+8)}{4}\right)\right) &= 0 \\
-xa_0 &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= pe^{\int \omega\, dx} \\
&= (x)\,\mathrm{e}^{\int \frac{x^2}{2} dx} \\
&= (x)\,\mathrm{e}^{\frac{x^3}{6}} \\
&= x\,\mathrm{e}^{\frac{x^3}{6}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2}{1}\, dx} \\
&= z_1 e^{\frac{x^3}{6}} \\
&= z_1\left(\mathrm{e}^{\frac{x^3}{6}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{\frac{x^3}{3}} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{x^3}{3}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{x^3}{3}}}{x^2} dx \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\frac{x^3}{3}} x \right) + c_2 \left(\mathrm{e}^{\frac{x^3}{3}} x \left(\int \frac{\mathrm{e}^{-\frac{x^3}{3}}}{x^2} dx \right) \right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) - x^2 \left(\frac{d}{dx}y(x) \right) - 3xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ◦ Convert $x \cdot y(x)$ to series expansion

  $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}$

- ◦ Shift index using $k- >k-1$

  $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^k$

- ◦ Convert $x^2 \cdot \left(\frac{d}{dx}y(x) \right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y(x) \right) = \sum_{k=0}^{\infty} a_k k\, x^{k+1}$

- ◦ Shift index using $k- >k-1$

  $x^2 \cdot \left(\frac{d}{dx}y(x) \right) = \sum_{k=1}^{\infty} a_{k-1}(k-1)\, x^k$

- ◦ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

$\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- Shift index using $k->k+2$

  $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_{k-1}(k+2)\right) x^k\right) = 0$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+2)\,(ka_{k+2} - a_{k-1} + a_{k+2}) = 0$

- Shift index using $k->k+1$

  $(k+3)\,((k+1)\,a_{k+3} - a_k + a_{k+3}) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{a_k}{k+2}, 2a_2 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 58

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x^2-3*x*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{9\,\mathrm{e}^{\frac{x^3}{6}}\,\mathrm{WhittakerM}\left(\frac{1}{3}, \frac{5}{6}, \frac{x^3}{3}\right) c_2 x^3 + 9c_1 x^2 \mathrm{e}^{\frac{x^3}{3}} + 5\, 3^{2/3} c_2 (x^3)^{1/3}\,(x^3+2)}{9x}$$

Mathematica DSolve solution

Solving time : 0.065 (sec)
Leaf size : 51

```
DSolve[{D[y[x],{x,2}]-x^2*D[y[x],x]-3*x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{9} e^{\frac{x^3}{3}} \left(9c_1 x - 3^{2/3} c_2 \sqrt[3]{x^3}\,\Gamma\left(-\frac{1}{3}, \frac{x^3}{3}\right)\right)$$

2.1.647 Problem 664

Internal problem ID [9819]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 664
Date solved : `Monday, January 27, 2025 at 06:14:37 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-4x^2+1\right)y''-20xy'-16y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.216 (sec)

Writing the ode as

$$\left(-4x^2+1\right)y''-20xy'-16y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -4x^2+1 \\ B &= -20x \\ C &= -16 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-4x^2+6}{\left(4x^2-1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4x^2+6 \\ t &= \left(4x^2-1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-4x^2+6}{\left(4x^2-1\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1237: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (4x^2 - 1)^2$. There is a pole at $x = \frac{1}{2}$ of order 2. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16\left(x - \frac{1}{2}\right)^2} - \frac{7}{8\left(x - \frac{1}{2}\right)} + \frac{5}{16\left(x + \frac{1}{2}\right)^2} + \frac{7}{8\left(x + \frac{1}{2}\right)}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4} \end{aligned}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x + \frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-4x^2+6}{(4x^2-1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-4x^2+6}{(4x^2-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{1}{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-\frac{1}{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{4\left(x-\frac{1}{2}\right)} - \frac{1}{4\left(x+\frac{1}{2}\right)} + (-)(0) \\
&= -\frac{1}{4\left(x-\frac{1}{2}\right)} - \frac{1}{4\left(x+\frac{1}{2}\right)} \\
&= -\frac{2x}{4x^2-1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{4\left(x-\frac{1}{2}\right)} - \frac{1}{4\left(x+\frac{1}{2}\right)}\right)(1) + \left(\left(\frac{1}{4\left(x-\frac{1}{2}\right)^2} + \frac{1}{4\left(x+\frac{1}{2}\right)^2}\right) + \left(-\frac{1}{4\left(x-\frac{1}{2}\right)} - \frac{1}{4\left(x+\frac{1}{2}\right)}\right)^2 -$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x)\, \mathrm{e}^{\int \left(-\frac{1}{4\left(x-\frac{1}{2}\right)} - \frac{1}{4\left(x+\frac{1}{2}\right)}\right) dx} \\ &= (x) \frac{1}{\left((2x-1)(2x+1)\right)^{1/4}} \\ &= \frac{x}{(4x^2-1)^{1/4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-20x}{-4x^2+1}\, dx} \\ &= z_1 e^{-\frac{5\ln(4x^2-1)}{4}} \\ &= z_1 \left(\frac{1}{(4x^2-1)^{5/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x}{(4x^2-1)^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-20x}{-4x^2+1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{5\ln(4x^2-1)}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{(4x^2-1)^{3/2}}{x} - 4x\sqrt{4x^2-1} + \ln\left(x\sqrt{4} + \sqrt{4x^2-1}\right)\sqrt{4}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x}{(4x^2-1)^{3/2}}\right) \\
&\quad + c_2\left(\frac{x}{(4x^2-1)^{3/2}}\left(\frac{(4x^2-1)^{3/2}}{x} - 4x\sqrt{4x^2-1} + \ln\left(x\sqrt{4}+\sqrt{4x^2-1}\right)\sqrt{4}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(-4x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) - 20x\left(\frac{d}{dx}y(x)\right) - 16y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{16y(x)}{4x^2-1} - \frac{20x\left(\frac{d}{dx}y(x)\right)}{4x^2-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{20x\left(\frac{d}{dx}y(x)\right)}{4x^2-1} + \frac{16y(x)}{4x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{20x}{4x^2-1}, P_3(x) = \frac{16}{4x^2-1}\right]$

  ◦ $\left(x+\frac{1}{2}\right)\cdot P_2(x)$ is analytic at $x = -\frac{1}{2}$

  $\left(\left(x+\frac{1}{2}\right)\cdot P_2(x)\right)\bigg|_{x=-\frac{1}{2}} = \frac{5}{2}$

  ◦ $\left(x+\frac{1}{2}\right)^2\cdot P_3(x)$ is analytic at $x = -\frac{1}{2}$

  $\left(\left(x+\frac{1}{2}\right)^2\cdot P_3(x)\right)\bigg|_{x=-\frac{1}{2}} = 0$

  ◦ $x = -\frac{1}{2}$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -\frac{1}{2}$

- Multiply by denominators

  $(4x^2-1)\left(\frac{d^2}{dx^2}y(x)\right) + 20x\left(\frac{d}{dx}y(x)\right) + 16y(x) = 0$

- Change variables using $x = u - \frac{1}{2}$ so that the regular singular point is at $u = 0$

  $(4u^2-4u)\left(\frac{d^2}{du^2}y(u)\right) + (20u-10)\left(\frac{d}{du}y(u)\right) + 16y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

$$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- ◦ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

$$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$$

- ◦ Shift index using $k->k+2-m$

$$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0r(3+2r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)(2k+5+2r)+4a_k(k+r+2)^2\right)u^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2r(3+2r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,-\frac{3}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$4a_k(k+r+2)^2-4(k+1+r)\,a_{k+1}\left(k+r+\frac{5}{2}\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{2a_k(k+r+2)^2}{(k+1+r)(2k+5+2r)}$$

- Recursion relation for $r=0$

$$a_{k+1}=\frac{2a_k(k+2)^2}{(k+1)(2k+5)}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty}a_ku^k,a_{k+1}=\frac{2a_k(k+2)^2}{(k+1)(2k+5)}\right]$$

- Revert the change of variables $u=x+\frac{1}{2}$

$$\left[y(x)=\sum_{k=0}^{\infty}a_k\left(x+\frac{1}{2}\right)^k,a_{k+1}=\frac{2a_k(k+2)^2}{(k+1)(2k+5)}\right]$$

- Recursion relation for $r=-\frac{3}{2}$

$$a_{k+1}=\frac{2a_k\left(k+\frac{1}{2}\right)^2}{\left(k-\frac{1}{2}\right)(2k+2)}$$

- Solution for $r=-\frac{3}{2}$

$$\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k-\frac{3}{2}},a_{k+1}=\frac{2a_k\left(k+\frac{1}{2}\right)^2}{\left(k-\frac{1}{2}\right)(2k+2)}\right]$$

- Revert the change of variables $u=x+\frac{1}{2}$

$$\left[y(x)=\sum_{k=0}^{\infty}a_k\left(x+\frac{1}{2}\right)^{k-\frac{3}{2}},a_{k+1}=\frac{2a_k\left(k+\frac{1}{2}\right)^2}{\left(k-\frac{1}{2}\right)(2k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty}a_k\left(x+\frac{1}{2}\right)^k\right)+\left(\sum_{k=0}^{\infty}b_k\left(x+\frac{1}{2}\right)^{k-\frac{3}{2}}\right),a_{k+1}=\frac{2a_k(k+2)^2}{(k+1)(2k+5)},b_{k+1}=\frac{2b_k\left(k+\frac{1}{2}\right)^2}{\left(k-\frac{1}{2}\right)(2k+2)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.038 (sec)
Leaf size : 48

```
dsolve((-4*x^2+1)*diff(diff(y(x),x),x)-20*diff(y(x),x)*x-16*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{2 \ln \left(2x + \sqrt{4x^2 - 1}\right) c_2 x + c_1 x - \sqrt{4x^2 - 1}\, c_2}{\left(4x^2 - 1\right)^{3/2}}$$

Mathematica DSolve solution

Solving time : 0.101 (sec)
Leaf size : 57

```
DSolve[{(1-4*x^2)*D[y[x],{x,2}]-20*x*D[y[x],x]-16*y[x]==0,{}},y[x],x,IncludeSingularSolutions->
```

$$y(x) \to \frac{-2c_2 x \arcsin(2x) - c_2 \sqrt{1 - 4x^2} + c_1 x}{\sqrt[4]{1 - 4x^2}\,\left(4x^2 - 1\right)^{5/4}}$$

2.1.648 Problem 665

Internal problem ID [9820]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 665
Date solved : `Monday, January 27, 2025 at 06:14:38 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(x^2-1\right) y''-6 x y'+12 y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.203 (sec)

Writing the ode as

$$\left(x^2-1\right) y''-6 x y'+12 y=0 \tag{1}$$
$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2-1 \\ B &= -6 x \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2 A} d x}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2 A B'-2 B A'+B^2-4 A C}{4 A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{15}{\left(x^2-1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15 \\ t &= \left(x^2-1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{15}{\left(x^2-1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1239: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{15}{4\,(x-1)} + \frac{15}{4\,(x+1)} + \frac{15}{4\,(x-1)^2} + \frac{15}{4\,(x+1)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{15}{\left(x^2-1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
-1	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{2\,(x-1)} + \frac{5}{2\,(x+1)} + (-)\,(0) \\
&= -\frac{3}{2\,(x-1)} + \frac{5}{2\,(x+1)} \\
&= \frac{x-4}{x^2-1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{3}{2(x-1)}+\frac{5}{2(x+1)}\right)(0)+\left(\left(\frac{3}{2(x-1)^2}-\frac{5}{2(x+1)^2}\right)+\left(-\frac{3}{2(x-1)}+\frac{5}{2(x+1)}\right)^2-\left(\frac{}{(x^2}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\, dx}\\
&= e^{\int \left(-\frac{3}{2(x-1)}+\frac{5}{2(x+1)}\right)dx}\\
&= \frac{(x+1)^{5/2}}{(x-1)^{3/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}\\
&= z_1 e^{-\int \frac{1}{2}\frac{-6x}{x^2-1}\, dx}\\
&= z_1 e^{\frac{3\ln(x-1)}{2}+\frac{3\ln(x+1)}{2}}\\
&= z_1\left((x-1)^{3/2}(x+1)^{3/2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = (x+1)^4$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-6x}{x^2-1}\, dx}}{(y_1)^2}\, dx\\
&= y_1 \int \frac{e^{3\ln(x-1)+3\ln(x+1)}}{(y_1)^2}\, dx\\
&= y_1\left(-\frac{x(x^2+1)\, e^{3\ln(x-1)+3\ln(x+1)}}{(x+1)^7 (x-1)^3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2\\
&= c_1\left((x+1)^4\right)+c_2\left((x+1)^4\left(-\frac{x(x^2+1)\, e^{3\ln(x-1)+3\ln(x+1)}}{(x+1)^7 (x-1)^3}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x^2-1)\left(\frac{d^2}{dx^2}y(x)\right)-6x\left(\frac{d}{dx}y(x)\right)+12y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{12y(x)}{x^2-1}+\frac{6\left(\frac{d}{dx}y(x)\right)x}{x^2-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)-\frac{6\left(\frac{d}{dx}y(x)\right)x}{x^2-1}+\frac{12y(x)}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{6x}{x^2-1}, P_3(x)=\frac{12}{x^2-1}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=-3$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $(x^2-1)\left(\frac{d^2}{dx^2}y(x)\right)-6x\left(\frac{d}{dx}y(x)\right)+12y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(-6u+6)\left(\frac{d}{du}y(u)\right)+12y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  ◦ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

  ◦ Shift index using $k->k+2-m$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r(-4+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)(k+r-3)+a_k(k+r-3)(k+r-4)\right)u^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-2r(-4+r)=0$

- Values of r that satisfy the indicial equation
  $r\in\{0,4\}$
- Each term in the series must be 0, giving the recursion relation
  $((-2k-2r-2)\,a_{k+1}+a_k(k+r-4))\,(k+r-3)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{a_k(k+r-4)}{2(k+1+r)}$
- Recursion relation for $r=0$; series terminates at $k=4$
  $a_{k+1}=\frac{a_k(k-4)}{2(k+1)}$
- Apply recursion relation for $k=0$
  $a_1=-2a_0$
- Apply recursion relation for $k=1$
  $a_2=-\frac{3a_1}{4}$
- Express in terms of a_0
  $a_2=\frac{3a_0}{2}$
- Apply recursion relation for $k=2$
  $a_3=-\frac{a_2}{3}$
- Express in terms of a_0
  $a_3=-\frac{a_0}{2}$
- Apply recursion relation for $k=3$
  $a_4=-\frac{a_3}{8}$
- Express in terms of a_0
  $a_4=\frac{a_0}{16}$
- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second linea
  $y(u)=a_0\cdot\left(1-2u+\frac{3}{2}u^2-\frac{1}{2}u^3+\frac{1}{16}u^4\right)$
- Revert the change of variables $u=x+1$
  $\left[y(x)=\frac{a_0(x-1)^4}{16}\right]$
- Recursion relation for $r=4$
  $a_{k+1}=\frac{a_k k}{2(k+5)}$
- Solution for $r=4$
  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+4},a_{k+1}=\frac{a_k k}{2(k+5)}\right]$
- Revert the change of variables $u=x+1$
  $\left[y(x)=\sum_{k=0}^{\infty}a_k(x+1)^{k+4},a_{k+1}=\frac{a_k k}{2(k+5)}\right]$
- Combine solutions and rename parameters
  $\left[y(x)=\frac{a_0(x-1)^4}{16}+\left(\sum_{k=0}^{\infty}b_k(x+1)^{4+k}\right),b_{k+1}=\frac{b_k k}{2(5+k)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 25

```
dsolve((x^2-1)*diff(diff(y(x),x),x)-6*diff(y(x),x)*x+12*y(x) = 0,y(x),singsol=all)
```

$$y = c_2x^4 + c_1x^3 + 6c_2x^2 + c_1x + c_2$$

Mathematica DSolve solution

Solving time : 0.366 (sec)
Leaf size : 75

```
DSolve[{(x^2-1)*D[y[x],{x,2}]-6*x*D[y[x],x]+12*y[x]==0,{}},y[x],x,IncludeSingularSolutions->
```

$$y(x) \to (x^2 - 1)^{3/2} \exp\left(\int_1^x \frac{K[1]+4}{K[1]^2-1} dK[1]\right) \left(c_2 \int_1^x \exp\left(-2\int_1^{K[2]} \frac{K[1]+4}{K[1]^2-1} dK[1]\right) dK[2] + c_1\right)$$

2.1.649 Problem 666

Internal problem ID [9821]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 666
Date solved : `Monday, January 27, 2025 at 06:14:38 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + xy' + (2 + x)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.276 (sec)

Writing the ode as

$$y'' + xy' + (2 + x)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 2 + x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x - 6}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 4x - 6 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 - x - \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1241: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - 1 - \frac{5}{2x} - \frac{5}{x^2} - \frac{65}{4x^3} - \frac{115}{2x^4} - \frac{885}{4x^5} - \frac{1785}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} - 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 - x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x - 6}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 - x - \frac{3}{2}\right) + (0) \\ &= \frac{1}{4}x^2 - x - \frac{3}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{2}\right) - (1) \\ &= -\frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} - 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= -3 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= 2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 - x - \frac{3}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2} - 1$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2} - 1\right) \\ &= 1 - \frac{x}{2} \\ &= 1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (2) + 2\left(1 - \frac{x}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2}\right) + \left(1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 - x - \frac{3}{2}\right)\right) &= 0 \\ (2 + x)\, a_1 + 4x + 2a_0 + 2 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 3, a_1 = -4\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 4x + 3$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (x^2 - 4x + 3)\, e^{\int (1 - \frac{x}{2}) dx} \\ &= (x^2 - 4x + 3)\, e^{x - \frac{1}{4}x^2} \\ &= (x^2 - 4x + 3)\, e^{-\frac{x(-4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{1}\, dx} \\ &= z_1 e^{-\frac{x^2}{4}} \\ &= z_1 \left(e^{-\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \left(x^2 - 4x + 3\right) \mathrm{e}^{-\frac{x(-2+x)}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}} \mathrm{e}^{x(-2+x)}}{(x^2-4x+3)^2} dx \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\left(x^2 - 4x + 3\right) \mathrm{e}^{-\frac{x(-2+x)}{2}} \right) + c_2 \left(\left(x^2 - 4x + 3\right) \mathrm{e}^{-\frac{x(-2+x)}{2}} \left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}} \mathrm{e}^{x(-2+x)}}{(x^2-4x+3)^2} dx \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2} y(x) + x\left(\frac{d}{dx} y(x)\right) + (x+2)\, y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2} y(x)$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ○ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$
    $x^m \cdot y(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$
  ○ Shift index using $k->k-m$
    $x^m \cdot y(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$
  ○ Convert $x \cdot \left(\frac{d}{dx} y(x)\right)$ to series expansion
    $x \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$
  ○ Convert $\frac{d^2}{dx^2} y(x)$ to series expansion
    $\frac{d^2}{dx^2} y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$
  ○ Shift index using $k->k+2$

$\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$

Rewrite ODE with series expansions

$2a_2 + 2a_0 + \left(\sum_{k=1}^{\infty} (a_{k+2}(k+2)(k+1) + a_k(k+2) + a_{k-1})x^k\right) = 0$

- Each term must be 0

  $2a_2 + 2a_0 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)a_{k+2} + a_k k + 2a_k + a_{k-1} = 0$

- Shift index using $k- >k+1$

  $\left((k+1)^2 + 3k + 5\right)a_{k+3} + a_{k+1}(k+1) + 2a_{k+1} + a_k = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = -\frac{k a_{k+1} + a_k + 3a_{k+1}}{k^2 + 5k + 6}, 2a_2 + 2a_0 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.019 (sec)
Leaf size : 78

```
dsolve(diff(diff(y(x),x),x)+diff(y(x),x)*x+(x+2)*y(x) = 0,y(x),singsol=all)
```

$$y = \left(c_2 \mathrm{e}^{-\frac{(x-2)^2}{2}} (x-1)(x-3)\left(\operatorname{erf}\left(\frac{\sqrt{2}\,\sqrt{-(x-2)^2}}{2}\right)-1\right)\sqrt{\pi}-\sqrt{2}\,\sqrt{-(x-2)^2}\,c_2 - c_1 \mathrm{e}^{-\frac{(x-2)^2}{2}}(x-1)(x-3)\right)\mathrm{e}^{-x}$$

Mathematica DSolve solution

Solving time : 0.248 (sec)
Leaf size : 63

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]+(2+x)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{x-\frac{x^2}{2}}(x^2-4x+3)\left(c_2\int_1^x \frac{e^{\frac{1}{2}(K[1]-4)K[1]}}{(K[1]-3)^2(K[1]-1)^2}dK[1]+c_1\right)$$

2.1.650 Problem 667

Internal problem ID [9822]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 667
Date solved : `Monday, January 27, 2025 at 06:14:39 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(2x^2+1\right) y''+7xy'+2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.323 (sec)

Writing the ode as

$$\left(2x^2+1\right) y''+7xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2+1 \\ B &= 7x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{5x^2+6}{4\left(2x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 5x^2+6 \\ t &= 4\left(2x^2+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{5x^2+6}{4\left(2x^2+1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1243: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2+1)^2$. There is a pole at $x = \frac{i\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{i\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{64\left(x - \frac{i\sqrt{2}}{2}\right)^2} - \frac{7}{64\left(x + \frac{i\sqrt{2}}{2}\right)^2} - \frac{17i\sqrt{2}}{64\left(x - \frac{i\sqrt{2}}{2}\right)} + \frac{17i\sqrt{2}}{64\left(x + \frac{i\sqrt{2}}{2}\right)}$$

For the pole at $x = \frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8} \end{aligned}$$

For the pole at $x=-\frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{7}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{7}{8} \\
\alpha_c^- &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{8}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty=0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r=\frac{s}{t}=\frac{5x^2+6}{4\left(2x^2+1\right)^2}
$$

Since the $\gcd(s,t)=1$. This gives $b=\frac{5}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{4} \\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{5x^2+6}{4\left(2x^2+1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{i\sqrt{2}}{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$
$-\frac{i\sqrt{2}}{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=\frac{5}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+-\left(\alpha_{c_1}^-+\alpha_{c_2}^-\right) \\
&= \frac{5}{4}-\left(\frac{1}{4}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{8x-4i\sqrt{2}} + \frac{1}{8x+4i\sqrt{2}} + (0) \\
&= \frac{1}{8x-4i\sqrt{2}} + \frac{1}{8x+4i\sqrt{2}} \\
&= \frac{x}{4x^2+2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{8x-4i\sqrt{2}} + \frac{1}{8x+4i\sqrt{2}}\right)(1) + \left(\left(-\frac{1}{8\left(x-\frac{i\sqrt{2}}{2}\right)^2} - \frac{1}{8\left(x+\frac{i\sqrt{2}}{2}\right)^2}\right) + \left(\frac{1}{8x-4i\sqrt{2}} + \frac{1}{8x+4i\sqrt{}}\right.\right.
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (x)\, e^{\int \left(\frac{1}{8x-4i\sqrt{2}} + \frac{1}{8x+4i\sqrt{2}}\right) dx} \\
&= (x)\left(\left(i\sqrt{2}-2x\right)\left(2x+i\sqrt{2}\right)\right)^{1/8} \\
&= x\left(-4x^2-2\right)^{1/8}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7x}{2x^2+1}\, dx} \\
&= z_1 e^{-\frac{7\ln(2x^2+1)}{8}} \\
&= z_1 \left(\frac{1}{(2x^2+1)^{7/8}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{2^{7/8} x(-4x^2-2)^{1/8}}{(4x^2+2)^{7/8}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{7x}{2x^2+1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{7\ln\left(2x^2+1\right)}{4}}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{2^{1/4}(4x^2+2)^{7/4}}{4\left(2x^2+1\right)^{7/4} x^2 \left(-4x^2-2\right)^{1/4}} dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{2^{7/8} x(-4x^2-2)^{1/8}}{(4x^2+2)^{7/8}}\right) + c_2 \left(\frac{2^{7/8} x(-4x^2-2)^{1/8}}{(4x^2+2)^{7/8}} \left(\int \frac{2^{1/4}(4x^2+2)^{7/4}}{4\left(2x^2+1\right)^{7/4} x^2 \left(-4x^2-2\right)^{1/4}} dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Legendre successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special functio
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.041 (sec)
Leaf size : 37

```
dsolve((2*x^2+1)*diff(diff(y(x),x),x)+7*diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_1 \operatorname{LegendreP}\left(\frac{1}{4}, \frac{3}{4}, i\sqrt{2}\, x\right) + c_2 \operatorname{LegendreQ}\left(\frac{1}{4}, \frac{3}{4}, i\sqrt{2}\, x\right)}{\left(2x^2+1\right)^{3/8}}$$

Mathematica DSolve solution

Solving time : 0.061 (sec)
Leaf size : 66

```
DSolve[{(1+2*x^2)*D[y[x],{x,2}]+7*x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to \frac{c_2 Q_{\frac{1}{4}}^{\frac{3}{4}}\left(i\sqrt{2}x\right)}{\left(2x^2+1\right)^{3/8}} + \frac{2i\sqrt[4]{2}c_1 x}{\left(2x^2+1\right)^{3/4} \operatorname{Gamma}\left(\frac{1}{4}\right)}$$

2.1.651 Problem 668

Internal problem ID [9823]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 668
Date solved : `Monday, January 27, 2025 at 06:14:40 PM`
CAS classification : `[_Lienard]`

Solve

$$4y'' + xy' + 4y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.247 (sec)

Writing the ode as

$$4y'' + xy' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4 \\ B &= x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 56}{64} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 56 \\ t &= 64 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{64} - \frac{7}{8}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1244: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{8} - \frac{7}{2x} - \frac{49}{x^3} - \frac{1372}{x^5} - \frac{48020}{x^7} - \frac{1882384}{x^9} - \frac{79060128}{x^{11}} - \frac{3478645632}{x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{8}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{8} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{64}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 56}{64} \\ &= Q + \frac{R}{64} \\ &= \left(\frac{x^2}{64} - \frac{7}{8}\right) + (0) \\ &= \frac{x^2}{64} - \frac{7}{8} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{7}{8}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{7}{8}\right) - (0) \\ &= -\frac{7}{8} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{8} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{7}{8}}{\frac{1}{8}} - 1\right) &&= -4 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{7}{8}}{\frac{1}{8}} - 1\right) &&= 3 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{64} - \frac{7}{8}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{8}$	-4	3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 3$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 3 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{8}\right) \\ &= -\frac{x}{8} \\ &= -\frac{x}{8} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 3$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (6x + 2a_2) + 2\left(-\frac{x}{8}\right)\left(3x^2 + 2xa_2 + a_1\right) + \left(\left(-\frac{1}{8}\right) + \left(-\frac{x}{8}\right)^2 - \left(\frac{x^2}{64} - \frac{7}{8}\right)\right) &= 0 \\ 6x + 2a_2 + \frac{1}{4}a_2x^2 + \frac{1}{2}a_1x + \frac{3}{4}a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0, a_1 = -12, a_2 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^3 - 12x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= \left(x^3 - 12x\right)e^{\int -\frac{x}{8} dx} \\ &= \left(x^3 - 12x\right)e^{-\frac{x^2}{16}} \\ &= x\left(x^2 - 12\right)e^{-\frac{x^2}{16}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{4}\, dx} \\ &= z_1 e^{-\frac{x^2}{16}} \\ &= z_1\left(e^{-\frac{x^2}{16}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{-\frac{x^2}{8}} x(x^2 - 12)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{4}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{8}}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{e^{\frac{x^2}{8}}}{x^2 (x^2 - 12)^2} dx \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(e^{-\frac{x^2}{8}} x(x^2 - 12) \right) + c_2 \left(e^{-\frac{x^2}{8}} x(x^2 - 12) \left(\int \frac{e^{\frac{x^2}{8}}}{x^2 (x^2 - 12)^2} dx \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$4\frac{d^2}{dx^2}y(x) + x\left(\frac{d}{dx}y(x)\right) + 4y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{x\left(\frac{d}{dx}y(x)\right)}{4} - y(x)$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x) + \frac{x\left(\frac{d}{dx}y(x)\right)}{4} + y(x) = 0$
- Multiply by denominators
  $4\frac{d^2}{dx^2}y(x) + x\left(\frac{d}{dx}y(x)\right) + 4y(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions
  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion
    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$
  - Convert $\frac{d^2}{dx^2}y(x)$ to series expansion
    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$
  - Shift index using $k- >k+2$

$\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$

Rewrite DE with series expansions

$\sum_{k=0}^{\infty} (4a_{k+2}(k+2)(k+1) + a_k(k+4))x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $4(k^2+3k+2)a_{k+2} + a_k(k+4) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(k+4)}{4(k^2+3k+2)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solut
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special fu
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.022 (sec)
Leaf size : 34

```
dsolve(4*diff(diff(y(x),x),x)+diff(y(x),x)*x+4*y(x) = 0,y(x),singsol=all)
```

$$y = -\frac{e^{-\frac{x^2}{8}}\left(-12\,\text{hypergeom}\left(\left[-\frac{3}{2}\right], \left[\frac{1}{2}\right], \frac{x^2}{8}\right)c_2 + xc_1(x^2-12)\right)}{12}$$

Mathematica DSolve solution

Solving time : 0.096 (sec)
Leaf size : 122

```
DSolve[{4*D[y[x],{x,2}]+x*D[y[x],x]+4*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-\frac{x^2}{8}}\left(\sqrt{2\pi}c_2(x^2-12)\,x^2\operatorname{erfi}\left(\frac{\sqrt{x^2}}{2\sqrt{2}}\right)+4\sqrt{x^2}\left(2\sqrt{2}c_1x^3-c_2e^{\frac{x^2}{8}}x^2+8c_2e^{\frac{x^2}{8}}-24\sqrt{2}c_1x\right)\right)}{32\sqrt{x^2}}$$

2.1.652 Problem 669

Internal problem ID [9824]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 669
Date solved : `Monday, January 27, 2025 at 06:14:40 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + xy' - 4y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.217 (sec)

Writing the ode as

$$y'' + xy' - 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 18}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 18 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} + \frac{9}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1246: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + \frac{9}{2x} - \frac{81}{4x^3} + \frac{729}{4x^5} - \frac{32805}{16x^7} + \frac{413343}{16x^9} - \frac{11160261}{32x^{11}} + \frac{157837977}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 18}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} + \frac{9}{2}\right) + (0) \\ &= \frac{x^2}{4} + \frac{9}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{9}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{9}{2}\right) - (0) \\ &= \frac{9}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{9}{2}}{\frac{1}{2}} - 1\right) &= 4 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{9}{2}}{\frac{1}{2}} - 1\right) &= -5 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} + \frac{9}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	4	-5

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 4$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 4 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{x}{2}\right) \\ &= \frac{x}{2} \\ &= \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(12x^2 + 6xa_3 + 2a_2\right) + 2\left(\frac{x}{2}\right)\left(4x^3 + 3x^2a_3 + 2xa_2 + a_1\right) + \left(\left(\frac{1}{2}\right) + \left(\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} + \frac{9}{2}\right)\right) = 0$$

$$-a_3x^3 + (-2a_2 + 12)\,x^2 + (-3a_1 + 6a_3)\,x - 4a_0 + 2a_2 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 3, a_1 = 0, a_2 = 6, a_3 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 + 6x^2 + 3$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= \left(x^4 + 6x^2 + 3\right) e^{\int \frac{x}{2} dx} \\ &= \left(x^4 + 6x^2 + 3\right) e^{\frac{x^2}{4}} \\ &= \left(x^4 + 6x^2 + 3\right) e^{\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{1}\, dx} \\ &= z_1 e^{-\frac{x^2}{4}} \\ &= z_1\left(e^{-\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^4 + 6x^2 + 3$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2}\,dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{(x^4 + 6x^2 + 3)^2} dx \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (x^4 + 6x^2 + 3) + c_2 \left(x^4 + 6x^2 + 3 \left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{(x^4 + 6x^2 + 3)^2} dx \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.040 (sec)
Leaf size : 47

```
dsolve(diff(diff(y(x),x),x)+diff(y(x),x)*x-4*y(x) = 0,y(x),singsol=all)
```

$$y = c_1\sqrt{2}\,x\left(x^2+5\right)\mathrm{e}^{-\frac{x^2}{2}}+\left(x^4+6x^2+3\right)\left(\sqrt{\pi}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right)c_1+c_2\right)$$

Mathematica DSolve solution

Solving time : 0.019 (sec)
Leaf size : 43

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]-4*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{-\frac{x^2}{2}}\,\mathrm{HermiteH}\left(-5,\frac{x}{\sqrt{2}}\right)+\frac{1}{3}c_2\left(x^4+6x^2+3\right)$$

2.1.653 Problem 670

Internal problem ID [9825]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 670
Date solved : `Monday, January 27, 2025 at 06:14:41 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4xy'' - xy' + 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.211 (sec)

Writing the ode as

$$4xy'' - xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x \\ B &= -x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x - 32}{64x} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x - 32 \\ t &= 64x \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x - 32}{64x}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1247: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 1 - 1 \\
&= 0
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64x$. There is a pole at $x = 0$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= 1 \\
\alpha_c^- &= 1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{8} - \frac{2}{x} - \frac{16}{x^2} - \frac{256}{x^3} - \frac{5120}{x^4} - \frac{114688}{x^5} - \frac{2752512}{x^6} - \frac{69206016}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{8}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{8} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{64}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x - 32}{64x} \\ &= Q + \frac{R}{64x} \\ &= \left(\frac{1}{64}\right) + \left(-\frac{1}{2x}\right) \\ &= \frac{1}{64} - \frac{1}{2x} \end{aligned}$$

Since the degree of t is 1, then we see that the coefficient of the term 1 in the remainder R is -32. Dividing this by leading coefficient in t which is 64 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{8} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{8}} - 0\right) &&= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{8}} - 0\right) &&= 2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x - 32}{64x}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{8}$	-2	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= 2 - (1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{x} + (-)\left(\frac{1}{8}\right) \\ &= \frac{1}{x} - \frac{1}{8} \\ &= \frac{1}{x} - \frac{1}{8} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x} - \frac{1}{8}\right)(1) + \left(\left(-\frac{1}{x^2}\right) + \left(\frac{1}{x} - \frac{1}{8}\right)^2 - \left(\frac{x-32}{64x}\right) \right) = 0$$

$$\frac{8+a_0}{4x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -8\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = -8 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (-8+x)\, e^{\int \left(\frac{1}{x} - \frac{1}{8}\right) dx} \\ &= (-8+x)\, e^{-\frac{x}{8} + \ln(x)} \\ &= (-8+x)\, x\, e^{-\frac{x}{8}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{4x}\,dx} \\
&= z_1 e^{\frac{x}{8}} \\
&= z_1 \left(\mathrm{e}^{\frac{x}{8}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (-8 + x)\, x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{4x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{x}{4}}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\mathrm{Ei}_1\left(-\frac{x}{4}\right)}{128} - \frac{\mathrm{e}^{\frac{x}{4}}}{256\left(-2+\frac{x}{4}\right)} - \frac{\mathrm{e}^{\frac{x}{4}}}{64x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1((-8+x)\,x) + c_2\left((-8+x)\,x\left(-\frac{\mathrm{Ei}_1\left(-\frac{x}{4}\right)}{128} - \frac{\mathrm{e}^{\frac{x}{4}}}{256\left(-2+\frac{x}{4}\right)} - \frac{\mathrm{e}^{\frac{x}{4}}}{64x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 33

```
dsolve(4*x*diff(diff(y(x),x),x)-diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{x c_2 (x-8) \operatorname{Ei}_1\left(-\frac{x}{4}\right)}{16} + \frac{c_2 (x-4) e^{\frac{x}{4}}}{4} + c_1 x (x-8)$$

Mathematica DSolve solution

Solving time : 0.194 (sec)
Leaf size : 42

```
DSolve[{4*x*D[y[x],{x,2}]-x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to (x-8)x\left(c_2 \int_1^x \frac{e^{\frac{K[1]}{4}}}{(K[1]-8)^2 K[1]^2} dK[1] + c_1\right)$$

2.1.654 Problem 671

Internal problem ID [9826]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 671
Date solved : `Monday, January 27, 2025 at 06:14:42 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$6x^2y'' + x(1+18x)\,y' + (1+12x)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.273 (sec)

Writing the ode as

$$6x^2y'' + (18x^2+x)\,y' + (1+12x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 6x^2 \\ B &= 18x^2 + x \\ C &= 1 + 12x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{324x^2 - 252x - 35}{144x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 324x^2 - 252x - 35 \\ t &= 144x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{324x^2 - 252x - 35}{144x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1248: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 144x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{4} - \frac{35}{144x^2} - \frac{7}{4x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3}{2} - \frac{7}{12x} - \frac{7}{36x^2} - \frac{49}{648x^3} - \frac{245}{5832x^4} - \frac{343}{13122x^5} - \frac{66199}{3779136x^6} - \frac{837949}{68024448x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{3}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{324x^2 - 252x - 35}{144x^2} \\ &= Q + \frac{R}{144x^2} \\ &= \left(\frac{9}{4}\right) + \left(\frac{-252x - 35}{144x^2}\right) \\ &= \frac{9}{4} + \frac{-252x - 35}{144x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -252. Dividing this by leading coefficient in t which is 144 gives $-\frac{7}{4}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{7}{4}\right) - (0) \\ &= -\frac{7}{4} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{3}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{7}{4}}{\frac{3}{2}} - 0\right) &= -\frac{7}{12} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{7}{4}}{\frac{3}{2}} - 0\right) &= \frac{7}{12} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{324x^2 - 252x - 35}{144x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{12}$	$\frac{5}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{3}{2}$	$-\frac{7}{12}$	$\frac{7}{12}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{7}{12}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{7}{12} - \left(\frac{7}{12}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{7}{12x} + (-)\left(\frac{3}{2}\right) \\ &= \frac{7}{12x} - \frac{3}{2} \\ &= \frac{7}{12x} - \frac{3}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{7}{12x} - \frac{3}{2}\right)(0) + \left(\left(-\frac{7}{12x^2}\right) + \left(\frac{7}{12x} - \frac{3}{2}\right)^2 - \left(\frac{324x^2 - 252x - 35}{144x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{7}{12x} - \frac{3}{2}\right) dx} \\ &= x^{7/12} e^{-\frac{3x}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{18x^2+x}{6x^2}\,dx} \\ &= z_1 e^{-\frac{3x}{2}-\frac{\ln(x)}{12}} \\ &= z_1 \left(\frac{\mathrm{e}^{-\frac{3x}{2}}}{x^{1/12}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}\,\mathrm{e}^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{18x^2+x}{6x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-3x-\frac{\ln(x)}{6}}}{(y_1)^2}\,dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{-3x-\frac{\ln(x)}{6}}\mathrm{e}^{6x}}{x}dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\sqrt{x}\,\mathrm{e}^{-3x}\right) + c_2\left(\sqrt{x}\,\mathrm{e}^{-3x}\left(\int \frac{\mathrm{e}^{-3x-\frac{\ln(x)}{6}}\mathrm{e}^{6x}}{x}dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$6x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(1+18x)\left(\frac{d}{dx}y(x)\right) + (1+12x)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{(1+12x)y(x)}{6x^2} - \frac{(1+18x)\left(\frac{d}{dx}y(x)\right)}{6x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) + \frac{(1+18x)\left(\frac{d}{dx}y(x)\right)}{6x} + \frac{(1+12x)y(x)}{6x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point
  ◦ Define functions
    $\left[P_2(x) = \frac{1+18x}{6x}, P_3(x) = \frac{1+12x}{6x^2}\right]$

- ∘ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{1}{6}$
- ∘ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{1}{6}$
- ∘ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$
- Multiply by denominators

  $6x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(1+18x)\left(\frac{d}{dx}y(x)\right) + (1+12x)\,y(x) = 0$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- □ Rewrite ODE with series expansions
  - ∘ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - ∘ Shift index using $k- >k - m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - ∘ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$
  - ∘ Shift index using $k- >k + 1 - m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - ∘ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+3r)(-1+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(3k+3r-1)(2k+2r-1) + 6a_{k-1}(3k+3r-1)\right) x^{k+r}\right) =$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+3r)(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2}, \frac{1}{3}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $6\left(\left(k+r-\frac{1}{2}\right) a_k + 3a_{k-1}\right)\left(k - \frac{1}{3} + r\right) = 0$
- Shift index using $k- >k + 1$

  $6\left(\left(k+\frac{1}{2}+r\right) a_{k+1} + 3a_k\right)\left(k + \frac{2}{3} + r\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{6a_k}{2k+1+2r}$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = -\frac{6a_k}{2k+2}$
- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{6a_k}{2k+2}\right]$

- Recursion relation for $r = \frac{1}{3}$

  $a_{k+1} = -\frac{6a_k}{2k+\frac{5}{3}}$
- Solution for $r = \frac{1}{3}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+1} = -\frac{6a_k}{2k+\frac{5}{3}}\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+1} = -\frac{6a_k}{2k+2}, b_{k+1} = -\frac{6b_k}{2k+\frac{5}{3}}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.025 (sec)
Leaf size : 40

```
dsolve(6*x^2*diff(diff(y(x),x),x)+x*(1+18*x)*diff(y(x),x)+(1+12*x)*y(x) = 0,y(x),singsol
```

$$y = \frac{-\frac{c_2(-x)^{5/6}3^{5/6}}{3} + x\,\mathrm{e}^{-3x}\left(c_2\Gamma\left(\frac{5}{6}\right) - c_2\Gamma\left(\frac{5}{6}, -3x\right) + c_1\right)}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.064 (sec)
Leaf size : 47

```
DSolve[{6*x^2*D[y[x],{x,2}]+x*(1+18*x)*D[y[x],x]+(1+12*x)*y[x]==0,{}},y[x],x,IncludeSingularSol
```

$$y(x) \to e^{-3x}\left(\frac{\sqrt[6]{3}c_2x^{4/3}\Gamma\left(-\frac{1}{6}, -3x\right)}{(-x)^{5/6}} + c_1\sqrt{x}\right)$$

2.1.655 Problem 672

Internal problem ID [9827]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 672
Date solved : `Monday, January 27, 2025 at 06:14:42 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$3x^2y'' - x(x+8)\,y' + 6y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 3.536 (sec)

Writing the ode as

$$3x^2y'' + \left(-x^2 - 8x\right) y' + 6y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^2 \\ B &= -x^2 - 8x \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 16x + 40}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 16x + 40 \\ t &= 36x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 16x + 40}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1250: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{36} + \frac{4}{9x} + \frac{10}{9x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{10}{9}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{3} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{2}{3} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{6} + \frac{4}{3x} - \frac{2}{x^2} + \frac{16}{x^3} - \frac{140}{x^4} + \frac{1312}{x^5} - \frac{12944}{x^6} + \frac{132736}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{6} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{36}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 16x + 40}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{1}{36}\right) + \left(\frac{16x + 40}{36x^2}\right) \\ &= \frac{1}{36} + \frac{16x + 40}{36x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 16. Dividing this by leading coefficient in t which is 36 gives $\frac{4}{9}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{4}{9}\right) - (0) \\ &= \frac{4}{9} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{6} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{4}{9}}{\frac{1}{6}} - 0\right) &= \frac{4}{3} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{4}{9}}{\frac{1}{6}} - 0\right) &= -\frac{4}{3} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 16x + 40}{36x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{3}$	$-\frac{2}{3}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{6}$	$\frac{4}{3}$	$-\frac{4}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{4}{3}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\
&= \frac{4}{3} - \left(-\frac{2}{3}\right) \\
&= 2
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{2}{3x} + \left(\frac{1}{6}\right) \\
&= -\frac{2}{3x} + \frac{1}{6} \\
&= \frac{-4+x}{6x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(-\frac{2}{3x} + \frac{1}{6}\right)(2x + a_1) + \left(\left(\frac{2}{3x^2}\right) + \left(-\frac{2}{3x} + \frac{1}{6}\right)^2 - \left(\frac{x^2 + 16x + 40}{36x^2}\right)\right) = 0$$

$$\frac{(-a_1 - 2)\,x - 2a_0 - 4a_1}{3x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 4, a_1 = -2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 2x + 4$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= pe^{\int \omega \, dx} \\
&= \left(x^2 - 2x + 4\right) \mathrm{e}^{\int \left(-\frac{2}{3x} + \frac{1}{6}\right) dx} \\
&= \left(x^2 - 2x + 4\right) \mathrm{e}^{\frac{x}{6} - \frac{2\ln(x)}{3}} \\
&= \frac{\left(x^2 - 2x + 4\right) \mathrm{e}^{\frac{x}{6}}}{x^{2/3}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x^2 - 8x}{3x^2} \, dx} \\
&= z_1 e^{\frac{x}{6} + \frac{4\ln(x)}{3}} \\
&= z_1 \left(x^{4/3} \mathrm{e}^{\frac{x}{6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^{2/3} \mathrm{e}^{\frac{x}{3}} \left(x^2 - 2x + 4\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2 - 8x}{3x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{x}{3} + \frac{8\ln(x)}{3}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{x}{3} + \frac{8\ln(x)}{3}} \mathrm{e}^{-\frac{2x}{3}}}{x^{4/3} \left(x^2 - 2x + 4\right)^2} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{2/3} \mathrm{e}^{\frac{x}{3}} \left(x^2 - 2x + 4\right)\right) + c_2 \left(x^{2/3} \mathrm{e}^{\frac{x}{3}} \left(x^2 - 2x + 4\right) \left(\int \frac{\mathrm{e}^{\frac{x}{3} + \frac{8\ln(x)}{3}} \mathrm{e}^{-\frac{2x}{3}}}{x^{4/3} \left(x^2 - 2x + 4\right)^2} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$3x^2\left(\frac{d^2}{dx^2}y(x)\right)-x(x+8)\left(\frac{d}{dx}y(x)\right)+6y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{2y(x)}{x^2}+\frac{(x+8)\left(\frac{d}{dx}y(x)\right)}{3x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x)-\frac{(x+8)\left(\frac{d}{dx}y(x)\right)}{3x}+\frac{2y(x)}{x^2}=0$

☐ Check to see if $x_0=0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x)=-\frac{x+8}{3x},P_3(x)=\frac{2}{x^2}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=-\frac{8}{3}$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=2$

- ○ $x=0$is a regular singular point

Check to see if $x_0=0$ is a regular singular point

$x_0=0$

- Multiply by denominators

$3x^2\left(\frac{d^2}{dx^2}y(x)\right)-x(x+8)\left(\frac{d}{dx}y(x)\right)+6y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ○ Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

Rewrite ODE with series expansions

$a_0(-2+3r)\,(-3+r)\,x^r+\left(\sum_{k=1}^{\infty}\left(a_k(3k+3r-2)\,(k+r-3)-a_{k-1}(k+r-1)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(-2+3r)\,(-3+r)=0$

- Values of r that satisfy the indicial equation

$r\in\left\{3,\frac{2}{3}\right\}$

- Each term in the series must be 0, giving the recursion relation

$3\left(k+r-\frac{2}{3}\right)(k+r-3)\,a_k-a_{k-1}(k+r-1)=0$

- Shift index using $k->k+1$

$$3\left(k+\frac{1}{3}+r\right)(k-2+r)\,a_{k+1}-a_k(k+r)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{a_k(k+r)}{(3k+1+3r)(k-2+r)}$$

- Recursion relation for $r=3$

$$a_{k+1}=\frac{a_k(k+3)}{(3k+10)(k+1)}$$

- Solution for $r=3$

$$\left[y(x)=\sum_{k=0}^{\infty}a_k x^{k+3},a_{k+1}=\frac{a_k(k+3)}{(3k+10)(k+1)}\right]$$

- Recursion relation for $r=\frac{2}{3}$

$$a_{k+1}=\frac{a_k\left(k+\frac{2}{3}\right)}{(3k+3)\left(k-\frac{4}{3}\right)}$$

- Solution for $r=\frac{2}{3}$

$$\left[y(x)=\sum_{k=0}^{\infty}a_k x^{k+\frac{2}{3}},a_{k+1}=\frac{a_k\left(k+\frac{2}{3}\right)}{(3k+3)\left(k-\frac{4}{3}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty}a_k x^{k+3}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{2}{3}}\right),a_{k+1}=\frac{a_k(k+3)}{(3k+10)(k+1)},b_{k+1}=\frac{b_k\left(k+\frac{2}{3}\right)}{(3k+3)\left(k-\frac{4}{3}\right)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric sol
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special functio
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.056 (sec)
Leaf size : 38

```
dsolve(3*x^2*diff(diff(y(x),x),x)-x*(x+8)*diff(y(x),x)+6*y(x) = 0,y(x),singsol=all)
```

$$y = c_2 \left(x^{2/3} - \frac{x^{5/3}}{2} + \frac{x^{8/3}}{4}\right) e^{\frac{x}{3}} + c_1 \,\mathrm{hypergeom}\left([3], \left[\frac{10}{3}\right], \frac{x}{3}\right) x^3$$

Mathematica DSolve solution

Solving time : 0.591 (sec)
Leaf size : 99

```
DSolve[{3*x^2*D[y[x],{x,2}]-x*(x+8)*D[y[x],x]+6*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tr
```

$$y(x) \to x^{4/3}(x^2 - 2x + 4) \exp\left(\frac{1}{6}\left(6\int_1^x \frac{K[1]-4}{6K[1]} dK[1] + x + 8\right)\right) \left(c_2 \int_1^x \frac{\exp\left(-2\int_1^{K[2]} \frac{K[1]-4}{6K[1]} dK[1]\right)}{(K[2]^2 - 2K[2] + 4)^2} dK[2] + c_1\right)$$

2.1.656 Problem 673

Internal problem ID [9828]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 673
Date solved : `Monday, January 27, 2025 at 06:14:46 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2y'' - x(1+2x)\,y' + 2(4x-1)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.683 (sec)

Writing the ode as

$$2x^2y'' + \left(-2x^2 - x\right)y' + (8x-2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= -2x^2 - x \\ C &= 8x - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 - 60x + 21}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2 - 60x + 21 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 - 60x + 21}{16x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1252: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{15}{4x} + \frac{21}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{15}{4x} - \frac{51}{4x^2} - \frac{765}{8x^3} - \frac{3519}{4x^4} - \frac{144585}{16x^5} - \frac{6358527}{64x^6} - \frac{146409525}{128x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{4x^2 - 60x + 21}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-60x + 21}{16x^2}\right) \\ &= \frac{1}{4} + \frac{-60x + 21}{16x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -60. Dividing this by leading coefficient in t which is 16 gives $-\frac{15}{4}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{15}{4}\right) - (0) \\ &= -\frac{15}{4}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{15}{4}}{\frac{1}{2}} - 0\right) &&= -\frac{15}{4} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{15}{4}}{\frac{1}{2}} - 0\right) &&= \frac{15}{4}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^2 - 60x + 21}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{15}{4}$	$\frac{15}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{15}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{15}{4} - \left(\frac{7}{4}\right) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{7}{4x} + (-)\left(\frac{1}{2}\right) \\ &= \frac{7}{4x} - \frac{1}{2} \\ &= \frac{7}{4x} - \frac{1}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{7}{4x} - \frac{1}{2}\right)(2x + a_1) + \left(\left(-\frac{7}{4x^2}\right) + \left(\frac{7}{4x} - \frac{1}{2}\right)^2 - \left(\frac{4x^2 - 60x + 21}{16x^2}\right) \right) = 0$$
$$\frac{2(9 + a_1)x + 4a_0 + 7a_1}{2x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{ a_0 = \frac{63}{4}, a_1 = -9 \right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 9x + \frac{63}{4}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 9x + \frac{63}{4}\right) e^{\int \left(\frac{7}{4x} - \frac{1}{2}\right) dx} \\
&= \left(x^2 - 9x + \frac{63}{4}\right) e^{-\frac{x}{2} + \frac{7 \ln(x)}{4}} \\
&= \frac{(4x^2 - 36x + 63)\, x^{7/4} e^{-\frac{x}{2}}}{4}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-2x^2 - x}{2x^2} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x)}{4}} \\
&= z_1 \left(x^{1/4} e^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^4 - 9x^3 + \frac{63}{4} x^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^2 - x}{2x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{x + \frac{\ln(x)}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{e^{x + \frac{\ln(x)}{2}}}{\left(x^4 - 9x^3 + \frac{63}{4} x^2\right)^2} dx \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^4 - 9x^3 + \frac{63}{4} x^2\right) + c_2 \left(x^4 - 9x^3 + \frac{63}{4} x^2 \left(\int \frac{e^{x + \frac{\ln(x)}{2}}}{\left(x^4 - 9x^3 + \frac{63}{4} x^2\right)^2} dx \right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2\left(\frac{d^2}{dx^2}y(x)\right)-x(2x+1)\left(\frac{d}{dx}y(x)\right)+2(4x-1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(4x-1)y(x)}{x^2}+\frac{(2x+1)\left(\frac{d}{dx}y(x)\right)}{2x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)-\frac{(2x+1)\left(\frac{d}{dx}y(x)\right)}{2x}+\frac{(4x-1)y(x)}{x^2}=0$

- ☐ Check to see if $x_0=0$ is a regular singular point
  - Define functions

    $\left[P_2(x)=-\frac{2x+1}{2x},P_3(x)=\frac{4x-1}{x^2}\right]$
  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=-\frac{1}{2}$
  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=-1$
  - $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2x^2\left(\frac{d^2}{dx^2}y(x)\right)-x(2x+1)\left(\frac{d}{dx}y(x)\right)+(8x-2)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

    $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$
  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-2+r)\,x^r+\left(\sum_{k=1}^{\infty}\left(a_k(2k+2r+1)\,(k+r-2)-2a_{k-1}(k-5+r)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(1+2r)(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{2, -\frac{1}{2}\}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k+r+\frac{1}{2}\right)(k+r-2)\,a_k - 2a_{k-1}(k-5+r) = 0$

- Shift index using $k->k+1$

  $2\left(k+\frac{3}{2}+r\right)(k+r-1)\,a_{k+1} - 2a_k(k+r-4) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k(k+r-4)}{(2k+3+2r)(k+r-1)}$

- Recursion relation for $r=2$; series terminates at $k=2$

  $a_{k+1} = \frac{2a_k(k-2)}{(2k+7)(k+1)}$

- Apply recursion relation for $k=0$

  $a_1 = -\frac{4a_0}{7}$

- Apply recursion relation for $k=1$

  $a_2 = -\frac{a_1}{9}$

- Express in terms of a_0

  $a_2 = \frac{4a_0}{63}$

- Terminating series solution of the ODE for $r=2$. Use reduction of order to find the second li

  $y(x) = a_0 \cdot \left(1 - \frac{4}{7}x + \frac{4}{63}x^2\right)$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+1} = \frac{2a_k\left(k-\frac{9}{2}\right)}{(2k+2)\left(k-\frac{3}{2}\right)}$

- Solution for $r = -\frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = \frac{2a_k\left(k-\frac{9}{2}\right)}{(2k+2)\left(k-\frac{3}{2}\right)}\right]$

- Combine solutions and rename parameters

  $\left[y(x) = a_0 \cdot \left(1 - \frac{4}{7}x + \frac{4}{63}x^2\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{2}}\right), b_{k+1} = \frac{2b_k\left(k-\frac{9}{2}\right)}{(2k+2)\left(k-\frac{3}{2}\right)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric sol
```

```
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returnin
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.056 (sec)
Leaf size : 32

```
dsolve(2*x^2*diff(diff(y(x),x),x)-x*(2*x+1)*diff(y(x),x)+2*(-1+4*x)*y(x) = 0,y(x),singso
```

$$y = \frac{c_1 x^2(4x^2-36x+63)}{63} + \frac{c_2 \operatorname{hypergeom}\left(\left[-\frac{9}{2}\right], \left[-\frac{3}{2}\right], x\right)}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.277 (sec)
Leaf size : 61

```
DSolve[{2*x^2*D[y[x],{x,2}]-x*(1+2*x)*D[y[x],x]+2*(4*x-1)*y[x]==0,{}},y[x],x,IncludeSingularSol
```

$$y(x) \to \frac{1}{4}x^2(4x^2-36x+63)\left(c_2\int_1^x \frac{16e^{K[1]}}{K[1]^{7/2}\left(4K[1]^2-36K[1]+63\right)^2}dK[1]+c_1\right)$$

2.1.657 Problem 674

Internal problem ID [9829]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 674
Date solved : `Monday, January 27, 2025 at 06:14:47 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' - 4x^2y' + (1+2x)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.219 (sec)

Writing the ode as

$$4x^2y'' - 4x^2y' + (1+2x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x^2 \\ C &= 1+2x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 2x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 2x - 1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 2x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1254: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} - \frac{1}{2x^2} - \frac{1}{2x^3} - \frac{3}{4x^4} - \frac{5}{4x^5} - \frac{9}{4x^6} - \frac{17}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 2x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-2x - 1}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 2x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\left(\frac{1}{2}\right) \\ &= \frac{1}{2x} - \frac{1}{2} \\ &= -\frac{x-1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered} (0) + 2\left(\frac{1}{2x} - \frac{1}{2}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^2-2x-1}{4x^2}\right)\right) = 0 \\ 0 = 0 \end{gathered}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{1}{2x} - \frac{1}{2}\right) dx} \\ &= \sqrt{x}\, e^{-\frac{x}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x^2}{4x^2}\,dx} \\
&= z_1 e^{\frac{x}{2}} \\
&= z_1\left(e^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \sqrt{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^2}{4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^x}{(y_1)^2}\,dx \\
&= y_1(-\operatorname{Ei}_1(-x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x}\right) + c_2\left(\sqrt{x}(-\operatorname{Ei}_1(-x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2\left(\frac{d^2}{dx^2}y(x)\right) - 4x^2\left(\frac{d}{dx}y(x)\right) + (2x+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(2x+1)y(x)}{4x^2} + \frac{d}{dx}y(x)$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) - \frac{d}{dx}y(x) + \frac{(2x+1)y(x)}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -1, P_3(x) = \frac{2x+1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{1}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) - 4x^2\left(\frac{d}{dx}y(x)\right) + (2x+1)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^2 \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r+1}$

  - Shift index using $k->k-1$

    $x^2 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\,x^{k+r}$

  - Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)^2\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)^2 - 2a_{k-1}(2k-3+2r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = \frac{1}{2}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(2k+2r-1)^2 + (-4k-4r+6)\,a_{k-1} = 0$

- Shift index using $k->k+1$

  $a_{k+1}(2k+1+2r)^2 + a_k(-4k-4r+2) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k(2k+2r-1)}{(2k+1+2r)^2}$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = \frac{4a_k k}{(2k+2)^2}$

- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{4a_k k}{(2k+2)^2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 17

```
dsolve(4*x^2*diff(diff(y(x),x),x)-4*diff(y(x),x)*x^2+(2*x+1)*y(x) = 0,y(x),singsol=all
```

$$y = (\mathrm{Ei}_1(-x)\, c_2 + c_1)\sqrt{x}$$

Mathematica DSolve solution

Solving time : 0.206 (sec)
Leaf size : 33

```
DSolve[{4*x^2*D[y[x],{x,2}]-4*x^2*D[y[x],x]+(1+2*x)*y[x]==0,{}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to \sqrt{x}\left(c_2 \int_1^x \frac{e^{K[1]}}{K[1]} dK[1] + c_1\right)$$

2.1.658 Problem 675

Internal problem ID [9830]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 675
Date solved : Monday, January 27, 2025 at 06:14:48 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$x^2y'' + x(3-2x)\, y' + (1-2x)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.263 (sec)

Writing the ode as

$$x^2y'' + \left(-2x^2+3x\right) y' + (1-2x)\, y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x^2+3x \\ C &= 1-2x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2-4x-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2-4x-1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2-4x-1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1256: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 - \frac{1}{4x^2} - \frac{1}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 - \frac{1}{2x} - \frac{1}{4x^2} - \frac{1}{8x^3} - \frac{3}{32x^4} - \frac{5}{64x^5} - \frac{9}{128x^6} - \frac{17}{256x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 - 4x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (1) + \left(\frac{-4x - 1}{4x^2}\right) \\ &= 1 + \frac{-4x - 1}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{1} - 0\right) &&= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{1} - 0\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^2 - 4x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\,(1) \\
&= \frac{1}{2x} - 1 \\
&= \frac{1}{2x} - 1
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2x} - 1\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} - 1\right)^2 - \left(\frac{4x^2 - 4x - 1}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(\frac{1}{2x} - 1\right) dx} \\
&= \sqrt{x}\, e^{-x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x^2+3x}{x^2}\,dx} \\ &= z_1 e^{x-\frac{3\ln(x)}{2}} \\ &= z_1 \left(\frac{e^x}{x^{3/2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^2+3x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x-3\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(-\operatorname{Ei}_1(-2x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x}(-\operatorname{Ei}_1(-2x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(3-2x)\left(\frac{d}{dx}y(x)\right) + (-2x+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = \frac{(2x-1)y(x)}{x^2} + \frac{(2x-3)\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) - \frac{(2x-3)\left(\frac{d}{dx}y(x)\right)}{x} - \frac{(2x-1)y(x)}{x^2} = 0$
- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{2x-3}{x}, P_3(x) = -\frac{2x-1}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

$(x \cdot P_2(x))\Big|_{x=0} = 3$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 1$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) - (2x-3)\,x\left(\frac{d}{dx}y(x)\right) + (-2x+1)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ◦ Shift index using $k- >k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- ◦ Shift index using $k- >k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ◦ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)^2\,x^r + \left(\sum_{k=1}^{\infty}\left(a_k(k+r+1)^2 - 2a_{k-1}(k+r)\right)x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = -1$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)^2 - 2a_{k-1}(k+r) = 0$

- Shift index using $k- >k+1$

  $a_{k+1}(k+2+r)^2 - 2a_k(k+r+1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k(k+r+1)}{(k+2+r)^2}$

- Recursion relation for $r = -1$

  $a_{k+1} = \frac{2a_k k}{(k+1)^2}$

- Solution for $r = -1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = \frac{2a_k k}{(k+1)^2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(-2*x+3)*diff(y(x),x)+(1-2*x)*y(x) = 0,y(x),singsol=al
```

$$y = \frac{c_2 \operatorname{Ei}_1(-2x) + c_1}{x}$$

Mathematica DSolve solution

Solving time : 0.178 (sec)
Leaf size : 33

```
DSolve[{x^2*D[y[x],{x,2}]+x*(3-2*x)*D[y[x],x]+(1-2*x)*y[x]==0,{}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to \frac{c_2 \int_1^x \frac{e^{2K[1]}}{K[1]} dK[1] + c_1}{x}$$

2.1.659 Problem 676

Internal problem ID [9831]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 676
Date solved : `Monday, January 27, 2025 at 06:14:49 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x(3+x)\,y' + (4-x)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.270 (sec)

Writing the ode as

$$x^2y'' + \left(-x^2 - 3x\right)y' + (4-x)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 - 3x \\ C &= 4 - x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 10x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 10x - 1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 10x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1258: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{5}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{5}{2x} - \frac{13}{2x^2} + \frac{65}{2x^3} - \frac{819}{4x^4} + \frac{5785}{4x^5} - \frac{43797}{4x^6} + \frac{347425}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 10x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{10x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{10x - 1}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 10. Dividing this by leading coefficient in t which is 4 gives $\frac{5}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{5}{2}\right) - (0) \\ &= \frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{5}{2}}{\frac{1}{2}} - 0\right) &= \frac{5}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{5}{2}}{\frac{1}{2}} - 0\right) &= -\frac{5}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 10x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{5}{2}$	$-\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+\right) \\
&= \frac{5}{2} - \left(\frac{1}{2}\right) \\
&= 2
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2x} + \frac{1}{2} \\
&= \frac{1+x}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{1}{2x} + \frac{1}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} + \frac{1}{2}\right)^2 - \left(\frac{x^2 + 10x - 1}{4x^2}\right) \right) = 0$$

$$\frac{(-a_1 + 4)\, x - 2a_0 + a_1}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2, a_1 = 4\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + 4x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \left(x^2+4x+2\right) \mathrm{e}^{\int \left(\frac{1}{2x}+\frac{1}{2}\right) dx} \\
&= \left(x^2+4x+2\right) \mathrm{e}^{\frac{x}{2}+\frac{\ln(x)}{2}} \\
&= \left(x^2+4x+2\right) \sqrt{x}\, \mathrm{e}^{\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2-3x}{x^2}\, dx} \\
&= z_1 e^{\frac{x}{2}+\frac{3\ln(x)}{2}} \\
&= z_1 \left(x^{3/2} \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^2 \mathrm{e}^x \left(x^2+4x+2\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-3x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x+3\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{\mathrm{e}^{-x}(-x-3)}{4\left(x^2+4x+2\right)} - \frac{\mathrm{Ei}_1(x)}{4}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^2 \mathrm{e}^x \left(x^2+4x+2\right)\right) + c_2 \left(x^2 \mathrm{e}^x \left(x^2+4x+2\right) \left(-\frac{\mathrm{e}^{-x}(-x-3)}{4\left(x^2+4x+2\right)} - \frac{\mathrm{Ei}_1(x)}{4}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right) - x(x+3)\left(\frac{d}{dx}y(x)\right) + (4-x)\, y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = \frac{(-4+x)y(x)}{x^2} + \frac{(x+3)\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x) - \frac{(x+3)\left(\frac{d}{dx}y(x)\right)}{x} - \frac{(-4+x)y(x)}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = -\frac{x+3}{x}, P_3(x) = -\frac{-4+x}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = -3$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 4$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) - x(x+3)\left(\frac{d}{dx}y(x)\right) + (4-x)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-2)^2 - a_{k-1}(k+r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 2$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-2)^2 - a_{k-1}(k+r) = 0$

- Shift index using $k- >k+1$

  $a_{k+1}(k+r-1)^2 - a_k(k+r+1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r+1)}{(k+r-1)^2}$

- Recursion relation for $r = 2$

$a_{k+1} = \frac{a_k(k+3)}{(k+1)^2}$

- Solution for $r = 2$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k(k+3)}{(k+1)^2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 42

```
dsolve(x^2*diff(diff(y(x),x),x)-x*(x+3)*diff(y(x),x)+(-x+4)*y(x) = 0,y(x),singsol=all)
```

$$y = x^2\left(e^x c_2\left(x^2+4x+2\right)\operatorname{Ei}_1(x) + c_1\left(x^2+4x+2\right)e^x - c_2(x+3)\right)$$

Mathematica DSolve solution

Solving time : 0.259 (sec)
Leaf size : 60

```
DSolve[{x^2*D[y[x],{x,2}]-x*(3+x)*D[y[x],x]+(4-x)*y[x]==0,{}},y[x],x,IncludeSingularSolution
```

$$y(x) \to e^{x+2}x^2\left(x^2+4x+2\right)\left(c_2\int_1^x \frac{e^{-K[1]-1}}{K[1]\left(K[1]^2+4K[1]+2\right)^2}dK[1] + c_1\right)$$

2.1.660 Problem 677

Internal problem ID [9832]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 677
Date solved : `Monday, January 27, 2025 at 06:14:49 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x(3-x)\,y' + y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.272 (sec)

Writing the ode as

$$x^2y'' + \left(-x^2+3x\right)y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2+3x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2-6x-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2-6x-1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2-6x-1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1260: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{4x^2} - \frac{3}{2x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{3}{2x} - \frac{5}{2x^2} - \frac{15}{2x^3} - \frac{115}{4x^4} - \frac{495}{4x^5} - \frac{2285}{4x^6} - \frac{11055}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 6x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-6x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-6x - 1}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -6. Dividing this by leading coefficient in t which is 4 gives $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{2}\right) - (0) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &&= \frac{3}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 6x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^+) \\ &= \frac{3}{2} - \left(\frac{1}{2}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\left(\frac{1}{2}\right) \\ &= \frac{1}{2x} - \frac{1}{2} \\ &= -\frac{-1+x}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{2}\right)(1) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^2-6x-1}{4x^2}\right)\right) = 0$$

$$\frac{1+a_0}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = -1 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (-1+x)\, \mathrm{e}^{\int \left(\frac{1}{2x}-\frac{1}{2}\right) dx} \\ &= (-1+x)\, \mathrm{e}^{-\frac{x}{2}+\frac{\ln(x)}{2}} \\ &= (-1+x)\sqrt{x}\, \mathrm{e}^{-\frac{x}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x^2+3x}{x^2} \, dx} \\ &= z_1 e^{\frac{x}{2}-\frac{3\ln(x)}{2}} \\ &= z_1 \left(\frac{\mathrm{e}^{\frac{x}{2}}}{x^{3/2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{-1+x}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+3x}{x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{x-3\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{x}}{-1+x} - \mathrm{Ei}_1(-x)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{-1+x}{x}\right) + c_2 \left(\frac{-1+x}{x}\left(-\frac{\mathrm{e}^{x}}{-1+x} - \mathrm{Ei}_1(-x)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right)+x(-x+3)\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{y(x)}{x^2}+\frac{(x-3)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)-\frac{(x-3)\left(\frac{d}{dx}y(x)\right)}{x}+\frac{y(x)}{x^2}=0$

☐ Check to see if $x_0=0$ is a regular singular point

  - Define functions

  $\left[P_2(x)=-\frac{x-3}{x}, P_3(x)=\frac{1}{x^2}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=3$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=1$

  - $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

$x^2\left(\frac{d^2}{dx^2}y(x)\right)-x(x-3)\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)^2 x^r+\left(\sum_{k=1}^{\infty}\left(a_k(k+r+1)^2-a_{k-1}(k+r-1)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(1+r)^2=0$

- Values of r that satisfy the indicial equation

$r=-1$

- Each term in the series must be 0, giving the recursion relation

$a_k(k+r+1)^2-a_{k-1}(k+r-1)=0$

- Shift index using $k->k+1$

$a_{k+1}(k+2+r)^2 - a_k(k+r) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r)}{(k+2+r)^2}$

- Recursion relation for $r=-1$; series terminates at $k=1$

  $a_{k+1} = \frac{a_k(k-1)}{(k+1)^2}$

- Apply recursion relation for $k=0$

  $a_1 = -a_0$

- Terminating series solution of the ODE for $r=-1$. Use reduction of order to find the second lin

  $y(x) = a_0 \cdot (1-x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 28

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(-x+3)*diff(y(x),x)+y(x) = 0,y(x),singsol=all)
```

$$y = \frac{\mathrm{Ei}_1(-x)\, c_2(x-1) + \mathrm{e}^x c_2 + c_1(x-1)}{x}$$

Mathematica DSolve solution

Solving time : 0.202 (sec)
Leaf size : 40

```
DSolve[{x^2*D[y[x],{x,2}]+x*(3-x)*D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{(x-1)\left(c_2 \int_1^x \frac{e^{K[1]}}{(K[1]-1)^2 K[1]} dK[1] + c_1\right)}{x}$$

2.1.661 Problem 678

Internal problem ID [9833]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 678
Date solved : `Monday, January 27, 2025 at 06:14:50 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - \left(2\sqrt{5}-1\right)xy' + \left(\frac{19}{4}-3x^2\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.145 (sec)

Writing the ode as

$$x^2y'' + \left(-2x\sqrt{5}+x\right)y' + \left(\frac{19}{4}-3x^2\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x\sqrt{5}+x \\ C &= \frac{19}{4}-3x^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 3z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1262: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 3$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\sqrt{3}\, x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x\sqrt{5}+x}{x^2}\, dx} \\ &= z_1 e^{\ln(x)\sqrt{5}-\frac{\ln(x)}{2}} \\ &= z_1 \left(x^{\sqrt{5}-\frac{1}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^{\sqrt{5}-\frac{1}{2}} \mathrm{e}^{-\sqrt{3}\, x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x\sqrt{5}+x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)\left(2\sqrt{5}-1\right)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{e^{\ln(x)\left(2\sqrt{5}-1\right)} x^{1-2\sqrt{5}} e^{2\sqrt{3}\,x}\sqrt{3}}{6}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^{\sqrt{5}-\frac{1}{2}} e^{-\sqrt{3}\,x}\right) + c_2\left(x^{\sqrt{5}-\frac{1}{2}} e^{-\sqrt{3}\,x}\left(\frac{e^{\ln(x)\left(2\sqrt{5}-1\right)} x^{1-2\sqrt{5}} e^{2\sqrt{3}\,x}\sqrt{3}}{6}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) - \left(-1+2\sqrt{5}\right)x\left(\frac{d}{dx}y(x)\right) + \left(\frac{19}{4} - 3x^2\right)y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{(12x^2-19)y(x)}{4x^2} + \frac{\left(-1+2\sqrt{5}\right)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) - \frac{\left(-1+2\sqrt{5}\right)\left(\frac{d}{dx}y(x)\right)}{x} - \frac{(12x^2-19)y(x)}{4x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{-1+2\sqrt{5}}{x}, P_3(x) = -\frac{12x^2-19}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1 - 2\sqrt{5}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{19}{4}$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) - 4\left(-1+2\sqrt{5}\right)x\left(\frac{d}{dx}y(x)\right) + (-12x^2+19)y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

  - Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $\left(1+2\sqrt{5}-2r\right)\left(-1+2\sqrt{5}-2r\right) a_0 x^r + \left(-1+2\sqrt{5}-2r\right)\left(-3+2\sqrt{5}-2r\right) a_1 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(\left(-2k\right.\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $\left(1+2\sqrt{5}-2r\right)\left(-1+2\sqrt{5}-2r\right) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}+\sqrt{5}, \frac{1}{2}+\sqrt{5}\right\}$

- Each term must be 0

  $\left(-1+2\sqrt{5}-2r\right)\left(-3+2\sqrt{5}-2r\right) a_1 = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $-8a_k(k+r)\sqrt{5} + (4k^2+8kr+4r^2+19)\, a_k - 12a_{k-2} = 0$

- Shift index using $k- >k+2$

  $-8a_{k+2}(k+2+r)\sqrt{5} + \left(4(k+2)^2 + 8(k+2)\, r + 4r^2 + 19\right) a_{k+2} - 12a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{12a_k}{-35+8k\sqrt{5}+8\sqrt{5}\, r-4k^2-8kr-4r^2+16\sqrt{5}-16k-16r}$

- Recursion relation for $r = -\frac{1}{2}+\sqrt{5}$

  $a_{k+2} = -\frac{12a_k}{-27+8k\sqrt{5}+8\sqrt{5}\left(-\frac{1}{2}+\sqrt{5}\right)-4k^2-8k\left(-\frac{1}{2}+\sqrt{5}\right)-4\left(-\frac{1}{2}+\sqrt{5}\right)^2-16k}$

- Solution for $r = -\frac{1}{2}+\sqrt{5}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}+\sqrt{5}}, a_{k+2} = -\frac{12a_k}{-27+8k\sqrt{5}+8\sqrt{5}\left(-\frac{1}{2}+\sqrt{5}\right)-4k^2-8k\left(-\frac{1}{2}+\sqrt{5}\right)-4\left(-\frac{1}{2}+\sqrt{5}\right)^2-16k}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{2}+\sqrt{5}$

  $a_{k+2} = -\frac{12a_k}{-43+8k\sqrt{5}+8\sqrt{5}\left(\frac{1}{2}+\sqrt{5}\right)-4k^2-8k\left(\frac{1}{2}+\sqrt{5}\right)-4\left(\frac{1}{2}+\sqrt{5}\right)^2-16k}$

- Solution for $r = \frac{1}{2}+\sqrt{5}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}+\sqrt{5}}, a_{k+2} = -\frac{12a_k}{-43+8k\sqrt{5}+8\sqrt{5}\left(\frac{1}{2}+\sqrt{5}\right)-4k^2-8k\left(\frac{1}{2}+\sqrt{5}\right)-4\left(\frac{1}{2}+\sqrt{5}\right)^2-16k}, a_1 = 0\right]$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}+\sqrt{5}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}+\sqrt{5}}\right), a_{k+2} = -\frac{12a_k}{-27+8k\sqrt{5}+8\sqrt{5}\left(-\frac{1}{2}+\sqrt{5}\right)-4k^2-8k\left(-\frac{1}{2}+\sqrt{5}\right)-4\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 29

```
dsolve(x^2*diff(diff(y(x),x),x)-(-1+2*5^(1/2))*x*diff(y(x),x)+(19/4-3*x^2)*y(x) = 0,y(
```

$$y = x^{-\frac{1}{2}+\sqrt{5}}\left(c_1 \sinh\left(\sqrt{3}\,x\right) + c_2 \cosh\left(\sqrt{3}\,x\right)\right)$$

Mathematica DSolve solution

Solving time : 0.076 (sec)
Leaf size : 53

```
DSolve[{x^2*D[y[x],{x,2}]-(2*Sqrt[5]-1)*x*D[y[x],x]+(19/4-3*x^2)*y[x]==0,{}},y[x],x,IncludeS
```

$$y(x) \to \frac{1}{6}e^{-\sqrt{3}x}x^{\sqrt{5}-\frac{1}{2}}\left(\sqrt{3}c_2e^{2\sqrt{3}x} + 6c_1\right)$$

2.1.662 Problem 679

Internal problem ID [9834]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 679
Date solved : `Monday, January 27, 2025 at 06:14:51 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x(x-3)\,y' + (4-x)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.227 (sec)

Writing the ode as

$$x^2y'' + \left(x^2-3x\right)y' + (4-x)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2-3x \\ C &= 4-x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2-2x-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2-2x-1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2-2x-1}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1264: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} - \frac{1}{2x^2} - \frac{1}{2x^3} - \frac{3}{4x^4} - \frac{5}{4x^5} - \frac{9}{4x^6} - \frac{17}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 2x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-2x - 1}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 2x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\left(\frac{1}{2}\right) \\ &= \frac{1}{2x} - \frac{1}{2} \\ &= -\frac{x-1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2x} - \frac{1}{2}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^2-2x-1}{4x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{1}{2x} - \frac{1}{2}\right) dx} \\ &= \sqrt{x}\, e^{-\frac{x}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^2-3x}{x^2}\,dx} \\ &= z_1 e^{-\frac{x}{2}+\frac{3\ln(x)}{2}} \\ &= z_1\left(x^{3/2}\mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^2\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2-3x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-x+3\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(-\operatorname{Ei}_1(-x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(x^2\mathrm{e}^{-x}\right) + c_2\left(x^2\mathrm{e}^{-x}(-\operatorname{Ei}_1(-x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(x-3)\left(\frac{d}{dx}y(x)\right) + (4-x)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{(-4+x)y(x)}{x^2} - \frac{(x-3)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(x-3)\left(\frac{d}{dx}y(x)\right)}{x} - \frac{(-4+x)y(x)}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{x-3}{x}, P_3(x) = -\frac{-4+x}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -3$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^{2} \cdot P_{3}(x)\right)\Big|_{x=0} = 4$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^{2}\left(\frac{d^{2}}{d x^{2}}y(x)\right)+x(x-3)\left(\frac{d}{d x}y(x)\right)+(4-x)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty} a_{k} x^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $x^{m} \cdot y(x)$ to series expansion for $m = 0..1$

    $x^{m} \cdot y(x)=\sum_{k=0}^{\infty} a_{k} x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^{m} \cdot y(x)=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^{m} \cdot\left(\frac{d}{d x}y(x)\right)$ to series expansion for $m = 1..2$

    $x^{m} \cdot\left(\frac{d}{d x}y(x)\right)=\sum_{k=0}^{\infty} a_{k}(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^{m} \cdot\left(\frac{d}{d x}y(x)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^{2} \cdot\left(\frac{d^{2}}{d x^{2}}y(x)\right)$ to series expansion

    $x^{2} \cdot\left(\frac{d^{2}}{d x^{2}}y(x)\right)=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(-2+r)^{2} x^{r}+\left(\sum_{k=1}^{\infty}\left(a_{k}(k+r-2)^{2}+a_{k-1}(k+r-2)\right) x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)^{2}=0$

- Values of r that satisfy the indicial equation

  $r = 2$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)\left(a_{k}(k+r-2)+a_{k-1}\right)=0$

- Shift index using $k->k+1$

  $(k+r-1)\left(a_{k+1}(k+r-1)+a_{k}\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{a_{k}}{k+r-1}$

- Recursion relation for $r = 2$

  $a_{k+1}=-\frac{a_{k}}{k+1}$

- Solution for $r = 2$

  $\left[y(x)=\sum_{k=0}^{\infty} a_{k} x^{k+2}, a_{k+1}=-\frac{a_{k}}{k+1}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 21

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(x-3)*diff(y(x),x)+(-x+4)*y(x) = 0,y(x),singsol=all)
```

$$y = e^{-x} x^2 (\mathrm{Ei}_1(-x) c_2 + c_1)$$

Mathematica DSolve solution

Solving time : 0.376 (sec)
Leaf size : 60

```
DSolve[{x^2*D[y[x],{x,2}]+x*(x-3)*D[y[x],x]+(4-x)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->
```

$$y(x) \to \sqrt{x} \left(c_2 \int_1^x \frac{e^{K[2]}}{K[2]} dK[2] + c_1 \right) \exp \left(\frac{1}{2} \left(-\int_1^x \left(1 - \frac{3}{K[1]} \right) dK[1] - x \right) \right)$$

2.1.663 Problem 680

Internal problem ID [9835]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 680
Date solved : `Monday, January 27, 2025 at 06:14:51 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x^2y' - (2 + x)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.194 (sec)

Writing the ode as

$$x^2y'' + x^2y' + (-x - 2)\, y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2 \\ C &= -x - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x + 8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 4x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1266: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{x^2} + \frac{1}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \qquad (8) \end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{x} + \frac{1}{x^2} - \frac{2}{x^3} + \frac{3}{x^4} - \frac{2}{x^5} - \frac{6}{x^6} + \frac{28}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 + 4x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{4x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{4x + 8}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$\begin{aligned}b &= (1) - (0) \\ &= 1\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{1}{\frac{1}{2}} - 0\right) &&= 1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{1}{\frac{1}{2}} - 0\right) &&= -1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 4x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	1	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + (-)\left(\frac{1}{2}\right) \\ &= -\frac{1}{x} - \frac{1}{2} \\ &= -\frac{2+x}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{x} - \frac{1}{2}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - \frac{1}{2}\right)^2 - \left(\frac{x^2+4x+8}{4x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{x} - \frac{1}{2}\right) dx} \\ &= \frac{e^{-\frac{x}{2}}}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2}{x^2}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{-x}}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1 \left(\left(x^2 - 2x + 2\right) \mathrm{e}^{x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{-x}}{x}\right) + c_2 \left(\frac{\mathrm{e}^{-x}}{x}\left(\left(x^2 - 2x + 2\right) \mathrm{e}^{x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x^2\left(\frac{d}{dx}y(x)\right) - (x+2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = \frac{(x+2)y(x)}{x^2} - \frac{d}{dx}y(x)$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x) + \frac{d}{dx}y(x) - \frac{(x+2)y(x)}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = 1, P_3(x) = -\frac{x+2}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = 0$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -2$$

- $x = 0$is a regular singular point
  Check to see if $x_0 = 0$ is a regular singular point
  $x_0 = 0$
- Multiply by denominators
  $$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x^2\left(\frac{d}{dx}y(x)\right) + (-x-2)\,y(x) = 0$$
- Assume series solution for $y(x)$
  $$y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$$
- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$
    $$x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$
  - Shift index using $k- >k - m$
    $$x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$
  - Convert $x^2 \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion
    $$x^2 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r+1}$$
  - Shift index using $k- >k - 1$
    $$x^2 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\,x^{k+r}$$
  - Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion
    $$x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\,x^{k+r}$$

  Rewrite ODE with series expansions
  $$a_0(1+r)\,(-2+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-2) + a_{k-1}(k+r-2)\right)x^{k+r}\right) = 0$$
- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(1+r)\,(-2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-1, 2\}$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-2)\,(a_k(k+r+1) + a_{k-1}) = 0$
- Shift index using $k- >k + 1$
  $(k-1+r)\,(a_{k+1}(k+2+r) + a_k) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = -\frac{a_k}{k+2+r}$
- Recursion relation for $r = -1$
  $a_{k+1} = -\frac{a_k}{k+1}$
- Solution for $r = -1$
  $$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = -\frac{a_k}{k+1}\right]$$
- Recursion relation for $r = 2$
  $a_{k+1} = -\frac{a_k}{k+4}$
- Solution for $r = 2$
  $$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = -\frac{a_k}{k+4}\right]$$

- Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+1}=-\frac{a_k}{k+1}, b_{k+1}=-\frac{b_k}{4+k}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 25

```
dsolve(x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x^2-(x+2)*y(x) = 0,y(x),singsol=all)
```

$$y=\frac{c_1 \mathrm{e}^{-x}+c_2(x^2-2x+2)}{x}$$

Mathematica DSolve solution

Solving time : 0.208 (sec)
Leaf size : 40

```
DSolve[{x^2*D[y[x],{x,2}]+x^2*D[y[x],x]-(2+x)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x)\to \frac{e^{-x-1}\left(c_2 \int_1^x e^{K[1]+2}K[1]^2dK[1]+c_1\right)}{x}$$

2.1.664 Problem 681

Internal problem ID [9836]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 681
Date solved : `Monday, January 27, 2025 at 06:14:52 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + 2x^2y' + \left(x - \frac{3}{4}\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.211 (sec)

Writing the ode as

$$x^2y'' + 2x^2y' + \left(x - \frac{3}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 2x^2 \\ C &= x - \frac{3}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 - 4x + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2 - 4x + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 - 4x + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1268: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{3}{4x^2} - \frac{1}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 - \frac{1}{2x} + \frac{1}{4x^2} + \frac{1}{8x^3} + \frac{1}{32x^4} - \frac{1}{64x^5} - \frac{3}{128x^6} - \frac{3}{256x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 - 4x + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (1) + \left(\frac{-4x + 3}{4x^2}\right) \\ &= 1 + \frac{-4x + 3}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{1} - 0\right) &&= -\frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{1} - 0\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^2 - 4x + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (1) \\
&= 1 - \frac{1}{2x} \\
&= 1 - \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(1 - \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(1 - \frac{1}{2x}\right)^2 - \left(\frac{4x^2 - 4x + 3}{4x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(1-\frac{1}{2x}\right) dx} \\ &= \frac{e^x}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x^2}{x^2}\, dx} \\ &= z_1 e^{-x} \\ &= z_1\left(e^{-x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2}\, dx \\ &= y_1\left(-\frac{(1+2x)\, e^{-2x}}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{1}{\sqrt{x}}\right) + c_2\left(\frac{1}{\sqrt{x}}\left(-\frac{(1+2x)\, e^{-2x}}{4}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) + 2x^2\left(\frac{d}{dx}y(x)\right) + \left(x - \frac{3}{4}\right) y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(4x-3)y(x)}{4x^2} - 2\frac{d}{dx}y(x)$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x) + 2\frac{d}{dx}y(x) + \frac{(4x-3)y(x)}{4x^2} = 0$

- Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = 2, P_3(x) = \frac{4x-3}{4x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 0$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{3}{4}$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) + 8x^2\left(\frac{d}{dx}y(x)\right) + (4x-3)\,y(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$
    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k - m$
    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^2 \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion
    $x^2 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r+1}$
  - Shift index using $k- >k - 1$
    $x^2 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\,x^{k+r}$
  - Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion
    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\,x^{k+r}$
    Rewrite ODE with series expansions
    $a_0(1+2r)\,(-3+2r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-3) + 4a_{k-1}(2k-1+2r)\right)x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(1+2r)\,(-3+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{-\frac{1}{2}, \frac{3}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $4\left(k+r+\frac{1}{2}\right)\left(k+r-\frac{3}{2}\right)a_k + 8\left(k+r-\frac{1}{2}\right)a_{k-1} = 0$
- Shift index using $k- >k + 1$
  $4\left(k+\frac{3}{2}+r\right)\left(k+r-\frac{1}{2}\right)a_{k+1} + 8\left(k+r+\frac{1}{2}\right)a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = -\frac{4(2k+2r+1)a_k}{(2k+3+2r)(2k-1+2r)}$
- Recursion relation for $r = -\frac{1}{2}$

$a_{k+1} = -\frac{8ka_k}{(2k+2)(2k-2)}$

- Series not valid for $r = -\frac{1}{2}$, division by 0 in the recursion relation at $k = 1$

$a_{k+1} = -\frac{8ka_k}{(2k+2)(2k-2)}$

- Recursion relation for $r = \frac{3}{2}$

$a_{k+1} = -\frac{4(2k+4)a_k}{(2k+6)(2k+2)}$

- Solution for $r = \frac{3}{2}$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1} = -\frac{4(2k+4)a_k}{(2k+6)(2k+2)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.024 (sec)
Leaf size : 24

```
dsolve(x^2*diff(diff(y(x),x),x)+2*diff(y(x),x)*x^2+(x-3/4)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{2c_2 \mathrm{e}^{-2x} x + c_2 \mathrm{e}^{-2x} + c_1}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.208 (sec)
Leaf size : 33

```
DSolve[{x^2*D[y[x],{x,2}]+2*x^2*D[y[x],x]+(x-3/4)*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to \frac{c_2 \int_1^x e^{-2K[1]} K[1] dK[1] + c_1}{\sqrt{x}}$$

2.1.665 Problem 682

Internal problem ID [9837]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 682
Date solved : `Monday, January 27, 2025 at 06:14:53 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+x)\,y''+x^2y'-2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.193 (sec)

Writing the ode as

$$x^2(1+x)\,y''+x^2y'-2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(1+x) \\ B &= x^2 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-x^2+8x+8}{4\left(x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2+8x+8 \\ t &= 4\left(x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-x^2+8x+8}{4\left(x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1270: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{1+x} - \frac{1}{4\,(1+x)^2} + \frac{2}{x^2} - \frac{2}{x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 + 8x + 8}{4\left(x^2 + x\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 + 8x + 8}{4\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2 + 2x} - \frac{1}{x} + (-)\,(0) \\
&= \frac{1}{2 + 2x} - \frac{1}{x} \\
&= -\frac{x + 2}{2x\,(1 + x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2+2x} - \frac{1}{x}\right)(1) + \left(\left(-\frac{1}{2\,(1+x)^2} + \frac{1}{x^2}\right) + \left(\frac{1}{2+2x} - \frac{1}{x}\right)^2 - \left(\frac{-x^2+8x+8}{4\,(x^2+x)^2}\right)\right) = 0$$

$$\frac{-2+a_0}{x\,(1+x)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (x+2)\, e^{\int \left(\frac{1}{2+2x} - \frac{1}{x}\right) dx} \\ &= (x+2)\, e^{-\ln(x) + \frac{\ln(1+x)}{2}} \\ &= \frac{(x+2)\sqrt{1+x}}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^2}{x^2(1+x)}\, dx} \\ &= z_1 e^{-\frac{\ln(1+x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{1+x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x+2}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2}{x^2(1+x)}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\ln(1+x)}}{(y_1)^2}\, dx \\ &= y_1 \left(\ln(1+x) + \frac{4}{x+2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x+2}{x}\right) + c_2\left(\frac{x+2}{x}\left(\ln(1+x) + \frac{4}{x+2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right) + x^2\left(\frac{d}{dx}y(x)\right) - 2y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{2y(x)}{(x+1)x^2} - \frac{\frac{d}{dx}y(x)}{x+1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{\frac{d}{dx}y(x)}{x+1} - \frac{2y(x)}{(x+1)x^2} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x+1}, P_3(x) = -\frac{2}{(x+1)x^2}\right]$

  - $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((x+1) \cdot P_2(x)\right)\Big|_{x=-1} = 1$

  - $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $\left((x+1)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$

  - $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right) + x^2\left(\frac{d}{dx}y(x)\right) - 2y(x) = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(u^3 - 2u^2 + u)\left(\frac{d^2}{du^2}y(u)\right) + (u^2 - 2u + 1)\left(\frac{d}{du}y(u)\right) - 2y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 u^{-1+r} + \left(a_1(1+r)^2 - 2a_0(r^2+1)\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)^2 - 2a_k(k^2+2kr+r^2+1) + a_k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 0$
- Each term must be 0
  $a_1(1+r)^2 - 2a_0(r^2+1) = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1)^2 - 2a_k(k^2+1) + a_{k-1}(k-1)^2 = 0$
- Shift index using $k->k+1$
  $a_{k+2}(k+2)^2 - 2a_{k+1}\left((k+1)^2+1\right) + k^2 a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 4k a_{k+1} - 4a_{k+1}}{(k+2)^2}$
- Recursion relation for $r = 0$
  $a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 4k a_{k+1} - 4a_{k+1}}{(k+2)^2}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 4k a_{k+1} - 4a_{k+1}}{(k+2)^2}, a_1 - 2a_0 = 0\right]$
- Revert the change of variables $u = x+1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x+1)^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 4k a_{k+1} - 4a_{k+1}}{(k+2)^2}, a_1 - 2a_0 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 27

```
dsolve(x^2*(x+1)*diff(diff(y(x),x),x)+diff(y(x),x)*x^2-2*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_2(x+2)\ln(x+1) + c_1 x + 2c_1 + 4c_2}{x}$$

Mathematica DSolve solution

Solving time : 0.451 (sec)
Leaf size : 87

```
DSolve[{x^2*(1+x)*D[y[x],{x,2}]+x^2*D[y[x],x]-2*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to \frac{(x+2)\exp\left(\int_1^x \left(\frac{1}{2K[1]+2} - \frac{1}{K[1]}\right) dK[1]\right)\left(c_2 \int_1^x \frac{\exp\left(-2\int_1^{K[2]}\left(\frac{1}{2K[1]+2} - \frac{1}{K[1]}\right) dK[1]\right)}{(K[2]+2)^2} dK[2] + c_1\right)}{\sqrt{x+1}}$$

2.1.666 Problem 683

Internal problem ID [9838]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 683
Date solved : `Monday, January 27, 2025 at 06:14:53 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x(x^2+6)\,y' + 6y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.323 (sec)

Writing the ode as

$$x^2y'' + (x^3+6x)\,y' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^3+6x \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2+14}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2+14 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} + \frac{7}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1272: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + \frac{7}{2x} - \frac{49}{4x^3} + \frac{343}{4x^5} - \frac{12005}{16x^7} + \frac{117649}{16x^9} - \frac{2470629}{32x^{11}} + \frac{27176919}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 + 14}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} + \frac{7}{2}\right) + (0) \\ &= \frac{x^2}{4} + \frac{7}{2}\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{7}{2}$. Now b can be found.

$$\begin{aligned}b &= \left(\frac{7}{2}\right) - (0) \\ &= \frac{7}{2}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{7}{2}}{\frac{1}{2}} - 1\right) &= 3 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{7}{2}}{\frac{1}{2}} - 1\right) &= -4\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} + \frac{7}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}$	3	-4

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 3$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 3 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{x}{2}\right) \\ &= \frac{x}{2} \\ &= \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 3$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (6x + 2a_2) + 2\left(\frac{x}{2}\right)(3x^2 + 2xa_2 + a_1) + \left(\left(\frac{1}{2}\right) + \left(\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} + \frac{7}{2}\right)\right) &= 0 \\ -a_2x^2 + (-2a_1 + 6)\,x - 3a_0 + 2a_2 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0, a_1 = 3, a_2 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^3 + 3x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= (x^3 + 3x)\, e^{\int \frac{x}{2} dx} \\ &= (x^3 + 3x)\, e^{\frac{x^2}{4}} \\ &= x(x^2 + 3)\, e^{\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^3+6x}{x^2}\, dx} \\ &= z_1 e^{-\frac{x^2}{4} - 3\ln(x)} \\ &= z_1\left(\frac{e^{-\frac{x^2}{4}}}{x^3}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2+3}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^3+6x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}-6\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{e^{-\frac{x^2}{2}-6\ln(x)} x^4}{(x^2+3)^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^2+3}{x^2}\right) + c_2 \left(\frac{x^2+3}{x^2} \left(\int \frac{e^{-\frac{x^2}{2}-6\ln(x)} x^4}{(x^2+3)^2} dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2 \left(\frac{d^2}{dx^2} y(x)\right) + x(x^2+6) \left(\frac{d}{dx} y(x)\right) + 6y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2} y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2} y(x) = -\frac{6y(x)}{x^2} - \frac{(x^2+6)\left(\frac{d}{dx} y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2} y(x) + \frac{(x^2+6)\left(\frac{d}{dx} y(x)\right)}{x} + \frac{6y(x)}{x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{x^2+6}{x}, P_3(x) = \frac{6}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 6$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 6$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators
  $x^2\left(\frac{d^2}{dx^2}y(x)\right)+x(x^2+6)\left(\frac{d}{dx}y(x)\right)+6y(x)=0$
- Assume series solution for $y(x)$
  $y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions
  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$
    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k->k+1-m$
    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion
    $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions
  $a_0(3+r)\,(2+r)\,x^r+a_1(4+r)\,(3+r)\,x^{1+r}+\left(\sum_{k=2}^{\infty}(a_k(k+r+3)\,(k+r+2)+a_{k-2}(k-2+r)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(3+r)\,(2+r)=0$
- Values of r that satisfy the indicial equation
  $r\in\{-3,-2\}$
- Each term must be 0
  $a_1(4+r)\,(3+r)=0$
- Solve for the dependent coefficient(s)
  $a_1=0$
- Each term in the series must be 0, giving the recursion relation
  $a_k(k+r+3)\,(k+r+2)+a_{k-2}(k-2+r)=0$
- Shift index using $k->k+2$
  $a_{k+2}(k+5+r)\,(k+4+r)+a_k(k+r)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+2}=-\frac{a_k(k+r)}{(k+5+r)(k+4+r)}$
- Recursion relation for $r=-3$
  $a_{k+2}=-\frac{a_k(k-3)}{(k+2)(k+1)}$
- Solution for $r=-3$
  $\left[y(x)=\sum_{k=0}^{\infty}a_k x^{k-3},a_{k+2}=-\frac{a_k(k-3)}{(k+2)(k+1)},a_1=0\right]$
- Recursion relation for $r=-2$; series terminates at $k=2$
  $a_{k+2}=-\frac{a_k(k-2)}{(k+3)(k+2)}$
- Solution for $r=-2$
  $\left[y(x)=\sum_{k=0}^{\infty}a_k x^{k-2},a_{k+2}=-\frac{a_k(k-2)}{(k+3)(k+2)},a_1=0\right]$
- Combine solutions and rename parameters
  $\left[y(x)=\left(\sum_{k=0}^{\infty}a_k x^{k-3}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k-2}\right),a_{k+2}=-\frac{a_k(k-3)}{(k+2)(k+1)},a_1=0,b_{k+2}=-\frac{b_k(k-2)}{(k+3)(k+2)},b_1=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solut
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returnin
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.010 (sec)
Leaf size : 35

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(x^2+6)*diff(y(x),x)+6*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_2 \mathrm{e}^{-\frac{x^2}{2}} \operatorname{hypergeom}\left(\left[2\right], \left[\frac{1}{2}\right], \frac{x^2}{2}\right) + c_1 (x^2+3)\, x}{x^3}$$

Mathematica DSolve solution

Solving time : 0.248 (sec)
Leaf size : 50

```
DSolve[{x^2*D[y[x],{x,2}]+x*(6+x^2)*D[y[x],x]+6*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to \frac{(x^2+3)\left(c_2 \int_1^x \frac{e^{-\frac{1}{2}K[1]^2}}{K[1]^2 (K[1]^2+3)^2} dK[1] + c_1\right)}{x^2}$$

2.1.667 Problem 684

Internal problem ID [9839]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 684
Date solved : `Monday, January 27, 2025 at 06:14:54 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x(1-x)\,y' - y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.209 (sec)

Writing the ode as

$$x^2y'' + \left(-x^2+x\right)y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2+x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2-2x+3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2-2x+3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2-2x+3}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1274: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4x^2} - \frac{1}{2x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{2x^2} + \frac{1}{2x^3} + \frac{1}{4x^4} - \frac{1}{4x^5} - \frac{3}{4x^6} - \frac{3}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 2x + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 3}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-2x + 3}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 2x + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2} - \frac{1}{2x} \\ &= \frac{x-1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2} - \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(\frac{1}{2} - \frac{1}{2x}\right)^2 - \left(\frac{x^2-2x+3}{4x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{1}{2} - \frac{1}{2x}\right) dx} \\ &= \frac{e^{\frac{x}{2}}}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2+x}{x^2}\,dx} \\
&= z_1 e^{\frac{x}{2}-\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{e^{\frac{x}{2}}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{e^x}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(-(1+x)\,x\,e^{x-\ln(x)}e^{-2x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{e^x}{x}\right) + c_2\left(\frac{e^x}{x}\left(-(1+x)\,x\,e^{x-\ln(x)}e^{-2x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(1-x)\left(\frac{d}{dx}y(x)\right) - y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{y(x)}{x^2} + \frac{\left(\frac{d}{dx}y(x)\right)(x-1)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lir

  $\frac{d^2}{dx^2}y(x) - \frac{\left(\frac{d}{dx}y(x)\right)(x-1)}{x} - \frac{y(x)}{x^2} = 0$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{x-1}{x}, P_3(x) = -\frac{1}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

$$(x \cdot P_2(x))\bigg|_{x=0} = 1$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$(x^2 \cdot P_3(x))\bigg|_{x=0} = -1$$

- $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$$x^2\left(\frac{d^2}{dx^2}y(x)\right) - x(x-1)\left(\frac{d}{dx}y(x)\right) - y(x) = 0$$

- Assume series solution for $y(x)$

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- Rewrite ODE with series expansions
  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

$$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

  - Shift index using $k- >k+1-m$

$$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

  - Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

$$x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-1) - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(1+r)\,(-1+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{-1, 1\}$

- Each term in the series must be 0, giving the recursion relation

$(k+r-1)\,(a_k(k+r+1) - a_{k-1}) = 0$

- Shift index using $k- >k+1$

$(k+r)\,(a_{k+1}(k+2+r) - a_k) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = \frac{a_k}{k+2+r}$

- Recursion relation for $r = -1$

$a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = -1$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 1$

$a_{k+1} = \frac{a_k}{k+3}$

- Solution for $r = 1$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(1-x)*diff(y(x),x)-y(x) = 0,y(x),singsol=all)
```

$$y = \frac{e^{x} c_2 + c_1 x + c_1}{x}$$

Mathematica DSolve solution

Solving time : 0.254 (sec)
Leaf size : 80

```
DSolve[{x^2*D[y[x],{x,2}]+x*(1-x)*D[y[x],x]-y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tru
```

$$y(x) \to \exp\left(\int_1^x \left(1 - \frac{1}{K[1]}\right) dK[1]\right) \left(\int_1^x \exp\left(-\int_1^{K[2]} \left(1 - \frac{1}{K[1]}\right) dK[1]\right) c_1 dK[2] + c_2\right)$$

$$y(x) \to c_2 \exp\left(\int_1^x \left(1 - \frac{1}{K[1]}\right) dK[1]\right)$$

2.1.668 Problem 685

Internal problem ID [9840]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 685
Date solved : `Monday, January 27, 2025 at 06:14:54 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x(x+3)\,y' + 4y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.262 (sec)

Writing the ode as

$$x^2y'' + \left(-x^2 - 3x\right)y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 - 3x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 6x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 6x - 1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 6x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1276: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{4x^2} + \frac{3}{2x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{3}{2x} - \frac{5}{2x^2} + \frac{15}{2x^3} - \frac{115}{4x^4} + \frac{495}{4x^5} - \frac{2285}{4x^6} + \frac{11055}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 + 6x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{6x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{6x - 1}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 6. Dividing this by leading coefficient in t which is 4 gives $\frac{3}{2}$. Now b can be found.

$$\begin{aligned}b &= \left(\frac{3}{2}\right) - (0) \\ &= \frac{3}{2}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{3}{2}}{\frac{1}{2}} - 0\right) &= \frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{3}{2}}{\frac{1}{2}} - 0\right) &= -\frac{3}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 6x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{3}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+\right) \\ &= \frac{3}{2} - \left(\frac{1}{2}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2x} + \frac{1}{2} \\ &= \frac{1+x}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{1}{2}\right)(1) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} + \frac{1}{2}\right)^2 - \left(\frac{x^2+6x-1}{4x^2}\right) \right) = 0$$
$$\frac{1-a_0}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 1 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (1+x)\, e^{\int \left(\frac{1}{2x}+\frac{1}{2}\right) dx} \\
&= (1+x)\, e^{\frac{x}{2}+\frac{\ln(x)}{2}} \\
&= (1+x)\sqrt{x}\, e^{\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2-3x}{x^2}\, dx} \\
&= z_1 e^{\frac{x}{2}+\frac{3\ln(x)}{2}} \\
&= z_1 \left(x^{3/2} e^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^2 e^x (1+x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-3x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x+3\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\operatorname{Ei}_1(x) - \frac{e^{-x}}{-1-x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^2 e^x (1+x)\right) + c_2\left(x^2 e^x (1+x)\left(-\operatorname{Ei}_1(x) - \frac{e^{-x}}{-1-x}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) - x(x+3)\left(\frac{d}{dx}y(x)\right) + 4y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{4y(x)}{x^2} + \frac{(x+3)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x) - \frac{(x+3)\left(\frac{d}{dx}y(x)\right)}{x} + \frac{4y(x)}{x^2} = 0$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - ◦ Define functions

    $\left[P_2(x) = -\frac{x+3}{x}, P_3(x) = \frac{4}{x^2}\right]$
  - ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -3$
  - ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 4$
  - ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) - x(x+3)\left(\frac{d}{dx}y(x)\right) + 4y(x) = 0$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- □ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
  - ◦ Shift index using $k->k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - ◦ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r}$

    Rewrite ODE with series expansions

    $a_0(-2+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-2)^2 - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 2$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-2)^2 - a_{k-1}(k+r-1) = 0$
- Shift index using $k->k+1$

  $a_{k+1}(k+r-1)^2 - a_k(k+r) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r)}{(k+r-1)^2}$
- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k(k+2)}{(k+1)^2}$
- Solution for $r = 2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k(k+2)}{(k+1)^2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 29

```
dsolve(x^2*diff(diff(y(x),x),x)-x*(x+3)*diff(y(x),x)+4*y(x) = 0,y(x),singsol=all)
```

$$y = x^2(\mathrm{e}^x c_2(x+1)\,\mathrm{Ei}_1(x) + (x+1)\,c_1\mathrm{e}^x - c_2)$$

Mathematica DSolve solution

Solving time : 0.225 (sec)
Leaf size : 49

```
DSolve[{x^2*D[y[x],{x,2}]-x*(x+3)*D[y[x],x]+4*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True
```

$$y(x) \to e^{x+2}x^2(x+1)\left(c_2\int_1^x \frac{e^{-K[1]-1}}{K[1](K[1]+1)^2}dK[1] + c_1\right)$$

2.1.669 Problem 686

Internal problem ID [9841]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 686
Date solved : `Monday, January 27, 2025 at 06:14:55 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x^2y' - 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.220 (sec)

Writing the ode as

$$x^2y'' - x^2y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1278: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{2}{x^2} - \frac{4}{x^4} + \frac{16}{x^6} - \frac{80}{x^8} + \frac{448}{x^{10}} - \frac{2688}{x^{12}} + \frac{16896}{x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{2}{x^2}\right) \\ &= \frac{1}{4} + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 4 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{\frac{1}{2}} - 0\right) &= 0 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{\frac{1}{2}} - 0\right) &= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + (-)\left(\frac{1}{2}\right) \\ &= -\frac{1}{x} - \frac{1}{2} \\ &= -\frac{2 + x}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x} - \frac{1}{2}\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - \frac{1}{2}\right)^2 - \left(\frac{x^2 + 8}{4x^2}\right)\right) = 0$$

$$\frac{-2 + a_0}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (2+x)\, e^{\int \left(-\frac{1}{x}-\frac{1}{2}\right) dx} \\ &= (2+x)\, e^{-\frac{x}{2}-\ln(x)} \\ &= \frac{(2+x)\, e^{-\frac{x}{2}}}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x^2}{x^2} \, dx} \\ &= z_1 e^{\frac{x}{2}} \\ &= z_1 \left(e^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{2+x}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^x}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{(-2+x)\, e^x}{2+x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{2+x}{x}\right) + c_2 \left(\frac{2+x}{x} \left(\frac{(-2+x)\, e^x}{2+x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2 \left(\frac{d^2}{dx^2} y(x)\right) - x^2 \left(\frac{d}{dx} y(x)\right) - 2y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2} y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2} y(x) = \frac{2y(x)}{x^2} + \frac{d}{dx} y(x)$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x) - \frac{d}{dx}y(x) - \frac{2y(x)}{x^2} = 0$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - ◦ Define functions

    $\left[P_2(x) = -1, P_3(x) = -\frac{2}{x^2}\right]$
  - ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 0$
  - ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -2$
  - ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) - x^2\left(\frac{d}{dx}y(x)\right) - 2y(x) = 0$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- □ Rewrite ODE with series expansions
  - ◦ Convert $x^2 \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r+1}$
  - ◦ Shift index using $k- >k-1$

    $x^2 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, x^{k+r}$
  - ◦ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\, x^{k+r}$

    Rewrite ODE with series expansions

    $a_0(1+r)\,(-2+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-2) - a_{k-1}(k-1+r)\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-1, 2\}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)\,(k+r-2) - a_{k-1}(k-1+r) = 0$
- Shift index using $k- >k+1$

  $a_{k+1}(k+2+r)\,(k-1+r) - a_k(k+r) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r)}{(k+2+r)(k-1+r)}$
- Recursion relation for $r = -1$; series terminates at $k = 1$

  $a_{k+1} = \frac{a_k(k-1)}{(k+1)(k-2)}$
- Apply recursion relation for $k = 0$

  $a_1 = \frac{a_0}{2}$
- Terminating series solution of the ODE for $r = -1$. Use reduction of order to find the second lin

  $y(x) = a_0 \cdot \left(1 + \frac{x}{2}\right)$
- Recursion relation for $r = 2$

$a_{k+1} = \frac{a_k(k+2)}{(k+4)(k+1)}$

- Solution for $r = 2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k(k+2)}{(k+4)(k+1)}\right]$

- Combine solutions and rename parameters

  $\left[y(x) = a_0 \cdot \left(1 + \frac{x}{2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), b_{k+1} = \frac{b_k(k+2)}{(4+k)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.022 (sec)
Leaf size : 21

```
dsolve(x^2*diff(diff(y(x),x),x)-diff(y(x),x)*x^2-2*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_2(x-2)\,\mathrm{e}^{x} + c_1(x+2)}{x}$$

Mathematica DSolve solution

Solving time : 0.075 (sec)
Leaf size : 74

```
DSolve[{x^2*D[y[x],{x,2}]-x^2*D[y[x],x]-2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{2e^{\frac{x+1}{2}}\left((c_1x + 2ic_2)\cosh\left(\frac{x}{2}\right) - (ic_2x + 2c_1)\sinh\left(\frac{x}{2}\right)\right)}{\sqrt{\pi}\sqrt{-ix}\sqrt{x}}$$

2.1.670 Problem 687

Internal problem ID [9842]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 687
Date solved : `Monday, January 27, 2025 at 06:14:56 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x^2y' - (3x+2)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.262 (sec)

Writing the ode as

$$x^2y'' - x^2y' + (-3x-2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 \\ C &= -3x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2+12x+8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2+12x+8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2+12x+8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1280: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{x} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{3}{x} - \frac{7}{x^2} + \frac{42}{x^3} - \frac{301}{x^4} + \frac{2394}{x^5} - \frac{20342}{x^6} + \frac{180852}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 12x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{12x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{12x + 8}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 12. Dividing this by leading coefficient in t which is 4 gives 3. Now b can be found.

$$\begin{aligned} b &= (3) - (0) \\ &= 3 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{3}{\frac{1}{2}} - 0\right) &= 3 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{3}{\frac{1}{2}} - 0\right) &= -3 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 12x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	3	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 3$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^+) \\ &= 3 - (2) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{2}{x} + \left(\frac{1}{2}\right) \\ &= \frac{2}{x} + \frac{1}{2} \\ &= \frac{4+x}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{2}{x} + \frac{1}{2}\right)(1) + \left(\left(-\frac{2}{x^2}\right) + \left(\frac{2}{x} + \frac{1}{2}\right)^2 - \left(\frac{x^2+12x+8}{4x^2}\right)\right) = 0$$
$$\frac{4-a_0}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 4\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 4 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (4+x)\,\mathrm{e}^{\int \left(\frac{2}{x}+\frac{1}{2}\right)dx} \\ &= (4+x)\,\mathrm{e}^{\frac{x}{2}+2\ln(x)} \\ &= (4+x)\,x^2\mathrm{e}^{\frac{x}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x^2}{x^2}\,dx} \\ &= z_1 e^{\frac{x}{2}} \\ &= z_1\left(\mathrm{e}^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}(4+x)\,x^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{x}}{(y_1)^2}\,dx \\ &= y_1\left(-\frac{\mathrm{e}^{-x}(x^3+3x^2-2x+2)}{24\,(4+x)\,x^3}+\frac{\mathrm{Ei}_1(x)}{24}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{x}(4+x)\,x^2\right) + c_2\left(\mathrm{e}^{x}(4+x)\,x^2\left(-\frac{\mathrm{e}^{-x}(x^3+3x^2-2x+2)}{24\,(4+x)\,x^3}+\frac{\mathrm{Ei}_1(x)}{24}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) - x^2\left(\frac{d}{dx}y(x)\right) - (3x+2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{(3x+2)y(x)}{x^2} + \frac{d}{dx}y(x)$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x) - \frac{d}{dx}y(x) - \frac{(3x+2)y(x)}{x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = -1, P_3(x) = -\frac{3x+2}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 0$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -2$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) - x^2\left(\frac{d}{dx}y(x)\right) + (-3x-2)\,y(x) = 0$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^2 \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r+1}$
  - Shift index using $k->k-1$

    $x^2 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\,x^{k+r}$
  - Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-2+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-2) - a_{k-1}(k+2+r)\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-1, 2\}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)\,(k+r-2) - a_{k-1}(k+2+r) = 0$
- Shift index using $k->k+1$

  $a_{k+1}(k+2+r)\,(k-1+r) - a_k(k+r+3) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r+3)}{(k+2+r)(k-1+r)}$
- Recursion relation for $r = -1$

$a_{k+1} = \frac{a_k(k+2)}{(k+1)(k-2)}$

- Series not valid for $r = -1$, division by 0 in the recursion relation at $k = 2$

  $a_{k+1} = \frac{a_k(k+2)}{(k+1)(k-2)}$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k(k+5)}{(k+4)(k+1)}$

- Solution for $r = 2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k(k+5)}{(k+4)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 48

```
dsolve(x^2*diff(diff(y(x),x),x)-diff(y(x),x)*x^2-(2+3*x)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{-c_2 e^x x^3 (x+4) \operatorname{Ei}_1(x) + c_1 x^3 (x+4) e^x + c_2 (x^3 + 3x^2 - 2x + 2)}{x}$$

Mathematica DSolve solution

Solving time : 0.192 (sec)
Leaf size : 45

```
DSolve[{x^2*D[y[x],{x,2}]-x^2*D[y[x],x]-(3*x+2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to e^x x^2 (x+4) \left(c_2 \int_1^x \frac{e^{-K[1]}}{K[1]^4 (K[1]+4)^2} dK[1] + c_1\right)$$

2.1.671 Problem 688

Internal problem ID [9843]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 688
Date solved : `Monday, January 27, 2025 at 06:14:56 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x(5-x)\,y' + 4y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.338 (sec)

Writing the ode as

$$x^2y'' + \left(-x^2+5x\right)y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2+5x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2-10x-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2-10x-1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2-10x-1}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1282: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{5}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{5}{2x} - \frac{13}{2x^2} - \frac{65}{2x^3} - \frac{819}{4x^4} - \frac{5785}{4x^5} - \frac{43797}{4x^6} - \frac{347425}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 10x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-10x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-10x - 1}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -10. Dividing this by leading coefficient in t which is 4 gives $-\frac{5}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{5}{2}\right) - (0) \\ &= -\frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{5}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 0\right) &&= \frac{5}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 10x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{5}{2}$	$\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{5}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\
&= \frac{5}{2} - \left(\frac{1}{2}\right) \\
&= 2
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{1}{2x} - \frac{1}{2} \\
&= -\frac{-1+x}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(2) + 2\left(\frac{1}{2x} - \frac{1}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^2 - 10x - 1}{4x^2}\right) \right) &= 0 \\
\frac{(a_1 + 4)\, x + 2a_0 + a_1}{x} &= 0
\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2, a_1 = -4\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 4x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 4x + 2\right) e^{\int \left(\frac{1}{2x} - \frac{1}{2}\right) dx} \\
&= \left(x^2 - 4x + 2\right) e^{-\frac{x}{2} + \frac{\ln(x)}{2}} \\
&= \left(x^2 - 4x + 2\right) \sqrt{x}\, e^{-\frac{x}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x^2+5x}{x^2} \, dx} \\
&= z_1 e^{\frac{x}{2} - \frac{5\ln(x)}{2}} \\
&= z_1 \left(\frac{e^{\frac{x}{2}}}{x^{5/2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^2 - 4x + 2}{x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+5x}{x^2} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{x - 5\ln(x)}}{\left(y_1\right)^2} \, dx \\
&= y_1 \left(-\frac{e^x (x-3)}{4\left(x^2 - 4x + 2\right)} - \frac{\operatorname{Ei}_1(-x)}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^2 - 4x + 2}{x^2}\right) + c_2 \left(\frac{x^2 - 4x + 2}{x^2} \left(-\frac{e^x (x-3)}{4\left(x^2 - 4x + 2\right)} - \frac{\operatorname{Ei}_1(-x)}{4}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right)+x(5-x)\left(\frac{d}{dx}y(x)\right)+4y(x)=0$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
$\frac{d^2}{dx^2}y(x)=-\frac{4y(x)}{x^2}+\frac{(x-5)\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
$\frac{d^2}{dx^2}y(x)-\frac{(x-5)\left(\frac{d}{dx}y(x)\right)}{x}+\frac{4y(x)}{x^2}=0$

☐ Check to see if $x_0=0$ is a regular singular point
  ◦ Define functions
  $\left[P_2(x)=-\frac{x-5}{x},P_3(x)=\frac{4}{x^2}\right]$
  ◦ $x\cdot P_2(x)$ is analytic at $x=0$
  $\left.(x\cdot P_2(x))\right|_{x=0}=5$
  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$
  $\left.(x^2\cdot P_3(x))\right|_{x=0}=4$
  ◦ $x=0$is a regular singular point
  Check to see if $x_0=0$ is a regular singular point
  $x_0=0$

- Multiply by denominators
$x^2\left(\frac{d^2}{dx^2}y(x)\right)-x(x-5)\left(\frac{d}{dx}y(x)\right)+4y(x)=0$
- Assume series solution for $y(x)$
$y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

☐ Rewrite ODE with series expansions
  ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$
  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$
  ◦ Shift index using $k->k+1-m$
  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  ◦ Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion
  $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$
  Rewrite ODE with series expansions
  $a_0(2+r)^2\,x^r+\left(\sum_{k=1}^{\infty}\left(a_k(k+r+2)^2-a_{k-1}(k+r-1)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation
$(2+r)^2=0$
- Values of r that satisfy the indicial equation
$r=-2$
- Each term in the series must be 0, giving the recursion relation
$a_k(k+r+2)^2-a_{k-1}(k+r-1)=0$
- Shift index using $k->k+1$

$a_{k+1}(k+3+r)^2 - a_k(k+r) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r)}{(k+3+r)^2}$

- Recursion relation for $r = -2$; series terminates at $k = 2$

  $a_{k+1} = \frac{a_k(k-2)}{(k+1)^2}$

- Apply recursion relation for $k = 0$

  $a_1 = -2a_0$

- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{4}$

- Express in terms of a_0

  $a_2 = \frac{a_0}{2}$

- Terminating series solution of the ODE for $r = -2$. Use reduction of order to find the second

  $y(x) = a_0 \cdot \left(1 - 2x + \frac{1}{2}x^2\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 41

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(-x+5)*diff(y(x),x)+4*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{(x^2 - 4x + 2)\, c_2 \operatorname{Ei}_1(-x) + c_2(x-3)\, e^x + c_1(x^2 - 4x + 2)}{x^2}$$

Mathematica DSolve solution

Solving time : 0.247 (sec)
Leaf size : 51

```
DSolve[{x^2*D[y[x],{x,2}]+x*(5-x)*D[y[x],x]+4*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to \frac{(x^2 - 4x + 2)\left(c_2 \int_1^x \frac{e^{K[1]}}{K[1](K[1]^2 - 4K[1] + 2)^2} dK[1] + c_1\right)}{x^2}$$

2.1.672 Problem 689

Internal problem ID [9844]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 689
Date solved : `Monday, January 27, 2025 at 06:14:57 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' + 4x(1-x)\,y' + (2x-9)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.223 (sec)

Writing the ode as

$$4x^2y'' + \left(-4x^2+4x\right)y' + (2x-9)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x^2+4x \\ C &= 2x-9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2-4x+8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2-4x+8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2-4x+8}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1284: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{x} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{1}{x^2} + \frac{2}{x^3} + \frac{3}{x^4} + \frac{2}{x^5} - \frac{6}{x^6} - \frac{28}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-4x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-4x + 8}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned}b &= (-1) - (0) \\ &= -1\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 0\right) &&= -1 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 0\right) &&= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -1$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} - \frac{1}{x} \\
&= \frac{x-2}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2} - \frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(\frac{1}{2} - \frac{1}{x}\right)^2 - \left(\frac{x^2-4x+8}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(\frac{1}{2} - \frac{1}{x}\right) dx} \\
&= \frac{e^{\frac{x}{2}}}{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x^2+4x}{4x^2}\,dx} \\ &= z_1 e^{\frac{x}{2}-\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{e^{\frac{x}{2}}}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^x}{x^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^2+4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{x-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(-\left(x^2+2x+2\right)x\,e^{x-\ln(x)}e^{-2x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{e^x}{x^{3/2}}\right) + c_2\left(\frac{e^x}{x^{3/2}}\left(-\left(x^2+2x+2\right)x\,e^{x-\ln(x)}e^{-2x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$4x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x(1-x)\left(\frac{d}{dx}y(x)\right) + (2x-9)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{(2x-9)y(x)}{4x^2} + \frac{\left(\frac{d}{dx}y(x)\right)(x-1)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) - \frac{\left(\frac{d}{dx}y(x)\right)(x-1)}{x} + \frac{(2x-9)y(x)}{4x^2} = 0$
- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{x-1}{x}, P_3(x) = \frac{2x-9}{4x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

$(x \cdot P_2(x))\Big|_{x=0} = 1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{9}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) - 4x(x-1)\left(\frac{d}{dx}y(x)\right) + (2x-9)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(3+2r)\,(-3+2r)\,x^r + \left(\sum_{k=1}^{\infty} (a_k(2k+2r+3)\,(2k+2r-3) - 2a_{k-1}(2k+2r-3))\,x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(3+2r)\,(-3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{3}{2}, \frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $4\left(\left(k+r+\frac{3}{2}\right)a_k - a_{k-1}\right)\left(k+r-\frac{3}{2}\right) = 0$

- Shift index using $k- >k+1$

  $4\left(\left(k+\frac{5}{2}+r\right)a_{k+1} - a_k\right)\left(k-\frac{1}{2}+r\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k}{2k+5+2r}$

- Recursion relation for $r = -\frac{3}{2}$

  $a_{k+1} = \frac{2a_k}{2k+2}$

- Solution for $r = -\frac{3}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}, a_{k+1} = \frac{2a_k}{2k+2}\right]$

- Recursion relation for $r = \frac{3}{2}$

$a_{k+1} = \frac{2a_k}{2k+8}$

- Solution for $r = \frac{3}{2}$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1} = \frac{2a_k}{2k+8}\right]$$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+1} = \frac{2a_k}{2k+2}, b_{k+1} = \frac{2b_k}{2k+8}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.043 (sec)
Leaf size : 23

```
dsolve(4*x^2*diff(diff(y(x),x),x)+4*x*(1-x)*diff(y(x),x)+(2*x-9)*y(x) = 0,y(x),singsol=a
```

$$y = \frac{e^x c_1 + c_2(x^2 + 2x + 2)}{x^{3/2}}$$

Mathematica DSolve solution

Solving time : 0.231 (sec)
Leaf size : 38

```
DSolve[{4*x^2*D[y[x],{x,2}]+4*x*(1-x)*D[y[x],x]+(2*x-9)*y[x]==0,{}},y[x],x,IncludeSingularSolu
```

$$y(x) \to \frac{e^x \left(c_2 \int_1^x e^{-K[1]} K[1]^2 dK[1] + c_1\right)}{x^{3/2}}$$

2.1.673 Problem 690

Internal problem ID [9845]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 690
Date solved : `Monday, January 27, 2025 at 06:14:58 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + 2x(2 + x)\, y' + 2(1 + x)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.251 (sec)

Writing the ode as

$$x^2y'' + (2x^2 + 4x)\, y' + (2x + 2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 2x^2 + 4x \\ C &= 2x + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2 + x}{x} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 + x \\ t &= x \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2 + x}{x}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1286: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x$. There is a pole at $x = 0$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{2x^3} - \frac{5}{8x^4} + \frac{7}{8x^5} - \frac{21}{16x^6} + \frac{33}{16x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2 + x}{x} \\ &= Q + \frac{R}{x} \\ &= (1) + \left(\frac{2}{x}\right) \\ &= 1 + \frac{2}{x} \end{aligned}$$

Since the degree of t is 1, then we see that the coefficient of the term 1 in the remainder R is 2. Dividing this by leading coefficient in t which is 1 gives 2. Now b can be found.

$$\begin{aligned} b &= (2) - (0) \\ &= 2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{2}{1} - 0\right) &&= 1 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{2}{1} - 0\right) &&= -1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2 + x}{x}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	1	1	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 1$ then

$$\begin{aligned} d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{x} + (1) \\ &= 1 + \frac{1}{x} \\ &= 1 + \frac{1}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(1 + \frac{1}{x}\right)(0) + \left(\left(-\frac{1}{x^2}\right) + \left(1 + \frac{1}{x}\right)^2 - \left(\frac{2+x}{x}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int (1+\frac{1}{x}) dx} \\ &= x\, \mathrm{e}^{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x^2+4x}{x^2}\, dx} \\ &= z_1 e^{-x-2\ln(x)} \\ &= z_1 \left(\frac{\mathrm{e}^{-x}}{x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2+4x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2x-4\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{e^{-2x}}{3x} + \frac{e^{-2x}}{3} - \frac{2x\, e^{-2x}}{3} + \frac{4x^2 \operatorname{Ei}_1(2x)}{3} \right. \\
&\qquad \left. - \frac{4 \operatorname{Ei}_1(2x)\, x^3 - 2\, e^{-2x} x^2 + x\, e^{-2x} - 6x \operatorname{Ei}_1(2x) + 2\, e^{-2x}}{3x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x}\left(-\frac{e^{-2x}}{3x} + \frac{e^{-2x}}{3} - \frac{2x\, e^{-2x}}{3} + \frac{4x^2 \operatorname{Ei}_1(2x)}{3} \right.\right. \\
&\qquad \left.\left. - \frac{4 \operatorname{Ei}_1(2x)\, x^3 - 2\, e^{-2x} x^2 + x\, e^{-2x} - 6x \operatorname{Ei}_1(2x) + 2\, e^{-2x}}{3x}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 27

```
dsolve(x^2*diff(diff(y(x),x),x)+2*x*(x+2)*diff(y(x),x)+2*(x+1)*y(x) = 0,y(x),singsol=a
```

$$y = \frac{-2 \operatorname{Ei}_1(2x)\, c_2 x + c_2 e^{-2x} + c_1 x}{x^2}$$

Mathematica DSolve solution

Solving time : 0.199 (sec)
Leaf size : 33

```
DSolve[{x^2*D[y[x],{x,2}]+2*x*(2+x)*D[y[x],x]+2*(1+x)*y[x]==0,{}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to \frac{c_2 \int_1^x \frac{e^{-2K[1]}}{K[1]^2} dK[1] + c_1}{x}$$

2.1.674 Problem 691

Internal problem ID [9846]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 691
Date solved : `Monday, January 27, 2025 at 06:14:58 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x(1-x)\,y' + (1-x)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.247 (sec)

Writing the ode as

$$x^2y'' + \left(x^2-x\right)y' + (1-x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2 - x \\ C &= 1 - x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 2x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 2x - 1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 2x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1287: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{1}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{2x} - \frac{1}{2x^2} + \frac{1}{2x^3} - \frac{3}{4x^4} + \frac{5}{4x^5} - \frac{9}{4x^6} + \frac{17}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 2x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{2x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{2x - 1}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 2. Dividing this by leading coefficient in t which is 4 gives $\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{2}\right) - (0) \\ &= \frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 2x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2} + \frac{1}{2x} \\ &= \frac{x+1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2} + \frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2} + \frac{1}{2x}\right)^2 - \left(\frac{x^2+2x-1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{1}{2} + \frac{1}{2x}\right) dx} \\ &= \sqrt{x}\, e^{\frac{x}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^2-x}{x^2}\,dx} \\ &= z_1 e^{-\frac{x}{2}+\frac{\ln(x)}{2}} \\ &= z_1\left(\sqrt{x}\,\mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2-x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-x+\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(-\operatorname{Ei}_1(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(x) + c_2(x(-\operatorname{Ei}_1(x))) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right) - x(1-x)\left(\frac{d}{dx}y(x)\right) + (1-x)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
$\frac{d^2}{dx^2}y(x) = \frac{(x-1)y(x)}{x^2} - \frac{\left(\frac{d}{dx}y(x)\right)(x-1)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
$\frac{d^2}{dx^2}y(x) + \frac{\left(\frac{d}{dx}y(x)\right)(x-1)}{x} - \frac{(x-1)y(x)}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
  $\left[P_2(x) = \frac{x-1}{x}, P_3(x) = -\frac{x-1}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
  $\left.(x \cdot P_2(x))\right|_{x=0} = -1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\Bigg|_{x=0} = 1$$

- $x = 0$is a regular singular point
  Check to see if $x_0 = 0$ is a regular singular point
  $x_0 = 0$

- Multiply by denominators
  $x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(x-1)\left(\frac{d}{dx}y(x)\right) + (1-x)\,y(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$
  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- Shift index using $k- >k-m$
  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$
  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
- Shift index using $k- >k+1-m$
  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion
  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions
  $a_0(-1+r)^2\,x^r + \left(\sum_{k=1}^{\infty}\left(a_k(k+r-1)^2 + a_{k-1}(k-2+r)\right)x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 1$
- Each term in the series must be 0, giving the recursion relation
  $a_k(k+r-1)^2 + a_{k-1}(k-2+r) = 0$
- Shift index using $k- >k+1$
  $a_{k+1}(k+r)^2 + a_k(k+r-1) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = -\frac{a_k(k+r-1)}{(k+r)^2}$
- Recursion relation for $r = 1$
  $a_{k+1} = -\frac{a_k k}{(k+1)^2}$
- Solution for $r = 1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{a_k k}{(k+1)^2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 13

```
dsolve(x^2*diff(diff(y(x),x),x)-x*(1-x)*diff(y(x),x)+(1-x)*y(x) = 0,y(x),singsol=all)
```

$$y = x(\mathrm{Ei}_1(x)\, c_2 + c_1)$$

Mathematica DSolve solution

Solving time : 0.44 (sec)
Leaf size : 63

```
DSolve[{x^2*D[y[x],{x,2}]-x*(1-x)*D[y[x],x]+(1-x)*y[x]==0,{}},y[x],x,IncludeSingularSolution
```

$$y(x) \to \sqrt{x}\left(c_2 \int_1^x \frac{e^{-K[2]-1}}{K[2]} dK[2] + c_1\right) \exp\left(\frac{1}{2}\left(-\int_1^x \left(1 - \frac{1}{K[1]}\right) dK[1] + x + 1\right)\right)$$

2.1.675 Problem 692

Internal problem ID [9847]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 692
Date solved : Monday, January 27, 2025 at 06:14:59 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$4x^2y'' + 4x(1+2x)\,y' + (4x-1)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.099 (sec)

Writing the ode as

$$4x^2y'' + \left(8x^2+4x\right)y' + (4x-1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= 8x^2+4x \\ C &= 4x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1289: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{8x^2+4x}{4x^2}\, dx} \\ &= z_1 e^{-x-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{\mathrm{e}^{-x}}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-2x}}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{8x^2+4x}{4x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-2x-\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{e^{-2x-\ln(x)} x\, e^{4x}}{2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{e^{-2x}}{\sqrt{x}} \right) + c_2 \left(\frac{e^{-2x}}{\sqrt{x}} \left(\frac{e^{-2x-\ln(x)} x\, e^{4x}}{2} \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2 \left(\frac{d^2}{dx^2} y(x) \right) + 4x(2x+1) \left(\frac{d}{dx} y(x) \right) + (4x-1)\, y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2} y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2} y(x) = -\frac{(4x-1)y(x)}{4x^2} - \frac{(2x+1)\left(\frac{d}{dx} y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2} y(x) + \frac{(2x+1)\left(\frac{d}{dx} y(x)\right)}{x} + \frac{(4x-1)y(x)}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{2x+1}{x}, P_3(x) = \frac{4x-1}{4x^2} \right]$
- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left. (x \cdot P_2(x)) \right|_{x=0} = 1$
- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left. (x^2 \cdot P_3(x)) \right|_{x=0} = -\frac{1}{4}$
- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2 \left(\frac{d^2}{dx^2} y(x) \right) + 4x(2x+1) \left(\frac{d}{dx} y(x) \right) + (4x-1)\, y(x) = 0$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-1}(2k+2r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r-\frac{1}{2}\right)\left(a_k\left(k+r+\frac{1}{2}\right) + 2a_{k-1}\right) = 0$

- Shift index using $k->k+1$

  $4\left(k+r+\frac{1}{2}\right)\left(a_{k+1}\left(k+\frac{3}{2}+r\right) + 2a_k\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{4a_k}{2k+3+2r}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+1} = -\frac{4a_k}{2k+2}$

- Solution for $r = -\frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = -\frac{4a_k}{2k+2}\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = -\frac{4a_k}{2k+4}$

- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{4a_k}{2k+4}\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = -\frac{4a_k}{2k+2}, b_{k+1} = -\frac{4b_k}{2k+4}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.022 (sec)
Leaf size : 16

```
dsolve(4*x^2*diff(diff(y(x),x),x)+4*x*(2*x+1)*diff(y(x),x)+(-1+4*x)*y(x) = 0,y(x),singso
```

$$y = \frac{c_1 + c_2 \mathrm{e}^{-2x}}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.039 (sec)
Leaf size : 26

```
DSolve[{4*x^2*D[y[x],{x,2}]+4*x*(1+2*x)*D[y[x],x]+(4*x-1)*y[x]==0,{}},y[x],x,IncludeSingularSo
```

$$y(x) \to \frac{2c_1 e^{-2x} + c_2}{2\sqrt{x}}$$

2.1.676 Problem 693

Internal problem ID [9848]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 693
Date solved : `Monday, January 27, 2025 at 06:15:00 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x(4+x)\,y' + (2+x)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.250 (sec)

Writing the ode as

$$x^2y'' + \left(x^2+4x\right)y' + (2+x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2+4x \\ C &= 2+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4+x}{4x} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4+x \\ t &= 4x \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4+x}{4x}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1291: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x$. There is a pole at $x = 0$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \frac{5}{x^4} + \frac{14}{x^5} - \frac{42}{x^6} + \frac{132}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4+x}{4x} \\ &= Q + \frac{R}{4x} \\ &= \left(\frac{1}{4}\right) + \left(\frac{1}{x}\right) \\ &= \frac{1}{4} + \frac{1}{x} \end{aligned}$$

Since the degree of t is 1, then we see that the coefficient of the term 1 in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$\begin{aligned} b &= (1) - (0) \\ &= 1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{1}{\frac{1}{2}} - 0\right) &&= 1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{1}{\frac{1}{2}} - 0\right) &&= -1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4+x}{4x}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	1	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} + \frac{1}{x} \\
&= \frac{1}{2} + \frac{1}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2} + \frac{1}{x}\right)(0) + \left(\left(-\frac{1}{x^2}\right) + \left(\frac{1}{2} + \frac{1}{x}\right)^2 - \left(\frac{4 + x}{4x}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{2} + \frac{1}{x}\right) dx} \\
&= x\, e^{\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2+4x}{x^2}\, dx} \\
&= z_1 e^{-\frac{x}{2} - 2\ln(x)} \\
&= z_1 \left(\frac{e^{-\frac{x}{2}}}{x^2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+4x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-x-4\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{e^{-x}}{3x} + \frac{e^{-x}}{6} - \frac{x\, e^{-x}}{6} + \frac{x^2 \operatorname{Ei}_1(x)}{6} - \frac{\operatorname{Ei}_1(x)\, x^3 - e^{-x}x^2 - 6x \operatorname{Ei}_1(x) + x\, e^{-x} + 4\, e^{-x}}{6x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x}\left(-\frac{e^{-x}}{3x} + \frac{e^{-x}}{6} - \frac{x\, e^{-x}}{6} + \frac{x^2 \operatorname{Ei}_1(x)}{6} \right.\right. \\
&\qquad \left.\left. - \frac{\operatorname{Ei}_1(x)\, x^3 - e^{-x}x^2 - 6x \operatorname{Ei}_1(x) + x\, e^{-x} + 4\, e^{-x}}{6x}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 25

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(x+4)*diff(y(x),x)+(x+2)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_2 e^{-x} + x(-\operatorname{Ei}_1(x)\, c_2 + c_1)}{x^2}$$

Mathematica DSolve solution

Solving time : 0.217 (sec)
Leaf size : 36

```
DSolve[{x^2*D[y[x],{x,2}]+x*(4+x)*D[y[x],x]+(2+x)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->
```

$$y(x) \to \frac{c_2 \int_1^x \frac{e^{-K[1]}}{K[1]^2} dK[1] + c_1}{e^2 x}$$

2.1.677 Problem 694

Internal problem ID [9849]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 694
Date solved : Monday, January 27, 2025 at 06:15:00 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{9}{4}\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.274 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{9}{4}\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{9}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2 + 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2+2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1292: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2-2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1,2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -1 + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx i - \frac{i}{x^2} - \frac{i}{2x^4} - \frac{i}{2x^6} - \frac{5i}{8x^8} - \frac{7i}{8x^{10}} - \frac{21i}{16x^{12}} - \frac{33i}{16x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= i \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-x^2 + 2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-1) + \left(\frac{2}{x^2}\right) \\ &= -1 + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= i \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{0}{i} - 0\right) = 0 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{0}{i} - 0\right) = 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 + 2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	i	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + (-)\,(i) \\ &= -\frac{1}{x} - i \\ &= -\frac{1}{x} - i \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x} - i\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - i\right)^2 - \left(\frac{-x^2+2}{x^2}\right) \right) = 0$$

$$\frac{2ia_0 - 2}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -i\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - i$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x - i) \, \mathrm{e}^{\int (-\frac{1}{x} - i) dx} \\ &= (x - i) \, \mathrm{e}^{-\ln(x) - ix} \\ &= \frac{(x - i) \, \mathrm{e}^{-ix}}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x}{x^2} \, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}} \right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x - i) \, \mathrm{e}^{-ix}}{x^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{(ix - 1) \, \mathrm{e}^{2ix}}{-2x + 2i} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{(x - i) \, \mathrm{e}^{-ix}}{x^{3/2}} \right) + c_2 \left(\frac{(x - i) \, \mathrm{e}^{-ix}}{x^{3/2}} \left(\frac{(ix - 1) \, \mathrm{e}^{2ix}}{-2x + 2i} \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right)+x\left(\frac{d}{dx}y(x)\right)+\left(x^2-\frac{9}{4}\right)y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(4x^2-9)y(x)}{4x^2}-\frac{\frac{d}{dx}y(x)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{\frac{d}{dx}y(x)}{x}+\frac{(4x^2-9)y(x)}{4x^2}=0$

- ☐ Check to see if $x_0=0$ is a regular singular point
  - Define functions

    $\left[P_2(x)=\frac{1}{x}, P_3(x)=\frac{4x^2-9}{4x^2}\right]$
  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=1$
  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=-\frac{9}{4}$
  - $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right)+4x\left(\frac{d}{dx}y(x)\right)+(4x^2-9)y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

    $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_k x^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$
  - Convert $x\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r}$
  - Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(3+2r)\,(-3+2r)\,x^r+a_1(5+2r)\,(-1+2r)\,x^{1+r}+\left(\sum_{k=2}^{\infty}(a_k(2k+2r+3)\,(2k+2r-3)+4a_{k-2}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(3+2r)\,(-3+2r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{-\frac{3}{2},\frac{3}{2}\right\}$

- Each term must be 0
  $a_1(5+2r)(-1+2r)=0$
- Solve for the dependent coefficient(s)
  $a_1=0$
- Each term in the series must be 0, giving the recursion relation
  $a_k(4k^2+8kr+4r^2-9)+4a_{k-2}=0$
- Shift index using $k->k+2$
  $a_{k+2}\left(4(k+2)^2+8(k+2)r+4r^2-9\right)+4a_k=0$
- Recursion relation that defines series solution to ODE
  $a_{k+2}=-\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+7}$
- Recursion relation for $r=-\frac{3}{2}$
  $a_{k+2}=-\frac{4a_k}{4k^2+4k-8}$
- Solution for $r=-\frac{3}{2}$
  $\left[y(x)=\sum_{k=0}^{\infty}a_k x^{k-\frac{3}{2}}, a_{k+2}=-\frac{4a_k}{4k^2+4k-8}, a_1=0\right]$
- Recursion relation for $r=\frac{3}{2}$
  $a_{k+2}=-\frac{4a_k}{4k^2+28k+40}$
- Solution for $r=\frac{3}{2}$
  $\left[y(x)=\sum_{k=0}^{\infty}a_k x^{k+\frac{3}{2}}, a_{k+2}=-\frac{4a_k}{4k^2+28k+40}, a_1=0\right]$
- Combine solutions and rename parameters
  $\left[y(x)=\left(\sum_{k=0}^{\infty}a_k x^{k-\frac{3}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{3}{2}}\right), a_{k+2}=-\frac{4a_k}{4k^2+4k-8}, a_1=0, b_{k+2}=-\frac{4b_k}{4k^2+28k+40}, b_1=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.050` (sec)
Leaf size : 30

```
dsolve(x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x+(x^2-9/4)*y(x) = 0,y(x),singsol=all)
```

$$y=\frac{-(-x+i)\,c_2\mathrm{e}^{-ix}+(x+i)\,c_1\mathrm{e}^{ix}}{x^{3/2}}$$

Mathematica DSolve solution

Solving time : 0.059 (sec)
Leaf size : 44

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-9/4)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tr
```

$$y(x) \to -\frac{\sqrt{\frac{2}{\pi}}((c_1x + c_2)\cos(x) + (c_2x - c_1)\sin(x))}{x^{3/2}}$$

2.1.678 Problem 695

Internal problem ID [9850]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 695
Date solved : Monday, January 27, 2025 at 06:15:01 PM
CAS classification : [_Lienard]

Solve

$$xy'' + 2y' + xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.122 (sec)

Writing the ode as

$$xy'' + 2y' + xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1294: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\, dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1\left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\, dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{x}\right) + c_2 \left(\frac{\cos(x)}{x}(\tan(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dx^2}y(x)\right)x + 2\frac{d}{dx}y(x) + xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -y(x) - \frac{2\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{2\left(\frac{d}{dx}y(x)\right)}{x} + y(x) = 0$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = 1\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $\left(\frac{d^2}{dx^2}y(x)\right)x + 2\frac{d}{dx}y(x) + xy(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x \cdot y(x)$ to series expansion

    $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+1}$
  - Shift index using $k- >k-1$

$x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

- ○ Convert $\frac{d}{dx}y(x)$ to series expansion

  $\frac{d}{dx}y(x) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

- ○ Shift index using $k- >k+1$

  $\frac{d}{dx}y(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$

- ○ Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

- ○ Shift index using $k- >k+1$

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $a_1(1+r)\,(2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1} = 0$

- Shift index using $k- >k+1$

  $a_{k+2}(k+2+r)\,(k+3+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{(k+1)(k+2)}$

- Solution for $r = -1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+3)}$

- Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, 0 = 0, b_{k+2} = -\frac{b_k}{(k+3)(k+2)}, 2b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 17

```
dsolve(x*diff(diff(y(x),x),x)+2*diff(y(x),x)+x*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{\sin(x) c_1 + \cos(x) c_2}{x}$$

Mathematica DSolve solution

Solving time : 0.027 (sec)
Leaf size : 37

```
DSolve[{x*D[y[x],{x,2}]+2*D[y[x],x]+x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2c_1 e^{-ix} - ic_2 e^{ix}}{2x}$$

2.1.679 Problem 696

Internal problem ID [9851]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 696
Date solved : `Monday, January 27, 2025 at 06:15:01 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2xy'' + 5(1-2x)\,y' - 5y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.424 (sec)

Writing the ode as

$$2xy'' + (-10x+5)\,y' - 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x \\ B &= -10x+5 \\ C &= -5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{100x^2 - 60x + 5}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 100x^2 - 60x + 5 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{100x^2 - 60x + 5}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1296: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{25}{4} + \frac{5}{16x^2} - \frac{15}{4x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{5}{2} - \frac{3}{4x} - \frac{1}{20x^2} - \frac{3}{200x^3} - \frac{1}{200x^4} - \frac{9}{5000x^5} - \frac{137}{200000x^6} - \frac{543}{2000000x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{5}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{5}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{25}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{100x^2 - 60x + 5}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{25}{4}\right) + \left(\frac{-60x + 5}{16x^2}\right) \\ &= \frac{25}{4} + \frac{-60x + 5}{16x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -60. Dividing this by leading coefficient in t which is 16 gives $-\frac{15}{4}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{15}{4}\right) - (0) \\ &= -\frac{15}{4} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{5}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{15}{4}}{\frac{5}{2}} - 0\right) &&= -\frac{3}{4} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{15}{4}}{\frac{5}{2}} - 0\right) &&= \frac{3}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{100x^2 - 60x + 5}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{5}{2}$	$-\frac{3}{4}$	$\frac{3}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{3}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{3}{4} - \left(-\frac{1}{4}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{4x} + (-)\left(\frac{5}{2}\right) \\ &= -\frac{1}{4x} - \frac{5}{2} \\ &= -\frac{1}{4x} - \frac{5}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{4x} - \frac{5}{2}\right)(1) + \left(\left(\frac{1}{4x^2}\right) + \left(-\frac{1}{4x} - \frac{5}{2}\right)^2 - \left(\frac{100x^2 - 60x + 5}{16x^2}\right) \right) &= 0 \\ \frac{-1 + 10a_0}{2x} &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{ a_0 = \frac{1}{10} \right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{1}{10}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{1}{10}\right) e^{\int \left(-\frac{1}{4x} - \frac{5}{2}\right) dx} \\
&= \left(x + \frac{1}{10}\right) e^{-\frac{5x}{2} - \frac{\ln(x)}{4}} \\
&= \frac{(1 + 10x) e^{-\frac{5x}{2}}}{10x^{1/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-10x+5}{2x} \, dx} \\
&= z_1 e^{\frac{5x}{2} - \frac{5 \ln(x)}{4}} \\
&= z_1 \left(\frac{e^{\frac{5x}{2}}}{x^{5/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1 + 10x}{10x^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-10x+5}{2x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{5x - \frac{5 \ln(x)}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{100 \, e^{5x - \frac{5 \ln(x)}{2}} x^3}{(1 + 10x)^2} dx \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1 + 10x}{10x^{3/2}}\right) + c_2 \left(\frac{1 + 10x}{10x^{3/2}} \left(\int \frac{100 \, e^{5x - \frac{5 \ln(x)}{2}} x^3}{(1 + 10x)^2} dx \right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2\left(\frac{d^2}{dx^2}y(x)\right)x+5(-2x+1)\left(\frac{d}{dx}y(x)\right)-5y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{5y(x)}{2x}+\frac{5(2x-1)\left(\frac{d}{dx}y(x)\right)}{2x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)-\frac{5(2x-1)\left(\frac{d}{dx}y(x)\right)}{2x}-\frac{5y(x)}{2x}=0$

□ Check to see if $x_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(x)=-\frac{5(2x-1)}{2x}, P_3(x)=-\frac{5}{2x}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=\frac{5}{2}$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=0$

  - $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2\left(\frac{d^2}{dx^2}y(x)\right)x+(-10x+5)\left(\frac{d}{dx}y(x)\right)-5y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..1$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-1}$

  - Shift index using $k->k+1$

    $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(3+2r)\,x^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(2k+5+2r)-5a_k(2k+2r+1)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(3+2r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{0,-\frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2(k+1+r)\, a_{k+1}\left(k+r+\frac{5}{2}\right)-10 a_k\left(k+r+\frac{1}{2}\right)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{5 a_k(2k+2r+1)}{(k+1+r)(2k+5+2r)}$
- Recursion relation for $r=0$

  $a_{k+1}=\frac{5 a_k(2k+1)}{(k+1)(2k+5)}$
- Solution for $r=0$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^k, a_{k+1}=\frac{5 a_k(2k+1)}{(k+1)(2k+5)}\right]$
- Recursion relation for $r=-\frac{3}{2}$; series terminates at $k=1$

  $a_{k+1}=\frac{5 a_k(2k-2)}{\left(k-\frac{1}{2}\right)(2k+2)}$
- Apply recursion relation for $k=0$

  $a_1=10 a_0$
- Terminating series solution of the ODE for $r=-\frac{3}{2}$. Use reduction of order to find the second lin

  $y(x)=a_0\cdot(1+10x)$
- Combine solutions and rename parameters

  $\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^k\right)+b_0\cdot(1+10x), a_{k+1}=\frac{5 a_k(2k+1)}{(k+1)(2k+5)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.051 (sec)
Leaf size : 44

```
dsolve(2*x*diff(diff(y(x),x),x)+5*(1-2*x)*diff(y(x),x)-5*y(x) = 0,y(x),singsol=all)
```

$$y=\frac{10\sqrt{5}\, c_1\sqrt{\pi}\left(\frac{1}{10}+x\right)\operatorname{erfi}\left(\sqrt{5}\,\sqrt{x}\right)-10\,\mathrm{e}^{5x} c_1\sqrt{x}+10 c_2\left(\frac{1}{10}+x\right)}{x^{3/2}}$$

Mathematica DSolve solution

Solving time : 0.032 (sec)
Leaf size : 40

```
DSolve[{2*x*D[y[x],{x,2}]+5*(1-2*x)*D[y[x],x]-5*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to c_2 L_{-\frac{1}{2}}^{\frac{3}{2}}(5x) + \frac{c_1(10x+1)}{10\sqrt{5}x^{3/2}}$$

2.1.680 Problem 697

Internal problem ID [9852]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 697
Date solved : `Monday, January 27, 2025 at 06:15:02 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.140 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1298: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\, dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2\left(\frac{\cos(x)}{\sqrt{x}}(\tan(x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x\left(\frac{d}{dx}y(x)\right) + \left(x^2 - \frac{1}{4}\right)y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(4x^2-1)y(x)}{4x^2} - \frac{\frac{d}{dx}y(x)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{\frac{d}{dx}y(x)}{x} + \frac{(4x^2-1)y(x)}{4x^2} = 0$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - ○ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

  - ○ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  - ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$

  - ○ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x\left(\frac{d}{dx}y(x)\right) + (4x^2 - 1)y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$
- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$
- Shift index using $k- >k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$
- Solution for $r = -\frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$
- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.039 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x+(x^2-1/4)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{\sin(x) c_1 + \cos(x) c_2}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.043 (sec)
Leaf size : 39

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-1/4)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to \frac{e^{-ix}(2c_1 - ic_2 e^{2ix})}{2\sqrt{x}}$$

2.1.681 Problem 698

Internal problem ID [9853]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 698
Date solved : `Monday, January 27, 2025 at 06:15:03 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + (x+n)\,y' + (n+1)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.372 (sec)

Writing the ode as

$$xy'' + (x+n)\,y' + (n+1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= x+n \\ C &= n+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{n^2 - 2xn + x^2 - 2n - 4x}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= n^2 - 2xn + x^2 - 2n - 4x \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{n^2 - 2xn + x^2 - 2n - 4x}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1300: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{\frac{1}{4}n^2 - \frac{1}{2}n}{x^2} + \frac{-\frac{n}{2} - 1}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{1}{4}n^2 - \frac{1}{2}n$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{n}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = 1 - \frac{n}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} - \frac{n}{2x} - \frac{3n^6}{2x^7} - \frac{3n^5}{2x^6} - \frac{3n^4}{2x^5} - \frac{3n^3}{2x^4} - \frac{3n^2}{2x^3} - \frac{3n}{2x^2} - \frac{77n^5}{2x^7} - \frac{53n^4}{2x^6} - \frac{67n^3}{4x^5} - \frac{37n^2}{4x^4} - \frac{4n}{x^3} - \frac{1075n^4}{4x^7} - \frac{491n^3}{4x^6} - \cdots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{n^2 - 2xn + x^2 - 2n - 4x}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{(-2n-4)\,x + n^2 - 2n}{4x^2}\right) \\ &= \frac{1}{4} + \frac{(-2n-4)\,x + n^2 - 2n}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is $-2n - 4$. Dividing this by leading coefficient in t which is 4 gives $-\frac{n}{2} - 1$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{n}{2} - 1\right) - (0) \\ &= -\frac{n}{2} - 1 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{n}{2}-1}{\frac{1}{2}} - 0\right) &= -\frac{n}{2} - 1 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{n}{2}-1}{\frac{1}{2}} - 0\right) &= \frac{n}{2} + 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{n^2 - 2xn + x^2 - 2n - 4x}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{n}{2}$	$1 - \frac{n}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{n}{2} - 1$	$\frac{n}{2} + 1$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{n}{2} + 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{n}{2} + 1 - \left(\frac{n}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{n}{2x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{n}{2x} - \frac{1}{2} \\
&= \frac{n - x}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{n}{2x}-\frac{1}{2}\right)(1)+\left(\left(-\frac{n}{2x^2}\right)+\left(\frac{n}{2x}-\frac{1}{2}\right)^2-\left(\frac{n^2-2xn+x^2-2n-4x}{4x^2}\right)\right)=0$$

$$\frac{n+a_0}{x}=0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0=-n\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x)=x-n$$

Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\,dx}\\
&= (x-n)\,\mathrm{e}^{\int \left(\frac{n}{2x}-\frac{1}{2}\right)dx}\\
&= (x-n)\,\mathrm{e}^{-\frac{x}{2}+\frac{n\ln(x)}{2}}\\
&= -(n-x)\,x^{\frac{n}{2}}\mathrm{e}^{-\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\
&= z_1 e^{-\int \frac{1}{2}\frac{x+n}{x}\,dx}\\
&= z_1 e^{-\frac{x}{2}-\frac{n\ln(x)}{2}}\\
&= z_1\left(x^{-\frac{n}{2}}\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1=(x-n)\,\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1\int \frac{e^{\int -\frac{x+n}{x}\,dx}}{(y_1)^2}\,dx\\
&= y_1\int \frac{\mathrm{e}^{-n\ln(x)-x}}{(y_1)^2}\,dx\\
&= y_1\left(\int \frac{\mathrm{e}^{-n\ln(x)-x}\mathrm{e}^{2x}}{(x-n)^2}dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1y_1+c_2y_2\\
&= c_1\left((x-n)\,\mathrm{e}^{-x}\right)+c_2\left((x-n)\,\mathrm{e}^{-x}\left(\int \frac{\mathrm{e}^{-n\ln(x)-x}\mathrm{e}^{2x}}{(x-n)^2}dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dx^2}y(x)\right)x+(x+n)\left(\frac{d}{dx}y(x)\right)+(n+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{(n+1)y(x)}{x}-\frac{(x+n)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x)+\frac{(x+n)\left(\frac{d}{dx}y(x)\right)}{x}+\frac{(n+1)y(x)}{x}=0$

- ☐ Check to see if $x_0=0$ is a regular singular point
    - Define functions

      $\left[P_2(x)=\frac{x+n}{x},P_3(x)=\frac{n+1}{x}\right]$
    - $x\cdot P_2(x)$ is analytic at $x=0$

      $(x\cdot P_2(x))\Big|_{x=0}=n$
    - $x^2\cdot P_3(x)$ is analytic at $x=0$

      $(x^2\cdot P_3(x))\Big|_{x=0}=0$
    - $x=0$is a regular singular point

Check to see if $x_0=0$ is a regular singular point

$x_0=0$

- Multiply by denominators

$\left(\frac{d^2}{dx^2}y(x)\right)x+(x+n)\left(\frac{d}{dx}y(x)\right)+(n+1)\,y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

- ☐ Rewrite ODE with series expansions
    - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..1$

      $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$
    - Shift index using $k->k+1-m$

      $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$
    - Convert $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

      $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-1}$
    - Shift index using $k->k+1$

      $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

Rewrite ODE with series expansions

$a_0r(-1+r+n)\,x^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(k+r+n)+a_k(n+k+r+1)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-1+r+n)=0$

- Values of r that satisfy the indicial equation

$r\in\{0,1-n\}$

- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)(k+r+n)+a_k(n+k+r+1)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=-\frac{a_k(n+k+r+1)}{(k+1+r)(k+r+n)}$
- Recursion relation for $r=0$
  $a_{k+1}=-\frac{a_k(n+k+1)}{(k+1)(k+n)}$
- Solution for $r=0$
  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^k, a_{k+1}=-\frac{a_k(n+k+1)}{(k+1)(k+n)}\right]$
- Recursion relation for $r=1-n$
  $a_{k+1}=-\frac{a_k(k+2)}{(k+2-n)(k+1)}$
- Solution for $r=1-n$
  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+1-n}, a_{k+1}=-\frac{a_k(k+2)}{(k+2-n)(k+1)}\right]$
- Combine solutions and rename parameters
  $\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+1-n}\right), a_{k+1}=-\frac{a_k(n+k+1)}{(k+1)(k+n)}, b_{k+1}=-\frac{b_k(k+2)}{(k+2-n)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric sol
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.023` (sec)
Leaf size : 42

```
dsolve(x*diff(diff(y(x),x),x)+(x+n)*diff(y(x),x)+(n+1)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{\left(c_2 x^{-n+1} \operatorname{hypergeom}\left([-n], [-n+2], x\right) n + (-x+n) c_1\right) e^{-x}}{n}$$

Mathematica DSolve solution

Solving time : `0.362` (sec)
Leaf size : 77

```
DSolve[{x*D[y[x],{x,2}]+(x+n)*D[y[x],x]+(n+1)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True
```

$$y(x) \to e^{\int \frac{-n+x-1}{n-x} dx} \left(c_2 \int_1^x \exp\left(\int_1^{K[2]} -\frac{n^2 - 2K[1]n + (K[1]-2)K[1]}{(n-K[1])K[1]} dK[1]\right) dK[2] + c_1\right)$$

2.1.682 Problem 699

Internal problem ID [9854]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 699
Date solved : `Monday, January 27, 2025 at 06:15:04 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^4y'' + xy' + y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.335 (sec)

Writing the ode as

$$x^4y'' + xy' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4 \\ B &= x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-10x^2 + 1}{4x^6} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -10x^2 + 1 \\ t &= 4x^6 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-10x^2 + 1}{4x^6}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1302: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^6$. There is a pole at $x = 0$ of order 6. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned} \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) \end{aligned}$$

The partial fraction decomposition of r is

$$r = \frac{1}{4x^6} - \frac{5}{2x^4}$$

There is pole in r at $x = 0$ of order 6, hence $v = 3$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{1}{2x^3} - \frac{5}{2x} - \frac{25x}{4} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 3$ the above becomes

$$[\sqrt{r}]_c = \frac{1}{2x^3} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^3}$ is

$$a = \frac{1}{2}$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^4}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be $-\frac{5}{2}$. Therefore

$$\begin{aligned} b &= \left(-\frac{5}{2}\right) - (0) \\ &= -\frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{1}{2x^3} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} + 3\right) &= -1 \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} + 3\right) &= 4 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-10x^2 + 1}{4x^6}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	6	$\frac{1}{2x^3}$	-1	4

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= 1 - (-1) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x^3} - \frac{1}{x} + (-)\,(0) \\
&= \frac{1}{2x^3} - \frac{1}{x} \\
&= \frac{1}{2x^3} - \frac{1}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{1}{2x^3} - \frac{1}{x}\right)(2x + a_1) + \left(\left(-\frac{3}{2x^4} + \frac{1}{x^2} \right) + \left(\frac{1}{2x^3} - \frac{1}{x} \right)^2 - \left(\frac{-10x^2+1}{4x^6} \right) \right) = 0$$

$$\frac{(2a_0+2)\,x + a_1}{x^3} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 1\right) e^{\int \left(\frac{1}{2x^3} - \frac{1}{x}\right) dx} \\
&= \left(x^2 - 1\right) e^{-\frac{1}{4x^2} - \ln(x)} \\
&= \frac{\left(x^2 - 1\right) e^{-\frac{1}{4x^2}}}{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{x^4} \, dx} \\
&= z_1 e^{\frac{1}{4x^2}} \\
&= z_1 \left(e^{\frac{1}{4x^2}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2 - 1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^4}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{1}{2x^2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{e^{\frac{1}{2x^2}} x^2}{(x^2-1)^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^2-1}{x}\right) + c_2 \left(\frac{x^2-1}{x}\left(\int \frac{e^{\frac{1}{2x^2}} x^2}{(x^2-1)^2} dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special functio
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.030 (sec)
Leaf size : 50

```
dsolve(diff(diff(y(x),x),x)*x^4+diff(y(x),x)*x+y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_1\sqrt{2}\,\sqrt{\pi}\,(x-1)\,(x+1)\,\mathrm{erfi}\left(\frac{\sqrt{2}}{2x}\right)+c_2x^2+2\,\mathrm{e}^{\frac{1}{2x^2}}c_1x-c_2}{x}$$

Mathematica DSolve solution

Solving time : 0.317 (sec)
Leaf size : 57

```
DSolve[{x^4*D[y[x],{x,2}]+x*D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{3/2}(x^2-1)\left(c_2\int_1^x \frac{e^{\frac{1}{2K[1]^2}-3}K[1]^2}{(K[1]^2-1)^2}dK[1]+c_1\right)}{x}$$

2.1.683 Problem 700

Internal problem ID [9855]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 700
Date solved : `Monday, January 27, 2025 at 06:15:04 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + \left(2x^2 + x\right) y' - 4y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.255 (sec)

Writing the ode as

$$x^2y'' + \left(2x^2 + x\right) y' - 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 2x^2 + x \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 + 4x + 15}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2 + 4x + 15 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 + 4x + 15}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1303: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{1}{x} + \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{1}{2x} + \frac{7}{4x^2} - \frac{7}{8x^3} - \frac{35}{32x^4} + \frac{133}{64x^5} + \frac{63}{128x^6} - \frac{1239}{256x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 + 4x + 15}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (1) + \left(\frac{4x + 15}{4x^2}\right) \\ &= 1 + \frac{4x + 15}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$\begin{aligned} b &= (1) - (0) \\ &= 1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{1}{1} - 0\right) = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{1}{1} - 0\right) = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^2 + 4x + 15}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{3}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{2x} + (-)(1) \\
&= -\frac{3}{2x} - 1 \\
&= -\frac{3}{2x} - 1
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{3}{2x} - 1\right)(1) + \left(\left(\frac{3}{2x^2}\right) + \left(-\frac{3}{2x} - 1\right)^2 - \left(\frac{4x^2 + 4x + 15}{4x^2}\right) \right) &= 0 \\
\frac{-3 + 2a_0}{x} &= 0
\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{ a_0 = \frac{3}{2} \right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{3}{2}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{3}{2}\right) \mathrm{e}^{\int \left(-\frac{3}{2x} - 1\right) dx} \\
&= \left(x + \frac{3}{2}\right) \mathrm{e}^{-x - \frac{3\ln(x)}{2}} \\
&= \frac{(3 + 2x)\, \mathrm{e}^{-x}}{2x^{3/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{2x^2 + x}{x^2} \, dx} \\
&= z_1 e^{-x - \frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{-x}}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-2x}(3 + 2x)}{2x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2 + x}{x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-2x - \ln(x)}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{(2x^2 - 4x + 3)\, x\, \mathrm{e}^{-2x - \ln(x)} \mathrm{e}^{4x}}{6 + 4x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{-2x}(3 + 2x)}{2x^2}\right) + c_2 \left(\frac{\mathrm{e}^{-2x}(3 + 2x)}{2x^2} \left(\frac{(2x^2 - 4x + 3)\, x\, \mathrm{e}^{-2x - \ln(x)} \mathrm{e}^{4x}}{6 + 4x}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right)+(2x^2+x)\left(\frac{d}{dx}y(x)\right)-4y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=\frac{4y(x)}{x^2}-\frac{(2x+1)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x)+\frac{(2x+1)\left(\frac{d}{dx}y(x)\right)}{x}-\frac{4y(x)}{x^2}=0$

☐ Check to see if $x_0=0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x)=\frac{2x+1}{x},P_3(x)=-\frac{4}{x^2}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=1$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=-4$

- ○ $x=0$is a regular singular point

Check to see if $x_0=0$ is a regular singular point

$x_0=0$

- Multiply by denominators

$x^2\left(\frac{d^2}{dx^2}y(x)\right)+x(2x+1)\left(\frac{d}{dx}y(x)\right)-4y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ○ Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

Rewrite ODE with series expansions

$a_0(2+r)\,(-2+r)\,x^r+\left(\sum_{k=1}^{\infty}\left(a_k(k+r+2)\,(k+r-2)+2a_{k-1}(k+r-1)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(2+r)\,(-2+r)=0$

- Values of r that satisfy the indicial equation

$r\in\{-2,2\}$

- Each term in the series must be 0, giving the recursion relation

$a_k(k+r+2)\,(k+r-2)+2a_{k-1}(k+r-1)=0$

- Shift index using $k->k+1$

$a_{k+1}(k+3+r)(k+r-1)+2a_k(k+r)=0$

- Recursion relation that defines series solution to ODE
  $a_{k+1}=-\frac{2a_k(k+r)}{(k+3+r)(k+r-1)}$
- Recursion relation for $r=-2$; series terminates at $k=2$
  $a_{k+1}=-\frac{2a_k(k-2)}{(k+1)(k-3)}$
- Apply recursion relation for $k=0$
  $a_1=-\frac{4a_0}{3}$
- Apply recursion relation for $k=1$
  $a_2=-\frac{a_1}{2}$
- Express in terms of a_0
  $a_2=\frac{2a_0}{3}$
- Terminating series solution of the ODE for $r=-2$. Use reduction of order to find the second
  $y(x)=a_0\cdot\left(1-\frac{4}{3}x+\frac{2}{3}x^2\right)$
- Recursion relation for $r=2$
  $a_{k+1}=-\frac{2a_k(k+2)}{(k+5)(k+1)}$
- Solution for $r=2$
  $\left[y(x)=\sum_{k=0}^{\infty}a_k x^{k+2}, a_{k+1}=-\frac{2a_k(k+2)}{(k+5)(k+1)}\right]$
- Combine solutions and rename parameters
  $\left[y(x)=a_0\cdot\left(1-\frac{4}{3}x+\frac{2}{3}x^2\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+2}\right), b_{k+1}=-\frac{2b_k(k+2)}{(5+k)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 31

```
dsolve(x^2*diff(diff(y(x),x),x)+(2*x^2+x)*diff(y(x),x)-4*y(x) = 0,y(x),singsol=all)
```

$$y=\frac{c_2\mathrm{e}^{-2x}(2x+3)+2c_1\left(x^2-2x+\frac{3}{2}\right)}{x^2}$$

Mathematica DSolve solution

Solving time : 0.267 (sec)
Leaf size : 71

```
DSolve[{x^2*D[y[x],{x,2}]+(x+2*x^2)*D[y[x],x]-4*y[x]==2,{}},y[x],x,IncludeSingularSolutions->Tr
```

$$y(x) \rightarrow \frac{e^{-2x}\left(c_2(2x+3)\int_1^x \frac{4e^{2K[1]}K[1]^3}{(2K[1]+3)^2}dK[1] - e^{2x}x^2 + c_1(2x+3)\right)}{2x^2}$$

2.1.684 Problem 701

Internal problem ID [9856]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 701
Date solved : `Monday, January 27, 2025 at 06:15:05 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(4x^3-14x^2-2x\right)y''-\left(6x^2-7x+1\right)y'+\left(6x-1\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.444 (sec)

Writing the ode as

$$\left(4x^3-14x^2-2x\right)y''+\left(-6x^2+7x-1\right)y'+\left(6x-1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=4x^3-14x^2-2x\\B&=-6x^2+7x-1\\C&=6x-1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-12x^4+156x^3+297x^2-78x-3}{16\left(2x^3-7x^2-x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=-12x^4+156x^3+297x^2-78x-3\\t&=16\left(2x^3-7x^2-x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-12x^4+156x^3+297x^2-78x-3}{16\left(2x^3-7x^2-x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1305: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(2x^3 - 7x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{7}{4} + \frac{\sqrt{57}}{4}$ of order 2. There is a pole at $x = \frac{7}{4} - \frac{\sqrt{57}}{4}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{9}{4x} - \frac{3}{16x^2} + \frac{3}{4\left(x - \frac{7}{4} - \frac{\sqrt{57}}{4}\right)^2} + \frac{3}{4\left(x - \frac{7}{4} + \frac{\sqrt{57}}{4}\right)^2} + \frac{\frac{9}{8} - \frac{29\sqrt{57}}{152}}{x - \frac{7}{4} - \frac{\sqrt{57}}{4}} + \frac{\frac{9}{8} + \frac{29\sqrt{57}}{152}}{x - \frac{7}{4} + \frac{\sqrt{57}}{4}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x=\frac{7}{4}+\frac{\sqrt{57}}{4}$ let b be the coefficient of $\frac{1}{\left(x-\frac{7}{4}-\frac{\sqrt{57}}{4}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

For the pole at $x=\frac{7}{4}-\frac{\sqrt{57}}{4}$ let b be the coefficient of $\frac{1}{\left(x-\frac{7}{4}+\frac{\sqrt{57}}{4}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty=0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r=\frac{s}{t}=\frac{-12x^4+156x^3+297x^2-78x-3}{16\left(2x^3-7x^2-x\right)^2}
$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-12x^4+156x^3+297x^2-78x-3}{16\left(2x^3-7x^2-x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
$\frac{7}{4}+\frac{\sqrt{57}}{4}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$\frac{7}{4}-\frac{\sqrt{57}}{4}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=\frac{1}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{1}{4}-\left(-\frac{3}{4}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{4x} - \frac{1}{2\left(x-\frac{7}{4}-\frac{\sqrt{57}}{4}\right)} - \frac{1}{2\left(x-\frac{7}{4}+\frac{\sqrt{57}}{4}\right)} + (-)(0) \\
&= \frac{1}{4x} - \frac{1}{2\left(x-\frac{7}{4}-\frac{\sqrt{57}}{4}\right)} - \frac{1}{2\left(x-\frac{7}{4}+\frac{\sqrt{57}}{4}\right)} \\
&= \frac{-6x^2+7x-1}{8x^3-28x^2-4x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{4x} - \frac{1}{2\left(x-\frac{7}{4}-\frac{\sqrt{57}}{4}\right)} - \frac{1}{2\left(x-\frac{7}{4}+\frac{\sqrt{57}}{4}\right)}\right)(1) + \left(\left(-\frac{1}{4x^2} + \frac{1}{2\left(x-\frac{7}{4}-\frac{\sqrt{57}}{4}\right)^2} + \frac{1}{2\left(x-\frac{7}{4}+\frac{\sqrt{5}}{4}\right.}\right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\,dx} \\
&= (x-1)\,\mathrm{e}^{\int \left(\frac{1}{4x} - \frac{1}{2\left(x-\frac{7}{4}-\frac{\sqrt{57}}{4}\right)} - \frac{1}{2\left(x-\frac{7}{4}+\frac{\sqrt{57}}{4}\right)}\right)dx} \\
&= (x-1)\,\mathrm{e}^{\frac{2\ln(x)}{(7+\sqrt{57})(-7+\sqrt{57})} + \frac{(-57+7\sqrt{57})\sqrt{57}\ln(4x-7+\sqrt{57})}{-798+114\sqrt{57}} - \frac{(57+7\sqrt{57})\sqrt{57}\ln(4x-7-\sqrt{57})}{2(399+57\sqrt{57})}} \\
&= \frac{(x-1)\,x^{1/4}}{\sqrt{4x-7+\sqrt{57}}\,\sqrt{4x-7-\sqrt{57}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-6x^2+7x-1}{4x^3-14x^2-2x}\,dx} \\
&= z_1 e^{\frac{\ln(2x^2-7x-1)}{2} - \frac{\ln(x)}{4}} \\
&= z_1\left(\frac{\sqrt{2x^2-7x-1}}{x^{1/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x-1)\sqrt{2}}{4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-6x^2+7x-1}{4x^3-14x^2-2x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(2x^2-7x-1)-\frac{\ln(x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{16x(2x+1)\,e^{\ln(2x^2-7x-1)-\frac{\ln(x)}{2}}}{(x-1)(2x^2-7x-1)}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(x-1)\sqrt{2}}{4}\right) + c_2\left(\frac{(x-1)\sqrt{2}}{4}\left(\frac{16x(2x+1)\,e^{\ln(2x^2-7x-1)-\frac{\ln(x)}{2}}}{(x-1)(2x^2-7x-1)}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$(4x^3-14x^2-2x)\left(\frac{d^2}{dx^2}y(x)\right)-(6x^2-7x+1)\left(\frac{d}{dx}y(x)\right)+(6x-1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{(6x-1)y(x)}{2x(2x^2-7x-1)} + \frac{(6x^2-7x+1)\left(\frac{d}{dx}y(x)\right)}{2x(2x^2-7x-1)}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x) - \frac{(6x^2-7x+1)\left(\frac{d}{dx}y(x)\right)}{2x(2x^2-7x-1)} + \frac{(6x-1)y(x)}{2x(2x^2-7x-1)} = 0$

☐ Check to see if x_0 is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{6x^2-7x+1}{2x(2x^2-7x-1)}, P_3(x) = \frac{6x-1}{2x(2x^2-7x-1)}\right]$
  - $x \cdot P_2(x)$ is analytic at $x=0$
    $\left.(x\cdot P_2(x))\right|_{x=0} = \frac{1}{2}$
  - $x^2 \cdot P_3(x)$ is analytic at $x=0$
    $\left.(x^2\cdot P_3(x))\right|_{x=0} = 0$
  - $x=0$is a regular singular point
    Check to see if x_0 is a regular singular point
    $x_0 = 0$

- Multiply by denominators
  $2x(2x^2-7x-1)\left(\frac{d^2}{dx^2}y(x)\right)+(-6x^2+7x-1)\left(\frac{d}{dx}y(x)\right)+(6x-1)\,y(x)=0$
- Assume series solution for $y(x)$
  $y(x)=\sum_{k=0}^{\infty} a_k x^{k+r}$
- □ Rewrite ODE with series expansions
  - Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$
    $x^m\cdot y(x)=\sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k->k-m$
    $x^m\cdot y(x)=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..2$
    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k->k+1-m$
    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=1..3$
    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$
  - Shift index using $k->k+2-m$
    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions
  $-a_0r(-1+2r)\,x^{-1+r}+(-a_1(1+r)\,(1+2r)-a_0(14r^2-21r+1))\,x^r+\left(\sum_{k=1}^{\infty}(-a_{k+1}(k+1+r)\,(2\right.$
- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-r(-1+2r)=0$
- Values of r that satisfy the indicial equation
  $r\in\left\{0,\frac{1}{2}\right\}$
- Each term must be 0
  $-a_1(1+r)\,(1+2r)-a_0(14r^2-21r+1)=0$
- Each term in the series must be 0, giving the recursion relation
  $(-14a_k+4a_{k-1}-2a_{k+1})\,k^2+((-28a_k+8a_{k-1}-4a_{k+1})\,r+21a_k-18a_{k-1}-3a_{k+1})\,k+(-14a_k+$
- Shift index using $k->k+1$
  $(-14a_{k+1}+4a_k-2a_{k+2})\,(k+1)^2+((-28a_{k+1}+8a_k-4a_{k+2})\,r+21a_{k+1}-18a_k-3a_{k+2})\,(k+1)$
- Recursion relation that defines series solution to ODE
  $a_{k+2}=\frac{4k^2a_k-14k^2a_{k+1}+8kra_k-28kra_{k+1}+4r^2a_k-14r^2a_{k+1}-10ka_k-7ka_{k+1}-10ra_k-7ra_{k+1}+6a_k+6a_{k+1}}{2k^2+4kr+2r^2+7k+7r+6}$
- Recursion relation for $r=0$
  $a_{k+2}=\frac{4k^2a_k-14k^2a_{k+1}-10ka_k-7ka_{k+1}+6a_k+6a_{k+1}}{2k^2+7k+6}$
- Solution for $r=0$
  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=\frac{4k^2a_k-14k^2a_{k+1}-10ka_k-7ka_{k+1}+6a_k+6a_{k+1}}{2k^2+7k+6}, -a_1-a_0=0\right]$
- Recursion relation for $r=\frac{1}{2}$
  $a_{k+2}=\frac{4k^2a_k-14k^2a_{k+1}-6ka_k-21ka_{k+1}+2a_k-a_{k+1}}{2k^2+9k+10}$
- Solution for $r=\frac{1}{2}$

$$\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2}=\frac{4k^2a_k-14k^2a_{k+1}-6ka_k-21ka_{k+1}+2a_k-a_{k+1}}{2k^2+9k+10}, -3a_1+6a_0=0\right]$$

- Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2}=\frac{4k^2a_k-14k^2a_{k+1}-10ka_k-7ka_{k+1}+6a_k+6a_{k+1}}{2k^2+7k+6}, -a_1-a_0=0$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.009` (sec)
Leaf size : 21

```
dsolve((4*x^3-14*x^2-2*x)*diff(diff(y(x),x),x)-(6*x^2-7*x+1)*diff(y(x),x)+(6*x-1)*y(x
```

$$y=c_2\sqrt{x}+c_1(x-1)+2c_2x^{3/2}$$

Mathematica DSolve solution

Solving time : `1.956` (sec)
Leaf size : 155

```
DSolve[{(4*x^3-14*x^2-2*x)*D[y[x],{x,2}]-(6*x^2-7*x+1)*D[y[x],x]+(6*x-1)*y[x]==0,{}},y[x],x,
```

$$y(x)\to(x-1)\exp\left(\int_1^x \frac{6K[1]^2-7K[1]+1}{-8K[1]^3+28K[1]^2+4K[1]}dK[1]-\frac{1}{2}\int_1^x\left(\frac{7-4K[2]}{K[2](2K[2]-7)-1}+\frac{1}{2K[2]}\right)dK[2]\right)\left(c_2\int_1^x \frac{\exp\left(-2\int_1^{K[3]}\frac{6K[1]^2-7K[1]+1}{-8K[1]^3+28K[1]^2+4K[1]}dK[1]\right)}{(K[3]-1)^2}dK[3]+c_1\right)$$

2.1.685 Problem 702

Internal problem ID [9857]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 702
Date solved : `Monday, January 27, 2025 at 06:15:06 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x^2y' + (x-2)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.206 (sec)

Writing the ode as

$$x^2y'' + x^2y' + (x-2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2 \\ C &= x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 4x + 8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1307: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{x} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{1}{x^2} + \frac{2}{x^3} + \frac{3}{x^4} + \frac{2}{x^5} - \frac{6}{x^6} - \frac{28}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-4x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-4x + 8}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned}b &= (-1) - (0) \\ &= -1\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 0\right) &&= -1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 0\right) &&= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -1$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2} - \frac{1}{x} \\ &= \frac{x-2}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2} - \frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(\frac{1}{2} - \frac{1}{x}\right)^2 - \left(\frac{x^2-4x+8}{4x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{1}{2} - \frac{1}{x}\right) dx} \\ &= \frac{e^{\frac{x}{2}}}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^2}{x^2}\,dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\ &= y_1\left(-\left(x^2+2x+2\right)\mathrm{e}^{-x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x}\left(-\left(x^2+2x+2\right)\mathrm{e}^{-x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x^2\left(\frac{d}{dx}y(x)\right) + (x-2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(x-2)y(x)}{x^2} - \frac{d}{dx}y(x)$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{d}{dx}y(x) + \frac{(x-2)y(x)}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = 1, P_3(x) = \frac{x-2}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 0$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -2$$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$
- • Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) + x^2\left(\frac{d}{dx}y(x)\right) + (x-2)\,y(x) = 0$
- • Assume series solution for $y(x)$

  $y(x) = \sum\limits_{k=0}^{\infty} a_k x^{k+r}$
- □ Rewrite ODE with series expansions
  - ○ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

    $x^m \cdot y(x) = \sum\limits_{k=0}^{\infty} a_k x^{k+r+m}$
  - ○ Shift index using $k->k-m$

    $x^m \cdot y(x) = \sum\limits_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - ○ Convert $x^2 \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d}{dx}y(x)\right) = \sum\limits_{k=0}^{\infty} a_k(k+r)\,x^{k+r+1}$
  - ○ Shift index using $k->k-1$

    $x^2 \cdot \left(\frac{d}{dx}y(x)\right) = \sum\limits_{k=1}^{\infty} a_{k-1}(k-1+r)\,x^{k+r}$
  - ○ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum\limits_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-2+r)\,x^r + \left(\sum\limits_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-2) + a_{k-1}(k+r)\right) x^{k+r}\right) = 0$
- • $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-2+r) = 0$
- • Values of r that satisfy the indicial equation

  $r \in \{-1, 2\}$
- • Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)\,(k+r-2) + a_{k-1}(k+r) = 0$
- • Shift index using $k->k+1$

  $a_{k+1}(k+2+r)\,(k-1+r) + a_k(k+r+1) = 0$
- • Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r+1)}{(k+2+r)(k-1+r)}$
- • Recursion relation for $r = -1$

  $a_{k+1} = -\frac{a_k k}{(k+1)(k-2)}$
- • Series not valid for $r = -1$, division by 0 in the recursion relation at $k = 2$

  $a_{k+1} = -\frac{a_k k}{(k+1)(k-2)}$
- • Recursion relation for $r = 2$

  $a_{k+1} = -\frac{a_k(k+3)}{(k+4)(k+1)}$
- • Solution for $r = 2$

  $\left[y(x) = \sum\limits_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = -\frac{a_k(k+3)}{(k+4)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 24

```
dsolve(x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x^2+(x-2)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_2(x^2 + 2x + 2)\,\mathrm{e}^{-x} + c_1}{x}$$

Mathematica DSolve solution

Solving time : 0.195 (sec)
Leaf size : 33

```
DSolve[{x^2*D[y[x],{x,2}]+x^2*D[y[x],x]+(x-2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tru
```

$$y(x) \to \frac{c_2 \int_1^x e^{-K[1]}K[1]^2 dK[1] + c_1}{x}$$

2.1.686 Problem 703

Internal problem ID [9858]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 703
Date solved : `Monday, January 27, 2025 at 06:15:07 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x^2y' + (x-2)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.199 (sec)

Writing the ode as

$$x^2y'' - x^2y' + (x-2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 \\ C &= x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 4x + 8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1309: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{x^2} - \frac{1}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{1}{x^2} + \frac{2}{x^3} + \frac{3}{x^4} + \frac{2}{x^5} - \frac{6}{x^6} - \frac{28}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-4x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-4x + 8}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned}b &= (-1) - (0) \\ &= -1\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 0\right) &&= -1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 0\right) &&= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -1$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2} - \frac{1}{x} \\ &= \frac{x-2}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2} - \frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(\frac{1}{2} - \frac{1}{x}\right)^2 - \left(\frac{x^2-4x+8}{4x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{1}{2} - \frac{1}{x}\right) dx} \\ &= \frac{e^{\frac{x}{2}}}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x^2}{x^2}\,dx} \\ &= z_1 e^{\frac{x}{2}} \\ &= z_1\left(\mathrm{e}^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^x}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^x}{(y_1)^2}\,dx \\ &= y_1\left(-\left(x^2+2x+2\right)\mathrm{e}^{-x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\mathrm{e}^x}{x}\right) + c_2\left(\frac{\mathrm{e}^x}{x}\left(-\left(x^2+2x+2\right)\mathrm{e}^{-x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) - x^2\left(\frac{d}{dx}y(x)\right) + (x-2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(x-2)y(x)}{x^2} + \frac{d}{dx}y(x)$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) - \frac{d}{dx}y(x) + \frac{(x-2)y(x)}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -1, P_3(x) = \frac{x-2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^{2}\cdot P_{3}(x)\right)\Big|_{x=0} = -2$

- $x = 0$is a regular singular point

  Check to see if $x_{0} = 0$ is a regular singular point

  $x_{0} = 0$

- Multiply by denominators

  $x^{2}\left(\frac{d^{2}}{dx^{2}}y(x)\right)-x^{2}\left(\frac{d}{dx}y(x)\right)+(x-2)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_{k}x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^{m}\cdot y(x)$ to series expansion for $m = 0..1$

  $x^{m}\cdot y(x) = \sum_{k=0}^{\infty} a_{k}x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y(x) = \sum_{k=m}^{\infty} a_{k-m}x^{k+r}$

- Convert $x^{2}\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x^{2}\cdot\left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_{k}(k+r)\,x^{k+r+1}$

- Shift index using $k->k-1$

  $x^{2}\cdot\left(\frac{d}{dx}y(x)\right) = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\,x^{k+r}$

- Convert $x^{2}\cdot\left(\frac{d^{2}}{dx^{2}}y(x)\right)$ to series expansion

  $x^{2}\cdot\left(\frac{d^{2}}{dx^{2}}y(x)\right) = \sum_{k=0}^{\infty} a_{k}(k+r)\,(k-1+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(1+r)\,(-2+r)\,x^{r}+\left(\sum_{k=1}^{\infty}\left(a_{k}(k+r+1)\,(k+r-2)-a_{k-1}(k+r-2)\right)x^{k+r}\right) = 0$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r\in\{-1,2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)\,(a_{k}(k+r+1)-a_{k-1}) = 0$

- Shift index using $k->k+1$

  $(k-1+r)\,(a_{k+1}(k+2+r)-a_{k}) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_{k}}{k+2+r}$

- Recursion relation for $r = -1$

  $a_{k+1} = \frac{a_{k}}{k+1}$

- Solution for $r = -1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_{k}x^{k-1}, a_{k+1} = \frac{a_{k}}{k+1}\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_{k}}{k+4}$

- Solution for $r = 2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_{k}x^{k+2}, a_{k+1} = \frac{a_{k}}{k+4}\right]$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{4+k}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 23

```
dsolve(x^2*diff(diff(y(x),x),x)-diff(y(x),x)*x^2+(x-2)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{e^x c_1 + c_2(x^2 + 2x + 2)}{x}$$

Mathematica DSolve solution

Solving time : 0.202 (sec)
Leaf size : 36

```
DSolve[{x^2*D[y[x],{x,2}]-x^2*D[y[x],x]+(x-2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to \frac{e^x \left(c_2 \int_1^x e^{-K[1]} K[1]^2 dK[1] + c_1\right)}{x}$$

2.1.687 Problem 704

Internal problem ID [9859]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 704
Date solved : `Monday, January 27, 2025 at 06:15:07 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1-4x)\,y''+\left(-\frac{1}{4}x-x^2\right)y'-\frac{5xy}{16}=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.347 (sec)

Writing the ode as

$$\left(-4x^3+x^2\right)y''+\left(-\frac{1}{4}x-x^2\right)y'-\frac{5xy}{16}=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=-4x^3+x^2\\ B&=-\frac{1}{4}x-x^2\\ C&=-\frac{5x}{16}\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-192x^2-36x+9}{64\left(4x^2-x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=-192x^2-36x+9\\ t&=64\left(4x^2-x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-192x^2 - 36x + 9}{64\,(4x^2 - x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1311: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64(4x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{1}{4}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{16x} + \frac{9}{64x^2} - \frac{3}{16\left(x - \frac{1}{4}\right)^2} - \frac{9}{16\left(x - \frac{1}{4}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{9}{64}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \left\{2, -\frac{1}{2}, \frac{9}{2}\right\} \end{aligned}$$

For the pole at $x = \frac{1}{4}$ let b be the coefficient of $\frac{1}{\left(x-\frac{1}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-192x^2 - 36x + 9}{64\left(4x^2 - x\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_\infty &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{2, -\frac{1}{2}, \frac{9}{2}\}$
$\frac{1}{4}$	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
2	$\{1, 2, 3\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 2,\ e_2 = 1,\ e_\infty = 3$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(3 - (2 + (1))) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{2}{(x-(0))} + \frac{1}{\left(x-\left(\frac{1}{4}\right)\right)}\right) \\ &= \frac{1}{x} + \frac{1}{2x-\frac{1}{2}} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p}\\ &= \frac{1}{x} + \frac{1}{2x - \frac{1}{2}}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{x} + \frac{1}{2x - \frac{1}{2}}\right) w + \frac{576x^2 - 92x - 9}{64x^2 \left(-1 + 4x\right)^2} = 0$$

Solving for ω gives

$$\omega = \frac{24x - 4 + 5\sqrt{1 - 4x}}{8x \left(-1 + 4x\right)}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega \, dx}\\ &= e^{\int \frac{24x-4+5\sqrt{1-4x}}{8x(-1+4x)} dx}\\ &= \frac{\left(-1 + 4x\right)^{1/4} \sqrt{x}\, 2^{3/4} \left(\frac{\sqrt{1-4x}+1}{\sqrt{x}}\right)^{5/4}}{4}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-\frac{1}{4}x - x^2}{-4x^3 + x^2} \, dx}\\ &= z_1 e^{-\frac{\ln(-1+4x)}{4} + \frac{\ln(x)}{8}}\\ &= z_1 \left(\frac{x^{1/8}}{\left(-1 + 4x\right)^{1/4}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/8} 2^{3/4} \left(\sqrt{1 - 4x} + 1\right) \left(\frac{\sqrt{1-4x}+1}{\sqrt{x}}\right)^{1/4}}{4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-\frac{1}{4}x - x^2}{-4x^3 + x^2} \, dx}}{\left(y_1\right)^2} \, dx\\ &= y_1 \int \frac{e^{-\frac{\ln(-1+4x)}{2} + \frac{\ln(x)}{4}}}{\left(y_1\right)^2} \, dx\\ &= y_1 \left(\int \frac{4\, \mathrm{e}^{-\frac{\ln(-1+4x)}{2} + \frac{\ln(x)}{4}} \sqrt{2}}{x^{1/4} \left(\sqrt{1 - 4x} + 1\right)^2 \sqrt{\frac{\sqrt{1-4x}+1}{\sqrt{x}}}} dx\right)\end{aligned}$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\frac{x^{1/8} 2^{3/4} \left(\sqrt{1-4x}+1\right) \left(\frac{\sqrt{1-4x}+1}{\sqrt{x}}\right)^{1/4}}{4}\right) + c_2 \left(\frac{x^{1/8} 2^{3/4} \left(\sqrt{1-4x}+1\right) \left(\frac{\sqrt{1-4x}+1}{\sqrt{x}}\right)^{1/4}}{4} \left(\int \frac{4\,e^{-\frac{\ln(-1}{}}}{x^{1/4}\left(\sqrt{1-4}\right.}\right.\right.$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(1-4x)\left(\frac{d^2}{dx^2}y(x)\right) + \left(-\frac{1}{4}x - x^2\right)\left(\frac{d}{dx}y(x)\right) - \frac{5xy(x)}{16} = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{5y(x)}{16x(4x-1)} - \frac{(4x+1)\left(\frac{d}{dx}y(x)\right)}{4x(4x-1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(4x+1)\left(\frac{d}{dx}y(x)\right)}{4x(4x-1)} + \frac{5y(x)}{16x(4x-1)} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{4x+1}{4x(4x-1)}, P_3(x) = \frac{5}{16x(4x-1)}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -\frac{1}{4}$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $16x(4x-1)\left(\frac{d^2}{dx^2}y(x)\right) + (16x+4)\left(\frac{d}{dx}y(x)\right) + 5y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..1$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- ◦ Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 1..2$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$

- ∘ Shift index using $k->k+2-m$

  $$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$-4a_0r(-5+4r)\,x^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-4a_{k+1}(k+1+r)\,(4k-1+4r)+a_k(8k+8r-1)\,(8k+8r-5\right)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-4r(-5+4r)=0$$

- Values of r that satisfy the indicial equation

  $$r\in\left\{0,\frac{5}{4}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$-16(k+1+r)\left(k+r-\frac{1}{4}\right)a_{k+1}+64\left(k+r-\frac{1}{8}\right)\left(k+r-\frac{5}{8}\right)a_k=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1}=\frac{(8k+8r-1)(8k+8r-5)a_k}{4(k+1+r)(4k-1+4r)}$$

- Recursion relation for $r=0$

  $$a_{k+1}=\frac{(8k-1)(8k-5)a_k}{4(k+1)(4k-1)}$$

- Solution for $r=0$

  $$\left[y(x)=\sum_{k=0}^{\infty}a_kx^k,a_{k+1}=\frac{(8k-1)(8k-5)a_k}{4(k+1)(4k-1)}\right]$$

- Recursion relation for $r=\frac{5}{4}$

  $$a_{k+1}=\frac{(8k+9)(8k+5)a_k}{4\left(k+\frac{9}{4}\right)(4k+4)}$$

- Solution for $r=\frac{5}{4}$

  $$\left[y(x)=\sum_{k=0}^{\infty}a_kx^{k+\frac{5}{4}},a_{k+1}=\frac{(8k+9)(8k+5)a_k}{4\left(k+\frac{9}{4}\right)(4k+4)}\right]$$

- Combine solutions and rename parameters

  $$\left[y(x)=\left(\sum_{k=0}^{\infty}a_kx^k\right)+\left(\sum_{k=0}^{\infty}b_kx^{k+\frac{5}{4}}\right),a_{k+1}=\frac{(8k-1)(8k-5)a_k}{4(k+1)(4k-1)},b_{k+1}=\frac{(8k+9)(8k+5)b_k}{4\left(k+\frac{9}{4}\right)(4k+4)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.010 (sec)
Leaf size : 55

```
dsolve(x^2*(1-4*x)*diff(diff(y(x),x),x)+(-1/4*x-x^2)*diff(y(x),x)-5/16*x*y(x) = 0,y(x),s
```

$$y = -\frac{2^{1/4}\left(c_1\sqrt{2}\left(x-\frac{\sqrt{1-4x}}{2}-\frac{1}{2}\right)\sqrt{1+\sqrt{1-4x}}-2c_2x^{5/4}\right)}{\left(1+\sqrt{1-4x}\right)^{5/4}}$$

Mathematica DSolve solution

Solving time : 4.189 (sec)
Leaf size : 129

```
DSolve[{x^2*(1-4*x)*D[y[x],{x,2}]+((1-(5/4))*x-(6-4*(5/4))*x^2)*D[y[x],x]+(5/4)*(1-(5/4))*x*y[:
```

$$y(x) \to \frac{\sqrt[4]{4x-1}\left(5c_1\left(\sqrt{4x-1}-i\right)^{5/4}+ic_2\left(\sqrt{4x-1}+i\right)^{5/4}\right)\exp\left(-\frac{1}{2}\int_1^x\left(\frac{2}{4K[1]-1}-\frac{1}{4K[1]}\right)dK[1]\right)}{5\sqrt[8]{\sqrt{4x-1}-i}\sqrt[8]{\sqrt{4x-1}+i}}$$

2.1.688 Problem 705

Internal problem ID [9860]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 705
Date solved : Monday, January 27, 2025 at 06:15:08 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$x^2y'' + \left(x^2+x\right)y' + (x-9)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.259 (sec)

Writing the ode as

$$x^2y'' + \left(x^2+x\right)y' + (x-9)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2+x \\ C &= x-9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2-2x+35}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2-2x+35 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2-2x+35}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1313: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{35}{4x^2} - \frac{1}{2x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{17}{2x^2} + \frac{17}{2x^3} - \frac{255}{4x^4} - \frac{833}{4x^5} + \frac{3213}{4x^6} + \frac{21709}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 2x + 35}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 35}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-2x + 35}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 2x + 35}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\
&= -\frac{1}{2} - \left(-\frac{5}{2}\right) \\
&= 2
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{5}{2x} + \left(\frac{1}{2}\right) \\
&= -\frac{5}{2x} + \frac{1}{2} \\
&= \frac{-5+x}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(-\frac{5}{2x} + \frac{1}{2}\right)(2x + a_1) + \left(\left(\frac{5}{2x^2}\right) + \left(-\frac{5}{2x} + \frac{1}{2}\right)^2 - \left(\frac{x^2 - 2x + 35}{4x^2}\right) \right) = 0$$

$$\frac{(-a_1 - 8)\,x - 2a_0 - 5a_1}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 20, a_1 = -8\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 8x + 20$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 8x + 20\right) \mathrm{e}^{\int \left(-\frac{5}{2x} + \frac{1}{2}\right) dx} \\
&= \left(x^2 - 8x + 20\right) \mathrm{e}^{\frac{x}{2} - \frac{5 \ln(x)}{2}} \\
&= \frac{\left(x^2 - 8x + 20\right) \mathrm{e}^{\frac{x}{2}}}{x^{5/2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x^2 + x}{x^2} \, dx} \\
&= z_1 e^{-\frac{x}{2} - \frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{2}}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^2 - 8x + 20}{x^3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2 + x}{x^2} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{-x - \ln(x)}}{\left(y_1\right)^2} \, dx \\
&= y_1 \left(-\frac{\left(x^3 + 9x^2 + 36x + 60\right) x \, \mathrm{e}^{-x - \ln(x)}}{x^2 - 8x + 20}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^2 - 8x + 20}{x^3}\right) + c_2 \left(\frac{x^2 - 8x + 20}{x^3} \left(-\frac{\left(x^3 + 9x^2 + 36x + 60\right) x \, \mathrm{e}^{-x - \ln(x)}}{x^2 - 8x + 20}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right)+(x^2+x)\left(\frac{d}{dx}y(x)\right)+(x-9)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(x-9)y(x)}{x^2}-\frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{x}+\frac{(x-9)y(x)}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{x+1}{x},P_3(x)=\frac{x-9}{x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=-9$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right)+x(x+1)\left(\frac{d}{dx}y(x)\right)+(x-9)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  ◦ Shift index using $k->k-m$

  $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(3+r)\,(-3+r)\,x^r+\left(\sum_{k=1}^{\infty}\left(a_k(k+r+3)\,(k+r-3)+a_{k-1}(k+r)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(3+r)(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-3, 3\}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+3)(k+r-3) + a_{k-1}(k+r) = 0$
- Shift index using $k- > k+1$

  $a_{k+1}(k+4+r)(k-2+r) + a_k(k+r+1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r+1)}{(k+4+r)(k-2+r)}$
- Recursion relation for $r = -3$; series terminates at $k = 2$

  $a_{k+1} = -\frac{a_k(k-2)}{(k+1)(k-5)}$
- Apply recursion relation for $k = 0$

  $a_1 = -\frac{2a_0}{5}$
- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{8}$
- Express in terms of a_0

  $a_2 = \frac{a_0}{20}$
- Terminating series solution of the ODE for $r = -3$. Use reduction of order to find the second

  $y(x) = a_0 \cdot \left(1 - \frac{2}{5}x + \frac{1}{20}x^2\right)$
- Recursion relation for $r = 3$

  $a_{k+1} = -\frac{a_k(k+4)}{(k+7)(k+1)}$
- Solution for $r = 3$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = -\frac{a_k(k+4)}{(k+7)(k+1)}\right]$
- Combine solutions and rename parameters

  $\left[y(x) = a_0 \cdot \left(1 - \frac{2}{5}x + \frac{1}{20}x^2\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), b_{k+1} = -\frac{b_k(4+k)}{(k+7)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 38

```
dsolve(x^2*diff(diff(y(x),x),x)+(x^2+x)*diff(y(x),x)+(x-9)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_2(x^3 + 9x^2 + 36x + 60)\,\mathrm{e}^{-x} + c_1(x^2 - 8x + 20)}{x^3}$$

Mathematica DSolve solution

Solving time : 0.501 (sec)
Leaf size : 96

```
DSolve[{x^2*D[y[x],{x,2}]+(x+x^2)*D[y[x],x]+(x-9)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->
```

$$y(x) \to \frac{(x^2 - 8x + 20)\exp\left(\int_1^x \frac{K[1]-5}{2K[1]}dK[1] - \frac{x}{2}\right)\left(c_2\int_1^x \frac{\exp\left(-2\int_1^{K[2]} \frac{K[1]-5}{2K[1]}dK[1]\right)}{(K[2]^2-8K[2]+20)^2}dK[2] + c_1\right)}{\sqrt{x}}$$

2.1.689 Problem 706

Internal problem ID [9861]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 706
Date solved : `Monday, January 27, 2025 at 06:15:09 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x(x+1)\,y' + (3x-1)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.264 (sec)

Writing the ode as

$$x^2y'' + \left(x^2+x\right)y' + (3x-1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2+x \\ C &= 3x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 10x + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 10x + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 10x + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1315: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{5}{2x} + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{5}{2x} - \frac{11}{2x^2} - \frac{55}{2x^3} - \frac{671}{4x^4} - \frac{4565}{4x^5} - \frac{33231}{4x^6} - \frac{253275}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 10x + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-10x + 3}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-10x + 3}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -10. Dividing this by leading coefficient in t which is 4 gives $-\frac{5}{2}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{5}{2}\right) - (0) \\ &= -\frac{5}{2}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{5}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 0\right) &&= \frac{5}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 10x + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{5}{2}$	$\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{5}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{5}{2} - \left(\frac{3}{2}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\ &= \frac{3}{2x} + (-)\left(\frac{1}{2}\right) \\ &= \frac{3}{2x} - \frac{1}{2} \\ &= -\frac{-3+x}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{2x} - \frac{1}{2}\right)(1) + \left(\left(-\frac{3}{2x^2}\right) + \left(\frac{3}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^2-10x+3}{4x^2}\right)\right) = 0$$

$$\frac{3+a_0}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -3\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = -3 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (-3+x)\, \mathrm{e}^{\int \left(\frac{3}{2x}-\frac{1}{2}\right)dx} \\ &= (-3+x)\, \mathrm{e}^{-\frac{x}{2}+\frac{3\ln(x)}{2}} \\ &= (-3+x)\, x^{3/2} \mathrm{e}^{-\frac{x}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^2+x}{x^2}\, dx} \\ &= z_1 e^{-\frac{x}{2}-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{2}}}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x\, \mathrm{e}^{-x}(-3+x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-x-\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{x}}{27\,(-3+x)} - \frac{\mathrm{Ei}_1(-x)}{6} - \frac{\mathrm{e}^{x}}{18x^2} - \frac{7\,\mathrm{e}^{x}}{54x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(x\, \mathrm{e}^{-x}(-3+x)\right) + c_2\left(x\, \mathrm{e}^{-x}(-3+x)\left(-\frac{\mathrm{e}^{x}}{27\,(-3+x)} - \frac{\mathrm{Ei}_1(-x)}{6} - \frac{\mathrm{e}^{x}}{18x^2} - \frac{7\,\mathrm{e}^{x}}{54x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right)+x(x+1)\left(\frac{d}{dx}y(x)\right)+(3x-1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(3x-1)y(x)}{x^2}-\frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{x}+\frac{(3x-1)y(x)}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{x+1}{x},P_3(x)=\frac{3x-1}{x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=-1$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right)+x(x+1)\left(\frac{d}{dx}y(x)\right)+(3x-1)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

    $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  ◦ Shift index using $k->k-m$

    $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-1+r)\,x^r+\left(\sum_{k=1}^{\infty}\left(a_k(k+r+1)\,(k+r-1)+a_{k-1}(k+2+r)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(1+r)(-1+r)=0$

- Values of r that satisfy the indicial equation
  $r\in\{-1,1\}$
- Each term in the series must be 0, giving the recursion relation
  $a_k(k+r+1)(k+r-1)+a_{k-1}(k+2+r)=0$
- Shift index using $k->k+1$
  $a_{k+1}(k+2+r)(k+r)+a_k(k+r+3)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=-\frac{a_k(k+r+3)}{(k+2+r)(k+r)}$
- Recursion relation for $r=-1$
  $a_{k+1}=-\frac{a_k(k+2)}{(k+1)(k-1)}$
- Series not valid for $r=-1$, division by 0 in the recursion relation at $k=1$
  $a_{k+1}=-\frac{a_k(k+2)}{(k+1)(k-1)}$
- Recursion relation for $r=1$
  $a_{k+1}=-\frac{a_k(k+4)}{(k+3)(k+1)}$
- Solution for $r=1$
  $\left[y(x)=\sum_{k=0}^{\infty}a_kx^{k+1},a_{k+1}=-\frac{a_k(k+4)}{(k+3)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 48

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(x+1)*diff(y(x),x)+(3*x-1)*y(x) = 0,y(x),singsol=all)
```

$$y=\frac{x^2c_2e^{-x}(x-3)\operatorname{Ei}_1(-x)+x^2c_1(x-3)e^{-x}+c_2(x^2-2x-1)}{x}$$

Mathematica DSolve solution

Solving time : 0.218 (sec)
Leaf size : 43

```
DSolve[{x^2*D[y[x],{x,2}]+x*(x+1)*D[y[x],x]+(3*x-1)*y[x]==0,{}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to e^{-x}(x-3)x\left(c_2\int_1^x \frac{e^{K[1]}}{(K[1]-3)^2K[1]^3}dK[1]+c_1\right)$$

2.1.690 Problem 707

Internal problem ID [9862]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 707
Date solved : `Monday, January 27, 2025 at 06:15:09 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - \left(x^2 + 4x\right) y' + 4y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.238 (sec)

Writing the ode as

$$x^2y'' + \left(-x^2 - 4x\right) y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 - 4x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 8x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 8x + 8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 8x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1317: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{x^2} + \frac{2}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{2}{x} - \frac{2}{x^2} + \frac{8}{x^3} - \frac{36}{x^4} + \frac{176}{x^5} - \frac{912}{x^6} + \frac{4928}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 8x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{8x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{8x + 8}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 8. Dividing this by leading coefficient in t which is 4 gives 2. Now b can be found.

$$\begin{aligned} b &= (2) - (0) \\ &= 2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{2}{\frac{1}{2}} - 0\right) &&= 2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{2}{\frac{1}{2}} - 0\right) &&= -2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 8x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	2	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+\right) \\ &= 2 - (2) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{2}{x} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2} + \frac{2}{x} \\ &= \frac{x+4}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2} + \frac{2}{x}\right)(0) + \left(\left(-\frac{2}{x^2}\right) + \left(\frac{1}{2} + \frac{2}{x}\right)^2 - \left(\frac{x^2+8x+8}{4x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{1}{2} + \frac{2}{x}\right) dx} \\ &= x^2 e^{\frac{x}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2-4x}{x^2}\,dx} \\
&= z_1 e^{\frac{x}{2}+2\ln(x)} \\
&= z_1 \left(x^2 \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^4 \mathrm{e}^x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-4x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x+4\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\mathrm{e}^{-x}}{3x^3}+\frac{\mathrm{e}^{-x}}{6x^2}-\frac{\mathrm{e}^{-x}}{6x}+\frac{\mathrm{Ei}_1(x)}{6}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^4\mathrm{e}^x\right) + c_2\left(x^4\mathrm{e}^x\left(-\frac{\mathrm{e}^{-x}}{3x^3}+\frac{\mathrm{e}^{-x}}{6x^2}-\frac{\mathrm{e}^{-x}}{6x}+\frac{\mathrm{Ei}_1(x)}{6}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) - (x^2+4x)\left(\frac{d}{dx}y(x)\right) + 4y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{4y(x)}{x^2} + \frac{(x+4)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) - \frac{(x+4)\left(\frac{d}{dx}y(x)\right)}{x} + \frac{4y(x)}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = -\frac{x+4}{x}, P_3(x) = \frac{4}{x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -4$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 4$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) - x(x+4)\left(\frac{d}{dx}y(x)\right) + 4y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  - ◦ Shift index using $k- >k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - ◦ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)\,(-4+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)\,(k+r-4) - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-4+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{1, 4\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_k(k+r-4) - a_{k-1}\right) = 0$

- Shift index using $k- >k+1$

  $(k+r)\left(a_{k+1}(k-3+r) - a_k\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k-3+r}$

- Recursion relation for $r = 1$

  $a_{k+1} = \frac{a_k}{k-2}$

- Series not valid for $r = 1$, division by 0 in the recursion relation at $k = 2$

  $a_{k+1} = \frac{a_k}{k-2}$

- Recursion relation for $r = 4$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 4$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+4}, a_{k+1} = \frac{a_k}{k+1}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 35

```
dsolve(x^2*diff(diff(y(x),x),x)-(x^2+4*x)*diff(y(x),x)+4*y(x) = 0,y(x),singsol=all)
```

$$y = x\left(\mathrm{Ei}_1\left(x\right) \mathrm{e}^{x} c_2 x^3 + \mathrm{e}^{x} x^3 c_1 - c_2\left(x^2 - x + 2\right)\right)$$

Mathematica DSolve solution

Solving time : 60.027 (sec)
Leaf size : 39

```
DSolve[{x^2*D[y[x],{x,2}]-(x^2+4*x)*D[y[x],x]+4*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to e^{x+4} x^4 \left(\int_1^x \frac{e^{-K[1]-4} c_1}{K[1]^4} dK[1] + c_2\right)$$

2.1.691 Problem 708

Internal problem ID [9863]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 708
Date solved : `Monday, January 27, 2025 at 06:15:10 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2y'' - (3x+2)\,y' + \frac{(2x-1)\,y}{x} = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.484 (sec)

Writing the ode as

$$2x^2y'' + (-3x-2)\,y' + \left(2-\frac{1}{x}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= -3x-2 \\ C &= 2-\frac{1}{x} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5x^2+36x+4}{16x^4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 5x^2+36x+4 \\ t &= 16x^4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5x^2 + 36x + 4}{16x^4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1319: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^4$. There is a pole at $x = 0$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned} \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) \end{aligned}$$

The partial fraction decomposition of r is

$$r = \frac{1}{4x^4} + \frac{9}{4x^3} + \frac{5}{16x^2}$$

There is pole in r at $x = 0$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{1}{2x^2} + \frac{9}{4x} - \frac{19}{4} + \frac{171x}{8} - \frac{475x^2}{4} + \frac{11799x^3}{16} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{1}{2x^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^2}$ is

$$a = \frac{1}{2}$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be $\frac{9}{4}$. Therefore

$$\begin{aligned} b &= \left(\frac{9}{4}\right) - (0) \\ &= \frac{9}{4} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{1}{2x^2} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &&= \frac{1}{2}\left(\frac{\frac{9}{4}}{\frac{1}{2}} + 2\right) &&= \frac{13}{4} \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &&= \frac{1}{2}\left(-\frac{\frac{9}{4}}{\frac{1}{2}} + 2\right) &&= -\frac{5}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{5x^2 + 36x + 4}{16x^4}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{5x^2 + 36x + 4}{16x^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	4	$\frac{1}{2x^2}$	$\frac{13}{4}$	$-\frac{5}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{4} - \left(-\frac{5}{4}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x^2} - \frac{5}{4x} + (-)\,(0) \\ &= -\frac{1}{2x^2} - \frac{5}{4x} \\ &= \frac{-2-5x}{4x^2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x^2} - \frac{5}{4x}\right)(1) + \left(\left(\frac{1}{x^3} + \frac{5}{4x^2}\right) + \left(-\frac{1}{2x^2} - \frac{5}{4x}\right)^2 - \left(\frac{5x^2+36x+4}{16x^4}\right) \right) = 0$$

$$\frac{-2+5a_0}{2x^2} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{ a_0 = \frac{2}{5} \right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{2}{5}$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= pe^{\int \omega \, dx} \\
&= \left(x + \frac{2}{5}\right) \mathrm{e}^{\int \left(-\frac{1}{2x^2} - \frac{5}{4x}\right)dx} \\
&= \left(x + \frac{2}{5}\right) \mathrm{e}^{\frac{1}{2x} - \frac{5\ln(x)}{4}} \\
&= \frac{(2 + 5x)\,\mathrm{e}^{\frac{1}{2x}}}{5x^{5/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3x-2}{2x^2}\, dx} \\
&= z_1 e^{-\frac{1}{2x} + \frac{3\ln(x)}{4}} \\
&= z_1 \left(x^{3/4} \mathrm{e}^{-\frac{1}{2x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{2 + 5x}{5\sqrt{x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x-2}{2x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{1}{x} + \frac{3\ln(x)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{25\,\mathrm{e}^{-\frac{1}{x} + \frac{3\ln(x)}{2}}\, x}{(2 + 5x)^2} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{2 + 5x}{5\sqrt{x}}\right) + c_2 \left(\frac{2 + 5x}{5\sqrt{x}} \left(\int \frac{25\,\mathrm{e}^{-\frac{1}{x} + \frac{3\ln(x)}{2}}\, x}{(2 + 5x)^2} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric sol
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returr
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.108 (sec)
Leaf size : 35

```
dsolve(2*x^2*diff(diff(y(x),x),x)-(2+3*x)*diff(y(x),x)+(-1+2*x)/x*y(x) = 0,y(x),singsc
```

$$y = \frac{c_2 \mathrm{e}^{-\frac{1}{x}} \operatorname{hypergeom}\left([2], \left[-\frac{1}{2}\right], \frac{1}{x}\right) x^{5/2} + 5c_1 x + 2c_1}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.393 (sec)
Leaf size : 65

```
DSolve[{2*x^2*D[y[x],{x,2}]-(3*x+2)*D[y[x],x]+(2*x-1)/x*y[x]==0,{}},y[x],x,IncludeSingularSo
```

$$y(x) \to \frac{\sqrt{e}(5x+2)\left(c_2 \int_1^x \frac{25e^{-\frac{5}{2}-\frac{1}{K[1]}} K[1]^{5/2}}{(5K[1]+2)^2} dK[1] + c_1\right)}{5\sqrt{x}}$$

2.1.692 Problem 709

Internal problem ID [9864]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 709
Date solved : Monday, January 27, 2025 at 06:15:11 PM
CAS classification : [_Jacobi]

Solve

$$x(1-x)\,y''+\left(\frac{3}{2}-2x\right)y'-\frac{y}{4}=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.215 (sec)

Writing the ode as

$$\left(-x^2+x\right)y''+\left(\frac{3}{2}-2x\right)y'-\frac{y}{4}=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+x \\ B &= \frac{3}{2}-2x \\ C &= -\frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-4x^2+4x-3}{16\left(x^2-x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4x^2+4x-3 \\ t &= 16\left(x^2-x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-4x^2+4x-3}{16\left(x^2-x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1320: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2-x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{8x} + \frac{1}{-8+8x} - \frac{3}{16x^2} - \frac{3}{16\left(-1+x\right)^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-4x^2 + 4x - 3}{16\,(x^2 - x)^2}$$

Since the gcd$(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-4x^2 + 4x - 3}{16\,(x^2 - x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
1	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{4x} + \frac{1}{-4+4x} + (-)\,(0) \\
&= \frac{1}{4x} + \frac{1}{-4+4x} \\
&= \frac{2x-1}{4x\,(-1+x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{4x} + \frac{1}{-4+4x}\right)(0) + \left(\left(-\frac{1}{4x^2} - \frac{1}{4\,(-1+x)^2}\right) + \left(\frac{1}{4x} + \frac{1}{-4+4x}\right)^2 - \left(\frac{-4x^2+4x-3}{16\,(x^2-x)^2}\right)\right) = 0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(\frac{1}{4x} + \frac{1}{-4+4x}\right) dx} \\
&= (x(-1+x))^{1/4}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{\frac{3}{2}-2x}{-x^2+x}\, dx} \\
&= z_1 e^{-\frac{\ln(-1+x)}{4} - \frac{3\ln(x)}{4}} \\
&= z_1 \left(\frac{1}{(-1+x)^{1/4}\, x^{3/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(x(-1+x))^{1/4}}{(-1+x)^{1/4}\, x^{3/4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{\frac{3}{2}-2x}{-x^2+x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{\ln(-1+x)}{2} - \frac{3\ln(x)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\ln\left(-\frac{1}{2} + x + \sqrt{x^2 - x}\right)\right)
\end{aligned}
$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x(-1+x))^{1/4}}{(-1+x)^{1/4} x^{3/4}} \right) + c_2 \left(\frac{(x(-1+x))^{1/4}}{(-1+x)^{1/4} x^{3/4}} \left(\ln \left(-\frac{1}{2} + x + \sqrt{x^2 - x} \right) \right) \right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x(1-x)\left(\frac{d^2}{dx^2}y(x)\right) + \left(\frac{3}{2} - 2x\right)\left(\frac{d}{dx}y(x)\right) - \frac{y(x)}{4} = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{y(x)}{4x(x-1)} - \frac{(4x-3)\left(\frac{d}{dx}y(x)\right)}{2x(x-1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(4x-3)\left(\frac{d}{dx}y(x)\right)}{2x(x-1)} + \frac{y(x)}{4x(x-1)} = 0$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{4x-3}{2x(x-1)}, P_3(x) = \frac{1}{4x(x-1)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = \frac{3}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

  - $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x(x-1)\left(\frac{d^2}{dx^2}y(x)\right) + (8x-6)\left(\frac{d}{dx}y(x)\right) + y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..1$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 1..2$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$

  - Shift index using $k- >k+2-m$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0r(1+2r)\,x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)\,(2k+3+2r) + a_k(2k+2r+1)^2\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{0, -\frac{1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(2k+2r+1)^2 - 4\left(k+r+\frac{3}{2}\right)(k+1+r)\,a_{k+1} = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(2k+2r+1)^2}{2(2k+3+2r)(k+1+r)}$
- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k(2k+1)^2}{2(2k+3)(k+1)}$
- Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k(2k+1)^2}{2(2k+3)(k+1)}\right]$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+1} = \frac{2a_k k^2}{(2k+2)\left(k+\frac{1}{2}\right)}$
- Solution for $r = -\frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = \frac{2a_k k^2}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{2}}\right), a_{k+1} = \frac{a_k(2k+1)^2}{2(2k+3)(k+1)}, b_{k+1} = \frac{2b_k k^2}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.025 (sec)
Leaf size : 32

```
dsolve(x*(1-x)*diff(diff(y(x),x),x)+(3/2-2*x)*diff(y(x),x)-1/4*y(x) = 0,y(x),singsol=all
```

$$y = \frac{-c_2 \ln(2) + c_2 \ln\left(2x - 1 + 2\sqrt{(x-1)\,x}\right) + c_1}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.273 (sec)
Leaf size : 104

```
DSolve[{x*(1-x)*D[y[x],{x,2}]+(3/2-2*x)*D[y[x],x]-1/4*y[x]==0,{}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to \exp\left(\int_1^x \frac{1}{4}\left(\frac{1}{K[1]} + \frac{1}{K[1]-1}\right) dK[1] - \frac{1}{2}\int_1^x \frac{1}{2}\left(\frac{3}{K[2]} + \frac{1}{K[2]-1}\right) dK[2]\right)\left(c_2 \int_1^x \exp\left(-2\int_1^{K[3]} \frac{1-2K[1]}{4K[1]-4K[1]^2} dK[1]\right) dK[3] + c_1\right)$$

2.1.693 Problem 710

Internal problem ID [9865]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 710
Date solved : `Monday, January 27, 2025 at 06:15:11 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x(1-x)\,y''+xy'-y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.228 (sec)

Writing the ode as

$$\left(-2x^2+2x\right)y''+xy'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-2x^2+2x\\B&=x\\C&=-1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-3x+8}{16x\left(-1+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=-3x+8\\t&=16x(-1+x)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-3x+8}{16x\left(-1+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1322: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x(-1+x)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16(-1+x)^2} - \frac{1}{2(-1+x)} + \frac{1}{2x}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x+8}{16x\left(-1+x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x+8}{16x\left(-1+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1
1	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{3}{4} - \left(\frac{3}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{x} - \frac{1}{4\left(-1+x\right)} + (0) \\
&= \frac{1}{x} - \frac{1}{4\left(-1+x\right)} \\
&= \frac{1}{x} - \frac{1}{-4+4x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x} - \frac{1}{4(-1+x)}\right)(0) + \left(\left(-\frac{1}{x^2} + \frac{1}{4(-1+x)^2}\right) + \left(\frac{1}{x} - \frac{1}{4(-1+x)}\right)^2 - \left(\frac{-3x+8}{16x(-1+x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(\frac{1}{x} - \frac{1}{4(-1+x)}\right) dx} \\
&= \frac{x}{(-1+x)^{1/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{-2x^2+2x}\, dx} \\
&= z_1 e^{\frac{\ln(-1+x)}{4}} \\
&= z_1\left((-1+x)^{1/4}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{-2x^2+2x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{\ln(-1+x)}{2}}}{(y_1)^2}\, dx \\
&= y_1\left(-\frac{\sqrt{-1+x}}{x} + \arctan\left(\sqrt{-1+x}\right)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2\left(x\left(-\frac{\sqrt{-1+x}}{x} + \arctan\left(\sqrt{-1+x}\right)\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 25

```
dsolve(2*x*(1-x)*diff(diff(y(x),x),x)+diff(y(x),x)*x-y(x) = 0,y(x),singsol=all)
```

$$y = c_1 x + \arctan\left(\sqrt{x-1}\right) x c_2 - \sqrt{x-1}\, c_2$$

Mathematica DSolve solution

Solving time : `0.411` (sec)
Leaf size : 75

```
DSolve[{2*x*(1-x)*D[y[x],{x,2}]+x*D[y[x],x]-y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tru
```

$$y(x) \to \sqrt[4]{2-2x} \exp\left(\int_1^x \left(\frac{1}{K[1]} + \frac{1}{4-4K[1]}\right) dK[1]\right) \left(c_2 \int_1^x \exp\left(-2\int_1^{K[2]} \left(\frac{1}{K[1]} + \frac{1}{4-4K[1]}\right) dK[1]\right) dK[2] + c_1\right)$$

2.1.694 Problem 711

Internal problem ID [9866]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 711
Date solved : `Monday, January 27, 2025 at 06:15:12 PM`
CAS classification : `[_Jacobi]`

Solve

$$2x(1-x)\,y''+(1-11x)\,y'-10y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.209 (sec)

Writing the ode as

$$\left(-2x^2+2x\right)y''+(1-11x)\,y'-10y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=-2x^2+2x\\ B&=1-11x \\ C&=-10 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-3x^2+66x-3}{16\left(x^2-x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=-3x^2+66x-3\\ t&=16\left(x^2-x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-3x^2+66x-3}{16\left(x^2-x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1323: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{15}{4\,(-1+x)} + \frac{15}{4\,(-1+x)^2} + \frac{15}{4x} - \frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 + 66x - 3}{16\left(x^2 - x\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x^2 + 66x - 3}{16\left(x^2 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
1	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= \frac{1}{4} - \left(-\frac{3}{4}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{3}{4x} - \frac{3}{2(-1 + x)} + (-)(0) \\
&= \frac{3}{4x} - \frac{3}{2(-1 + x)} \\
&= -\frac{3(x + 1)}{4x(-1 + x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{4x} - \frac{3}{2(-1+x)}\right)(1) + \left(\left(-\frac{3}{4x^2} + \frac{3}{2(-1+x)^2}\right) + \left(\frac{3}{4x} - \frac{3}{2(-1+x)}\right)^2 - \left(\frac{-3x^2+66x-3}{16(x^2-x)^2}\right.\right.$$
$$\frac{-3+3a_0}{2x(-1+x)}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (x+1)\, e^{\int \left(\frac{3}{4x} - \frac{3}{2(-1+x)}\right) dx} \\
&= (x+1)\, e^{\frac{3\ln(x)}{4} - \frac{3\ln(-1+x)}{2}} \\
&= \frac{(x+1)\, x^{3/4}}{(-1+x)^{3/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1-11x}{-2x^2+2x}\, dx} \\
&= z_1 e^{-\frac{\ln(x)}{4} - \frac{5\ln(-1+x)}{2}} \\
&= z_1 \left(\frac{1}{x^{1/4}(-1+x)^{5/2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}\,(x+1)}{(-1+x)^4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1-11x}{-2x^2+2x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x)}{2} - 5\ln(-1+x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{2(x^2+6x+1)(-1+x)^5\, e^{-\frac{\ln(x)}{2} - 5\ln(-1+x)}}{x+1}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{\sqrt{x}\,(x+1)}{(-1+x)^4}\right) + c_2\left(\frac{\sqrt{x}\,(x+1)}{(-1+x)^4}\left(\frac{2(x^2+6x+1)\,(-1+x)^5\,\mathrm{e}^{-\frac{\ln(x)}{2}-5\ln(-1+x)}}{x+1}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x(1-x)\left(\frac{d^2}{dx^2}y(x)\right) + (1-11x)\left(\frac{d}{dx}y(x)\right) - 10y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{5y(x)}{x(x-1)} - \frac{(11x-1)\left(\frac{d}{dx}y(x)\right)}{2x(x-1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(11x-1)\left(\frac{d}{dx}y(x)\right)}{2x(x-1)} + \frac{5y(x)}{x(x-1)} = 0$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{11x-1}{2x(x-1)}, P_3(x) = \frac{5}{x(x-1)}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = \frac{1}{2}$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$
  - $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x(x-1)\left(\frac{d^2}{dx^2}y(x)\right) + (11x-1)\left(\frac{d}{dx}y(x)\right) + 10y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..1$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k- >k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 1..2$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$
  - Shift index using $k- >k+2-m$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0 r(-1+2r)\,x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(2k+1+2r) + a_k(2k+2r+5)\,(k+r+2)\right) x^{k-}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $-2(k+1+r)\left(k+r+\frac{1}{2}\right)a_{k+1} + 2(k+r+2)\,a_k\left(k+r+\frac{5}{2}\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{(k+r+2)a_k(2k+2r+5)}{(k+1+r)(2k+1+2r)}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{(k+2)a_k(2k+5)}{(k+1)(2k+1)}$

- Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{(k+2)a_k(2k+5)}{(k+1)(2k+1)}\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = \frac{\left(k+\frac{5}{2}\right)a_k(2k+6)}{\left(k+\frac{3}{2}\right)(2k+2)}$

- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{\left(k+\frac{5}{2}\right)a_k(2k+6)}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = \frac{(k+2)a_k(2k+5)}{(k+1)(2k+1)}, b_{k+1} = \frac{\left(k+\frac{5}{2}\right)b_k(2k+6)}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.025` (sec)
Leaf size : 29

```
dsolve(2*x*(1-x)*diff(diff(y(x),x),x)+(1-11*x)*diff(y(x),x)-10*y(x) = 0,y(x),singsol=a
```

$$y = \frac{c_1(x^2+6x+1) + c_2\sqrt{x}\,(x+1)}{(x-1)^4}$$

Mathematica DSolve solution

Solving time : 0.534 (sec)
Leaf size : 119

```
DSolve[{2*x*(1-x)*D[y[x],{x,2}]+(1-11*x)*D[y[x],x]-10*y[x]==0,{}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to (x+1)\exp\left(\int_1^x \frac{3K[1]+3}{4K[1]-4K[1]^2}dK[1]-\frac{1}{2}\int_1^x\left(\frac{1}{2K[2]}+\frac{5}{K[2]-1}\right)dK[2]\right)\left(c_2\int_1^x \frac{\exp\left(-2\int_1^{K[3]}\frac{3K[1]+3}{4K[1]-4K[1]^2}dK[1]\right)}{(K[3]+1)^2}dK[3]+c_1\right)$$

2.1.695 Problem 712

Internal problem ID [9867]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 712
Date solved : `Monday, January 27, 2025 at 06:15:13 PM`
CAS classification : `[_Jacobi]`

Solve

$$x(1-x)\,y''+\frac{(1-2x)\,y'}{3}+\frac{20y}{9}=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.223 (sec)

Writing the ode as

$$\left(-x^2+x\right)y''+\left(-\frac{2x}{3}+\frac{1}{3}\right)y'+\frac{20y}{9}=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+x \\ B &= -\frac{2x}{3}+\frac{1}{3} \\ C &= \frac{20}{9} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{72x^2-72x-5}{36\left(x^2-x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 72x^2-72x-5 \\ t &= 36\left(x^2-x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{72x^2 - 72x - 5}{36\left(x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1325: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{36\left(-1+x\right)^2} - \frac{41}{18x} - \frac{5}{36x^2} + \frac{41}{18\left(-1+x\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6} \end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{72x^2 - 72x - 5}{36\left(x^2 - x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{72x^2 - 72x - 5}{36\left(x^2 - x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{5}{6}$	$\frac{1}{6}$
1	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 2$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{6x} + \frac{5}{6(-1+x)} + (0) \\
&= \frac{1}{6x} + \frac{5}{6(-1+x)} \\
&= \frac{-1+6x}{6x(-1+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{6x} + \frac{5}{6(-1+x)}\right)(1) + \left(\left(-\frac{1}{6x^2} - \frac{5}{6(-1+x)^2}\right) + \left(\frac{1}{6x} + \frac{5}{6(-1+x)}\right)^2 - \left(\frac{72x^2-72x-5}{36(x^2-x)^2}\right)\right) \\ \frac{-1-6a_0}{3x(-1+x)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{1}{6}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = -\frac{1}{6} + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(-\frac{1}{6} + x\right) e^{\int \left(\frac{1}{6x} + \frac{5}{6(-1+x)}\right) dx} \\
&= \left(-\frac{1}{6} + x\right) e^{\frac{\ln(x)}{6} + \frac{5\ln(-1+x)}{6}} \\
&= \left(-\frac{1}{6} + x\right) x^{1/6}(-1+x)^{5/6}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-\frac{2x}{3}+\frac{1}{3}}{-x^2+x} \, dx} \\
&= z_1 e^{-\frac{\ln(x(-1+x))}{6}} \\
&= z_1 \left(\frac{1}{(x(-1+x))^{1/6}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(-1+6x)\, x^{1/6}(-1+x)^{5/6}}{6(x(-1+x))^{1/6}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-\frac{2x}{3}+\frac{1}{3}}{-x^2+x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{\ln(x(-1+x))}{3}}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{54x^{2/3}(-5+6x)}{5(-1+6x)(-1+x)^{2/3}}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{(-1+6x)\, x^{1/6}(-1+x)^{5/6}}{6(x(-1+x))^{1/6}}\right) + c_2 \left(\frac{(-1+6x)\, x^{1/6}(-1+x)^{5/6}}{6(x(-1+x))^{1/6}} \left(-\frac{54x^{2/3}(-5+6x)}{5(-1+6x)(-1+x)^{2/3}}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x(1-x)\left(\frac{d^2}{dx^2}y(x)\right) + \frac{(-2x+1)\left(\frac{d}{dx}y(x)\right)}{3} + \frac{20y(x)}{9} = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{20y(x)}{9x(x-1)} - \frac{(2x-1)\left(\frac{d}{dx}y(x)\right)}{3x(x-1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(2x-1)\left(\frac{d}{dx}y(x)\right)}{3x(x-1)} - \frac{20y(x)}{9x(x-1)} = 0$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{2x-1}{3x(x-1)}, P_3(x) = -\frac{20}{9x(x-1)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x=0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{1}{3}$

  - $x^2 \cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $9x(x-1)\left(\frac{d^2}{dx^2}y(x)\right) + (6x-3)\left(\frac{d}{dx}y(x)\right) - 20y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..1$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  ◦ Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 1..2$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

  ◦ Shift index using $k- >k+2-m$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-3a_0 r(-2+3r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} (-3a_{k+1}(k+1+r)\,(3k+1+3r) + a_k(3k+3r+4)\,(3k+3r-5))\, x$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-3r(-2+3r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{2}{3}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $-9(k+1+r)\left(k+r+\frac{1}{3}\right) a_{k+1} + 9\left(k+r-\frac{5}{3}\right)\left(k+\frac{4}{3}+r\right) a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{(3k+3r-5)(3k+3r+4)a_k}{3(k+1+r)(3k+1+3r)}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{(3k-5)(3k+4)a_k}{3(k+1)(3k+1)}$

- Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{(3k-5)(3k+4)a_k}{3(k+1)(3k+1)}\right]$

- Recursion relation for $r = \frac{2}{3}$; series terminates at $k = 1$

  $a_{k+1} = \frac{(3k-3)(3k+6)a_k}{3\left(k+\frac{5}{3}\right)(3k+3)}$

- Apply recursion relation for $k = 0$

  $a_1 = -\frac{6a_0}{5}$

- Terminating series solution of the ODE for $r = \frac{2}{3}$. Use reduction of order to find the second linea

  $y(x) = a_0 \cdot \left(-\frac{6x}{5}+1\right)$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^k\right) + b_0 \cdot \left(-\frac{6x}{5}+1\right), a_{k+1} = \frac{(3k-5)(3k+4)a_k}{3(k+1)(3k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 27

```
dsolve(x*(1-x)*diff(diff(y(x),x),x)+1/3*(1-2*x)*diff(y(x),x)+20/9*y(x) = 0,y(x),singsol
```

$$y = c_1(6x-5)\,x^{2/3} + c_2(6x-1)\,(x-1)^{2/3}$$

Mathematica DSolve solution

Solving time : 0.494 (sec)
Leaf size : 93

```
DSolve[{x*(1-x)*D[y[x],{x,2}]+1/3*(1-2*x)*D[y[x],x]+20/9*y[x]==0,{}},y[x],x,IncludeSingularS
```

$$y(x) \to \frac{\left(3c_2(x-1)\,\text{Gamma}\left(\frac{4}{3}\right)Q_1^{\frac{2}{3}}(2x-1) + c_1(5-6x)(1-x)^{2/3}\sqrt[3]{x}\right)\exp\left(\int_1^x \frac{1-2K[1]}{3K[1]-3K[1]^2}dK[1]\right)}{3(x-1)\,\text{Gamma}\left(\frac{4}{3}\right)}$$

2.1.696 Problem 713

Internal problem ID [9868]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 713
Date solved : `Monday, January 27, 2025 at 06:15:13 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4y'' + \frac{3(-x^2+2)\,y}{(-x^2+1)^2} = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.187 (sec)

Writing the ode as

$$4y'' + \frac{(-3x^2+6)\,y}{(x^2-1)^2} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4 \\ B &= 0 \\ C &= \frac{-3x^2+6}{(x^2-1)^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^2-6}{4\,(x^2-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3x^2 - 6 \\ t &= 4(x^2-1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^2 - 6}{4(x^2 - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1327: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16(x+1)^2} - \frac{3}{16(x-1)^2} - \frac{9}{16(x+1)} + \frac{9}{16(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x^2-6}{4\left(x^2-1\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3x^2-6}{4\left(x^2-1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{4}$	$\frac{1}{4}$
-1	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{2}$ then

$$\begin{aligned}d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right)\\ &= \frac{3}{2} - \left(\frac{3}{2}\right)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{3}{4(x-1)} + \frac{3}{4(x+1)} + (0) \\
&= \frac{3}{4(x-1)} + \frac{3}{4(x+1)} \\
&= \frac{3x}{2x^2-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{4(x-1)} + \frac{3}{4(x+1)}\right)(0) + \left(\left(-\frac{3}{4(x-1)^2} - \frac{3}{4(x+1)^2}\right) + \left(\frac{3}{4(x-1)} + \frac{3}{4(x+1)}\right)^2 - \left(\frac{3}{4(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{3}{4(x-1)} + \frac{3}{4(x+1)}\right) dx} \\
&= \left(x^2-1\right)^{3/4}
\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}$$

Since $B=0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= \left(x^2-1\right)^{3/4}
\end{aligned}$$

Which simplifies to

$$y_1 = \left(x^2-1\right)^{3/4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B=0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\
&= \left(x^2-1\right)^{3/4} \int \frac{1}{\left(x^2-1\right)^{3/2}} \, dx \\
&= \left(x^2-1\right)^{3/4} \left(-\frac{(x-1)(x+1)x}{\left(x^2-1\right)^{3/2}}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\left(x^2-1\right)^{3/4} \right) + c_2 \left(\left(x^2-1\right)^{3/4} \left(-\frac{(x-1)(x+1)x}{\left(x^2-1\right)^{3/2}} \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4\frac{d^2}{dx^2}y(x) + \frac{3(-x^2+2)y(x)}{(-x^2+1)^2} = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{3(x^2-2)y(x)}{4(x^2-1)^2}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) - \frac{3(x^2-2)y(x)}{4(x^2-1)^2} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = 0, P_3(x) = -\frac{3(x^2-2)}{4(x^2-1)^2}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1} = 0$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1} = \frac{3}{16}$

  ◦ $x=-1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -1$

- Multiply by denominators

  $4(x^2-1)^2\left(\frac{d^2}{dx^2}y(x)\right) + (-3x^2+6)\,y(x) = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(4u^4-16u^3+16u^2)\left(\frac{d^2}{du^2}y(u)\right) + (-3u^2+6u+3)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot y(u)$ to series expansion for $m=0..2$

    $u^m\cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

  ◦ Shift index using $k->k-m$

    $u^m\cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  ◦ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=2..4$

$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+4r)(-3+4r)\, u^r + (a_1(3+4r)(1+4r) - 2a_0(8r^2-8r-3))\, u^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(4k+4r -\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+4r)(-3+4r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{4}, \frac{3}{4}\right\}$
- Each term must be 0

  $a_1(3+4r)(1+4r) - 2a_0(8r^2-8r-3) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = \frac{2a_0(8r^2-8r-3)}{16r^2+16r+3}$
- Each term in the series must be 0, giving the recursion relation

  $4(4a_k + a_{k-2} - 4a_{k-1})\, k^2 + 4(2(4a_k + a_{k-2} - 4a_{k-1})\, r - 4a_k - 5a_{k-2} + 12a_{k-1})\, k + 4(4a_k + a_{k-2}$
- Shift index using $k->k+2$

  $4(4a_{k+2} + a_k - 4a_{k+1})(k+2)^2 + 4(2(4a_{k+2} + a_k - 4a_{k+1})\, r - 4a_{k+2} - 5a_k + 12a_{k+1})(k+2) + 4$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4k^2a_k - 16k^2a_{k+1} + 8kra_k - 32kra_{k+1} + 4r^2a_k - 16r^2a_{k+1} - 4ka_k - 16ka_{k+1} - 4ra_k - 16ra_{k+1} - 3a_k + 6a_{k+1}}{16k^2+32kr+16r^2+48k+48r+35}$
- Recursion relation for $r = \frac{1}{4}$

  $a_{k+2} = -\frac{4k^2a_k - 16k^2a_{k+1} - 2ka_k - 24ka_{k+1} - \frac{15}{4}a_k + a_{k+1}}{16k^2+56k+48}$
- Solution for $r = \frac{1}{4}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{4}}, a_{k+2} = -\frac{4k^2a_k - 16k^2a_{k+1} - 2ka_k - 24ka_{k+1} - \frac{15}{4}a_k + a_{k+1}}{16k^2+56k+48}, a_1 = -\frac{9a_0}{8}\right]$
- Revert the change of variables $u = x+1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^{k+\frac{1}{4}}, a_{k+2} = -\frac{4k^2a_k - 16k^2a_{k+1} - 2ka_k - 24ka_{k+1} - \frac{15}{4}a_k + a_{k+1}}{16k^2+56k+48}, a_1 = -\frac{9a_0}{8}\right]$
- Recursion relation for $r = \frac{3}{4}$

  $a_{k+2} = -\frac{4k^2a_k - 16k^2a_{k+1} + 2ka_k - 40ka_{k+1} - \frac{15}{4}a_k - 15a_{k+1}}{16k^2+72k+80}$
- Solution for $r = \frac{3}{4}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{3}{4}}, a_{k+2} = -\frac{4k^2a_k - 16k^2a_{k+1} + 2ka_k - 40ka_{k+1} - \frac{15}{4}a_k - 15a_{k+1}}{16k^2+72k+80}, a_1 = -\frac{3a_0}{8}\right]$
- Revert the change of variables $u = x+1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^{k+\frac{3}{4}}, a_{k+2} = -\frac{4k^2a_k - 16k^2a_{k+1} + 2ka_k - 40ka_{k+1} - \frac{15}{4}a_k - 15a_{k+1}}{16k^2+72k+80}, a_1 = -\frac{3a_0}{8}\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k(x+1)^{k+\frac{1}{4}}\right) + \left(\sum_{k=0}^{\infty} b_k(x+1)^{k+\frac{3}{4}}\right), a_{k+2} = -\frac{4k^2a_k - 16k^2a_{k+1} - 2ka_k - 24ka_{k+1} - \frac{15}{4}a_k + a_k}{16k^2+56k+48}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 24

```
dsolve(4*diff(diff(y(x),x),x)+3*(-x^2+2)/(-x^2+1)^2*y(x) = 0,y(x),singsol=all)
```

$$y = c_1 \left(x^2-1\right)^{3/4}+c_2 \left(x^2-1\right)^{1/4} x$$

Mathematica DSolve solution

Solving time : 0.04 (sec)
Leaf size : 51

```
DSolve[{4*D[y[x],{x,2}]+3*(2-x^2)/(1-x^2)^2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to \sqrt{x^2-1}\left(c_2 Q_{\frac{1}{2}}^{\frac{1}{2}}(x)+\frac{\sqrt{\frac{2}{\pi}}c_1 x}{\sqrt[4]{1-x^2}}\right)$$

2.1.697 Problem 714

Internal problem ID [9869]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 714
Date solved : Monday, January 27, 2025 at 06:15:14 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$u'' - \frac{2u'}{x} - a^2 u = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.276 (sec)

Writing the ode as

$$u'' - \frac{2u'}{x} - a^2 u = 0 \tag{1}$$
$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -\frac{2}{x} \\ C &= -a^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = u e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2x^2 + 2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= a^2x^2 + 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{a^2x^2+2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1329: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2-2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2} + a^2$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx a + \frac{1}{a\,x^2} - \frac{1}{2a^3x^4} + \frac{1}{2a^5x^6} - \frac{5}{8a^7x^8} + \frac{7}{8a^9x^{10}} - \frac{21}{16a^{11}x^{12}} + \frac{33}{16a^{13}x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = a$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= a \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = a^2$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{a^2x^2 + 2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (a^2) + \left(\frac{2}{x^2}\right) \\ &= \frac{2}{x^2} + a^2 \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= a \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{0}{a} - 0\right) = 0 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{0}{a} - 0\right) = 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{a^2x^2+2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	a	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=0$ then

$$\begin{aligned}d&=\alpha_\infty^- - \left(\alpha_{c_1}^-\right)\\&=0-(-1)\\&=1\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega&=\left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^-}{x-c_1}\right)+(-)[\sqrt{r}]_\infty\\&=-\frac{1}{x}+(-)\left(a\right)\\&=-\frac{1}{x}-a\\&=\frac{-ax-1}{x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=x+a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{x}-a\right)(1)+\left(\left(\frac{1}{x^2}\right)+\left(-\frac{1}{x}-a\right)^2-\left(\frac{a^2x^2+2}{x^2}\right)\right)=0$$
$$\frac{2aa_0-2}{x}=0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0=\frac{1}{a}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{1}{a}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x + \frac{1}{a}\right) e^{\int \left(-\frac{1}{x} - a\right) dx} \\ &= \left(x + \frac{1}{a}\right) e^{-ax - \ln(x)} \\ &= \frac{(ax+1)\, e^{-ax}}{ax} \end{aligned}$$

The first solution to the original ode in u is found from

$$\begin{aligned} u_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-\frac{2}{x}}{1} \, dx} \\ &= z_1 e^{\ln(x)} \\ &= z_1(x) \end{aligned}$$

Which simplifies to

$$u_1 = \frac{(ax+1)\, e^{-ax}}{a}$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{u_1^2} \, dx$$

Substituting gives

$$\begin{aligned} u_2 &= u_1 \int \frac{e^{\int -\frac{-\frac{2}{x}}{1} \, dx}}{(u_1)^2} \, dx \\ &= u_1 \int \frac{e^{2\ln(x)}}{(u_1)^2} \, dx \\ &= u_1 \left(\frac{(ax-1)\, e^{2ax}}{2a\,(ax+1)}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} u &= c_1 u_1 + c_2 u_2 \\ &= c_1 \left(\frac{(ax+1)\, e^{-ax}}{a}\right) + c_2 \left(\frac{(ax+1)\, e^{-ax}}{a} \left(\frac{(ax-1)\, e^{2ax}}{2a\,(ax+1)}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}u(x) - \frac{2\left(\frac{d}{dx}u(x)\right)}{x} - a^2 u(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}u(x)$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{2}{x}, P_3(x) = -a^2\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$
  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $-a^2 u(x)\, x + \left(\frac{d^2}{dx^2}u(x)\right) x - 2\frac{d}{dx}u(x) = 0$

- Assume series solution for $u(x)$

  $u(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x \cdot u(x)$ to series expansion

    $x \cdot u(x) = \sum_{k=0}^{\infty} a_k x^{k+r+1}$
  - Shift index using $k- >k-1$

    $x \cdot u(x) = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$
  - Convert $\frac{d}{dx}u(x)$ to series expansion

    $\frac{d}{dx}u(x) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$
  - Shift index using $k- >k+1$

    $\frac{d}{dx}u(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$
  - Convert $x \cdot \left(\frac{d^2}{dx^2}u(x)\right)$ to series expansion

    $x \cdot \left(\frac{d^2}{dx^2}u(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-1}$
  - Shift index using $k- >k+1$

    $x \cdot \left(\frac{d^2}{dx^2}u(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-3+r)\, x^{-1+r} + a_1(1+r)(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)(k-2+r) - a^2 a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{0, 3\}$

- Each term must be 0

  $a_1(1+r)(-2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)(k-2+r) - a^2 a_{k-1} = 0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)(k+r-1) - a^2 a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a^2 a_k}{(k+2+r)(k+r-1)}$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{a^2 a_k}{(k+2)(k-1)}$

- Solution for $r = 0$

  $\left[u(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a^2 a_k}{(k+2)(k-1)}, -2a_1 = 0\right]$

- Recursion relation for $r = 3$

  $a_{k+2} = \frac{a^2 a_k}{(k+5)(k+2)}$

- Solution for $r = 3$

  $\left[u(x) = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = \frac{a^2 a_k}{(k+5)(k+2)}, 4a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[u(x) = \left(\sum_{k=0}^{\infty} b_k x^k\right) + \left(\sum_{k=0}^{\infty} c_k x^{k+3}\right), b_{k+2} = \frac{a^2 b_k}{(k+2)(k-1)}, -2b_1 = 0, c_{k+2} = \frac{a^2 c_k}{(5+k)(k+2)}, 4c_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 28

```
dsolve(diff(diff(u(x),x),x)-2/x*diff(u(x),x)-a^2*u(x) = 0,u(x),singsol=all)
```

$$u = c_1 e^{ax}(ax-1) + c_2 e^{-ax}(ax+1)$$

Mathematica DSolve solution

Solving time : 0.093 (sec)
Leaf size : 68

```
DSolve[{D[u[x],{x,2}]-2/x*D[u[x],x]-a^2*u[x]==0,{}},u[x],x,IncludeSingularSolutions->True]
```

$$u(x) \to \frac{\sqrt{\frac{2}{\pi}}\sqrt{x}((iac_2x + c_1)\sinh(ax) - (ac_1x + ic_2)\cosh(ax))}{a\sqrt{-iax}}$$

2.1.698 Problem 715

Internal problem ID [9870]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 715
Date solved : `Monday, January 27, 2025 at 06:15:15 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$u''+\frac{2u'}{x}-a^2u=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.104 (sec)

Writing the ode as

$$u''+\frac{2u'}{x}-a^2u=0 \tag{1}$$

$$Au''+Bu'+Cu=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=1\\ B&=\frac{2}{x} \\ C&=-a^2 \end{aligned}\tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ue^{\int\frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned}\tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{a^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=a^2\\ t&=1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = (a^2)\, z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1331: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = a^2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{\sqrt{a^2}\, x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in u is found from

$$\begin{aligned} u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{\frac{2}{x}}{1}\, dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1 \left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$u_1 = \frac{\mathrm{e}^{\mathrm{csgn}(a) a x}}{x}$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{u_1^2}\, dx$$

Substituting gives

$$\begin{aligned} u_2 &= u_1 \int \frac{e^{\int -\frac{\frac{2}{x}}{1}\, dx}}{(u_1)^2}\, dx \\ &= u_1 \int \frac{e^{-2\ln(x)}}{(u_1)^2}\, dx \\ &= u_1 \left(-\frac{\mathrm{e}^{-2\,\mathrm{csgn}(a)ax}}{2\,\mathrm{csgn}(a)\, a}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} u &= c_1 u_1 + c_2 u_2 \\ &= c_1 \left(\frac{\mathrm{e}^{\mathrm{csgn}(a)ax}}{x}\right) + c_2 \left(\frac{\mathrm{e}^{\mathrm{csgn}(a)ax}}{x}\left(-\frac{\mathrm{e}^{-2\,\mathrm{csgn}(a)ax}}{2\,\mathrm{csgn}(a)\, a}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}u(x) + \frac{2\left(\frac{d}{dx}u(x)\right)}{x} - a^2u(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}u(x)$
- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{2}{x}, P_3(x) = -a^2\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $-a^2u(x)\, x + \left(\frac{d^2}{dx^2}u(x)\right) x + 2\frac{d}{dx}u(x) = 0$
- Assume series solution for $u(x)$
  $u(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- □ Rewrite ODE with series expansions
  - Convert $x \cdot u(x)$ to series expansion
    $x \cdot u(x) = \sum_{k=0}^{\infty} a_k x^{k+r+1}$
  - Shift index using $k- >k-1$

$x \cdot u(x) = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

- Convert $\frac{d}{dx}u(x)$ to series expansion

  $\frac{d}{dx}u(x) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $\frac{d}{dx}u(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$

- Convert $x \cdot \left(\frac{d^2}{dx^2}u(x)\right)$ to series expansion

  $x \cdot \left(\frac{d^2}{dx^2}u(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $x \cdot \left(\frac{d^2}{dx^2}u(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) - a^2 a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $a_1(1+r)\,(2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)\,(k+2+r) - a^2 a_{k-1} = 0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)\,(k+3+r) - a^2 a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a^2 a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r=-1$

  $a_{k+2} = \frac{a^2 a_k}{(k+1)(k+2)}$

- Solution for $r=-1$

  $\left[u(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = \frac{a^2 a_k}{(k+1)(k+2)}, 0 = 0\right]$

- Recursion relation for $r=0$

  $a_{k+2} = \frac{a^2 a_k}{(k+2)(k+3)}$

- Solution for $r=0$

  $\left[u(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a^2 a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[u(x) = \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} c_k x^k\right), b_{k+2} = \frac{a^2 b_k}{(k+1)(k+2)}, 0 = 0, c_{k+2} = \frac{a^2 c_k}{(k+2)(k+3)}, 2c_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 21

```
dsolve(diff(diff(u(x),x),x)+2/x*diff(u(x),x)-a^2*u(x) = 0,u(x),singsol=all)
```

$$u = \frac{c_1 \sinh (ax) + c_2 \cosh (ax)}{x}$$

Mathematica DSolve solution

Solving time : 0.03 (sec)
Leaf size : 35

```
DSolve[{D[u[x],{x,2}]+2/x*D[u[x],x]-a^2*u[x]==0,{}},u[x],x,IncludeSingularSolutions->True]
```

$$u(x) \to \frac{2ac_1e^{-ax} + c_2e^{ax}}{2ax}$$

2.1.699 Problem 716

Internal problem ID [9871]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 716
Date solved : `Monday, January 27, 2025 at 06:15:15 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$u'' + \frac{2u'}{x} + a^2 u = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.118 (sec)

Writing the ode as

$$u'' + \frac{2u'}{x} + a^2 u = 0 \tag{1}$$
$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{2}{x} \\ C &= a^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = u e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-a^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -a^2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-a^2\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1333: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -a^2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{\sqrt{-a^2}\, x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in u is found from

$$\begin{aligned} u_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{\frac{2}{x}}{1}\, dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1 \left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$u_1 = \frac{e^{\sqrt{-a^2}\, x}}{x}$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{u_1^2}\, dx$$

Substituting gives

$$\begin{aligned} u_2 &= u_1 \int \frac{e^{\int -\frac{\frac{2}{x}}{1}\, dx}}{(u_1)^2}\, dx \\ &= u_1 \int \frac{e^{-2\ln(x)}}{(u_1)^2}\, dx \\ &= u_1 \left(\frac{\sqrt{-a^2}\, \mathrm{e}^{-2\sqrt{-a^2}\, x}}{2a^2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} u &= c_1 u_1 + c_2 u_2 \\ &= c_1 \left(\frac{\mathrm{e}^{\sqrt{-a^2}\, x}}{x}\right) + c_2 \left(\frac{\mathrm{e}^{\sqrt{-a^2}\, x}}{x} \left(\frac{\sqrt{-a^2}\, \mathrm{e}^{-2\sqrt{-a^2}\, x}}{2a^2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}u(x) + \frac{2\left(\frac{d}{dx}u(x)\right)}{x} + a^2u(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}u(x)$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = a^2\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $a^2u(x)\, x + \left(\frac{d^2}{dx^2}u(x)\right) x + 2\frac{d}{dx}u(x) = 0$
- Assume series solution for $u(x)$

  $u(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- □ Rewrite ODE with series expansions
  - Convert $x \cdot u(x)$ to series expansion

    $x \cdot u(x) = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

- Shift index using $k->k-1$

  $x \cdot u(x) = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

- Convert $\frac{d}{dx}u(x)$ to series expansion

  $\frac{d}{dx}u(x) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $\frac{d}{dx}u(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$

- Convert $x \cdot \left(\frac{d^2}{dx^2}u(x)\right)$ to series expansion

  $x \cdot \left(\frac{d^2}{dx^2}u(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $x \cdot \left(\frac{d^2}{dx^2}u(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + a^2 a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $a_1(1+r)\,(2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)\,(k+2+r) + a^2 a_{k-1} = 0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)\,(k+3+r) + a^2 a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a^2 a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a^2 a_k}{(k+1)(k+2)}$

- Solution for $r = -1$

  $\left[u(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a^2 a_k}{(k+1)(k+2)}, 0 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{a^2 a_k}{(k+2)(k+3)}$

- Solution for $r = 0$

  $\left[u(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a^2 a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[u(x) = \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} c_k x^k\right), b_{k+2} = -\frac{a^2 b_k}{(k+1)(k+2)}, 0 = 0, c_{k+2} = -\frac{a^2 c_k}{(k+2)(k+3)}, 2c_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 21

```
dsolve(diff(diff(u(x),x),x)+2/x*diff(u(x),x)+a^2*u(x) = 0,u(x),singsol=all)
```

$$u = \frac{c_1 \sin(ax) + c_2 \cos(ax)}{x}$$

Mathematica DSolve solution

Solving time : 0.034 (sec)
Leaf size : 42

```
DSolve[{D[u[x],{x,2}]+2/x*D[u[x],x]+a^2*u[x]==0,{}},u[x],x,IncludeSingularSolutions->True]
```

$$u(x) \to \frac{e^{-iax}\left(2c_1 - \frac{ic_2 e^{2iax}}{a}\right)}{2x}$$

2.1.700 Problem 717

Internal problem ID [9872]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 717
Date solved : `Monday, January 27, 2025 at 06:15:16 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$u'' + \frac{4u'}{x} - a^2 u = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.269 (sec)

Writing the ode as

$$u'' + \frac{4u'}{x} - a^2 u = 0 \tag{1}$$

$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{4}{x} \\ C &= -a^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = u e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2x^2 + 2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= a^2x^2 + 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{a^2x^2+2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1335: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2-2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1,2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2} + a^2$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx a + \frac{1}{a\,x^2} - \frac{1}{2a^3x^4} + \frac{1}{2a^5x^6} - \frac{5}{8a^7x^8} + \frac{7}{8a^9x^{10}} - \frac{21}{16a^{11}x^{12}} + \frac{33}{16a^{13}x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = a$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= a
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = a^2$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{a^2x^2 + 2}{x^2} \\
&= Q + \frac{R}{x^2} \\
&= (a^2) + \left(\frac{2}{x^2}\right) \\
&= \frac{2}{x^2} + a^2
\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned}
b &= (0) - (0) \\
&= 0
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= a \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{a} - 0\right) &= 0 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{a} - 0\right) &= 0
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{a^2x^2+2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	a	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + (-)\,(a) \\ &= -\frac{1}{x} - a \\ &= \frac{-ax-1}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x} - a\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - a\right)^2 - \left(\frac{a^2x^2+2}{x^2}\right) \right) = 0$$

$$\frac{2aa_0 - 2}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{1}{a}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{1}{a}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x + \frac{1}{a}\right) e^{\int \left(-\frac{1}{x} - a\right) dx} \\ &= \left(x + \frac{1}{a}\right) e^{-ax - \ln(x)} \\ &= \frac{(ax+1)\, e^{-ax}}{ax} \end{aligned}$$

The first solution to the original ode in u is found from

$$\begin{aligned} u_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{\frac{4}{x}}{1} \, dx} \\ &= z_1 e^{-2\ln(x)} \\ &= z_1 \left(\frac{1}{x^2}\right) \end{aligned}$$

Which simplifies to

$$u_1 = \frac{(ax+1)\, e^{-ax}}{x^3 a}$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{u_1^2} \, dx$$

Substituting gives

$$\begin{aligned} u_2 &= u_1 \int \frac{e^{\int -\frac{\frac{4}{x}}{1} \, dx}}{(u_1)^2} \, dx \\ &= u_1 \int \frac{e^{-4\ln(x)}}{(u_1)^2} \, dx \\ &= u_1 \left(\frac{(ax-1)\, e^{2ax}}{2a\,(ax+1)}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} u &= c_1 u_1 + c_2 u_2 \\ &= c_1 \left(\frac{(ax+1)\, e^{-ax}}{x^3 a}\right) + c_2 \left(\frac{(ax+1)\, e^{-ax}}{x^3 a} \left(\frac{(ax-1)\, e^{2ax}}{2a\,(ax+1)}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}u(x)+\frac{4\left(\frac{d}{dx}u(x)\right)}{x}-a^2u(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}u(x)$

- □ Check to see if $x_0=0$ is a regular singular point
  - Define functions

    $\left[P_2(x)=\frac{4}{x}, P_3(x)=-a^2\right]$
  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\Big|_{x=0}=4$
  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=0$
  - $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $-a^2u(x)\,x+\left(\frac{d^2}{dx^2}u(x)\right)x+4\frac{d}{dx}u(x)=0$

- Assume series solution for $u(x)$

  $u(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x\cdot u(x)$ to series expansion

    $x\cdot u(x)=\sum_{k=0}^{\infty}a_kx^{k+r+1}$
  - Shift index using $k->k-1$

    $x\cdot u(x)=\sum_{k=1}^{\infty}a_{k-1}x^{k+r}$
  - Convert $\frac{d}{dx}u(x)$ to series expansion

    $\frac{d}{dx}u(x)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1}$
  - Shift index using $k->k+1$

    $\frac{d}{dx}u(x)=\sum_{k=-1}^{\infty}a_{k+1}(k+r+1)\,x^{k+r}$
  - Convert $x\cdot\left(\frac{d^2}{dx^2}u(x)\right)$ to series expansion

    $x\cdot\left(\frac{d^2}{dx^2}u(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-1}$
  - Shift index using $k->k+1$

    $x\cdot\left(\frac{d^2}{dx^2}u(x)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+r+1)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(3+r)\,x^{-1+r}+a_1(1+r)\,(4+r)\,x^r+\left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+r+1)\,(k+4+r)-a^2a_{k-1}\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(3+r)=0$

- Values of r that satisfy the indicial equation

$r \in \{-3, 0\}$

- Each term must be 0

  $a_1(1+r)(4+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)(k+4+r) - a^2 a_{k-1} = 0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)(k+5+r) - a^2 a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a^2 a_k}{(k+2+r)(k+5+r)}$

- Recursion relation for $r = -3$

  $a_{k+2} = \frac{a^2 a_k}{(k-1)(k+2)}$

- Solution for $r = -3$

  $\left[u(x) = \sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2} = \frac{a^2 a_k}{(k-1)(k+2)}, -2a_1 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{a^2 a_k}{(k+2)(k+5)}$

- Solution for $r = 0$

  $\left[u(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a^2 a_k}{(k+2)(k+5)}, 4a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[u(x) = \left(\sum_{k=0}^{\infty} b_k x^{k-3}\right) + \left(\sum_{k=0}^{\infty} c_k x^k\right), b_{k+2} = \frac{a^2 b_k}{(k+2)(k-1)}, -2b_1 = 0, c_{k+2} = \frac{a^2 c_k}{(5+k)(k+2)}, 4c_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 32

```
dsolve(diff(diff(u(x),x),x)+4/x*diff(u(x),x)-a^2*u(x) = 0,u(x),singsol=all)
```

$$u = \frac{c_1 e^{ax}(ax-1) + c_2 e^{-ax}(ax+1)}{x^3}$$

Mathematica DSolve solution

Solving time : 0.072 (sec)
Leaf size : 68

```
DSolve[{D[u[x],{x,2}]+4/x*D[u[x],x]-a^2*u[x]==0,{}},u[x],x,IncludeSingularSolutions->True]
```

$$u(x) \to \frac{\sqrt{\frac{2}{\pi}}((iac_2x + c_1)\sinh(ax) - (ac_1x + ic_2)\cosh(ax))}{ax^{5/2}\sqrt{-iax}}$$

2.1.701 Problem 718

Internal problem ID [9873]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 718
Date solved : `Monday, January 27, 2025 at 06:15:16 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$u''+\frac{4u'}{x}+a^2u=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.318 (sec)

Writing the ode as

$$u''+\frac{4u'}{x}+a^2u=0 \tag{1}$$
$$Au''+Bu'+Cu=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=1\\ B&=\frac{4}{x}\\ C&=a^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ue^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-a^2x^2+2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=-a^2x^2+2\\ t&=x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-a^2x^2+2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1337: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2} - a^2$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx ia - \frac{i}{a\,x^2} - \frac{i}{2a^3x^4} - \frac{i}{2a^5x^6} - \frac{5i}{8a^7x^8} - \frac{7i}{8a^9x^{10}} - \frac{21i}{16a^{11}x^{12}} - \frac{33i}{16a^{13}x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = ia$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= ia \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -a^2$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-a^2x^2 + 2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= \left(-a^2\right) + \left(\frac{2}{x^2}\right) \\ &= \frac{2}{x^2} - a^2 \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= ia \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{ia} - 0\right) &&= 0 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{ia} - 0\right) &&= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-a^2x^2+2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	ia	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + (-)(ia) \\ &= -\frac{1}{x} - ia \\ &= -\frac{1}{x} - ia \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x} - ia\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - ia\right)^2 - \left(\frac{-a^2x^2+2}{x^2}\right)\right) = 0$$

$$\frac{2iaa_0 - 2}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{i}{a}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - \frac{i}{a}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{i}{a}\right) e^{\int \left(-\frac{1}{x} - ia\right) dx} \\
&= \left(x - \frac{i}{a}\right) e^{-\ln(x) - iax} \\
&= \frac{(ax - i) e^{-iax}}{xa}
\end{aligned}$$

The first solution to the original ode in u is found from

$$\begin{aligned}
u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{\frac{4}{x}}{1} \, dx} \\
&= z_1 e^{-2\ln(x)} \\
&= z_1 \left(\frac{1}{x^2}\right)
\end{aligned}$$

Which simplifies to

$$u_1 = \frac{(ax - i) e^{-iax}}{x^3 a}$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{u_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
u_2 &= u_1 \int \frac{e^{\int -\frac{\frac{4}{x}}{1} \, dx}}{(u_1)^2} \, dx \\
&= u_1 \int \frac{e^{-4\ln(x)}}{(u_1)^2} \, dx \\
&= u_1 \left(\frac{(iax - 1) e^{2iax}}{2a(-ax + i)}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1 \left(\frac{(ax - i) e^{-iax}}{x^3 a}\right) + c_2 \left(\frac{(ax - i) e^{-iax}}{x^3 a} \left(\frac{(iax - 1) e^{2iax}}{2a(-ax + i)}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}u(x)+\frac{4\left(\frac{d}{dx}u(x)\right)}{x}+a^2u(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}u(x)$

- □ Check to see if $x_0=0$ is a regular singular point
  - Define functions

    $\left[P_2(x)=\frac{4}{x},P_3(x)=a^2\right]$
  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=4$
  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=0$
  - $x=0$is a regular singular point

    Check to see if $x_0=0$ is a regular singular point

    $x_0=0$

- Multiply by denominators

  $a^2u(x)\,x+\left(\frac{d^2}{dx^2}u(x)\right)x+4\frac{d}{dx}u(x)=0$

- Assume series solution for $u(x)$

  $u(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x\cdot u(x)$ to series expansion

    $x\cdot u(x)=\sum_{k=0}^{\infty}a_kx^{k+r+1}$
  - Shift index using $k->k-1$

    $x\cdot u(x)=\sum_{k=1}^{\infty}a_{k-1}x^{k+r}$
  - Convert $\frac{d}{dx}u(x)$ to series expansion

    $\frac{d}{dx}u(x)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1}$
  - Shift index using $k->k+1$

    $\frac{d}{dx}u(x)=\sum_{k=-1}^{\infty}a_{k+1}(k+r+1)\,x^{k+r}$
  - Convert $x\cdot\left(\frac{d^2}{dx^2}u(x)\right)$ to series expansion

    $x\cdot\left(\frac{d^2}{dx^2}u(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-1}$
  - Shift index using $k->k+1$

    $x\cdot\left(\frac{d^2}{dx^2}u(x)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+r+1)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(3+r)\,x^{-1+r}+a_1(1+r)\,(4+r)\,x^r+\left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+r+1)\,(k+4+r)+a^2a_{k-1}\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(3+r)=0$

- Values of r that satisfy the indicial equation

$r \in \{-3, 0\}$

- Each term must be 0

  $a_1(1+r)(4+r) = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)(k+4+r) + a^2 a_{k-1} = 0$
- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)(k+5+r) + a^2 a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a^2 a_k}{(k+2+r)(k+5+r)}$
- Recursion relation for $r = -3$

  $a_{k+2} = -\frac{a^2 a_k}{(k-1)(k+2)}$
- Solution for $r = -3$

  $\left[u(x) = \sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2} = -\frac{a^2 a_k}{(k-1)(k+2)}, -2a_1 = 0\right]$
- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{a^2 a_k}{(k+2)(k+5)}$
- Solution for $r = 0$

  $\left[u(x) = \sum_{k=0}^{\infty} a_k x^{k}, a_{k+2} = -\frac{a^2 a_k}{(k+2)(k+5)}, 4a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[u(x) = \left(\sum_{k=0}^{\infty} b_k x^{k-3}\right) + \left(\sum_{k=0}^{\infty} c_k x^{k}\right), b_{k+2} = -\frac{a^2 b_k}{(k+2)(k-1)}, -2b_1 = 0, c_{k+2} = -\frac{a^2 c_k}{(5+k)(k+2)}, 4c_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.032` (sec)
Leaf size : 33

```
dsolve(diff(diff(u(x),x),x)+4/x*diff(u(x),x)+a^2*u(x) = 0,u(x),singsol=all)
```

$$u = \frac{(ac_1x + c_2)\cos(ax) + \sin(ax)(ac_2x - c_1)}{x^3}$$

Mathematica DSolve solution

Solving time : 0.071 (sec)
Leaf size : 57

```
DSolve[{D[u[x],{x,2}]+4/x*D[u[x],x]+a^2*u[x]==0,{}},u[x],x,IncludeSingularSolutions->True]
```

$$u(x) \to -\frac{\sqrt{\frac{2}{\pi}}((ac_1x + c_2)\cos(ax) + (ac_2x - c_1)\sin(ax))}{x^{3/2}(ax)^{3/2}}$$

2.1.702 Problem 719

Internal problem ID [9874]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 719
Date solved : `Monday, January 27, 2025 at 06:15:17 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - a^2 y = \frac{6y}{x^2}$$

Solved as second order ode using Kovacic algorithm

Time used: 0.290 (sec)

Writing the ode as

$$y'' + \left(-a^2 - \frac{6}{x^2}\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -a^2 - \frac{6}{x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2 x^2 + 6}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= a^2 x^2 + 6 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{a^2x^2+6}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1339: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = a^2 + \frac{6}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx a + \frac{3}{a\,x^2} - \frac{9}{2a^3x^4} + \frac{27}{2a^5x^6} - \frac{405}{8a^7x^8} + \frac{1701}{8a^9x^{10}} - \frac{15309}{16a^{11}x^{12}} + \frac{72171}{16a^{13}x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = a$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= a \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = a^2$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{a^2x^2 + 6}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (a^2) + \left(\frac{6}{x^2}\right) \\ &= a^2 + \frac{6}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= a \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{0}{a} - 0\right) = 0 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{0}{a} - 0\right) = 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{a^2x^2 + 6}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	a	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= 0 - (-2) \\
&= 2
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{2}{x} + (-)\,(a) \\
&= -\frac{2}{x} - a \\
&= \frac{-ax-2}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(-\frac{2}{x} - a\right)(2x + a_1) + \left(\left(\frac{2}{x^2}\right) + \left(-\frac{2}{x} - a\right)^2 - \left(\frac{a^2x^2+6}{x^2}\right) \right) = 0$$

$$\frac{2axa_1 + 4aa_0 - 6x - 4a_1}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{ a_0 = \frac{3}{a^2}, a_1 = \frac{3}{a} \right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + \frac{3x}{a} + \frac{3}{a^2}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \left(x^2 + \frac{3x}{a} + \frac{3}{a^2}\right) e^{\int \left(-\frac{2}{x} - a\right) dx} \\ &= \left(x^2 + \frac{3x}{a} + \frac{3}{a^2}\right) e^{-ax - 2\ln(x)} \\ &= \frac{(a^2x^2 + 3ax + 3)\, e^{-ax}}{a^2x^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{(a^2x^2 + 3ax + 3)\, e^{-ax}}{a^2x^2} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(a^2x^2 + 3ax + 3)\, e^{-ax}}{a^2x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \frac{(a^2x^2 + 3ax + 3)\, e^{-ax}}{a^2x^2} \int \frac{1}{\frac{(a^2x^2+3ax+3)^2 e^{-2ax}}{a^4x^4}}\, dx \\ &= \frac{(a^2x^2 + 3ax + 3)\, e^{-ax}}{a^2x^2} \left(\frac{(a^2x^2 - 3ax + 3)\, e^{2ax}}{2a\,(a^2x^2 + 3ax + 3)}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{(a^2x^2 + 3ax + 3)\, e^{-ax}}{a^2x^2}\right) + c_2 \left(\frac{(a^2x^2 + 3ax + 3)\, e^{-ax}}{a^2x^2} \left(\frac{(a^2x^2 - 3ax + 3)\, e^{2ax}}{2a\,(a^2x^2 + 3ax + 3)}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2}y(x) - a^2 y(x) = \frac{6y(x)}{x^2}$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
$\frac{d^2}{dx^2}y(x) = \frac{(a^2x^2+6)y(x)}{x^2}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
$\frac{d^2}{dx^2}y(x) - \frac{(a^2x^2+6)y(x)}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
  $\left[P_2(x) = 0, P_3(x) = -\frac{a^2x^2+6}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
  $\left.(x \cdot P_2(x))\right|_{x=0} = 0$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -6$
  - $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point
$x_0 = 0$

- Multiply by denominators
$x^2\left(\frac{d^2}{dx^2}y(x)\right) + (-a^2x^2 - 6)\,y(x) = 0$
- Assume series solution for $y(x)$
$y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$
  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k - m$
  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion
  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

Rewrite ODE with series expansions
$a_0(2+r)\,(-3+r)\,x^r + a_1(3+r)\,(-2+r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+2)\,(k+r-3) - a_{k-2}a^2)\,x^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation
$(2+r)\,(-3+r) = 0$
- Values of r that satisfy the indicial equation
$r \in \{-2, 3\}$
- Each term must be 0
$a_1(3+r)\,(-2+r) = 0$
- Solve for the dependent coefficient(s)
$a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)(k+r-3) - a_{k-2}a^2 = 0$
- Shift index using $k->k+2$

  $a_{k+2}(k+4+r)(k+r-1) - a_k a^2 = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k a^2}{(k+4+r)(k+r-1)}$
- Recursion relation for $r = -2$

  $a_{k+2} = \frac{a_k a^2}{(k+2)(k-3)}$
- Solution for $r = -2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = \frac{a_k a^2}{(k+2)(k-3)}, a_1 = 0\right]$
- Recursion relation for $r = 3$

  $a_{k+2} = \frac{a_k a^2}{(k+7)(k+2)}$
- Solution for $r = 3$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = \frac{a_k a^2}{(k+7)(k+2)}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} b_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} c_k x^{k+3}\right), b_{k+2} = \frac{b_k a^2}{(k+2)(k-3)}, b_1 = 0, c_{k+2} = \frac{c_k a^2}{(k+7)(k+2)}, c_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 48

```
dsolve(diff(diff(y(x),x),x)-a^2*y(x) = 6/x^2*y(x),y(x),singsol=all)
```

$$y = \frac{c_2(a^2x^2 + 3ax + 3)\,\mathrm{e}^{-ax} + \mathrm{e}^{ax} c_1(a^2x^2 - 3ax + 3)}{x^2}$$

Mathematica DSolve solution

Solving time : 0.134 (sec)
Leaf size : 90

```
DSolve[{D[y[x],{x,2}]-a^2*y[x]==6*y[x]/x^2,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt{\frac{2}{\pi}}((a^2c_2x^2 - 3iac_1x + 3c_2)\cosh(ax) + i(c_1(a^2x^2 + 3) + 3iac_2x)\sinh(ax))}{a^2x^{3/2}\sqrt{-iax}}$$

2.1.703 Problem 720

Internal problem ID [9875]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 720
Date solved : `Monday, January 27, 2025 at 06:15:18 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + n^2 y = \frac{6y}{x^2}$$

Solved as second order ode using Kovacic algorithm

Time used: 0.344 (sec)

Writing the ode as

$$y'' + \left(n^2 - \frac{6}{x^2}\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= n^2 - \frac{6}{x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-n^2 x^2 + 6}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -n^2 x^2 + 6 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-n^2x^2+6}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1341: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2-2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1,2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -n^2 + \frac{6}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx in - \frac{3i}{n\,x^2} - \frac{9i}{2n^3x^4} - \frac{27i}{2n^5x^6} - \frac{405i}{8n^7x^8} - \frac{1701i}{8n^9x^{10}} - \frac{15309i}{16n^{11}x^{12}} - \frac{72171i}{16n^{13}x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = in$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= in \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -n^2$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-n^2x^2 + 6}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-n^2) + \left(\frac{6}{x^2}\right) \\ &= -n^2 + \frac{6}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= in \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{in} - 0\right) &&= 0 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{in} - 0\right) &&= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-n^2x^2+6}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	in	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= 0 - (-2) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{2}{x} + (-)(in) \\ &= -\frac{2}{x} - in \\ &= -\frac{2}{x} - in \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(-\frac{2}{x} - in\right)(2x + a_1) + \left(\left(\frac{2}{x^2}\right) + \left(-\frac{2}{x} - in\right)^2 - \left(\frac{-n^2x^2+6}{x^2}\right) \right) = 0$$

$$\frac{(2ina_1 - 6)\, x + 4ina_0 - 4a_1}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{ a_0 = -\frac{3}{n^2}, a_1 = -\frac{3i}{n} \right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - \frac{3ix}{n} - \frac{3}{n^2}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x^2 - \frac{3ix}{n} - \frac{3}{n^2}\right) e^{\int \left(-\frac{2}{x} - in\right) dx} \\ &= \left(x^2 - \frac{3ix}{n} - \frac{3}{n^2}\right) e^{-2\ln(x) - inx} \\ &= \frac{(n^2x^2 - 3inx - 3)\, e^{-inx}}{x^2n^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{(n^2x^2 - 3inx - 3)\, e^{-inx}}{x^2n^2} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(n^2x^2 - 3inx - 3)\, e^{-inx}}{x^2n^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= \frac{(n^2x^2 - 3inx - 3)\, e^{-inx}}{x^2n^2} \int \frac{1}{\frac{(n^2x^2 - 3inx - 3)^2 e^{-2inx}}{x^4n^4}} \, dx \\ &= \frac{(n^2x^2 - 3inx - 3)\, e^{-inx}}{x^2n^2} \left(\frac{(in^2x^2 - 3nx - 3i)\, e^{2inx}}{2n\,(-n^2x^2 + 3inx + 3)}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1y_1 + c_2y_2 \\ &= c_1\left(\frac{(n^2x^2 - 3inx - 3)\, e^{-inx}}{x^2n^2}\right) + c_2\left(\frac{(n^2x^2 - 3inx - 3)\, e^{-inx}}{x^2n^2}\left(\frac{(in^2x^2 - 3nx - 3i)\, e^{2inx}}{2n\,(-n^2x^2 + 3inx + 3)}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + n^2y(x) = \frac{6y(x)}{x^2}$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(n^2x^2-6)y(x)}{x^2}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(n^2x^2-6)y(x)}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = 0, P_3(x) = \frac{n^2x^2-6}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = -6$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) + (n^2x^2 - 6)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ◦ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)(-3+r)\,x^r + a_1(3+r)(-2+r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+2)(k+r-3) + a_{k-2}n^2\right) x^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 3\}$

- Each term must be 0

  $a_1(3+r)(-2+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation
  $a_k(k+r+2)(k+r-3)+a_{k-2}n^2=0$
- Shift index using $k->k+2$
  $a_{k+2}(k+4+r)(k+r-1)+a_k n^2=0$
- Recursion relation that defines series solution to ODE
  $a_{k+2}=-\frac{a_k n^2}{(k+4+r)(k+r-1)}$
- Recursion relation for $r=-2$
  $a_{k+2}=-\frac{a_k n^2}{(k+2)(k-3)}$
- Solution for $r=-2$
  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2}=-\frac{a_k n^2}{(k+2)(k-3)}, a_1=0\right]$
- Recursion relation for $r=3$
  $a_{k+2}=-\frac{a_k n^2}{(k+7)(k+2)}$
- Solution for $r=3$
  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2}=-\frac{a_k n^2}{(k+7)(k+2)}, a_1=0\right]$
- Combine solutions and rename parameters
  $\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^{k-2}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), a_{k+2}=-\frac{a_k n^2}{(k+2)(k-3)}, a_1=0, b_{k+2}=-\frac{b_k n^2}{(k+7)(k+2)}, b_1=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.034 (sec)
Leaf size : 53

```
dsolve(diff(diff(y(x),x),x)+n^2*y(x) = 6/x^2*y(x),y(x),singsol=all)
```

$$y=\frac{(c_1 n^2 x^2+3c_2 n x-3c_1)\cos(nx)+\sin(nx)(c_2 n^2 x^2-3c_1 n x-3c_2)}{x^2}$$

Mathematica DSolve solution

Solving time : 0.132 (sec)
Leaf size : 79

```
DSolve[{D[y[x],{x,2}]+n^2*y[x]==6*y[x]/x^2,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{\sqrt{\frac{2}{\pi}}\sqrt{x}((c_2(-n^2)\,x^2+3c_1nx+3c_2)\cos(nx)+(c_1(n^2x^2-3)+3c_2nx)\sin(nx))}{(nx)^{5/2}}$$

2.1.704 Problem 721

Internal problem ID [9876]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 721
Date solved : `Monday, January 27, 2025 at 06:15:19 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' - \left(x^2 + \frac{1}{4}\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.086 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(-x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= -x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1343: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-x}}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{e^{2x}}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{-x}}{\sqrt{x}}\right) + c_2 \left(\frac{e^{-x}}{\sqrt{x}} \left(\frac{e^{2x}}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x\left(\frac{d}{dx}y(x)\right) - \left(x^2 + \frac{1}{4}\right)y(x) = 0$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
$\frac{d^2}{dx^2}y(x) = \frac{(4x^2+1)y(x)}{4x^2} - \frac{\frac{d}{dx}y(x)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
$\frac{d^2}{dx^2}y(x) + \frac{\frac{d}{dx}y(x)}{x} - \frac{(4x^2+1)y(x)}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
  $\left[P_2(x) = \frac{1}{x}, P_3(x) = -\frac{4x^2+1}{4x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
  $\left.(x \cdot P_2(x))\right|_{x=0} = 1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$
  - $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point
$x_0 = 0$

- Multiply by denominators
$4x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x\left(\frac{d}{dx}y(x)\right) + (-4x^2 - 1)y(x) = 0$
- Assume series solution for $y(x)$
$y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- > k - m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$
  - Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) - 4a_{k-2})\right.$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) - 4a_{k-2} = 0$
- Shift index using $k- > k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) - 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = \frac{4a_k}{4k^2+12k+8}$
- Solution for $r = -\frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = \frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = \frac{4a_k}{4k^2+20k+24}$
- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = \frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = \frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.042 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x-(x^2+1/4)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_1 \sinh(x) + c_2 \cosh(x)}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.031 (sec)
Leaf size : 32

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]-(x^2+1/4)*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to \frac{e^{-x}(c_2 e^{2x} + 2c_1)}{2\sqrt{x}}$$

2.1.705 Problem 722

Internal problem ID [9877]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 722
Date solved : `Monday, January 27, 2025 at 06:15:19 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \frac{(-9a^2 + 4x^2)\, y}{4a^2} = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.314 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(-\frac{9}{4} + \frac{x^2}{a^2}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= -\frac{9}{4} + \frac{x^2}{a^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2a^2 - x^2}{x^2a^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2a^2 - x^2 \\ t &= x^2a^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2a^2 - x^2}{x^2 a^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1345: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2 a^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{a^2} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx -\frac{33ia^{13}}{16x^{14}} - \frac{21ia^{11}}{16x^{12}} - \frac{7ia^9}{8x^{10}} - \frac{5ia^7}{8x^8} - \frac{ia^5}{2x^6} - \frac{ia^3}{2x^4} - \frac{ia}{x^2} + \frac{i}{a} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{i}{a}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{i}{a} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -\frac{1}{a^2}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2a^2 - x^2}{x^2 a^2} \\ &= Q + \frac{R}{x^2 a^2} \\ &= \left(-\frac{1}{a^2}\right) + \left(\frac{2}{x^2}\right) \\ &= -\frac{1}{a^2} + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{i}{a} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{\frac{i}{a}} - 0\right) &= 0 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{\frac{i}{a}} - 0\right) &= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2a^2 - x^2}{x^2 a^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{i}{a}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + (-)\left(\frac{i}{a}\right) \\ &= -\frac{1}{x} - \frac{i}{a} \\ &= -\frac{ix+a}{xa} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x} - \frac{i}{a}\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - \frac{i}{a}\right)^2 - \left(\frac{2a^2 - x^2}{x^2 a^2}\right) \right) = 0$$
$$\frac{2ia_0 - 2a}{xa} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -ia\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = -ia + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (-ia + x)\, e^{\int \left(-\frac{1}{x} - \frac{i}{a}\right) dx} \\
&= (-ia + x)\, e^{-\ln(x) - \frac{ix}{a}} \\
&= \frac{(-ia + x)\, e^{-\frac{ix}{a}}}{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{x^2} \, dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(-ia + x)\, e^{-\frac{ix}{a}}}{x^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2} \, dx \\
&= y_1 \left(-\frac{(ix + a)\, a(ia + x)\, e^{\frac{2ix}{a}}}{2\,(ia - x)^2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(-ia + x)\, e^{-\frac{ix}{a}}}{x^{3/2}}\right) + c_2 \left(\frac{(-ia + x)\, e^{-\frac{ix}{a}}}{x^{3/2}} \left(-\frac{(ix + a)\, a(ia + x)\, e^{\frac{2ix}{a}}}{2\,(ia - x)^2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right)+x\left(\frac{d}{dx}y(x)\right)+\frac{(-9a^2+4x^2)y(x)}{4a^2}=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{(9a^2-4x^2)y(x)}{4a^2x^2}-\frac{\frac{d}{dx}y(x)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{\frac{d}{dx}y(x)}{x}-\frac{(9a^2-4x^2)y(x)}{4a^2x^2}=0$

- Multiply by denominators of the ODE

  $4\left(\frac{d^2}{dx^2}y(x)\right)x^2a^2+4\left(\frac{d}{dx}y(x)\right)x\,a^2-(9a^2-4x^2)\,y(x)=0$

- Make a change of variables

  $t=\ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $\frac{d}{dx}y(x)=\left(\frac{d}{dt}y(t)\right)\left(\frac{d}{dx}t(x)\right)$

  - Compute derivative

    $\frac{d}{dx}y(x)=\frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $\frac{d^2}{dx^2}y(x)=\left(\frac{d^2}{dt^2}y(t)\right)\left(\frac{d}{dx}t(x)\right)^2+\left(\frac{d^2}{dx^2}t(x)\right)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $\frac{d^2}{dx^2}y(x)=\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $4\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}\right)x^2a^2+4\left(\frac{d}{dt}y(t)\right)a^2-(9a^2-4x^2)\,y(t)=0$

- Simplify

  $4a^2\left(\frac{d^2}{dt^2}y(t)\right)-9y(t)\,a^2+4y(t)\,x^2=0$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t)=\frac{(9a^2-4x^2)y(t)}{4a^2}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dt^2}y(t)-\frac{(9a^2-4x^2)y(t)}{4a^2}=0$

- Characteristic polynomial of ODE

  $r^2-\frac{9a^2-4x^2}{4a^2}=0$

- Factor the characteristic polynomial

  $\frac{4r^2a^2-9a^2+4x^2}{4a^2}=0$

- Roots of the characteristic polynomial

  $r=\left(\frac{\sqrt{9a^2-4x^2}}{2a},-\frac{\sqrt{9a^2-4x^2}}{2a}\right)$

- 1st solution of the ODE

  $y_1(t)=e^{\frac{\sqrt{9a^2-4x^2}\,t}{2a}}$

- 2nd solution of the ODE

  $y_2(t)=e^{-\frac{\sqrt{9a^2-4x^2}\,t}{2a}}$

- General solution of the ODE

  $y(t)=C1y_1(t)+C2y_2(t)$

- Substitute in solutions

  $y(t) = C1\,\mathrm{e}^{\frac{\sqrt{9a^2-4x^2}\,t}{2a}} + C2\,\mathrm{e}^{-\frac{\sqrt{9a^2-4x^2}\,t}{2a}}$
- Change variables back using $t = \ln(x)$

  $y(x) = C1\,\mathrm{e}^{\frac{\sqrt{9a^2-4x^2}\,\ln(x)}{2a}} + C2\,\mathrm{e}^{-\frac{\sqrt{9a^2-4x^2}\,\ln(x)}{2a}}$
- Simplify

  $y(x) = C1\,x^{\frac{\sqrt{9a^2-4x^2}}{2a}} + C2\,x^{-\frac{\sqrt{9a^2-4x^2}}{2a}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.085 (sec)
Leaf size : 37

```
dsolve(x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x+1/4*(-9*a^2+4*x^2)/a^2*y(x) = 0,y(x),sing
```

$$y = \frac{(ix+a)\,c_2\mathrm{e}^{-\frac{ix}{a}} + (-ix+a)\,c_1\mathrm{e}^{\frac{ix}{a}}}{x^{3/2}}$$

Mathematica DSolve solution

Solving time : 0.079 (sec)
Leaf size : 62

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(4*x^2-9*a^2)/(4*a^2)*y[x]==0,{}},y[x],x,IncludeSingular
```

$$y(x) \to -\frac{\sqrt{\frac{2}{\pi}}\left((ac_2 + c_1x)\cos\left(\frac{x}{a}\right) + (c_2x - ac_1)\sin\left(\frac{x}{a}\right)\right)}{x\sqrt{\frac{x}{a}}}$$

2.1.706 Problem 723

Internal problem ID [9878]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 723
Date solved : `Monday, January 27, 2025 at 06:15:20 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{25}{4}\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.291 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{25}{4}\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{25}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 6}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2 + 6 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2+6}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1347: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -1 + \frac{6}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx i - \frac{3i}{x^2} - \frac{9i}{2x^4} - \frac{27i}{2x^6} - \frac{405i}{8x^8} - \frac{1701i}{8x^{10}} - \frac{15309i}{16x^{12}} - \frac{72171i}{16x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= i
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{-x^2 + 6}{x^2} \\
&= Q + \frac{R}{x^2} \\
&= (-1) + \left(\frac{6}{x^2}\right) \\
&= -1 + \frac{6}{x^2}
\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned}
b &= (0) - (0) \\
&= 0
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= i \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{i} - 0\right) &= 0 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{i} - 0\right) &= 0
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2+6}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	i	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= 0 - (-2) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{2}{x} + (-)\,(i) \\ &= -\frac{2}{x} - i \\ &= -\frac{2}{x} - i \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (2) + 2\left(-\frac{2}{x} - i\right)(2x + a_1) + \left(\left(\frac{2}{x^2}\right) + \left(-\frac{2}{x} - i\right)^2 - \left(\frac{-x^2+6}{x^2}\right) \right) &= 0 \\ \frac{2ixa_1 + 4ia_0 - 6x - 4a_1}{x} &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -3, a_1 = -3i\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 3ix - 3$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= \left(x^2 - 3ix - 3\right) e^{\int \left(-\frac{2}{x} - i\right) dx} \\ &= \left(x^2 - 3ix - 3\right) e^{-2\ln(x) - ix} \\ &= \frac{(x^2 - 3ix - 3)\, e^{-ix}}{x^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2 - 3ix - 3)\, e^{-ix}}{x^{5/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{(ix^2 - 3x - 3i)\, e^{2ix}}{-2x^2 + 6ix + 6}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{(x^2 - 3ix - 3)\, e^{-ix}}{x^{5/2}}\right) + c_2 \left(\frac{(x^2 - 3ix - 3)\, e^{-ix}}{x^{5/2}} \left(\frac{(ix^2 - 3x - 3i)\, e^{2ix}}{-2x^2 + 6ix + 6}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x\left(\frac{d}{dx}y(x)\right) + \left(x^2 - \frac{25}{4}\right)y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(4x^2-25)y(x)}{4x^2} - \frac{\frac{d}{dx}y(x)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{\frac{d}{dx}y(x)}{x} + \frac{(4x^2-25)y(x)}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-25}{4x^2}\right]$
- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1$
- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{25}{4}$
- ○ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x\left(\frac{d}{dx}y(x)\right) + (4x^2 - 25)y(x) = 0$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- ○ Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- ○ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$
- ○ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0(5+2r)(-5+2r)\, x^r + a_1(7+2r)(-3+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+5)(2k+2r-5) + 4a_{k-2}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(5+2r)(-5+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{5}{2}, \frac{5}{2}\right\}$

- Each term must be 0
  $a_1(7+2r)(-3+2r)=0$
- Solve for the dependent coefficient(s)
  $a_1=0$
- Each term in the series must be 0, giving the recursion relation
  $a_k(2k+2r+5)(2k+2r-5)+4a_{k-2}=0$
- Shift index using $k->k+2$
  $a_{k+2}(2k+9+2r)(2k-1+2r)+4a_k=0$
- Recursion relation that defines series solution to ODE
  $a_{k+2}=-\frac{4a_k}{(2k+9+2r)(2k-1+2r)}$
- Recursion relation for $r=-\frac{5}{2}$
  $a_{k+2}=-\frac{4a_k}{(2k+4)(2k-6)}$
- Solution for $r=-\frac{5}{2}$
  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}, a_{k+2}=-\frac{4a_k}{(2k+4)(2k-6)}, a_1=0\right]$
- Recursion relation for $r=\frac{5}{2}$
  $a_{k+2}=-\frac{4a_k}{(2k+14)(2k+4)}$
- Solution for $r=\frac{5}{2}$
  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+\frac{5}{2}}, a_{k+2}=-\frac{4a_k}{(2k+14)(2k+4)}, a_1=0\right]$
- Combine solutions and rename parameters
  $\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{5}{2}}\right), a_{k+2}=-\frac{4a_k}{(2k+4)(2k-6)}, a_1=0, b_{k+2}=-\frac{4b_k}{(2k+14)(2k+4)}, b_1=\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.051 (sec)
Leaf size : 43

```
dsolve(x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x+(x^2-25/4)*y(x) = 0,y(x),singsol=all)
```

$$y=\frac{-3c_2\left(ix-\frac{1}{3}x^2+1\right)e^{-ix}+3\left(ix+\frac{1}{3}x^2-1\right)c_1e^{ix}}{x^{5/2}}$$

Mathematica DSolve solution

Solving time : 0.075 (sec)
Leaf size : 59

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-25/4)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to -\frac{\sqrt{\frac{2}{\pi}}((-c_2x^2+3c_1x+3c_2)\cos(x)+(c_1(x^2-3)+3c_2x)\sin(x))}{x^{5/2}}$$

2.1.707 Problem 724

Internal problem ID [9879]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 724
Date solved : `Monday, January 27, 2025 at 06:15:20 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + qy' = \frac{2y}{x^2}$$

Solved as second order ode using Kovacic algorithm

Time used: 0.268 (sec)

Writing the ode as

$$y'' + qy' - \frac{2y}{x^2} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= q \\ C &= -\frac{2}{x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{q^2x^2 + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= q^2x^2 + 8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{q^2x^2 + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1349: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{q^2}{4} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{q}{2} + \frac{2}{q\,x^2} - \frac{4}{q^3x^4} + \frac{16}{q^5x^6} - \frac{80}{q^7x^8} + \frac{448}{q^9x^{10}} - \frac{2688}{q^{11}x^{12}} + \frac{16896}{q^{13}x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{q}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{q}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{q^2}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{q^2x^2 + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{q^2}{4}\right) + \left(\frac{2}{x^2}\right) \\ &= \frac{q^2}{4} + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 4 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{q}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{\frac{q}{2}} - 0\right) &&= 0 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{\frac{q}{2}} - 0\right) &&= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{q^2x^2+8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{q}{2}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= 0 - (-1) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x} + (-)\left(\frac{q}{2}\right) \\
&= -\frac{1}{x} - \frac{q}{2} \\
&= -\frac{qx+2}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x} - \frac{q}{2}\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - \frac{q}{2}\right)^2 - \left(\frac{q^2x^2+8}{4x^2}\right) \right) = 0$$

$$\frac{qa_0 - 2}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{ a_0 = \frac{2}{q} \right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{2}{q}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x + \frac{2}{q}\right) e^{\int \left(-\frac{1}{x} - \frac{q}{2}\right) dx} \\ &= \left(x + \frac{2}{q}\right) e^{-\frac{qx}{2} - \ln(x)} \\ &= \frac{(qx+2)\, e^{-\frac{qx}{2}}}{qx} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{q}{1} \, dx} \\ &= z_1 e^{-\frac{qx}{2}} \\ &= z_1 \left(e^{-\frac{qx}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{-qx}(qx+2)}{qx}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{q}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-qx}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{(qx-2)\, e^{qx}}{q\,(qx+2)}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{e^{-qx}(qx+2)}{qx}\right) + c_2 \left(\frac{e^{-qx}(qx+2)}{qx} \left(\frac{(qx-2)\, e^{qx}}{q\,(qx+2)}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x)+q\left(\frac{d}{dx}y(x)\right)=\frac{2y(x)}{x^2}$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-q\left(\frac{d}{dx}y(x)\right)+\frac{2y(x)}{x^2}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+q\left(\frac{d}{dx}y(x)\right)-\frac{2y(x)}{x^2}=0$

- □ Check to see if $x_0=0$ is a regular singular point
  - ○ Define functions

    $\left[P_2(x)=q, P_3(x)=-\frac{2}{x^2}\right]$
  - ○ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=0$
  - ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=-2$
  - ○ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $q\left(\frac{d}{dx}y(x)\right)x^2+x^2\left(\frac{d^2}{dx^2}y(x)\right)-2y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - ○ Convert $x^2\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x^2\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r+1}$
  - ○ Shift index using $k->k-1$

    $x^2\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=1}^{\infty}a_{k-1}(k-1+r)\,x^{k+r}$
  - ○ Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k-1+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-2+r)\,x^r+\left(\sum_{k=1}^{\infty}\left(a_k(k+r+1)\,(k+r-2)+qa_{k-1}(k-1+r)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{-1,2\}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)\,(k+r-2)+qa_{k-1}(k-1+r)=0$

- Shift index using $k->k+1$

  $a_{k+1}(k+2+r)\,(k-1+r)+qa_k(k+r)=0$

- Recursion relation that defines series solution to ODE
  $a_{k+1} = -\frac{q a_k (k+r)}{(k+2+r)(k-1+r)}$
- Recursion relation for $r = -1$; series terminates at $k = 1$
  $a_{k+1} = -\frac{q a_k (k-1)}{(k+1)(k-2)}$
- Apply recursion relation for $k = 0$
  $a_1 = -\frac{q a_0}{2}$
- Terminating series solution of the ODE for $r = -1$. Use reduction of order to find the second
  $y(x) = a_0 \cdot \left(-\frac{qx}{2} + 1\right)$
- Recursion relation for $r = 2$
  $a_{k+1} = -\frac{q a_k (k+2)}{(k+4)(k+1)}$
- Solution for $r = 2$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = -\frac{q a_k (k+2)}{(k+4)(k+1)}\right]$
- Combine solutions and rename parameters
  $\left[y(x) = a_0 \cdot \left(-\frac{qx}{2} + 1\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), b_{k+1} = -\frac{q b_k (k+2)}{(4+k)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 28

```
dsolve(diff(diff(y(x),x),x)+q*diff(y(x),x) = 2/x^2*y(x),y(x),singsol=all)
```

$$y = \frac{c_2 (qx + 2) \,\mathrm{e}^{-qx} + c_1 (qx - 2)}{x}$$

Mathematica DSolve solution

Solving time : 0.081 (sec)
Leaf size : 83

```
DSolve[{D[y[x],{x,2}]+q*D[y[x],x]==2*y[x]/x^2,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2 q x^{3/2} e^{\frac{1}{2} - \frac{qx}{2}} \left((c_1 q x + 2 i c_2) \cosh\left(\frac{qx}{2}\right) - (i c_2 q x + 2 c_1) \sinh\left(\frac{qx}{2}\right)\right)}{\sqrt{\pi} (-iqx)^{5/2}}$$

2.1.708 Problem 725

Internal problem ID [9880]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 725
Date solved : `Monday, January 27, 2025 at 06:15:21 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$xy'' + 3y' + 4x^3y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.268 (sec)

Writing the ode as

$$xy'' + 3y' + 4x^3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 3 \\ C &= 4x^3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16x^4 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -16x^4 + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-16x^4 + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1351: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -4x^2 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 2ix - \frac{3i}{16x^3} - \frac{9i}{1024x^7} - \frac{27i}{32768x^{11}} - \frac{405i}{4194304x^{15}} - \frac{1701i}{134217728x^{19}} - \frac{15309i}{8589934592x^{23}} - \frac{72171i}{274877906944x^{27}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 2i$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= 2ix \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -4x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-16x^4 + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (-4x^2) + \left(\frac{3}{4x^2}\right) \\ &= -4x^2 + \frac{3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 2ix \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{2i} - 1\right) &&= -\frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{2i} - 1\right) &&= -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-16x^4 + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$2ix$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)\,(2ix) \\
&= -\frac{1}{2x} - 2ix \\
&= -\frac{1}{2x} - 2ix
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{2x} - 2ix\right)(0) + \left(\left(\frac{1}{2x^2} - 2i\right) + \left(-\frac{1}{2x} - 2ix\right)^2 - \left(\frac{-16x^4+3}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{1}{2x} - 2ix\right) dx} \\
&= \frac{e^{-ix^2}}{\sqrt{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3}{x}\,dx} \\ &= z_1 e^{-\frac{3\ln(x)}{2}} \\ &= z_1\left(\frac{1}{x^{3/2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{-i x^2}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3}{x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-3\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(-\frac{i e^{2 i x^2}}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{e^{-i x^2}}{x^2}\right) + c_2\left(\frac{e^{-i x^2}}{x^2}\left(-\frac{i e^{2 i x^2}}{4}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dx^2}y(x)\right)x + 3\frac{d}{dx}y(x) + 4x^3 y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -4x^2 y(x) - \frac{3\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) + \frac{3\left(\frac{d}{dx}y(x)\right)}{x} + 4x^2 y(x) = 0$
- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{3}{x}, P_3(x) = 4x^2\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

$(x \cdot P_2(x))\bigg|_{x=0} = 3$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $\left(\frac{d^2}{dx^2}y(x)\right)x + 3\frac{d}{dx}y(x) + 4x^3y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^3 \cdot y(x)$ to series expansion

  $x^3 \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+3}$

- Shift index using $k- > k-3$

  $x^3 \cdot y(x) = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$

- Convert $\frac{d}{dx}y(x)$ to series expansion

  $\frac{d}{dx}y(x) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

- Shift index using $k- > k+1$

  $\frac{d}{dx}y(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$

- Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k- > k+1$

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(2+r)\, x^{-1+r} + a_1(1+r)\,(3+r)\, x^r + a_2(2+r)\,(4+r)\, x^{1+r} + a_3(3+r)\,(5+r)\, x^{2+r} + \left(\sum_{k=3}^{\infty} (\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 0\}$

- The coefficients of each power of x must be 0

  $[a_1(1+r)\,(3+r) = 0, a_2(2+r)\,(4+r) = 0, a_3(3+r)\,(5+r) = 0]$

- Solve for the dependent coefficient(s)

  $\{a_1 = 0, a_2 = 0, a_3 = 0\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k+r+3) + 4a_{k-3} = 0$

- Shift index using $k- > k+3$

  $a_{k+4}(k+4+r)\,(k+6+r) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+4} = -\frac{4a_k}{(k+4+r)(k+6+r)}$

- Recursion relation for $r=-2$

  $a_{k+4}=-\frac{4a_k}{(k+2)(k+4)}$
- Solution for $r=-2$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+4}=-\frac{4a_k}{(k+2)(k+4)}, a_1=0, a_2=0, a_3=0\right]$
- Recursion relation for $r=0$

  $a_{k+4}=-\frac{4a_k}{(k+4)(k+6)}$
- Solution for $r=0$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k}, a_{k+4}=-\frac{4a_k}{(k+4)(k+6)}, a_1=0, a_2=0, a_3=0\right]$
- Combine solutions and rename parameters

  $\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^{k-2}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k}\right), a_{4+k}=-\frac{4a_k}{(k+2)(4+k)}, a_1=0, a_2=0, a_3=0, b_{4+k}=-\frac{4b_k}{(4+k)(k+6)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 21

```
dsolve(x*diff(diff(y(x),x),x)+3*diff(y(x),x)+4*x^3*y(x) = 0,y(x),singsol=all)
```

$$y=\frac{c_1 \sin\left(x^2\right)+c_2 \cos\left(x^2\right)}{x^2}$$

Mathematica DSolve solution

Solving time : 0.051 (sec)
Leaf size : 41

```
DSolve[{x*D[y[x],{x,2}]+3*D[y[x],x]+4*x^3*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to \frac{4c_1 e^{-ix^2}-ic_2 e^{ix^2}}{4x^2}$$

2.1.709 Problem 726

Internal problem ID [9881]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 726
Date solved : `Monday, January 27, 2025 at 06:15:22 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(2-x)\,y''+2xy'-2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.202 (sec)

Writing the ode as

$$\left(-x^3+2x^2\right)y''+2xy'-2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-x^3+2x^2\\ B&=2x\\ C&=-2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{3}{\left(x^2-2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=3\\ t&=\left(x^2-2x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{3}{\left(x^2-2x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1353: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(-2+x)^2} + \frac{3}{4x} + \frac{3}{4x^2} - \frac{3}{4(-2+x)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = 2$ let b be the coefficient of $\frac{1}{(-2+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^{+} &= 0\\ \alpha_\infty^{-} &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{(x^2-2x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
2	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$\begin{aligned}d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right)\\ &= 1 - (1)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (-)[\sqrt{r}]_\infty\\ &= -\frac{1}{2x} + \frac{3}{2(-2+x)} + (-)(0)\\ &= -\frac{1}{2x} + \frac{3}{2(-2+x)}\\ &= \frac{1+x}{x(-2+x)}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2x}+\frac{3}{2(-2+x)}\right)(0)+\left(\left(\frac{1}{2x^2}-\frac{3}{2(-2+x)^2}\right)+\left(-\frac{1}{2x}+\frac{3}{2(-2+x)}\right)^2-\left(\frac{3}{(x^2-2x)^2}\right)\right)=$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\, dx} \\
&= e^{\int \left(-\frac{1}{2x}+\frac{3}{2(-2+x)}\right)dx} \\
&= \frac{(-2+x)^{3/2}}{\sqrt{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x}{-x^3+2x^2}\, dx} \\
&= z_1 e^{\frac{\ln(-2+x)}{2}-\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{\sqrt{-2+x}}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\frac{(-2+x)^2}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1\int \frac{e^{\int -\frac{2x}{-x^3+2x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1\int \frac{e^{\ln(-2+x)-\ln(x)}}{(y_1)^2}\, dx \\
&= y_1\left(-\frac{(x-1)\, x\, e^{\ln(-2+x)-\ln(x)}}{(-2+x)^3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(-2+x)^2}{x}\right)+c_2\left(\frac{(-2+x)^2}{x}\left(-\frac{(x-1)\, x\, e^{\ln(-2+x)-\ln(x)}}{(-2+x)^3}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(-x+2)\left(\frac{d^2}{dx^2}y(x)\right)+2x\left(\frac{d}{dx}y(x)\right)-2y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{2y(x)}{x^2(x-2)}+\frac{2\left(\frac{d}{dx}y(x)\right)}{x(x-2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)-\frac{2\left(\frac{d}{dx}y(x)\right)}{x(x-2)}+\frac{2y(x)}{x^2(x-2)}=0$

□ Check to see if x_0 is a regular singular point

- ∘ Define functions

$\left[P_2(x)=-\frac{2}{x(x-2)},P_3(x)=\frac{2}{x^2(x-2)}\right]$

- ∘ $x\cdot P_2(x)$ is analytic at $x=0$

$\left.(x\cdot P_2(x))\right|_{x=0}=1$

- ∘ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left.(x^2\cdot P_3(x))\right|_{x=0}=-1$

- ∘ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$x^2(x-2)\left(\frac{d^2}{dx^2}y(x)\right)-2x\left(\frac{d}{dx}y(x)\right)+2y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ∘ Convert $x\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion

$x\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r}$

- ∘ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..3$

$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- ∘ Shift index using $k->k+2-m$

$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

Rewrite ODE with series expansions

$-2a_0(1+r)\,(-1+r)\,x^r+\left(\sum_{k=1}^{\infty}\left(-2a_k(k+r+1)\,(k+r-1)+a_{k-1}(k+r-1)\,(k-2+r)\right)x^{k+}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-2(1+r)\,(-1+r)=0$

- Values of r that satisfy the indicial equation

$r\in\{-1,1\}$

- Each term in the series must be 0, giving the recursion relation

$-2\left(\left(-\frac{k}{2}-\frac{r}{2}+1\right)a_{k-1}+a_k(k+r+1)\right)(k+r-1)=0$

- Shift index using $k->k+1$
  $-2\left(\left(-\frac{k}{2}+\frac{1}{2}-\frac{r}{2}\right)a_k+a_{k+1}(k+2+r)\right)(k+r)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{(k+r-1)a_k}{2(k+2+r)}$
- Recursion relation for $r=-1$; series terminates at $k=2$
  $a_{k+1}=\frac{(k-2)a_k}{2(k+1)}$
- Apply recursion relation for $k=0$
  $a_1=-a_0$
- Apply recursion relation for $k=1$
  $a_2=-\frac{a_1}{4}$
- Express in terms of a_0
  $a_2=\frac{a_0}{4}$
- Terminating series solution of the ODE for $r=-1$. Use reduction of order to find the second lin
  $y(x)=a_0\cdot\left(1-x+\frac{1}{4}x^2\right)$
- Recursion relation for $r=1$
  $a_{k+1}=\frac{ka_k}{2(k+3)}$
- Solution for $r=1$
  $\left[y(x)=\sum_{k=0}^{\infty}a_kx^{k+1},a_{k+1}=\frac{ka_k}{2(k+3)}\right]$
- Combine solutions and rename parameters
  $\left[y(x)=a_0\cdot\left(1-x+\frac{1}{4}x^2\right)+\left(\sum_{k=0}^{\infty}b_kx^{k+1}\right),b_{k+1}=\frac{kb_k}{2(k+3)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 19

```
dsolve(x^2*(-x+2)*diff(diff(y(x),x),x)+2*diff(y(x),x)*x-2*y(x) = 0,y(x),singsol=all)
```

$$y=\frac{c_1x^2+c_2(x-1)}{x}$$

Mathematica DSolve solution

Solving time : 0.191 (sec)
Leaf size : 94

```
DSolve[{x^2*(2-x)*D[y[x],{x,2}]+2*x*D[y[x],x]-2*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to \exp\left(\int_1^x \frac{K[1]+1}{(K[1]-2)K[1]}dK[1] - \frac{1}{2}\int_1^x -\frac{2}{(K[2]-2)K[2]}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{K[1]+1}{(K[1]-2)K[1]}dK[1]\right)dK[3] + c_1\right)$$

2.1.710 Problem 727

Internal problem ID [9882]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 727
Date solved : Monday, January 27, 2025 at 06:15:22 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$\left(x^2+1\right) y''-2xy'+2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.267 (sec)

Writing the ode as

$$\left(x^2+1\right) y''-2xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= -2x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-3}{\left(x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= \left(x^2+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{3}{\left(x^2+1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1355: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{3i}{4(x-i)} - \frac{3i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{3}{(x^2+1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)} + (-)\left(0\right) \\
&= -\frac{1}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)} \\
&= \frac{x-2i}{x^2+1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2(x-i)}+\frac{3}{2(x+i)}\right)(0)+\left(\left(\frac{1}{2(x-i)^2}-\frac{3}{2(x+i)^2}\right)+\left(-\frac{1}{2(x-i)}+\frac{3}{2(x+i)}\right)^2-\left(-\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\,dx} \\
&= \mathrm{e}^{\int\left(-\frac{1}{2(x-i)}+\frac{3}{2(x+i)}\right)dx} \\
&= \frac{(x^2+1)^{3/2}}{(ix+1)^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2+1}\,dx} \\
&= z_1 e^{\frac{\ln(x^2+1)}{2}} \\
&= z_1\left(\sqrt{x^2+1}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2+1)^2}{(ix+1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x^2+1)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{x}{(x+i)^2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(x^2+1)^2}{(ix+1)^2}\right) + c_2\left(\frac{(x^2+1)^2}{(ix+1)^2}\left(-\frac{x}{(x+i)^2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 16

```
dsolve((x^2+1)*diff(diff(y(x),x),x)-2*diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y = c_2 x^2 + c_1 x - c_2$$

Mathematica DSolve solution

Solving time : 0.319 (sec)
Leaf size : 79

```
DSolve[{(x^2+1)*D[y[x],{x,2}]-2*x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \sqrt{x^2+1} \exp \left(\int_1^x \frac{K[1]+2i}{K[1]^2+1} dK[1]\right) \left(c_2 \int_1^x \exp \left(-2 \int_1^{K[2]} \frac{K[1]+2i}{K[1]^2+1} dK[1]\right) dK[2] + c_1\right)$$

2.1.711 Problem 728

Internal problem ID [9883]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 728
Date solved : `Monday, January 27, 2025 at 06:15:23 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' - 2(x+1)\, y' + (x+2)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.124 (sec)

Writing the ode as

$$xy'' + (-2x-2)\, y' + (x+2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -2x-2 \\ C &= x+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1356: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the gcd$(s,t)=1$. This gives $b=2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b} = 2\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b} = -1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{2}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=-1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right)\\
&= -1-(-1)\\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_\infty\\
&= -\frac{1}{x} + (-)(0)\\
&= -\frac{1}{x}\\
&= -\frac{1}{x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) &= 0\\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x-2}{x} \, dx} \\ &= z_1 e^{x+\ln(x)} \\ &= z_1(x\,\mathrm{e}^x) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-2}{x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{2x+2\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{x\,\mathrm{e}^{2x+2\ln(x)}\mathrm{e}^{-2x}}{3} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\mathrm{e}^x) + c_2 \left(\mathrm{e}^x \left(\frac{x\,\mathrm{e}^{2x+2\ln(x)}\mathrm{e}^{-2x}}{3} \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dx^2}y(x)\right) x - 2(x+1)\left(\frac{d}{dx}y(x)\right) + (x+2)\, y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(x+2)y(x)}{x} + \frac{2(x+1)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x) - \frac{2(x+1)\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(x+2)y(x)}{x} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - ◦ Define functions

    $\left[P_2(x) = -\frac{2(x+1)}{x}, P_3(x) = \frac{x+2}{x}\right]$
  - ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -2$
  - ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$
  - ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $\left(\frac{d^2}{dx^2}y(x)\right)x + (-2 - 2x)\left(\frac{d}{dx}y(x)\right) + (x+2)\,y(x) = 0$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - ◦ Shift index using $k- > k - m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..1$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$
  - ◦ Shift index using $k- > k + 1 - m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - ◦ Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-1}$
  - ◦ Shift index using $k- > k + 1$

    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$

    Rewrite ODE with series expansions

    $a_0 r(-3+r)\, x^{-1+r} + (a_1(1+r)(-2+r) - 2a_0(-1+r))\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)(k-2+r)\right.$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 3\}$
- Each term must be 0

  $a_1(1+r)(-2+r) - 2a_0(-1+r) = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k-2+r) - 2a_k k - 2a_k r + 2a_k + a_{k-1} = 0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)(k+r-1)-2a_{k+1}(k+1)-2ra_{k+1}+2a_{k+1}+a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{2ka_{k+1}+2ra_{k+1}-a_k}{(k+2+r)(k+r-1)}$
- Recursion relation for $r=0$

  $a_{k+2}=\frac{2ka_{k+1}-a_k}{(k+2)(k-1)}$
- Series not valid for $r=0$, division by 0 in the recursion relation at $k=1$

  $a_{k+2}=\frac{2ka_{k+1}-a_k}{(k+2)(k-1)}$
- Recursion relation for $r=3$

  $a_{k+2}=\frac{2ka_{k+1}-a_k+6a_{k+1}}{(k+5)(k+2)}$
- Solution for $r=3$

  $\left[y(x)=\sum_{k=0}^{\infty}a_kx^{k+3}, a_{k+2}=\frac{2ka_{k+1}-a_k+6a_{k+1}}{(k+5)(k+2)}, 4a_1-4a_0=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 14

```
dsolve(x*diff(diff(y(x),x),x)-2*(x+1)*diff(y(x),x)+(x+2)*y(x) = 0,y(x),singsol=all)
```

$$y=e^{x}\left(c_2x^3+c_1\right)$$

Mathematica DSolve solution

Solving time : 0.056 (sec)
Leaf size : 25

```
DSolve[{x*D[y[x],{x,2}]-2*(x+1)*D[y[x],x]+(x+2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x)\to\frac{1}{3}e^{x+1}\left(c_2x^3+3c_1\right)$$

2.1.712 Problem 729

Internal problem ID [9884]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 729
Date solved : `Monday, January 27, 2025 at 06:15:24 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$3xy'' - 2(3x-1)\,y' + (3x-2)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.131 (sec)

Writing the ode as

$$3xy'' + (-6x+2)\,y' + (3x-2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x \\ B &= -6x+2 \\ C &= 3x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{9x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -2 \\ t &= 9x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{2}{9x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1358: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 9x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{9x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{2}{9}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{2}{3} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{3} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{2}{9x^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{2}{9}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{2}{3} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{3}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=-\frac{2}{9x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{2}{3}$	$\frac{1}{3}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{2}{3}$	$\frac{1}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{3}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{3}-\left(\frac{1}{3}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^{-}}{x-c_1}\right)+(-)[\sqrt{r}]_\infty \\
&= \frac{1}{3x}+(-)\,(0) \\
&= \frac{1}{3x} \\
&= \frac{1}{3x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}
$$

Let

$$
p(x)=1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{1}{3x}\right)(0)+\left(\left(-\frac{1}{3x^2}\right)+\left(\frac{1}{3x}\right)^2-\left(-\frac{2}{9x^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}z_1(x)&=pe^{\int \omega\,dx}\\&=\mathrm{e}^{\int \frac{1}{3x}dx}\\&=x^{1/3}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1&=z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\&=z_1e^{-\int \frac{1}{2}\frac{-6x+2}{3x}\,dx}\\&=z_1e^{x-\frac{\ln(x)}{3}}\\&=z_1\left(\frac{\mathrm{e}^x}{x^{1/3}}\right)\end{aligned}$$

Which simplifies to

$$y_1=\mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int\frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2&=y_1\int\frac{e^{\int -\frac{-6x+2}{3x}\,dx}}{(y_1)^2}\,dx\\&=y_1\int\frac{e^{2x-\frac{2\ln(x)}{3}}}{(y_1)^2}\,dx\\&=y_1\left(3x\,\mathrm{e}^{2x-\frac{2\ln(x)}{3}}\mathrm{e}^{-2x}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y&=c_1y_1+c_2y_2\\&=c_1(\mathrm{e}^x)+c_2\left(\mathrm{e}^x\left(3x\,\mathrm{e}^{2x-\frac{2\ln(x)}{3}}\mathrm{e}^{-2x}\right)\right)\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$3\left(\frac{d^2}{dx^2}y(x)\right)x - 2(3x-1)\left(\frac{d}{dx}y(x)\right) + (3x-2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(3x-2)y(x)}{3x} + \frac{2(3x-1)\left(\frac{d}{dx}y(x)\right)}{3x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) - \frac{2(3x-1)\left(\frac{d}{dx}y(x)\right)}{3x} + \frac{(3x-2)y(x)}{3x} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = -\frac{2(3x-1)}{3x}, P_3(x) = \frac{3x-2}{3x}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{2}{3}$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $3\left(\frac{d^2}{dx^2}y(x)\right)x + (2-6x)\left(\frac{d}{dx}y(x)\right) + (3x-2)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ○ Shift index using $k- >k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ○ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..1$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- ○ Shift index using $k- >k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- ○ Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-1}$

- ○ Shift index using $k- >k+1$

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

$$a_0 r(-1+3r)\, x^{-1+r} + (a_1(1+r)\,(2+3r) - 2a_0(1+3r))\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(3k+2+3r) -\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
$$r(-1+3r) = 0$$
- Values of r that satisfy the indicial equation
$$r \in \left\{0, \frac{1}{3}\right\}$$
- Each term must be 0
$$a_1(1+r)\,(2+3r) - 2a_0(1+3r) = 0$$
- Each term in the series must be 0, giving the recursion relation
$$3\left(k+\frac{2}{3}+r\right)(k+1+r)\, a_{k+1} - 6a_k k - 6a_k r - 2a_k + 3a_{k-1} = 0$$
- Shift index using $k->k+1$
$$3\left(k+\frac{5}{3}+r\right)(k+2+r)\, a_{k+2} - 6a_{k+1}(k+1) - 6ra_{k+1} - 2a_{k+1} + 3a_k = 0$$
- Recursion relation that defines series solution to ODE
$$a_{k+2} = \frac{6ka_{k+1}+6ra_{k+1}-3a_k+8a_{k+1}}{(3k+5+3r)(k+2+r)}$$
- Recursion relation for $r = 0$
$$a_{k+2} = \frac{6ka_{k+1}-3a_k+8a_{k+1}}{(3k+5)(k+2)}$$
- Solution for $r = 0$
$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{6ka_{k+1}-3a_k+8a_{k+1}}{(3k+5)(k+2)}, 2a_1 - 2a_0 = 0\right]$$
- Recursion relation for $r = \frac{1}{3}$
$$a_{k+2} = \frac{6ka_{k+1}-3a_k+10a_{k+1}}{(3k+6)\left(k+\frac{7}{3}\right)}$$
- Solution for $r = \frac{1}{3}$
$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = \frac{6ka_{k+1}-3a_k+10a_{k+1}}{(3k+6)\left(k+\frac{7}{3}\right)}, 4a_1 - 4a_0 = 0\right]$$
- Combine solutions and rename parameters
$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = \frac{6ka_{k+1}-3a_k+8a_{k+1}}{(3k+5)(k+2)}, 2a_1 - 2a_0 = 0, b_{k+2} = \frac{6kb_{k+1}-3b_k+10b_k}{(3k+6)\left(k+\frac{7}{3}\right)}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 14

```
dsolve(3*x*diff(diff(y(x),x),x)-2*(3*x-1)*diff(y(x),x)+(3*x-2)*y(x) = 0,y(x),singsol=a
```

$$y = e^{x}\left(x^{1/3}c_2 + c_1\right)$$

Mathematica DSolve solution

Solving time : 0.03 (sec)
Leaf size : 21

```
DSolve[{3*x*D[y[x],{x,2}]-2*(3*x-1)*D[y[x],x]+(3*x-2)*y[x]==0,{}},y[x],x,IncludeSingularSolu
```

$$y(x) \to e^{x}\left(3c_2\sqrt[3]{x} + c_1\right)$$

2.1.713 Problem 730

Internal problem ID [9885]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 730
Date solved : Monday, January 27, 2025 at 06:15:24 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$x(x+1)\, y'' - (x-1)\, y' + y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.219 (sec)

Writing the ode as

$$\left(x^2+x\right) y'' + (1-x)\, y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+x \\ B &= 1-x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2-10x-1}{4\left(x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2-10x-1 \\ t &= 4\left(x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2-10x-1}{4\left(x^2+x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1360: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x+1} - \frac{1}{4x^2} + \frac{2}{(x+1)^2} - \frac{2}{x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 10x - 1}{4\left(x^2 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 - 10x - 1}{4\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	2	-1
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x+1} + \frac{1}{2x} + (-)(0) \\
&= -\frac{1}{x+1} + \frac{1}{2x} \\
&= -\frac{x-1}{2x(x+1)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x+1} + \frac{1}{2x}\right)(1) + \left(\left(\frac{1}{(x+1)^2} - \frac{1}{2x^2}\right) + \left(-\frac{1}{x+1} + \frac{1}{2x}\right)^2 - \left(\frac{-x^2 - 10x - 1}{4(x^2+x)^2}\right)\right) = 0$$

$$\frac{1 + a_0}{x(x+1)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x-1) e^{\int \left(-\frac{1}{x+1} + \frac{1}{2x}\right) dx} \\
&= (x-1) e^{\frac{\ln(x)}{2} - \ln(x+1)} \\
&= \frac{(x-1)\sqrt{x}}{x+1}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1-x}{x^2+x} \, dx} \\
&= z_1 e^{-\frac{\ln(x)}{2} + \ln(x+1)} \\
&= z_1 \left(\frac{x+1}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x - 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1-x}{x^2+x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\ln(x) + 2\ln(x+1)}}{(y_1)^2} \, dx \\
&= y_1 \left(\ln(x) - \frac{4}{x-1}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x-1) + c_2\left(x - 1\left(\ln(x) - \frac{4}{x-1}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x(x+1)\left(\frac{d^2}{dx^2}y(x)\right) - (x-1)\left(\frac{d}{dx}y(x)\right) + y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{y(x)}{x(x+1)} + \frac{(x-1)\left(\frac{d}{dx}y(x)\right)}{x(x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) - \frac{(x-1)\left(\frac{d}{dx}y(x)\right)}{x(x+1)} + \frac{y(x)}{x(x+1)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{x-1}{x(x+1)}, P_3(x) = \frac{1}{x(x+1)}\right]$

  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1} = -2$

  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1} = 0$

  - $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $x(x+1)\left(\frac{d^2}{dx^2}y(x)\right) + (1-x)\left(\frac{d}{dx}y(x)\right) + y(x) = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(u^2-u)\left(\frac{d^2}{du^2}y(u)\right) + (2-u)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-a_0 r(-3+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)(k-2+r) + a_k(k+r-1)^2\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
$$-r(-3+r) = 0$$
- Values of r that satisfy the indicial equation
$$r \in \{0, 3\}$$
- Each term in the series must be 0, giving the recursion relation
$$-a_{k+1}(k+1+r)(k-2+r) + a_k(k+r-1)^2 = 0$$
- Recursion relation that defines series solution to ODE
$$a_{k+1} = \frac{a_k(k+r-1)^2}{(k+1+r)(k-2+r)}$$
- Recursion relation for $r=0$; series terminates at $k=1$
$$a_{k+1} = \frac{a_k(k-1)^2}{(k+1)(k-2)}$$
- Apply recursion relation for $k=0$
$$a_1 = -\frac{a_0}{2}$$
- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li
$$y(u) = a_0 \cdot \left(1 - \frac{u}{2}\right)$$
- Revert the change of variables $u = x+1$
$$\left[y(x) = a_0\left(-\frac{x}{2} + \frac{1}{2}\right)\right]$$
- Recursion relation for $r=3$
$$a_{k+1} = \frac{a_k(k+2)^2}{(k+4)(k+1)}$$
- Solution for $r=3$
$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+3}, a_{k+1} = \frac{a_k(k+2)^2}{(k+4)(k+1)}\right]$$
- Revert the change of variables $u = x+1$
$$\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^{k+3}, a_{k+1} = \frac{a_k(k+2)^2}{(k+4)(k+1)}\right]$$
- Combine solutions and rename parameters
$$\left[y(x) = a_0\left(-\frac{x}{2} + \frac{1}{2}\right) + \left(\sum_{k=0}^{\infty} b_k(x+1)^{k+3}\right), b_{k+1} = \frac{b_k(k+2)^2}{(4+k)(k+1)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 20

```
dsolve(x*(x+1)*diff(diff(y(x),x),x)-(x-1)*diff(y(x),x)+y(x) = 0,y(x),singsol=all)
```

$$y = c_2(x-1)\ln(x) - 4c_2 + c_1(x-1)$$

Mathematica DSolve solution

Solving time : 0.444 (sec)
Leaf size : 112

```
DSolve[{x*(x+1)*D[y[x],{x,2}]-(x-1)*D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolutions->True
```

$$y(x) \to (x-1)\exp\left(\int_1^x \left(\frac{1}{2K[1]} - \frac{1}{K[1]+1}\right) dK[1] - \frac{1}{2}\int_1^x \left(\frac{1}{K[2]} - \frac{2}{K[2]+1}\right) dK[2]\right)\left(c_2\int_1^x \frac{\exp\left(-2\int_1^{K[3]} \frac{1-K[1]}{2K[1]^2+2K[1]} dK[1]\right)}{(K[3]-1)^2} dK[3] + c_1\right)$$

2.1.714 Problem 731

Internal problem ID [9886]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 731
Date solved : `Monday, January 27, 2025 at 06:15:25 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(x^2+2x)\,y''-2(x+1)\,y'+2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.204 (sec)

Writing the ode as

$$\begin{align*}(x^2+2x)\,y''+(-2x-2)\,y'+2y&=0 \tag{1}\\ Ay''+By'+Cy&=0 \tag{2}\end{align*}$$

Comparing (1) and (2) shows that

$$\begin{align*}A&=x^2+2x\\ B&=-2x-2 \tag{3}\\ C&=2\end{align*}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{align*}r&=\frac{s}{t} \tag{5}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{align*}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{3}{(x^2+2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{align*}s&=3\\ t&=(x^2+2x)^2\end{align*}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{3}{(x^2+2x)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1362: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{4x} + \frac{3}{4(x+2)^2} + \frac{3}{4(x+2)} + \frac{3}{4x^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3}{(x^2+2x)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-2	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\,(x+2)} + \frac{3}{2x} + (-)\,(0) \\
&= -\frac{1}{2\,(x+2)} + \frac{3}{2x} \\
&= \frac{x+3}{x\,(x+2)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2(x+2)}+\frac{3}{2x}\right)(0)+\left(\left(\frac{1}{2(x+2)^2}-\frac{3}{2x^2}\right)+\left(-\frac{1}{2(x+2)}+\frac{3}{2x}\right)^2-\left(\frac{3}{(x^2+2x)^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x+2)}+\frac{3}{2x}\right)dx} \\
&= \frac{x^{3/2}}{\sqrt{x+2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x-2}{x^2+2x}\, dx} \\
&= z_1 e^{\frac{\ln(x(x+2))}{2}} \\
&= z_1\left(\sqrt{x(x+2)}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\frac{\sqrt{x(x+2)}\,x^{3/2}}{\sqrt{x+2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1\int \frac{e^{\int -\frac{-2x-2}{x^2+2x}\, dx}}{(y_1)^2}\, dx \\
&= y_1\int \frac{e^{\ln(x(x+2))}}{(y_1)^2}\, dx \\
&= y_1\left(-\frac{1}{x^2}-\frac{1}{x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\sqrt{x(x+2)}\,x^{3/2}}{\sqrt{x+2}}\right)+c_2\left(\frac{\sqrt{x(x+2)}\,x^{3/2}}{\sqrt{x+2}}\left(-\frac{1}{x^2}-\frac{1}{x}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x^2+2x)\left(\frac{d^2}{dx^2}y(x)\right)-2(x+1)\left(\frac{d}{dx}y(x)\right)+2y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{2y(x)}{x(x+2)}+\frac{2(x+1)\left(\frac{d}{dx}y(x)\right)}{x(x+2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)-\frac{2(x+1)\left(\frac{d}{dx}y(x)\right)}{x(x+2)}+\frac{2y(x)}{x(x+2)}=0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

$\left[P_2(x)=-\frac{2(x+1)}{x(x+2)},P_3(x)=\frac{2}{x(x+2)}\right]$

- ○ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

$\left((x+2)\cdot P_2(x)\right)\Big|_{x=-2}=-1$

- ○ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

$\left((x+2)^2\cdot P_3(x)\right)\Big|_{x=-2}=0$

- ○ $x=-2$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-2$

- Multiply by denominators

$x(x+2)\left(\frac{d^2}{dx^2}y(x)\right)+(-2-2x)\left(\frac{d}{dx}y(x)\right)+2y(x)=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

$(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(2-2u)\left(\frac{d}{du}y(u)\right)+2y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ○ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

- ○ Shift index using $k->k+2-m$

$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

Rewrite ODE with series expansions

$-2a_0r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)(k+r-1)+a_k(k+r-1)(k+r-2)\right)u^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-2r(-2+r)=0$

- Values of r that satisfy the indicial equation
  $r \in \{0,2\}$
- Each term in the series must be 0, giving the recursion relation
  $((-2k-2r-2)\, a_{k+1}+a_k(k+r-2))\,(k+r-1)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{a_k(k+r-2)}{2(k+1+r)}$
- Recursion relation for $r=0$; series terminates at $k=2$
  $a_{k+1}=\frac{a_k(k-2)}{2(k+1)}$
- Apply recursion relation for $k=0$
  $a_1=-a_0$
- Apply recursion relation for $k=1$
  $a_2=-\frac{a_1}{4}$
- Express in terms of a_0
  $a_2=\frac{a_0}{4}$
- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second linea
  $y(u)=a_0\cdot\left(1-u+\frac{1}{4}u^2\right)$
- Revert the change of variables $u=x+2$
  $\left[y(x)=\frac{a_0x^2}{4}\right]$
- Recursion relation for $r=2$
  $a_{k+1}=\frac{a_k k}{2(k+3)}$
- Solution for $r=2$
  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1}=\frac{a_k k}{2(k+3)}\right]$
- Revert the change of variables $u=x+2$
  $\left[y(x)=\sum_{k=0}^{\infty} a_k(x+2)^{k+2}, a_{k+1}=\frac{a_k k}{2(k+3)}\right]$
- Combine solutions and rename parameters
  $\left[y(x)=\frac{a_0x^2}{4}+\left(\sum_{k=0}^{\infty} b_k(x+2)^{k+2}\right), b_{k+1}=\frac{kb_k}{2(k+3)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 14

```
dsolve((x^2+2*x)*diff(diff(y(x),x),x)-2*(x+1)*diff(y(x),x)+2*y(x) = 0,y(x),singsol=all
```

$$y = c_1x^2 + c_2x + c_2$$

Mathematica DSolve solution

Solving time : 0.182 (sec)
Leaf size : 100

```
DSolve[{(x^2+2*x)*D[y[x],{x,2}]-2*(x+1)*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to \exp\left(\int_1^x \frac{K[1]+3}{K[1]^2+2K[1]}dK[1] - \frac{1}{2}\int_1^x -\frac{2(K[2]+1)}{K[2](K[2]+2)}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{K[1]+3}{K[1]^2+2K[1]}dK[1]\right)dK[3] + c_1\right)$$

2.1.715 Problem 732

Internal problem ID [9887]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 732
Date solved : `Monday, January 27, 2025 at 06:15:25 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+2x\right)y''-2(x+1)\,y'+2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.214 (sec)

Writing the ode as

$$\left(x^2+2x\right)y''+(-2x-2)\,y'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2+2x\\B&=-2x-2\\C&=2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{3}{\left(x^2+2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=3\\t&=\left(x^2+2x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{3}{\left(x^2+2x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1364: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x+2)^2} + \frac{3}{4x^2} - \frac{3}{4x} + \frac{3}{4(x+2)}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3}{\left(x^2+2x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-2	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(x+2\right)} + \frac{3}{2x} + (-)\left(0\right) \\
&= -\frac{1}{2\left(x+2\right)} + \frac{3}{2x} \\
&= \frac{x+3}{x\left(x+2\right)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2\left(x+2\right)}+\frac{3}{2x}\right)(0)+\left(\left(\frac{1}{2\left(x+2\right)^{2}}-\frac{3}{2x^{2}}\right)+\left(-\frac{1}{2\left(x+2\right)}+\frac{3}{2x}\right)^{2}-\left(\frac{3}{\left(x^{2}+2x\right)^{2}}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2(x+2)}+\frac{3}{2x}\right)dx} \\ &= \frac{x^{3/2}}{\sqrt{x+2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x-2}{x^2+2x}\, dx} \\ &= z_1 e^{\frac{\ln(x(x+2))}{2}} \\ &= z_1\left(\sqrt{x\left(x+2\right)}\right) \end{aligned}$$

Which simplifies to

$$y_1=\frac{\sqrt{x\left(x+2\right)}\, x^{3/2}}{\sqrt{x+2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-2}{x^2+2x}\, dx}}{\left(y_1\right)^2}\, dx \\ &= y_1 \int \frac{e^{\ln(x(x+2))}}{\left(y_1\right)^2}\, dx \\ &= y_1\left(-\frac{1}{x}-\frac{1}{x^2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\sqrt{x\left(x+2\right)}\, x^{3/2}}{\sqrt{x+2}}\right)+c_2\left(\frac{\sqrt{x\left(x+2\right)}\, x^{3/2}}{\sqrt{x+2}}\left(-\frac{1}{x}-\frac{1}{x^2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x^2+2x)\left(\frac{d^2}{dx^2}y(x)\right)-2(x+1)\left(\frac{d}{dx}y(x)\right)+2y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{2y(x)}{x(x+2)}+\frac{2(x+1)\left(\frac{d}{dx}y(x)\right)}{x(x+2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x)-\frac{2(x+1)\left(\frac{d}{dx}y(x)\right)}{x(x+2)}+\frac{2y(x)}{x(x+2)}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=-\frac{2(x+1)}{x(x+2)}, P_3(x)=\frac{2}{x(x+2)}\right]$

- ○ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

  $\left((x+2)\cdot P_2(x)\right)\bigg|_{x=-2}=-1$

- ○ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

  $\left((x+2)^2\cdot P_3(x)\right)\bigg|_{x=-2}=0$

- ○ $x=-2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-2$

- Multiply by denominators

$x(x+2)\left(\frac{d^2}{dx^2}y(x)\right)+(-2-2x)\left(\frac{d}{dx}y(x)\right)+2y(x)=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

$(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(2-2u)\left(\frac{d}{du}y(u)\right)+2y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ○ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

- ○ Shift index using $k->k+2-m$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)(k+r-1)+a_k(k+r-1)(k+r-2)\right)u^{k+r}\right)=$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-2r(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{0,2\}$

- Each term in the series must be 0, giving the recursion relation

  $((-2k-2r-2)\,a_{k+1}+a_k(k+r-2))\,(k+r-1)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k(k+r-2)}{2(k+1+r)}$

- Recursion relation for $r=0$; series terminates at $k=2$

  $a_{k+1}=\frac{a_k(k-2)}{2(k+1)}$

- Apply recursion relation for $k=0$

  $a_1=-a_0$

- Apply recursion relation for $k=1$

  $a_2=-\frac{a_1}{4}$

- Express in terms of a_0

  $a_2=\frac{a_0}{4}$

- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li

  $y(u)=a_0\cdot\left(1-u+\frac{1}{4}u^2\right)$

- Revert the change of variables $u=x+2$

  $\left[y(x)=\frac{a_0x^2}{4}\right]$

- Recursion relation for $r=2$

  $a_{k+1}=\frac{a_kk}{2(k+3)}$

- Solution for $r=2$

  $\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k+2},a_{k+1}=\frac{a_kk}{2(k+3)}\right]$

- Revert the change of variables $u=x+2$

  $\left[y(x)=\sum_{k=0}^{\infty}a_k(x+2)^{k+2},a_{k+1}=\frac{a_kk}{2(k+3)}\right]$

- Combine solutions and rename parameters

  $\left[y(x)=\frac{a_0x^2}{4}+\left(\sum_{k=0}^{\infty}b_k(x+2)^{k+2}\right),b_{k+1}=\frac{kb_k}{2(k+3)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 14

```
dsolve((x^2+2*x)*diff(diff(y(x),x),x)-2*(x+1)*diff(y(x),x)+2*y(x) = 0,y(x),singsol=all)
```

$$y = c_1 x^2 + c_2 x + c_2$$

Mathematica DSolve solution

Solving time : 0.163 (sec)
Leaf size : 100

```
DSolve[{(x^2+2*x)*D[y[x],{x,2}]-2*(x+1)*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \exp\left(\int_1^x \frac{K[1]+3}{K[1]^2+2K[1]}dK[1] - \frac{1}{2}\int_1^x -\frac{2(K[2]+1)}{K[2](K[2]+2)}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{K[1]+3}{K[1]^2+2K[1]}dK[1]\right)dK[3] + c_1\right)$$

2.1.716 Problem 733

Internal problem ID [9888]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 733
Date solved : `Monday, January 27, 2025 at 06:15:26 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+1\right) y''-2xy'+2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.259 (sec)

Writing the ode as

$$\left(x^2+1\right) y''-2xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= -2x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-3}{\left(x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= \left(x^2+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{3}{\left(x^2+1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1366: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{3i}{4(x-i)} - \frac{3i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= 0\\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right)\\ &= 1 - (1)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (-)[\sqrt{r}]_\infty\\ &= -\frac{1}{2\,(x-i)} + \frac{3}{2\,(x+i)} + (-)\,(0)\\ &= -\frac{1}{2\,(x-i)} + \frac{3}{2\,(x+i)}\\ &= \frac{x-2i}{x^2+1}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2(x-i)}+\frac{3}{2(x+i)}\right)(0)+\left(\left(\frac{1}{2(x-i)^2}-\frac{3}{2(x+i)^2}\right)+\left(-\frac{1}{2(x-i)}+\frac{3}{2(x+i)}\right)^2-\left(-\frac{\quad}{(x^2\ \cdots}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x-i)}+\frac{3}{2(x+i)}\right)dx} \\
&= \frac{(x^2+1)^{3/2}}{(ix+1)^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2+1}\, dx} \\
&= z_1 e^{\frac{\ln(x^2+1)}{2}} \\
&= z_1\left(\sqrt{x^2+1}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2+1)^2}{(ix+1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2+1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\ln(x^2+1)}}{(y_1)^2}\, dx \\
&= y_1\left(-\frac{x}{(x+i)^2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(x^2+1)^2}{(ix+1)^2}\right) + c_2\left(\frac{(x^2+1)^2}{(ix+1)^2}\left(-\frac{x}{(x+i)^2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 16

```
dsolve((x^2+1)*diff(diff(y(x),x),x)-2*diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y = c_2 x^2 + c_1 x - c_2$$

Mathematica DSolve solution

Solving time : 0.341 (sec)
Leaf size : 79

```
DSolve[{(x^2+1)*D[y[x],{x,2}]-2*x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to \sqrt{x^2+1}\exp\left(\int_1^x \frac{K[1]+2i}{K[1]^2+1}dK[1]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[2]} \frac{K[1]+2i}{K[1]^2+1}dK[1]\right)dK[2] + c_1\right)$$

2.1.717 Problem 734

Internal problem ID [9889]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 734
Date solved : `Monday, January 27, 2025 at 06:15:27 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+1\right)y''-2xy'+2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.267 (sec)

Writing the ode as

$$\left(x^2+1\right)y''-2xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2+1\\ B&=-2x\\ C&=2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-3}{\left(x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=-3\\ t&=\left(x^2+1\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{3}{\left(x^2+1\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1367: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 + 1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{3i}{4(x-i)} - \frac{3i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{3}{(x^2+1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\,(x-i)} + \frac{3}{2\,(x+i)} + (-)\,(0) \\
&= -\frac{1}{2\,(x-i)} + \frac{3}{2\,(x+i)} \\
&= \frac{x-2i}{x^2+1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2\left(x-i\right)}+\frac{3}{2\left(x+i\right)}\right)(0)+\left(\left(\frac{1}{2\left(x-i\right)^{2}}-\frac{3}{2\left(x+i\right)^{2}}\right)+\left(-\frac{1}{2\left(x-i\right)}+\frac{3}{2\left(x+i\right)}\right)^{2}-\left(-\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2(x-i)}+\frac{3}{2(x+i)}\right)dx} \\ &= \frac{\left(x^2+1\right)^{3/2}}{\left(ix+1\right)^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2+1}\, dx} \\ &= z_1 e^{\frac{\ln\left(x^2+1\right)}{2}} \\ &= z_1\left(\sqrt{x^2+1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(x^2+1\right)^2}{\left(ix+1\right)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2+1}\, dx}}{\left(y_1\right)^2}\, dx \\ &= y_1 \int \frac{e^{\ln\left(x^2+1\right)}}{\left(y_1\right)^2}\, dx \\ &= y_1\left(-\frac{x}{\left(x+i\right)^2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\left(x^2+1\right)^2}{\left(ix+1\right)^2}\right) + c_2\left(\frac{\left(x^2+1\right)^2}{\left(ix+1\right)^2}\left(-\frac{x}{\left(x+i\right)^2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 16

```
dsolve((x^2+1)*diff(diff(y(x),x),x)-2*diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y = c_2x^2 + c_1x - c_2$$

Mathematica DSolve solution

Solving time : 0.344 (sec)
Leaf size : 79

```
DSolve[{(x^2+1)*D[y[x],{x,2}]-2*x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \sqrt{x^2+1}\exp\left(\int_1^x \frac{K[1]+2i}{K[1]^2+1}dK[1]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[2]}\frac{K[1]+2i}{K[1]^2+1}dK[1]\right)dK[2] + c_1\right)$$

2.1.718 Problem 735

Internal problem ID [9890]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 735
Date solved : `Monday, January 27, 2025 at 06:15:27 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - 4xy' + \left(4x^2 - 2\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.054 (sec)

Writing the ode as

$$y'' - 4xy' + \left(4x^2 - 2\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4x \\ C &= 4x^2 - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1368: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{1}\, dx} \\ &= z_1 e^{x^2} \\ &= z_1 \left(e^{x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x^2}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{x^2}\right) + c_2\left(\mathrm{e}^{x^2}(x)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) - 4x\left(\frac{d}{dx}y(x)\right) + (4x^2 - 2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  - ◦ Shift index using $k- >k - m$

    $x^m \cdot y(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  - ◦ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  - ◦ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - ◦ Shift index using $k- >k + 2$

    $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 - 2a_0 + (6a_3 - 6a_1)\,x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - 2a_k(2k+1) + 4a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 - 2a_0 = 0, 6a_3 - 6a_1 = 0]$

- Solve for the dependent coefficient(s)

  $\{a_2 = a_0, a_3 = a_1\}$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\,a_{k+2} - 4a_k k - 2a_k + 4a_{k-2} = 0$

- Shift index using $k->k+2$

  $$\left((k+2)^2+3k+8\right)a_{k+4}-4a_{k+2}(k+2)-2a_{k+2}+4a_k=0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y(x)=\sum_{k=0}^{\infty}a_kx^k, a_{k+4}=\frac{2(2ka_{k+2}-2a_k+5a_{k+2})}{k^2+7k+12}, a_2=a_0, a_3=a_1\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 14

```
dsolve(diff(diff(y(x),x),x)-4*diff(y(x),x)*x+(4*x^2-2)*y(x) = 0,y(x),singsol=all)
```

$$y=e^{x^2}(c_2x+c_1)$$

Mathematica DSolve solution

Solving time : 0.022 (sec)
Leaf size : 18

```
DSolve[{D[y[x],{x,2}]-4*x*D[y[x],x]+(4*x^2-2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True
```

$$y(x)\to e^{x^2}(c_2x+c_1)$$

2.1.719 Problem 736

Internal problem ID [9891]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 736
Date solved : Monday, January 27, 2025 at 06:15:28 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$y'' - 4xy' + \left(4x^2 - 2\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.051 (sec)

Writing the ode as

$$y'' - 4xy' + \left(4x^2 - 2\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4x \\ C &= 4x^2 - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1370: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{1}\, dx} \\ &= z_1 e^{x^2} \\ &= z_1 \left(e^{x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x^2}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{x^2}\right) + c_2\left(\mathrm{e}^{x^2}(x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) - 4x\left(\frac{d}{dx}y(x)\right) + (4x^2-2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k->k-m$

  $x^m \cdot y(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

  $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k->k+2$

  $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 - 2a_0 + (6a_3 - 6a_1)\,x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - 2a_k(2k+1) + 4a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 - 2a_0 = 0, 6a_3 - 6a_1 = 0]$

- Solve for the dependent coefficient(s)

  $\{a_2 = a_0, a_3 = a_1\}$

- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)\,a_{k+2} - 4a_k k - 2a_k + 4a_{k-2} = 0$

- Shift index using $k->k+2$

  $$\left((k+2)^2+3k+8\right)a_{k+4}-4a_{k+2}(k+2)-2a_{k+2}+4a_k=0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y(x)=\sum_{k=0}^{\infty}a_k x^k, a_{k+4}=\frac{2(2ka_{k+2}-2a_k+5a_{k+2})}{k^2+7k+12}, a_2=a_0, a_3=a_1\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 14

```
dsolve(diff(diff(y(x),x),x)-4*diff(y(x),x)*x+(4*x^2-2)*y(x) = 0,y(x),singsol=all)
```

$$y=\mathrm{e}^{x^2}(c_2x+c_1)$$

Mathematica DSolve solution

Solving time : 0.019 (sec)
Leaf size : 18

```
DSolve[{D[y[x],{x,2}]-4*x*D[y[x],x]+(4*x^2-2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tru
```

$$y(x)\to e^{x^2}(c_2x+c_1)$$

2.1.720 Problem 737

Internal problem ID [9892]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 737
Date solved : `Monday, January 27, 2025 at 06:15:28 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(2x-3)\,y''-xy'+y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.335 (sec)

Writing the ode as

$$(2x-3)\,y''-xy'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x-3\\ B &= -x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{x^2-8x+18}{4\,(2x-3)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2-8x+18 \\ t &= 4(2x-3)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x^2-8x+18}{4\,(2x-3)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1372: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x-3)^2$. There is a pole at $x = \frac{3}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16} + \frac{33}{64\left(x - \frac{3}{2}\right)^2} - \frac{5}{16\left(x - \frac{3}{2}\right)}$$

For the pole at $x = \frac{3}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{3}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{33}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{8} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{4} - \frac{5}{8x} - \frac{11}{16x^2} - \frac{1}{32x^3} + \frac{245}{64x^4} + \frac{2591}{128x^5} + \frac{21117}{256x^6} + \frac{154743}{512x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{4} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{16}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 8x + 18}{16x^2 - 48x + 36} \\ &= Q + \frac{R}{16x^2 - 48x + 36} \\ &= \left(\frac{1}{16}\right) + \left(\frac{-5x + \frac{63}{4}}{16x^2 - 48x + 36}\right) \\ &= \frac{1}{16} + \frac{-5x + \frac{63}{4}}{16x^2 - 48x + 36} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -5. Dividing this by leading coefficient in t which is 16 gives $-\frac{5}{16}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{5}{16}\right) - (0) \\ &= -\frac{5}{16} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{4} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{16}}{\frac{1}{4}} - 0\right) &&= -\frac{5}{8} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{16}}{\frac{1}{4}} - 0\right) &&= \frac{5}{8}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 8x + 18}{4\left(2x - 3\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$\frac{3}{2}$	2	0	$\frac{11}{8}$	$-\frac{3}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{4}$	$-\frac{5}{8}$	$\frac{5}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{5}{8}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{5}{8} - \left(-\frac{3}{8}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{8\left(x - \frac{3}{2}\right)} + (-)\left(\frac{1}{4}\right) \\
&= -\frac{3}{8\left(x - \frac{3}{2}\right)} - \frac{1}{4} \\
&= -\frac{x}{4x - 6}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{3}{8\left(x-\frac{3}{2}\right)}-\frac{1}{4}\right)(1)+\left(\left(\frac{3}{8\left(x-\frac{3}{2}\right)^{2}}\right)+\left(-\frac{3}{8\left(x-\frac{3}{2}\right)}-\frac{1}{4}\right)^{2}-\left(\frac{x^{2}-8x+18}{4\left(2x-3\right)^{2}}\right)\right)=0$$

$$\frac{a_0}{2x-3}=0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0=0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x)=x$$

Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (x)\, \mathrm{e}^{\int \left(-\frac{3}{8\left(x-\frac{3}{2}\right)}-\frac{1}{4}\right) dx} \\
&= (x)\, \mathrm{e}^{-\frac{x}{4}-\frac{3\ln(2x-3)}{8}} \\
&= \frac{x\, \mathrm{e}^{-\frac{x}{4}}}{(2x-3)^{3/8}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{2x-3}\, dx} \\
&= z_1 e^{\frac{x}{4}+\frac{3\ln(2x-3)}{8}} \\
&= z_1\left((2x-3)^{3/8}\, \mathrm{e}^{\frac{x}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{2x-3}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{x}{2}+\frac{3\ln(2x-3)}{4}}}{(y_1)^2}\, dx \\
&= y_1\left(\int \frac{\mathrm{e}^{\frac{x}{2}+\frac{3\ln(2x-3)}{4}}}{x^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2\left(x\left(\int \frac{\mathrm{e}^{\frac{x}{2}+\frac{3\ln(2x-3)}{4}}}{x^2} dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(2x-3)\left(\frac{d^2}{dx^2}y(x)\right)-x\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{y(x)}{2x-3}+\frac{\left(\frac{d}{dx}y(x)\right)x}{2x-3}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x)-\frac{\left(\frac{d}{dx}y(x)\right)x}{2x-3}+\frac{y(x)}{2x-3}=0$

□ Check to see if $x_0=\frac{3}{2}$ is a regular singular point

- ○ Define functions

$\left[P_2(x)=-\frac{x}{2x-3}, P_3(x)=\frac{1}{2x-3}\right]$

- ○ $\left(x-\frac{3}{2}\right)\cdot P_2(x)$ is analytic at $x=\frac{3}{2}$

$\left(\left(x-\frac{3}{2}\right)\cdot P_2(x)\right)\bigg|_{x=\frac{3}{2}}=-\frac{3}{4}$

- ○ $\left(x-\frac{3}{2}\right)^2\cdot P_3(x)$ is analytic at $x=\frac{3}{2}$

$\left(\left(x-\frac{3}{2}\right)^2\cdot P_3(x)\right)\bigg|_{x=\frac{3}{2}}=0$

- ○ $x=\frac{3}{2}$is a regular singular point

Check to see if $x_0=\frac{3}{2}$ is a regular singular point

$x_0=\frac{3}{2}$

- Multiply by denominators

$(2x-3)\left(\frac{d^2}{dx^2}y(x)\right)-x\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Change variables using $x=u+\frac{3}{2}$ so that the regular singular point is at $u=0$

$2u\left(\frac{d^2}{du^2}y(u)\right)+\left(-u-\frac{3}{2}\right)\left(\frac{d}{du}y(u)\right)+y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ○ Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-1}$

- ○ Shift index using $k->k+1$

$u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$

Rewrite ODE with series expansions

$\frac{a_0r(-7+4r)u^{-1+r}}{2}+\left(\sum_{k=0}^{\infty}\left(\frac{a_{k+1}(k+1+r)(4k-3+4r)}{2}-a_k(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $\frac{r(-7+4r)}{2} = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, \frac{7}{4}\}$
- Each term in the series must be 0, giving the recursion relation
  $2\left(k-\frac{3}{4}+r\right)(k+1+r)\,a_{k+1} - a_k(k+r-1) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{2a_k(k+r-1)}{(4k-3+4r)(k+1+r)}$
- Recursion relation for $r = 0$; series terminates at $k = 1$
  $a_{k+1} = \frac{2a_k(k-1)}{(4k-3)(k+1)}$
- Apply recursion relation for $k = 0$
  $a_1 = \frac{2a_0}{3}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li
  $y(u) = a_0 \cdot \left(1+\frac{2u}{3}\right)$
- Revert the change of variables $u = x - \frac{3}{2}$
  $\left[y(x) = \frac{2a_0x}{3}\right]$
- Recursion relation for $r = \frac{7}{4}$
  $a_{k+1} = \frac{2a_k\left(k+\frac{3}{4}\right)}{(4k+4)\left(k+\frac{11}{4}\right)}$
- Solution for $r = \frac{7}{4}$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{7}{4}}, a_{k+1} = \frac{2a_k\left(k+\frac{3}{4}\right)}{(4k+4)\left(k+\frac{11}{4}\right)}\right]$
- Revert the change of variables $u = x - \frac{3}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k \left(x-\frac{3}{2}\right)^{k+\frac{7}{4}}, a_{k+1} = \frac{2a_k\left(k+\frac{3}{4}\right)}{(4k+4)\left(k+\frac{11}{4}\right)}\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \frac{2a_0x}{3} + \left(\sum_{k=0}^{\infty} b_k \left(x-\frac{3}{2}\right)^{k+\frac{7}{4}}\right), b_{k+1} = \frac{2b_k\left(k+\frac{3}{4}\right)}{(4k+4)\left(k+\frac{11}{4}\right)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
```

```
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returnin
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.033 (sec)
Leaf size : 29

```
dsolve((2*x-3)*diff(diff(y(x),x),x)-diff(y(x),x)*x+y(x) = 0,y(x),singsol=all)
```

$$y = 2(2x-3)^{3/4}\left(x-\frac{3}{2}\right)c_1\,\text{KummerM}\left(\frac{3}{4},\frac{11}{4},\frac{x}{2}-\frac{3}{4}\right)+c_2x$$

Mathematica DSolve solution

Solving time : 0.083 (sec)
Leaf size : 63

```
DSolve[{(2*x-3)*D[y[x],{x,2}]-x*D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to 2\ 2^{3/4}(2x-3)\left(c_2(2x-3)^{3/4}L_{-\frac{3}{4}}^{\frac{7}{4}}\left(\frac{x}{2}-\frac{3}{4}\right)+\frac{4\sqrt{2}c_1x}{2x-3}\right)$$

2.1.721 Problem 738

Internal problem ID [9893]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 738
Date solved : `Monday, January 27, 2025 at 06:15:29 PM`
CAS classification : `[_Hermite]`

Solve

$$y'' - xy' - 3y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.223 (sec)

Writing the ode as

$$y'' - xy' - 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 10}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 10 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} + \frac{5}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1374: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + \frac{5}{2x} - \frac{25}{4x^3} + \frac{125}{4x^5} - \frac{3125}{16x^7} + \frac{21875}{16x^9} - \frac{328125}{32x^{11}} + \frac{2578125}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 10}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} + \frac{5}{2}\right) + (0) \\ &= \frac{x^2}{4} + \frac{5}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{5}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{5}{2}\right) - (0) \\ &= \frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= 2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= -3 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} + \frac{5}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	2	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{x}{2}\right) \\ &= \frac{x}{2} \\ &= \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (2) + 2\left(\frac{x}{2}\right)(2x + a_1) + \left(\left(\frac{1}{2}\right) + \left(\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} + \frac{5}{2}\right)\right) &= 0 \\ -a_1 x - 2a_0 + 2 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (x^2+1)\, e^{\int \frac{x}{2} dx} \\ &= (x^2+1)\, e^{\frac{x^2}{4}} \\ &= (x^2+1)\, e^{\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1 \left(e^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{\frac{x^2}{2}}\left(x^2+1\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{e^{-\frac{x^2}{2}}}{(x^2+1)^2}dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(e^{\frac{x^2}{2}}\left(x^2+1\right)\right) + c_2\left(e^{\frac{x^2}{2}}\left(x^2+1\right)\left(\int \frac{e^{-\frac{x^2}{2}}}{(x^2+1)^2}dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) - x\left(\frac{d}{dx}y(x)\right) - 3y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

  $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

  $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_k(k+3)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)\, a_{k+2} - a_k(k+3) = 0$

- Recursion relation that defines the series solution to the ODE

$$\left[y(x)=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=\frac{a_k(k+3)}{k^2+3k+2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special fu
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.038 (sec)
Leaf size : 37

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x-3*y(x) = 0,y(x),singsol=all)
```

$$y=\left(x^2+1\right)\left(\sqrt{\pi}\,\operatorname{erf}\left(\frac{\sqrt{2}\,x}{2}\right)c_1+c_2\right)\mathrm{e}^{\frac{x^2}{2}}+\sqrt{2}\,c_1 x$$

Mathematica DSolve solution

Solving time : 0.02 (sec)
Leaf size : 35

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-3*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to c_1\,\text{HermiteH}\left(-3,\frac{x}{\sqrt{2}}\right)+c_2e^{\frac{x^2}{2}}\left(x^2+1\right)$$

2.1.722 Problem 739

Internal problem ID [9894]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 739
Date solved : `Monday, January 27, 2025 at 06:15:30 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+1\right) y''-x y'+y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.245 (sec)

Writing the ode as

$$\left(x^2+1\right) y''-x y'+y=0 \tag{1}$$
$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= -x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{-x^2-6}{4\left(x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2-6 \\ t &= 4\left(x^2+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-x^2-6}{4\left(x^2+1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1376: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16\,(x-i)^2} + \frac{5}{16\,(x+i)^2} + \frac{7i}{16\,(x-i)} - \frac{7i}{16\,(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 6}{4\left(x^2+1\right)^2}$$

Since the gcd$(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 - 6}{4\left(x^2+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-i$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{4\left(x-i\right)} - \frac{1}{4\left(x+i\right)} + (-)\left(0\right) \\
&= -\frac{1}{4\left(x-i\right)} - \frac{1}{4\left(x+i\right)} \\
&= -\frac{x}{2x^2+2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{4(x-i)} - \frac{1}{4(x+i)}\right)(1) + \left(\left(\frac{1}{4(x-i)^2} + \frac{1}{4(x+i)^2}\right) + \left(-\frac{1}{4(x-i)} - \frac{1}{4(x+i)}\right)^2 - \left(\frac{-x^2}{4(x^2 \cdots}\right.\right. \cdots \frac{(x^2+1)\, a_{\cdots}}{(-x+i)^2 (x \cdots}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (x)\, e^{\int \left(-\frac{1}{4(x-i)} - \frac{1}{4(x+i)}\right) dx} \\ &= (x) \frac{1}{((-x+i)(x+i))^{1/4}} \\ &= \frac{x}{(-x^2-1)^{1/4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{x^2+1}\, dx} \\ &= z_1 e^{\frac{\ln(x^2+1)}{4}} \\ &= z_1\left((x^2+1)^{1/4}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \left(\frac{1}{2} - \frac{i}{2}\right) x\sqrt{2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x^2+1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\frac{\ln(x^2+1)}{2}}}{(y_1)^2}\, dx \\ &= y_1\left(i\left(-\frac{(x^2+1)^{3/2}}{x} + x\sqrt{x^2+1} + \operatorname{arcsinh}(x)\right)\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\left(\frac{1}{2}-\frac{i}{2}\right)x\sqrt{2}\right)+c_2\left(\left(\frac{1}{2}-\frac{i}{2}\right)x\sqrt{2}\left(i\left(-\frac{(x^2+1)^{3/2}}{x}+x\sqrt{x^2+1}+\operatorname{arcsinh}(x)\right)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 23

```
dsolve((x^2+1)*diff(diff(y(x),x),x)-diff(y(x),x)*x+y(x) = 0,y(x),singsol=all)
```

$$
y = -\sqrt{x^2+1}\, c_2 + x(\operatorname{arcsinh}(x)\, c_2 + c_1)
$$

Mathematica DSolve solution

Solving time : 0.05 (sec)
Leaf size : 29

```
DSolve[{(1+x^2)*D[y[x],{x,2}]-x*D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$
y(x) \to c_2 x \operatorname{arcsinh}(x) - c_2\sqrt{x^2+1} + c_1 x
$$

2.1.723 Problem 740

Internal problem ID [9895]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 740
Date solved : `Monday, January 27, 2025 at 06:15:30 PM`
CAS classification : `[_Hermite]`

Solve

$$y'' - xy' + 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.232 (sec)

Writing the ode as

$$y'' - xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 10}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 10 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{5}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1377: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{5}{2x} - \frac{25}{4x^3} - \frac{125}{4x^5} - \frac{3125}{16x^7} - \frac{21875}{16x^9} - \frac{328125}{32x^{11}} - \frac{2578125}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 10}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} - \frac{5}{2}\right) + (0) \\ &= \frac{x^2}{4} - \frac{5}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{5}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{5}{2}\right) - (0) \\ &= -\frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= -3 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= 2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{5}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (2) + 2\left(-\frac{x}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{5}{2}\right)\right) &= 0 \\ a_1 x + 2a_0 + 2 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \left(x^2 - 1\right) e^{\int -\frac{x}{2} dx} \\ &= \left(x^2 - 1\right) e^{-\frac{x^2}{4}} \\ &= \left(x^2 - 1\right) e^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1\left(e^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^2 - 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\,dx \\ &= y_1 \left(\int \frac{e^{\frac{x^2}{2}}}{(x^2-1)^2}\,dx \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (x^2 - 1) + c_2 \left(x^2 - 1 \left(\int \frac{e^{\frac{x^2}{2}}}{(x^2-1)^2}\,dx \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2}y(x) - x\left(\frac{d}{dx}y(x)\right) + 2y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite DE with series expansions

  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion
    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$
  - Convert $\frac{d^2}{dx^2}y(x)$ to series expansion
    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$
  - Shift index using $k- >k+2$
    $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite DE with series expansions
  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_k(k-2)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation
  $(k^2+3k+2)\, a_{k+2} - a_k(k-2) = 0$
- Recursion relation; series terminates at $k = 2$

$a_{k+2} = \frac{a_k(k-2)}{k^2+3k+2}$

- Apply recursion relation for $k = 0$

  $a_2 = -a_0$

- Terminating series solution of the ODE. Use reduction of order to find the second linearly ind

  $y(x) = A_2x^2 + A_1x - a_0$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special functio
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.029 (sec)
Leaf size : 42

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y = 2c_1 e^{\frac{x^2}{2}} x - (x-1)(x+1)\left(c_1\sqrt{\pi}\,\mathrm{erfi}\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2} - c_2\right)$$

Mathematica DSolve solution

Solving time : 0.231 (sec)
Leaf size : 43

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to (x^2-1)\left(c_2\int_1^x \frac{e^{\frac{K[1]^2}{2}}}{(K[1]^2-1)^2}dK[1] + c_1\right)$$

2.1.724 Problem 741

Internal problem ID [9896]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 741
Date solved : Monday, January 27, 2025 at 06:15:31 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$\left(-x^2+1\right) y''-y'+y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.662 (sec)

Writing the ode as

$$\left(-x^2+1\right) y''-y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+1 \\ B &= -1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{4x^2-4x-3}{4\left(x^2-1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2-4x-3 \\ t &= 4\left(x^2-1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{4x^2-4x-3}{4\left(x^2-1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1379: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16\,(x+1)^2} - \frac{3}{16\,(x-1)^2} - \frac{7}{16\,(x+1)} + \frac{7}{16\,(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{4x^2 - 4x - 3}{4\left(x^2 - 1\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 1$. Hence

$$\begin{aligned} E_\infty &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{2\} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
1	2	$\{1, 2, 3\}$
-1	2	$\{-1, 2, 5\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1,\ e_2 = -1,\ e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(2 - (1 + (-1))) \\ &= 1 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x - c} \\ &= \frac{1}{2}\left(\frac{1}{(x - (1))} + \frac{-1}{(x - (-1))}\right) \\ &= \frac{1}{2x - 2} - \frac{1}{2\left(x + 1\right)} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 1$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 1$, then letting

$$p = x + a_0 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$\frac{4a_0 - 6}{\left(x + 1\right)^2\left(x - 1\right)} = 0$$

And solving for p gives

$$p = x + \frac{3}{2}$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p}\\ &= \frac{1}{x+\frac{3}{2}} + \frac{1}{2x-2} - \frac{1}{2(x+1)}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{x+\frac{3}{2}} + \frac{1}{2x-2} - \frac{1}{2(x+1)}\right) w + \frac{-8x^3 - 4x^2 + 10x + 7}{4(x^2-1)^2(2x+3)} = 0$$

Solving for ω gives

$$\omega = \frac{2\sqrt{5}\,\sqrt{(x-1)(x+1)}\,x + 2\sqrt{5}\,\sqrt{(x-1)(x+1)} + 2x^2 + 2x + 1}{2(2x+3)(x-1)(x+1)}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega\,dx}\\ &= e^{\int \frac{2\sqrt{5}\,\sqrt{(x-1)(x+1)}\,x + 2\sqrt{5}\,\sqrt{(x-1)(x+1)} + 2x^2 + 2x + 1}{2(2x+3)(x-1)(x+1)}dx}\\ &= \frac{(x-1)^{1/4}\sqrt{2x+3}\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}} 5^{1/4}}{(x+1)^{1/4}\sqrt{\frac{5\sqrt{x^2-1}+(2+3x)\sqrt{5}}{\sqrt{x^2-1}\sqrt{-\frac{(2x+3)^2}{x^2-1}}}}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-1}{-x^2+1}\,dx}\\ &= z_1 e^{\frac{\operatorname{arctanh}(x)}{2}}\\ &= z_1\left(\sqrt{\frac{x+1}{\sqrt{-x^2+1}}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{\frac{x+1}{\sqrt{-x^2+1}}}\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}}\sqrt{2x+3}\,(5x-5)^{1/4}}{\sqrt{\frac{i\left(3\sqrt{5}\,x+5\sqrt{x^2-1}+2\sqrt{5}\right)}{2x+3}}\,(x+1)^{1/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{-x^2+1}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{\operatorname{arctanh}(x)}}{(y_1)^2}\,dx\\ &= y_1\left(\int \frac{i\sqrt{x+1}\left(x+\sqrt{x^2-1}\right)^{-\sqrt{5}}\left(3\sqrt{5}\,x+5\sqrt{x^2-1}+2\sqrt{5}\right)}{(2x+3)^2\sqrt{5x-5}}dx\right)\end{aligned}$$

Therefore the solution is

$$y = c_1y_1 + c_2y_2$$

$$= c_1\left(\frac{\sqrt{\frac{x+1}{\sqrt{-x^2+1}}}\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}}\sqrt{2x+3}\,(5x-5)^{1/4}}{\sqrt{\frac{i\left(3\sqrt{5}\,x+5\sqrt{x^2-1}+2\sqrt{5}\right)}{2x+3}}\,(x+1)^{1/4}}\right)$$

$$+c_2\left(\frac{\sqrt{\frac{x+1}{\sqrt{-x^2+1}}}\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}}\sqrt{2x+3}\,(5x-5)^{1/4}}{\sqrt{\frac{i\left(3\sqrt{5}\,x+5\sqrt{x^2-1}+2\sqrt{5}\right)}{2x+3}}\,(x+1)^{1/4}}\left(\int \frac{i\sqrt{x+1}\left(x+\sqrt{x^2-1}\right)^{-\sqrt{5}}\left(3\sqrt{5}\,x+5\sqrt{x^2-\right.}{(2x+3)^2\sqrt{5x-5}}\right.\right.$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$(-x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) - \frac{d}{dx}y(x) + y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = \frac{y(x)}{x^2-1} - \frac{\frac{d}{dx}y(x)}{x^2-1}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) + \frac{\frac{d}{dx}y(x)}{x^2-1} - \frac{y(x)}{x^2-1} = 0$
- □ Check to see if x_0 is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{1}{x^2-1}, P_3(x) = -\frac{1}{x^2-1}\right]$
  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$
    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1} = -\frac{1}{2}$
  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$
    $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$
  - $x=-1$is a regular singular point
  Check to see if x_0 is a regular singular point
  $x_0 = -1$
- Multiply by denominators
  $(x^2-1)\left(\frac{d^2}{dx^2}y(x)\right) + \frac{d}{dx}y(x) - y(x) = 0$
- Change variables using $x = u-1$ so that the regular singular point is at $u=0$
  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + \frac{d}{du}y(u) - y(u) = 0$
- Assume series solution for $y(u)$
  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$
- □ Rewrite ODE with series expansions
  - Convert $\frac{d}{du}y(u)$ to series expansion
    $\frac{d}{du}y(u) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1}$
  - Shift index using $k->k+1$

$\frac{d}{du}y(u) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-a_0 r(-3+2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(2k-1+2r) + a_k(k^2+2kr+r^2-k-r-1)\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-3+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{3}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $-2(k+1+r)\left(k+r-\frac{1}{2}\right) a_{k+1} + \left(k^2+(2r-1)\,k+r^2-r-1\right) a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{(k^2+2kr+r^2-k-r-1)a_k}{(k+1+r)(2k-1+2r)}$
- Recursion relation for $r = 0$

  $a_{k+1} = \frac{(k^2-k-1)a_k}{(k+1)(2k-1)}$
- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{(k^2-k-1)a_k}{(k+1)(2k-1)}\right]$
- Revert the change of variables $u = x+1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+1} = \frac{(k^2-k-1)a_k}{(k+1)(2k-1)}\right]$
- Recursion relation for $r = \frac{3}{2}$

  $a_{k+1} = \frac{(k^2+2k-\frac{1}{4})a_k}{(k+\frac{5}{2})(2k+2)}$
- Solution for $r = \frac{3}{2}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{3}{2}}, a_{k+1} = \frac{(k^2+2k-\frac{1}{4})a_k}{(k+\frac{5}{2})(2k+2)}\right]$
- Revert the change of variables $u = x+1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+1)^{k+\frac{3}{2}}, a_{k+1} = \frac{(k^2+2k-\frac{1}{4})a_k}{(k+\frac{5}{2})(2k+2)}\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k(x+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k(x+1)^{k+\frac{3}{2}}\right), a_{k+1} = \frac{(k^2-k-1)a_k}{(k+1)(2k-1)}, b_{k+1} = \frac{(k^2+2k-\frac{1}{4})b_k}{(k+\frac{5}{2})(2k+2)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.029 (sec)
Leaf size : 66

```
dsolve((-x^2+1)*diff(diff(y(x),x),x)-diff(y(x),x)+y(x) = 0,y(x),singsol=all)
```

$$y = c_1 \,\mathrm{hypergeom}\left(\left[\frac{\sqrt{5}}{2}-\frac{1}{2}, -\frac{1}{2}-\frac{\sqrt{5}}{2}\right], \left[-\frac{1}{2}\right], \frac{1}{2}+\frac{x}{2}\right) + 2c_2\sqrt{2+2x}\,\mathrm{hypergeom}\left(\left[1-\frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{2}+1\right], \left[\frac{5}{2}\right], \frac{1}{2}+\frac{x}{2}\right)(x+1)$$

Mathematica DSolve solution

Solving time : 1.314 (sec)
Leaf size : 210

```
DSolve[{(1-x^2)*D[y[x],{x,2}]-D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \left(\sqrt{x-1}-\sqrt{x+1}\right)^{-\frac{1}{2}-\frac{\sqrt{5}}{2}}\left(\sqrt{x-1}+\sqrt{x+1}\right)^{\frac{1}{2}\left(\sqrt{5}-1\right)}\left(\sqrt{x-1}-\sqrt{5}\sqrt{x+1}\right)\left(c_2\int_1^x -\frac{2e^{\operatorname{arctanh}(K[2])}\left(\sqrt{K[2]-1}-\sqrt{K[2]+1}\right)^{\sqrt{5}}\left(\sqrt{K[2]-1}+\sqrt{K[2]+1}\right)^{-\sqrt{5}}}{\left(\sqrt{K[2]-1}-\sqrt{5}\sqrt{K[2]+1}\right)^2}dK[2] + c_1\right)\exp\left(-\frac{1}{2}\int_1^x \frac{1}{K[1]^2-1}dK[1]-\frac{\operatorname{arctanh}(x)}{2}\right)$$

2.1.725 Problem 742

Internal problem ID [9897]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 742
Date solved : `Monday, January 27, 2025 at 06:15:32 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x(x+1)^2 y''+\left(-x^2+1\right) y'+(x-1) y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.163 (sec)

Writing the ode as

$$x(x+1)^2 y''+\left(-x^2+1\right) y'+(x-1) y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=x(x+1)^2\\ B&=-x^2+1\\ C&=x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=-1\\ t&=4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1381: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the gcd$(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0)+2\left(\frac{1}{2x}\right)(0)+\left(\left(-\frac{1}{2x^2}\right)+\left(\frac{1}{2x}\right)^2-\left(-\frac{1}{4x^2}\right)\right)=0
$$
$$
0=0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$
\begin{aligned}
z_1(x) &= pe^{\int \omega\,dx} \\
&= \mathrm{e}^{\int \frac{1}{2x}dx} \\
&= \sqrt{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2+1}{x(x+1)^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}+\ln(x+1)} \\
&= z_1\left(\frac{x+1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x+1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1\int \frac{e^{\int -\frac{-x^2+1}{x(x+1)^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1\int \frac{e^{-\ln(x)+2\ln(x+1)}}{(y_1)^2}\,dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1+c_2y_2 \\
&= c_1(x+1)+c_2(x+1(\ln(x)))
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x(x+1)^2\left(\frac{d^2}{dx^2}y(x)\right)+(-x^2+1)\left(\frac{d}{dx}y(x)\right)+(x-1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{(x-1)y(x)}{x(x+1)^2}+\frac{(x-1)\left(\frac{d}{dx}y(x)\right)}{x(x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)-\frac{(x-1)\left(\frac{d}{dx}y(x)\right)}{x(x+1)}+\frac{(x-1)y(x)}{x(x+1)^2}=0$

☐ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=-\frac{x-1}{x(x+1)}, P_3(x)=\frac{x-1}{x(x+1)^2}\right]$

- ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1}=-2$

- ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1}=2$

- ◦ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1$

- Multiply by denominators

$x(x+1)^2\left(\frac{d^2}{dx^2}y(x)\right)-(x-1)(x+1)\left(\frac{d}{dx}y(x)\right)+(x-1)\,y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(u^3-u^2)\left(\frac{d^2}{du^2}y(u)\right)+(-u^2+2u)\left(\frac{d}{du}y(u)\right)+(u-2)\,y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

- ◦ Shift index using $k->k-m$

  $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=1..2$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ◦ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=2..3$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

- ◦ Shift index using $k->k+2-m$

$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

Rewrite ODE with series expansions

$-a_0(-1+r)(-2+r)\, u^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r-1)(k+r-2) + a_{k-1}(k+r-2)^2\right) u^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-(-1+r)(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$
- Each term in the series must be 0, giving the recursion relation

  $-a_k(k+r-1)(k+r-2) + a_{k-1}(k+r-2)^2 = 0$
- Shift index using $k- >k+1$

  $-a_{k+1}(k+r)(k+r-1) + a_k(k+r-1)^2 = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r-1)}{k+r}$
- Recursion relation for $r = 1$

  $a_{k+1} = \frac{a_k k}{k+1}$
- Solution for $r = 1$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+1}, a_{k+1} = \frac{a_k k}{k+1}\right]$
- Revert the change of variables $u = x+1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x+1)^{k+1}, a_{k+1} = \frac{a_k k}{k+1}\right]$
- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k(k+1)}{k+2}$
- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k(k+1)}{k+2}\right]$
- Revert the change of variables $u = x+1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x+1)^{k+2}, a_{k+1} = \frac{a_k(k+1)}{k+2}\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k (x+1)^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k+2}\right), a_{k+1} = \frac{a_k k}{k+1}, b_{k+1} = \frac{b_k(k+1)}{k+2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 14

```
dsolve(x*(x+1)^2*diff(diff(y(x),x),x)+(-x^2+1)*diff(y(x),x)+(x-1)*y(x) = 0,y(x),singsc
```

$$y = (x+1)(c_2 \ln(x) + c_1)$$

Mathematica DSolve solution

Solving time : 0.247 (sec)
Leaf size : 45

```
DSolve[{x*(x+1)^2*D[y[x],{x,2}]+(1-x^2)*D[y[x],x]+(x-1)*y[x]==0,{}},y[x],x,IncludeSingularSo
```

$$y(x) \to \sqrt{x}(c_2 \log(x) + c_1) \exp\left(-\frac{1}{2}\int_1^x \left(\frac{1}{K[1]} - \frac{2}{K[1]+1}\right) dK[1]\right)$$

2.1.726 Problem 743

Internal problem ID [9898]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 743
Date solved : `Monday, January 27, 2025 at 06:15:32 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$2xy'' - y' + 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.204 (sec)

Writing the ode as

$$2xy'' - y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x \\ B &= -1 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5 - 16x}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 5 - 16x \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5 - 16x}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1383: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16x^2} - \frac{1}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{-1, 2, 5\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = -1, \; e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(1 - (-1)) \\ &= 1 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x - c} \\ &= \frac{1}{2}\left(\frac{-1}{(x - (0))}\right) \\ &= -\frac{1}{2x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 1$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 1$, then letting

$$p = x + a_0 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$\frac{1 - 4a_0}{x^2} = 0$$

And solving for p gives

$$p = x + \frac{1}{4}$$

Now that $p(x)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{x + \frac{1}{4}} - \frac{1}{2x} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{x + \frac{1}{4}} - \frac{1}{2x}\right)w + \frac{64x^2 - 12x + 1}{64x^3 + 16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{16x\sqrt{-x} + 4x - 1}{4\,(4x + 1)\,x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{16x\sqrt{-x}+4x-1}{4(4x+1)x} dx} \\
&= \frac{\left(2\sqrt{-x}-1\right) \mathrm{e}^{2\sqrt{-x}}}{(-x)^{1/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{2x} \, dx} \\
&= z_1 e^{\frac{\ln(x)}{4}} \\
&= z_1 \left(x^{1/4}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/4}\left(2\sqrt{-x}-1\right) \mathrm{e}^{2\sqrt{-x}}}{(-x)^{1/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{2x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{\ln(x)}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{\mathrm{e}^{-4\sqrt{-x}}}{8} + \frac{\mathrm{e}^{-4\sqrt{-x}}}{8\sqrt{-x}-4}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{1/4}\left(2\sqrt{-x}-1\right) \mathrm{e}^{2\sqrt{-x}}}{(-x)^{1/4}}\right) + c_2 \left(\frac{x^{1/4}\left(2\sqrt{-x}-1\right) \mathrm{e}^{2\sqrt{-x}}}{(-x)^{1/4}} \left(\frac{\mathrm{e}^{-4\sqrt{-x}}}{8} + \frac{\mathrm{e}^{-4\sqrt{-x}}}{8\sqrt{-x}-4}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2\left(\frac{d^2}{dx^2}y(x)\right)x - \frac{d}{dx}y(x) + 2y(x) = 0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x) = -\frac{y(x)}{x} + \frac{\frac{d}{dx}y(x)}{2x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x) - \frac{\frac{d}{dx}y(x)}{2x} + \frac{y(x)}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

$\left[P_2(x) = -\frac{1}{2x}, P_3(x) = \frac{1}{x}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$(x \cdot P_2(x))\Big|_{x=0} = -\frac{1}{2}$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

- ◦ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$2\left(\frac{d^2}{dx^2}y(x)\right)x - \frac{d}{dx}y(x) + 2y(x) = 0$

- Assume series solution for $y(x)$

$y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $\frac{d}{dx}y(x)$ to series expansion

$\frac{d}{dx}y(x) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

- ◦ Shift index using $k- >k+1$

$\frac{d}{dx}y(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$

- ◦ Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

$x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-1}$

- ◦ Shift index using $k- >k+1$

$x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0 r(-3+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(2k-1+2r) + 2a_k\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-3+2r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{0, \frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation
  $2(k+1+r)\left(k+r-\frac{1}{2}\right)a_{k+1}+2a_k=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=-\frac{2a_k}{(k+1+r)(2k-1+2r)}$
- Recursion relation for $r=0$
  $a_{k+1}=-\frac{2a_k}{(k+1)(2k-1)}$
- Solution for $r=0$
  $\left[y(x)=\sum_{k=0}^{\infty}a_kx^k, a_{k+1}=-\frac{2a_k}{(k+1)(2k-1)}\right]$
- Recursion relation for $r=\frac{3}{2}$
  $a_{k+1}=-\frac{2a_k}{\left(k+\frac{5}{2}\right)(2k+2)}$
- Solution for $r=\frac{3}{2}$
  $\left[y(x)=\sum_{k=0}^{\infty}a_kx^{k+\frac{3}{2}}, a_{k+1}=-\frac{2a_k}{\left(k+\frac{5}{2}\right)(2k+2)}\right]$
- Combine solutions and rename parameters
  $\left[y(x)=\left(\sum_{k=0}^{\infty}a_kx^k\right)+\left(\sum_{k=0}^{\infty}b_kx^{k+\frac{3}{2}}\right), a_{k+1}=-\frac{2a_k}{(k+1)(2k-1)}, b_{k+1}=-\frac{2b_k}{\left(k+\frac{5}{2}\right)(2k+2)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.017 (sec)
Leaf size : 36

```
dsolve(2*x*diff(diff(y(x),x),x)-diff(y(x),x)+2*y(x) = 0,y(x),singsol=all)
```

$$y=\left(2\sqrt{x}\,c_1+c_2\right)\cos\left(2\sqrt{x}\right)-\sin\left(2\sqrt{x}\right)\left(-2c_2\sqrt{x}+c_1\right)$$

Mathematica DSolve solution

Solving time : 0.206 (sec)
Leaf size : 74

```
DSolve[{2*x*D[y[x],{x,2}]-D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to e^{2i\sqrt{x}}\left(2\sqrt{x}+i\right)\left(c_2\int_1^x\frac{e^{-4i\sqrt{K[1]}}\sqrt{K[1]}}{\left(2\sqrt{K[1]}+i\right)^2}dK[1]+c_1\right)$$

2.1.727 Problem 744

Internal problem ID [9899]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 744
Date solved : `Monday, January 27, 2025 at 06:15:33 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + xy' - 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.217 (sec)

Writing the ode as

$$xy'' + xy' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x+8}{4x} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x + 8 \\ t &= 4x \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x+8}{4x}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1385: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x$. There is a pole at $x = 0$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{2}{x} - \frac{4}{x^2} + \frac{16}{x^3} - \frac{80}{x^4} + \frac{448}{x^5} - \frac{2688}{x^6} + \frac{16896}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x+8}{4x} \\ &= Q + \frac{R}{4x} \\ &= \left(\frac{1}{4}\right) + \left(\frac{2}{x}\right) \\ &= \frac{1}{4} + \frac{2}{x} \end{aligned}$$

Since the degree of t is 1, then we see that the coefficient of the term 1 in the remainder R is 8. Dividing this by leading coefficient in t which is 4 gives 2. Now b can be found.

$$\begin{aligned} b &= (2) - (0) \\ &= 2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{2}{\frac{1}{2}} - 0\right) &= 2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{2}{\frac{1}{2}} - 0\right) &= -2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x+8}{4x}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	2	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= 2 - (1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{x} + \left(\frac{1}{2}\right) \\ &= \frac{1}{x} + \frac{1}{2} \\ &= \frac{1}{x} + \frac{1}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x} + \frac{1}{2}\right)(1) + \left(\left(-\frac{1}{x^2}\right) + \left(\frac{1}{x} + \frac{1}{2}\right)^2 - \left(\frac{x+8}{4x}\right) \right) = 0$$

$$\frac{2 - a_0}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (2 + x) e^{\int \left(\frac{1}{x} + \frac{1}{2}\right) dx} \\ &= (2 + x) e^{\frac{x}{2} + \ln(x)} \\ &= (2 + x) x e^{\frac{x}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (2+x)\,x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{\mathrm{e}^{-x}}{4x} + \frac{\mathrm{Ei}_1(x)}{2} + \frac{\mathrm{e}^{-x}}{-8-4x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1((2+x)\,x) + c_2\left((2+x)\,x\left(-\frac{\mathrm{e}^{-x}}{4x} + \frac{\mathrm{Ei}_1(x)}{2} + \frac{\mathrm{e}^{-x}}{-8-4x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 28

```
dsolve(x*diff(diff(y(x),x),x)+diff(y(x),x)*x-2*y(x) = 0,y(x),singsol=all)
```

$$y = -\frac{(x+1)\, c_2 \mathrm{e}^{-x}}{2} + x(x+2)\left(c_1 + \frac{\mathrm{Ei}_1(x)\, c_2}{2}\right)$$

Mathematica DSolve solution

Solving time : 0.226 (sec)
Leaf size : 40

```
DSolve[{x*D[y[x],{x,2}]+x*D[y[x],x]-2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to x(x+2)\left(c_2 \int_1^x \frac{e^{-K[1]}}{K[1]^2(K[1]+2)^2} dK[1] + c_1\right)$$

2.1.728 Problem 745

Internal problem ID [9900]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 745
Date solved : `Monday, January 27, 2025 at 06:15:34 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x(x-1)^2 y'' - 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.146 (sec)

Writing the ode as

$$x(x-1)^2 y'' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x(x-1)^2 \\ B &= 0 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x\,(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= x(x-1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x\,(x-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1386: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 0 \\ &= 3 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x(x-1)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 3 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 3 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{x-1} + \frac{2}{x} + \frac{2}{(x-1)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $3 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= 0\\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x(x-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1
1	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
3	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$ then

$$\begin{aligned}d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right)\\ &= 0 - (0)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (+)[\sqrt{r}]_\infty\\ &= \frac{1}{x} - \frac{1}{x-1} + (0)\\ &= \frac{1}{x} - \frac{1}{x-1}\\ &= -\frac{1}{x(x-1)}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(\frac{1}{x} - \frac{1}{x-1}\right)(0) + \left(\left(-\frac{1}{x^2} + \frac{1}{(x-1)^2}\right) + \left(\frac{1}{x} - \frac{1}{x-1}\right)^2 - \left(\frac{2}{x(x-1)^2}\right)\right) &= 0\\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= pe^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{x} - \frac{1}{x-1}\right) dx} \\
&= \frac{x}{x-1}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \frac{x}{x-1}
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x}{x-1}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\
&= \frac{x}{x-1} \int \frac{1}{\frac{x^2}{(x-1)^2}} \, dx \\
&= \frac{x}{x-1}\left(x - \frac{1}{x} - 2\ln(x)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x}{x-1}\right) + c_2\left(\frac{x}{x-1}\left(x - \frac{1}{x} - 2\ln(x)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x(x-1)^2 \left(\frac{d^2}{dx^2}y(x)\right) - 2y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{2y(x)}{x(x-1)^2}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) - \frac{2y(x)}{x(x-1)^2} = 0$

☐ Check to see if x_0 is a regular singular point

  ○ Define functions

  $\left[P_2(x) = 0, P_3(x) = -\frac{2}{x(x-1)^2}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 0$

  ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

  ○ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x(x-1)^2 \left(\frac{d^2}{dx^2}y(x)\right) - 2y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ○ Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 1..3$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$

  ○ Shift index using $k- >k+2-m$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-1+r)\, x^{-1+r} + (a_1(1+r)\, r - 2a_0(r^2-r+1))\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)(k+r) - 2a_k(k^2 +\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 1\}$

- Each term must be 0

  $a_1(1+r)\, r - 2a_0(r^2-r+1) = 0$

- Each term in the series must be 0, giving the recursion relation

  $(-2a_k + a_{k-1} + a_{k+1})\, k^2 + ((-4a_k + 2a_{k-1} + 2a_{k+1})\, r + 2a_k - 3a_{k-1} + a_{k+1})\, k + (-2a_k + a_{k-1} + a$

- Shift index using $k- >k+1$

  $(-2a_{k+1} + a_k + a_{k+2})\, (k+1)^2 + ((-4a_{k+1} + 2a_k + 2a_{k+2})\, r + 2a_{k+1} - 3a_k + a_{k+2})\, (k+1) + (-2a_k$

- Recursion relation that defines series solution to ODE
$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2kr a_k - 4kr a_{k+1} + r^2 a_k - 2r^2 a_{k+1} - k a_k - 2k a_{k+1} - r a_k - 2r a_{k+1} - 2a_{k+1}}{k^2 + 2kr + r^2 + 3k + 3r + 2}$$
- Recursion relation for $r = 0$
$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - k a_k - 2k a_{k+1} - 2a_{k+1}}{k^2 + 3k + 2}$$
- Solution for $r = 0$
$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - k a_k - 2k a_{k+1} - 2a_{k+1}}{k^2 + 3k + 2}, -2a_0 = 0\right]$$
- Recursion relation for $r = 1$
$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + k a_k - 6k a_{k+1} - 6a_{k+1}}{k^2 + 5k + 6}$$
- Solution for $r = 1$
$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + k a_k - 6k a_{k+1} - 6a_{k+1}}{k^2 + 5k + 6}, 2a_1 - 2a_0 = 0\right]$$
- Combine solutions and rename parameters
$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - k a_k - 2k a_{k+1} - 2a_{k+1}}{k^2 + 3k + 2}, -2a_0 = 0, b_{k+2} = -\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 27

```
dsolve(x*(x-1)^2*diff(diff(y(x),x),x)-2*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{2\ln(x) c_2 x - c_2 x^2 + c_1 x + c_2}{x - 1}$$

Mathematica DSolve solution

Solving time : 0.059 (sec)
Leaf size : 62

```
DSolve[{x*(x-1)^2*D[y[x],{x,2}]-2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \exp\left(\int_1^x \frac{1}{K[1] - K[1]^2} dK[1]\right) \left(c_2 \int_1^x \exp\left(-2 \int_1^{K[2]} \frac{1}{K[1] - K[1]^2} dK[1]\right) dK[2] + c_1\right)$$

2.1.729 Problem 746

Internal problem ID [9901]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 746
Date solved : `Monday, January 27, 2025 at 06:15:34 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - 2xy' + x^2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.148 (sec)

Writing the ode as

$$y'' - 2xy' + x^2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2x \\ C &= x^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1388: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{1}\, dx} \\ &= z_1 e^{\frac{x^2}{2}} \\ &= z_1 \left(e^{\frac{x^2}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{\frac{x^2}{2}} \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x^2}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{\frac{x^2}{2}}\cos(x)\right) + c_2\left(\mathrm{e}^{\frac{x^2}{2}}\cos(x)(\tan(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2}y(x) - 2x\left(\frac{d}{dx}y(x)\right) + x^2 y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^k$
- □ Rewrite ODE with series expansions
  - Convert $x^2 \cdot y(x)$ to series expansion
    $x^2 \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}$
  - Shift index using $k- >k-2$
    $x^2 \cdot y(x) = \sum_{k=2}^{\infty} a_{k-2} x^k$
  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion
    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$
  - Convert $\frac{d^2}{dx^2}y(x)$ to series expansion
    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$
  - Shift index using $k- >k+2$
    $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite ODE with series expansions
  $2a_2 + (6a_3 - 2a_1)x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)(k+1) - 2a_k k + a_{k-2})\, x^k\right) = 0$
- The coefficients of each power of x must be 0
  $[2a_2 = 0, 6a_3 - 2a_1 = 0]$
- Solve for the dependent coefficient(s)
  $\left\{a_2 = 0, a_3 = \frac{a_1}{3}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $(k^2 + 3k + 2)\, a_{k+2} - 2a_k k + a_{k-2} = 0$

- Shift index using $k- > k+2$

  $\left((k+2)^2+3k+8\right)a_{k+4}-2a_{k+2}(k+2)+a_k=0$
- Recursion relation that defines the series solution to the ODE

  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^k, a_{k+4}=\frac{2ka_{k+2}-a_k+4a_{k+2}}{k^2+7k+12}, a_2=0, a_3=\frac{a_1}{3}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.014` (sec)
Leaf size : 20

```
dsolve(diff(diff(y(x),x),x)-2*diff(y(x),x)*x+x^2*y(x) = 0,y(x),singsol=all)
```

$$y=\mathrm{e}^{\frac{x^2}{2}}\left(\cos\left(x\right)c_1+\sin\left(x\right)c_2\right)$$

Mathematica DSolve solution

Solving time : `0.033` (sec)
Leaf size : 39

```
DSolve[{D[y[x],{x,2}]-2*x*D[y[x],x]+x^2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to\frac{1}{2}e^{\frac{1}{2}x(x-2i)}\left(2c_1-ic_2e^{2ix}\right)$$

2.1.730 Problem 747

Internal problem ID [9902]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 747
Date solved : `Monday, January 27, 2025 at 06:15:35 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x\left(-x^2+2\right) y''-\left(x^2+4x+2\right)\left((1-x)\, y'+y\right)=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.509 (sec)

Writing the ode as

$$\left(-x^3+2x\right) y''+\left(x^3+3x^2-2x-2\right) y'+\left(-x^2-4x-2\right) y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^3+2x \\ B &= x^3+3x^2-2x-2 \\ C &= -x^2-4x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{x^6+2x^5-5x^4-16x^3+24x^2+24x+12}{4\left(x^3-2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^6+2x^5-5x^4-16x^3+24x^2+24x+12 \\ t &= 4\left(x^3-2x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x^6+2x^5-5x^4-16x^3+24x^2+24x+12}{4\left(x^3-2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1390: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 6 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 - 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \sqrt{2}$ of order 2. There is a pole at $x = -\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4\left(x-\sqrt{2}\right)^2} + \frac{3}{4\left(x+\sqrt{2}\right)^2} + \frac{-\frac{5\sqrt{2}}{8} - \frac{1}{2}}{x-\sqrt{2}} + \frac{\frac{5\sqrt{2}}{8} - \frac{1}{2}}{x+\sqrt{2}} + \frac{3}{2x} + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = \sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x-\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x+\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}
$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r} \approx \frac{1}{2} + \frac{1}{2x} - \frac{1}{2x^2} - \frac{3}{2x^3} + \frac{21}{4x^4} - \frac{43}{4x^5} + \frac{135}{4x^6} - \frac{147}{4x^7} + \dots \qquad (9)
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a = \frac{1}{2}
$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i\\
&= \frac{1}{2} \qquad (10)
\end{aligned}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}
$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t}\\
&= \frac{x^6 + 2x^5 - 5x^4 - 16x^3 + 24x^2 + 24x + 12}{4x^6 - 16x^4 + 16x^2}\\
&= Q + \frac{R}{4x^6 - 16x^4 + 16x^2}\\
&= \left(\frac{1}{4}\right) + \left(\frac{2x^5 - x^4 - 16x^3 + 20x^2 + 24x + 12}{4x^6 - 16x^4 + 16x^2}\right)\\
&= \frac{1}{4} + \frac{2x^5 - x^4 - 16x^3 + 20x^2 + 24x + 12}{4x^6 - 16x^4 + 16x^2}
\end{aligned}
$$

Since the degree of t is 6, then we see that the coefficient of the term x^5 in the remainder R is 2. Dividing this by leading coefficient in t which is 4 gives $\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{2}\right) - (0) \\ &= \frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^6 + 2x^5 - 5x^4 - 16x^3 + 24x^2 + 24x + 12}{4\left(x^3 - 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$\sqrt{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-\sqrt{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\ &= \frac{3}{2x} - \frac{1}{2\left(x-\sqrt{2}\right)} - \frac{1}{2\left(x+\sqrt{2}\right)} + \left(\frac{1}{2}\right) \\ &= \frac{3}{2x} - \frac{1}{2\left(x-\sqrt{2}\right)} - \frac{1}{2\left(x+\sqrt{2}\right)} + \frac{1}{2} \\ &= \frac{x^3 + x^2 - 2x - 6}{2x^3 - 4x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{2x} - \frac{1}{2\left(x-\sqrt{2}\right)} - \frac{1}{2\left(x+\sqrt{2}\right)} + \frac{1}{2}\right)(0) + \left(\left(-\frac{3}{2x^2} + \frac{1}{2\left(x-\sqrt{2}\right)^2} + \frac{1}{2\left(x+\sqrt{2}\right)^2}\right) + \left(\frac{3}{2x} - \right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(\frac{3}{2x} - \frac{1}{2\left(x-\sqrt{2}\right)} - \frac{1}{2\left(x+\sqrt{2}\right)} + \frac{1}{2}\right) dx} \\ &= \frac{x^{3/2} \mathrm{e}^{\frac{x}{2}}}{\sqrt{x+\sqrt{2}}\,\sqrt{x-\sqrt{2}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^3+3x^2-2x-2}{-x^3+2x} \, dx} \\ &= z_1 e^{\frac{x}{2} + \frac{\ln(x)}{2} + \frac{\ln\left(x^2-2\right)}{2}} \\ &= z_1 \left(\sqrt{x}\,\sqrt{x^2-2}\,\mathrm{e}^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^2 \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^3+3x^2-2x-2}{-x^3+2x} \, dx}}{\left(y_1\right)^2} \, dx \\ &= y_1 \int \frac{e^{x+\ln(x)+\ln\left(x^2-2\right)}}{\left(y_1\right)^2} \, dx \\ &= y_1 \left(-\frac{(x-1)\,\mathrm{e}^{x+\ln(x)+\ln\left(x^2-2\right)}\mathrm{e}^{-2x}}{x^3\left(x^2-2\right)}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(x^2 \mathrm{e}^x\right) + c_2\left(x^2 \mathrm{e}^x\left(-\frac{(x-1)\,\mathrm{e}^{x+\ln(x)+\ln\left(x^2-2\right)}\mathrm{e}^{-2x}}{x^3\left(x^2-2\right)}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x(-x^2+2)\left(\frac{d^2}{dx^2}y(x)\right)-(x^2+4x+2)\left((1-x)\left(\frac{d}{dx}y(x)\right)+y(x)\right)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{(x^2+4x+2)y(x)}{x(x^2-2)}+\frac{(x^2+4x+2)(x-1)\left(\frac{d}{dx}y(x)\right)}{x(x^2-2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)-\frac{(x^2+4x+2)(x-1)\left(\frac{d}{dx}y(x)\right)}{x(x^2-2)}+\frac{(x^2+4x+2)y(x)}{x(x^2-2)}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

$\left[P_2(x)=-\frac{(x-1)(x^2+4x+2)}{x(x^2-2)},P_3(x)=\frac{x^2+4x+2}{x(x^2-2)}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

$\left.(x\cdot P_2(x))\right|_{x=0}=-1$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

- ○ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$x(x^2-2)\left(\frac{d^2}{dx^2}y(x)\right)-(x-1)(x^2+4x+2)\left(\frac{d}{dx}y(x)\right)+(x^2+4x+2)y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

$x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ○ Shift index using $k->k-m$

$x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ○ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..3$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ○ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=1..3$

$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,x^{k+r-2+m}$

- ○ Shift index using $k->k+2-m$

$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)\,x^{k+r}$

Rewrite ODE with series expansions

$$-2a_0r(-2+r)\,x^{-1+r}+\left(-2a_1(1+r)\,(-1+r)+2a_0(1+r)\right)x^r+\left(-2a_2(2+r)\,r+2a_1(2+r)+a_0\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-2r(-2+r)=0$
- Values of r that satisfy the indicial equation
  $r\in\{0,2\}$
- The coefficients of each power of x must be 0
  $\left[-2a_1(1+r)\,(-1+r)+2a_0(1+r)=0,-2a_2(2+r)\,r+2a_1(2+r)+a_0(-2+r)^2=0\right]$
- Solve for the dependent coefficient(s)
  $\left\{a_1=\frac{a_0}{-1+r},a_2=\frac{a_0(r^2-5r+10)}{2(r^2+r-2)}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $a_{k-1}(k-3+r)^2-2a_{k+1}(k+r+1)\,(k+r-1)+(2a_k-a_{k-2})\,k+(2a_k-a_{k-2})\,r+2a_k+3a_{k-2}=$
- Shift index using $k->k+2$
  $a_{k+1}(k+r-1)^2-2a_{k+3}(k+3+r)\,(k+r+1)+(2a_{k+2}-a_k)\,(k+2)+(2a_{k+2}-a_k)\,r+2a_{k+2}+$
- Recursion relation that defines series solution to ODE
  $a_{k+3}=\frac{k^2a_{k+1}+2kra_{k+1}+r^2a_{k+1}-a_kk-2ka_{k+1}+2ka_{k+2}-a_kr-2ra_{k+1}+2ra_{k+2}+a_k+a_{k+1}+6a_{k+2}}{2(k+3+r)(k+r+1)}$
- Recursion relation for $r=0$
  $a_{k+3}=\frac{k^2a_{k+1}-a_kk-2ka_{k+1}+2ka_{k+2}+a_k+a_{k+1}+6a_{k+2}}{2(k+3)(k+1)}$
- Solution for $r=0$
  $\left[y(x)=\sum_{k=0}^{\infty}a_kx^k,a_{k+3}=\frac{k^2a_{k+1}-a_kk-2ka_{k+1}+2ka_{k+2}+a_k+a_{k+1}+6a_{k+2}}{2(k+3)(k+1)},a_1=-a_0,a_2=-\frac{5a_0}{2}\right]$
- Recursion relation for $r=2$
  $a_{k+3}=\frac{k^2a_{k+1}-a_kk+2ka_{k+1}+2ka_{k+2}-a_k+a_{k+1}+10a_{k+2}}{2(k+5)(k+3)}$
- Solution for $r=2$
  $\left[y(x)=\sum_{k=0}^{\infty}a_kx^{k+2},a_{k+3}=\frac{k^2a_{k+1}-a_kk+2ka_{k+1}+2ka_{k+2}-a_k+a_{k+1}+10a_{k+2}}{2(k+5)(k+3)},a_1=a_0,a_2=\frac{a_0}{2}\right]$
- Combine solutions and rename parameters
  $\left[y(x)=\left(\sum_{k=0}^{\infty}a_kx^k\right)+\left(\sum_{k=0}^{\infty}b_kx^{k+2}\right),a_{k+3}=\frac{k^2a_{k+1}-ka_k-2ka_{k+1}+2ka_{k+2}+a_k+a_{k+1}+6a_{k+2}}{2(k+3)(k+1)},a_1=-a_0,a_2=\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 17

```
dsolve(x*(-x^2+2)*diff(diff(y(x),x),x)-(x^2+4*x+2)*((1-x)*diff(y(x),x)+y(x)) = 0,y(x),
```

$$y = c_1(x-1) + c_2 e^x x^2$$

Mathematica DSolve solution

Solving time : 0.329 (sec)
Leaf size : 126

```
DSolve[{x*(2-x^2)*D[y[x],{x,2}]-(x^2+4*x+2)*((1-x)*D[y[x],x]+y[x])==0,{}},y[x],x,IncludeSing
```

$$y(x) \to \exp\left(\int_1^x \left(-\frac{K[1]}{K[1]^2-2} + \frac{1}{2} + \frac{3}{2K[1]}\right) dK[1] - \frac{1}{2}\int_1^x \left(-\frac{2K[2]}{K[2]^2-2} - 1 - \frac{1}{K[2]}\right) dK[2]\right) \left(c_2 \int_1^x \exp\left(-2\int_1^{K[3]} \frac{K[1]^3+K[1]^2-2K[1]-6}{2K[1]\left(K[1]^2-2\right)} dK[1]\right) dK[3] + c_1\right)$$

2.1.731 Problem 748

Internal problem ID [9903]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 748
Date solved : `Monday, January 27, 2025 at 06:15:36 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+x)\,y'' - (1+2x)\,(xy' - y) = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.211 (sec)

Writing the ode as

$$x^2(1+x)\,y'' + \left(-2x^2 - x\right)y' + (1+2x)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(1+x) \\ B &= -2x^2 - x \\ C &= 1+2x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4x - 1}{4\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4x - 1 \\ t &= 4\left(x^2 + x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-4x - 1}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1392: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 3 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 3 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(1+x)^2} + \frac{1}{2+2x} - \frac{1}{4x^2} - \frac{1}{2x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $3 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= 0\\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-4x-1}{4\left(x^2+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
3	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$ then

$$\begin{aligned}d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right)\\ &= 0 - (0)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (+)[\sqrt{r}]_\infty\\ &= -\frac{1}{2\left(1+x\right)} + \frac{1}{2x} + (0)\\ &= -\frac{1}{2\left(1+x\right)} + \frac{1}{2x}\\ &= \frac{1}{2x\left(1+x\right)}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2\left(1+x\right)}+\frac{1}{2x}\right)(0)+\left(\left(\frac{1}{2\left(1+x\right)^{2}}-\frac{1}{2x^{2}}\right)+\left(-\frac{1}{2\left(1+x\right)}+\frac{1}{2x}\right)^{2}-\left(\frac{-4x-1}{4\left(x^{2}+x\right)^{2}}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x)&=pe^{\int \omega\,dx}\\
&=\mathrm{e}^{\int \left(-\frac{1}{2(1+x)}+\frac{1}{2x}\right)dx}\\
&=\frac{\sqrt{x}}{\sqrt{1+x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1&=z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\
&=z_1e^{-\int \frac{1}{2}\frac{-2x^2-x}{x^2(1+x)}\,dx}\\
&=z_1e^{\frac{\ln(x(1+x))}{2}}\\
&=z_1\left(\sqrt{x\left(1+x\right)}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\frac{\sqrt{x\left(1+x\right)}\,\sqrt{x}}{\sqrt{1+x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2&=y_1\int \frac{e^{\int -\frac{-2x^2-x}{x^2(1+x)}\,dx}}{\left(y_1\right)^2}\,dx\\
&=y_1\int \frac{e^{\ln(x(1+x))}}{\left(y_1\right)^2}\,dx\\
&=y_1(x+\ln\left(x\right))
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y&=c_1y_1+c_2y_2\\
&=c_1\left(\frac{\sqrt{x\left(1+x\right)}\,\sqrt{x}}{\sqrt{1+x}}\right)+c_2\left(\frac{\sqrt{x\left(1+x\right)}\,\sqrt{x}}{\sqrt{1+x}}(x+\ln\left(x\right))\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)-(2x+1)\left(x\left(\frac{d}{dx}y(x)\right)-y(x)\right)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(2x+1)y(x)}{(x+1)x^2}+\frac{(2x+1)\left(\frac{d}{dx}y(x)\right)}{x(x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)-\frac{(2x+1)\left(\frac{d}{dx}y(x)\right)}{x(x+1)}+\frac{(2x+1)y(x)}{(x+1)x^2}=0$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x)=-\frac{2x+1}{x(x+1)},P_3(x)=\frac{2x+1}{(x+1)x^2}\right]$

  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1}=-1$

  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1}=0$

  - $x=-1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=-1$

- Multiply by denominators

  $x^2(x+1)\left(\frac{d^2}{dx^2}y(x)\right)-x(2x+1)\left(\frac{d}{dx}y(x)\right)+(2x+1)y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^3-2u^2+u)\left(\frac{d^2}{du^2}y(u)\right)+(-2u^2+3u-1)\left(\frac{d}{du}y(u)\right)+(2u-1)y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

    $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

  - Shift index using $k->k-m$

    $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

Rewrite ODE with series expansions

$a_0 r(-2+r)\, u^{-1+r} + (a_1(1+r)(-1+r) - a_0(2r^2-5r+1))\, u^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)(k+r -\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 2\}$
- Each term must be 0
  $a_1(1+r)(-1+r) - a_0(2r^2-5r+1) = 0$
- Each term in the series must be 0, giving the recursion relation
  $(-2a_k + a_{k-1} + a_{k+1})\, k^2 + ((-4a_k + 2a_{k-1} + 2a_{k+1})\, r + 5a_k - 5a_{k-1})\, k + (-2a_k + a_{k-1} + a_{k+1})$
- Shift index using $k- > k+1$
  $(-2a_{k+1} + a_k + a_{k+2})\,(k+1)^2 + ((-4a_{k+1} + 2a_k + 2a_{k+2})\, r + 5a_{k+1} - 5a_k)\,(k+1) + (-2a_{k+1} +$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + 2kra_k - 4kra_{k+1} + r^2a_k - 2r^2a_{k+1} - 3ka_k + ka_{k+1} - 3ra_k + ra_{k+1} + 2a_k + 2a_{k+1}}{k^2+2kr+r^2+2k+2r}$
- Recursion relation for $r = 0$
  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} - 3ka_k + ka_{k+1} + 2a_k + 2a_{k+1}}{k^2+2k}$
- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$
  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} - 3ka_k + ka_{k+1} + 2a_k + 2a_{k+1}}{k^2+2k}$
- Recursion relation for $r = 2$
  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + ka_k - 7ka_{k+1} - 4a_{k+1}}{k^2+6k+8}$
- Solution for $r = 2$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + ka_k - 7ka_{k+1} - 4a_{k+1}}{k^2+6k+8}, 3a_1 + a_0 = 0\right]$
- Revert the change of variables $u = x+1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x+1)^{k+2}, a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + ka_k - 7ka_{k+1} - 4a_{k+1}}{k^2+6k+8}, 3a_1 + a_0 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 15

```
dsolve(x^2*(x+1)*diff(diff(y(x),x),x)-(2*x+1)*(diff(y(x),x)*x-y(x)) = 0,y(x),singsol=all
```

$$y = x(c_2 \ln (x) + c_2 x + c_1)$$

Mathematica DSolve solution

Solving time : 0.183 (sec)
Leaf size : 132

```
DSolve[{x^2*(1+x)*D[y[x],{x,2}]-(1+2*x)*(x*D[y[x],x]+y[x])==0,{}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to c_2 x^{1+\sqrt{2}} \text{Hypergeometric2F1}\left(-\frac{1}{2}+\sqrt{2}-\frac{\sqrt{17}}{2}, -\frac{1}{2}+\sqrt{2}+\frac{\sqrt{17}}{2}, 1+2\sqrt{2}, -x\right) + c_1 x^{1-\sqrt{2}} \text{Hypergeometric2F1}\left(\frac{1}{2}\left(-1-2\sqrt{2}-\sqrt{17}\right), \frac{1}{2}\left(-1-2\sqrt{2}+\sqrt{17}\right), 1 - 2\sqrt{2}, -x\right)$$

2.1.732 Problem 749

Internal problem ID [9904]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 749
Date solved : `Monday, January 27, 2025 at 06:15:36 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2(2-x)\,x^2y''-(4-x)\,xy'+(3-x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.160 (sec)

Writing the ode as

$$\left(-2x^3+4x^2\right)y''+\left(x^2-4x\right)y'+(3-x)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -2x^3+4x^2 \\ B &= x^2-4x \\ C &= 3-x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-3}{16\,(-2+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 16(-2+x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{3}{16\,(-2+x)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1394: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(-2 + x)^2$. There is a pole at $x = 2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16(-2 + x)^2}$$

For the pole at $x = 2$ let b be the coefficient of $\frac{1}{(-2+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{3}{16(-2 + x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=-\frac{3}{16\left(-2+x\right)^{2}}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
2	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{4}-\left(\frac{1}{4}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^{-}}{x-c_1}\right)+(-)[\sqrt{r}]_\infty \\
&= \frac{1}{-8+4x}+(-)\left(0\right) \\
&= \frac{1}{-8+4x} \\
&= \frac{1}{-8+4x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p''+2\omega p'+\left(\omega'+\omega^{2}-r\right)p=0 \tag{1A}
$$

Let

$$
p(x)=1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{1}{-8+4x}\right)(0)+\left(\left(-\frac{1}{4\left(-2+x\right)^{2}}\right)+\left(\frac{1}{-8+4x}\right)^{2}-\left(-\frac{3}{16\left(-2+x\right)^{2}}\right)\right)=0$$

$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\,dx} \\ &= \mathrm{e}^{\int \frac{1}{-8+4x}dx} \\ &= \left(-2+x\right)^{1/4} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^2-4x}{-2x^3+4x^2}\,dx} \\ &= z_1 e^{\frac{\ln(x)}{2}-\frac{\ln(-2+x)}{4}} \\ &= z_1\left(\frac{\sqrt{x}}{\left(-2+x\right)^{1/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1=\sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1\int \frac{e^{\int -\frac{x^2-4x}{-2x^3+4x^2}\,dx}}{\left(y_1\right)^2}\,dx \\ &= y_1\int \frac{e^{\ln(x)-\frac{\ln(-2+x)}{2}}}{\left(y_1\right)^2}\,dx \\ &= y_1\left(\frac{2\,\mathrm{e}^{\ln(x)-\frac{\ln(-2+x)}{2}}\left(-2+x\right)}{x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1y_1+c_2y_2 \\ &= c_1\left(\sqrt{x}\right)+c_2\left(\sqrt{x}\left(\frac{2\,\mathrm{e}^{\ln(x)-\frac{\ln(-2+x)}{2}}\left(-2+x\right)}{x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2(-x+2)\left(\frac{d^2}{dx^2}y(x)\right)-x(4-x)\left(\frac{d}{dx}y(x)\right)+(-x+3)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(x-3)y(x)}{2x^2(x-2)}+\frac{(-4+x)\left(\frac{d}{dx}y(x)\right)}{2x(x-2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)-\frac{(-4+x)\left(\frac{d}{dx}y(x)\right)}{2x(x-2)}+\frac{(x-3)y(x)}{2x^2(x-2)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{-4+x}{2x(x-2)},P_3(x)=\frac{x-3}{2x^2(x-2)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=-1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=\frac{3}{4}$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2x^2(x-2)\left(\frac{d^2}{dx^2}y(x)\right)-x(-4+x)\left(\frac{d}{dx}y(x)\right)+(x-3)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

  ◦ Shift index using $k->k-m$

  $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..3$

  $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  ◦ Shift index using $k->k+2-m$

  $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$-a_0(-1+2r)(-3+2r)x^r+\left(\sum_{k=1}^{\infty}\left(-a_k(2k+2r-1)(2k+2r-3)+a_{k-1}(2k+2r-3)(k-2+r\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-(-1+2r)(-3+2r)=0$
- Values of r that satisfy the indicial equation
  $r\in\left\{\frac{1}{2},\frac{3}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $-4\left(\left(-\frac{k}{2}-\frac{r}{2}+1\right)a_{k-1}+a_k\left(k+r-\frac{1}{2}\right)\right)\left(k+r-\frac{3}{2}\right)=0$
- Shift index using $k->k+1$
  $-4\left(\left(-\frac{k}{2}+\frac{1}{2}-\frac{r}{2}\right)a_k+a_{k+1}\left(k+\frac{1}{2}+r\right)\right)\left(k+r-\frac{1}{2}\right)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{(k+r-1)a_k}{2k+1+2r}$
- Recursion relation for $r=\frac{1}{2}$
  $a_{k+1}=\frac{\left(k-\frac{1}{2}\right)a_k}{2k+2}$
- Solution for $r=\frac{1}{2}$
  $\left[y(x)=\sum_{k=0}^{\infty}a_kx^{k+\frac{1}{2}},a_{k+1}=\frac{\left(k-\frac{1}{2}\right)a_k}{2k+2}\right]$
- Recursion relation for $r=\frac{3}{2}$
  $a_{k+1}=\frac{\left(k+\frac{1}{2}\right)a_k}{2k+4}$
- Solution for $r=\frac{3}{2}$
  $\left[y(x)=\sum_{k=0}^{\infty}a_kx^{k+\frac{3}{2}},a_{k+1}=\frac{\left(k+\frac{1}{2}\right)a_k}{2k+4}\right]$
- Combine solutions and rename parameters
  $\left[y(x)=\left(\sum_{k=0}^{\infty}a_kx^{k+\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_kx^{k+\frac{3}{2}}\right),a_{k+1}=\frac{\left(k-\frac{1}{2}\right)a_k}{2k+2},b_{k+1}=\frac{\left(k+\frac{1}{2}\right)b_k}{2k+4}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 19

```
dsolve(2*x^2*(-x+2)*diff(diff(y(x),x),x)-(-x+4)*x*diff(y(x),x)+(-x+3)*y(x) = 0,y(x),sing
```

$$y=\sqrt{x}\,c_1+c_2\sqrt{x\,(x-2)}$$

Mathematica DSolve solution

Solving time : 0.288 (sec)
Leaf size : 57

```
DSolve[{2*(2-x)*x^2*D[y[x],{x,2}]-(4-x)*x*D[y[x],x]+(3-x)*y[x]==0,{}},y[x],x,IncludeSingular
```

$$y(x) \to \sqrt[4]{x-2}\left(2c_2\sqrt{x-2}+c_1\right)\exp\left(-\frac{1}{2}\int_1^x\left(\frac{1}{2(K[1]-2)}-\frac{1}{K[1]}\right)dK[1]\right)$$

2.1.733 Problem 750

Internal problem ID [9905]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 750
Date solved : `Monday, January 27, 2025 at 06:15:37 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(1-x)\,x^2y''+(5x-4)\,xy'+(6-9x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.186 (sec)

Writing the ode as

$$\left(-x^3+x^2\right)y''+\left(5x^2-4x\right)y'+(6-9x)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^3+x^2\\ B &= 5x^2-4x\\ C &= 6-9x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t}\\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-x+4}{4x\,(-1+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x+4\\ t &= 4x(-1+x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-x+4}{4x\,(-1+x)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1396: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x(-1+x)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4\,(-1+x)^2} - \frac{1}{-1+x} + \frac{1}{x}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x+4}{4x(-1+x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x+4}{4x(-1+x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right)\\ &= \frac{1}{2} - \left(\frac{1}{2}\right)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (-)[\sqrt{r}]_\infty\\ &= \frac{1}{x} - \frac{1}{2(-1+x)} + (-)(0)\\ &= \frac{1}{x} - \frac{1}{2(-1+x)}\\ &= \frac{-2+x}{2(-1+x)x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x} - \frac{1}{2(-1+x)}\right)(0) + \left(\left(-\frac{1}{x^2} + \frac{1}{2(-1+x)^2}\right) + \left(\frac{1}{x} - \frac{1}{2(-1+x)}\right)^2 - \left(\frac{-x+4}{4x(-1+x)^2}\right)\right) =$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{1}{x} - \frac{1}{2(-1+x)}\right) dx} \\ &= \frac{x}{\sqrt{-1+x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{5x^2-4x}{-x^3+x^2} \, dx} \\ &= z_1 e^{2\ln(x) + \frac{\ln(-1+x)}{2}} \\ &= z_1 \left(x^2 \sqrt{-1+x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^3$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{5x^2-4x}{-x^3+x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{4\ln(x) + \ln(-1+x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\ln(x) + \frac{1}{x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(x^3\right) + c_2 \left(x^3 \left(\ln(x) + \frac{1}{x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 18

```
dsolve((1-x)*x^2*diff(diff(y(x),x),x)+(5*x-4)*x*diff(y(x),x)+(6-9*x)*y(x) = 0,y(x),sings
```

$$y = x^2(\ln(x)\, c_2 x + c_1 x + c_2)$$

Mathematica DSolve solution

Solving time : 0.227 (sec)
Leaf size : 98

```
DSolve[{(1-x)*x^2*D[y[x],{x,2}]+(5*x-4)*x*D[y[x],x]+(6-9*x)*y[x]==0,{}},y[x],x,IncludeSingular
```

$$y(x) \to \exp\left(\int_1^x \left(\frac{1}{K[1]} + \frac{1}{2-2K[1]}\right) dK[1] - \frac{1}{2}\int_1^x \left(\frac{1}{1-K[2]} - \frac{4}{K[2]}\right) dK[2]\right)\left(c_2 \int_1^x \exp\left(-2\int_1^{K[3]} \frac{K[1]-2}{2(K[1]-1)K[1]} dK[1]\right) dK[3] + c_1\right)$$

2.1.734 Problem 751

Internal problem ID [9906]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 751
Date solved : Monday, January 27, 2025 at 06:15:37 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$xy'' + (4x^2 + 1)\, y' + 4x(x^2 + 1)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.153 (sec)

Writing the ode as

$$xy'' + (4x^2 + 1)\, y' + (4x^3 + 4x)\, y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 4x^2 + 1 \\ C &= 4x^3 + 4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1397: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=-\frac{1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{2}-\left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^{-}}{x-c_1}\right)+(-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x}+(-)\,(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}
$$

Let

$$
p(x)=1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x^2+1}{x} \, dx} \\ &= z_1 e^{-x^2 - \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{\mathrm{e}^{-x^2}}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x^2+1}{x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-2x^2 - \ln(x)}}{(y_1)^2} \, dx \\ &= y_1 (\ln(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-x^2}\right) + c_2 \left(\mathrm{e}^{-x^2} (\ln(x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dx^2}y(x)\right)x+(4x^2+1)\left(\frac{d}{dx}y(x)\right)+4x(x^2+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=(-4x^2-4)\,y(x)-\frac{(4x^2+1)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)+\frac{(4x^2+1)\left(\frac{d}{dx}y(x)\right)}{x}+(4x^2+4)\,y(x)=0$

☐ Check to see if $x_0=0$ is a regular singular point

- ∘ Define functions

$\left[P_2(x)=\frac{4x^2+1}{x},P_3(x)=4x^2+4\right]$

- ∘ $x\cdot P_2(x)$ is analytic at $x=0$

$(x\cdot P_2(x))\bigg|_{x=0}=1$

- ∘ $x^2\cdot P_3(x)$ is analytic at $x=0$

$(x^2\cdot P_3(x))\bigg|_{x=0}=0$

- ∘ $x=0$is a regular singular point

Check to see if $x_0=0$ is a regular singular point

$x_0=0$

- Multiply by denominators

$\left(\frac{d^2}{dx^2}y(x)\right)x+(4x^2+1)\left(\frac{d}{dx}y(x)\right)+4x(x^2+1)\,y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

☐ Rewrite ODE with series expansions

- ∘ Convert $x^m\cdot y(x)$ to series expansion for $m=1..3$

$x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ∘ Shift index using $k->k-m$

$x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ∘ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..2$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ∘ Shift index using $k->k+1-m$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ∘ Convert $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

$x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-1}$

- ∘ Shift index using $k->k+1$

$x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

Rewrite ODE with series expansions

$$a_0 r^2 x^{-1+r} + a_1(1+r)^2 x^r + \left(a_2(2+r)^2 + 4a_0(1+r)\right) x^{1+r} + \left(a_3(3+r)^2 + 4a_1(2+r)\right) x^{2+r} + \left(\sum_{k=}^{\infty}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 0$
- The coefficients of each power of x must be 0
  $\left[a_1(1+r)^2 = 0, a_2(2+r)^2 + 4a_0(1+r) = 0, a_3(3+r)^2 + 4a_1(2+r) = 0\right]$
- Solve for the dependent coefficient(s)
  $\left\{a_1 = 0, a_2 = -\frac{4a_0(1+r)}{r^2+4r+4}, a_3 = 0\right\}$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1)^2 + 4a_{k-1}k + 4a_{k-3} = 0$
- Shift index using $k- >k+3$
  $a_{k+4}(k+4)^2 + 4a_{k+2}(k+3) + 4a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+4} = -\frac{4(ka_{k+2}+a_k+3a_{k+2})}{(k+4)^2}$
- Recursion relation for $r = 0$
  $a_{k+4} = -\frac{4(ka_{k+2}+a_k+3a_{k+2})}{(k+4)^2}$
- Solution for $r = 0$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{4(ka_{k+2}+a_k+3a_{k+2})}{(k+4)^2}, a_1 = 0, a_2 = -a_0, a_3 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 17

```
dsolve(x*diff(diff(y(x),x),x)+(4*x^2+1)*diff(y(x),x)+4*x*(x^2+1)*y(x) = 0,y(x),singsol=a
```

$$y = e^{-x^2}(c_2 \ln(x) + c_1)$$

Mathematica DSolve solution

Solving time : 0.033 (sec)
Leaf size : 21

```
DSolve[{x*D[y[x],{x,2}]+(4*x^2+1)*D[y[x],x]+4*x*(x^2+1)*y[x]==0,{}},y[x],x,IncludeSingularSo
```

$$y(x) \rightarrow e^{-x^2}(c_2 \log(x) + c_1)$$

2.1.735 Problem 754

Internal problem ID [9907]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 754
Date solved : `Monday, January 27, 2025 at 06:15:38 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-x^2+1\right) y''-2xy'+12y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.272 (sec)

Writing the ode as

$$\left(-x^2+1\right) y''-2xy'+12y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+1 \\ B &= -2x \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{12x^2-13}{\left(x^2-1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 12x^2-13 \\ t &= \left(x^2-1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{12x^2-13}{\left(x^2-1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1399: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{25}{4(x+1)} + \frac{25}{4(x-1)} - \frac{1}{4(x+1)^2} - \frac{1}{4(x-1)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{12x^2 - 13}{(x^2 - 1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 12$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 4 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -3
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{12x^2 - 13}{(x^2 - 1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	4	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 4$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= 4 - (1) \\
&= 3
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x-2} + \frac{1}{2x+2} + (0) \\
&= \frac{1}{2x-2} + \frac{1}{2x+2} \\
&= \frac{x}{x^2-1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 3$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(6x + 2a_2) + 2\left(\frac{1}{2x-2} + \frac{1}{2x+2}\right)(3x^2 + 2xa_2 + a_1) + \left(\left(-\frac{1}{2(x-1)^2} - \frac{1}{2(x+1)^2}\right) + \left(\frac{1}{2x-2} + \frac{1}{2x \cdots}\right.\right.$$

$$\frac{-6a_2x^2 + (-10a_1 \cdots}{x^{\cdots}}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = 0, a_1 = -\frac{3}{5}, a_2 = 0\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^3 - \frac{3}{5}x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^3 - \frac{3}{5}x\right) e^{\int \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right) dx} \\
&= \left(x^3 - \frac{3}{5}x\right) \sqrt{(x-1)(x+1)} \\
&= \frac{(5x^3 - 3x)\sqrt{x^2-1}}{5}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{-x^2+1} \, dx} \\
&= z_1 e^{-\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x-1}\sqrt{x+1}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(5x^3 - 3x)\sqrt{x^2-1}}{5\sqrt{x-1}\sqrt{x+1}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{-x^2+1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\ln(x-1) - \ln(x+1)}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{25}{9x} + \frac{25\ln(x-1)}{8} + \frac{125x}{36\left(x^2 - \frac{3}{5}\right)} - \frac{25\ln(x+1)}{8}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(5x^3-3x)\sqrt{x^2-1}}{5\sqrt{x-1}\,\sqrt{x+1}}\right) \\
&\quad + c_2\left(\frac{(5x^3-3x)\sqrt{x^2-1}}{5\sqrt{x-1}\,\sqrt{x+1}}\left(\frac{25}{9x}+\frac{25\ln(x-1)}{8}+\frac{125x}{36\left(x^2-\frac{3}{5}\right)}-\frac{25\ln(x+1)}{8}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(-x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)-2x\left(\frac{d}{dx}y(x)\right)+12y(x)=0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x)=\frac{12y(x)}{x^2-1}-\frac{2\left(\frac{d}{dx}y(x)\right)x}{x^2-1}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x)+\frac{2\left(\frac{d}{dx}y(x)\right)x}{x^2-1}-\frac{12y(x)}{x^2-1}=0$
- □ Check to see if x_0 is a regular singular point
  - Define functions
    $\left[P_2(x)=\frac{2x}{x^2-1}, P_3(x)=-\frac{12}{x^2-1}\right]$
  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$
    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=1$
  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$
    $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$
  - $x=-1$is a regular singular point
    Check to see if x_0 is a regular singular point
    $x_0=-1$
- Multiply by denominators
  $(x^2-1)\left(\frac{d^2}{dx^2}y(x)\right)+2x\left(\frac{d}{dx}y(x)\right)-12y(x)=0$
- Change variables using $x=u-1$ so that the regular singular point is at $u=0$
  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(2u-2)\left(\frac{d}{du}y(u)\right)-12y(u)=0$
- Assume series solution for $y(u)$
  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$
- □ Rewrite ODE with series expansions
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$
    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$
    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ◦ Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$
- ◦ Shift index using $k- > k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r^2u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)^2 + a_k(k+r+4)(k+r-3)\right)u^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 0$
- Each term in the series must be 0, giving the recursion relation

  $-2a_{k+1}(k+1)^2 + a_k(k+4)(k-3) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+4)(k-3)}{2(k+1)^2}$
- Recursion relation for $r = 0$; series terminates at $k = 3$

  $a_{k+1} = \frac{a_k(k+4)(k-3)}{2(k+1)^2}$
- Apply recursion relation for $k = 0$

  $a_1 = -6a_0$
- Apply recursion relation for $k = 1$

  $a_2 = -\frac{5a_1}{4}$
- Express in terms of a_0

  $a_2 = \frac{15a_0}{2}$
- Apply recursion relation for $k = 2$

  $a_3 = -\frac{a_2}{3}$
- Express in terms of a_0

  $a_3 = -\frac{5a_0}{2}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - 6u + \frac{15}{2}u^2 - \frac{5}{2}u^3\right)$
- Revert the change of variables $u = x + 1$

  $\left[y(x) = a_0\left(\frac{3}{2}x - \frac{5}{2}x^3\right)\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 55

```
dsolve((-x^2+1)*diff(diff(y(x),x),x)-2*diff(y(x),x)*x+12*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_2(5x^3 - 3x)\ln(x-1)}{24} + \frac{(-5x^3 + 3x)\,c_2\ln(x+1)}{24} - \frac{5c_1x^3}{3} + \frac{5c_2x^2}{12} + c_1x - \frac{c_2}{9}$$

Mathematica DSolve solution

Solving time : `0.024` (sec)
Leaf size : 59

```
DSolve[{(1-x^2)*D[y[x],{x,2}]-2*x*D[y[x],x]+12*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tru
```

$$y(x) \to \frac{1}{2}c_1x(5x^2 - 3) + c_2\left(-\frac{5x^2}{2} - \frac{1}{4}(5x^2 - 3)\,x(\log(1-x) - \log(x+1)) + \frac{2}{3}\right)$$

2.1.736 Problem 755

Internal problem ID [9908]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 755
Date solved : `Monday, January 27, 2025 at 06:15:39 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x(x+2)\, y''+2(x+1)\, y'-2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.223 (sec)

Writing the ode as

$$\left(x^{2}+2x\right) y''+(2x+2)\, y'-2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^{2}+2x \\ B &= 2x+2 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^{2}-4AC}{4A^{2}} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{2x^{2}+4x-1}{\left(x^{2}+2x\right)^{2}} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2x^{2}+4x-1 \\ t &= \left(x^{2}+2x\right)^{2} \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{2x^{2}+4x-1}{\left(x^{2}+2x\right)^{2}}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1401: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(x+2)^2} - \frac{5}{4(x+2)} - \frac{1}{4x^2} + \frac{5}{4x}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^2+4x-1}{(x^2+2x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2x^2+4x-1}{(x^2+2x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{1}{2}$	$\frac{1}{2}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x+4} + \frac{1}{2x} + (0) \\
&= \frac{1}{2x+4} + \frac{1}{2x} \\
&= \frac{x+1}{x(x+2)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x+4} + \frac{1}{2x}\right)(1) + \left(\left(-\frac{1}{2(x+2)^2} - \frac{1}{2x^2}\right) + \left(\frac{1}{2x+4} + \frac{1}{2x}\right)^2 - \left(\frac{2x^2+4x-1}{(x^2+2x)^2}\right)\right) = 0$$

$$\frac{2-2a_0}{x(x+2)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+1)\, e^{\int \left(\frac{1}{2x+4} + \frac{1}{2x}\right) dx} \\
&= (x+1)\sqrt{x(x+2)} \\
&= (x+1)\sqrt{x(x+2)}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x+2}{x^2+2x}\, dx} \\
&= z_1 e^{-\frac{\ln(x(x+2))}{2}} \\
&= z_1\left(\frac{1}{\sqrt{x(x+2)}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x + 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x+2}{x^2+2x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\ln(x(x+2))}}{(y_1)^2}\, dx \\
&= y_1\left(\frac{\ln(x)}{2} + \frac{1}{x+1} - \frac{\ln(x+2)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x+1) + c_2\left(x+1\left(\frac{\ln(x)}{2} + \frac{1}{x+1} - \frac{\ln(x+2)}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x(x+2)\left(\frac{d^2}{dx^2}y(x)\right)+2(x+1)\left(\frac{d}{dx}y(x)\right)-2y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{2(x+1)\left(\frac{d}{dx}y(x)\right)}{x(x+2)}+\frac{2y(x)}{x(x+2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)-\frac{2y(x)}{x(x+2)}+\frac{2(x+1)\left(\frac{d}{dx}y(x)\right)}{x(x+2)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=\frac{2(x+1)}{x(x+2)}, P_3(x)=-\frac{2}{x(x+2)}\right]$

- ◦ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

$\left.\left((x+2)\cdot P_2(x)\right)\right|_{x=-2}=1$

- ◦ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

$\left.\left((x+2)^2\cdot P_3(x)\right)\right|_{x=-2}=0$

- ◦ $x=-2$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-2$

- Multiply by denominators

$x(x+2)\left(\frac{d^2}{dx^2}y(x)\right)+(2x+2)\left(\frac{d}{dx}y(x)\right)-2y(x)=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

$(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(2u-2)\left(\frac{d}{du}y(u)\right)-2y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ◦ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

- ◦ Shift index using $k->k+2-m$

$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

Rewrite ODE with series expansions

$-2a_0r^2u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)^2+a_k(k+r+2)(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-2r^2 = 0$

- Values of r that satisfy the indicial equation
  $r = 0$
- Each term in the series must be 0, giving the recursion relation
  $-2a_{k+1}(k+1)^2 + a_k(k+2)(k-1) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$
- Recursion relation for $r = 0$; series terminates at $k = 1$
  $a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$
- Apply recursion relation for $k = 0$
  $a_1 = -a_0$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second linea
  $y(u) = a_0 \cdot (-u+1)$
- Revert the change of variables $u = x+2$
  $[y(x) = a_0(-x-1)]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 28

```
dsolve(x*(x+2)*diff(diff(y(x),x),x)+2*(x+1)*diff(y(x),x)-2*y(x) = 0,y(x),singsol=all)
```

$$y = -\frac{(x+1)\, c_2 \ln(x+2)}{2} + \frac{c_2(x+1)\ln(x)}{2} + c_1 x + c_1 + c_2$$

Mathematica DSolve solution

Solving time : 0.025 (sec)
Leaf size : 37

```
DSolve[{x*(x+2)*D[y[x],{x,2}]+2*(x+1)*D[y[x],x]-2*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to c_1(x+1) - \frac{1}{2}c_2((x+1)\log(-x) - (x+1)\log(x+2) + 2)$$

2.1.737 Problem 757

Internal problem ID [9909]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 757
Date solved : Monday, January 27, 2025 at 06:15:39 PM
CAS classification :
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, '_with_symmetry_[0,F(x)]

Solve

$$x(x+2)\, y'' + (x+1)\, y' - 4y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.204 (sec)

Writing the ode as

$$\left(x^2+2x\right) y'' + (x+1)\, y' - 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+2x \\ B &= x+1 \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15x^2+30x-3}{4\left(x^2+2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15x^2+30x-3 \\ t &= 4\left(x^2+2x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15x^2 + 30x - 3}{4\left(x^2 + 2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1403: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2} - \frac{3}{16\left(x + 2\right)^2} + \frac{33}{16x} - \frac{33}{16\left(x + 2\right)}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15x^2 + 30x - 3}{4\left(x^2 + 2x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2}\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15x^2 + 30x - 3}{4\left(x^2 + 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{3}{4}$	$\frac{1}{4}$
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{2}$ then

$$\begin{aligned}d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right)\\ &= \frac{5}{2} - \left(\frac{3}{2}\right)\\ &= 1\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{3}{4(x+2)} + \frac{3}{4x} + (0) \\
&= \frac{3}{4(x+2)} + \frac{3}{4x} \\
&= \frac{\frac{3x}{2}+\frac{3}{2}}{x(x+2)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{4(x+2)} + \frac{3}{4x}\right)(1) + \left(\left(-\frac{3}{4(x+2)^2} - \frac{3}{4x^2}\right) + \left(\frac{3}{4(x+2)} + \frac{3}{4x}\right)^2 - \left(\frac{15x^2+30x-3}{4(x^2+2x)^2}\right)\right) = 0$$

$$\frac{3-3a_0}{x(x+2)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (x+1)\, e^{\int \left(\frac{3}{4(x+2)} + \frac{3}{4x}\right) dx} \\
&= (x+1)\,(x(x+2))^{3/4} \\
&= (x+1)\,(x(x+2))^{3/4}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x+1}{x^2+2x}\, dx} \\
&= z_1 e^{-\frac{\ln(x(x+2))}{4}} \\
&= z_1 \left(\frac{1}{(x(x+2))^{1/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x(x+2)}\,(x+1)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x+1}{x^2+2x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x(x+2))}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(-\frac{2x^2+4x+1}{\sqrt{x\,(x+2)}\,(x+1)} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\sqrt{x\,(x+2)}\,(x+1) \right) + c_2 \left(\sqrt{x\,(x+2)}\,(x+1) \left(-\frac{2x^2+4x+1}{\sqrt{x\,(x+2)}\,(x+1)} \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x(x+2)\left(\frac{d^2}{dx^2}y(x)\right) + (x+1)\left(\frac{d}{dx}y(x)\right) - 4y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{4y(x)}{x(x+2)} - \frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{x(x+2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{x(x+2)} - \frac{4y(x)}{x(x+2)} = 0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{x+1}{x(x+2)}, P_3(x) = -\frac{4}{x(x+2)}\right]$

  - $(x+2) \cdot P_2(x)$ is analytic at $x = -2$

    $\left((x+2) \cdot P_2(x)\right)\Big|_{x=-2} = \frac{1}{2}$

  - $(x+2)^2 \cdot P_3(x)$ is analytic at $x = -2$

    $\left((x+2)^2 \cdot P_3(x)\right)\Big|_{x=-2} = 0$

  - $x = -2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -2$

- Multiply by denominators

  $x(x+2)\left(\frac{d^2}{dx^2}y(x)\right) + (x+1)\left(\frac{d}{dx}y(x)\right) - 4y(x) = 0$

- Change variables using $x = u - 2$ so that the regular singular point is at $u = 0$

  $(u^2 - 2u)\left(\frac{d^2}{du^2}y(u)\right) + (u-1)\left(\frac{d}{du}y(u)\right) - 4y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

Rewrite ODE with series expansions

$-a_0 r(-1+2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)(2k+1+2r) + a_k(k+r+2)(k+r-2)\right) u^{k+r}\right) =$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $-2(k+1+r)\left(k+r+\frac{1}{2}\right) a_{k+1} + a_k(k+r+2)(k+r-2) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r+2)(k+r-2)}{(k+1+r)(2k+1+2r)}$

- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+1} = \frac{a_k(k+2)(k-2)}{(k+1)(2k+1)}$

- Apply recursion relation for $k = 0$

  $a_1 = -4a_0$

- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{2}$

- Express in terms of a_0

  $a_2 = 2a_0$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second linea

  $y(u) = a_0 \cdot (2u^2 - 4u + 1)$

- Revert the change of variables $u = x + 2$

  $[y(x) = a_0(2x^2 + 4x + 1)]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}$

- Solution for $r = \frac{1}{2}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$

- Revert the change of variables $u = x + 2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k(x+2)^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$

- Combine solutions and rename parameters

$$\left[y(x)=a_0(2x^2+4x+1)+\left(\sum_{k=0}^{\infty} b_k (x+2)^{k+\frac{1}{2}}\right), b_{k+1}=\frac{b_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
   Solution is available but has compositions of trig with ln functions of radicals. A
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- linear_1 successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 28

```
dsolve(x*(x+2)*diff(diff(y(x),x),x)+(x+1)*diff(y(x),x)-4*y(x) = 0,y(x),singsol=all)
```

$$y=c_2(x+1)\sqrt{x(x+2)}+2\left(x^2+2x+\frac{1}{2}\right)c_1$$

Mathematica DSolve solution

Solving time : 0.061 (sec)
Leaf size : 45

```
DSolve[{x*(x+2)*D[y[x],{x,2}]+(x+1)*D[y[x],x]-4*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x)\to c_1\cosh\left(4\operatorname{arctanh}\left(\frac{1}{\sqrt{\frac{x}{x+2}}}\right)\right)+ic_2\sinh\left(4\operatorname{arctanh}\left(\frac{1}{\sqrt{\frac{x}{x+2}}}\right)\right)$$

2.1.738 Problem 758

Internal problem ID [9910]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 758
Date solved : `Monday, January 27, 2025 at 06:15:40 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(x-1)\,y''-xy'+y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.226 (sec)

Writing the ode as

$$(x-1)\,y''-xy'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x-1\\ B &= -x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{x^2-4x+6}{4\,(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s=x^2-4x+6$$
$$t=4(x-1)^2$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x^2-4x+6}{4\,(x-1)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1405: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4\,(x-1)^2} - \frac{1}{2\,(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 4x + 6}{4(x-1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(x-1)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2(x-1)} + \frac{1}{2} \\
&= \frac{x-2}{2x-2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2(x-1)^2}\right) + \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)^2 - \left(\frac{x^2-4x+6}{4(x-1)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{x-1} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x-1)}{2}} \\
&= z_1\left(\sqrt{x-1}\,\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x-1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{x + \ln(x-1)}}{(y_1)^2} \, dx \\
&= y_1 \left(-\frac{x\,\mathrm{e}^{x+\ln(x-1)}\mathrm{e}^{-2x}}{x-1}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(-\frac{x\,\mathrm{e}^{x+\ln(x-1)}\mathrm{e}^{-2x}}{x-1}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x-1)\left(\frac{d^2}{dx^2}y(x)\right)-x\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{y(x)}{x-1}+\frac{x\left(\frac{d}{dx}y(x)\right)}{x-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)-\frac{x\left(\frac{d}{dx}y(x)\right)}{x-1}+\frac{y(x)}{x-1}=0$

□ Check to see if $x_0=1$ is a regular singular point

  - Define functions

    $\left[P_2(x)=-\frac{x}{x-1}, P_3(x)=\frac{1}{x-1}\right]$

  - $(x-1)\cdot P_2(x)$ is analytic at $x=1$

    $\left((x-1)\cdot P_2(x)\right)\Big|_{x=1}=-1$

  - $(x-1)^2\cdot P_3(x)$ is analytic at $x=1$

    $\left((x-1)^2\cdot P_3(x)\right)\Big|_{x=1}=0$

  - $x=1$is a regular singular point

  Check to see if $x_0=1$ is a regular singular point

  $x_0=1$

- Multiply by denominators

  $(x-1)\left(\frac{d^2}{dx^2}y(x)\right)-x\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Change variables using $x=u+1$ so that the regular singular point is at $u=0$

  $u\left(\frac{d^2}{du^2}y(u)\right)+(-u-1)\left(\frac{d}{du}y(u)\right)+y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

    $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-1}$

  - Shift index using $k->k+1$

    $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(k+r-1)-a_k(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r)=0$

- Values of r that satisfy the indicial equation
  $r\in\{0,2\}$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-1)\,(a_{k+1}(k+1+r)-a_k)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{a_k}{k+1+r}$
- Recursion relation for $r=0$
  $a_{k+1}=\frac{a_k}{k+1}$
- Solution for $r=0$
  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+1}=\frac{a_k}{k+1}\right]$
- Revert the change of variables $u=x-1$
  $\left[y(x)=\sum_{k=0}^{\infty} a_k (x-1)^k, a_{k+1}=\frac{a_k}{k+1}\right]$
- Recursion relation for $r=2$
  $a_{k+1}=\frac{a_k}{k+3}$
- Solution for $r=2$
  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$
- Revert the change of variables $u=x-1$
  $\left[y(x)=\sum_{k=0}^{\infty} a_k (x-1)^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$
- Combine solutions and rename parameters
  $\left[y(x)=\left(\sum_{k=0}^{\infty} a_k (x-1)^k\right)+\left(\sum_{k=0}^{\infty} b_k (x-1)^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+3}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 12

```
dsolve((x-1)*diff(diff(y(x),x),x)-diff(y(x),x)*x+y(x) = 0,y(x),singsol=all)
```

$$y=c_1x+\mathrm{e}^{x}c_2$$

Mathematica DSolve solution

Solving time : 0.156 (sec)
Leaf size : 90

```
DSolve[{(x-1)*D[y[x],{x,2}]-x*D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \exp\left(\int_1^x \frac{K[1]-2}{2(K[1]-1)}dK[1] - \frac{1}{2}\int_1^x -\frac{K[2]}{K[2]-1}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{K[1]-2}{2(K[1]-1)}dK[1]\right)dK[3] + c_1\right)$$

2.1.739 Problem 759

Internal problem ID [9911]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 759
Date solved : `Monday, January 27, 2025 at 06:15:41 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+1\right) y''-2xy'+2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.268 (sec)

Writing the ode as

$$\left(x^2+1\right) y''-2xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= -2x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-3}{\left(x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= \left(x^2+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{3}{\left(x^2+1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1407: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4\,(x-i)^2} + \frac{3}{4\,(x+i)^2} + \frac{3i}{4\,(x-i)} - \frac{3i}{4\,(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{3}{(x^2+1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\,(x-i)} + \frac{3}{2\,(x+i)} + (-)\,(0) \\
&= -\frac{1}{2\,(x-i)} + \frac{3}{2\,(x+i)} \\
&= \frac{x-2i}{x^2+1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)(0) + \left(\left(\frac{1}{2(x-i)^2} - \frac{3}{2(x+i)^2}\right) + \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)^2 - \left(-\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right) dx} \\ &= \frac{(x^2+1)^{3/2}}{(ix+1)^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2+1}\, dx} \\ &= z_1 e^{\frac{\ln(x^2+1)}{2}} \\ &= z_1\left(\sqrt{x^2+1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2+1)^2}{(ix+1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2+1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\ln(x^2+1)}}{(y_1)^2}\, dx \\ &= y_1\left(-\frac{x}{(x+i)^2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{(x^2+1)^2}{(ix+1)^2}\right) + c_2\left(\frac{(x^2+1)^2}{(ix+1)^2}\left(-\frac{x}{(x+i)^2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 16

```
dsolve((x^2+1)*diff(diff(y(x),x),x)-2*diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y = c_2 x^2 + c_1 x - c_2$$

Mathematica DSolve solution

Solving time : 0.307 (sec)
Leaf size : 79

```
DSolve[{(1+x^2)*D[y[x],{x,2}]-2*x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True
```

$$y(x) \to \sqrt{x^2+1}\exp\left(\int_1^x \frac{K[1]+2i}{K[1]^2+1}dK[1]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[2]}\frac{K[1]+2i}{K[1]^2+1}dK[1]\right)dK[2] + c_1\right)$$

2.1.740 Problem 760

Internal problem ID [9912]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 760
Date solved : `Monday, January 27, 2025 at 06:15:41 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2-2x+10\right)y''+xy'-4y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.356 (sec)

Writing the ode as

$$\left(x^2-2x+10\right)y''+xy'-4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2-2x+10 \\ B &= x \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{15x^2-32x+180}{4\left(x^2-2x+10\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15x^2-32x+180 \\ t &= 4\left(x^2-2x+10\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{15x^2-32x+180}{4\left(x^2-2x+10\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1408: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 2x + 10)^2$. There is a pole at $x = 1 + 3i$ of order 2. There is a pole at $x = 1 - 3i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{-\frac{7}{36} + \frac{i}{24}}{(x - 1 - 3i)^2} + \frac{-\frac{7}{36} - \frac{i}{24}}{(x - 1 + 3i)^2} - \frac{149i}{216\,(x - 1 - 3i)} + \frac{149i}{216\,(x - 1 + 3i)}$$

For the pole at $x = 1 + 3i$ let b be the coefficient of $\frac{1}{(x-1-3i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{36} + \frac{i}{24}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} + \frac{i}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} - \frac{i}{12} \end{aligned}$$

For the pole at $x = 1 - 3i$ let b be the coefficient of $\frac{1}{(x-1+3i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{36} - \frac{i}{24}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} - \frac{i}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} + \frac{i}{12} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15x^2 - 32x + 180}{4\left(x^2 - 2x + 10\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15x^2 - 32x + 180}{4\left(x^2 - 2x + 10\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$1+3i$	2	0	$\frac{3}{4} + \frac{i}{12}$	$\frac{1}{4} - \frac{i}{12}$
$1-3i$	2	0	$\frac{3}{4} - \frac{i}{12}$	$\frac{1}{4} + \frac{i}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{5}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{+}\right) \\
&= \frac{5}{2} - \left(\frac{3}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{\frac{3}{4} + \frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4} - \frac{i}{12}}{x-1+3i} + (0) \\
&= \frac{\frac{3}{4} + \frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4} - \frac{i}{12}}{x-1+3i} \\
&= \frac{3x-4}{2x^2 - 4x + 20}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{3}{4}+\frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4}-\frac{i}{12}}{x-1+3i}\right)(1) + \left(\left(\frac{-\frac{3}{4}-\frac{i}{12}}{(x-1-3i)^2} + \frac{-\frac{3}{4}+\frac{i}{12}}{(x-1+3i)^2}\right) + \left(\frac{\frac{3}{4}+\frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4}-\frac{i}{12}}{x-1+3i}\right)^2 - \frac{3(x^2}{(-x+}\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{4}{3}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - \frac{4}{3}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{4}{3}\right) e^{\int \left(\frac{\frac{3}{4}+\frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4}-\frac{i}{12}}{x-1+3i}\right) dx} \\
&= \left(x - \frac{4}{3}\right) e^{\frac{3\ln\left(x^2-2x+10\right)}{4} - \frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{6}} \\
&= \frac{(3x-4)\left(x^2-2x+10\right)^{3/4} e^{-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{6}}}{3}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2-2x+10} \, dx} \\
&= z_1 e^{-\frac{\ln\left(x^2-2x+10\right)}{4} - \frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{6}} \\
&= z_1 \left(\frac{e^{-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{6}}}{\left(x^2-2x+10\right)^{1/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x^2-2x+10}\, e^{-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}} (3x-4)}{3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{x}{x^2-2x+10}\,dx}}{(y_1)^2}\,dx$$

$$= y_1 \int \frac{e^{-\frac{\ln(x^2-2x+10)}{2}-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}}{(y_1)^2}\,dx$$

$$= y_1 \left(-\frac{9(3x^2-4x+15)\,\mathrm{e}^{-\frac{\ln(x^2-2x+10)}{2}-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}\,\mathrm{e}^{\frac{2\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}}{410\,(3x-4)}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\frac{\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}(3x-4)}{3}\right)$$

$$+ c_2 \left(\frac{\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}(3x-4)}{3}\left(-\frac{9(3x^2-4x+15)\,\mathrm{e}^{-\frac{\ln(x^2-2x+10)}{2}-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}\,\mathrm{e}^{\frac{2\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}}{410\,(3x-4)}\right)\right)$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.013 (sec)
Leaf size : 31

```
dsolve((x^2-2*x+10)*diff(diff(y(x),x),x)+diff(y(x),x)*x-4*y(x) = 0,y(x),singsol=all)
```

$$y = 3c_2(x-1+3i)^{\frac{1}{2}-\frac{i}{6}}\left(x-\frac{4}{3}\right)(x-1-3i)^{\frac{1}{2}+\frac{i}{6}} + c_1\left(x^2-\frac{4}{3}x+5\right)$$

Mathematica DSolve solution

Solving time : 0.707 (sec)
Leaf size : 125

```
DSolve[{(x^2-2*x+10)*D[y[x],{x,2}]+x*D[y[x],x]-4*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{3}(3x-4)\exp\left(\int_1^x \frac{3K[1]-4}{2(K[1]-2)K[1]+20}dK[1] - \frac{1}{2}\int_1^x \frac{K[2]}{(K[2]-2)K[2]+10}dK[2]\right)\left(c_2\int_1^x \frac{9\exp\left(-2\int_1^{K[3]} \frac{3K[1]-4}{2(K[1]^2-2K[1]+10)}dK[1]\right)}{(4-3K[3])^2}dK[3] + c_1\right)$$

2.1.741 Problem 761

Internal problem ID [9913]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 761
Date solved : `Monday, January 27, 2025 at 06:15:42 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2-2x+10\right)y''+xy'-4y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.417 (sec)

Writing the ode as

$$\left(x^2-2x+10\right)y''+xy'-4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2-2x+10\\ B &= x\\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t}\\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{15x^2-32x+180}{4\left(x^2-2x+10\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15x^2-32x+180\\ t &= 4\left(x^2-2x+10\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{15x^2-32x+180}{4\left(x^2-2x+10\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1409: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 2x + 10)^2$. There is a pole at $x = 1 + 3i$ of order 2. There is a pole at $x = 1 - 3i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{-\frac{7}{36} + \frac{i}{24}}{(x - 1 - 3i)^2} + \frac{-\frac{7}{36} - \frac{i}{24}}{(x - 1 + 3i)^2} - \frac{149i}{216\,(x - 1 - 3i)} + \frac{149i}{216\,(x - 1 + 3i)}$$

For the pole at $x = 1 + 3i$ let b be the coefficient of $\frac{1}{(x-1-3i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{36} + \frac{i}{24}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} + \frac{i}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} - \frac{i}{12} \end{aligned}$$

For the pole at $x = 1 - 3i$ let b be the coefficient of $\frac{1}{(x-1+3i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{36} - \frac{i}{24}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} - \frac{i}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} + \frac{i}{12} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15x^2 - 32x + 180}{4\left(x^2 - 2x + 10\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15x^2 - 32x + 180}{4\left(x^2 - 2x + 10\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$1+3i$	2	0	$\frac{3}{4} + \frac{i}{12}$	$\frac{1}{4} - \frac{i}{12}$
$1-3i$	2	0	$\frac{3}{4} - \frac{i}{12}$	$\frac{1}{4} + \frac{i}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\
&= \frac{5}{2} - \left(\frac{3}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{\frac{3}{4} + \frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4} - \frac{i}{12}}{x-1+3i} + (0) \\
&= \frac{\frac{3}{4} + \frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4} - \frac{i}{12}}{x-1+3i} \\
&= \frac{3x-4}{2x^2-4x+20}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{3}{4} + \frac{i}{12}}{x - 1 - 3i} + \frac{\frac{3}{4} - \frac{i}{12}}{x - 1 + 3i}\right)(1) + \left(\left(\frac{-\frac{3}{4} - \frac{i}{12}}{(x - 1 - 3i)^2} + \frac{-\frac{3}{4} + \frac{i}{12}}{(x - 1 + 3i)^2}\right) + \left(\frac{\frac{3}{4} + \frac{i}{12}}{x - 1 - 3i} + \frac{\frac{3}{4} - \frac{i}{12}}{x - 1 + 3i}\right)^2 - \frac{3(x^2}{(-x +}\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{4}{3}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - \frac{4}{3}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{4}{3}\right) e^{\int \left(\frac{\frac{3}{4} + \frac{i}{12}}{x - 1 - 3i} + \frac{\frac{3}{4} - \frac{i}{12}}{x - 1 + 3i}\right) dx} \\
&= \left(x - \frac{4}{3}\right) e^{\frac{3\ln\left(x^2 - 2x + 10\right)}{4} - \frac{\arctan\left(\frac{x}{3} - \frac{1}{3}\right)}{6}} \\
&= \frac{(3x - 4)\left(x^2 - 2x + 10\right)^{3/4} e^{-\frac{\arctan\left(\frac{x}{3} - \frac{1}{3}\right)}{6}}}{3}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2 - 2x + 10} \, dx} \\
&= z_1 e^{-\frac{\ln\left(x^2 - 2x + 10\right)}{4} - \frac{\arctan\left(\frac{x}{3} - \frac{1}{3}\right)}{6}} \\
&= z_1 \left(\frac{e^{-\frac{\arctan\left(\frac{x}{3} - \frac{1}{3}\right)}{6}}}{\left(x^2 - 2x + 10\right)^{1/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x^2 - 2x + 10}\, e^{-\frac{\arctan\left(\frac{x}{3} - \frac{1}{3}\right)}{3}} (3x - 4)}{3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{x}{x^2-2x+10} dx}}{(y_1)^2} dx$$
$$= y_1 \int \frac{e^{-\frac{\ln(x^2-2x+10)}{2} - \frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}}{(y_1)^2} dx$$
$$= y_1 \left(-\frac{9(3x^2-4x+15)\, e^{-\frac{\ln(x^2-2x+10)}{2} - \frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}} e^{\frac{2\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}}{410\,(3x-4)} \right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{\sqrt{x^2-2x+10}\, e^{-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}} (3x-4)}{3} \right)$$
$$+ c_2 \left(\frac{\sqrt{x^2-2x+10}\, e^{-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}} (3x-4)}{3} \left(-\frac{9(3x^2-4x+15)\, e^{-\frac{\ln(x^2-2x+10)}{2} - \frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}} e^{\frac{2\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}}{410\,(3x-4)} \right) \right)$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.010 (sec)
Leaf size : 31

```
dsolve((x^2-2*x+10)*diff(diff(y(x),x),x)+diff(y(x),x)*x-4*y(x) = 0,y(x),singsol=all)
```

$$y = 3c_2(x-1+3i)^{\frac{1}{2}-\frac{i}{6}} \left(x - \frac{4}{3}\right) (x-1-3i)^{\frac{1}{2}+\frac{i}{6}} + c_1\left(x^2 - \frac{4}{3}x + 5\right)$$

Mathematica DSolve solution

Solving time : `0.543` (sec)
Leaf size : 125

```
DSolve[{(x^2-2*x+10)*D[y[x],{x,2}]+x*D[y[x],x]-4*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to \frac{1}{3}(3x-4)\exp\left(\int_1^x \frac{3K[1]-4}{2(K[1]-2)K[1]+20}dK[1] - \frac{1}{2}\int_1^x \frac{K[2]}{(K[2]-2)K[2]+10}dK[2]\right)\left(c_2\int_1^x \frac{9\exp\left(-2\int_1^{K[3]} \frac{3K[1]-4}{2(K[1]^2-2K[1]+10)}dK[1]\right)}{(4-3K[3])^2}dK[3] + c_1\right)$$

2.1.742 Problem 762

Internal problem ID [9914]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 762
Date solved : `Monday, January 27, 2025 at 06:15:43 PM`
CAS classification : `[_Hermite]`

Solve

$$y'' - xy' + 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.232 (sec)

Writing the ode as

$$y'' - xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 10}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 10 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{5}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1410: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{5}{2x} - \frac{25}{4x^3} - \frac{125}{4x^5} - \frac{3125}{16x^7} - \frac{21875}{16x^9} - \frac{328125}{32x^{11}} - \frac{2578125}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{2}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}
$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 10}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} - \frac{5}{2}\right) + (0) \\
&= \frac{x^2}{4} - \frac{5}{2}
\end{aligned}
$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{5}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{5}{2}\right) - (0) \\
&= -\frac{5}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &= -3 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &= 2
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2}{4} - \frac{5}{2}
$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (2) + 2\left(-\frac{x}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{5}{2}\right)\right) &= 0 \\ a_1 x + 2a_0 + 2 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \left(x^2 - 1\right) e^{\int -\frac{x}{2} dx} \\ &= \left(x^2 - 1\right) e^{-\frac{x^2}{4}} \\ &= \left(x^2 - 1\right) e^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1\left(e^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^2 - 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{e^{\frac{x^2}{2}}}{(x^2-1)^2} dx \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (x^2 - 1) + c_2 \left(x^2 - 1 \left(\int \frac{e^{\frac{x^2}{2}}}{(x^2-1)^2} dx \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2}y(x) - x\left(\frac{d}{dx}y(x)\right) + 2y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite DE with series expansions

  ◦ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion
  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$
  ◦ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion
  $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$
  ◦ Shift index using $k- >k+2$
  $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions
  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k(k-2)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation
  $(k^2 + 3k + 2)\, a_{k+2} - a_k(k-2) = 0$
- Recursion relation; series terminates at $k = 2$

$a_{k+2} = \frac{a_k(k-2)}{k^2+3k+2}$

- Apply recursion relation for $k = 0$

  $a_2 = -a_0$

- Terminating series solution of the ODE. Use reduction of order to find the second linearly indepe

  $y(x) = A_2x^2 + A_1x - a_0$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.030` (sec)
Leaf size : 42

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y = 2c_1 \mathrm{e}^{\frac{x^2}{2}} x - (x-1)(x+1)\left(c_1\sqrt{\pi}\,\mathrm{erfi}\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2} - c_2\right)$$

Mathematica DSolve solution

Solving time : `0.207` (sec)
Leaf size : 43

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to (x^2-1)\left(c_2 \int_1^x \frac{e^{\frac{K[1]^2}{2}}}{(K[1]^2-1)^2} dK[1] + c_1\right)$$

2.1.743 Problem 763

Internal problem ID [9915]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 763
Date solved : `Monday, January 27, 2025 at 06:15:43 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(x+2)\,y''+xy'-y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.248 (sec)

Writing the ode as

$$(x+2)\,y''+xy'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x+2\\ B &= x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{x^2+4x+12}{4\left(x+2\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2+4x+12\\ t &= 4(x+2)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x^2+4x+12}{4\left(x+2\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1412: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x+2)^2$. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{(x+2)^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{2}{x^2} - \frac{8}{x^3} + \frac{20}{x^4} - \frac{32}{x^5} + \frac{16}{x^6} + \frac{64}{x^7} - \frac{80}{x^8} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2} \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 4x + 12}{4x^2 + 16x + 16} \\
&= Q + \frac{R}{4x^2 + 16x + 16} \\
&= \left(\frac{1}{4}\right) + \left(\frac{8}{4x^2 + 16x + 16}\right) \\
&= \frac{1}{4} + \frac{8}{4x^2 + 16x + 16}
\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 4 gives 0. Now b can be found.

$$\begin{aligned}
b &= (0) - (0) \\
&= 0
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{0}{\frac{1}{2}} - 0\right) = 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{0}{\frac{1}{2}} - 0\right) = 0
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 4x + 12}{4\left(x+2\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x+2} + (-)\left(\frac{1}{2}\right) \\ &= -\frac{1}{x+2} - \frac{1}{2} \\ &= -\frac{4+x}{2\left(x+2\right)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x+2} - \frac{1}{2}\right)(1) + \left(\left(\frac{1}{\left(x+2\right)^2}\right) + \left(-\frac{1}{x+2} - \frac{1}{2}\right)^2 - \left(\frac{x^2+4x+12}{4\left(x+2\right)^2}\right)\right) = 0$$

$$\frac{a_0 - 4}{x+2} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 4\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 4 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (4 + x) \, \mathrm{e}^{\int \left(-\frac{1}{x+2} - \frac{1}{2}\right) dx} \\
&= (4 + x) \, \mathrm{e}^{-\frac{x}{2} - \ln(x+2)} \\
&= \frac{(4 + x) \, \mathrm{e}^{-\frac{x}{2}}}{x + 2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{x+2} \, dx} \\
&= z_1 e^{-\frac{x}{2} + \ln(x+2)} \\
&= z_1 \left((x + 2) \, \mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}(4 + x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x+2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-x + 2\ln(x+2)}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{x \, \mathrm{e}^{-x + 2\ln(x+2)} \mathrm{e}^{2x}}{(4 + x) \, (x + 2)^2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}(4 + x)\right) + c_2 \left(\mathrm{e}^{-x}(4 + x) \left(\frac{x \, \mathrm{e}^{-x + 2\ln(x+2)} \mathrm{e}^{2x}}{(4 + x) \, (x + 2)^2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x+2)\left(\frac{d^2}{dx^2}y(x)\right)+x\left(\frac{d}{dx}y(x)\right)-y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=\frac{y(x)}{x+2}-\frac{x\left(\frac{d}{dx}y(x)\right)}{x+2}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x)+\frac{x\left(\frac{d}{dx}y(x)\right)}{x+2}-\frac{y(x)}{x+2}=0$

□ Check to see if $x_0=-2$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{x}{x+2}, P_3(x)=-\frac{1}{x+2}\right]$

  ◦ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

  $\left((x+2)\cdot P_2(x)\right)\Big|_{x=-2}=-2$

  ◦ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

  $\left((x+2)^2\cdot P_3(x)\right)\Big|_{x=-2}=0$

  ◦ $x=-2$is a regular singular point

  Check to see if $x_0=-2$ is a regular singular point

  $x_0=-2$

- Multiply by denominators

$(x+2)\left(\frac{d^2}{dx^2}y(x)\right)+x\left(\frac{d}{dx}y(x)\right)-y(x)=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

$u\left(\frac{d^2}{du^2}y(u)\right)+(u-2)\left(\frac{d}{du}y(u)\right)-y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  ◦ Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-1}$

  ◦ Shift index using $k->k+1$

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)(k+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-3+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)(k-2+r)+a_k(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-3+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{0,3\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k-2+r)+a_k(k+r-1)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{a_k(k+r-1)}{(k+1+r)(k-2+r)}$

- Recursion relation for $r=0$; series terminates at $k=1$

  $a_{k+1}=-\frac{a_k(k-1)}{(k+1)(k-2)}$

- Apply recursion relation for $k=0$

  $a_1=-\frac{a_0}{2}$

- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li

  $y(u)=a_0\cdot\left(1-\frac{u}{2}\right)$

- Revert the change of variables $u=x+2$

  $\left[y(x)=-\frac{a_0x}{2}\right]$

- Recursion relation for $r=3$

  $a_{k+1}=-\frac{a_k(k+2)}{(k+4)(k+1)}$

- Solution for $r=3$

  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+3}, a_{k+1}=-\frac{a_k(k+2)}{(k+4)(k+1)}\right]$

- Revert the change of variables $u=x+2$

  $\left[y(x)=\sum_{k=0}^{\infty}a_k(x+2)^{k+3}, a_{k+1}=-\frac{a_k(k+2)}{(k+4)(k+1)}\right]$

- Combine solutions and rename parameters

  $\left[y(x)=-\frac{a_0x}{2}+\left(\sum_{k=0}^{\infty}b_k(x+2)^{k+3}\right), b_{k+1}=-\frac{b_k(k+2)}{(4+k)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 17

```
dsolve((x+2)*diff(diff(y(x),x),x)+diff(y(x),x)*x-y(x) = 0,y(x),singsol=all)
```

$$y=c_1x+c_2e^{-x}(x+4)$$

Mathematica DSolve solution

Solving time : 0.334 (sec)
Leaf size : 96

```
DSolve[{(x+2)*D[y[x],{x,2}]+x*D[y[x],x]-y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2i\left((c_1(x+2)+2ic_2)\cosh\left(\frac{x+2}{2}\right)-(ic_2(x+2)+2c_1)\sinh\left(\frac{x+2}{2}\right)\right)\exp\left(\int_1^x \frac{1-K[1]}{2K[1]+4}dK[1]\right)}{\sqrt{\pi}(-i(x+2))^{3/2}}$$

2.1.744 Problem 764

Internal problem ID [9916]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 764
Date solved : Monday, January 27, 2025 at 06:15:44 PM
CAS classification : [[_Emden, _Fowler]]

Solve

$$\left(x^2+1\right) y''-6y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.216 (sec)

Writing the ode as

$$\left(x^2+1\right) y''-6y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= 0 \\ C &= -6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{6}{x^2+1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 6 \\ t &= x^2+1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{6}{x^2+1}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1414: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2 + 1$. There is a pole at $x = i$ of order 1. There is a pole at $x = -i$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = i$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{6}{x^2 + 1}$$

Since the $\gcd(s, t) = 1$. This gives $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{6}{x^2 + 1}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	3	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 3$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= 3 - (1) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{x-i} + (0) \\ &= \frac{1}{x-i} \\ &= \frac{1}{x-i} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{1}{x-i}\right)(2x + a_1) + \left(\left(-\frac{1}{(x-i)^2}\right) + \left(\frac{1}{x-i}\right)^2 - \left(\frac{6}{x^2+1}\right)\right) = 0$$

$$2 + \frac{-4x - 2a_1}{-x+i} + \frac{-6x^2 - 6a_1 x - 6a_0}{x^2+1} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0, a_1 = i\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + ix$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 + ix\right) e^{\int \frac{1}{x-i} dx} \\
&= \left(x^2 + ix\right) e^{\frac{\ln\left(x^2+1\right)}{2} + i\arctan(x)} \\
&= x(x+i)\,(ix+1)
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= x(x+i)\,(ix+1)
\end{aligned}
$$

Which simplifies to

$$
y_1 = ix^3 + ix
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\
&= ix^3 + ix \int \frac{1}{\left(ix^3 + ix\right)^2} \, dx \\
&= ix^3 + ix \left(\frac{x}{2x^2+2} + \frac{3\arctan\left(x\right)}{2} + \frac{1}{x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(ix^3 + ix\right) + c_2 \left(ix^3 + ix \left(\frac{x}{2x^2+2} + \frac{3\arctan\left(x\right)}{2} + \frac{1}{x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
```

```
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 31

```
dsolve((x^2+1)*diff(diff(y(x),x),x)-6*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{3xc_2(x^2+1)\arctan(x)}{2} + c_1x^3 + \frac{3c_2x^2}{2} + c_1x + c_2$$

Mathematica DSolve solution

Solving time : 0.148 (sec)
Leaf size : 33

```
DSolve[{(x^2+1)*D[y[x],{x,2}]-6*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to (x^3+x)\left(c_2\int_1^x \frac{1}{(K[1]^3+K[1])^2}dK[1] + c_1\right)$$

2.1.745 Problem 765

Internal problem ID [9917]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 765
Date solved : `Monday, January 27, 2025 at 06:15:45 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+2\right) y''+3xy'-y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.357 (sec)

Writing the ode as

$$\left(x^2+2\right) y''+3xy'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+2 \\ B &= 3x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{7x^2+20}{4\left(x^2+2\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 7x^2+20 \\ t &= 4\left(x^2+2\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{7x^2+20}{4\left(x^2+2\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1415: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+2)^2$. There is a pole at $x = i\sqrt{2}$ of order 2. There is a pole at $x = -i\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16\left(x - i\sqrt{2}\right)^2} - \frac{3}{16\left(x + i\sqrt{2}\right)^2} - \frac{17i\sqrt{2}}{32\left(x - i\sqrt{2}\right)} + \frac{17i\sqrt{2}}{32\left(x + i\sqrt{2}\right)}$$

For the pole at $x = i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x-i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

For the pole at $x = -i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x+i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{7x^2 + 20}{4(x^2 + 2)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{7}{4}$. Hence

$$\begin{aligned} E_\infty &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{2\} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
$i\sqrt{2}$	2	$\{1, 2, 3\}$
$-i\sqrt{2}$	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1,\ e_2 = 1,\ e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(2 - (1 + (1))) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{1}{(x-(i\sqrt{2}))} + \frac{1}{(x-(-i\sqrt{2}))}\right) \\ &= \frac{1}{2x - 2i\sqrt{2}} + \frac{1}{2x + 2i\sqrt{2}} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + (3\theta^2 + 3\theta' - 4r)\,p' + (\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r')\,p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p}\\ &= \frac{1}{2x-2i\sqrt{2}} + \frac{1}{2x+2i\sqrt{2}}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{2x-2i\sqrt{2}} + \frac{1}{2x+2i\sqrt{2}}\right) w + \frac{7x^2+16}{4\left(\sqrt{2}+ix\right)^2\left(x+i\sqrt{2}\right)^2} = 0$$

Solving for ω gives

$$\omega = \frac{x+2\sqrt{2x^2+4}}{2x^2+4}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega\,dx}\\ &= e^{\int \frac{x+2\sqrt{2x^2+4}}{2x^2+4}dx}\\ &= \left(x^2+2\right)^{1/4} e^{\sqrt{2}\,\operatorname{arcsinh}\left(\frac{\sqrt{2}\,x}{2}\right)}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{3x}{x^2+2}\,dx}\\ &= z_1 e^{-\frac{3\ln\left(x^2+2\right)}{4}}\\ &= z_1\left(\frac{1}{\left(x^2+2\right)^{3/4}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{\sqrt{2}\,\operatorname{arcsinh}\left(\frac{\sqrt{2}\,x}{2}\right)}}{\sqrt{x^2+2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{x^2+2}\,dx}}{\left(y_1\right)^2}\,dx\\ &= y_1 \int \frac{e^{-\frac{3\ln\left(x^2+2\right)}{2}}}{\left(y_1\right)^2}\,dx\\ &= y_1\left(\int \frac{e^{-2\sqrt{2}\,\operatorname{arcsinh}\left(\frac{\sqrt{2}\,x}{2}\right)}}{\sqrt{x^2+2}}dx\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}}{\sqrt{x^2+2}} \right) + c_2 \left(\frac{e^{\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}}{\sqrt{x^2+2}} \left(\int \frac{e^{-2\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}}{\sqrt{x^2+2}} dx \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.059 (sec)
Leaf size : 45

```
dsolve((x^2+2)*diff(diff(y(x),x),x)+3*diff(y(x),x)*x-y(x) = 0,y(x),singsol=all)
```

$$
y = \frac{c_2 \left(\sqrt{x^2+2}+x\right)^{-\sqrt{2}} + c_1 \left(\sqrt{x^2+2}+x\right)^{\sqrt{2}}}{\sqrt{x^2+2}}
$$

Mathematica DSolve solution

Solving time : 0.082 (sec)
Leaf size : 92

```
DSolve[{(x^2+2)*D[y[x],{x,2}]+3*x*D[y[x],x]-y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$
y(x) \to \frac{2^{3/4} c_1 \cos\left(2\sqrt{2} \arcsin\left(\frac{1}{2}\sqrt{2 - i\sqrt{2}x}\right)\right)}{\sqrt{\pi}\sqrt{x^2+2}} + \frac{c_2 Q_{-\frac{1}{2}+\sqrt{2}}^{\frac{1}{2}}\left(\frac{ix}{\sqrt{2}}\right)}{\sqrt[4]{x^2+2}}
$$

2.1.746 Problem 766

Internal problem ID [9918]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 766
Date solved : Monday, January 27, 2025 at 06:15:45 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$(x-1)\,y''-xy'+y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.228 (sec)

Writing the ode as

$$(x-1)\,y''-xy'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=x-1\\ B&=-x \\ C&=1 \end{aligned}\tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned}\tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{x^2-4x+6}{4\,(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=x^2-4x+6\\ t&=4(x-1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x^2-4x+6}{4\,(x-1)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1416: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2(x-1)} + \frac{3}{4(x-1)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 4x + 6}{4(x-1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(x-1)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2(x-1)} + \frac{1}{2} \\
&= \frac{x-2}{2x-2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2(x-1)}+\frac{1}{2}\right)(0)+\left(\left(\frac{1}{2(x-1)^{2}}\right)+\left(-\frac{1}{2(x-1)}+\frac{1}{2}\right)^{2}-\left(\frac{x^{2}-4x+6}{4(x-1)^{2}}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \,dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2(x-1)}+\frac{1}{2}\right)dx} \\ &= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x-1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{x-1}\,dx} \\ &= z_1 e^{\frac{x}{2}+\frac{\ln(x-1)}{2}} \\ &= z_1\left(\sqrt{x-1}\,\mathrm{e}^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x-1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{x+\ln(x-1)}}{(y_1)^2}\,dx \\ &= y_1\left(-\frac{x\,\mathrm{e}^{x+\ln(x-1)}\mathrm{e}^{-2x}}{x-1}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\mathrm{e}^{x}) + c_2\left(\mathrm{e}^{x}\left(-\frac{x\,\mathrm{e}^{x+\ln(x-1)}\mathrm{e}^{-2x}}{x-1}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x-1)\left(\frac{d^2}{dx^2}y(x)\right)-x\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{y(x)}{x-1}+\frac{x\left(\frac{d}{dx}y(x)\right)}{x-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)-\frac{x\left(\frac{d}{dx}y(x)\right)}{x-1}+\frac{y(x)}{x-1}=0$

- □ Check to see if $x_0=1$ is a regular singular point
  - Define functions

    $\left[P_2(x)=-\frac{x}{x-1}, P_3(x)=\frac{1}{x-1}\right]$
  - $(x-1)\cdot P_2(x)$ is analytic at $x=1$

    $\left((x-1)\cdot P_2(x)\right)\Big|_{x=1}=-1$
  - $(x-1)^2\cdot P_3(x)$ is analytic at $x=1$

    $\left((x-1)^2\cdot P_3(x)\right)\Big|_{x=1}=0$
  - $x=1$is a regular singular point

  Check to see if $x_0=1$ is a regular singular point

  $x_0=1$

- Multiply by denominators

  $(x-1)\left(\frac{d^2}{dx^2}y(x)\right)-x\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Change variables using $x=u+1$ so that the regular singular point is at $u=0$

  $u\left(\frac{d^2}{du^2}y(u)\right)+(-u-1)\left(\frac{d}{du}y(u)\right)+y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

    $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-1}$
  - Shift index using $k->k+1$

    $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)(k+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)(k+r-1)-a_k(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{0,2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_{k+1}(k+1+r)-a_k\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k}{k+1+r}$

- Recursion relation for $r=0$

  $a_{k+1}=\frac{a_k}{k+1}$

- Solution for $r=0$

  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+1}=\frac{a_k}{k+1}\right]$

- Revert the change of variables $u=x-1$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k(x-1)^k, a_{k+1}=\frac{a_k}{k+1}\right]$

- Recursion relation for $r=2$

  $a_{k+1}=\frac{a_k}{k+3}$

- Solution for $r=2$

  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$

- Revert the change of variables $u=x-1$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k(x-1)^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$

- Combine solutions and rename parameters

  $\left[y(x)=\left(\sum_{k=0}^{\infty} a_k(x-1)^k\right)+\left(\sum_{k=0}^{\infty} b_k(x-1)^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+3}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 12

```
dsolve((x-1)*diff(diff(y(x),x),x)-diff(y(x),x)*x+y(x) = 0,y(x),singsol=all)
```

$$y = c_1 x + e^x c_2$$

Mathematica DSolve solution

Solving time : 0.149 (sec)
Leaf size : 90

```
DSolve[{(x-1)*D[y[x],{x,2}]-x*D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \exp\left(\int_1^x \frac{K[1]-2}{2(K[1]-1)}dK[1] - \frac{1}{2}\int_1^x -\frac{K[2]}{K[2]-1}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{K[1]-2}{2(K[1]-1)}dK[1]\right)dK[3] + c_1\right)$$

2.1.747 Problem 769

Internal problem ID [9919]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 769
Date solved : `Monday, January 27, 2025 at 06:15:46 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + \left(\frac{5}{3}x + x^2\right)y' - \frac{y}{3} = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.335 (sec)

Writing the ode as

$$x^2y'' + \left(\frac{5}{3}x + x^2\right)y' - \frac{y}{3} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= \frac{5}{3}x + x^2 \\ C &= -\frac{1}{3} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9x^2 + 30x + 7}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9x^2 + 30x + 7 \\ t &= 36x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{9x^2 + 30x + 7}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1418: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{7}{36x^2} + \frac{5}{6x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{6} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{5}{6x} - \frac{1}{2x^2} + \frac{5}{6x^3} - \frac{59}{36x^4} + \frac{385}{108x^5} - \frac{2681}{324x^6} + \frac{19525}{972x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{9x^2 + 30x + 7}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{30x + 7}{36x^2}\right) \\ &= \frac{1}{4} + \frac{30x + 7}{36x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 30. Dividing this by leading coefficient in t which is 36 gives $\frac{5}{6}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{5}{6}\right) - (0) \\ &= \frac{5}{6} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{5}{6}}{\frac{1}{2}} - 0\right) &= \frac{5}{6} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{5}{6}}{\frac{1}{2}} - 0\right) &= -\frac{5}{6}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{9x^2 + 30x + 7}{36x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$\frac{5}{6}$	$-\frac{5}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{5}{6}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{5}{6} - \left(-\frac{1}{6}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{6x} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{6x} + \frac{1}{2} \\
&= -\frac{1}{6x} + \frac{1}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{6x}+\frac{1}{2}\right)(1)+\left(\left(\frac{1}{6x^2}\right)+\left(-\frac{1}{6x}+\frac{1}{2}\right)^2-\left(\frac{9x^2+30x+7}{36x^2}\right)\right)=0$$

$$\frac{-1-3a_0}{3x}=0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0=-\frac{1}{3}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x)=x-\frac{1}{3}$$

Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\,dx} \\
&= \left(x-\frac{1}{3}\right)e^{\int \left(-\frac{1}{6x}+\frac{1}{2}\right)dx} \\
&= \left(x-\frac{1}{3}\right)e^{\frac{x}{2}-\frac{\ln(x)}{6}} \\
&= \frac{(-1+3x)\,e^{\frac{x}{2}}}{3x^{1/6}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{\frac{5}{3}x+x^2}{x^2}\,dx} \\
&= z_1 e^{-\frac{x}{2}-\frac{5\ln(x)}{6}} \\
&= z_1\left(\frac{e^{-\frac{x}{2}}}{x^{5/6}}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\frac{-1+3x}{3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1\int \frac{e^{\int -\frac{\frac{5}{3}x+x^2}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1\int \frac{e^{-x-\frac{5\ln(x)}{3}}}{(y_1)^2}\,dx \\
&= y_1\left(\int \frac{9\,e^{-x-\frac{5\ln(x)}{3}}x^2}{(-1+3x)^2}dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1y_1+c_2y_2 \\
&= c_1\left(\frac{-1+3x}{3x}\right)+c_2\left(\frac{-1+3x}{3x}\left(\int \frac{9\,e^{-x-\frac{5\ln(x)}{3}}x^2}{(-1+3x)^2}dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right)+\left(\frac{5}{3}x+x^2\right)\left(\frac{d}{dx}y(x)\right)-\frac{y(x)}{3}=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{y(x)}{3x^2}-\frac{(5+3x)\left(\frac{d}{dx}y(x)\right)}{3x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{(5+3x)\left(\frac{d}{dx}y(x)\right)}{3x}-\frac{y(x)}{3x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{5+3x}{3x},P_3(x)=-\frac{1}{3x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{5}{3}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=-\frac{1}{3}$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $3x^2\left(\frac{d^2}{dx^2}y(x)\right)+x(5+3x)\left(\frac{d}{dx}y(x)\right)-y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..2$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-1+3r)\,x^r+\left(\sum_{k=1}^{\infty}\left(a_k(k+r+1)\,(3k+3r-1)+3a_{k-1}(k+r-1)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-1+3r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{-1,\frac{1}{3}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $3(k+r+1)\left(k-\frac{1}{3}+r\right)a_k+3a_{k-1}(k+r-1)=0$

- Shift index using $k->k+1$

$3(k+2+r)\left(k+\frac{2}{3}+r\right)a_{k+1}+3a_k(k+r)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{3a_k(k+r)}{(k+2+r)(3k+2+3r)}$

- Recursion relation for $r=-1$; series terminates at $k=1$

  $a_{k+1}=-\frac{3a_k(k-1)}{(k+1)(3k-1)}$

- Apply recursion relation for $k=0$

  $a_1=-3a_0$

- Terminating series solution of the ODE for $r=-1$. Use reduction of order to find the second

  $y(x)=a_0\cdot(1-3x)$

- Recursion relation for $r=\frac{1}{3}$

  $a_{k+1}=-\frac{3a_k\left(k+\frac{1}{3}\right)}{\left(k+\frac{7}{3}\right)(3k+3)}$

- Solution for $r=\frac{1}{3}$

  $\left[y(x)=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{3}}, a_{k+1}=-\frac{3a_k\left(k+\frac{1}{3}\right)}{\left(k+\frac{7}{3}\right)(3k+3)}\right]$

- Combine solutions and rename parameters

  $\left[y(x)=a_0\cdot(1-3x)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{3}}\right), b_{k+1}=-\frac{3b_k\left(k+\frac{1}{3}\right)}{\left(k+\frac{7}{3}\right)(3k+3)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric sol
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special functio
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.054` (sec)
Leaf size : 29

```
dsolve(x^2*diff(diff(y(x),x),x)+(5/3*x+x^2)*diff(y(x),x)-1/3*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_1 x^{4/3} \operatorname{hypergeom}\left([2], \left[\frac{7}{3}\right], x\right) e^{-x} - 3c_2 x + c_2}{x}$$

Mathematica DSolve solution

Solving time : `0.285` (sec)
Leaf size : 52

```
DSolve[{x^2*D[y[x],{x,2}]+(5/3*x+x^2)*D[y[x],x]-1/3*y[x]==0,{}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{(3x-1)\left(c_2 \int_1^x \frac{9e^{-K[1]}\sqrt[3]{K[1]}}{(1-3K[1])^2} dK[1] + c_1\right)}{3x}$$

2.1.748 Problem 770

Internal problem ID [9920]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 770
Date solved : `Monday, January 27, 2025 at 06:15:47 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$2xy'' - y' + 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.204 (sec)

Writing the ode as

$$2xy'' - y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x \\ B &= -1 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5 - 16x}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 5 - 16x \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5 - 16x}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1420: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{x} + \frac{5}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{-1, 2, 5\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = -1, \ e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(1-(-1)) \\ &= 1 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{-1}{(x-(0))}\right) \\ &= -\frac{1}{2x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d=1$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d=1$, then letting

$$p = x + a_0 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$\frac{1-4a_0}{x^2} = 0$$

And solving for p gives

$$p = x + \frac{1}{4}$$

Now that $p(x)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{x+\frac{1}{4}} - \frac{1}{2x} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{x+\frac{1}{4}} - \frac{1}{2x}\right)w + \frac{64x^2 - 12x + 1}{64x^3 + 16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{16x\sqrt{-x} + 4x - 1}{4\,(4x+1)\,x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= e^{\int \omega \, dx} \\
&= e^{\int \frac{16x\sqrt{-x}+4x-1}{4(4x+1)x} dx} \\
&= \frac{\left(2\sqrt{-x}-1\right) e^{2\sqrt{-x}}}{\left(-x\right)^{1/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{2x} \, dx} \\
&= z_1 e^{\frac{\ln(x)}{4}} \\
&= z_1 \left(x^{1/4}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^{1/4}\left(2\sqrt{-x}-1\right) e^{2\sqrt{-x}}}{\left(-x\right)^{1/4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{2x} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{\ln(x)}{2}}}{\left(y_1\right)^2} \, dx \\
&= y_1 \left(\frac{e^{-4\sqrt{-x}}}{8} + \frac{e^{-4\sqrt{-x}}}{8\sqrt{-x}-4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{1/4}\left(2\sqrt{-x}-1\right) e^{2\sqrt{-x}}}{\left(-x\right)^{1/4}}\right) + c_2 \left(\frac{x^{1/4}\left(2\sqrt{-x}-1\right) e^{2\sqrt{-x}}}{\left(-x\right)^{1/4}} \left(\frac{e^{-4\sqrt{-x}}}{8} + \frac{e^{-4\sqrt{-x}}}{8\sqrt{-x}-4}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2\left(\frac{d^2}{dx^2}y(x)\right)x-\frac{d}{dx}y(x)+2y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{y(x)}{x}+\frac{\frac{d}{dx}y(x)}{2x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lir

  $\frac{d^2}{dx^2}y(x)-\frac{\frac{d}{dx}y(x)}{2x}+\frac{y(x)}{x}=0$

☐ Check to see if $x_0=0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=-\frac{1}{2x},P_3(x)=\frac{1}{x}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=-\frac{1}{2}$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=0$

- ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2\left(\frac{d^2}{dx^2}y(x)\right)x-\frac{d}{dx}y(x)+2y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $\frac{d}{dx}y(x)$ to series expansion

  $\frac{d}{dx}y(x)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1}$

- ◦ Shift index using $k->k+1$

  $\frac{d}{dx}y(x)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,x^{k+r}$

- ◦ Convert $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-1}$

- ◦ Shift index using $k->k+1$

  $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-3+2r)\,x^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(2k-1+2r)+2a_k\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+2r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{0,\frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation
  $2(k+1+r)\left(k+r-\frac{1}{2}\right)a_{k+1}+2a_k=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=-\frac{2a_k}{(k+1+r)(2k-1+2r)}$
- Recursion relation for $r=0$
  $a_{k+1}=-\frac{2a_k}{(k+1)(2k-1)}$
- Solution for $r=0$
  $\left[y(x)=\sum_{k=0}^{\infty}a_k x^k, a_{k+1}=-\frac{2a_k}{(k+1)(2k-1)}\right]$
- Recursion relation for $r=\frac{3}{2}$
  $a_{k+1}=-\frac{2a_k}{\left(k+\frac{5}{2}\right)(2k+2)}$
- Solution for $r=\frac{3}{2}$
  $\left[y(x)=\sum_{k=0}^{\infty}a_k x^{k+\frac{3}{2}}, a_{k+1}=-\frac{2a_k}{\left(k+\frac{5}{2}\right)(2k+2)}\right]$
- Combine solutions and rename parameters
  $\left[y(x)=\left(\sum_{k=0}^{\infty}a_k x^k\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{3}{2}}\right), a_{k+1}=-\frac{2a_k}{(k+1)(2k-1)}, b_{k+1}=-\frac{2b_k}{\left(k+\frac{5}{2}\right)(2k+2)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.019 (sec)
Leaf size : 36

```
dsolve(2*x*diff(diff(y(x),x),x)-diff(y(x),x)+2*y(x) = 0,y(x),singsol=all)
```

$$y=\left(2\sqrt{x}\,c_1+c_2\right)\cos\left(2\sqrt{x}\right)-\sin\left(2\sqrt{x}\right)\left(-2c_2\sqrt{x}+c_1\right)$$

Mathematica DSolve solution

Solving time : 0.205 (sec)
Leaf size : 74

```
DSolve[{2*x*D[y[x],{x,2}]-D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to e^{2i\sqrt{x}}\left(2\sqrt{x}+i\right)\left(c_2\int_1^x\frac{e^{-4i\sqrt{K[1]}}\sqrt{K[1]}}{\left(2\sqrt{K[1]}+i\right)^2}dK[1]+c_1\right)$$

2.1.749 Problem 771

Internal problem ID [9921]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 771
Date solved : `Monday, January 27, 2025 at 06:15:47 PM`
CAS classification : `[_Laguerre]`

Solve

$$2xy'' - (3+2x)\,y' + y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.339 (sec)

Writing the ode as

$$2xy'' + (-3-2x)\,y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x \\ B &= -3-2x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 + 4x + 21}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2 + 4x + 21 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 + 4x + 21}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1422: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{21}{16x^2} + \frac{1}{4x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{4} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{4x} + \frac{5}{4x^2} - \frac{5}{8x^3} - \frac{5}{4x^4} + \frac{35}{16x^5} + \frac{105}{64x^6} - \frac{1005}{128x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{4x^2 + 4x + 21}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{4x + 21}{16x^2}\right) \\ &= \frac{1}{4} + \frac{4x + 21}{16x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 16 gives $\frac{1}{4}$. Now b can be found.

$$\begin{aligned}b &= \left(\frac{1}{4}\right) - (0) \\ &= \frac{1}{4}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{4}}{\frac{1}{2}} - 0\right) &&= \frac{1}{4} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{4}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{4}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^2 + 4x + 21}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= \frac{1}{4} - \left(-\frac{3}{4}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{3}{4x} + \left(\frac{1}{2}\right) \\ &= -\frac{3}{4x} + \frac{1}{2} \\ &= -\frac{3}{4x} + \frac{1}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{3}{4x} + \frac{1}{2}\right)(1) + \left(\left(\frac{3}{4x^2}\right) + \left(-\frac{3}{4x} + \frac{1}{2}\right)^2 - \left(\frac{4x^2 + 4x + 21}{16x^2}\right) \right) &= 0 \\ \frac{-3 - 2a_0}{2x} &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{ a_0 = -\frac{3}{2} \right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - \frac{3}{2}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{3}{2}\right) e^{\int \left(-\frac{3}{4x} + \frac{1}{2}\right) dx} \\
&= \left(x - \frac{3}{2}\right) e^{\frac{x}{2} - \frac{3\ln(x)}{4}} \\
&= \frac{(-3+2x)\, e^{\frac{x}{2}}}{2x^{3/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-3-2x}{2x} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{3\ln(x)}{4}} \\
&= z_1 \left(x^{3/4} e^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^x(-3+2x)}{2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3-2x}{2x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{x + \frac{3\ln(x)}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{4\, e^{x + \frac{3\ln(x)}{2}} e^{-2x}}{(-3+2x)^2} dx \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^x(-3+2x)}{2} \right) + c_2 \left(\frac{e^x(-3+2x)}{2} \left(\int \frac{4\, e^{x + \frac{3\ln(x)}{2}} e^{-2x}}{(-3+2x)^2} dx \right) \right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2\left(\frac{d^2}{dx^2}y(x)\right)x-(2x+3)\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{y(x)}{2x}+\frac{(2x+3)\left(\frac{d}{dx}y(x)\right)}{2x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)-\frac{(2x+3)\left(\frac{d}{dx}y(x)\right)}{2x}+\frac{y(x)}{2x}=0$

☐ Check to see if $x_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(x)=-\frac{2x+3}{2x},P_3(x)=\frac{1}{2x}\right]$
  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=-\frac{3}{2}$
  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$
  - $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$
- Multiply by denominators

  $2\left(\frac{d^2}{dx^2}y(x)\right)x+(-2x-3)\left(\frac{d}{dx}y(x)\right)+y(x)=0$
- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..1$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,x^{k+r-1}$
  - Shift index using $k->k+1$

    $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-5+2r)\,x^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)(2k-3+2r)-a_k(2k+2r-1)\right)x^{k+r}\right)=0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-5+2r)=0$
- Values of r that satisfy the indicial equation

  $r\in\left\{0,\frac{5}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation
  $2(k+1+r)\left(k+r-\frac{3}{2}\right)a_{k+1}-2a_k\left(k+r-\frac{1}{2}\right)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{a_k(2k+2r-1)}{(k+1+r)(2k-3+2r)}$
- Recursion relation for $r=0$
  $a_{k+1}=\frac{a_k(2k-1)}{(k+1)(2k-3)}$
- Solution for $r=0$
  $\left[y(x)=\sum_{k=0}^{\infty}a_kx^k, a_{k+1}=\frac{a_k(2k-1)}{(k+1)(2k-3)}\right]$
- Recursion relation for $r=\frac{5}{2}$
  $a_{k+1}=\frac{a_k(2k+4)}{\left(k+\frac{7}{2}\right)(2k+2)}$
- Solution for $r=\frac{5}{2}$
  $\left[y(x)=\sum_{k=0}^{\infty}a_kx^{k+\frac{5}{2}}, a_{k+1}=\frac{a_k(2k+4)}{\left(k+\frac{7}{2}\right)(2k+2)}\right]$
- Combine solutions and rename parameters
  $\left[y(x)=\left(\sum_{k=0}^{\infty}a_kx^k\right)+\left(\sum_{k=0}^{\infty}b_kx^{k+\frac{5}{2}}\right), a_{k+1}=\frac{a_k(2k-1)}{(k+1)(2k-3)}, b_{k+1}=\frac{b_k(2k+4)}{\left(k+\frac{7}{2}\right)(2k+2)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric sol
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returr
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.041 (sec)
Leaf size : 24

```
dsolve(2*x*diff(diff(y(x),x),x)-(2*x+3)*diff(y(x),x)+y(x) = 0,y(x),singsol=all)
```

$$y = c_1 \,\mathrm{hypergeom}\left([2], \left[\frac{7}{2}\right], x\right) x^{5/2} - \frac{2c_2 \mathrm{e}^{x}\left(x-\frac{3}{2}\right)}{3}$$

Mathematica DSolve solution

Solving time : 0.225 (sec)
Leaf size : 52

```
DSolve[{2*x*D[y[x],{x,2}]-(3+2*x)*D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{x}(2x-3)\left(c_2 \int_1^x \frac{4e^{-K[1]}K[1]^{3/2}}{(3-2K[1])^2}dK[1] + c_1\right)$$

2.1.750 Problem 772

Internal problem ID [9922]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 772
Date solved : `Monday, January 27, 2025 at 06:15:48 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2y'' + 3xy' + (2x - 1)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.202 (sec)

Writing the ode as

$$2x^2y'' + 3xy' + (2x - 1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= 3x \\ C &= 2x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5 - 16x}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 5 - 16x \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5 - 16x}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1424: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{x} + \frac{5}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{-1,2,5\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = -1, \; e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(1-(-1)) \\ &= 1 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{-1}{(x-(0))}\right) \\ &= -\frac{1}{2x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d=1$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d=1$, then letting

$$p = x + a_0 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$\frac{1-4a_0}{x^2} = 0$$

And solving for p gives

$$p = x + \frac{1}{4}$$

Now that $p(x)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{x+\frac{1}{4}} - \frac{1}{2x} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{x+\frac{1}{4}} - \frac{1}{2x}\right)w + \frac{64x^2 - 12x + 1}{64x^3 + 16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{16x\sqrt{-x} + 4x - 1}{4\,(4x+1)\,x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= e^{\int \omega \,dx} \\
&= \mathrm{e}^{\int \frac{16x\sqrt{-x}+4x-1}{4(4x+1)x}dx} \\
&= \frac{\left(2\sqrt{-x}-1\right)\mathrm{e}^{2\sqrt{-x}}}{\left(-x\right)^{1/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x}{2x^2}\,dx} \\
&= z_1 e^{-\frac{3\ln(x)}{4}} \\
&= z_1 \left(\frac{1}{x^{3/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\left(2\sqrt{-x}-1\right)\mathrm{e}^{2\sqrt{-x}}}{x^{3/4}\left(-x\right)^{1/4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{2x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{3\ln(x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{-4\sqrt{-x}}}{8}+\frac{\mathrm{e}^{-4\sqrt{-x}}}{8\sqrt{-x}-4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\left(2\sqrt{-x}-1\right)\mathrm{e}^{2\sqrt{-x}}}{x^{3/4}\left(-x\right)^{1/4}}\right) + c_2 \left(\frac{\left(2\sqrt{-x}-1\right)\mathrm{e}^{2\sqrt{-x}}}{x^{3/4}\left(-x\right)^{1/4}}\left(\frac{\mathrm{e}^{-4\sqrt{-x}}}{8}+\frac{\mathrm{e}^{-4\sqrt{-x}}}{8\sqrt{-x}-4}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2\left(\frac{d^2}{dx^2}y(x)\right)+3x\left(\frac{d}{dx}y(x)\right)+(2x-1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(2x-1)y(x)}{2x^2}-\frac{3\left(\frac{d}{dx}y(x)\right)}{2x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{3\left(\frac{d}{dx}y(x)\right)}{2x}+\frac{(2x-1)y(x)}{2x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{3}{2x},P_3(x)=\frac{2x-1}{2x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{3}{2}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=-\frac{1}{2}$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2x^2\left(\frac{d^2}{dx^2}y(x)\right)+3x\left(\frac{d}{dx}y(x)\right)+(2x-1)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum\limits_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

  $x^m\cdot y(x)=\sum\limits_{k=0}^{\infty}a_kx^{k+r+m}$

  ◦ Shift index using $k->k-m$

  $x^m\cdot y(x)=\sum\limits_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x\cdot\left(\frac{d}{dx}y(x)\right)=\sum\limits_{k=0}^{\infty}a_k(k+r)\,x^{k+r}$

  ◦ Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum\limits_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-1+2r)\,x^r+\left(\sum\limits_{k=1}^{\infty}\left(a_k(k+r+1)\,(2k+2r-1)+2a_{k-1}\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-1+2r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{-1,\frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation
  $2(k+r+1)\left(k+r-\frac{1}{2}\right)a_k+2a_{k-1}=0$
- Shift index using $k->k+1$
  $2(k+2+r)\left(k+\frac{1}{2}+r\right)a_{k+1}+2a_k=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=-\frac{2a_k}{(k+2+r)(2k+1+2r)}$
- Recursion relation for $r=-1$
  $a_{k+1}=-\frac{2a_k}{(k+1)(2k-1)}$
- Solution for $r=-1$
  $\left[y(x)=\sum_{k=0}^{\infty}a_k x^{k-1}, a_{k+1}=-\frac{2a_k}{(k+1)(2k-1)}\right]$
- Recursion relation for $r=\frac{1}{2}$
  $a_{k+1}=-\frac{2a_k}{\left(k+\frac{5}{2}\right)(2k+2)}$
- Solution for $r=\frac{1}{2}$
  $\left[y(x)=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}, a_{k+1}=-\frac{2a_k}{\left(k+\frac{5}{2}\right)(2k+2)}\right]$
- Combine solutions and rename parameters
  $\left[y(x)=\left(\sum_{k=0}^{\infty}a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{2}}\right), a_{k+1}=-\frac{2a_k}{(k+1)(2k-1)}, b_{k+1}=-\frac{2b_k}{\left(k+\frac{5}{2}\right)(2k+2)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.329 (sec)
Leaf size : 73

```
dsolve(2*x^2*diff(diff(y(x),x),x)+3*diff(y(x),x)*x+(-1+2*x)*y(x) = 0,y(x),singsol=all)
```

$$y=\frac{c_2\sqrt{\frac{(-2\sqrt{x}+i)(4x+1)}{2\sqrt{x}+i}}\,\mathrm{e}^{-2i\sqrt{x}}+c_1\sqrt{\frac{(2\sqrt{x}+i)(4x+1)}{-2\sqrt{x}+i}}\,\mathrm{e}^{2i\sqrt{x}}}{x}$$

Mathematica DSolve solution

Solving time : 0.213 (sec)
Leaf size : 77

```
DSolve[{2*x^2*D[y[x],{x,2}]+3*x*D[y[x],x]+(2*x-1)*y[x]==0,{}},y[x],x,IncludeSingularSolution
```

$$y(x) \to \frac{e^{2i\sqrt{x}}(2\sqrt{x}+i)\left(c_2\int_1^x \frac{e^{-4i\sqrt{K[1]}}\sqrt{K[1]}}{\left(2\sqrt{K[1]}+i\right)^2}dK[1]+c_1\right)}{x}$$

2.1.751 Problem 773

Internal problem ID [9923]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 773
Date solved : Monday, January 27, 2025 at 06:15:49 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$xy'' + 2y' - xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.061 (sec)

Writing the ode as

$$xy'' + 2y' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1426: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\, dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1 \left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-x}}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{e^{2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{-x}}{x}\right) + c_2 \left(\frac{e^{-x}}{x}\left(\frac{e^{2x}}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dx^2}y(x)\right) x + 2\frac{d}{dx}y(x) - xy(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = y(x) - \frac{2\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) + \frac{2\left(\frac{d}{dx}y(x)\right)}{x} - y(x) = 0$
- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{2}{x}, P_3(x) = -1\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $\left(\frac{d^2}{dx^2}y(x)\right) x + 2\frac{d}{dx}y(x) - xy(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x \cdot y(x)$ to series expansion
    $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

- Shift index using $k->k-1$

  $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

- Convert $\frac{d}{dx}y(x)$ to series expansion

  $\frac{d}{dx}y(x) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $\frac{d}{dx}y(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$

- Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) - a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $a_1(1+r)\,(2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)\,(k+2+r) - a_{k-1} = 0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)\,(k+3+r) - a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r = -1$

  $a_{k+2} = \frac{a_k}{(k+1)(k+2)}$

- Solution for $r = -1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = \frac{a_k}{(k+1)(k+2)}, 0 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{a_k}{(k+2)(k+3)}$

- Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = \frac{a_k}{(k+1)(k+2)}, 0 = 0, b_{k+2} = \frac{b_k}{(k+2)(k+3)}, 2b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 17

```
dsolve(x*diff(diff(y(x),x),x)+2*diff(y(x),x)-x*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_1 \sinh(x) + c_2 \cosh(x)}{x}$$

Mathematica DSolve solution

Solving time : 0.026 (sec)
Leaf size : 28

```
DSolve[{x*D[y[x],{x,2}]+2*D[y[x],x]-x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2c_1 e^{-x} + c_2 e^{x}}{2x}$$

2.1.752 Problem 774

Internal problem ID [9924]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 774
Date solved : Monday, January 27, 2025 at 06:15:49 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.138 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1428: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\, dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos(x)}{\sqrt{x}}(\tan(x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2 \left(\frac{d^2}{dx^2} y(x)\right) + x\left(\frac{d}{dx} y(x)\right) + \left(x^2 - \frac{1}{4}\right) y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2} y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2} y(x) = -\frac{(4x^2-1)y(x)}{4x^2} - \frac{\frac{d}{dx} y(x)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2} y(x) + \frac{\frac{d}{dx} y(x)}{x} + \frac{(4x^2-1)y(x)}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $(x \cdot P_2(x))\Big|_{x=0} = 1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{1}{4}$
  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point
  $x_0 = 0$

- Multiply by denominators
  $4x^2\left(\frac{d^2}{dx^2} y(x)\right) + 4x\left(\frac{d}{dx} y(x)\right) + (4x^2 - 1)\, y(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- Shift index using $k->k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$
- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-2})\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$
- Shift index using $k->k+2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$
- Solution for $r = -\frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$
- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.042 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x+(x^2-1/4)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{\sin(x) c_1 + \cos(x) c_2}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.034 (sec)
Leaf size : 39

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-1/4)*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to \frac{e^{-ix}(2c_1 - ic_2e^{2ix})}{2\sqrt{x}}$$

2.1.753 Problem 775

Internal problem ID [9925]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 775
Date solved : `Monday, January 27, 2025 at 06:15:50 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + (x-6)\,y' - 3y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.255 (sec)

Writing the ode as

$$xy'' + (x-6)\,y' - 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= x-6 \\ C &= -3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2+48}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2+48 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2+48}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1430: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{12}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 12$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 4 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -3 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{12}{x^2} - \frac{144}{x^4} + \frac{3456}{x^6} - \frac{103680}{x^8} + \frac{3483648}{x^{10}} - \frac{125411328}{x^{12}} + \frac{4729798656}{x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 + 48}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{12}{x^2}\right) \\ &= \frac{1}{4} + \frac{12}{x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 4 gives 0. Now b can be found.

$$\begin{aligned}b &= (0) - (0) \\ &= 0\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{0}{\frac{1}{2}} - 0\right) = 0 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{0}{\frac{1}{2}} - 0\right) = 0\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 48}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	4	-3

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= 0 - (-3) \\ &= 3 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{3}{x} + (-)\left(\frac{1}{2}\right) \\ &= -\frac{3}{x} - \frac{1}{2} \\ &= -\frac{6+x}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 3$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(6x + 2a_2) + 2\left(-\frac{3}{x} - \frac{1}{2}\right)(3x^2 + 2a_2 x + a_1) + \left(\left(\frac{3}{x^2}\right) + \left(-\frac{3}{x} - \frac{1}{2}\right)^2 - \left(\frac{x^2+48}{4x^2}\right) \right) = 0$$

$$\frac{(a_2 - 12)\,x^2 + 2(a_1 - 5a_2)\,x + 3a_0 - 6a_1}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 120, a_1 = 60, a_2 = 12\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^3 + 12x^2 + 60x + 120$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^3 + 12x^2 + 60x + 120\right) e^{\int \left(-\frac{3}{x} - \frac{1}{2}\right) dx} \\
&= \left(x^3 + 12x^2 + 60x + 120\right) e^{-\frac{x}{2} - 3\ln(x)} \\
&= \frac{\left(x^3 + 12x^2 + 60x + 120\right) e^{-\frac{x}{2}}}{x^3}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x-6}{x} \, dx} \\
&= z_1 e^{-\frac{x}{2} + 3\ln(x)} \\
&= z_1 \left(x^3 e^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = e^{-x} \left(x^3 + 12x^2 + 60x + 120\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x-6}{x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-x + 6\ln(x)}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{\left(x^3 - 12x^2 + 60x - 120\right) e^{-x + 6\ln(x)} e^{2x}}{\left(x^3 + 12x^2 + 60x + 120\right) x^6} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{-x} \left(x^3 + 12x^2 + 60x + 120\right)\right) \\
&\quad + c_2 \left(e^{-x} \left(x^3 + 12x^2 + 60x + 120\right) \left(\frac{\left(x^3 - 12x^2 + 60x - 120\right) e^{-x + 6\ln(x)} e^{2x}}{\left(x^3 + 12x^2 + 60x + 120\right) x^6} \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dx^2}y(x)\right)x+(-6+x)\left(\frac{d}{dx}y(x)\right)-3y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=\frac{3y(x)}{x}-\frac{(-6+x)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)+\frac{(-6+x)\left(\frac{d}{dx}y(x)\right)}{x}-\frac{3y(x)}{x}=0$

□ Check to see if $x_0=0$ is a regular singular point

- ○ Define functions

$\left[P_2(x)=\frac{-6+x}{x}, P_3(x)=-\frac{3}{x}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

$\left.(x\cdot P_2(x))\right|_{x=0}=-6$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

- ○ $x=0$is a regular singular point

Check to see if $x_0=0$ is a regular singular point

$x_0=0$

- Multiply by denominators

$\left(\frac{d^2}{dx^2}y(x)\right)x+(-6+x)\left(\frac{d}{dx}y(x)\right)-3y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..1$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ○ Convert $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

$x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-1}$

- ○ Shift index using $k->k+1$

$x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

Rewrite ODE with series expansions

$a_0r(-7+r)\,x^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(k-6+r)+a_k(k+r-3)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-7+r)=0$

- Values of r that satisfy the indicial equation

$r\in\{0,7\}$

- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)\,(k-6+r)+a_k(k+r-3)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=-\frac{a_k(k+r-3)}{(k+1+r)(k-6+r)}$
- Recursion relation for $r=0$; series terminates at $k=3$
  $a_{k+1}=-\frac{a_k(k-3)}{(k+1)(k-6)}$
- Apply recursion relation for $k=0$
  $a_1=-\frac{a_0}{2}$
- Apply recursion relation for $k=1$
  $a_2=-\frac{a_1}{5}$
- Express in terms of a_0
  $a_2=\frac{a_0}{10}$
- Apply recursion relation for $k=2$
  $a_3=-\frac{a_2}{12}$
- Express in terms of a_0
  $a_3=-\frac{a_0}{120}$
- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second linea
  $y(x)=a_0\cdot\left(1-\frac{1}{2}x+\frac{1}{10}x^2-\frac{1}{120}x^3\right)$
- Recursion relation for $r=7$
  $a_{k+1}=-\frac{a_k(k+4)}{(k+8)(k+1)}$
- Solution for $r=7$
  $\left[y(x)=\sum_{k=0}^{\infty}a_k x^{k+7}, a_{k+1}=-\frac{a_k(k+4)}{(k+8)(k+1)}\right]$
- Combine solutions and rename parameters
  $\left[y(x)=a_0\cdot\left(1-\frac{1}{2}x+\frac{1}{10}x^2-\frac{1}{120}x^3\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+7}\right), b_{k+1}=-\frac{b_k(4+k)}{(k+8)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 39

```
dsolve(x*diff(diff(y(x),x),x)+(x-6)*diff(y(x),x)-3*y(x) = 0,y(x),singsol=all)
```

$$y=c_1\left(x^3-12x^2+60x-120\right)+c_2\mathrm{e}^{-x}\left(x^3+12x^2+60x+120\right)$$

Mathematica DSolve solution

Solving time : 0.078 (sec)
Leaf size : 98

```
DSolve[{x*D[y[x],{x,2}]+(x-6)*D[y[x],x]-3*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2e^{-x/2}\sqrt{x}\left((c_1x^3+12ic_2x^2+60c_1x+120ic_2)\cosh\left(\frac{x}{2}\right)-(12c_1(x^2+10)+ic_2x(x^2+60))\sinh\left(\frac{x}{2}\right)\right)}{\sqrt{\pi}\sqrt{-ix}}$$

2.1.754 Problem 776

Internal problem ID [9926]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 776
Date solved : `Monday, January 27, 2025 at 06:15:50 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$x^4y'' + \lambda y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.236 (sec)

Writing the ode as

$$x^4y'' + \lambda y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4 \\ B &= 0 \\ C &= \lambda \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-\lambda}{x^4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -\lambda \\ t &= x^4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{\lambda}{x^4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1432: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^4$. There is a pole at $x = 0$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned} \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) \end{aligned}$$

The partial fraction decomposition of r is

$$r = -\frac{\lambda}{x^4}$$

There is pole in r at $x = 0$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{i\sqrt{\lambda}}{x^2} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{i\sqrt{\lambda}}{x^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^2}$ is

$$a = i\sqrt{\lambda}$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be 0. Therefore

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{i\sqrt{\lambda}}{x^2} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &&= \frac{1}{2}\left(\frac{0}{i\sqrt{\lambda}} + 2\right) &&= 1 \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &&= \frac{1}{2}\left(-\frac{0}{i\sqrt{\lambda}} + 2\right) &&= 1 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{\lambda}{x^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	4	$\frac{i\sqrt{\lambda}}{x^2}$	1	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{i\sqrt{\lambda}}{x^2} + \frac{1}{x} + (-)\,(0) \\ &= -\frac{i\sqrt{\lambda}}{x^2} + \frac{1}{x} \\ &= \frac{-i\sqrt{\lambda}+x}{x^2}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{i\sqrt{\lambda}}{x^2} + \frac{1}{x}\right)(0) + \left(\left(\frac{2i\sqrt{\lambda}}{x^3} - \frac{1}{x^2}\right) + \left(-\frac{i\sqrt{\lambda}}{x^2} + \frac{1}{x}\right)^2 - \left(-\frac{\lambda}{x^4}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\,dx} \\ &= \mathrm{e}^{\int \left(-\frac{i\sqrt{\lambda}}{x^2} + \frac{1}{x}\right)dx} \\ &= x\,\mathrm{e}^{\frac{i\sqrt{\lambda}}{x}}\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1 \\ &= x\,\mathrm{e}^{\frac{i\sqrt{\lambda}}{x}}\end{aligned}$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{\frac{i\sqrt{\lambda}}{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= x\,\mathrm{e}^{\frac{i\sqrt{\lambda}}{x}} \int \frac{1}{x^2 \mathrm{e}^{\frac{2i\sqrt{\lambda}}{x}}}\, dx \\
&= x\,\mathrm{e}^{\frac{i\sqrt{\lambda}}{x}} \left(-\frac{i\mathrm{e}^{-\frac{2i\sqrt{\lambda}}{x}}}{2\sqrt{\lambda}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x\,\mathrm{e}^{\frac{i\sqrt{\lambda}}{x}}\right) + c_2 \left(x\,\mathrm{e}^{\frac{i\sqrt{\lambda}}{x}} \left(-\frac{i\mathrm{e}^{-\frac{2i\sqrt{\lambda}}{x}}}{2\sqrt{\lambda}}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 31

```
dsolve(diff(diff(y(x),x),x)*x^4+lambda*y(x) = 0,y(x),singsol=all)
```

$$
y = x\left(c_1 \sinh\left(\frac{\sqrt{-\lambda}}{x}\right) + c_2 \cosh\left(\frac{\sqrt{-\lambda}}{x}\right)\right)
$$

Mathematica DSolve solution

Solving time : 0.108 (sec)
Leaf size : 56

```
DSolve[{x^4*D[y[x],{x,2}]+\[Lambda]*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$
y(x) \to c_1 x e^{-1+\frac{i\sqrt{\lambda}}{x}} - \frac{i c_2 x e^{1-\frac{i\sqrt{\lambda}}{x}}}{2\sqrt{\lambda}}
$$

2.1.755 Problem 777

Internal problem ID [9927]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 777
Date solved : `Monday, January 27, 2025 at 06:15:51 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' + 4xy' + \left(4x^2 - 25\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.290 (sec)

Writing the ode as

$$4x^2y'' + 4xy' + \left(4x^2 - 25\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= 4x \\ C &= 4x^2 - 25 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 6}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2 + 6 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 6}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1433: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -1 + \frac{6}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx i - \frac{3i}{x^2} - \frac{9i}{2x^4} - \frac{27i}{2x^6} - \frac{405i}{8x^8} - \frac{1701i}{8x^{10}} - \frac{15309i}{16x^{12}} - \frac{72171i}{16x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= i \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-x^2 + 6}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-1) + \left(\frac{6}{x^2}\right) \\ &= -1 + \frac{6}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= i \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{0}{i} - 0\right) = 0 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{0}{i} - 0\right) = 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 + 6}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	i	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= 0 - (-2) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{2}{x} + (-)(i) \\ &= -\frac{2}{x} - i \\ &= -\frac{2}{x} - i \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(-\frac{2}{x} - i\right)(2x + a_1) + \left(\left(\frac{2}{x^2}\right) + \left(-\frac{2}{x} - i\right)^2 - \left(\frac{-x^2+6}{x^2}\right) \right) = 0$$

$$\frac{2ixa_1 + 4ia_0 - 6x - 4a_1}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -3, a_1 = -3i\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 3ix - 3$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (x^2 - 3ix - 3)\, e^{\int (-\frac{2}{x} - i) dx} \\ &= (x^2 - 3ix - 3)\, e^{-2\ln(x) - ix} \\ &= \frac{(x^2 - 3ix - 3)\, e^{-ix}}{x^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x}{4x^2}\,dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{5/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{(ix^2 - 3x - 3i)\,\mathrm{e}^{2ix}}{-2x^2 + 6ix + 6}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{5/2}}\right) + c_2\left(\frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{5/2}}\left(\frac{(ix^2 - 3x - 3i)\,\mathrm{e}^{2ix}}{-2x^2 + 6ix + 6}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x\left(\frac{d}{dx}y(x)\right) + (4x^2 - 25)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(4x^2-25)y(x)}{4x^2} - \frac{\frac{d}{dx}y(x)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{\frac{d}{dx}y(x)}{x} + \frac{(4x^2-25)y(x)}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-25}{4x^2}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

$(x \cdot P_2(x))\bigg|_{x=0} = 1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{25}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x\left(\frac{d}{dx}y(x)\right) + (4x^2 - 25)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

  - Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(5+2r)\,(-5+2r)\,x^r + a_1(7+2r)\,(-3+2r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+5)\,(2k+2r-5) + 4a_{k-2}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(5+2r)\,(-5+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{5}{2}, \frac{5}{2}\right\}$

- Each term must be 0

  $a_1(7+2r)\,(-3+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(2k+2r+5)\,(2k+2r-5) + 4a_{k-2} = 0$

- Shift index using $k- >k + 2$

  $a_{k+2}(2k+9+2r)\,(2k-1+2r) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{(2k+9+2r)(2k-1+2r)}$

- Recursion relation for $r = -\frac{5}{2}$

  $a_{k+2} = -\frac{4a_k}{(2k+4)(2k-6)}$

- Solution for $r = -\frac{5}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}, a_{k+2} = -\frac{4a_k}{(2k+4)(2k-6)}, a_1 = 0\right]$

- Recursion relation for $r=\frac{5}{2}$

  $a_{k+2}=-\frac{4a_k}{(2k+14)(2k+4)}$
- Solution for $r=\frac{5}{2}$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+\frac{5}{2}}, a_{k+2}=-\frac{4a_k}{(2k+14)(2k+4)}, a_1=0\right]$
- Combine solutions and rename parameters

  $\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{5}{2}}\right), a_{k+2}=-\frac{4a_k}{(2k+4)(2k-6)}, a_1=0, b_{k+2}=-\frac{4b_k}{(2k+14)(2k+4)}, b_1=\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.058 (sec)
Leaf size : 43

```
dsolve(4*x^2*diff(diff(y(x),x),x)+4*diff(y(x),x)*x+(4*x^2-25)*y(x) = 0,y(x),singsol=al
```

$$y=\frac{-3c_2\left(ix-\frac{1}{3}x^2+1\right)\mathrm{e}^{-ix}+3\left(ix+\frac{1}{3}x^2-1\right)c_1\mathrm{e}^{ix}}{x^{5/2}}$$

Mathematica DSolve solution

Solving time : 0.059 (sec)
Leaf size : 59

```
DSolve[{4*x^2*D[y[x],{x,2}]+4*x*D[y[x],x]+(4*x^2-25)*y[x]==0,{}},y[x],x,IncludeSingularSolut
```

$$y(x)\to-\frac{\sqrt{\frac{2}{\pi}}\left((-c_2x^2+3c_1x+3c_2)\cos(x)+(c_1(x^2-3)+3c_2x)\sin(x)\right)}{x^{5/2}}$$

2.1.756 Problem 778

Internal problem ID [9928]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 778
Date solved : `Monday, January 27, 2025 at 06:15:52 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(36x^2 - \frac{1}{4}\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.152 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(36x^2 - \frac{1}{4}\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= 36x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-36}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -36 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -36z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1435: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -36$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(6x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(6x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{\tan(6x)}{6}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\cos(6x)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos(6x)}{\sqrt{x}} \left(\frac{\tan(6x)}{6}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x\left(\frac{d}{dx}y(x)\right) + \left(36x^2 - \frac{1}{4}\right)y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(144x^2-1)y(x)}{4x^2} - \frac{\frac{d}{dx}y(x)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{\frac{d}{dx}y(x)}{x} + \frac{(144x^2-1)y(x)}{4x^2} = 0$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{144x^2-1}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{1}{4}$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x\left(\frac{d}{dx}y(x)\right) + (144x^2 - 1)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- > k - m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x \cdot \left(\frac{d}{dx} y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$
  - Convert $x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\,x^r + a_1(3+2r)\,(1+2r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) + 144a\right.$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 144a_{k-2} = 0$
- Shift index using $k- > k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\,r + 4r^2 - 1\right) + 144a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{144a_k}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{144a_k}{4k^2+12k+8}$
- Solution for $r = -\frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{144a_k}{4k^2+12k+8}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{144a_k}{4k^2+20k+24}$
- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{144a_k}{4k^2+20k+24}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{144a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{144b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.044 (sec)
Leaf size : 21

```
dsolve(x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x+(36*x^2-1/4)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_1 \sin(6x) + c_2 \cos(6x)}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.041 (sec)
Leaf size : 39

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(36*x^2-1/4)*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to \frac{e^{-6ix}(12c_1 - ic_2 e^{12ix})}{12\sqrt{x}}$$

2.1.757 Problem 779

Internal problem ID [9929]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 779
Date solved : `Monday, January 27, 2025 at 06:15:52 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + \left(x^2 - 2\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.269 (sec)

Writing the ode as

$$x^2y'' + \left(x^2 - 2\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 0 \\ C &= x^2 - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2 + 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1437: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -1 + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx i - \frac{i}{x^2} - \frac{i}{2x^4} - \frac{i}{2x^6} - \frac{5i}{8x^8} - \frac{7i}{8x^{10}} - \frac{21i}{16x^{12}} - \frac{33i}{16x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= i\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-x^2+2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-1) + \left(\frac{2}{x^2}\right) \\ &= -1 + \frac{2}{x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= i \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{0}{i} - 0\right) = 0 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{0}{i} - 0\right) = 0\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2+2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	i	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + (-)\,(i) \\ &= -\frac{1}{x} - i \\ &= -\frac{1}{x} - i \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{x} - i\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - i\right)^2 - \left(\frac{-x^2+2}{x^2}\right)\right) &= 0 \\ \frac{2ia_0 - 2}{x} &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -i\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - i$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (x - i)\, e^{\int \left(-\frac{1}{x} - i\right) dx} \\ &= (x - i)\, e^{-\ln(x) - ix} \\ &= \frac{(x - i)\, e^{-ix}}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{(x-i)\,\mathrm{e}^{-ix}}{x} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x-i)\,\mathrm{e}^{-ix}}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \frac{(x-i)\,\mathrm{e}^{-ix}}{x} \int \frac{1}{\frac{(x-i)^2\mathrm{e}^{-2ix}}{x^2}}\,dx \\ &= \frac{(x-i)\,\mathrm{e}^{-ix}}{x}\left(\frac{(ix-1)\,\mathrm{e}^{2ix}}{-2x+2i}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{(x-i)\,\mathrm{e}^{-ix}}{x}\right) + c_2\left(\frac{(x-i)\,\mathrm{e}^{-ix}}{x}\left(\frac{(ix-1)\,\mathrm{e}^{2ix}}{-2x+2i}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) + (x^2-2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(x^2-2)y(x)}{x^2}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(x^2-2)y(x)}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

  $\left[P_2(x) = 0, P_3(x) = \frac{x^2-2}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 0$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -2$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) + (x^2 - 2)\, y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- > k - m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-2+r)\, x^r + a_1(2+r)\,(-1+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+1)\,(k+r-2) + a_{k-2}\right) x^{k+r}\right) =$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 2\}$

- Each term must be 0

  $a_1(2+r)\,(-1+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)\,(k+r-2) + a_{k-2} = 0$

- Shift index using $k- > k + 2$

  $a_{k+2}(k+3+r)\,(k+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+3+r)(k+r)}$

- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{(k+2)(k-1)}$

- Solution for $r = -1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+2)(k-1)}, a_1 = 0\right]$

- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{a_k}{(k+5)(k+2)}$

- Solution for $r = 2$

$$\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2}=-\frac{a_k}{(k+5)(k+2)}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2}=-\frac{a_k}{(k+2)(k-1)}, a_1=0, b_{k+2}=-\frac{b_k}{(5+k)(k+2)}, b_1=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.023 (sec)
Leaf size : 27

```
dsolve(x^2*diff(diff(y(x),x),x)+(x^2-2)*y(x) = 0,y(x),singsol=all)
```

$$y=\frac{(c_1 x+c_2)\cos(x)+\sin(x)(c_2 x-c_1)}{x}$$

Mathematica DSolve solution

Solving time : 0.02 (sec)
Leaf size : 21

```
DSolve[{x^2*D[y[x],{x,2}]+(x^2-2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to x(c_1 j_1(x)-c_2 y_1(x))$$

2.1.758 Problem 780

Internal problem ID [9930]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 780
Date solved : `Monday, January 27, 2025 at 06:15:53 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$xy'' + 3y' + x^3y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.268 (sec)

Writing the ode as

$$xy'' + 3y' + x^3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 3 \\ C &= x^3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4x^4 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4x^4 + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-4x^4 + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1439: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -x^2 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx ix - \frac{3i}{8x^3} - \frac{9i}{128x^7} - \frac{27i}{1024x^{11}} - \frac{405i}{32768x^{15}} - \frac{1701i}{262144x^{19}} - \frac{15309i}{4194304x^{23}} - \frac{72171i}{33554432x^{27}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= ix \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-4x^4 + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (-x^2) + \left(\frac{3}{4x^2}\right) \\ &= -x^2 + \frac{3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= ix \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{i} - 1\right) &&= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{i} - 1\right) &&= -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-4x^4 + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	ix	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)\,(ix) \\ &= -\frac{1}{2x} - ix \\ &= -\frac{1}{2x} - ix \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{2x} - ix\right)(0) + \left(\left(\frac{1}{2x^2} - i\right) + \left(-\frac{1}{2x} - ix\right)^2 - \left(\frac{-4x^4+3}{4x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{2x} - ix\right) dx} \\ &= \frac{e^{-\frac{i x^2}{2}}}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3}{x}\,dx} \\ &= z_1 e^{-\frac{3\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{x^{3/2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{-\frac{i x^2}{2}}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3}{x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-3\ln(x)}}{(y_1)^2}\,dx \\ &= y_1 \left(-\frac{i e^{i x^2}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{e^{-\frac{i x^2}{2}}}{x^2}\right) + c_2 \left(\frac{e^{-\frac{i x^2}{2}}}{x^2}\left(-\frac{i e^{i x^2}}{2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dx^2} y(x)\right) x + 3\frac{d}{dx} y(x) + x^3 y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2} y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2} y(x) = -x^2 y(x) - \frac{3\left(\frac{d}{dx} y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2} y(x) + \frac{3\left(\frac{d}{dx} y(x)\right)}{x} + x^2 y(x) = 0$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - ◦ Define functions

    $\left[P_2(x) = \frac{3}{x}, P_3(x) = x^2\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 3$
- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$
- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$
- Multiply by denominators

  $\left(\frac{d^2}{dx^2}y(x)\right)x + 3\frac{d}{dx}y(x) + x^3y(x) = 0$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- □ Rewrite ODE with series expansions
  - ◦ Convert $x^3 \cdot y(x)$ to series expansion

    $x^3 \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+3}$
  - ◦ Shift index using $k- > k-3$

    $x^3 \cdot y(x) = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$
  - ◦ Convert $\frac{d}{dx}y(x)$ to series expansion

    $\frac{d}{dx}y(x) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$
  - ◦ Shift index using $k- > k+1$

    $\frac{d}{dx}y(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$
  - ◦ Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-1}$
  - ◦ Shift index using $k- > k+1$

    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(2+r)\, x^{-1+r} + a_1(1+r)(3+r)\, x^r + a_2(2+r)(4+r)\, x^{1+r} + a_3(3+r)(5+r)\, x^{2+r} + \left(\sum_{k=3}^{\infty} (\right.$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-2, 0\}$
- The coefficients of each power of x must be 0

  $[a_1(1+r)(3+r) = 0, a_2(2+r)(4+r) = 0, a_3(3+r)(5+r) = 0]$
- Solve for the dependent coefficient(s)

  $\{a_1 = 0, a_2 = 0, a_3 = 0\}$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+r+3) + a_{k-3} = 0$
- Shift index using $k- > k+3$

  $a_{k+4}(k+4+r)(k+6+r) + a_k = 0$
- Recursion relation that defines series solution to ODE

$a_{k+4} = -\frac{a_k}{(k+4+r)(k+6+r)}$

- Recursion relation for $r = -2$

  $a_{k+4} = -\frac{a_k}{(k+2)(k+4)}$

- Solution for $r = -2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+4} = -\frac{a_k}{(k+2)(k+4)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+4} = -\frac{a_k}{(k+4)(k+6)}$

- Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{a_k}{(k+4)(k+6)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{4+k} = -\frac{a_k}{(k+2)(4+k)}, a_1 = 0, a_2 = 0, a_3 = 0, b_{4+k} = -\frac{b_k}{(4+k)(k+6)}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 25

```
dsolve(x*diff(diff(y(x),x),x)+3*diff(y(x),x)+x^3*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_1 \sin\left(\frac{x^2}{2}\right) + c_2 \cos\left(\frac{x^2}{2}\right)}{x^2}$$

Mathematica DSolve solution

Solving time : 0.053 (sec)
Leaf size : 43

```
DSolve[{x*D[y[x],{x,2}]+3*D[y[x],x]+x^3*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-\frac{ix^2}{2}}\left(2c_1 - ic_2 e^{ix^2}\right)}{2x^2}$$

2.1.759 Problem 781

Internal problem ID [9931]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 781
Date solved : `Monday, January 27, 2025 at 06:15:54 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + 4xy' + \left(x^2+2\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.126 (sec)

Writing the ode as

$$x^2y'' + 4xy' + \left(x^2+2\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 4x \\ C &= x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1441: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x}{x^2}\, dx} \\ &= z_1 e^{-2\ln(x)} \\ &= z_1 \left(\frac{1}{x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-4\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\cos(x)}{x^2}\right) + c_2\left(\frac{\cos(x)}{x^2}(\tan(x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x\left(\frac{d}{dx}y(x)\right) + (x^2+2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(x^2+2)y(x)}{x^2} - \frac{4\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{4\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(x^2+2)y(x)}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{4}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 4$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 2$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x\left(\frac{d}{dx}y(x)\right) + (x^2+2)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- > k-m$

  $x^{m}\cdot y(x)=\sum_{k=m}^{\infty} a_{k-m}x^{k+r}$
- Convert $x\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r}$
- Convert $x^{2}\cdot\left(\frac{d^{2}}{dx^{2}}y(x)\right)$ to series expansion

  $x^{2}\cdot\left(\frac{d^{2}}{dx^{2}}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)\,(1+r)\,x^{r}+a_1(3+r)\,(2+r)\,x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_k(k+r+2)\,(k+r+1)+a_{k-2}\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(1+r)=0$
- Values of r that satisfy the indicial equation

  $r\in\{-2,-1\}$
- Each term must be 0

  $a_1(3+r)\,(2+r)=0$
- Solve for the dependent coefficient(s)

  $a_1=0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)\,(k+r+1)+a_{k-2}=0$
- Shift index using $k- > k+2$

  $a_{k+2}(k+4+r)\,(k+3+r)+a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_k}{(k+4+r)(k+3+r)}$
- Recursion relation for $r=-2$

  $a_{k+2}=-\frac{a_k}{(k+2)(k+1)}$
- Solution for $r=-2$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2}=-\frac{a_k}{(k+2)(k+1)}, a_1=0\right]$
- Recursion relation for $r=-1$

  $a_{k+2}=-\frac{a_k}{(k+3)(k+2)}$
- Solution for $r=-1$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2}=-\frac{a_k}{(k+3)(k+2)}, a_1=0\right]$
- Combine solutions and rename parameters

  $\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^{k-2}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), a_{k+2}=-\frac{a_k}{(k+1)(k+2)}, a_1=0, b_{k+2}=-\frac{b_k}{(k+2)(k+3)}, b_1=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+4*diff(y(x),x)*x+(x^2+2)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{\sin(x) c_1 + \cos(x) c_2}{x^2}$$

Mathematica DSolve solution

Solving time : 0.03 (sec)
Leaf size : 37

```
DSolve[{x^2*D[y[x],{x,2}]+4*x*D[y[x],x]+(x^2+2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to \frac{2c_1 e^{-ix} - ic_2 e^{ix}}{2x^2}$$

2.1.760 Problem 782

Internal problem ID [9932]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 782
Date solved : `Monday, January 27, 2025 at 06:15:54 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$16x^2y'' + 32xy' + \left(x^4 - 12\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.263 (sec)

Writing the ode as

$$16x^2y'' + 32xy' + \left(x^4 - 12\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 16x^2 \\ B &= 32x \\ C &= x^4 - 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^4 + 12}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^4 + 12 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^4 + 12}{16x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1443: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{x^2}{16} + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{ix}{4} - \frac{3i}{2x^3} - \frac{9i}{2x^7} - \frac{27i}{x^{11}} - \frac{405i}{2x^{15}} - \frac{1701i}{x^{19}} - \frac{15309i}{x^{23}} - \frac{144342i}{x^{27}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{i}{4}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{ix}{4} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -\frac{x^2}{16}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-x^4 + 12}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(-\frac{x^2}{16}\right) + \left(\frac{3}{4x^2}\right) \\ &= -\frac{x^2}{16} + \frac{3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{ix}{4} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{\frac{i}{4}} - 1\right) &= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{\frac{i}{4}} - 1\right) &= -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^4 + 12}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{ix}{4}$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)\left(\frac{ix}{4}\right) \\ &= -\frac{1}{2x} - \frac{ix}{4} \\ &= -\frac{1}{2x} - \frac{ix}{4} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{2x} - \frac{ix}{4}\right)(0) + \left(\left(\frac{1}{2x^2} - \frac{i}{4}\right) + \left(-\frac{1}{2x} - \frac{ix}{4}\right)^2 - \left(\frac{-x^4 + 12}{16x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{2x} - \frac{ix}{4}\right) dx} \\ &= \frac{e^{-\frac{i x^2}{8}}}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{32x}{16x^2}\,dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1\left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{-\frac{i x^2}{8}}}{x^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{32x}{16x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(-2i e^{\frac{i x^2}{4}}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{e^{-\frac{i x^2}{8}}}{x^{3/2}}\right) + c_2\left(\frac{e^{-\frac{i x^2}{8}}}{x^{3/2}}\left(-2i e^{\frac{i x^2}{4}}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$16x^2\left(\frac{d^2}{dx^2}y(x)\right) + 32x\left(\frac{d}{dx}y(x)\right) + (x^4 - 12)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{(x^4-12)y(x)}{16x^2} - \frac{2\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) + \frac{2\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(x^4-12)y(x)}{16x^2} = 0$
- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{2}{x}, P_3(x) = \frac{x^4-12}{16x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

$(x \cdot P_2(x)) \bigg|_{x=0} = 2$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x)) \bigg|_{x=0} = -\frac{3}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $16x^2\left(\frac{d^2}{dx^2}y(x)\right) + 32x\left(\frac{d}{dx}y(x)\right) + (x^4 - 12)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..4$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $4a_0(3+2r)\,(-1+2r)\,x^r + 4a_1(5+2r)\,(1+2r)\,x^{1+r} + 4a_2(7+2r)\,(3+2r)\,x^{2+r} + 4a_3(9+2r)\,($

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $4(3+2r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{3}{2}, \frac{1}{2}\right\}$

- The coefficients of each power of x must be 0

  $[4a_1(5+2r)\,(1+2r) = 0, 4a_2(7+2r)\,(3+2r) = 0, 4a_3(9+2r)\,(5+2r) = 0]$

- Solve for the dependent coefficient(s)

  $\{a_1 = 0, a_2 = 0, a_3 = 0\}$

- Each term in the series must be 0, giving the recursion relation

  $16\left(k+r+\frac{3}{2}\right)\left(k+r-\frac{1}{2}\right)a_k + a_{k-4} = 0$

- Shift index using $k- >k + 4$

  $16\left(k+\frac{11}{2}+r\right)\left(k+\frac{7}{2}+r\right)a_{k+4} + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+4} = -\frac{a_k}{4(2k+11+2r)(2k+7+2r)}$

- Recursion relation for $r = -\frac{3}{2}$

  $a_{k+4} = -\frac{a_k}{4(2k+8)(2k+4)}$

- Solution for $r = -\frac{3}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}, a_{k+4} = -\frac{a_k}{4(2k+8)(2k+4)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+4} = -\frac{a_k}{4(2k+12)(2k+8)}$
- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+4} = -\frac{a_k}{4(2k+12)(2k+8)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{4+k} = -\frac{a_k}{4(2k+8)(2k+4)}, a_1 = 0, a_2 = 0, a_3 = 0, b_{4+k} = -\frac{}{4(2k}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.040 (sec)
Leaf size : 25

```
dsolve(16*x^2*diff(diff(y(x),x),x)+32*diff(y(x),x)*x+(x^4-12)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_1 \sin\left(\frac{x^2}{8}\right) + c_2 \cos\left(\frac{x^2}{8}\right)}{x^{3/2}}$$

Mathematica DSolve solution

Solving time : 0.103 (sec)
Leaf size : 48

```
DSolve[{16*x^2*D[y[x],{x,2}]+32*x*D[y[x],x]+(x^4-12)*y[x]==0,{}},y[x],x,IncludeSingularSolutio
```

$$y(x) \to \frac{e^{-\frac{1}{2}-\frac{ix^2}{8}}\left(c_1 - 2ic_2 e^{1+\frac{ix^2}{4}}\right)}{x^{3/2}}$$

2.1.761 Problem 783

Internal problem ID [9933]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 783
Date solved : `Monday, January 27, 2025 at 06:15:55 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y''-x^2y'+xy=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.322 (sec)

Writing the ode as

$$y''-x^2y'+xy=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x^2 \\ C &= x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{x(x^3-8)}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x(x^3-8) \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x(x^3-8)}{4}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1445: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 4 \\ &= -4 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{2} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x^2}{2} - \frac{2}{x} - \frac{4}{x^4} - \frac{16}{x^7} - \frac{80}{x^{10}} - \frac{448}{x^{13}} - \frac{2688}{x^{16}} - \frac{16896}{x^{19}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 2$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i x^i \\ &= \frac{x^2}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^1 = x$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^4}{4}$$

This shows that the coefficient of x in the above is 0. Now we need to find the coefficient of x in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x(x^3 - 8)}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^4 - 2x\right) + (0) \\ &= \frac{1}{4}x^4 - 2x \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is -2. Now b can be found.

$$\begin{aligned} b &= (-2) - (0) \\ &= -2 \end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{x^2}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-2}{\frac{1}{2}} - 2\right) &= -3 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-2}{\frac{1}{2}} - 2\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x(x^3 - 8)}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-4	$\frac{x^2}{2}$	-3	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x^2}{2}\right) \\ &= -\frac{x^2}{2} \\ &= -\frac{x^2}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{x^2}{2}\right)(1) + \left((-x) + \left(-\frac{x^2}{2}\right)^2 - \left(\frac{x(x^3-8)}{4}\right)\right) &= 0 \\ x a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (x)\, e^{\int -\frac{x^2}{2} dx} \\ &= (x)\, e^{-\frac{x^3}{6}} \\ &= x\, e^{-\frac{x^3}{6}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x^2}{1} dx} \\ &= z_1 e^{\frac{x^3}{6}} \\ &= z_1 \left(e^{\frac{x^3}{6}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{x^3}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{3^{2/3}(-1)^{1/3}\left(-\frac{3x^2(-1)^{2/3}\Gamma\left(\frac{2}{3}\right)}{(-x^3)^{2/3}} + \frac{3\,3^{1/3}(-1)^{2/3}e^{\frac{x^3}{3}}}{x} + \frac{3x^2(-1)^{2/3}\Gamma\left(\frac{2}{3},-\frac{x^3}{3}\right)}{(-x^3)^{2/3}}\right)}{9} \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2 \left(x \left(\frac{3^{2/3}(-1)^{1/3}\left(-\frac{3x^2(-1)^{2/3}\Gamma\left(\frac{2}{3}\right)}{(-x^3)^{2/3}} + \frac{3\,3^{1/3}(-1)^{2/3}e^{\frac{x^3}{3}}}{x} + \frac{3x^2(-1)^{2/3}\Gamma\left(\frac{2}{3},-\frac{x^3}{3}\right)}{(-x^3)^{2/3}}\right)}{9} \right) \right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) - x^2\left(\frac{d}{dx}y(x)\right) + xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

- ☐ Rewrite ODE with series expansions
  - Convert $x \cdot y(x)$ to series expansion

    $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k- >k-1$

    $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert $x^2 \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^{k+1}$

  - Shift index using $k- >k-1$

$$x^2 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=1}^{\infty} a_{k-1}(k-1)\, x^k$$

- Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

$$\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

$$\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_{k-1}(k-2)\right) x^k\right) = 0$$

- Each term must be 0

$$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$\left(k^2+3k+2\right) a_{k+2} - a_{k-1}(k-2) = 0$$

- Shift index using $k- >k+1$

$$\left((k+1)^2+3k+5\right) a_{k+3} - a_k(k-1) = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{a_k(k-1)}{k^2+5k+6}, 2a_2 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 51

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x^2+x*y(x) = 0,y(x),singsol=all)
```

$$y = c_2\left(-x^3\right)^{1/3} 3^{2/3}\Gamma\left(\frac{2}{3}\right) - c_2\left(-x^3\right)^{1/3} 3^{2/3}\Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right) + 3c_2 e^{\frac{x^3}{3}} + c_1 x$$

Mathematica DSolve solution

Solving time : 0.055 (sec)
Leaf size : 41

```
DSolve[{D[y[x],{x,2}]-x^2*D[y[x],x]+x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 x - \frac{c_2 \sqrt[3]{-x^3}\,\Gamma\left(-\frac{1}{3}, -\frac{x^3}{3}\right)}{3\sqrt[3]{3}}$$

2.1.762 Problem 784

Internal problem ID [9934]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 784
Date solved : `Monday, January 27, 2025 at 06:15:56 PM`
CAS classification : `[_Laguerre]`

Solve

$$xy'' - (x+2)\,y' + 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.204 (sec)

Writing the ode as

$$xy'' + (-x-2)\,y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -x-2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 4x + 8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1447: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{x} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{1}{x^2} + \frac{2}{x^3} + \frac{3}{x^4} + \frac{2}{x^5} - \frac{6}{x^6} - \frac{28}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-4x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-4x + 8}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned}b &= (-1) - (0) \\ &= -1\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 0\right) &= -1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 0\right) &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -1$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} - \frac{1}{x} \\
&= \frac{x-2}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2} - \frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(\frac{1}{2} - \frac{1}{x}\right)^2 - \left(\frac{x^2-4x+8}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(\frac{1}{2} - \frac{1}{x}\right) dx} \\
&= \frac{e^{\frac{x}{2}}}{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x-2}{x}\,dx} \\ &= z_1 e^{\frac{x}{2}+\ln(x)} \\ &= z_1\left(x\,\mathrm{e}^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x-2}{x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{x+2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(-\frac{(x^2+2x+2)\,\mathrm{e}^{x+2\ln(x)}\mathrm{e}^{-2x}}{x^2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1y_1 + c_2y_2 \\ &= c_1(\mathrm{e}^{x}) + c_2\left(\mathrm{e}^{x}\left(-\frac{(x^2+2x+2)\,\mathrm{e}^{x+2\ln(x)}\mathrm{e}^{-2x}}{x^2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dx^2}y(x)\right)x - (x+2)\left(\frac{d}{dx}y(x)\right) + 2y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{2y(x)}{x} + \frac{(x+2)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) - \frac{(x+2)\left(\frac{d}{dx}y(x)\right)}{x} + \frac{2y(x)}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{x+2}{x}, P_3(x) = \frac{2}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -2$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$
- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$
- • Multiply by denominators

  $\left(\frac{d^2}{dx^2}y(x)\right)x + (-x-2)\left(\frac{d}{dx}y(x)\right) + 2y(x) = 0$
- • Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- □ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..1$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
  - ◦ Shift index using $k- > k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - ◦ Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-1}$
  - ◦ Shift index using $k- > k+1$

    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-3+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-2) - a_k(k+r-2)\right) x^{k+r}\right) = 0$
- • $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+r) = 0$
- • Values of r that satisfy the indicial equation

  $r \in \{0, 3\}$
- • Each term in the series must be 0, giving the recursion relation

  $(k+r-2)\left(a_{k+1}(k+1+r) - a_k\right) = 0$
- • Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1+r}$
- • Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$
- • Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k}{k+1}\right]$
- • Recursion relation for $r = 3$

  $a_{k+1} = \frac{a_k}{k+4}$
- • Solution for $r = 3$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = \frac{a_k}{k+4}\right]$
- • Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{4+k}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 19

```
dsolve(x*diff(diff(y(x),x),x)-(x+2)*diff(y(x),x)+2*y(x) = 0,y(x),singsol=all)
```

$$y = e^{x} c_1 + c_2 \left(x^{2}+2 x +2\right)$$

Mathematica DSolve solution

Solving time : 0.222 (sec)
Leaf size : 35

```
DSolve[{x*D[y[x],{x,2}]-(x+2)*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{x+1}\left(c_2 \int_1^x e^{-K[1]} K[1]^2 dK[1] + c_1\right)$$

2.1.763 Problem 785

Internal problem ID [9935]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 785
Date solved : `Monday, January 27, 2025 at 06:15:56 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + xy' + 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.213 (sec)

Writing the ode as

$$y'' + xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 6 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1449: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{2x} - \frac{9}{4x^3} - \frac{27}{4x^5} - \frac{405}{16x^7} - \frac{1701}{16x^9} - \frac{15309}{32x^{11}} - \frac{72171}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$[\sqrt{r}]_\infty = \sum_{i=0}^{1} a_i x^i = \frac{x}{2} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 6}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} - \frac{3}{2}\right) + (0) \\ &= \frac{x^2}{4} - \frac{3}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{2}\right) - (0) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) = -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) = 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{3}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{3}{2}\right)\right) &= 0 \\ a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (x)\, e^{\int -\frac{x}{2} dx} \\ &= (x)\, e^{-\frac{x^2}{4}} \\ &= x\, e^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{1}\, dx} \\ &= z_1 e^{-\frac{x^2}{4}} \\ &= z_1\left(e^{-\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^2}{2}} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\, \mathrm{erf}\left(\frac{i\sqrt{2}\, x}{2}\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x^2}{2}} x\right) + c_2 \left(\mathrm{e}^{-\frac{x^2}{2}} x \left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\, \mathrm{erf}\left(\frac{i\sqrt{2}\, x}{2}\right)}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + x\left(\frac{d}{dx}y(x)\right) + 2y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

  $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

  $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+2)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

$(k+2)(k a_{k+2} + a_k + a_{k+2}) = 0$

- Recursion relation that defines the series solution to the ODE

$\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{k+1}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.007` (sec)
Leaf size : 34

```
dsolve(diff(diff(y(x),x),x)+diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y = x\left(c_2 \operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)\pi + c_1\right)\mathrm{e}^{-\frac{x^2}{2}} - i\sqrt{\pi}\,\sqrt{2}\,c_2$$

Mathematica DSolve solution

Solving time : `0.053` (sec)
Leaf size : 69

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\sqrt{\frac{\pi}{2}} c_2 e^{-\frac{x^2}{2}} \sqrt{x^2} \operatorname{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{2}}\right) + \sqrt{2} c_1 e^{-\frac{x^2}{2}} x + c_2$$

2.1.764 Problem 786

Internal problem ID [9936]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 786
Date solved : `Monday, January 27, 2025 at 06:15:57 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-x^{2}+1\right) y''-2 x y'+2 y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.250 (sec)

Writing the ode as

$$\left(-x^{2}+1\right) y''-2 x y'+2 y=0 \tag{1}$$
$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^{2}+1 \\ B &= -2 x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2 A} \, dx}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2 A B'-2 B A'+B^{2}-4 A C}{4 A^{2}} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{2 x^{2}-3}{\left(x^{2}-1\right)^{2}} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 x^{2}-3 \\ t &= \left(x^{2}-1\right)^{2} \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{2 x^{2}-3}{\left(x^{2}-1\right)^{2}}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1451: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(x-1)^2} - \frac{5}{4(x+1)} - \frac{1}{4(x+1)^2} + \frac{5}{4(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^2 - 3}{(x^2 - 1)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2x^2 - 3}{(x^2 - 1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\ &= 2 - (1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{2x - 2} + \frac{1}{2x + 2} + (0) \\ &= \frac{1}{2x - 2} + \frac{1}{2x + 2} \\ &= \frac{x}{x^2 - 1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x-2} + \frac{1}{2x+2}\right)(1) + \left(\left(-\frac{1}{2(x-1)^2} - \frac{1}{2(x+1)^2}\right) + \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right)^2 - \left(\frac{2x^2-3}{(x^2-1)^2}\right)\right) = -\frac{2a_0}{x^2-1} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x) \, e^{\int \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right) dx} \\
&= (x) \sqrt{(x-1)(x+1)} \\
&= x\sqrt{x^2-1}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-2x}{-x^2+1} \, dx} \\
&= z_1 e^{-\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x-1}\sqrt{x+1}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x\sqrt{x^2-1}}{\sqrt{x-1}\sqrt{x+1}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{-x^2+1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\ln(x-1) - \ln(x+1)}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + \frac{1}{x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x\sqrt{x^2-1}}{\sqrt{x-1}\sqrt{x+1}}\right) + c_2 \left(\frac{x\sqrt{x^2-1}}{\sqrt{x-1}\sqrt{x+1}} \left(\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + \frac{1}{x}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(-x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)-2x\left(\frac{d}{dx}y(x)\right)+2y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=\frac{2y(x)}{x^2-1}-\frac{2\left(\frac{d}{dx}y(x)\right)x}{x^2-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)+\frac{2\left(\frac{d}{dx}y(x)\right)x}{x^2-1}-\frac{2y(x)}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=\frac{2x}{x^2-1}, P_3(x)=-\frac{2}{x^2-1}\right]$

- ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

$\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=1$

- ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

$\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

- ◦ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1$

- Multiply by denominators

$(x^2-1)\left(\frac{d^2}{dx^2}y(x)\right)+2x\left(\frac{d}{dx}y(x)\right)-2y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(2u-2)\left(\frac{d}{du}y(u)\right)-2y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ◦ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- ◦ Shift index using $k->k+2-m$

$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

Rewrite ODE with series expansions

$-2a_0r^2u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)^2+a_k(k+r+2)\,(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-2r^2 = 0$

- Values of r that satisfy the indicial equation
  $r = 0$
- Each term in the series must be 0, giving the recursion relation
  $-2a_{k+1}(k+1)^2 + a_k(k+2)(k-1) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$
- Recursion relation for $r = 0$; series terminates at $k = 1$
  $a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$
- Apply recursion relation for $k = 0$
  $a_1 = -a_0$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second linea
  $y(u) = a_0 \cdot (-u+1)$
- Revert the change of variables $u = x+1$
  $[y(x) = -a_0 x]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 25

```
dsolve((-x^2+1)*diff(diff(y(x),x),x)-2*diff(y(x),x)*x+2*y(x) = 0,y(x),singsol=all)
```

$$y = -\frac{\ln(x+1)\, c_2 x}{2} + \frac{c_2 \ln(x-1)\, x}{2} + c_1 x + c_2$$

Mathematica DSolve solution

Solving time : 0.02 (sec)
Leaf size : 33

```
DSolve[{(1-x^2)*D[y[x],{x,2}]-2*x*D[y[x],x]+2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tru
```

$$y(x) \to c_1 x - \frac{1}{2} c_2 (x \log(1-x) - x \log(x+1) + 2)$$

2.1.765 Problem 787

Internal problem ID [9937]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 787
Date solved : `Monday, January 27, 2025 at 06:15:57 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - 4xy' + (4x^2 - 2)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.053 (sec)

Writing the ode as

$$y'' - 4xy' + (4x^2 - 2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4x \\ C &= 4x^2 - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1453: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{1}\, dx} \\ &= z_1 e^{x^2} \\ &= z_1 \left(e^{x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x^2}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(e^{x^2}\right) + c_2\left(e^{x^2}(x)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) - 4x\left(\frac{d}{dx}y(x)\right) + (4x^2 - 2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ∘ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ∘ Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ∘ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  ∘ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

  $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ∘ Shift index using $k- >k + 2$

  $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 - 2a_0 + (6a_3 - 6a_1)\,x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)(k+1) - 2a_k(2k+1) + 4a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 - 2a_0 = 0, 6a_3 - 6a_1 = 0]$

- Solve for the dependent coefficient(s)

  $\{a_2 = a_0, a_3 = a_1\}$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\,a_{k+2} - 4a_k k - 2a_k + 4a_{k-2} = 0$

- Shift index using $k->k+2$

  $\left((k+2)^2+3k+8\right)a_{k+4}-4a_{k+2}(k+2)-2a_{k+2}+4a_k=0$

- Recursion relation that defines the series solution to the ODE

  $\left[y(x)=\sum_{k=0}^{\infty}a_k x^k, a_{k+4}=\frac{2(2ka_{k+2}-2a_k+5a_{k+2})}{k^2+7k+12}, a_2=a_0, a_3=a_1\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 14

```
dsolve(diff(diff(y(x),x),x)-4*diff(y(x),x)*x+(4*x^2-2)*y(x) = 0,y(x),singsol=all)
```

$$y=\mathrm{e}^{x^2}(c_2x+c_1)$$

Mathematica DSolve solution

Solving time : 0.02 (sec)
Leaf size : 18

```
DSolve[{D[y[x],{x,2}]-4*x*D[y[x],x]+(4*x^2-2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True
```

$$y(x)\to e^{x^2}(c_2x+c_1)$$

2.1.766 Problem 788

Internal problem ID [9938]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 788
Date solved : Monday, January 27, 2025 at 06:15:58 PM
CAS classification : [_Gegenbauer]

Solve

$$\left(-x^2+1\right) y''-2xy'+30y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.300 (sec)

Writing the ode as

$$\left(-x^2+1\right) y''-2xy'+30y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+1 \\ B &= -2x \\ C &= 30 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{30x^2-31}{\left(x^2-1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 30x^2-31 \\ t &= \left(x^2-1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{30x^2-31}{\left(x^2-1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1455: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{61}{4(x+1)} + \frac{61}{4(x-1)} - \frac{1}{4(x-1)^2} - \frac{1}{4(x+1)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{30x^2 - 31}{(x^2 - 1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 30$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 6\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -5\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{30x^2 - 31}{(x^2 - 1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	6	-5

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 6$ then

$$\begin{aligned}d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right)\\ &= 6 - (1)\\ &= 5\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2}\right) + (+)[\sqrt{r}]_\infty\\ &= \frac{1}{2x-2} + \frac{1}{2x+2} + (0)\\ &= \frac{1}{2x-2} + \frac{1}{2x+2}\\ &= \frac{x}{x^2-1}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 5$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(20x^3 + 12x^2a_4 + 6xa_3 + 2a_2) + 2\left(\frac{1}{2x-2} + \frac{1}{2x+2}\right)(5x^4 + 4x^3a_4 + 3x^2a_3 + 2xa_2 + a_1) + \left(\left(-\frac{1}{2(x-1)}\right.\right.$$
$$\overline{-10a_4x^4 + (-18a_3 - 20)x^3}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = 0, a_1 = \frac{5}{21}, a_2 = 0, a_3 = -\frac{10}{9}, a_4 = 0\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^5 - \frac{10}{9}x^3 + \frac{5}{21}x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\, dx} \\
&= \left(x^5 - \frac{10}{9}x^3 + \frac{5}{21}x\right) e^{\int \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right) dx} \\
&= \left(x^5 - \frac{10}{9}x^3 + \frac{5}{21}x\right) \sqrt{(x-1)(x+1)} \\
&= \frac{(63x^5 - 70x^3 + 15x)\sqrt{x^2-1}}{63}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{-x^2+1}\, dx} \\
&= z_1 e^{-\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x-1}\sqrt{x+1}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(63x^5 - 70x^3 + 15x)\sqrt{x^2-1}}{63\sqrt{x-1}\sqrt{x+1}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{-x^2+1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\ln(x-1) - \ln(x+1)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{3087\left(-23x^3 + \frac{935}{63}x\right)}{1600\left(x^4 - \frac{10}{9}x^2 + \frac{5}{21}\right)} - \frac{3969\ln(x+1)}{128} + \frac{3969\ln(x-1)}{128} + \frac{441}{25x}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(63x^5 - 70x^3 + 15x)\sqrt{x^2-1}}{63\sqrt{x-1}\sqrt{x+1}} \right) \\
&\quad + c_2 \left(\frac{(63x^5 - 70x^3 + 15x)\sqrt{x^2-1}}{63\sqrt{x-1}\sqrt{x+1}} \left(-\frac{3087\left(-23x^3 + \frac{935}{63}x\right)}{1600\left(x^4 - \frac{10}{9}x^2 + \frac{5}{21}\right)} - \frac{3969\ln(x+1)}{128} \right.\right. \\
&\quad \left.\left. + \frac{3969\ln(x-1)}{128} + \frac{441}{25x} \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(-x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) - 2x\left(\frac{d}{dx}y(x)\right) + 30y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{30y(x)}{x^2-1} - \frac{2\left(\frac{d}{dx}y(x)\right)x}{x^2-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lir

  $\frac{d^2}{dx^2}y(x) + \frac{2\left(\frac{d}{dx}y(x)\right)x}{x^2-1} - \frac{30y(x)}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2x}{x^2-1}, P_3(x) = -\frac{30}{x^2-1}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1} = 1$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$

  ◦ $x=-1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -1$

- Multiply by denominators

  $(x^2-1)\left(\frac{d^2}{dx^2}y(x)\right) + 2x\left(\frac{d}{dx}y(x)\right) - 30y(x) = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (2u-2)\left(\frac{d}{du}y(u)\right) - 30y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum\limits_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum\limits_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum\limits_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

Rewrite ODE with series expansions

$-2a_0r^2u^{-1+r} + \left(\sum\limits_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)^2 + a_k(k+r+6)\,(k+r-5)\right) u^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-2r^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 0$
- Each term in the series must be 0, giving the recursion relation
  $-2a_{k+1}(k+1)^2 + a_k(k+6)\,(k-5) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_k(k+6)(k-5)}{2(k+1)^2}$
- Recursion relation for $r = 0$; series terminates at $k = 5$
  $a_{k+1} = \frac{a_k(k+6)(k-5)}{2(k+1)^2}$
- Apply recursion relation for $k = 0$
  $a_1 = -15a_0$
- Apply recursion relation for $k = 1$
  $a_2 = -\frac{7a_1}{2}$
- Express in terms of a_0
  $a_2 = \frac{105a_0}{2}$
- Apply recursion relation for $k = 2$
  $a_3 = -\frac{4a_2}{3}$
- Express in terms of a_0
  $a_3 = -70a_0$
- Apply recursion relation for $k = 3$
  $a_4 = -\frac{9a_3}{16}$
- Express in terms of a_0
  $a_4 = \frac{315a_0}{8}$
- Apply recursion relation for $k = 4$
  $a_5 = -\frac{a_4}{5}$
- Express in terms of a_0
  $a_5 = -\frac{63a_0}{8}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second linea
  $y(u) = a_0 \cdot \left(1 - 15u + \frac{105}{2}u^2 - 70u^3 + \frac{315}{8}u^4 - \frac{63}{8}u^5\right)$
- Revert the change of variables $u = x + 1$
  $\left[y(x) = a_0\left(-\frac{15}{8}x + \frac{35}{4}x^3 - \frac{63}{8}x^5\right)\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 71

```
dsolve((-x^2+1)*diff(diff(y(x),x),x)-2*diff(y(x),x)*x+30*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{21c_2x\left(x^4-\frac{10}{9}x^2+\frac{5}{21}\right)\ln\left(x-1\right)}{640} - \frac{21c_2x\left(x^4-\frac{10}{9}x^2+\frac{5}{21}\right)\ln\left(x+1\right)}{640} + \frac{21c_1x^5}{5} + \frac{21c_2x^4}{320} - \frac{14c_1x^3}{3} - \frac{49c_2x^2}{960} + c_1x + \frac{c_2}{225}$$

Mathematica DSolve solution

Solving time : 0.025 (sec)
Leaf size : 76

```
DSolve[{(1-x^2)*D[y[x],{x,2}]-2*x*D[y[x],x]+30*y[x]==0,{}},y[x],x,IncludeSingularSolutions->
```

$$y(x) \to \frac{1}{8}c_1x\left(63x^4-70x^2+15\right) + c_2\left(-\frac{63x^4}{8}+\frac{49x^2}{8}-\frac{1}{16}\left(63x^4-70x^2+15\right)x(\log(1-x)-\log(x+1))-\frac{8}{15}\right)$$

2.1.767 Problem 789

Internal problem ID [9939]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 789
Date solved : `Monday, January 27, 2025 at 06:15:59 PM`
CAS classification : `[_Lienard]`

Solve

$$xy'' + 2y' + xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.125 (sec)

Writing the ode as

$$xy'' + 2y' + xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1457: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s)\\
&= 0 - 0\\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}\\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\, dx}\\
&= z_1 e^{-\ln(x)}\\
&= z_1 \left(\frac{1}{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\, dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\cos(x)}{x}\right) + c_2\left(\frac{\cos(x)}{x}(\tan(x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dx^2}y(x)\right)x + 2\frac{d}{dx}y(x) + xy(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -y(x) - \frac{2\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) + \frac{2\left(\frac{d}{dx}y(x)\right)}{x} + y(x) = 0$
- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{2}{x}, P_3(x) = 1\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $\left(\frac{d^2}{dx^2}y(x)\right)x + 2\frac{d}{dx}y(x) + xy(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- □ Rewrite ODE with series expansions
  - Convert $x \cdot y(x)$ to series expansion
    $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+1}$
  - Shift index using $k- >k-1$

$x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

- Convert $\frac{d}{dx}y(x)$ to series expansion

  $\frac{d}{dx}y(x) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $\frac{d}{dx}y(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$

- Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $a_1(1+r)\,(2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1} = 0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)\,(k+3+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{(k+1)(k+2)}$

- Solution for $r = -1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+3)}$

- Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0, b_{k+2} = -\frac{b_k}{(k+2)(k+3)}, 2b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 17

```
dsolve(x*diff(diff(y(x),x),x)+2*diff(y(x),x)+x*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{\sin(x) c_1 + \cos(x) c_2}{x}$$

Mathematica DSolve solution

Solving time : 0.027 (sec)
Leaf size : 37

```
DSolve[{x*D[y[x],{x,2}]+2*D[y[x],x]+x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2c_1 e^{-ix} - ic_2 e^{ix}}{2x}$$

2.1.768 Problem 790

Internal problem ID [9940]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 790
Date solved : `Monday, January 27, 2025 at 06:15:59 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + (2x+1)\, y' + (x+1)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.155 (sec)

Writing the ode as

$$xy'' + (2x+1)\, y' + (x+1)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2x+1 \\ C &= x+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1459: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the gcd$(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\,(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0)+2\left(\frac{1}{2x}\right)(0)+\left(\left(-\frac{1}{2x^2}\right)+\left(\frac{1}{2x}\right)^2-\left(-\frac{1}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{1}{2x} dx} \\
&= \sqrt{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x+1}{x}\, dx} \\
&= z_1 e^{-x-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{-x}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x+1}{x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2x-\ln(x)}}{(y_1)^2}\, dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}(\ln(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dx^2}y(x)\right)x+(2x+1)\left(\frac{d}{dx}y(x)\right)+(x+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
$\frac{d^2}{dx^2}y(x)=-\frac{(x+1)y(x)}{x}-\frac{(2x+1)\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
$\frac{d^2}{dx^2}y(x)+\frac{(2x+1)\left(\frac{d}{dx}y(x)\right)}{x}+\frac{(x+1)y(x)}{x}=0$

□ Check to see if $x_0=0$ is a regular singular point
  - Define functions
  $\left[P_2(x)=\frac{2x+1}{x},P_3(x)=\frac{x+1}{x}\right]$
  - $x\cdot P_2(x)$ is analytic at $x=0$
  $\left(x\cdot P_2(x)\right)\Big|_{x=0}=1$
  - $x^2\cdot P_3(x)$ is analytic at $x=0$
  $\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=0$
  - $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point
  $x_0=0$

- Multiply by denominators
$\left(\frac{d^2}{dx^2}y(x)\right)x+(2x+1)\left(\frac{d}{dx}y(x)\right)+(x+1)\,y(x)=0$
- Assume series solution for $y(x)$
$y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions
  - Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$
  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$
  - Shift index using $k->k-m$
  $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$
  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..1$
  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k->k+1-m$
  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion
  $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-1}$
  - Shift index using $k->k+1$
  $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$a_0r^2x^{-1+r} + \left(a_1(1+r)^2 + a_0(1+2r)\right)x^r + \left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+1+r)^2 + a_k(2k+2r+1) + a_{k-1}\right)x^{k+r}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 0$
- Each term must be 0
  $a_1(1+r)^2 + a_0(1+2r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1)^2 + 2a_kk + a_k + a_{k-1} = 0$
- Shift index using $k- >k+1$
  $a_{k+2}(k+2)^2 + 2a_{k+1}(k+1) + a_{k+1} + a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{2ka_{k+1}+a_k+3a_{k+1}}{(k+2)^2}$
- Recursion relation for $r = 0$
  $a_{k+2} = -\frac{2ka_{k+1}+a_k+3a_{k+1}}{(k+2)^2}$
- Solution for $r = 0$
  $\left[y(x) = \sum_{k=0}^{\infty} a_kx^k, a_{k+2} = -\frac{2ka_{k+1}+a_k+3a_{k+1}}{(k+2)^2}, a_1 + a_0 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 15

```
dsolve(x*diff(diff(y(x),x),x)+(2*x+1)*diff(y(x),x)+(x+1)*y(x) = 0,y(x),singsol=all)
```

$$y = e^{-x}(c_2 \ln(x) + c_1)$$

Mathematica DSolve solution

Solving time : 0.029 (sec)
Leaf size : 19

```
DSolve[{x*D[y[x],{x,2}]+(2*x+1)*D[y[x],x]+(x+1)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to e^{-x}(c_2 \log(x) + c_1)$$

2.1.769 Problem 791

Internal problem ID [9941]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 791
Date solved : `Monday, January 27, 2025 at 06:16:00 PM`
CAS classification : `[_Jacobi]`

Solve

$$2x(x-1)\,y''-(x+1)\,y'+y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.189 (sec)

Writing the ode as

$$\left(2x^2-2x\right)y''+(-x-1)\,y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2-2x \\ B &= -x-1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-3x^2+18x-3}{16\left(x^2-x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3x^2+18x-3 \\ t &= 16\left(x^2-x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-3x^2+18x-3}{16\left(x^2-x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1461: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{4(x-1)} - \frac{3}{16x^2} + \frac{3}{4x} + \frac{3}{4(x-1)^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 + 18x - 3}{16\,(x^2 - x)^2}$$

Since the gcd$(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x^2 + 18x - 3}{16\,(x^2 - x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= \frac{1}{4} - \left(\frac{1}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{3}{4x} - \frac{1}{2\,(x-1)} + (-)\,(0) \\
&= \frac{3}{4x} - \frac{1}{2\,(x-1)} \\
&= \frac{x-3}{4x\,(x-1)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{4x} - \frac{1}{2(x-1)}\right)(0) + \left(\left(-\frac{3}{4x^2} + \frac{1}{2(x-1)^2}\right) + \left(\frac{3}{4x} - \frac{1}{2(x-1)}\right)^2 - \left(\frac{-3x^2 + 18x - 3}{16(x^2 - x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{3}{4x} - \frac{1}{2(x-1)}\right)dx} \\ &= \frac{x^{3/4}}{\sqrt{x-1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x-1}{2x^2-2x}\, dx} \\ &= z_1 e^{\frac{\ln(x-1)}{2} - \frac{\ln(x)}{4}} \\ &= z_1\left(\frac{\sqrt{x-1}}{x^{1/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x-1}{2x^2-2x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\ln(x-1) - \frac{\ln(x)}{2}}}{(y_1)^2}\, dx \\ &= y_1\left(\frac{2(x+1)\, e^{\ln(x-1) - \frac{\ln(x)}{2}}}{x-1}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\sqrt{x}\right) + c_2\left(\sqrt{x}\left(\frac{2(x+1)\, e^{\ln(x-1) - \frac{\ln(x)}{2}}}{x-1}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x(x-1)\left(\frac{d^2}{dx^2}y(x)\right)-(x+1)\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{y(x)}{2x(x-1)}+\frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{2x(x-1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)-\frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{2x(x-1)}+\frac{y(x)}{2x(x-1)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=-\frac{x+1}{2x(x-1)},P_3(x)=\frac{1}{2x(x-1)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\bigg|_{x=0}=\frac{1}{2}$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\bigg|_{x=0}=0$

  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2x(x-1)\left(\frac{d^2}{dx^2}y(x)\right)+(-x-1)\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..1$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=1..2$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0r(-1+2r)\,x^{-1+r}+\left(\sum_{k=0}^{\infty}(-a_{k+1}(k+1+r)\,(2k+1+2r)+a_k(2k+2r-1)\,(k+r-1))\,x^{k-}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-1+2r)=0$

- Values of r that satisfy the indicial equation

$r \in \left\{0, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation
  $-2(k+1+r)\left(k+r+\frac{1}{2}\right) a_{k+1} + 2\left(k+r-\frac{1}{2}\right)(k+r-1) a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{(2k+2r-1)(k+r-1)a_k}{(k+1+r)(2k+1+2r)}$
- Recursion relation for $r = 0$; series terminates at $k = 1$
  $a_{k+1} = \frac{(2k-1)(k-1)a_k}{(k+1)(2k+1)}$
- Apply recursion relation for $k = 0$
  $a_1 = a_0$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second linea
  $y(x) = a_0 \cdot (x+1)$
- Recursion relation for $r = \frac{1}{2}$
  $a_{k+1} = \frac{2k\left(k-\frac{1}{2}\right)a_k}{\left(k+\frac{3}{2}\right)(2k+2)}$
- Solution for $r = \frac{1}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{2k\left(k-\frac{1}{2}\right)a_k}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$
- Combine solutions and rename parameters
  $\left[y(x) = a_0 \cdot (x+1) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), b_{k+1} = \frac{2k\left(k-\frac{1}{2}\right)b_k}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 14

```
dsolve(2*x*(x-1)*diff(diff(y(x),x),x)-(x+1)*diff(y(x),x)+y(x) = 0,y(x),singsol=all)
```

$$y = c_2\sqrt{x} + c_1 x + c_1$$

Mathematica DSolve solution

Solving time : 0.252 (sec)
Leaf size : 106

```
DSolve[{2*x*(x-1)*D[y[x],{x,2}]-(x+1)*D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to \exp\left(\int_1^x \frac{K[1]-3}{4(K[1]-1)K[1]}dK[1] - \frac{1}{2}\int_1^x \frac{K[2]+1}{2K[2]-2K[2]^2}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{K[1]-3}{4(K[1]-1)K[1]}dK[1]\right)dK[3] + c_1\right)$$

2.1.770 Problem 792

Internal problem ID [9942]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 792
Date solved : `Monday, January 27, 2025 at 06:16:00 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + 2y' + 4xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.135 (sec)

Writing the ode as

$$xy'' + 2y' + 4xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= 4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1463: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\, dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1 \left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(2x)}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{2}{x} dx}}{(y_1)^2} dx$$
$$= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2} dx$$
$$= y_1 \left(\frac{\tan(2x)}{2}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{\cos(2x)}{x}\right) + c_2 \left(\frac{\cos(2x)}{x} \left(\frac{\tan(2x)}{2}\right)\right)$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dx^2}y(x)\right) x + 2\frac{d}{dx}y(x) + 4xy(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -4y(x) - \frac{2\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) + \frac{2\left(\frac{d}{dx}y(x)\right)}{x} + 4y(x) = 0$
- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{2}{x}, P_3(x) = 4\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $(x \cdot P_2(x))\Big|_{x=0} = 2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $\left(\frac{d^2}{dx^2}y(x)\right) x + 2\frac{d}{dx}y(x) + 4xy(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- □ Rewrite ODE with series expansions
  - Convert $x \cdot y(x)$ to series expansion
    $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

- Shift index using $k->k-1$

  $x\cdot y(x)=\sum_{k=1}^{\infty} a_{k-1}x^{k+r}$
- Convert $\frac{d}{dx}y(x)$ to series expansion

  $\frac{d}{dx}y(x)=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1}$
- Shift index using $k->k+1$

  $\frac{d}{dx}y(x)=\sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,x^{k+r}$
- Convert $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$
- Shift index using $k->k+1$

  $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(1+r)\,x^{-1+r}+a_1(1+r)\,(2+r)\,x^r+\left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+r+1)\,(k+2+r)+4a_{k-1}\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r)=0$
- Values of r that satisfy the indicial equation

  $r\in\{-1,0\}$
- Each term must be 0

  $a_1(1+r)\,(2+r)=0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)\,(k+2+r)+4a_{k-1}=0$
- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)\,(k+3+r)+4a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{4a_k}{(k+2+r)(k+3+r)}$
- Recursion relation for $r=-1$

  $a_{k+2}=-\frac{4a_k}{(k+1)(k+2)}$
- Solution for $r=-1$

  $\left[y(x)=\sum_{k=0}^{\infty} a_kx^{k-1},a_{k+2}=-\frac{4a_k}{(k+1)(k+2)},0=0\right]$
- Recursion relation for $r=0$

  $a_{k+2}=-\frac{4a_k}{(k+2)(k+3)}$
- Solution for $r=0$

  $\left[y(x)=\sum_{k=0}^{\infty} a_kx^{k},a_{k+2}=-\frac{4a_k}{(k+2)(k+3)},2a_1=0\right]$
- Combine solutions and rename parameters

  $\left[y(x)=\left(\sum_{k=0}^{\infty} a_kx^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_kx^{k}\right),a_{k+2}=-\frac{4a_k}{(k+1)(k+2)},0=0,b_{k+2}=-\frac{4b_k}{(k+2)(k+3)},2b_1=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 21

```
dsolve(x*diff(diff(y(x),x),x)+2*diff(y(x),x)+4*x*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_1 \sin(2x) + c_2 \cos(2x)}{x}$$

Mathematica DSolve solution

Solving time : 0.031 (sec)
Leaf size : 37

```
DSolve[{x*D[y[x],{x,2}]+2*D[y[x],x]+4*x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{4c_1 e^{-2ix} - ic_2 e^{2ix}}{4x}$$

2.1.771 Problem 793

Internal problem ID [9943]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 793
Date solved : `Monday, January 27, 2025 at 06:16:01 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + (2 - 2x)\, y' + (x - 2)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.062 (sec)

Writing the ode as

$$xy'' + (2 - 2x)\, y' + (x - 2)\, y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 - 2x \\ C &= x - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1465: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2-2x}{x}\, dx} \\ &= z_1 e^{x-\ln(x)} \\ &= z_1 \left(\frac{e^x}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^x}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2-2x}{x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x-2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{e^x}{x}\right) + c_2\left(\frac{e^x}{x}(x)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dx^2}y(x)\right)x + (-2x+2)\left(\frac{d}{dx}y(x)\right) + (x-2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(x-2)y(x)}{x} + \frac{2\left(\frac{d}{dx}y(x)\right)(x-1)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) - \frac{2\left(\frac{d}{dx}y(x)\right)(x-1)}{x} + \frac{(x-2)y(x)}{x} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - ◦ Define functions

    $\left[P_2(x) = -\frac{2(x-1)}{x}, P_3(x) = \frac{x-2}{x}\right]$

  - ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$

  - ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  - ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $\left(\frac{d^2}{dx^2}y(x)\right)x + (-2x+2)\left(\frac{d}{dx}y(x)\right) + (x-2)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $$x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..1$

  $$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $$x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

  $$x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(1+r)\, x^{-1+r} + \left(a_1(1+r)\,(2+r) - 2a_0(1+r)\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)\,(k+2+r) - 2a_k(k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$
- Each term must be 0

  $a_1(1+r)\,(2+r) - 2a_0(1+r) = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k+2+r) - 2a_k k - 2a_k r - 2a_k + a_{k-1} = 0$
- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)\,(k+3+r) - 2a_{k+1}(k+1) - 2r a_{k+1} - 2a_{k+1} + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2k a_{k+1} + 2r a_{k+1} - a_k + 4a_{k+1}}{(k+2+r)(k+3+r)}$
- Recursion relation for $r = -1$

  $a_{k+2} = \frac{2k a_{k+1} - a_k + 2a_{k+1}}{(k+1)(k+2)}$
- Solution for $r = -1$

  $$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = \frac{2k a_{k+1} - a_k + 2a_{k+1}}{(k+1)(k+2)}, 0 = 0\right]$$
- Recursion relation for $r = 0$

  $a_{k+2} = \frac{2k a_{k+1} - a_k + 4a_{k+1}}{(k+2)(k+3)}$
- Solution for $r = 0$

  $$\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2k a_{k+1} - a_k + 4a_{k+1}}{(k+2)(k+3)}, 2a_1 - 2a_0 = 0\right]$$
- Combine solutions and rename parameters

  $$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = \frac{2k a_{k+1} - a_k + 2a_{k+1}}{(k+1)(k+2)}, 0 = 0, b_{k+2} = \frac{2k b_{k+1} - b_k + 4b_{k+1}}{(k+2)(k+3)}, 2b_1 - 2b\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 15

```
dsolve(x*diff(diff(y(x),x),x)+(-2*x+2)*diff(y(x),x)+(x-2)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{e^{x}(c_1 x + c_2)}{x}$$

Mathematica DSolve solution

Solving time : 0.027 (sec)
Leaf size : 19

```
DSolve[{x*D[y[x],{x,2}]+(2-2*x)*D[y[x],x]+(x-2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to \frac{e^x(c_2 x + c_1)}{x}$$

2.1.772 Problem 794

Internal problem ID [9944]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 794
Date solved : `Monday, January 27, 2025 at 06:16:01 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + 6xy' + (4x^2 + 6)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.136 (sec)

Writing the ode as

$$x^2y'' + 6xy' + (4x^2 + 6)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 6x \\ C &= 4x^2 + 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1467: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{6x}{x^2}\, dx} \\ &= z_1 e^{-3\ln(x)} \\ &= z_1 \left(\frac{1}{x^3}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(2x)}{x^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{6x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-6\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\tan(2x)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(2x)}{x^3}\right) + c_2 \left(\frac{\cos(2x)}{x^3} \left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) + 6x\left(\frac{d}{dx}y(x)\right) + (4x^2+6)\, y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{2(2x^2+3)y(x)}{x^2} - \frac{6\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{6\left(\frac{d}{dx}y(x)\right)}{x} + \frac{2(2x^2+3)y(x)}{x^2} = 0$
- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{6}{x}, P_3(x) = \frac{2(2x^2+3)}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 6$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 6$
  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$
- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) + 6x\left(\frac{d}{dx}y(x)\right) + (4x^2+6)\, y(x) = 0$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

$x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(3+r)\,(2+r)\, x^r + a_1(4+r)\,(3+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+3)\,(k+r+2) + 4a_{k-2}\right) x^{k+r}\right) =$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(3+r)\,(2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-3, -2\}$

- Each term must be 0

  $a_1(4+r)\,(3+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+3)\,(k+r+2) + 4a_{k-2} = 0$

- Shift index using $k- >k+2$

  $a_{k+2}(k+5+r)\,(k+4+r) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{(k+5+r)(k+4+r)}$

- Recursion relation for $r = -3$

  $a_{k+2} = -\frac{4a_k}{(k+2)(k+1)}$

- Solution for $r = -3$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2} = -\frac{4a_k}{(k+2)(k+1)}, a_1 = 0\right]$

- Recursion relation for $r = -2$

  $a_{k+2} = -\frac{4a_k}{(k+3)(k+2)}$

- Solution for $r = -2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{4a_k}{(k+3)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-3}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-2}\right), a_{k+2} = -\frac{4a_k}{(k+1)(k+2)}, a_1 = 0, b_{k+2} = -\frac{4b_k}{(k+2)(k+3)}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 21

```
dsolve(x^2*diff(diff(y(x),x),x)+6*diff(y(x),x)*x+(4*x^2+6)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_1 \sin(2x) + c_2 \cos(2x)}{x^3}$$

Mathematica DSolve solution

Solving time : 0.032 (sec)
Leaf size : 37

```
DSolve[{x^2*D[y[x],{x,2}]+6*x*D[y[x],x]+(4*x^2+6)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->
```

$$y(x) \to \frac{4c_1 e^{-2ix} - ic_2 e^{2ix}}{4x^3}$$

2.1.773 Problem 795

Internal problem ID [9945]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 795
Date solved : `Monday, January 27, 2025 at 06:16:02 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + (1-2x)\,y' + (x-1)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.153 (sec)

Writing the ode as

$$xy'' + (1-2x)\,y' + (x-1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 1-2x \\ C &= x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1469: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=-\frac{1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2}-\left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^-}{x-c_1}\right)+(-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x}+(-)\,(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}
$$

Let

$$
p(x)=1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{1}{2x}\right)(0)+\left(\left(-\frac{1}{2x^2}\right)+\left(\frac{1}{2x}\right)^2-\left(-\frac{1}{4x^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}z_1(x)&=pe^{\int \omega\,dx}\\&=\mathrm{e}^{\int \frac{1}{2x}dx}\\&=\sqrt{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1&=z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\&=z_1e^{-\int \frac{1}{2}\frac{1-2x}{x}\,dx}\\&=z_1e^{x-\frac{\ln(x)}{2}}\\&=z_1\left(\frac{\mathrm{e}^x}{\sqrt{x}}\right)\end{aligned}$$

Which simplifies to

$$y_1=\mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2&=y_1\int \frac{e^{\int -\frac{1-2x}{x}\,dx}}{(y_1)^2}\,dx\\&=y_1\int \frac{e^{2x-\ln(x)}}{(y_1)^2}\,dx\\&=y_1(\ln(x))\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y&=c_1y_1+c_2y_2\\&=c_1(\mathrm{e}^x)+c_2(\mathrm{e}^x(\ln(x)))\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dx^2}y(x)\right)x+(-2x+1)\left(\frac{d}{dx}y(x)\right)+(x-1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
$\frac{d^2}{dx^2}y(x)=-\frac{(x-1)y(x)}{x}+\frac{(2x-1)\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
$\frac{d^2}{dx^2}y(x)-\frac{(2x-1)\left(\frac{d}{dx}y(x)\right)}{x}+\frac{(x-1)y(x)}{x}=0$

□ Check to see if $x_0=0$ is a regular singular point
  - Define functions
  $\left[P_2(x)=-\frac{2x-1}{x},P_3(x)=\frac{x-1}{x}\right]$
  - $x\cdot P_2(x)$ is analytic at $x=0$
  $\left.(x\cdot P_2(x))\right|_{x=0}=1$
  - $x^2\cdot P_3(x)$ is analytic at $x=0$
  $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$
  - $x=0$is a regular singular point
  Check to see if $x_0=0$ is a regular singular point
  $x_0=0$
- Multiply by denominators
$\left(\frac{d^2}{dx^2}y(x)\right)x+(-2x+1)\left(\frac{d}{dx}y(x)\right)+(x-1)\,y(x)=0$
- Assume series solution for $y(x)$
$y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions
  - Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$
  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$
  - Shift index using $k->k-m$
  $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$
  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..1$
  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k->k+1-m$
  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion
  $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-1}$
  - Shift index using $k->k+1$
  $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$
  Rewrite ODE with series expansions

$$a_0r^2x^{-1+r} + \left(a_1(1+r)^2 - a_0(1+2r)\right)x^r + \left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+1+r)^2 - a_k(2k+2r+1) + a_{k-1}\right)x^{k+r}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 0$
- Each term must be 0
  $a_1(1+r)^2 - a_0(1+2r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1)^2 + (-2k-1)\,a_k + a_{k-1} = 0$
- Shift index using $k- >k+1$
  $a_{k+2}(k+2)^2 + (-2k-3)\,a_{k+1} + a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{2ka_{k+1}-a_k+3a_{k+1}}{(k+2)^2}$
- Recursion relation for $r = 0$
  $a_{k+2} = \frac{2ka_{k+1}-a_k+3a_{k+1}}{(k+2)^2}$
- Solution for $r = 0$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2ka_{k+1}-a_k+3a_{k+1}}{(k+2)^2}, a_1 - a_0 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 13

```
dsolve(x*diff(diff(y(x),x),x)+(1-2*x)*diff(y(x),x)+(x-1)*y(x) = 0,y(x),singsol=all)
```

$$y = e^x(c_2 \ln(x) + c_1)$$

Mathematica DSolve solution

Solving time : 0.028 (sec)
Leaf size : 17

```
DSolve[{x*D[y[x],{x,2}]+(1-2*x)*D[y[x],x]+(x-1)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to e^x(c_2 \log(x) + c_1)$$

2.1.774 Problem 796

Internal problem ID [9946]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 796
Date solved : `Monday, January 27, 2025 at 06:16:02 PM`
CAS classification : `[_Jacobi]`

Solve

$$x(1-x)\,y''+\left(\frac{1}{2}+2x\right)y'-2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.253 (sec)

Writing the ode as

$$\left(-x^2+x\right)y''+\left(\frac{1}{2}+2x\right)y'-2y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=-x^2+x\\ B&=\frac{1}{2}+2x\\ C&=-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{48x-3}{16\left(x^2-x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=48x-3\\ t&=16\left(x^2-x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{48x - 3}{16\left(x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1471: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 3 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 3 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{45}{16\left(-1 + x\right)^2} + \frac{21}{8x} - \frac{21}{8\left(-1 + x\right)} - \frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{45}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{4}
\end{aligned}
$$

Since the order of r at ∞ is $3 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{48x - 3}{16\left(x^2 - x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
1	2	0	$\frac{9}{4}$	$-\frac{5}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
3	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= 0 - (-1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{4x} - \frac{5}{4(-1+x)} + (0) \\
&= \frac{1}{4x} - \frac{5}{4(-1+x)} \\
&= -\frac{4x+1}{4x(-1+x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{4x} - \frac{5}{4(-1+x)}\right)(1) + \left(\left(-\frac{1}{4x^2} + \frac{5}{4(-1+x)^2}\right) + \left(\frac{1}{4x} - \frac{5}{4(-1+x)}\right)^2 - \left(\frac{48x-3}{16(x^2-x)^2}\right)\right) = \frac{-1+4a_0}{2x(-1+x)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{1}{4}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{1}{4}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\,dx} \\
&= \left(x + \frac{1}{4}\right) e^{\int \left(\frac{1}{4x} - \frac{5}{4(-1+x)}\right)dx} \\
&= \left(x + \frac{1}{4}\right) e^{\frac{\ln(x)}{4} - \frac{5\ln(-1+x)}{4}} \\
&= \frac{\left(x + \frac{1}{4}\right) x^{1/4}}{(-1+x)^{5/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{\frac{1}{2}+2x}{-x^2+x}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{4} + \frac{5\ln(-1+x)}{4}} \\
&= z_1 \left(\frac{(-1+x)^{5/4}}{x^{1/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x + \frac{1}{4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{\frac{1}{2}+2x}{-x^2+x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x)}{2} + \frac{5\ln(-1+x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\sqrt{x}\,\sqrt{-1+x}\,\left(12\ln\left(-\frac{1}{2} + x + \sqrt{x(-1+x)}\right)x - 4\sqrt{x(-1+x)}\,x + 3\ln\left(-\frac{1}{2} + x + \sqrt{x(-1+x)}\right.\right.}{\sqrt{x(-1+x)}\,(4x+1)}\right.
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(x+\frac{1}{4}\right) + c_2\left(x \right. \\
&\quad +\frac{1}{4}\left(-\frac{\sqrt{x}\sqrt{-1+x}\left(12\ln\left(-\frac{1}{2}+x+\sqrt{x(-1+x)}\right)x - 4\sqrt{x(-1+x)}\,x + 3\ln\left(-\frac{1}{2}+x+\sqrt{x(-1\right.\right.}{\sqrt{x(-1+x)}\,(4x+1)}
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x(1-x)\left(\frac{d^2}{dx^2}y(x)\right) + \left(\frac{1}{2}+2x\right)\left(\frac{d}{dx}y(x)\right) - 2y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{2y(x)}{x(x-1)} + \frac{(4x+1)\left(\frac{d}{dx}y(x)\right)}{2x(x-1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) - \frac{(4x+1)\left(\frac{d}{dx}y(x)\right)}{2x(x-1)} + \frac{2y(x)}{x(x-1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{4x+1}{2x(x-1)}, P_3(x) = \frac{2}{x(x-1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = \frac{1}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $2x(x-1)\left(\frac{d^2}{dx^2}y(x)\right) + (-4x-1)\left(\frac{d}{dx}y(x)\right) + 4y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..1$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 1..2$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0r(-1+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)(2k+1+2r) + 2a_k(k+r-1)(k+r-2)\right) x^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-r(-1+2r)=0$
- Values of r that satisfy the indicial equation
  $r \in \left\{0, \frac{1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $-2(k+1+r)\left(k+r+\frac{1}{2}\right) a_{k+1} + 2a_k(k+r-1)(k+r-2) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{2a_k(k+r-1)(k+r-2)}{(k+1+r)(2k+1+2r)}$
- Recursion relation for $r=0$; series terminates at $k=1$
  $a_{k+1} = \frac{2a_k(k-1)(k-2)}{(k+1)(2k+1)}$
- Apply recursion relation for $k=0$
  $a_1 = 4a_0$
- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second linea
  $y(x) = a_0 \cdot (4x+1)$
- Recursion relation for $r=\frac{1}{2}$
  $a_{k+1} = \frac{2a_k\left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}$
- Solution for $r=\frac{1}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{2a_k\left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$
- Combine solutions and rename parameters
  $\left[y(x) = a_0 \cdot (4x+1) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), b_{k+1} = \frac{2b_k\left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 53

```
dsolve(x*(1-x)*diff(diff(y(x),x),x)+(1/2+2*x)*diff(y(x),x)-2*y(x) = 0,y(x),singsol=all
```

$$y = (-12x-3)\, c_2 \ln\left(2x-1+2\sqrt{(x-1)\, x}\right) + (4x+26)\, c_2 \sqrt{(x-1)\, x} + 4\left(x+\frac{1}{4}\right)\left(3c_2 \ln(2) + c_1\right)$$

Mathematica DSolve solution

Solving time : 0.528 (sec)
Leaf size : 130

```
DSolve[{x*(1-x)*D[y[x],{x,2}]+(1/2+2*x)*D[y[x],x]-2*y[x]==0,{}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to \frac{1}{4}(4x+1)\exp\left(\int_1^x \frac{4K[1]+1}{4K[1]-4K[1]^2}dK[1] - \frac{1}{2}\int_1^x \frac{4K[2]+1}{2K[2]-2K[2]^2}dK[2]\right)\left(c_2\int_1^x \frac{16\exp\left(-2\int_1^{K[3]}\frac{4K[1]+1}{4K[1]-4K[1]^2}dK[1]\right)}{(4K[3]+1)^2}dK[3] + c_1\right)$$

2.1.775 Problem 797

Internal problem ID [9947]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 797
Date solved : `Monday, January 27, 2025 at 06:16:03 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4\left(t^2-3t+2\right)y''-2y'+y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.241 (sec)

Writing the ode as

$$\left(4t^2-12t+8\right)y''-2y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=4t^2-12t+8\\B&=-2\\C&=1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-4t^2+20t-19}{16\left(t^2-3t+2\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=-4t^2+20t-19\\t&=16\left(t^2-3t+2\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(t)=\left(\frac{-4t^2+20t-19}{16\left(t^2-3t+2\right)^2}\right)z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1473: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(t^2 - 3t + 2)^2$. There is a pole at $t = 2$ of order 2. There is a pole at $t = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16\,(t-2)^2} + \frac{3}{8\,(t-1)} - \frac{3}{8\,(t-2)} - \frac{3}{16\,(t-1)^2}$$

For the pole at $t = 2$ let b be the coefficient of $\frac{1}{(t-2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

For the pole at $t = 1$ let b be the coefficient of $\frac{1}{(t-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-4t^2 + 20t - 19}{16\left(t^2 - 3t + 2\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-4t^2 + 20t - 19}{16\left(t^2 - 3t + 2\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
2	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
1	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{t-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{4(t-2)} + \frac{3}{4(t-1)} + (-)(0) \\
&= -\frac{1}{4(t-2)} + \frac{3}{4(t-1)} \\
&= \frac{2t-5}{4(t-1)(t-2)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{4(t-2)} + \frac{3}{4(t-1)}\right)(0) + \left(\left(\frac{1}{4(t-2)^2} - \frac{3}{4(t-1)^2}\right) + \left(-\frac{1}{4(t-2)} + \frac{3}{4(t-1)}\right)^2 - \left(\frac{-4\ldots}{16(\ldots}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega\, dt} \\ &= e^{\int \left(-\frac{1}{4(t-2)} + \frac{3}{4(t-1)}\right) dt} \\ &= \frac{(t-1)^{3/4}}{(t-2)^{1/4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{4t^2-12t+8}\, dt} \\ &= z_1 e^{-\frac{\ln(t-1)}{4} + \frac{\ln(t-2)}{4}} \\ &= z_1 \left(\frac{(t-2)^{1/4}}{(t-1)^{1/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{t-1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{4t^2-12t+8}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{-\frac{\ln(t-1)}{2} + \frac{\ln(t-2)}{2}}}{(y_1)^2}\, dt \\ &= y_1 \left(-\frac{2\sqrt{t-2}}{\sqrt{t-1}} + \frac{\ln\left(-\frac{3}{2} + t + \sqrt{t^2-3t+2}\right)\sqrt{(t-1)(t-2)}}{\sqrt{t-2}\,\sqrt{t-1}}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\sqrt{t-1}\right) + c_2\left(\sqrt{t-1}\left(-\frac{2\sqrt{t-2}}{\sqrt{t-1}} + \frac{\ln\left(-\frac{3}{2} + t + \sqrt{t^2-3t+2}\right)\sqrt{(t-1)(t-2)}}{\sqrt{t-2}\,\sqrt{t-1}}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4(t^2-3t+2)\left(\frac{d^2}{dt^2}y(t)\right)-2\frac{d}{dt}y(t)+y(t)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dt^2}y(t)$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t)=-\frac{y(t)}{4(t^2-3t+2)}+\frac{\frac{d}{dt}y(t)}{2(t^2-3t+2)}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d^2}{dt^2}y(t)-\frac{\frac{d}{dt}y(t)}{2(t^2-3t+2)}+\frac{y(t)}{4(t^2-3t+2)}=0$

□ Check to see if t_0 is a regular singular point

- ○ Define functions

  $\left[P_2(t)=-\frac{1}{2(t^2-3t+2)},P_3(t)=\frac{1}{4(t^2-3t+2)}\right]$

- ○ $(t-1)\cdot P_2(t)$ is analytic at $t=1$

  $\left((t-1)\cdot P_2(t)\right)\bigg|_{t=1}=\frac{1}{2}$

- ○ $(t-1)^2\cdot P_3(t)$ is analytic at $t=1$

  $\left((t-1)^2\cdot P_3(t)\right)\bigg|_{t=1}=0$

- ○ $t=1$is a regular singular point

  Check to see if t_0 is a regular singular point

  $t_0=1$

- Multiply by denominators

  $(4t^2-12t+8)\left(\frac{d^2}{dt^2}y(t)\right)-2\frac{d}{dt}y(t)+y(t)=0$

- Change variables using $t=u+1$ so that the regular singular point is at $u=0$

  $(4u^2-4u)\left(\frac{d^2}{du^2}y(u)\right)-2\frac{d}{du}y(u)+y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $\frac{d}{du}y(u)$ to series expansion

  $\frac{d}{du}y(u)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1}$

- ○ Shift index using $k->k+1$

  $\frac{d}{du}y(u)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,u^{k+r}$

- ○ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

- ○ Shift index using $k->k+2-m$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r(-1+2r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)(2k+1+2r)+a_k(2k+2r-1)^2\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2r(-1+2r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,\frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(2k+2r-1)^2-4(k+1+r)\,a_{k+1}\left(k+r+\frac{1}{2}\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{a_k(2k+2r-1)^2}{2(k+1+r)(2k+1+2r)}$$

- Recursion relation for $r=0$

$$a_{k+1}=\frac{a_k(2k-1)^2}{2(k+1)(2k+1)}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty}a_k u^k, a_{k+1}=\frac{a_k(2k-1)^2}{2(k+1)(2k+1)}\right]$$

- Revert the change of variables $u=t-1$

$$\left[y(t)=\sum_{k=0}^{\infty}a_k(t-1)^k, a_{k+1}=\frac{a_k(2k-1)^2}{2(k+1)(2k+1)}\right]$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+1}=\frac{2a_k k^2}{\left(k+\frac{3}{2}\right)(2k+2)}$$

- Solution for $r=\frac{1}{2}$

$$\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+\frac{1}{2}}, a_{k+1}=\frac{2a_k k^2}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$

- Revert the change of variables $u=t-1$

$$\left[y(t)=\sum_{k=0}^{\infty}a_k(t-1)^{k+\frac{1}{2}}, a_{k+1}=\frac{2a_k k^2}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y(t)=\left(\sum_{k=0}^{\infty}a_k(t-1)^k\right)+\left(\sum_{k=0}^{\infty}b_k(t-1)^{k+\frac{1}{2}}\right), a_{k+1}=\frac{a_k(2k-1)^2}{2(k+1)(2k+1)}, b_{k+1}=\frac{2b_k k^2}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.019 (sec)
Leaf size : 56

```
dsolve(4*(t^2-3*t+2)*diff(diff(y(t),t),t)-2*diff(y(t),t)+y(t) = 0,y(t),singsol=all)
```

$$y = c_1\sqrt{t-1} + \frac{c_2\left(\frac{\sqrt{t^2-3t+2}\left(\ln(2)-\ln\left(-3+2t+2\sqrt{(t-1)(t-2)}\right)\right)}{2}+t-2\right)}{\sqrt{t-2}}$$

Mathematica DSolve solution

Solving time : 0.178 (sec)
Leaf size : 112

```
DSolve[{4*(t^2-3*t+2)*D[y[t],{t,2}]-2*D[y[t],t]+y[t]==0,{}},y[t],t,IncludeSingularSolutions->T
```

$$y(t) \to \exp\left(\int_1^t \frac{2K[1]-5}{4\left(K[1]^2-3K[1]+2\right)}dK[1] - \frac{1}{2}\int_1^t -\frac{1}{2\left(K[2]^2-3K[2]+2\right)}dK[2]\right)\left(c_2\int_1^t \exp\left(-2\int_1^{K[3]} \frac{2K[1]-5}{4\left(K[1]^2-3K[1]+2\right)}dK[1]\right)dK[3] + c_1\right)$$

2.1.776 Problem 798

Internal problem ID [9948]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 798
Date solved : `Monday, January 27, 2025 at 06:16:04 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2\left(t^2-5t+6\right)y''+(2t-3)\,y'-8y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.240 (sec)

Writing the ode as

$$\left(2t^2-10t+12\right)y''+(2t-3)\,y'-8y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=2t^2-10t+12\\ B&=2t-3\\ C&=-8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{60t^2-308t+381}{16\left(t^2-5t+6\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=60t^2-308t+381\\ t&=16\left(t^2-5t+6\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t)=\left(\frac{60t^2-308t+381}{16\left(t^2-5t+6\right)^2}\right)z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1475: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(t^2 - 5t + 6)^2$. There is a pole at $t = 3$ of order 2. There is a pole at $t = 2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{29}{8(t-2)} + \frac{29}{8(t-3)} + \frac{5}{16(t-2)^2} - \frac{3}{16(t-3)^2}$$

For the pole at $t = 3$ let b be the coefficient of $\frac{1}{(t-3)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $t = 2$ let b be the coefficient of $\frac{1}{(t-2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{60t^2 - 308t + 381}{16\,(t^2 - 5t + 6)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{60t^2 - 308t + 381}{16\,(t^2 - 5t + 6)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
3	2	0	$\frac{3}{4}$	$\frac{1}{4}$
2	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= \frac{5}{2} - \left(\frac{3}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{t-c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{4t-12} + \frac{5}{4\,(t-2)} + (0) \\
&= \frac{1}{4t-12} + \frac{5}{4\,(t-2)} \\
&= \frac{6t-17}{4\,(t-2)\,(t-3)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 1$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{4t-12} + \frac{5}{4(t-2)}\right)(1) + \left(\left(-\frac{1}{4(t-3)^2} - \frac{5}{4(t-2)^2}\right) + \left(\frac{1}{4t-12} + \frac{5}{4(t-2)}\right)^2 - \left(\frac{60t^2 - 308}{16(t^2 - 5}\right.\right.$$

$$\frac{-6a_0}{2t^2 - 1(}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{17}{6}\right\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = t - \frac{17}{6}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \left(t - \frac{17}{6}\right) e^{\int \left(\frac{1}{4t-12} + \frac{5}{4(t-2)}\right) dt} \\
&= \left(t - \frac{17}{6}\right) e^{\frac{5\ln(t-2)}{4} + \frac{\ln(t-3)}{4}} \\
&= \left(t - \frac{17}{6}\right) (t-2)^{5/4} (t-3)^{1/4}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2t-3}{2t^2-10t+12} \, dt} \\
&= z_1 e^{\frac{\ln(t-2)}{4} - \frac{3\ln(t-3)}{4}} \\
&= z_1 \left(\frac{(t-2)^{1/4}}{(t-3)^{3/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(t-2)^{3/2}(6t-17)}{6\sqrt{t-3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2t-3}{2t^2-10t+12} \, dt}}{(y_1)^2} \, dt \\
&= y_1 \int \frac{e^{\frac{\ln(t-2)}{2} - \frac{3\ln(t-3)}{2}}}{(y_1)^2} \, dt \\
&= y_1 \left(\frac{24(t-3)^2 (24t^2 - 104t + 111) e^{\frac{\ln(t-2)}{2} - \frac{3\ln(t-3)}{2}}}{5(6t-17)(t-2)^2}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{(t-2)^{3/2}(6t-17)}{6\sqrt{t-3}}\right) \\
&\quad + c_2\left(\frac{(t-2)^{3/2}(6t-17)}{6\sqrt{t-3}}\left(\frac{24(t-3)^2(24t^2-104t+111)\,\mathrm{e}^{\frac{\ln(t-2)}{2}-\frac{3\ln(t-3)}{2}}}{5(6t-17)(t-2)^2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$2(t^2-5t+6)\left(\frac{d^2}{dt^2}y(t)\right)+(2t-3)\left(\frac{d}{dt}y(t)\right)-8y(t)=0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dt^2}y(t)$
- Isolate 2nd derivative
  $\frac{d^2}{dt^2}y(t)=\frac{4y(t)}{t^2-5t+6}-\frac{(2t-3)\left(\frac{d}{dt}y(t)\right)}{2(t^2-5t+6)}$
- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dt^2}y(t)+\frac{(2t-3)\left(\frac{d}{dt}y(t)\right)}{2(t^2-5t+6)}-\frac{4y(t)}{t^2-5t+6}=0$

□ Check to see if t_0 is a regular singular point
  - Define functions
    $\left[P_2(t)=\frac{2t-3}{2(t^2-5t+6)}, P_3(t)=-\frac{4}{t^2-5t+6}\right]$
  - $(t-2)\cdot P_2(t)$ is analytic at $t=2$
    $\left((t-2)\cdot P_2(t)\right)\Big|_{t=2}=-\frac{1}{2}$
  - $(t-2)^2\cdot P_3(t)$ is analytic at $t=2$
    $\left((t-2)^2\cdot P_3(t)\right)\Big|_{t=2}=0$
  - $t=2$is a regular singular point
    Check to see if t_0 is a regular singular point
    $t_0=2$
- Multiply by denominators
  $(2t^2-10t+12)\left(\frac{d^2}{dt^2}y(t)\right)+(2t-3)\left(\frac{d}{dt}y(t)\right)-8y(t)=0$
- Change variables using $t=u+2$ so that the regular singular point is at $u=0$
  $(2u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(2u+1)\left(\frac{d}{du}y(u)\right)-8y(u)=0$
- Assume series solution for $y(u)$
  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$
    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$

$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-a_0 r(-3+2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(2k-1+2r) + 2a_k(k+r+2)\,(k+r-2)\right) u^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $-2(k+1+r)\left(k+r-\frac{1}{2}\right) a_{k+1} + 2a_k(k+r+2)\,(k+r-2) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k(k+r+2)(k+r-2)}{(k+1+r)(2k-1+2r)}$

- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+1} = \frac{2a_k(k+2)(k-2)}{(k+1)(2k-1)}$

- Apply recursion relation for $k = 0$

  $a_1 = 8a_0$

- Apply recursion relation for $k = 1$

  $a_2 = -3a_1$

- Express in terms of a_0

  $a_2 = -24a_0$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second linea

  $y(u) = a_0 \cdot (-24u^2 + 8u + 1)$

- Revert the change of variables $u = t-2$

  $[y(t) = a_0(-24t^2 + 104t - 111)]$

- Recursion relation for $r = \frac{3}{2}$

  $a_{k+1} = \frac{2a_k\left(k+\frac{7}{2}\right)\left(k-\frac{1}{2}\right)}{\left(k+\frac{5}{2}\right)(2k+2)}$

- Solution for $r = \frac{3}{2}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{3}{2}}, a_{k+1} = \frac{2a_k\left(k+\frac{7}{2}\right)\left(k-\frac{1}{2}\right)}{\left(k+\frac{5}{2}\right)(2k+2)}\right]$

- Revert the change of variables $u = t-2$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k(t-2)^{k+\frac{3}{2}}, a_{k+1} = \frac{2a_k\left(k+\frac{7}{2}\right)\left(k-\frac{1}{2}\right)}{\left(k+\frac{5}{2}\right)(2k+2)}\right]$

- Combine solutions and rename parameters

  $\left[y(t) = a_0(-24t^2 + 104t - 111) + \left(\sum_{k=0}^{\infty} b_k(t-2)^{k+\frac{3}{2}}\right), b_{k+1} = \frac{2b_k\left(k+\frac{7}{2}\right)\left(k-\frac{1}{2}\right)}{\left(k+\frac{5}{2}\right)(2k+2)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.026 (sec)
Leaf size : 35

```
dsolve(2*(t^2-5*t+6)*diff(diff(y(t),t),t)+(2*t-3)*diff(y(t),t)-8*y(t) = 0,y(t),singsol
```

$$y = \frac{c_1(24t^2 - 104t + 111)}{24} + \frac{c_2(6t - 17)(t - 2)^{3/2}}{\sqrt{t - 3}}$$

Mathematica DSolve solution

Solving time : 0.524 (sec)
Leaf size : 130

```
DSolve[{2*(t^2-5*t+6)*D[y[t],{t,2}]+(2*t-3)*D[y[t],t]-8*y[t]==0,{}},y[t],t,IncludeSingularSo
```

$$y(t) \to \frac{1}{6}(6t-17)\exp\left(\int_1^t \frac{1}{4}\left(\frac{5}{K[1]-2} + \frac{1}{K[1]-3}\right)dK[1] - \frac{1}{2}\int_1^t \frac{2K[2]-3}{2(K[2]^2 - 5K[2] + 6)}dK[2]\right)\left(c_2\int_1^t \frac{36\exp\left(-2\int_1^{K[3]} \frac{1}{4}\left(\frac{5}{K[1]-2} + \frac{1}{K[1]-3}\right)dK[1]\right)}{(17-6K[3])^2}dK[3] + c_1\right)$$

2.1.777 Problem 799

Internal problem ID [9949]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 799
Date solved : `Monday, January 27, 2025 at 06:16:04 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$3t(1+t)\, y''+t y'-y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.289 (sec)

Writing the ode as

$$\left(3t^2+3t\right) y''+t y'-y=0 \tag{1}$$
$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3t^2+3t\\ B &= t \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=y e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t)=r z(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{7t+12}{36t\left(1+t\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 7t+12\\ t &= 36t(1+t)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t)=\left(\frac{7t+12}{36t\left(1+t\right)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1477: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36t(1+t)^2$. There is a pole at $t = 0$ of order 1. There is a pole at $t = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $t = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{3t} - \frac{5}{36\,(1+t)^2} - \frac{1}{3\,(1+t)}$$

For the pole at $t = -1$ let b be the coefficient of $\frac{1}{(1+t)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{7t + 12}{36t\left(1 + t\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{7}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{6} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{6}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{7t + 12}{36t\left(1 + t\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1
-1	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{6}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{7}{6} - \left(\frac{7}{6}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{t - c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{t} + \frac{1}{6 + 6t} + (0) \\
&= \frac{1}{t} + \frac{1}{6 + 6t} \\
&= \frac{1}{t} + \frac{1}{6 + 6t}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{t} + \frac{1}{6+6t}\right)(0) + \left(\left(-\frac{1}{t^2} - \frac{1}{6(1+t)^2}\right) + \left(\frac{1}{t} + \frac{1}{6+6t}\right)^2 - \left(\frac{7t+12}{36t(1+t)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= \mathrm{e}^{\int \left(\frac{1}{t} + \frac{1}{6+6t}\right) dt} \\ &= (1+t)^{1/6}\, t \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{t}{3t^2+3t}\, dt} \\ &= z_1 e^{-\frac{\ln(1+t)}{6}} \\ &= z_1 \left(\frac{1}{(1+t)^{1/6}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{t}{3t^2+3t}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{-\frac{\ln(1+t)}{3}}}{(y_1)^2}\, dt \\ &= y_1 \left(\frac{-2(1+t)^{1/3} - 1}{3(1+t)^{2/3} + 3(1+t)^{1/3} + 3} + \frac{\ln\left((1+t)^{2/3} + (1+t)^{1/3} + 1\right)}{6} \right. \\ &\quad \left. - \frac{\sqrt{3}\arctan\left(\frac{\left(1+2(1+t)^{1/3}\right)\sqrt{3}}{3}\right)}{3} - \frac{1}{3\left((1+t)^{1/3} - 1\right)} - \frac{\ln\left((1+t)^{1/3} - 1\right)}{3}\right) \end{aligned}$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1(t) + c_2\left(t\left(\frac{-2(1+t)^{1/3}-1}{3(1+t)^{2/3}+3(1+t)^{1/3}+3} + \frac{\ln\left((1+t)^{2/3}+(1+t)^{1/3}+1\right)}{6} - \frac{\sqrt{3}\arctan\left(\frac{\left(1+2(1+t)^{1/3}\right)\sqrt{3}}{3}\right)}{3} - \frac{1}{3\left((1+t)^{1/3}-1\right)} - \frac{\ln\left((1+t)^{1/3}-1\right)}{3}\right)\right)$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$3t(t+1)\left(\frac{d^2}{dt^2}y(t)\right) + t\left(\frac{d}{dt}y(t)\right) - y(t) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dt^2}y(t)$
- Isolate 2nd derivative
  $\frac{d^2}{dt^2}y(t) = \frac{y(t)}{3t(t+1)} - \frac{\frac{d}{dt}y(t)}{3(t+1)}$
- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d^2}{dt^2}y(t) + \frac{\frac{d}{dt}y(t)}{3(t+1)} - \frac{y(t)}{3t(t+1)} = 0$
- □ Check to see if t_0 is a regular singular point
  - Define functions
    $\left[P_2(t) = \frac{1}{3(t+1)}, P_3(t) = -\frac{1}{3t(t+1)}\right]$
  - $(t+1)\cdot P_2(t)$ is analytic at $t=-1$
    $\left((t+1)\cdot P_2(t)\right)\bigg|_{t=-1} = \frac{1}{3}$
  - $(t+1)^2\cdot P_3(t)$ is analytic at $t=-1$
    $\left((t+1)^2\cdot P_3(t)\right)\bigg|_{t=-1} = 0$
  - $t=-1$is a regular singular point
  Check to see if t_0 is a regular singular point
  $t_0 = -1$
- Multiply by denominators
  $3t(t+1)\left(\frac{d^2}{dt^2}y(t)\right) + t\left(\frac{d}{dt}y(t)\right) - y(t) = 0$
- Change variables using $t = u-1$ so that the regular singular point is at $u=0$
  $(3u^2-3u)\left(\frac{d^2}{du^2}y(u)\right) + (u-1)\left(\frac{d}{du}y(u)\right) - y(u) = 0$
- Assume series solution for $y(u)$
  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$
- □ Rewrite ODE with series expansions
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$
    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k- >k+1-m$

$u^{m}\cdot \left(\frac{d}{d u}y \left(u \right)\right)=\sum_{k =-1+m}^{\infty} a_{k +1-m} \left(k +1-m +r \right) u^{k +r}$

- Convert $u^{m}\cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right)$ to series expansion for $m =1..2$

  $u^{m}\cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right)=\sum_{k =0}^{\infty} a_{k} \left(k +r \right) \left(k +r -1\right) u^{k +r -2+m}$

- Shift index using $k->k +2-m$

  $u^{m}\cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right)=\sum_{k =-2+m}^{\infty} a_{k +2-m} \left(k +2-m +r \right) \left(k +1-m +r \right) u^{k +r}$

  Rewrite ODE with series expansions

  $-a_{0} r \left(-2+3 r \right) u^{-1+r}+\left(\sum_{k =0}^{\infty} \left(-a_{k +1} \left(k +1+r \right) \left(3 k +3 r +1\right)+a_{k} \left(3 k +3 r +1\right) \left(k +r -1\right)\right) u^{k-}$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $-r \left(-2+3 r \right)=0$
- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{2}{3}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $3 \left(k +r +\frac{1}{3}\right) \left(\left(-k -r -1\right) a_{k +1}+a_{k} \left(k +r -1\right)\right)=0$
- Recursion relation that defines series solution to ODE

  $a_{k +1}=\frac{a_{k} \left(k +r -1\right)}{k +1+r}$
- Recursion relation for $r =0$; series terminates at $k =1$

  $a_{k +1}=\frac{a_{k} \left(k -1\right)}{k +1}$
- Apply recursion relation for $k =0$

  $a_{1}=-a_{0}$
- Terminating series solution of the ODE for $r =0$. Use reduction of order to find the second li

  $y \left(u \right)=a_{0}\cdot \left(-u +1\right)$
- Revert the change of variables $u =t +1$

  $\left[y \left(t \right)=-a_{0} t \right]$
- Recursion relation for $r =\frac{2}{3}$

  $a_{k +1}=\frac{a_{k} \left(k -\frac{1}{3}\right)}{k +\frac{5}{3}}$
- Solution for $r =\frac{2}{3}$

  $\left[y \left(u \right)=\sum_{k =0}^{\infty} a_{k} u^{k +\frac{2}{3}}, a_{k +1}=\frac{a_{k} \left(k -\frac{1}{3}\right)}{k +\frac{5}{3}}\right]$
- Revert the change of variables $u =t +1$

  $\left[y \left(t \right)=\sum_{k =0}^{\infty} a_{k} \left(t +1\right)^{k +\frac{2}{3}}, a_{k +1}=\frac{a_{k} \left(k -\frac{1}{3}\right)}{k +\frac{5}{3}}\right]$
- Combine solutions and rename parameters

  $\left[y \left(t \right)=-a_{0} t +\left(\sum_{k =0}^{\infty} b_{k} \left(t +1\right)^{k +\frac{2}{3}}\right), b_{k +1}=\frac{b_{k} \left(k -\frac{1}{3}\right)}{k +\frac{5}{3}}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returnin
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.045 (sec)
Leaf size : 67

```
dsolve(3*t*(t+1)*diff(diff(y(t),t),t)+t*diff(y(t),t)-y(t) = 0,y(t),singsol=all)
```

$$y = c_1 t + 2\sqrt{3}\arctan\left(\frac{\left(2(t+1)^{1/3}+1\right)\sqrt{3}}{3}\right) t c_2 + 2\ln\left((t+1)^{1/3}-1\right) t c_2 + 6(t+1)^{2/3} c_2 - \ln\left((t+1)^{2/3}+(t+1)^{1/3}+1\right) t c_2$$

Mathematica DSolve solution

Solving time : 0.445 (sec)
Leaf size : 78

```
DSolve[{3*t*(1+t)*D[y[t],{t,2}]+t*D[y[t],t]-y[t]==0,{}},y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{\exp\left(\int_1^t \left(\frac{1}{6K[1]+6}+\frac{1}{K[1]}\right) dK[1]\right)\left(c_2 \int_1^t \exp\left(-2\int_1^{K[2]}\left(\frac{1}{6K[1]+6}+\frac{1}{K[1]}\right) dK[1]\right) dK[2] + c_1\right)}{\sqrt[6]{3}\sqrt[6]{t+1}}$$

2.1.778 Problem 800

Internal problem ID [9950]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 800
Date solved : `Monday, January 27, 2025 at 06:16:05 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2 y'' + \frac{\left(x + \frac{3}{4}\right) y}{4} = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.194 (sec)

Writing the ode as

$$x^2 y'' + \left(\frac{x}{4} + \frac{3}{16}\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 0 \\ C &= \frac{x}{4} + \frac{3}{16} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4x - 3}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4x - 3 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-4x-3}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1479: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x} - \frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{1,2,3\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1,2,3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1,\ e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(1-(1)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{1}{(x-(0))}\right) \\ &= \frac{1}{2x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d=0$ such that

$$p''' + 3\theta p'' + (3\theta^2 + 3\theta' - 4r)\,p' + (\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r')\,p = 0 \tag{1A}$$

Since $d=0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2x} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2x} + \frac{1+4x}{16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{1+2\sqrt{-x}}{4x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= e^{\int \omega \, dx} \\
&= e^{\int \frac{1+2\sqrt{-x}}{4x} dx} \\
&= x^{1/4} e^{\sqrt{-x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= x^{1/4} e^{\sqrt{-x}}
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^{1/4} e^{\sqrt{-x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\
&= x^{1/4} e^{\sqrt{-x}} \int \frac{1}{\sqrt{x}\, e^{2\sqrt{-x}}} \, dx \\
&= x^{1/4} e^{\sqrt{-x}} \left(-\frac{\sqrt{-x}\left(1 - e^{-2\sqrt{-x}}\right)}{\sqrt{x}} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{1/4} e^{\sqrt{-x}} \right) + c_2 \left(x^{1/4} e^{\sqrt{-x}} \left(-\frac{\sqrt{-x}\left(1 - e^{-2\sqrt{-x}}\right)}{\sqrt{x}} \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2 \left(\frac{d^2}{dx^2} y(x) \right) + \frac{\left(\frac{3}{4}+x\right) y(x)}{4} = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2} y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2} y(x) = -\frac{(3+4x) y(x)}{16x^2}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x) + \frac{(3+4x)y(x)}{16x^2} = 0$
- □ Check to see if $x_0 = 0$ is a regular singular point
  - ◦ Define functions
    $\left[P_2(x) = 0, P_3(x) = \frac{3+4x}{16x^2}\right]$
  - ◦ $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 0$
  - ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{3}{16}$
  - ◦ $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $16x^2\left(\frac{d^2}{dx^2}y(x)\right) + (3 + 4x)\,y(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- □ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$
    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - ◦ Shift index using $k- >k - m$
    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - ◦ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion
    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\,x^{k+r}$
    Rewrite ODE with series expansions
    $a_0(-1 + 4r)\,(-3 + 4r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(4k + 4r - 1)\,(4k + 4r - 3) + 4a_{k-1}\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1 + 4r)\,(-3 + 4r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{\frac{1}{4}, \frac{3}{4}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $16\left(k - \frac{3}{4} + r\right)\left(k + r - \frac{1}{4}\right) a_k + 4a_{k-1} = 0$
- Shift index using $k- >k + 1$
  $16\left(k + \frac{1}{4} + r\right)\left(k + \frac{3}{4} + r\right) a_{k+1} + 4a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = -\frac{4a_k}{(4k+1+4r)(4k+3+4r)}$
- Recursion relation for $r = \frac{1}{4}$
  $a_{k+1} = -\frac{4a_k}{(4k+2)(4k+4)}$
- Solution for $r = \frac{1}{4}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+1} = -\frac{4a_k}{(4k+2)(4k+4)}\right]$

- Recursion relation for $r = \frac{3}{4}$

  $a_{k+1} = -\frac{4a_k}{(4k+4)(4k+6)}$
- Solution for $r = \frac{3}{4}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{4}}, a_{k+1} = -\frac{4a_k}{(4k+4)(4k+6)}\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{4}}\right), a_{k+1} = -\frac{4a_k}{(4k+2)(4k+4)}, b_{k+1} = -\frac{4b_k}{(4k+4)(4k+6)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 21

```
dsolve(x^2*diff(diff(y(x),x),x)+1/4*(x+3/4)*y(x) = 0,y(x),singsol=all)
```

$$y = x^{1/4}\left(c_1 \sin\left(\sqrt{x}\right) + c_2 \cos\left(\sqrt{x}\right)\right)$$

Mathematica DSolve solution

Solving time : 0.046 (sec)
Leaf size : 43

```
DSolve[{x^2*D[y[x],{x,2}]+1/4*(x+3/4)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-i\sqrt{x}}\sqrt[4]{x}\left(c_1 e^{2i\sqrt{x}} + ic_2\right)$$

2.1.779 Problem 801

Internal problem ID [9951]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 801
Date solved : `Monday, January 27, 2025 at 06:16:06 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \frac{(x^2-1)\,y}{4} = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.151 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(\frac{x^2}{4} - \frac{1}{4}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= \frac{x^2}{4} - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1481: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos\left(\frac{x}{2}\right)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(2\tan\left(\frac{x}{2}\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\cos\left(\frac{x}{2}\right)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos\left(\frac{x}{2}\right)}{\sqrt{x}} \left(2\tan\left(\frac{x}{2}\right)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x\left(\frac{d}{dx}y(x)\right) + \frac{(x^2-1)y(x)}{4} = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{(x^2-1)y(x)}{4x^2} - \frac{\frac{d}{dx}y(x)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x) + \frac{\frac{d}{dx}y(x)}{x} + \frac{(x^2-1)y(x)}{4x^2} = 0$
- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{x^2-1}{4x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x\left(\frac{d}{dx}y(x)\right) + (x^2-1)\,y(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

$x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

$x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

$x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

$x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) + a_{k-2})\, x^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(1+2r)\,(-1+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0
  $a_1(3+2r)\,(1+2r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_k(4k^2 + 8kr + 4r^2 - 1) + a_{k-2} = 0$
- Shift index using $k- >k + 2$
  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{a_k}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r = -\frac{1}{2}$
  $a_{k+2} = -\frac{a_k}{4k^2+12k+8}$
- Solution for $r = -\frac{1}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{a_k}{4k^2+12k+8}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{2}$
  $a_{k+2} = -\frac{a_k}{4k^2+20k+24}$
- Solution for $r = \frac{1}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{a_k}{4k^2+20k+24}, a_1 = 0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.039 (sec)
Leaf size : 21

```
dsolve(x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x+1/4*(x^2-1)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_1 \sin\left(\frac{x}{2}\right) + c_2 \cos\left(\frac{x}{2}\right)}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.035 (sec)
Leaf size : 36

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+1/4*(x^2-1)*y[x]==0,{}},y[x],x,IncludeSingularSolution
```

$$y(x) \to \frac{e^{-\frac{ix}{2}}\left(c_1 - ic_2 e^{ix}\right)}{\sqrt{x}}$$

2.1.780 Problem 802

Internal problem ID [9952]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 802
Date solved : `Monday, January 27, 2025 at 06:16:06 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + (1-2x)\,y' + (x-1)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.146 (sec)

Writing the ode as

$$xy'' + (1-2x)\,y' + (x-1)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 1-2x \\ C &= x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1483: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=-\frac{1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{2}-\left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^{-}}{x-c_1}\right)+(-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x}+(-)\,(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}
$$

Let

$$
p(x)=1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0)+2\left(\frac{1}{2x}\right)(0)+\left(\left(-\frac{1}{2x^2}\right)+\left(\frac{1}{2x}\right)^2-\left(-\frac{1}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \frac{1}{2x} dx} \\
&= \sqrt{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1-2x}{x} dx} \\
&= z_1 e^{x-\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{\mathrm{e}^x}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1-2x}{x} dx}}{(y_1)^2} dx \\
&= y_1 \int \frac{e^{2x-\ln(x)}}{(y_1)^2} dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(\ln(x)))
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dx^2}y(x)\right)x+(-2x+1)\left(\frac{d}{dx}y(x)\right)+(x-1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
$\frac{d^2}{dx^2}y(x)=-\frac{(x-1)y(x)}{x}+\frac{(2x-1)\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
$\frac{d^2}{dx^2}y(x)-\frac{(2x-1)\left(\frac{d}{dx}y(x)\right)}{x}+\frac{(x-1)y(x)}{x}=0$

□ Check to see if $x_0=0$ is a regular singular point
  - Define functions
  $\left[P_2(x)=-\frac{2x-1}{x},P_3(x)=\frac{x-1}{x}\right]$
  - $x\cdot P_2(x)$ is analytic at $x=0$
  $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=1$
  - $x^2\cdot P_3(x)$ is analytic at $x=0$
  $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=0$
  - $x=0$is a regular singular point
  Check to see if $x_0=0$ is a regular singular point
  $x_0=0$
- Multiply by denominators
$\left(\frac{d^2}{dx^2}y(x)\right)x+(-2x+1)\left(\frac{d}{dx}y(x)\right)+(x-1)\,y(x)=0$
- Assume series solution for $y(x)$
$y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions
  - Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$
  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$
  - Shift index using $k->k-m$
  $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$
  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=0..1$
  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k->k+1-m$
  $x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion
  $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-1}$
  - Shift index using $k->k+1$
  $x\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$
  Rewrite ODE with series expansions

$$a_0r^2x^{-1+r}+\left(a_1(1+r)^2-a_0(1+2r)\right)x^r+\left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+1+r)^2-a_k(2k+2r+1)+a_{k-1}\right)x^k\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r^2=0$
- Values of r that satisfy the indicial equation
  $r=0$
- Each term must be 0
  $a_1(1+r)^2-a_0(1+2r)=0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1)^2+(-2k-1)\,a_k+a_{k-1}=0$
- Shift index using $k->k+1$
  $a_{k+2}(k+2)^2+(-2k-3)\,a_{k+1}+a_k=0$
- Recursion relation that defines series solution to ODE
  $a_{k+2}=\frac{2ka_{k+1}-a_k+3a_{k+1}}{(k+2)^2}$
- Recursion relation for $r=0$
  $a_{k+2}=\frac{2ka_{k+1}-a_k+3a_{k+1}}{(k+2)^2}$
- Solution for $r=0$
  $\left[y(x)=\sum_{k=0}^{\infty}a_kx^k,a_{k+2}=\frac{2ka_{k+1}-a_k+3a_{k+1}}{(k+2)^2},a_1-a_0=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 13

```
dsolve(x*diff(diff(y(x),x),x)+(1-2*x)*diff(y(x),x)+(x-1)*y(x) = 0,y(x),singsol=all)
```

$$y=e^x(c_2\ln(x)+c_1)$$

Mathematica DSolve solution

Solving time : 0.026 (sec)
Leaf size : 17

```
DSolve[{x*D[y[x],{x,2}]+(1-2*x)*D[y[x],x]+(x-1)*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x)\to e^x(c_2\log(x)+c_1)$$

2.1.781 Problem 803

Internal problem ID [9953]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 803
Date solved : `Monday, January 27, 2025 at 06:16:07 PM`
CAS classification : `[_Laguerre]`

Solve

$$xy'' - (x+1)\,y' + y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.216 (sec)

Writing the ode as

$$xy'' + (-x-1)\,y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -x-1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 2x + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 2x + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 2x + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1485: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4x^2} - \frac{1}{2x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{2x^2} + \frac{1}{2x^3} + \frac{1}{4x^4} - \frac{1}{4x^5} - \frac{3}{4x^6} - \frac{3}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 2x + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 3}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-2x + 3}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 2x + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2} - \frac{1}{2x} \\ &= \frac{x-1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2} - \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(\frac{1}{2} - \frac{1}{2x}\right)^2 - \left(\frac{x^2-2x+3}{4x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{1}{2} - \frac{1}{2x}\right) dx} \\ &= \frac{e^{\frac{x}{2}}}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x-1}{x}\,dx} \\ &= z_1 e^{\frac{x}{2}+\frac{\ln(x)}{2}} \\ &= z_1\left(\sqrt{x}\,e^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x-1}{x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{x+\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(-\frac{(x+1)\,e^{x+\ln(x)}e^{-2x}}{x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(e^x) + c_2\left(e^x\left(-\frac{(x+1)\,e^{x+\ln(x)}e^{-2x}}{x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dx^2}y(x)\right)x-(x+1)\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x)=-\frac{y(x)}{x}+\frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x)-\frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{x}+\frac{y(x)}{x}=0$

□ Check to see if $x_0=0$ is a regular singular point
  - Define functions
    $\left[P_2(x)=-\frac{x+1}{x}, P_3(x)=\frac{1}{x}\right]$
  - $x\cdot P_2(x)$ is analytic at $x=0$
    $\left.(x\cdot P_2(x))\right|_{x=0}=-1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $\left(\frac{d^2}{dx^2}y(x)\right)x + (-x-1)\left(\frac{d}{dx}y(x)\right) + y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..1$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  - Shift index using $k- > k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-1}$

  - Shift index using $k- > k+1$

    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-2+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1+r}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k}{k+1}\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+3}$

- Solution for $r = 2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 13

```
dsolve(x*diff(diff(y(x),x),x)-(x+1)*diff(y(x),x)+y(x) = 0,y(x),singsol=all)
```

$$y = e^{x} c_2 + c_1 x + c_1$$

Mathematica DSolve solution

Solving time : 0.387 (sec)
Leaf size : 78

```
DSolve[{x*D[y[x],{x,2}]-(x+1)*D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \sqrt{x} \exp\left(\frac{1}{2}\left(2\int_1^x \frac{K[1]-1}{2K[1]}dK[1] + x + 1\right)\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[2]} \frac{K[1]-1}{2K[1]}dK[1]\right)dK[2] + c_1\right)$$

2.1.782 Problem 804

Internal problem ID [9954]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 804
Date solved : `Monday, January 27, 2025 at 06:16:07 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$xy'' + 3y' + 4x^3y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.271 (sec)

Writing the ode as

$$xy'' + 3y' + 4x^3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 3 \\ C &= 4x^3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16x^4 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -16x^4 + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-16x^4 + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1487: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -4x^2 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 2ix - \frac{3i}{16x^3} - \frac{9i}{1024x^7} - \frac{27i}{32768x^{11}} - \frac{405i}{4194304x^{15}} - \frac{1701i}{134217728x^{19}} - \frac{15309i}{8589934592x^{23}} - \frac{72171i}{274877906944x^{27}} + \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 2i$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= 2ix \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -4x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-16x^4 + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(-4x^2\right) + \left(\frac{3}{4x^2}\right) \\ &= -4x^2 + \frac{3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 2ix \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{2i} - 1\right) &&= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{2i} - 1\right) &&= -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-16x^4 + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$2ix$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)(2ix) \\ &= -\frac{1}{2x} - 2ix \\ &= -\frac{1}{2x} - 2ix \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{2x} - 2ix\right)(0) + \left(\left(\frac{1}{2x^2} - 2i\right) + \left(-\frac{1}{2x} - 2ix\right)^2 - \left(\frac{-16x^4+3}{4x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{2x} - 2ix\right) dx} \\ &= \frac{e^{-ix^2}}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3}{x}\,dx} \\
&= z_1 e^{-\frac{3\ln(x)}{2}} \\
&= z_1\left(\frac{1}{x^{3/2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-ix^2}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{i\mathrm{e}^{2ix^2}}{4}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\mathrm{e}^{-ix^2}}{x^2}\right) + c_2\left(\frac{\mathrm{e}^{-ix^2}}{x^2}\left(-\frac{i\mathrm{e}^{2ix^2}}{4}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dx^2}y(x)\right)x + 3\frac{d}{dx}y(x) + 4x^3y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -4x^2y(x) - \frac{3\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x) + \frac{3\left(\frac{d}{dx}y(x)\right)}{x} + 4x^2y(x) = 0$
- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{3}{x}, P_3(x) = 4x^2\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

$(x \cdot P_2(x))\Big|_{x=0} = 3$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$
- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $\left(\frac{d^2}{dx^2}y(x)\right)x + 3\frac{d}{dx}y(x) + 4x^3y(x) = 0$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^3 \cdot y(x)$ to series expansion

  $x^3 \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+3}$
- Shift index using $k- > k - 3$

  $x^3 \cdot y(x) = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$
- Convert $\frac{d}{dx}y(x)$ to series expansion

  $\frac{d}{dx}y(x) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1}$
- Shift index using $k- > k + 1$

  $\frac{d}{dx}y(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,x^{k+r}$
- Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$
- Shift index using $k- > k + 1$

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(2+r)\,x^{-1+r} + a_1(1+r)\,(3+r)\,x^r + a_2(2+r)\,(4+r)\,x^{1+r} + a_3(3+r)\,(5+r)\,x^{2+r} + \left(\sum_{k=3}^{\infty} (a_{k+}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-2, 0\}$
- The coefficients of each power of x must be 0

  $[a_1(1+r)\,(3+r) = 0, a_2(2+r)\,(4+r) = 0, a_3(3+r)\,(5+r) = 0]$
- Solve for the dependent coefficient(s)

  $\{a_1 = 0, a_2 = 0, a_3 = 0\}$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k+r+3) + 4a_{k-3} = 0$
- Shift index using $k- > k + 3$

  $a_{k+4}(k+4+r)\,(k+6+r) + 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+4} = -\frac{4a_k}{(k+4+r)(k+6+r)}$

- Recursion relation for $r = -2$

  $a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}$

- Solution for $r = -2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+4} = -\frac{4a_k}{(k+4)(k+6)}$

- Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k}, a_{k+4} = -\frac{4a_k}{(k+4)(k+6)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k}\right), a_{4+k} = -\frac{4a_k}{(k+2)(4+k)}, a_1 = 0, a_2 = 0, a_3 = 0, b_{4+k} = -\frac{4b_k}{(4+k)(k\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 21

```
dsolve(x*diff(diff(y(x),x),x)+3*diff(y(x),x)+4*x^3*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_1 \sin\left(x^2\right) + c_2 \cos\left(x^2\right)}{x^2}$$

Mathematica DSolve solution

Solving time : `0.052` (sec)
Leaf size : 41

```
DSolve[{x*D[y[x],{x,2}]+3*D[y[x],x]+4*x^3*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{4c_1 e^{-ix^2} - ic_2 e^{ix^2}}{4x^2}$$

2.1.783 Problem 805

Internal problem ID [9955]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 805
Date solved : `Monday, January 27, 2025 at 06:16:08 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(-x^2+1\right)y''+2x\left(-x^2+1\right)y'-2y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.204 (sec)

Writing the ode as

$$\left(-x^4+x^2\right)y''+\left(-2x^3+2x\right)y'-2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^4+x^2\\ B &= -2x^3+2x\\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t}\\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-2}{x^2\left(x^2-1\right)} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -2\\ t &= x^2\left(x^2-1\right) \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{2}{x^2\left(x^2-1\right)}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1489: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2(x^2 - 1)$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 1. There is a pole at $x = -1$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 1$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{x+1} + \frac{2}{x^2} - \frac{1}{x-1}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= 0\\
\alpha_\infty^{-} &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{2}{x^2\left(x^2-1\right)}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	1	0	0	1
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-}\right)\\
&= 1 - (0)\\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + (-)[\sqrt{r}]_\infty\\
&= \frac{1}{x-1} - \frac{1}{x} + (-)\,(0)\\
&= \frac{1}{x-1} - \frac{1}{x}\\
&= \frac{1}{x^2-x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x-1} - \frac{1}{x}\right)(1) + \left(\left(-\frac{1}{(x-1)^2} + \frac{1}{x^2}\right) + \left(\frac{1}{x-1} - \frac{1}{x}\right)^2 - \left(-\frac{2}{x^2\left(x^2-1\right)}\right)\right) = 0$$

$$\frac{-2a_0 + 2}{x^3 - x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x+1)\, e^{\int \left(\frac{1}{x-1} - \frac{1}{x}\right) dx} \\ &= (x+1)\, e^{-\ln(x)+\ln(x-1)} \\ &= \frac{x^2-1}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2x^3+2x}{-x^4+x^2} \, dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1 \left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2-1}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^3+2x}{-x^4+x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{1}{4(x+1)} - \frac{\ln(x+1)}{4} - \frac{1}{4(x-1)} + \frac{\ln(x-1)}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{x^2-1}{x^2}\right) + c_2 \left(\frac{x^2-1}{x^2} \left(-\frac{1}{4(x+1)} - \frac{\ln(x+1)}{4} - \frac{1}{4(x-1)} + \frac{\ln(x-1)}{4}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.022 (sec)
Leaf size : 47

```
dsolve(x^2*(-x^2+1)*diff(diff(y(x),x),x)+2*x*(-x^2+1)*diff(y(x),x)-2*y(x) = 0,y(x),sings
```

$$y = \frac{c_2(x^2-1)\ln(x-1)+(-x^2+1)\,c_2\ln(x+1)+2c_1x^2-2c_2x-2c_1}{2x^2}$$

Mathematica DSolve solution

Solving time : 0.291 (sec)
Leaf size : 81

```
DSolve[{x^2*(1-x^2)*D[y[x],{x,2}]+2*x*(1-x^2)*D[y[x],x]-2*y[x]==0,{}},y[x],x,IncludeSingularSo
```

$$y(x)\to \frac{\exp\left(\int_1^x -\frac{K[1]^2+1}{K[1]-K[1]^3}dK[1]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[2]} -\frac{K[1]^2+1}{K[1]-K[1]^3}dK[1]\right)dK[2]+c_1\right)}{x}$$

2.1.784 Problem 806

Internal problem ID [9956]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 806
Date solved : Monday, January 27, 2025 at 06:16:09 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$2xy'' + (x-2)\,y' - y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.219 (sec)

Writing the ode as

$$2xy'' + (x-2)\,y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x \\ B &= x-2 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x + 12}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x + 12 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 4x + 12}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1490: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16} + \frac{1}{4x} + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{4} + \frac{1}{2x} + \frac{1}{x^2} - \frac{2}{x^3} + \frac{2}{x^4} + \frac{4}{x^5} - \frac{24}{x^6} + \frac{48}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{4}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{16}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 + 4x + 12}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{1}{16}\right) + \left(\frac{4x + 12}{16x^2}\right) \\ &= \frac{1}{16} + \frac{4x + 12}{16x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 16 gives $\frac{1}{4}$. Now b can be found.

$$\begin{aligned}b &= \left(\frac{1}{4}\right) - (0) \\ &= \frac{1}{4}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{4} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{4}}{\frac{1}{4}} - 0\right) &&= \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{4}}{\frac{1}{4}} - 0\right) &&= -\frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 4x + 12}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)\left(\frac{1}{4}\right) \\ &= -\frac{1}{2x} - \frac{1}{4} \\ &= -\frac{x+2}{4x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{2x} - \frac{1}{4}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x} - \frac{1}{4}\right)^2 - \left(\frac{x^2+4x+12}{16x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{2x} - \frac{1}{4}\right) dx} \\ &= \frac{e^{-\frac{x}{4}}}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x-2}{2x}\,dx} \\
&= z_1 e^{-\frac{x}{4}+\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\,\mathrm{e}^{-\frac{x}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x-2}{2x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x}{2}+\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{2(x-2)\,\mathrm{e}^{-\frac{x}{2}+\ln(x)}\mathrm{e}^{x}}{x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x}{2}}\right) + c_2\left(\mathrm{e}^{-\frac{x}{2}}\left(\frac{2(x-2)\,\mathrm{e}^{-\frac{x}{2}+\ln(x)}\mathrm{e}^{x}}{x}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2\left(\frac{d^2}{dx^2}y(x)\right)x + (x-2)\left(\frac{d}{dx}y(x)\right) - y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{y(x)}{2x} - \frac{(x-2)\left(\frac{d}{dx}y(x)\right)}{2x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(x-2)\left(\frac{d}{dx}y(x)\right)}{2x} - \frac{y(x)}{2x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{x-2}{2x}, P_3(x) = -\frac{1}{2x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -1$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- • Multiply by denominators

  $2\left(\frac{d^2}{dx^2}y(x)\right)x + (x-2)\left(\frac{d}{dx}y(x)\right) - y(x) = 0$

- • Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..1$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  - ◦ Shift index using $k- >k+1-m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - ◦ Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-1}$

  - ◦ Shift index using $k- >k+1$

    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $2a_0 r(-2+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(2a_{k+1}(k+1+r)(k+r-1) + a_k(k+r-1)\right) x^{k+r}\right) = 0$

- • $a_0$cannot be 0 by assumption, giving the indicial equation

  $2r(-2+r) = 0$

- • Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- • Each term in the series must be 0, giving the recursion relation

  $2\left(a_{k+1}(k+1+r) + \frac{a_k}{2}\right)(k+r-1) = 0$

- • Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{2(k+1+r)}$

- • Recursion relation for $r = 0$

  $a_{k+1} = -\frac{a_k}{2(k+1)}$

- • Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k}{2(k+1)}\right]$

- • Recursion relation for $r = 2$

  $a_{k+1} = -\frac{a_k}{2(k+3)}$

- • Solution for $r = 2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = -\frac{a_k}{2(k+3)}\right]$

- • Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+1} = -\frac{a_k}{2(k+1)}, b_{k+1} = -\frac{b_k}{2(k+3)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 16

```
dsolve(2*x*diff(diff(y(x),x),x)+(x-2)*diff(y(x),x)-y(x) = 0,y(x),singsol=all)
```

$$y = c_1(x-2) + \mathrm{e}^{-\frac{x}{2}} c_2$$

Mathematica DSolve solution

Solving time : 0.244 (sec)
Leaf size : 43

```
DSolve[{2*x*D[y[x],{x,2}]+(x-2)*D[y[x],x]-y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-\frac{x}{2}-\frac{1}{2}} \left(c_2 \int_1^x e^{\frac{K[1]}{2}+1} K[1] dK[1] + c_1 \right)$$

2.1.785 Problem 807

Internal problem ID [9957]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 807
Date solved : Monday, January 27, 2025 at 06:16:09 PM
CAS classification : [_Lienard]

Solve

$$xy'' + 2y' + xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.120 (sec)

Writing the ode as

$$xy'' + 2y' + xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1492: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\, dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1 \left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\, dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\cos(x)}{x}\right) + c_2\left(\frac{\cos(x)}{x}(\tan(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dx^2}y(x)\right)x + 2\frac{d}{dx}y(x) + xy(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -y(x) - \frac{2\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) + \frac{2\left(\frac{d}{dx}y(x)\right)}{x} + y(x) = 0$
- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{2}{x}, P_3(x) = 1\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $\left(\frac{d^2}{dx^2}y(x)\right)x + 2\frac{d}{dx}y(x) + xy(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x \cdot y(x)$ to series expansion
    $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+1}$
  - Shift index using $k- >k-1$

$x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

- ∘ Convert $\frac{d}{dx}y(x)$ to series expansion

  $\frac{d}{dx}y(x) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1}$

- ∘ Shift index using $k->k+1$

  $\frac{d}{dx}y(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,x^{k+r}$

- ∘ Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$

- ∘ Shift index using $k->k+1$

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(1+r)\,x^{-1+r} + a_1(1+r)\,(2+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $a_1(1+r)\,(2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1} = 0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)\,(k+3+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{(k+1)(k+2)}$

- Solution for $r = -1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+3)}$

- Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0, b_{k+2} = -\frac{b_k}{(k+2)(k+3)}, 2b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 17

```
dsolve(x*diff(diff(y(x),x),x)+2*diff(y(x),x)+x*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{\sin(x) c_1 + \cos(x) c_2}{x}$$

Mathematica DSolve solution

Solving time : 0.028 (sec)
Leaf size : 37

```
DSolve[{x*D[y[x],{x,2}]+2*D[y[x],x]+x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2c_1 e^{-ix} - ic_2 e^{ix}}{2x}$$

2.1.786 Problem 808

Internal problem ID [9958]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 808
Date solved : `Monday, January 27, 2025 at 06:16:10 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + 2x^2y' + \left(x^4 + 2x - 1\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.091 (sec)

Writing the ode as

$$y'' + 2x^2y' + \left(x^4 + 2x - 1\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2x^2 \\ C &= x^4 + 2x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1494: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x^2}{1}\, dx} \\ &= z_1 e^{-\frac{x^3}{3}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x^3}{3}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x\left(x^2+3\right)}{3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{2x^3}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{e^{-\frac{2x^3}{3}} e^{\frac{2x(x^2+3)}{3}}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{-\frac{x(x^2+3)}{3}}\right) + c_2 \left(e^{-\frac{x(x^2+3)}{3}} \left(\frac{e^{-\frac{2x^3}{3}} e^{\frac{2x(x^2+3)}{3}}}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2}y(x) + 2x^2\left(\frac{d}{dx}y(x)\right) + (x^4 + 2x - 1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..4$
    $x^m \cdot y(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$
  ◦ Shift index using $k- >k - m$
    $x^m \cdot y(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$
  ◦ Convert $x^2 \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion
    $x^2 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^{k+1}$
  ◦ Shift index using $k- >k - 1$
    $x^2 \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=1}^{\infty} a_{k-1}(k-1)\, x^k$
  ◦ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion
    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$
  ◦ Shift index using $k- >k + 2$
    $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite ODE with series expansions

$$2a_2 - a_0 + (6a_3 - a_1 + 2a_0)\,x + (12a_4 - a_2 + 4a_1)\,x^2 + (20a_5 - a_3 + 6a_2)\,x^3 + \left(\sum_{k=4}^{\infty} (a_{k+2}(k+2)\,(k$$

- The coefficients of each power of x must be 0
  $[2a_2 - a_0 = 0, 6a_3 - a_1 + 2a_0 = 0, 12a_4 - a_2 + 4a_1 = 0, 20a_5 - a_3 + 6a_2 = 0]$
- Solve for the dependent coefficient(s)
  $\left\{a_2 = \frac{a_0}{2}, a_3 = \frac{a_1}{6} - \frac{a_0}{3}, a_4 = \frac{a_0}{24} - \frac{a_1}{3}, a_5 = \frac{a_1}{120} - \frac{a_0}{6}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $(k^2 + 3k + 2)\,a_{k+2} + 2a_{k-1}k - a_k + a_{k-4} = 0$
- Shift index using $k- >k+4$
  $\left((k+4)^2 + 3k + 14\right) a_{k+6} + 2a_{k+3}(k+4) - a_{k+4} + a_k = 0$
- Recursion relation that defines the series solution to the ODE
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+6} = -\frac{2ka_{k+3} + a_k + 8a_{k+3} - a_{k+4}}{k^2 + 11k + 30}, a_2 = \frac{a_0}{2}, a_3 = \frac{a_1}{6} - \frac{a_0}{3}, a_4 = \frac{a_0}{24} - \frac{a_1}{3}, a_5 = \frac{a_1}{120} - \frac{a_0}{6}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 27

```
dsolve(diff(diff(y(x),x),x)+2*diff(y(x),x)*x^2+(x^4+2*x-1)*y(x) = 0,y(x),singsol=all)
```

$$y = c_1 \mathrm{e}^{-\frac{x(x^2-3)}{3}} + c_2 \mathrm{e}^{-\frac{x(x^2+3)}{3}}$$

Mathematica DSolve solution

Solving time : 0.043 (sec)
Leaf size : 34

```
DSolve[{D[y[x],{x,2}]+2*x^2*D[y[x],x]+(x^4+2*x-1)*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to \frac{1}{2} e^{-\frac{1}{3}x(x^2+3)} \left(c_2 e^{2x} + 2c_1\right)$$

2.1.787 Problem 809

Internal problem ID [9959]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 809
Date solved : `Monday, January 27, 2025 at 06:16:10 PM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$u'' + 2u' + u = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.049 (sec)

Writing the ode as

$$u'' + 2u' + u = 0 \tag{1}$$
$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ue^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1496: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in u is found from

$$\begin{aligned} u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\, dx} \\ &= z_1 e^{-x} \\ &= z_1 \left(\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$u_1 = \mathrm{e}^{-x}$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{u_1^2}\, dx$$

Substituting gives

$$\begin{aligned} u_2 &= u_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(u_1)^2}\,dx \\ &= u_1 \int \frac{e^{-2x}}{(u_1)^2}\,dx \\ &= u_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} u &= c_1 u_1 + c_2 u_2 \\ &= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}(x)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

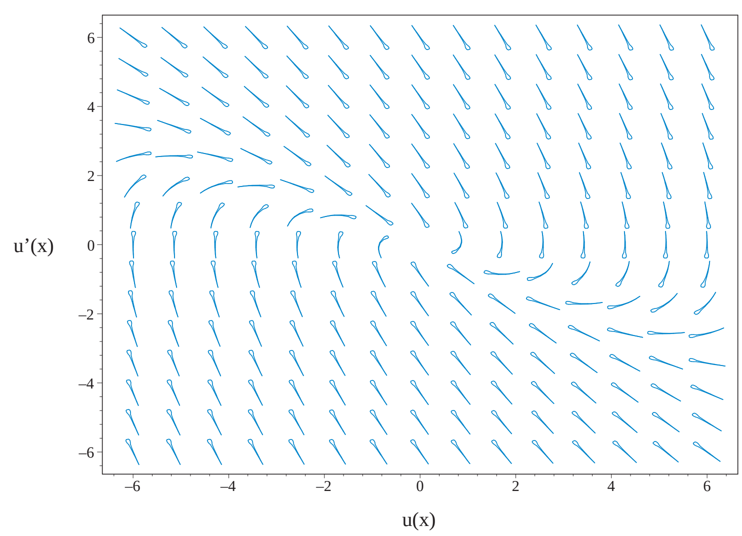

Figure 2.3: Slope field plot
$u'' + 2u' + u = 0$

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}u(x) + 2\frac{d}{dx}u(x) + u(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}u(x)$
- Characteristic polynomial of ODE
  $r^2 + 2r + 1 = 0$
- Factor the characteristic polynomial
  $(r+1)^2 = 0$
- Root of the characteristic polynomial
  $r = -1$
- 1st solution of the ODE
  $u_1(x) = \mathrm{e}^{-x}$
- Repeated root, multiply $u_1(x)$ by x to ensure linear independence
  $u_2(x) = x\,\mathrm{e}^{-x}$
- General solution of the ODE
  $u(x) = C1 u_1(x) + C2 u_2(x)$
- Substitute in solutions
  $u(x) = C1\,\mathrm{e}^{-x} + C2\,\mathrm{e}^{-x} x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 14

```
dsolve(diff(diff(u(x),x),x)+2*diff(u(x),x)+u(x) = 0,u(x),singsol=all)
```

$$u = e^{-x}(c_2x + c_1)$$

Mathematica DSolve solution

Solving time : 0.015 (sec)
Leaf size : 18

```
DSolve[{D[u[x],{x,2}]+2*D[u[x],x]+u[x]==0,{}},u[x],x,IncludeSingularSolutions->True]
```

$$u(x) \to e^{-x}(c_2x + c_1)$$

2.1.788 Problem 810

Internal problem ID [9960]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 810
Date solved : Monday, January 27, 2025 at 06:16:11 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$u'' - (2x+1)\,u' + \left(x^2+x-1\right)u = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.078 (sec)

Writing the ode as

$$u'' + (-2x-1)\,u' + \left(x^2+x-1\right)u = 0 \tag{1}$$
$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2x-1 \\ C &= x^2+x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = u e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1498: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in u is found from

$$\begin{aligned} u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x-1}{1}\, dx} \\ &= z_1 e^{\frac{1}{2}x^2+\frac{1}{2}x} \\ &= z_1 \left(\mathrm{e}^{\frac{x(x+1)}{2}}\right) \end{aligned}$$

Which simplifies to

$$u_1 = \mathrm{e}^{\frac{x^2}{2}}$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{u_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
u_2 &= u_1 \int \frac{e^{\int -\frac{-2x-1}{1}\,dx}}{(u_1)^2}\,dx \\
&= u_1 \int \frac{e^{x^2+x}}{(u_1)^2}\,dx \\
&= u_1 \left(\mathrm{e}^{x^2+x}\mathrm{e}^{-x^2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1 \left(\mathrm{e}^{\frac{x^2}{2}}\right) + c_2 \left(\mathrm{e}^{\frac{x^2}{2}} \left(\mathrm{e}^{x^2+x}\mathrm{e}^{-x^2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}u(x) - (2x+1)\left(\frac{d}{dx}u(x)\right) + (x^2+x-1)\,u(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}u(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}u(x) = (-x^2-x+1)\,u(x) + (2x+1)\left(\frac{d}{dx}u(x)\right)$

- Group terms with $u(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lir

  $\frac{d^2}{dx^2}u(x) + (-2x-1)\left(\frac{d}{dx}u(x)\right) + (x^2+x-1)\,u(x) = 0$

- Assume series solution for $u(x)$

  $u(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot u(x)$ to series expansion for $m = 0..2$

  $x^m \cdot u(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- >k-m$

  $x^m \cdot u(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert $x^m \cdot \left(\frac{d}{dx}u(x)\right)$ to series expansion for $m = 0..1$

  $x^m \cdot \left(\frac{d}{dx}u(x)\right) = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$

  ◦ Shift index using $k- >k+1-m$

  $x^m \cdot \left(\frac{d}{dx}u(x)\right) = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, x^k$

  ◦ Convert $\frac{d^2}{dx^2}u(x)$ to series expansion

  $\frac{d^2}{dx^2}u(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

  $\frac{d^2}{dx^2}u(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite ODE with series expansions

$$2a_2 - a_1 - a_0 + (6a_3 - 2a_2 - 3a_1 + a_0)\,x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_{k+1}(k+1) - a_k(2k+1) + \right.\right.$$

- The coefficients of each power of x must be 0

  $[2a_2 - a_1 - a_0 = 0, 6a_3 - 2a_2 - 3a_1 + a_0 = 0]$
- Solve for the dependent coefficient(s)

  $\left\{a_2 = \frac{a_1}{2} + \frac{a_0}{2}, a_3 = \frac{2a_1}{3}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $k^2 a_{k+2} + (-2a_k - a_{k+1} + 3a_{k+2})\,k - a_k + a_{k-2} + a_{k-1} - a_{k+1} + 2a_{k+2} = 0$
- Shift index using $k- >k+2$

  $(k+2)^2\,a_{k+4} + (-2a_{k+2} - a_{k+3} + 3a_{k+4})\,(k+2) - a_{k+2} + a_k + a_{k+1} - a_{k+3} + 2a_{k+4} = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[u(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{2ka_{k+2} + ka_{k+3} - a_k - a_{k+1} + 5a_{k+2} + 3a_{k+3}}{k^2 + 7k + 12}, a_2 = \frac{a_1}{2} + \frac{a_0}{2}, a_3 = \frac{2a_1}{3}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 22

```
dsolve(diff(diff(u(x),x),x)-(2*x+1)*diff(u(x),x)+(x^2+x-1)*u(x) = 0,u(x),singsol=all)
```

$$u = \mathrm{e}^{\frac{x^2}{2}} c_1 + c_2 \mathrm{e}^{\frac{x(x+2)}{2}}$$

Mathematica DSolve solution

Solving time : 0.026 (sec)
Leaf size : 24

```
DSolve[{D[u[x],{x,2}]-(2*x+1)*D[u[x],x]+(x^2+x-1)*u[x]==0,{}},u[x],x,IncludeSingularSolutions-
```

$$u(x) \to e^{\frac{x^2}{2}} (c_2 e^x + c_1)$$

2.1.789 Problem 811

Internal problem ID [9961]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 811
Date solved : `Monday, January 27, 2025 at 06:16:11 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + 2y' + \left(1 + \frac{2}{\left(1+3x\right)^2}\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.128 (sec)

Writing the ode as

$$y'' + 2y' + \left(1 + \frac{2}{\left(1+3x\right)^2}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 + \frac{2}{\left(1+3x\right)^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{\left(1+3x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -2 \\ t &= \left(1+3x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{2}{(1+3x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1500: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (1+3x)^2$. There is a pole at $x = -\frac{1}{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{9\left(x+\frac{1}{3}\right)^2}$$

For the pole at $x = -\frac{1}{3}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{2}{9}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{2}{3} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{3} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{2}{(1+3x)^2}$$

Since the gcd$(s,t) = 1$. This gives $b = -\frac{2}{9}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{2}{3}\\ \alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{3}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{2}{(1+3x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$-\frac{1}{3}$	2	0	$\frac{2}{3}$	$\frac{1}{3}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{2}{3}$	$\frac{1}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{3}$ then

$$\begin{aligned}d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right)\\ &= \frac{1}{3} - \left(\frac{1}{3}\right)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_\infty\\ &= \frac{1}{1+3x} + (-)(0)\\ &= \frac{1}{1+3x}\\ &= \frac{1}{1+3x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{1+3x}\right)(0) + \left(\left(-\frac{1}{3\left(x+\frac{1}{3}\right)^2}\right) + \left(\frac{1}{1+3x}\right)^2 - \left(-\frac{2}{(1+3x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\,dx} \\ &= \mathrm{e}^{\int \frac{1}{1+3x}dx} \\ &= (1+3x)^{1/3} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\ &= z_1 e^{-x} \\ &= z_1\left(\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}(1+3x)^{1/3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\ &= y_1\left((1+3x)^{1/3}\,\mathrm{e}^{-2x}\mathrm{e}^{2x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-x}(1+3x)^{1/3}\right) + c_2\left(\mathrm{e}^{-x}(1+3x)^{1/3}\left((1+3x)^{1/3}\,\mathrm{e}^{-2x}\mathrm{e}^{2x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + 2\frac{d}{dx}y(x) + \left(1 + \frac{2}{(3x+1)^2}\right)y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{3(3x^2+2x+1)y(x)}{(3x+1)^2} - 2\frac{d}{dx}y(x)$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + 2\frac{d}{dx}y(x) + \frac{3(3x^2+2x+1)y(x)}{(3x+1)^2} = 0$

☐ Check to see if $x_0 = -\frac{1}{3}$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = 2, P_3(x) = \frac{3(3x^2+2x+1)}{(3x+1)^2}\right]$

- ◦ $\left(x+\frac{1}{3}\right)\cdot P_2(x)$ is analytic at $x = -\frac{1}{3}$

  $\left(\left(x+\frac{1}{3}\right)\cdot P_2(x)\right)\bigg|_{x=-\frac{1}{3}} = 0$

- ◦ $\left(x+\frac{1}{3}\right)^2\cdot P_3(x)$ is analytic at $x = -\frac{1}{3}$

  $\left(\left(x+\frac{1}{3}\right)^2\cdot P_3(x)\right)\bigg|_{x=-\frac{1}{3}} = \frac{2}{9}$

- ◦ $x = -\frac{1}{3}$is a regular singular point

  Check to see if $x_0 = -\frac{1}{3}$ is a regular singular point

  $x_0 = -\frac{1}{3}$

- Multiply by denominators

  $(3x+1)^2\left(\frac{d^2}{dx^2}y(x)\right) + 2(3x+1)^2\left(\frac{d}{dx}y(x)\right) + (9x^2+6x+3)\,y(x) = 0$

- Change variables using $x = u - \frac{1}{3}$ so that the regular singular point is at $u = 0$

  $9u^2\left(\frac{d^2}{du^2}y(u)\right) + 18u^2\left(\frac{d}{du}y(u)\right) + (9u^2+2)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot y(u)$ to series expansion for $m = 0..2$

  $u^m\cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- ◦ Shift index using $k- >k-m$

  $u^m\cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- ◦ Convert $u^2\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion

  $u^2\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r+1}$

- ◦ Shift index using $k- >k-1$

  $u^2\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\,u^{k+r}$

- ◦ Convert $u^2\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u^2\cdot\left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\,u^{k+r}$

Rewrite ODE with series expansions

$$a_0(-1+3r)(-2+3r)u^r + (a_1(2+3r)(1+3r)+18a_0r)u^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(3k+3r-1)(3k+3r-\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+3r)(-2+3r)=0$
- Values of r that satisfy the indicial equation
  $r \in \left\{\frac{1}{3}, \frac{2}{3}\right\}$
- Each term must be 0
  $a_1(2+3r)(1+3r)+18a_0r=0$
- Solve for the dependent coefficient(s)
  $a_1 = -\frac{18a_0r}{9r^2+9r+2}$
- Each term in the series must be 0, giving the recursion relation
  $9\left(k+r-\frac{2}{3}\right)\left(k-\frac{1}{3}+r\right)a_k + 18a_{k-1}k + 18a_{k-1}r + 9a_{k-2} - 18a_{k-1} = 0$
- Shift index using $k->k+2$
  $9\left(k+\frac{4}{3}+r\right)\left(k+\frac{5}{3}+r\right)a_{k+2} + 18a_{k+1}(k+2) + 18a_{k+1}r + 9a_k - 18a_{k+1} = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{9(2ka_{k+1}+2a_{k+1}r+a_k+2a_{k+1})}{(3k+4+3r)(3k+5+3r)}$
- Recursion relation for $r=\frac{1}{3}$
  $a_{k+2} = -\frac{9\left(2ka_{k+1}+a_k+\frac{8}{3}a_{k+1}\right)}{(3k+5)(3k+6)}$
- Solution for $r=\frac{1}{3}$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{3}}, a_{k+2} = -\frac{9\left(2ka_{k+1}+a_k+\frac{8}{3}a_{k+1}\right)}{(3k+5)(3k+6)}, a_1 = -a_0\right]$
- Revert the change of variables $u = x+\frac{1}{3}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k \left(x+\frac{1}{3}\right)^{k+\frac{1}{3}}, a_{k+2} = -\frac{9\left(2ka_{k+1}+a_k+\frac{8}{3}a_{k+1}\right)}{(3k+5)(3k+6)}, a_1 = -a_0\right]$
- Recursion relation for $r=\frac{2}{3}$
  $a_{k+2} = -\frac{9\left(2ka_{k+1}+a_k+\frac{10}{3}a_{k+1}\right)}{(3k+6)(3k+7)}$
- Solution for $r=\frac{2}{3}$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{2}{3}}, a_{k+2} = -\frac{9\left(2ka_{k+1}+a_k+\frac{10}{3}a_{k+1}\right)}{(3k+6)(3k+7)}, a_1 = -a_0\right]$
- Revert the change of variables $u = x+\frac{1}{3}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k \left(x+\frac{1}{3}\right)^{k+\frac{2}{3}}, a_{k+2} = -\frac{9\left(2ka_{k+1}+a_k+\frac{10}{3}a_{k+1}\right)}{(3k+6)(3k+7)}, a_1 = -a_0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k \left(x+\frac{1}{3}\right)^{k+\frac{1}{3}}\right) + \left(\sum_{k=0}^{\infty} b_k \left(x+\frac{1}{3}\right)^{k+\frac{2}{3}}\right), a_{k+2} = -\frac{9\left(2ka_{k+1}+a_k+\frac{8}{3}a_{k+1}\right)}{(3k+5)(3k+6)}, a_1 = -a_0, b_{k+2} =\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
```

```
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 27

```
dsolve(diff(diff(y(x),x),x)+2*diff(y(x),x)+(1+2/(3*x+1)^2)*y(x) = 0,y(x),singsol=all)
```

$$y = \mathrm{e}^{-x}(3x+1)^{1/3}\left((3x+1)^{1/3}c_2+c_1\right)$$

Mathematica DSolve solution

Solving time : 0.05 (sec)
Leaf size : 35

```
DSolve[{D[y[x],{x,2}]+2*D[y[x],x]+(1+2/(1+3*x)^2)*y[x]==0,{}},y[x],x,IncludeSingularSolution
```

$$y(x) \to e^{-x}\sqrt[3]{3x+1}\left(c_2\sqrt[3]{3x+1}+c_1\right)$$

2.1.790 Problem 812

Internal problem ID [9962]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 812
Date solved : `Monday, January 27, 2025 at 06:16:12 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - 2xy' + \left(x^2+2\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.128 (sec)

Writing the ode as

$$x^2y'' - 2xy' + \left(x^2+2\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1502: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\, dx} \\ &= z_1 e^{\ln(x)} \\ &= z_1(x) \end{aligned}$$

Which simplifies to

$$y_1 = x\cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\, dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x\cos(x)) + c_2(x\cos(x)(\tan(x)))
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) - 2x\left(\frac{d}{dx}y(x)\right) + (x^2+2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(x^2+2)y(x)}{x^2} + \frac{2\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) - \frac{2\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(x^2+2)y(x)}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = -\frac{2}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -2$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) - 2x\left(\frac{d}{dx}y(x)\right) + (x^2+2)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ◦ Shift index using $k- >k - m$

$x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)\,(-2+r)\, x^r + a_1 r(-1+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-1)\,(k+r-2) + a_{k-2})\, x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+r)\,(-2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{1, 2\}$
- Each term must be 0
  $a_1 r(-1+r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_k(k+r-1)\,(k+r-2) + a_{k-2} = 0$
- Shift index using $k- >k+2$
  $a_{k+2}(k+1+r)\,(k+r) + a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{a_k}{(k+1+r)(k+r)}$
- Recursion relation for $r = 1$
  $a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$
- Solution for $r = 1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$
- Recursion relation for $r = 2$
  $a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$
- Solution for $r = 2$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+2)(k+3)}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
```

```
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 15

```
dsolve(x^2*diff(diff(y(x),x),x)-2*diff(y(x),x)*x+(x^2+2)*y(x) = 0,y(x),singsol=all)
```

$$y = x(\sin(x) c_1 + \cos(x) c_2)$$

Mathematica DSolve solution

Solving time : 0.029 (sec)
Leaf size : 33

```
DSolve[{x^2*D[y[x],{x,2}]-2*x*D[y[x],x]+(x^2+2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tr
```

$$y(x) \to c_1 e^{-ix} x - \frac{1}{2} i c_2 e^{ix} x$$

2.1.791 Problem 813

Internal problem ID [9963]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 813
Date solved : `Monday, January 27, 2025 at 06:16:12 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + \frac{2y'}{x} - \frac{2y}{(1+x)^2} = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.114 (sec)

Writing the ode as

$$y'' + \frac{2y'}{x} - \frac{2y}{(1+x)^2} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{2}{x} \\ C &= -\frac{2}{(1+x)^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{(1+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= (1+x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{(1+x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1504: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (1+x)^2$. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{(1+x)^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{(1+x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{(1+x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned}d &= \alpha_\infty^- - (\alpha_{c_1}^-)\\ &= -1 - (-1)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty\\ &= -\frac{1}{1+x} + (-)(0)\\ &= -\frac{1}{1+x}\\ &= -\frac{1}{1+x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{1+x}\right)(0) + \left(\left(\frac{1}{(1+x)^2}\right) + \left(-\frac{1}{1+x}\right)^2 - \left(\frac{2}{(1+x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int -\frac{1}{1+x} dx} \\ &= \frac{1}{1+x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{\frac{2}{x}}{1} \, dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1 \left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^2 + x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{\frac{2}{x}}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{(1+x)^3}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{1}{x^2 + x}\right) + c_2 \left(\frac{1}{x^2 + x} \left(\frac{(1+x)^3}{3}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + \frac{2\left(\frac{d}{dx}y(x)\right)}{x} - \frac{2y(x)}{(x+1)^2} = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = -\frac{2}{(x+1)^2}\right]$
  - $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((x+1) \cdot P_2(x)\right)\Big|_{x=-1} = 0$
  - $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $\left((x+1)^2 \cdot P_3(x)\right)\Big|_{x=-1} = -2$
  - $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $x(x+1)^2\left(\frac{d^2}{dx^2}y(x)\right) + 2(x+1)^2\left(\frac{d}{dx}y(x)\right) - 2xy(x) = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(u^3 - u^2)\left(\frac{d^2}{du^2}y(u)\right) + 2u^2\left(\frac{d}{du}y(u)\right) + (-2u + 2)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$
  - Shift index using $k- >k - m$

    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
  - Convert $u^2 \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion

    $u^2 \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r+1}$
  - Shift index using $k- >k - 1$

    $u^2 \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\,u^{k+r}$
  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 2..3$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k-1+r)\,u^{k+r-2+m}$
  - Shift index using $k- >k + 2 - m$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-a_0(1+r)(-2+r)\,u^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r+1)(k+r-2) + a_{k-1}(k+r+1)(k+r-2)\right) u^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-(1+r)(-2+r)=0$
- Values of r that satisfy the indicial equation
  $r\in\{-1,2\}$
- Each term in the series must be 0, giving the recursion relation
  $-(k+r+1)(k+r-2)(a_k-a_{k-1})=0$
- Shift index using $k->k+1$
  $-(k+r+2)(k-1+r)(a_{k+1}-a_k)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=a_k$
- Recursion relation for $r=-1$
  $a_{k+1}=a_k$
- Solution for $r=-1$
  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k-1}, a_{k+1}=a_k\right]$
- Revert the change of variables $u=x+1$
  $\left[y(x)=\sum_{k=0}^{\infty}a_k(x+1)^{k-1}, a_{k+1}=a_k\right]$
- Recursion relation for $r=2$
  $a_{k+1}=a_k$
- Solution for $r=2$
  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+2}, a_{k+1}=a_k\right]$
- Revert the change of variables $u=x+1$
  $\left[y(x)=\sum_{k=0}^{\infty}a_k(x+1)^{k+2}, a_{k+1}=a_k\right]$
- Combine solutions and rename parameters
  $\left[y(x)=\left(\sum_{k=0}^{\infty}a_k(x+1)^{k-1}\right)+\left(\sum_{k=0}^{\infty}b_k(x+1)^{k+2}\right), a_{k+1}=a_k, b_{k+1}=b_k\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 29

```
dsolve(diff(diff(y(x),x),x)+2/x*diff(y(x),x)-2/(x+1)^2*y(x) = 0,y(x),singsol=all)
```

$$y=\frac{(x^3+3x^2+3x)\,c_2+c_1}{x\,(x+1)}$$

Mathematica DSolve solution

Solving time : 0.033 (sec)
Leaf size : 34

```
DSolve[{D[y[x],{x,2}]+2/x*D[y[x],x]-2/(1+x)^2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to \frac{c_2 x(x^2+3x+3)+3c_1}{3x(x+1)}$$

2.1.792 Problem 815

Internal problem ID [9964]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 815
Date solved : `Monday, January 27, 2025 at 06:16:13 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.226 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1506: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x - 2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2} + 1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1-\frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= (2+x)\,e^{\int \left(-1-\frac{x}{2}\right)dx} \\ &= (2+x)\,e^{-x-\frac{1}{4}x^2} \\ &= (2+x)\,e^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1\left(e^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2 + x)\,\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\,dx \\ &= y_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) - x\left(\frac{d}{dx}y(x)\right) - xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

- ☐ Rewrite ODE with series expansions
  - Convert $x \cdot y(x)$ to series expansion

    $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k- > k-1$

    $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- > k+2$

$\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$

Rewrite ODE with series expansions

$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} - a_k k - a_{k-1} = 0$

- Shift index using $k- >k+1$

  $\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 52

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x-x*y(x) = 0,y(x),singsol=all)
```

$$y = -\pi c_2 \mathrm{e}^{-x-2}(x+2) \operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + i\mathrm{e}^{\frac{x(x+2)}{2}}\sqrt{\pi}\,\sqrt{2}\,c_2 + c_1\mathrm{e}^{-x}(x+2)$$

Mathematica DSolve solution

Solving time : 0.147 (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\operatorname{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2 e^{\frac{1}{2}(x+2)^2}\right)$$

2.1.793 Problem 816

Internal problem ID [9965]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 816
Date solved : `Monday, January 27, 2025 at 06:16:14 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.230 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1508: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v=1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2}+1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1}=x^0=1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2+x+1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v=0$ or not. Since $v=1$ which is not zero, then starting $r=\frac{s}{t}$, we do long division and write this in the form

$$r = Q+\frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2+4x-2}{4} \\ &= Q+\frac{R}{4} \\ &= \left(\frac{1}{4}x^2+x-\frac{1}{2}\right)+(0) \\ &= \frac{1}{4}x^2+x-\frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right)-(1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2}+1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a}-v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}}-1\right) &&= -2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a}-v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}}-1\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2+x-\frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered} (0) + 2\left(-1-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1-\frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) = 0 \\ -2 + a_0 = 0 \end{gathered}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= (2+x)\,\mathrm{e}^{\int (-1-\frac{x}{2})dx} \\ &= (2+x)\,\mathrm{e}^{-x-\frac{1}{4}x^2} \\ &= (2+x)\,\mathrm{e}^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1\left(\mathrm{e}^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2+x)\,\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) - x\left(\frac{d}{dx}y(x)\right) - xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite ODE with series expansions
  - Convert $x \cdot y(x)$ to series expansion

    $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k- >k-1$

    $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k+2$

$\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$

Rewrite ODE with series expansions

$2a_2 + \left(\sum_{k=1}^{\infty} (a_{k+2}(k+2)(k+1) - a_k k - a_{k-1})x^k\right) = 0$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)a_{k+2} - a_k k - a_{k-1} = 0$

- Shift index using $k- >k+1$

  $\left((k+1)^2+3k+5\right)a_{k+3} - a_{k+1}(k+1) - a_k = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{ka_{k+1}+a_k+a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 52

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x-x*y(x) = 0,y(x),singsol=all)
```

$$y = -\pi c_2 \mathrm{e}^{-x-2}(x+2)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + i\mathrm{e}^{\frac{x(x+2)}{2}}\sqrt{\pi}\,\sqrt{2}\,c_2 + c_1\mathrm{e}^{-x}(x+2)$$

Mathematica DSolve solution

Solving time : 0.091 (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

2.1.794 Problem 817

Internal problem ID [9966]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 817
Date solved : `Monday, January 27, 2025 at 06:16:14 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.227 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1510: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x - 2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2} + 1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$
\begin{aligned}
d &= \alpha_\infty^- \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= (-)[\sqrt{r}]_\infty \\
&= 0 + (-)\left(\frac{x}{2} + 1\right) \\
&= -1 - \frac{x}{2} \\
&= -1 - \frac{x}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\
-2 + a_0 &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 2\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = 2 + x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (2 + x)\, e^{\int \left(-1 - \frac{x}{2}\right) dx} \\
&= (2 + x)\, e^{-x - \frac{1}{4}x^2} \\
&= (2 + x)\, e^{-\frac{x(4+x)}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(e^{\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = (2+x)\,e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\,dx \\ &= y_1 \left(-\frac{e^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,e^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left((2+x)\,e^{-x}\right) + c_2\left((2+x)\,e^{-x}\left(-\frac{e^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,e^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) - x\left(\frac{d}{dx}y(x)\right) - xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite ODE with series expansions
  - Convert $x \cdot y(x)$ to series expansion

    $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k- >k-1$

    $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k+2$

$\frac{d^2}{dx^2}y(x) = \sum\limits_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\,x^k$

Rewrite ODE with series expansions

$2a_2 + \left(\sum\limits_{k=1}^{\infty} (a_{k+2}(k+2)(k+1) - a_k k - a_{k-1})\,x^k\right) = 0$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\,a_{k+2} - a_k k - a_{k-1} = 0$

- Shift index using $k- >k+1$

  $\left((k+1)^2 + 3k + 5\right)a_{k+3} - a_{k+1}(k+1) - a_k = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum\limits_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 52

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x-x*y(x) = 0,y(x),singsol=all)
```

$$y = -\pi c_2 \mathrm{e}^{-x-2}(x+2)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + i\mathrm{e}^{\frac{x(x+2)}{2}}\sqrt{\pi}\,\sqrt{2}\,c_2 + c_1\mathrm{e}^{-x}(x+2)$$

Mathematica DSolve solution

Solving time : 0.103 (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2 e^{\frac{1}{2}(x+2)^2}\right)$$

2.1.795 Problem 818

Internal problem ID [9967]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 818
Date solved : Monday, January 27, 2025 at 06:16:15 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$y'' - xy' - xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.231 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1512: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x - 2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2} + 1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered} (0) + 2\left(-1-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1-\frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) = 0 \\ -2 + a_0 = 0 \end{gathered}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (2+x)\,\mathrm{e}^{\int \left(-1-\frac{x}{2}\right)dx} \\ &= (2+x)\,\mathrm{e}^{-x-\frac{1}{4}x^2} \\ &= (2+x)\,\mathrm{e}^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1\left(\mathrm{e}^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2+x)\,\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) - x\left(\frac{d}{dx}y(x)\right) - xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

- ○ Convert $x \cdot y(x)$ to series expansion

  $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}$

- ○ Shift index using $k- >k-1$

  $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^k$

- ○ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

- ○ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

  $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ○ Shift index using $k- >k+2$

$$\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\,x^k$$

Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

$$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\,a_{k+2} - a_k k - a_{k-1} = 0$$

- Shift index using $k- > k+1$

$$\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 52

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x-x*y(x) = 0,y(x),singsol=all)
```

$$y = -\pi c_2 \mathrm{e}^{-x-2}(x+2)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + i\mathrm{e}^{\frac{x(x+2)}{2}}\sqrt{\pi}\,\sqrt{2}\,c_2 + c_1 \mathrm{e}^{-x}(x+2)$$

Mathematica DSolve solution

Solving time : 0.113 (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2 e^{\frac{1}{2}(x+2)^2}\right)$$

2.1.796 Problem 819

Internal problem ID [9968]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 819
Date solved : `Monday, January 27, 2025 at 06:16:15 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.224 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1514: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x - 2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2} + 1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1-\frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (2+x)\, e^{\int \left(-1-\frac{x}{2}\right) dx} \\ &= (2+x)\, e^{-x-\frac{1}{4}x^2} \\ &= (2+x)\, e^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1\left(e^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2+x)\,\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\,dx \\ &= y_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) - x\left(\frac{d}{dx}y(x)\right) - xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x \cdot y(x)$ to series expansion

    $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k- >k-1$

    $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k+2$

$\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$

Rewrite ODE with series expansions

$2a_2 + \left(\sum_{k=1}^{\infty} (a_{k+2}(k+2)(k+1) - a_k k - a_{k-1}) x^k\right) = 0$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)a_{k+2} - a_k k - a_{k-1} = 0$

- Shift index using $k- >k+1$

  $\left((k+1)^2+3k+5\right)a_{k+3} - a_{k+1}(k+1) - a_k = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 52

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x-x*y(x) = 0,y(x),singsol=all)
```

$$y = -\pi c_2 e^{-x-2}(x+2)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + i e^{\frac{x(x+2)}{2}}\sqrt{\pi}\,\sqrt{2}\,c_2 + c_1 e^{-x}(x+2)$$

Mathematica DSolve solution

Solving time : 0.104 (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2 e^{\frac{1}{2}(x+2)^2}\right)$$

2.1.797 Problem 820

Internal problem ID [9969]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 820
Date solved : `Monday, January 27, 2025 at 06:16:16 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.229 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1516: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v=1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2}+1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1}=x^0=1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2+x+1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v=0$ or not. Since $v=1$ which is not zero, then starting $r=\frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2+4x-2}{4} \\ &= Q+\frac{R}{4} \\ &= \left(\frac{1}{4}x^2+x-\frac{1}{2}\right)+(0) \\ &= \frac{1}{4}x^2+x-\frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right)-(1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2}+1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a}-v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}}-1\right) &= -2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a}-v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}}-1\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2+x-\frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered} (0) + 2\left(-1-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1-\frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) = 0 \\ -2 + a_0 = 0 \end{gathered}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (2+x)\, e^{\int \left(-1-\frac{x}{2}\right) dx} \\ &= (2+x)\, e^{-x-\frac{1}{4}x^2} \\ &= (2+x)\, e^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1\left(e^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2+x)\,\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\,dx \\ &= y_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) - x\left(\frac{d}{dx}y(x)\right) - xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

- ☐ Rewrite ODE with series expansions
  - Convert $x \cdot y(x)$ to series expansion

    $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k- > k-1$

    $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- > k+2$

$$\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\,x^k$$

Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $2a_2 = 0$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\,a_{k+2} - a_k k - a_{k-1} = 0$
- Shift index using $k- >k+1$

  $\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 52

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x-x*y(x) = 0,y(x),singsol=all)
```

$$y = -\pi c_2 \mathrm{e}^{-x-2}(x+2)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + i\mathrm{e}^{\frac{x(x+2)}{2}}\sqrt{\pi}\,\sqrt{2}\,c_2 + c_1\mathrm{e}^{-x}(x+2)$$

Mathematica DSolve solution

Solving time : 0.115 (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

2.1.798 Problem 821

Internal problem ID [9970]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 821
Date solved : `Monday, January 27, 2025 at 06:16:17 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.230 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1518: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x - 2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2} + 1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2} + 1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (2 + x)\, e^{\int \left(-1 - \frac{x}{2}\right) dx} \\ &= (2 + x)\, e^{-x - \frac{1}{4}x^2} \\ &= (2 + x)\, e^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1 \left(e^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2+x)\,\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) - x\left(\frac{d}{dx}y(x)\right) - xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

- ☐ Rewrite ODE with series expansions
  - Convert $x \cdot y(x)$ to series expansion

    $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k->k-1$

    $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k->k+2$

$$\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$$

Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2) a_{k+2} - a_k k - a_{k-1} = 0$

- Shift index using $k- >k+1$

  $\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 52

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x-x*y(x) = 0,y(x),singsol=all)
```

$$y = -\pi c_2 \mathrm{e}^{-x-2}(x+2)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + i\mathrm{e}^{\frac{x(x+2)}{2}}\sqrt{\pi}\,\sqrt{2}\,c_2 + c_1\mathrm{e}^{-x}(x+2)$$

Mathematica DSolve solution

Solving time : 0.112 (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\text{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

2.1.799 Problem 822

Internal problem ID [9971]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 822
Date solved : `Monday, January 27, 2025 at 06:16:17 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.221 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1520: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v=1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2}+1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1}=x^0=1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2+x+1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v=0$ or not. Since $v=1$ which is not zero, then starting $r=\frac{s}{t}$, we do long division and write this in the form

$$r = Q+\frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2+4x-2}{4} \\ &= Q+\frac{R}{4} \\ &= \left(\frac{1}{4}x^2+x-\frac{1}{2}\right)+(0) \\ &= \frac{1}{4}x^2+x-\frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right)-(1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2}+1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a}-v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}}-1\right) &&= -2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a}-v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}}-1\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2+x-\frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$
\begin{aligned}
d &= \alpha_\infty^- \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= (-)[\sqrt{r}]_\infty \\
&= 0 + (-)\left(\frac{x}{2} + 1\right) \\
&= -1 - \frac{x}{2} \\
&= -1 - \frac{x}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\
-2 + a_0 &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 2\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = 2 + x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (2 + x)\, \mathrm{e}^{\int \left(-1 - \frac{x}{2}\right) dx} \\
&= (2 + x)\, \mathrm{e}^{-x - \frac{1}{4}x^2} \\
&= (2 + x)\, \mathrm{e}^{-\frac{x(4+x)}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{1} \, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = (2+x)\,\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) - x\left(\frac{d}{dx}y(x)\right) - xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite ODE with series expansions
  - Convert $x \cdot y(x)$ to series expansion

    $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k- >k-1$

    $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k+2$

$$\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$$

Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_k k - a_{k-1}\right)x^k\right) = 0$$

- Each term must be 0

  $2a_2 = 0$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\,a_{k+2} - a_k k - a_{k-1} = 0$
- Shift index using $k- >k+1$

  $\left((k+1)^2 + 3k + 5\right)a_{k+3} - a_{k+1}(k+1) - a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 52

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x-x*y(x) = 0,y(x),singsol=all)
```

$$y = -\pi c_2 \mathrm{e}^{-x-2}(x+2)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + i\mathrm{e}^{\frac{x(x+2)}{2}}\sqrt{\pi}\,\sqrt{2}\,c_2 + c_1\mathrm{e}^{-x}(x+2)$$

Mathematica DSolve solution

Solving time : 0.101 (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

2.1.800 Problem 823

Internal problem ID [9972]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 823
Date solved : `Monday, January 27, 2025 at 06:16:18 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.222 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1522: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v=1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2}+1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1}=x^0=1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2+x+1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v=0$ or not. Since $v=1$ which is not zero, then starting $r=\frac{s}{t}$, we do long division and write this in the form

$$r = Q+\frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2+4x-2}{4} \\ &= Q+\frac{R}{4} \\ &= \left(\frac{1}{4}x^2+x-\frac{1}{2}\right)+(0) \\ &= \frac{1}{4}x^2+x-\frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right)-(1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2}+1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a}-v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}}-1\right) &&= -2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a}-v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}}-1\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2+x-\frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-1-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1-\frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) = 0$$
$$-2 + a_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (2+x)\, e^{\int \left(-1-\frac{x}{2}\right)dx} \\ &= (2+x)\, e^{-x-\frac{1}{4}x^2} \\ &= (2+x)\, e^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1\left(e^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2+x)\,\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\,dx \\ &= y_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) - x\left(\frac{d}{dx}y(x)\right) - xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- Convert $x \cdot y(x)$ to series expansion

  $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}$

- Shift index using $k->k-1$

  $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^k$

- Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

- Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

  $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- Shift index using $k->k+2$

$$\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\,x^k$$

Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2 + 3k + 2)\, a_{k+2} - a_k k - a_{k-1} = 0$$

- Shift index using $k- >k+1$

  $$\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 52

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x-x*y(x) = 0,y(x),singsol=all)
```

$$y = -\pi c_2 \mathrm{e}^{-x-2}(x+2)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + i\mathrm{e}^{\frac{x(x+2)}{2}}\sqrt{\pi}\,\sqrt{2}\,c_2 + c_1\mathrm{e}^{-x}(x+2)$$

Mathematica DSolve solution

Solving time : 0.094 (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2 e^{\frac{1}{2}(x+2)^2}\right)$$

2.1.801 Problem 824

Internal problem ID [9973]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 824
Date solved : `Monday, January 27, 2025 at 06:16:19 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.227 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1524: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x - 2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2} + 1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2} + 1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (2 + x)\, e^{\int \left(-1 - \frac{x}{2}\right) dx} \\ &= (2 + x)\, e^{-x - \frac{1}{4}x^2} \\ &= (2 + x)\, e^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1 \left(e^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2 + x)\,\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) - x\left(\frac{d}{dx}y(x)\right) - xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ◦ Convert $x \cdot y(x)$ to series expansion

  $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}$

- ◦ Shift index using $k- > k-1$

  $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^k$

- ◦ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

- ◦ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

  $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ◦ Shift index using $k- > k+2$

$\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$

Rewrite ODE with series expansions

$2a_2 + \left(\sum_{k=1}^{\infty} (a_{k+2}(k+2)(k+1) - a_k k - a_{k-1}) x^k\right) = 0$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2) a_{k+2} - a_k k - a_{k-1} = 0$

- Shift index using $k- >k+1$

  $((k+1)^2 + 3k + 5) a_{k+3} - a_{k+1}(k+1) - a_k = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 52

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x-x*y(x) = 0,y(x),singsol=all)
```

$$y = -\pi c_2 \mathrm{e}^{-x-2}(x+2)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + i\mathrm{e}^{\frac{x(x+2)}{2}}\sqrt{\pi}\,\sqrt{2}\,c_2 + c_1\mathrm{e}^{-x}(x+2)$$

Mathematica DSolve solution

Solving time : 0.094 (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

2.1.802 Problem 825

Internal problem ID [9974]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 825
Date solved : `Monday, January 27, 2025 at 06:16:19 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.223 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1526: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{2} + 1
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1
$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 4x - 2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}
$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{1}{4}x^2 + x - \frac{1}{2}
$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2} + 1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2} + 1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (2 + x)\, e^{\int \left(-1 - \frac{x}{2}\right) dx} \\ &= (2 + x)\, e^{-x - \frac{1}{4}x^2} \\ &= (2 + x)\, e^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1\left(e^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2+x)\,e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\,dx \\ &= y_1 \left(-\frac{e^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,e^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left((2+x)\,e^{-x}\right) + c_2\left((2+x)\,e^{-x}\left(-\frac{e^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,e^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) - x\left(\frac{d}{dx}y(x)\right) - xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ○ Convert $x \cdot y(x)$ to series expansion

  $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ○ Shift index using $k->k-1$

  $x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ○ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  ○ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

  $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ○ Shift index using $k->k+2$

$$\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$$

Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} (a_{k+2}(k+2)(k+1) - a_k k - a_{k-1})x^k\right) = 0$$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)a_{k+2} - a_k k - a_{k-1} = 0$

- Shift index using $k- > k+1$

  $((k+1)^2 + 3k + 5)a_{k+3} - a_{k+1}(k+1) - a_k = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.009 (sec)
Leaf size : 52

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x-x*y(x) = 0,y(x),singsol=all)
```

$$y = -\pi c_2 \mathrm{e}^{-x-2}(x+2)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + i\mathrm{e}^{\frac{x(x+2)}{2}}\sqrt{\pi}\,\sqrt{2}\,c_2 + c_1\mathrm{e}^{-x}(x+2)$$

Mathematica DSolve solution

Solving time : 0.095 (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2 e^{\frac{1}{2}(x+2)^2}\right)$$

2.1.803 Problem 826

Internal problem ID [9975]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 826
Date solved : `Monday, January 27, 2025 at 06:16:20 PM`
CAS classification : `[_Lienard]`

Solve

$$xy'' + 2y' + xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.124 (sec)

Writing the ode as

$$xy'' + 2y' + xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1528: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\, dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1 \left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2} dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\cos(x)}{x}\right) + c_2 \left(\frac{\cos(x)}{x}(\tan(x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dx^2}y(x)\right)x + 2\frac{d}{dx}y(x) + xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -y(x) - \frac{2\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{2\left(\frac{d}{dx}y(x)\right)}{x} + y(x) = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = 1\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $\left(\frac{d^2}{dx^2}y(x)\right)x + 2\frac{d}{dx}y(x) + xy(x) = 0$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x \cdot y(x)$ to series expansion

    $x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+1}$
  - Shift index using $k- >k-1$

$x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

- Convert $\frac{d}{dx}y(x)$ to series expansion

  $\frac{d}{dx}y(x) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

- Shift index using $k- >k+1$

  $\frac{d}{dx}y(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$

- Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k- >k+1$

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $a_1(1+r)\,(2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1} = 0$

- Shift index using $k- >k+1$

  $a_{k+2}(k+2+r)\,(k+3+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{(k+1)(k+2)}$

- Solution for $r = -1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+3)}$

- Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0, b_{k+2} = -\frac{b_k}{(k+2)(k+3)}, 2b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 17

```
dsolve(x*diff(diff(y(x),x),x)+2*diff(y(x),x)+x*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{\sin(x) c_1 + \cos(x) c_2}{x}$$

Mathematica DSolve solution

Solving time : 0.026 (sec)
Leaf size : 37

```
DSolve[{x*D[y[x],{x,2}]+2*D[y[x],x]+x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2c_1 e^{-ix} - ic_2 e^{ix}}{2x}$$

2.1.804 Problem 827

Internal problem ID [9976]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 827
Date solved : `Monday, January 27, 2025 at 06:16:20 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$2x^2y'' + 3xy' - xy = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.171 (sec)

Writing the ode as

$$2x^2y'' + 3xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= 3x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{8x - 3}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 8x - 3 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{8x - 3}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1530: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2} + \frac{1}{2x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1, \; e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(1 - (1)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{1}{(x-(0))}\right) \\ &= \frac{1}{2x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2x} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2x} + \frac{1-8x}{16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{1 + 2\sqrt{2}\,\sqrt{x}}{4x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= e^{\int \omega \, dx} \\ &= e^{\int \frac{1+2\sqrt{2}\,\sqrt{x}}{4x} dx} \\ &= x^{1/4} e^{\sqrt{2}\,\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3x}{2x^2}\,dx} \\ &= z_1 e^{-\frac{3\ln(x)}{4}} \\ &= z_1\left(\frac{1}{x^{3/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{\sqrt{2}\,\sqrt{x}}}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{2x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\frac{3\ln(x)}{2}}}{(y_1)^2}\,dx \\ &= y_1\left(-\frac{\mathrm{e}^{-2\sqrt{2}\,\sqrt{x}}\sqrt{2}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\mathrm{e}^{\sqrt{2}\,\sqrt{x}}}{\sqrt{x}}\right) + c_2\left(\frac{\mathrm{e}^{\sqrt{2}\,\sqrt{x}}}{\sqrt{x}}\left(-\frac{\mathrm{e}^{-2\sqrt{2}\,\sqrt{x}}\sqrt{2}}{2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$2x^2\left(\frac{d^2}{dx^2}y(x)\right) + 3x\left(\frac{d}{dx}y(x)\right) - xy(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = \frac{y(x)}{2x} - \frac{3\left(\frac{d}{dx}y(x)\right)}{2x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) + \frac{3\left(\frac{d}{dx}y(x)\right)}{2x} - \frac{y(x)}{2x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{3}{2x}, P_3(x) = -\frac{1}{2x}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{3}{2}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2\left(\frac{d^2}{dx^2}y(x)\right)x + 3\frac{d}{dx}y(x) - y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $\frac{d}{dx}y(x)$ to series expansion

    $\frac{d}{dx}y(x) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

  - Shift index using $k- >k+1$

    $\frac{d}{dx}y(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$

  - Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-1}$

  - Shift index using $k- >k+1$

    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(1+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(2k+3+2r) - a_k\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, -\frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k+r+\frac{3}{2}\right)(k+1+r)\, a_{k+1} - a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{(2k+3+2r)(k+1+r)}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{(2k+3)(k+1)}$

- Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k}{(2k+3)(k+1)}\right]$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+1} = \frac{a_k}{(2k+2)\left(k+\frac{1}{2}\right)}$

- Solution for $r = -\frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = \frac{a_k}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$

- Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{2}}\right), a_{k+1}=\frac{a_k}{(2k+3)(k+1)}, b_{k+1}=\frac{b_k}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.061 (sec)
Leaf size : 29

```
dsolve(2*x^2*diff(diff(y(x),x),x)+3*diff(y(x),x)*x-x*y(x) = 0,y(x),singsol=all)
```

$$y=\frac{c_1 \sinh\left(\sqrt{x}\,\sqrt{2}\right)+c_2 \cosh\left(\sqrt{x}\,\sqrt{2}\right)}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.058 (sec)
Leaf size : 56

```
DSolve[{2*x^2*D[y[x],{x,2}]+3*x*D[y[x],x]-x*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to\frac{e^{-\sqrt{2}\sqrt{x}}\left(2c_1 e^{2\sqrt{2}\sqrt{x}}-\sqrt{2}c_2\right)}{2\sqrt{x}}$$

2.1.805 Problem 828

Internal problem ID [9977]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 828
Date solved : `Monday, January 27, 2025 at 06:16:21 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + \left(3x^2+2x\right)y' - 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.212 (sec)

Writing the ode as

$$x^2y'' + \left(3x^2+2x\right)y' - 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 3x^2+2x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9x^2+12x+8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9x^2+12x+8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{9x^2+12x+8}{4x^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1532: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{4} + \frac{3}{x} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3}{2} + \frac{1}{x} + \frac{1}{3x^2} - \frac{2}{9x^3} + \frac{1}{9x^4} - \frac{2}{81x^5} - \frac{2}{81x^6} + \frac{28}{729x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{3}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{9x^2 + 12x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{9}{4}\right) + \left(\frac{12x + 8}{4x^2}\right) \\ &= \frac{9}{4} + \frac{12x + 8}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 12. Dividing this by leading coefficient in t which is 4 gives 3. Now b can be found.

$$\begin{aligned}b &= (3) - (0) \\ &= 3\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{3}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{3}{\frac{3}{2}} - 0\right) &&= 1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{3}{\frac{3}{2}} - 0\right) &&= -1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{9x^2 + 12x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{3}{2}$	1	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x} + (-)\left(\frac{3}{2}\right) \\
&= -\frac{1}{x} - \frac{3}{2} \\
&= -\frac{1}{x} - \frac{3}{2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{x} - \frac{3}{2}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - \frac{3}{2}\right)^2 - \left(\frac{9x^2 + 12x + 8}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{1}{x} - \frac{3}{2}\right) dx} \\
&= \frac{e^{-\frac{3x}{2}}}{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x^2+2x}{x^2}\,dx} \\
&= z_1 e^{-\frac{3x}{2}-\ln(x)} \\
&= z_1\left(\frac{\mathrm{e}^{-\frac{3x}{2}}}{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{-3x}}{x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^2+2x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-3x-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{(9x^2-6x+2)\,x^2\mathrm{e}^{-3x-2\ln(x)}\mathrm{e}^{6x}}{27}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{\mathrm{e}^{-3x}}{x^2}\right) + c_2\left(\frac{\mathrm{e}^{-3x}}{x^2}\left(\frac{(9x^2-6x+2)\,x^2\mathrm{e}^{-3x-2\ln(x)}\mathrm{e}^{6x}}{27}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) + (3x^2+2x)\left(\frac{d}{dx}y(x)\right) - 2y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{2y(x)}{x^2} - \frac{(3x+2)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(3x+2)\left(\frac{d}{dx}y(x)\right)}{x} - \frac{2y(x)}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{3x+2}{x}, P_3(x) = -\frac{2}{x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

$$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 2$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -2$$

- $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x(3x+2)\left(\frac{d}{dx}y(x)\right) - 2y(x) = 0$$

- Assume series solution for $y(x)$

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

$$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

$$x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(2+r)(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+2)(k+r-1) + 3a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(2+r)(-1+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{-2, 1\}$

- Each term in the series must be 0, giving the recursion relation

$(k+r-1)\left(a_k(k+r+2) + 3a_{k-1}\right) = 0$

- Shift index using $k- >k+1$

$(k+r)\left(a_{k+1}(k+3+r) + 3a_k\right) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = -\frac{3a_k}{k+3+r}$

- Recursion relation for $r = -2$

$a_{k+1} = -\frac{3a_k}{k+1}$

- Solution for $r = -2$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+1} = -\frac{3a_k}{k+1}\right]$$

- Recursion relation for $r = 1$

$a_{k+1} = -\frac{3a_k}{k+4}$

- Solution for $r = 1$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{3a_k}{k+4}\right]$$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+1} = -\frac{3a_k}{k+1}, b_{k+1} = -\frac{3b_k}{4+k}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 27

```
dsolve(x^2*diff(diff(y(x),x),x)+(3*x^2+2*x)*diff(y(x),x)-2*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_1 \mathrm{e}^{-3x} + c_2(9x^2 - 6x + 2)}{x^2}$$

Mathematica DSolve solution

Solving time : 0.858 (sec)
Leaf size : 52

```
DSolve[{x^2*D[y[x],{x,2}]+(2*x+3*x^2)*D[y[x],x]-2*y[x]==0,{}},y[x],x,IncludeSingularSolution
```

$$y(x) \to \frac{e^{-3x}\left(\int_1^x e^{3K[1]} c_1 K[1]^2 dK[1] + c_2\right)}{x^2}$$
$$y(x) \to \frac{c_2 e^{-3x}}{x^2}$$

2.1.806 Problem 829

Internal problem ID [9978]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 829
Date solved : `Monday, January 27, 2025 at 06:16:21 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2\left(x^2+x+1\right)y''+x\left(11x^2+11x+9\right)y'+\left(7x^2+10x+6\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 1.041 (sec)

Writing the ode as

$$\left(2x^4+2x^3+2x^2\right)y''+\left(11x^3+11x^2+9x\right)y'+\left(7x^2+10x+6\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x^4+2x^3+2x^2\\ B&=11x^3+11x^2+9x\\ C&=7x^2+10x+6\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{21x^4+18x^3+27x^2-2x-3}{16\left(x^3+x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=21x^4+18x^3+27x^2-2x-3\\ t&=16\left(x^3+x^2+x\right)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{21x^4+18x^3+27x^2-2x-3}{16\left(x^3+x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1534: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^3 + x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ of order 2. There is a pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{-\frac{5}{24} + \frac{i\sqrt{3}}{24}}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{5}{24} - \frac{i\sqrt{3}}{24}}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{1}{8} - \frac{43i\sqrt{3}}{72}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{-\frac{1}{8} + \frac{43i\sqrt{3}}{72}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} - \frac{3}{16x^2} + \frac{1}{4x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = -\frac{1}{2}+\frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{24}+\frac{i\sqrt{3}}{24}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{1}{2}+\frac{\sqrt{6+6i\sqrt{3}}}{12}\\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{2}-\frac{\sqrt{6+6i\sqrt{3}}}{12}
\end{aligned}
$$

For the pole at $x = -\frac{1}{2}-\frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{24}-\frac{i\sqrt{3}}{24}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{1}{2}+\frac{\sqrt{6-6i\sqrt{3}}}{12}\\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{2}-\frac{\sqrt{6-6i\sqrt{3}}}{12}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{21x^{4}+18x^{3}+27x^{2}-2x-3}{16\left(x^{3}+x^{2}+x\right)^{2}}
$$

Since the $\gcd(s,t)=1$. This gives $b = \frac{21}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{7}{4}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b} = -\frac{3}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{21x^{4}+18x^{3}+27x^{2}-2x-3}{16\left(x^{3}+x^{2}+x\right)^{2}}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
$-\frac{1}{2}+\frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2}+\frac{\sqrt{6+6i\sqrt{3}}}{12}$	$\frac{1}{2}-\frac{\sqrt{6+6i\sqrt{3}}}{12}$
$-\frac{1}{2}-\frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2}+\frac{\sqrt{6-6i\sqrt{3}}}{12}$	$\frac{1}{2}-\frac{\sqrt{6-6i\sqrt{3}}}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying

$\alpha_\infty^+ = \frac{7}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\ &= \frac{7}{4} - \left(\frac{7}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^+}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{4x} + \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} + (0) \\ &= \frac{1}{4x} + \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} \\ &= \frac{7x^2+3x+1}{4x\,(x^2+x+1)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{4x} + \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} \right)(0) + \left(\left(-\frac{1}{4x^2} - \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} - \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2} \right) + \left(\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{4x} + \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} \right) dx} \\ &= 2\left(x^2+x+1\right)^{3/4} \sqrt{2}\, x^{1/4} \mathrm{e}^{-\frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{11x^3+11x^2+9x}{2x^4+2x^3+2x^2}\, dx} \\ &= z_1 e^{-\frac{\ln\left(x^2+x+1\right)}{4} - \frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6} - \frac{9\ln(x)}{4}} \\ &= z_1 \left(\frac{\mathrm{e}^{-\frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}}{\left(x^2+x+1\right)^{1/4} x^{9/4}} \right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{2\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}\sqrt{2}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{11x^3+11x^2+9x}{2x^4+2x^3+2x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{\mathrm{e}^{-\frac{\ln(x^2+x+1)}{2}-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}-\frac{9\ln(x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{\ln(x^2+x+1)}{2}-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}-\frac{9\ln(x)}{2}}\,x^4\,\mathrm{e}^{\frac{2\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{8x^2+8x+8}\,dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{2\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}\sqrt{2}}{x^2}\right) \\
&\quad + c_2\left(\frac{2\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}\sqrt{2}}{x^2}\left(\int \frac{\mathrm{e}^{-\frac{\ln(x^2+x+1)}{2}-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}-\frac{9\ln(x)}{2}}\,x^4\,\mathrm{e}^{\frac{2\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{8x^2+8x+8}\,dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2(x^2+x+1)\left(\frac{d^2}{dx^2}y(x)\right) + x(11x^2+11x+9)\left(\frac{d}{dx}y(x)\right) + (7x^2+10x+6)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(7x^2+10x+6)y(x)}{2x^2(x^2+x+1)} - \frac{(11x^2+11x+9)\left(\frac{d}{dx}y(x)\right)}{2x(x^2+x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(11x^2+11x+9)\left(\frac{d}{dx}y(x)\right)}{2x(x^2+x+1)} + \frac{(7x^2+10x+6)y(x)}{2x^2(x^2+x+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{11x^2+11x+9}{2x(x^2+x+1)}, P_3(x) = \frac{7x^2+10x+6}{2x^2(x^2+x+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{9}{2}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 3$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x^2(x^2+x+1)\left(\frac{d^2}{dx^2}y(x)\right) + x(11x^2+11x+9)\left(\frac{d}{dx}y(x)\right) + (7x^2+10x+6)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k- >k + 1 - m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,x^{k+r-2+m}$

  - Shift index using $k- >k + 2 - m$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)(3+2r)\,x^r + \left(a_1(3+r)(5+2r) + a_0(5+2r)(2+r)\right)x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+2)(2k +$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)(3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-2, -\frac{3}{2}\right\}$

- Each term must be 0

  $a_1(3+r)(5+2r) + a_0(5+2r)(2+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = -\frac{(2+r)a_0}{3+r}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k+r+\frac{3}{2}\right)\left((a_k+a_{k-2}+a_{k-1})\,k + (a_k+a_{k-2}+a_{k-1})\,r + 2a_k - a_{k-2} + a_{k-1}\right) = 0$

- Shift index using $k- >k + 2$

  $2\left(k+\frac{7}{2}+r\right)\left((a_{k+2}+a_k+a_{k+1})(k+2) + (a_{k+2}+a_k+a_{k+1})\,r + 2a_{k+2} - a_k + a_{k+1}\right) = 0$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{ka_k + ka_{k+1} + ra_k + ra_{k+1} + a_k + 3a_{k+1}}{k + 4 + r}$$

- Recursion relation for $r = -2$

$$a_{k+2} = -\frac{ka_k + ka_{k+1} - a_k + a_{k+1}}{k + 2}$$

- Solution for $r = -2$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{ka_k + ka_{k+1} - a_k + a_{k+1}}{k + 2}, a_1 = 0\right]$$

- Recursion relation for $r = -\frac{3}{2}$

$$a_{k+2} = -\frac{ka_k + ka_{k+1} - \frac{1}{2}a_k + \frac{3}{2}a_{k+1}}{k + \frac{5}{2}}$$

- Solution for $r = -\frac{3}{2}$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}, a_{k+2} = -\frac{ka_k + ka_{k+1} - \frac{1}{2}a_k + \frac{3}{2}a_{k+1}}{k + \frac{5}{2}}, a_1 = -\frac{a_0}{3}\right]$$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{3}{2}}\right), a_{k+2} = -\frac{ka_k + ka_{k+1} - a_k + a_{k+1}}{k + 2}, a_1 = 0, b_{k+2} = -\frac{kb_k + kb_{k+1} - \frac{1}{2}b_k +}{k + \frac{5}{2}}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Mo
      <- Heun successful: received ODE is equivalent to the  HeunG  ODE, case  a <> 0, e
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 2.484 (sec)
Leaf size : 231

```
dsolve(2*x^2*(x^2+x+1)*diff(diff(y(x),x),x)+x*(11*x^2+11*x+9)*diff(y(x),x)+(7*x^2+10*x
```

$$y = \frac{\left(2x+i\sqrt{3}+1\right)^{\frac{5\sqrt{3}+3i}{6\sqrt{3}+6i}} \left(-2x+i\sqrt{3}-1\right)^{\frac{64i\sqrt{3}+2368}{\left(\sqrt{3}+i\right)^{3}\left(i-\sqrt{3}\right)^{4}\left(13\sqrt{3}+9i\right)}} \mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}} \left(\mathrm{HeunG}\left(\frac{\sqrt{3}+i}{i-\sqrt{3}}, 0, 0, \frac{5}{2}, \frac{1}{2}\right.\right.}{x^{5/2}\left(x^{2}+x+1\right)}$$

Mathematica DSolve solution

Solving time : 0.396 (sec)
Leaf size : 135

```
DSolve[{2*x^2*(1+x+x^2)*D[y[x],{x,2}] + x*(9+11*x+11*x^2)*D[y[x],x] + (6+10*x+7*x^2)*y[x] ==
```

$$y(x) \to \exp\left(\int_1^x \frac{K[1](7K[1]+3)+1}{4K[1]\left(K[1]^2+K[1]+1\right)}dK[1] - \frac{1}{2}\int_1^x \left(\frac{K[2]+1}{K[2]^2+K[2]+1} + \frac{9}{2K[2]}\right)dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]} \frac{7K[1]^2+3K[1]+1}{4K[1]\left(K[1]^2+K[1]+1\right)}dK[1]\right)dK[3] + c_1\right)$$

2.1.807 Problem 830

Internal problem ID [9979]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 830
Date solved : `Monday, January 27, 2025 at 06:16:23 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + (1+x)\,y' + 2y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.257 (sec)

Writing the ode as

$$xy'' + (1+x)\,y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 1+x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 6x - 1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 6x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1536: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2-2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1,2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{3}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{3}{2x} - \frac{5}{2x^2} - \frac{15}{2x^3} - \frac{115}{4x^4} - \frac{495}{4x^5} - \frac{2285}{4x^6} - \frac{11055}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 6x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-6x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-6x - 1}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -6. Dividing this by leading coefficient in t which is 4 gives $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{2}\right) - (0) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &= -\frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &= \frac{3}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 6x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^+) \\ &= \frac{3}{2} - \left(\frac{1}{2}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\left(\frac{1}{2}\right) \\ &= \frac{1}{2x} - \frac{1}{2} \\ &= -\frac{-1+x}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{2}\right)(1) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^2-6x-1}{4x^2}\right)\right) = 0$$

$$\frac{1+a_0}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = -1 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (-1+x)\, e^{\int \left(\frac{1}{2x}-\frac{1}{2}\right)dx} \\
&= (-1+x)\, e^{-\frac{x}{2}+\frac{\ln(x)}{2}} \\
&= (-1+x)\sqrt{x}\, e^{-\frac{x}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1+x}{x}\, dx} \\
&= z_1 e^{-\frac{x}{2}-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{e^{-\frac{x}{2}}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = e^{-x}(-1+x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1+x}{x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-x-\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\operatorname{Ei}_1(-x) - \frac{e^x}{-1+x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(e^{-x}(-1+x)\right) + c_2\left(e^{-x}(-1+x)\left(-\operatorname{Ei}_1(-x) - \frac{e^x}{-1+x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dx^2}y(x)\right)x + (x+1)\left(\frac{d}{dx}y(x)\right) + 2y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{2y(x)}{x} - \frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

$\frac{d^2}{dx^2}y(x) + \frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{x} + \frac{2y(x)}{x} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{x+1}{x}, P_3(x) = \frac{2}{x}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $\left(\frac{d^2}{dx^2}y(x)\right)x + (x+1)\left(\frac{d}{dx}y(x)\right) + 2y(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..1$
    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
  - Shift index using $k- >k+1-m$
    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion
    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-1}$
  - Shift index using $k- >k+1$
    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$
    Rewrite ODE with series expansions
    $a_0 r^2 x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)^2 + a_k(k+r+2)\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1)^2 + a_k(k+2) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = -\frac{a_k(k+2)}{(k+1)^2}$
- Recursion relation for $r = 0$
  $a_{k+1} = -\frac{a_k(k+2)}{(k+1)^2}$
- Solution for $r = 0$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k(k+2)}{(k+1)^2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 29

```
dsolve(x*diff(diff(y(x),x),x)+(x+1)*diff(y(x),x)+2*y(x) = 0,y(x),singsol=all)
```

$$y = c_2 \mathrm{e}^{-x}(x-1)\,\mathrm{Ei}_1(-x) + c_1 \mathrm{e}^{-x}(x-1) + c_2$$

Mathematica DSolve solution

Solving time : 0.207 (sec)
Leaf size : 42

```
DSolve[{x*D[y[x],{x,2}] +(1+x)*D[y[x],x]+2*y[x] == 0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-x}(x-1)\left(c_2 \int_1^x \frac{e^{K[1]}}{(K[1]-1)^2 K[1]} dK[1] + c_1\right)$$

2.1.808 Problem 831

Internal problem ID [9980]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 831
Date solved : `Monday, January 27, 2025 at 06:16:24 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(x^2-2x+1)\,y''-x(3+x)\,y'+(4+x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.303 (sec)

Writing the ode as

$$x^2(x-1)^2\,y''+\left(-x^2-3x\right)y'+(4+x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(x-1)^2 \\ B &= -x^2-3x \\ C &= 4+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{7x^2+10x-1}{4x^2\,(x-1)^4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 7x^2+10x-1 \\ t &= 4x^2(x-1)^4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{7x^2+10x-1}{4x^2\,(x-1)^4}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1538: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2(x-1)^4$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{(x-1)^3} + \frac{3}{2x} - \frac{1}{4x^2} + \frac{4}{(x-1)^4} + \frac{7}{4(x-1)^2} - \frac{3}{2(x-1)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\alpha_c^+ = \frac{1}{2}\left(\frac{b}{a} + v\right)$$
$$\alpha_c^- = \frac{1}{2}\left(-\frac{b}{a} + v\right)$$

The partial fraction decomposition of r is

$$r = -\frac{2}{(x-1)^3} + \frac{3}{2x} - \frac{1}{4x^2} + \frac{4}{(x-1)^4} + \frac{7}{4(x-1)^2} - \frac{3}{2(x-1)}$$

There is pole in r at $x = 1$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 1$ gives

$$[\sqrt{r}]_c \approx \frac{2}{(x-1)^2} - \frac{1}{2(x-1)} + \frac{21}{32} - \frac{9x}{32} + \frac{53(x-1)^2}{256} - \frac{149(x-1)^3}{1024} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{2}{(x-1)^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-1)^2}$ is

$$a = 2$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 1$. This term becomes $\frac{1}{(x-1)^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be -2. Therefore

$$\begin{aligned} b &= (-2) - (0) \\ &= -2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{2}{(x-1)^2} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) = \frac{1}{2}\left(\frac{-2}{2} + 2\right) = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) = \frac{1}{2}\left(-\frac{-2}{2} + 2\right) = \frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{7x^2 + 10x - 1}{4x^2(x-1)^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
1	4	$\frac{2}{(x-1)^2}$	$\frac{1}{2}$	$\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2} + (-)(0) \\ &= \frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2} \\ &= \frac{2x^2+x+1}{2x(x-1)^2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{4}{(x-1)^3} - \frac{1}{2(x-1)^2}\right) + \left(\frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2}\right)\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2}\right) dx} \\ &= \sqrt{x-1}\,\sqrt{x}\,e^{-\frac{2}{x-1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2-3x}{x^2(x-1)^2}\,dx} \\
&= z_1 e^{-\frac{2}{x-1}-\frac{3\ln(x-1)}{2}+\frac{3\ln(x)}{2}} \\
&= z_1 \left(\frac{x^{3/2}\mathrm{e}^{-\frac{2}{x-1}}}{(x-1)^{3/2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^{3/2}\mathrm{e}^{-\frac{4}{x-1}}\sqrt{x(x-1)}}{(x-1)^{3/2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-3x}{x^2(x-1)^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{4}{x-1}-3\ln(x-1)+3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\mathrm{e}^{-4}\,\mathrm{Ei}_1\left(-\frac{4}{x-1}-4\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{3/2}\mathrm{e}^{-\frac{4}{x-1}}\sqrt{x(x-1)}}{(x-1)^{3/2}}\right) + c_2 \left(\frac{x^{3/2}\mathrm{e}^{-\frac{4}{x-1}}\sqrt{x(x-1)}}{(x-1)^{3/2}}\left(\mathrm{e}^{-4}\,\mathrm{Ei}_1\left(-\frac{4}{x-1}-4\right)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(x^2-2x+1)\left(\frac{d^2}{dx^2}y(x)\right) - x(x+3)\left(\frac{d}{dx}y(x)\right) + (x+4)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(x+4)y(x)}{x^2(x^2-2x+1)} + \frac{(x+3)\left(\frac{d}{dx}y(x)\right)}{x(x^2-2x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) - \frac{(x+3)\left(\frac{d}{dx}y(x)\right)}{x(x^2-2x+1)} + \frac{(x+4)y(x)}{x^2(x^2-2x+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

$\left[P_2(x) = -\frac{x+3}{x(x^2-2x+1)}, P_3(x) = \frac{x+4}{x^2(x^2-2x+1)}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = -3$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 4$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2-2x+1)\left(\frac{d^2}{dx^2}y(x)\right) - x(x+3)\left(\frac{d}{dx}y(x)\right) + (x+4)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)^2\,x^r + \left(a_1(-1+r)^2 - a_0(1+2r)\,(-1+r)\right)x^{1+r} + \left(\sum_{k=2}^{\infty}\left(a_k(k+r-2)^2 - a_{k-1}(2k-1+\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 2$

- Each term must be 0

  $a_1(-1+r)^2 - a_0(1+2r)\,(-1+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = \frac{a_0(1+2r)}{-1+r}$

- Each term in the series must be 0, giving the recursion relation

  $((a_k + a_{k-2} - 2a_{k-1})\,k + (a_k + a_{k-2} - 2a_{k-1})\,r - 2a_k - 3a_{k-2} + a_{k-1})\,(k+r-2) = 0$

- Shift index using $k- >k+2$

$$((a_{k+2}+a_k-2a_{k+1})(k+2)+(a_{k+2}+a_k-2a_{k+1})r-2a_{k+2}-3a_k+a_{k+1})(k+r)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{ka_k-2ka_{k+1}+ra_k-2ra_{k+1}-a_k-3a_{k+1}}{k+r}$$

- Recursion relation for $r=2$

$$a_{k+2}=-\frac{ka_k-2ka_{k+1}+a_k-7a_{k+1}}{k+2}$$

- Solution for $r=2$

$$\left[y(x)=\sum_{k=0}^{\infty}a_k x^{k+2}, a_{k+2}=-\frac{ka_k-2ka_{k+1}+a_k-7a_{k+1}}{k+2}, a_1=5a_0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 45

```
dsolve(x^2*(x^2-2*x+1)*diff(diff(y(x),x),x)-x*(x+3)*diff(y(x),x)+(x+4)*y(x) = 0,y(x),s
```

$$y=\frac{x^2\left(\operatorname{Ei}_1\left(-\frac{4x}{x-1}\right)e^{-\frac{4x}{x-1}}c_2+e^{-\frac{4}{x-1}}c_1\right)}{x-1}$$

Mathematica DSolve solution

Solving time : 0.28 (sec)
Leaf size : 116

```
DSolve[{x^2*(1-2*x+x^2)*D[y[x],{x,2}] -x*(3+x)*D[y[x],x]+(4+x)*y[x] == 0,{}},y[x],x,IncludeS
```

$$y(x)\to \exp\left(\int_1^x \frac{2K[1]^2+K[1]+1}{2(K[1]-1)^2K[1]}dK[1]-\frac{1}{2}\int_1^x -\frac{K[2]+3}{(K[2]-1)^2K[2]}dK[2]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[3]}\frac{2K[1]^2+K[1]+1}{2(K[1]-1)^2K[1]}dK[1]\right)dK[3]+c_1\right)$$

2.1.809 Problem 832

Internal problem ID [9981]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 832
Date solved : `Monday, January 27, 2025 at 06:16:24 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(2+x)\,y''+5x^2y'+(1+x)\,y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.203 (sec)

Writing the ode as

$$\left(2x^3+4x^2\right)y''+5x^2y'+(1+x)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^3+4x^2 \\ B &= 5x^2 \\ C &= 1+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-3x^2-24x-16}{16\left(x^2+2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3x^2-24x-16 \\ t &= 16\left(x^2+2x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-3x^2-24x-16}{16\left(x^2+2x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1540: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s)\\ &= 4-2\\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2+2x)^2$. There is a pole at $x=0$ of order 2. There is a pole at $x=-2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1,2,4,6,12]$$

Attempting to find a solution using case $n=1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16\,(2+x)^2} - \frac{1}{8x} - \frac{1}{4x^2} + \frac{1}{16+8x}$$

For the pole at $x=-2$ let b be the coefficient of $\frac{1}{(2+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b} = \frac{5}{4}\\ \alpha_c^- &= \frac{1}{2}-\sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

For the pole at $x=0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 - 24x - 16}{16 (x^2 + 2x)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x^2 - 24x - 16}{16 (x^2 + 2x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= \frac{1}{4} - \left(\frac{1}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{4(2+x)} + \frac{1}{2x} + (-)(0) \\
&= -\frac{1}{4(2+x)} + \frac{1}{2x} \\
&= \frac{x+4}{4x(2+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{4\,(2+x)} + \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{4\,(2+x)^2} - \frac{1}{2x^2}\right) + \left(-\frac{1}{4\,(2+x)} + \frac{1}{2x}\right)^2 - \left(\frac{-3x^2 - 24x - 16}{16\,(x^2+2x)^2}\right)\right.$$

$$0 =$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(-\frac{1}{4(2+x)} + \frac{1}{2x}\right) dx} \\ &= \frac{\sqrt{x}}{(2+x)^{1/4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{5x^2}{2x^3+4x^2}\, dx} \\ &= z_1 e^{-\frac{5\ln(2+x)}{4}} \\ &= z_1 \left(\frac{1}{(2+x)^{5/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{(2+x)^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{5x^2}{2x^3+4x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{5\ln(2+x)}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(2\sqrt{2+x} - 2\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\sqrt{x}}{(2+x)^{3/2}}\right) + c_2 \left(\frac{\sqrt{x}}{(2+x)^{3/2}} \left(2\sqrt{2+x} - 2\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$2x^2(x+2)\left(\frac{d^2}{dx^2}y(x)\right)+5x^2\left(\frac{d}{dx}y(x)\right)+(x+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(x+1)y(x)}{2(x+2)x^2}-\frac{5\left(\frac{d}{dx}y(x)\right)}{2(x+2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)+\frac{5\left(\frac{d}{dx}y(x)\right)}{2(x+2)}+\frac{(x+1)y(x)}{2(x+2)x^2}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=\frac{5}{2(x+2)},P_3(x)=\frac{x+1}{2x^2(x+2)}\right]$

- ○ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

  $\left((x+2)\cdot P_2(x)\right)\Big|_{x=-2}=\frac{5}{2}$

- ○ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

  $\left((x+2)^2\cdot P_3(x)\right)\Big|_{x=-2}=0$

- ○ $x=-2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-2$

- Multiply by denominators

  $2x^2(x+2)\left(\frac{d^2}{dx^2}y(x)\right)+5x^2\left(\frac{d}{dx}y(x)\right)+(x+1)\,y(x)=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

  $(2u^3-8u^2+8u)\left(\frac{d^2}{du^2}y(u)\right)+(5u^2-20u+20)\left(\frac{d}{du}y(u)\right)+(u-1)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

- ○ Shift index using $k->k-m$

  $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

- ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ○ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- ○ Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$4a_0 r(3+2r)\, u^{-1+r} + \left(4a_1(1+r)(5+2r) - a_0(8r^2+12r+1)\right) u^r + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+r+1)(2k\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $4r(3+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{0, -\frac{3}{2}\right\}$
- Each term must be 0

  $4a_1(1+r)(5+2r) - a_0(8r^2+12r+1) = 0$
- Each term in the series must be 0, giving the recursion relation

  $2(-4a_k + a_{k-1} + 4a_{k+1})\, k^2 + (4(-4a_k + a_{k-1} + 4a_{k+1})\, r - 12a_k - a_{k-1} + 28a_{k+1})\, k + 2(-4a_k +$
- Shift index using $k- >k+1$

  $2(-4a_{k+1} + a_k + 4a_{k+2})\,(k+1)^2 + (4(-4a_{k+1} + a_k + 4a_{k+2})\, r - 12a_{k+1} - a_k + 28a_{k+2})\,(k+1)$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 4kr a_k - 16kr a_{k+1} + 2r^2 a_k - 8r^2 a_{k+1} + 3k a_k - 28k a_{k+1} + 3r a_k - 28r a_{k+1} + a_k - 21a_{k+1}}{4(2k^2 + 4kr + 2r^2 + 11k + 11r + 14)}$
- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 3k a_k - 28k a_{k+1} + a_k - 21a_{k+1}}{4(2k^2 + 11k + 14)}$
- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 3k a_k - 28k a_{k+1} + a_k - 21a_{k+1}}{4(2k^2 + 11k + 14)}, 20a_1 - a_0 = 0\right]$
- Revert the change of variables $u = x+2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x+2)^k, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 3k a_k - 28k a_{k+1} + a_k - 21a_{k+1}}{4(2k^2 + 11k + 14)}, 20a_1 - a_0 = 0\right]$
- Recursion relation for $r = -\frac{3}{2}$

  $a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - 3k a_k - 4k a_{k+1} + a_k + 3a_{k+1}}{4(2k^2 + 5k + 2)}$
- Solution for $r = -\frac{3}{2}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{3}{2}}, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - 3k a_k - 4k a_{k+1} + a_k + 3a_{k+1}}{4(2k^2 + 5k + 2)}, -4a_1 - a_0 = 0\right]$
- Revert the change of variables $u = x+2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k (x+2)^{k-\frac{3}{2}}, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - 3k a_k - 4k a_{k+1} + a_k + 3a_{k+1}}{4(2k^2 + 5k + 2)}, -4a_1 - a_0 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k (x+2)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+2)^{k-\frac{3}{2}}\right), a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 3k a_k - 28k a_{k+1} + a_k - 21a_{k+1}}{4(2k^2 + 11k + 14)}, \right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
```

```
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.043 (sec)
Leaf size : 39

```
dsolve(2*x^2*(x+2)*diff(diff(y(x),x),x)+5*diff(y(x),x)*x^2+(x+1)*y(x) = 0,y(x),singsol=a
```

$$y = \frac{\sqrt{x}\left(\sqrt{2}\,\sqrt{x+2}\,c_2 - 2\,\operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right) c_2 + c_1\right)}{(x+2)^{3/2}}$$

Mathematica DSolve solution

Solving time : 0.384 (sec)
Leaf size : 83

```
DSolve[{2*x^2*(2+x)*D[y[x],{x,2}] +5*x^2*D[y[x],x]+(1+x)*y[x] == 0,{}},y[x],x,IncludeSingularSo
```

$$y(x) \to \frac{\exp\left(\int_1^x \frac{K[1]+4}{4K[1]^2+8K[1]}dK[1]\right)\left(c_2\int_1^x \exp\left(-2\int_1^{K[2]} \frac{K[1]+4}{4K[1]^2+8K[1]}dK[1]\right)dK[2] + c_1\right)}{(x+2)^{5/4}}$$

2.1.810 Problem 833

Internal problem ID [9982]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 833
Date solved : `Monday, January 27, 2025 at 06:16:25 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + 4xy' + \left(x^2+2\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.130 (sec)

Writing the ode as

$$x^2y'' + 4xy' + \left(x^2+2\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 4x \\ C &= x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1542: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x}{x^2}\, dx} \\ &= z_1 e^{-2\ln(x)} \\ &= z_1 \left(\frac{1}{x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-4\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\cos(x)}{x^2}\right) + c_2\left(\frac{\cos(x)}{x^2}(\tan(x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x\left(\frac{d}{dx}y(x)\right) + (x^2+2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(x^2+2)y(x)}{x^2} - \frac{4\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{4\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(x^2+2)y(x)}{x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{4}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 4$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 2$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x\left(\frac{d}{dx}y(x)\right) + (x^2+2)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ◦ Shift index using $k->k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- ◦ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$
- ◦ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)\,(1+r)\, x^r + a_1(3+r)\,(2+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+2)\,(k+r+1) + a_{k-2})\, x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-2, -1\}$
- Each term must be 0

  $a_1(3+r)\,(2+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)\,(k+r+1) + a_{k-2} = 0$
- Shift index using $k->k+2$

  $a_{k+2}(k+4+r)\,(k+3+r) + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+4+r)(k+3+r)}$
- Recursion relation for $r = -2$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$
- Solution for $r = -2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$
- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$
- Solution for $r = -1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+2)(k+3)}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+4*diff(y(x),x)*x+(x^2+2)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{\sin(x) c_1 + \cos(x) c_2}{x^2}$$

Mathematica DSolve solution

Solving time : 0.043 (sec)
Leaf size : 37

```
DSolve[{x^2*D[y[x],{x,2}]+4*x*D[y[x],x]+(x^2+2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to \frac{2c_1 e^{-ix} - ic_2 e^{ix}}{2x^2}$$

2.1.811 Problem 834

Internal problem ID [9983]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 834
Date solved : `Monday, January 27, 2025 at 06:16:25 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.145 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1544: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\, dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos(x)}{\sqrt{x}}(\tan(x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x\left(\frac{d}{dx}y(x)\right) + \left(x^2 - \frac{1}{4}\right)y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{(4x^2-1)y(x)}{4x^2} - \frac{\frac{d}{dx}y(x)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) + \frac{\frac{d}{dx}y(x)}{x} + \frac{(4x^2-1)y(x)}{4x^2} = 0$
- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x\left(\frac{d}{dx}y(x)\right) + (4x^2 - 1)y(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- □ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx} y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$

- Shift index using $k- >k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$

- Solution for $r = -\frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$

- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.039` (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x+(x^2-1/4)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{\sin(x) c_1 + \cos(x) c_2}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : `0.033` (sec)
Leaf size : 39

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-1/4)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to \frac{e^{-ix}(2c_1 - ic_2 e^{2ix})}{2\sqrt{x}}$$

2.1.812 Problem 835

Internal problem ID [9984]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 835
Date solved : `Monday, January 27, 2025 at 06:16:26 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - xy' - \left(x^2 + \frac{5}{4}\right) y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.230 (sec)

Writing the ode as

$$x^2y'' - xy' + \left(-x^2 - \frac{5}{4}\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x \\ C &= -x^2 - \frac{5}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2+2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1546: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$[\sqrt{r}]_\infty = \sum_{i=0}^{v} a_i x^i = \sum_{i=0}^{0} a_i x^i \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{1}{x^2} - \frac{1}{2x^4} + \frac{1}{2x^6} - \frac{5}{8x^8} + \frac{7}{8x^{10}} - \frac{21}{16x^{12}} + \frac{33}{16x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$[\sqrt{r}]_\infty = \sum_{i=0}^{0} a_i x^i = 1 \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (1) + \left(\frac{2}{x^2}\right) \\ &= 1 + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{0}{1} - 0\right) = 0 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{0}{1} - 0\right) = 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2+2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + (-)(1) \\ &= -\frac{1}{x} - 1 \\ &= -\frac{1+x}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x} - 1\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - 1\right)^2 - \left(\frac{x^2+2}{x^2}\right) \right) = 0$$

$$\frac{-2 + 2a_0}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 1 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (1+x) \, e^{\int (-\frac{1}{x}-1) dx} \\ &= (1+x) \, e^{-x-\ln(x)} \\ &= \frac{(1+x) \, e^{-x}}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-x}{x^2} \, dx} \\ &= z_1 e^{\frac{\ln(x)}{2}} \\ &= z_1 \left(\sqrt{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(1+x) \, e^{-x}}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{(-1+x) \, e^{2x}}{2+2x} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{(1+x) \, e^{-x}}{\sqrt{x}} \right) + c_2 \left(\frac{(1+x) \, e^{-x}}{\sqrt{x}} \left(\frac{(-1+x) \, e^{2x}}{2+2x} \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right)-x\left(\frac{d}{dx}y(x)\right)-\left(x^2+\frac{5}{4}\right)y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{(4x^2+5)y(x)}{4x^2}+\frac{\frac{d}{dx}y(x)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)-\frac{\frac{d}{dx}y(x)}{x}-\frac{(4x^2+5)y(x)}{4x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=-\frac{1}{x},P_3(x)=-\frac{4x^2+5}{4x^2}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=-1$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=-\frac{5}{4}$

- ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right)-4x\left(\frac{d}{dx}y(x)\right)+(-4x^2-5)y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ◦ Shift index using $k->k-m$

  $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ◦ Convert $x\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r}$

- ◦ Convert $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-5+2r)\,x^r+a_1(3+2r)\,(-3+2r)\,x^{1+r}+\left(\sum_{k=2}^{\infty}(a_k(2k+2r+1)\,(2k+2r-5)-4a_{k-2}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-5+2r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{-\frac{1}{2},\frac{5}{2}\right\}$

- Each term must be 0
  $a_1(3+2r)(-3+2r)=0$
- Solve for the dependent coefficient(s)
  $a_1=0$
- Each term in the series must be 0, giving the recursion relation
  $4\left(k-\frac{5}{2}+r\right)\left(k+r+\frac{1}{2}\right)a_k-4a_{k-2}=0$
- Shift index using $k->k+2$
  $4\left(k-\frac{1}{2}+r\right)\left(k+\frac{5}{2}+r\right)a_{k+2}-4a_k=0$
- Recursion relation that defines series solution to ODE
  $a_{k+2}=\frac{4a_k}{(2k-1+2r)(2k+5+2r)}$
- Recursion relation for $r=-\frac{1}{2}$
  $a_{k+2}=\frac{4a_k}{(2k-2)(2k+4)}$
- Solution for $r=-\frac{1}{2}$
  $\left[y(x)=\sum_{k=0}^{\infty}a_kx^{k-\frac{1}{2}},a_{k+2}=\frac{4a_k}{(2k-2)(2k+4)},a_1=0\right]$
- Recursion relation for $r=\frac{5}{2}$
  $a_{k+2}=\frac{4a_k}{(2k+4)(2k+10)}$
- Solution for $r=\frac{5}{2}$
  $\left[y(x)=\sum_{k=0}^{\infty}a_kx^{k+\frac{5}{2}},a_{k+2}=\frac{4a_k}{(2k+4)(2k+10)},a_1=0\right]$
- Combine solutions and rename parameters
  $\left[y(x)=\left(\sum_{k=0}^{\infty}a_kx^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_kx^{k+\frac{5}{2}}\right),a_{k+2}=\frac{4a_k}{(2k-2)(2k+4)},a_1=0,b_{k+2}=\frac{4b_k}{(2k+4)(2k+10)},b_1=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.043 (sec)
Leaf size : 25

```
dsolve(x^2*diff(diff(y(x),x),x)-diff(y(x),x)*x-(x^2+5/4)*y(x) = 0,y(x),singsol=all)
```

$$y=\frac{(x+1)\,c_2\mathrm{e}^{-x}+c_1\mathrm{e}^{x}(x-1)}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.063 (sec)
Leaf size : 53

```
DSolve[{x^2*D[y[x],{x,2}]-x*D[y[x],x]-(x^2+5/4)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tr
```

$$y(x) \to \frac{\sqrt{\frac{2}{\pi}}((ic_2x + c_1)\sinh(x) - (c_1x + ic_2)\cosh(x))}{\sqrt{-ix}}$$

2.1.813 Problem 836

Internal problem ID [9985]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 836
Date solved : `Monday, January 27, 2025 at 06:16:27 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.140 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1548: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\, dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos(x)}{\sqrt{x}}(\tan(x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x\left(\frac{d}{dx}y(x)\right) + \left(x^2 - \frac{1}{4}\right)y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(4x^2-1)y(x)}{4x^2} - \frac{\frac{d}{dx}y(x)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{\frac{d}{dx}y(x)}{x} + \frac{(4x^2-1)y(x)}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$
- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 1$
- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$
- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$
- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x\left(\frac{d}{dx}y(x)\right) + (4x^2 - 1)y(x) = 0$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx} y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-2})\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$

- Shift index using $k- >k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$

- Solution for $r = -\frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$

- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.066 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x+(x^2-1/4)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{\sin(x) c_1 + \cos(x) c_2}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.029 (sec)
Leaf size : 39

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-1/4)*y[x]==0,{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to \frac{e^{-ix}(2c_1 - ic_2 e^{2ix})}{2\sqrt{x}}$$

2.1.814 Problem 837

Internal problem ID [9986]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 837
Date solved : `Monday, January 27, 2025 at 06:16:27 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$x^2y'' + 3xy' + 4x^4y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.272 (sec)

Writing the ode as

$$x^2y'' + 3xy' + 4x^4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 3x \\ C &= 4x^4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16x^4 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -16x^4 + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-16x^4 + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1550: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -4x^2 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 2ix - \frac{3i}{16x^3} - \frac{9i}{1024x^7} - \frac{27i}{32768x^{11}} - \frac{405i}{4194304x^{15}} - \frac{1701i}{134217728x^{19}} - \frac{15309i}{8589934592x^{23}} - \frac{72171i}{274877906944x^{27}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 2i$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= 2ix \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -4x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-16x^4 + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (-4x^2) + \left(\frac{3}{4x^2}\right) \\ &= -4x^2 + \frac{3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 2ix \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{0}{2i} - 1\right) = -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{0}{2i} - 1\right) = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-16x^4 + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$2ix$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)\,(2ix) \\ &= -\frac{1}{2x} - 2ix \\ &= -\frac{1}{2x} - 2ix \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{2x} - 2ix\right)(0) + \left(\left(\frac{1}{2x^2} - 2i\right) + \left(-\frac{1}{2x} - 2ix\right)^2 - \left(\frac{-16x^4+3}{4x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(-\frac{1}{2x} - 2ix\right) dx} \\ &= \frac{e^{-ix^2}}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{3x}{x^2}\,dx}\\ &= z_1 e^{-\frac{3\ln(x)}{2}}\\ &= z_1\left(\frac{1}{x^{3/2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-i x^2}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{x^2}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{-3\ln(x)}}{(y_1)^2}\,dx\\ &= y_1\left(-\frac{i\mathrm{e}^{2i x^2}}{4}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2\\ &= c_1\left(\frac{\mathrm{e}^{-i x^2}}{x^2}\right) + c_2\left(\frac{\mathrm{e}^{-i x^2}}{x^2}\left(-\frac{i\mathrm{e}^{2i x^2}}{4}\right)\right)\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) + 3x\left(\frac{d}{dx}y(x)\right) + 4x^4 y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -4x^2 y(x) - \frac{3\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{3\left(\frac{d}{dx}y(x)\right)}{x} + 4x^2 y(x) = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{3}{x}, P_3(x) = 4x^2\right]$
- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$(x \cdot P_2(x))\Big|_{x=0} = 3$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $\left(\frac{d^2}{dx^2}y(x)\right)x + 3\frac{d}{dx}y(x) + 4x^3y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^3 \cdot y(x)$ to series expansion

  $x^3 \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+3}$

- Shift index using $k- >k - 3$

  $x^3 \cdot y(x) = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$

- Convert $\frac{d}{dx}y(x)$ to series expansion

  $\frac{d}{dx}y(x) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

- Shift index using $k- >k + 1$

  $\frac{d}{dx}y(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$

- Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k- >k + 1$

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(2+r)\, x^{-1+r} + a_1(1+r)\,(3+r)\, x^r + a_2(2+r)\,(4+r)\, x^{1+r} + a_3(3+r)\,(5+r)\, x^{2+r} + \left(\sum_{k=3}^{\infty} (\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 0\}$

- The coefficients of each power of x must be 0

  $[a_1(1+r)\,(3+r) = 0, a_2(2+r)\,(4+r) = 0, a_3(3+r)\,(5+r) = 0]$

- Solve for the dependent coefficient(s)

  $\{a_1 = 0, a_2 = 0, a_3 = 0\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k+r+3) + 4a_{k-3} = 0$

- Shift index using $k- >k + 3$

  $a_{k+4}(k+4+r)\,(k+6+r) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+4} = -\frac{4a_k}{(k+4+r)(k+6+r)}$

- Recursion relation for $r = -2$

  $a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}$
- Solution for $r = -2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$
- Recursion relation for $r = 0$

  $a_{k+4} = -\frac{4a_k}{(k+4)(k+6)}$
- Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k}, a_{k+4} = -\frac{4a_k}{(k+4)(k+6)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k}\right), a_{4+k} = -\frac{4a_k}{(k+2)(4+k)}, a_1 = 0, a_2 = 0, a_3 = 0, b_{4+k} = -\frac{4b_k}{(4+k)(k+6)}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 21

```
dsolve(x^2*diff(diff(y(x),x),x)+3*diff(y(x),x)*x+4*x^4*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{c_1 \sin(x^2) + c_2 \cos(x^2)}{x^2}$$

Mathematica DSolve solution

Solving time : 0.054 (sec)
Leaf size : 41

```
DSolve[{x^2*D[y[x],{x,2}]+3*x*D[y[x],x]+4*x^4*y[x]==0,{}},y[x],x,IncludeSingularSolutions->Tru
```

$$y(x) \to \frac{4c_1 e^{-ix^2} - ic_2 e^{ix^2}}{4x^2}$$

2.1.815 Problem 838

Internal problem ID [9987]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 838
Date solved : `Monday, January 27, 2025 at 06:16:28 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' = \left(x^2+3\right) y$$

Solved as second order ode using Kovacic algorithm

Time used: 0.209 (sec)

Writing the ode as

$$y'' + \left(-x^2-3\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -x^2-3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2+3}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2+3 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(x^2+3\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1552: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x + \frac{3}{2x} - \frac{9}{8x^3} + \frac{27}{16x^5} - \frac{405}{128x^7} + \frac{1701}{256x^9} - \frac{15309}{1024x^{11}} + \frac{72171}{2048x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2+3}{1} \\ &= Q + \frac{R}{1} \\ &= (x^2+3) + (0) \\ &= x^2+3 \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is 3. Now b can be found.

$$\begin{aligned} b &= (3) - (0) \\ &= 3 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= x \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{3}{1} - 1\right) &= 1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{3}{1} - 1\right) &= -2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = x^2 + 3$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	x	1	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 1$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + (x) \\ &= x \\ &= x\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2(x)(1) + \left((1) + (x)^2 - (x^2 + 3)\right) &= 0 \\ -2a_0 &= 0\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= (x)\, e^{\int x dx} \\ &= (x)\, e^{\frac{x^2}{2}} \\ &= x\, e^{\frac{x^2}{2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1 \\ &= x\, e^{\frac{x^2}{2}}\end{aligned}$$

Which simplifies to

$$y_1 = x\, e^{\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= x \, \mathrm{e}^{\frac{x^2}{2}} \int \frac{1}{x^2 \mathrm{e}^{x^2}} \, dx \\ &= x \, \mathrm{e}^{\frac{x^2}{2}} \left(-\frac{\mathrm{e}^{-x^2}}{x} - \sqrt{\pi} \, \mathrm{erf}\,(x) \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(x \, \mathrm{e}^{\frac{x^2}{2}} \right) + c_2 \left(x \, \mathrm{e}^{\frac{x^2}{2}} \left(-\frac{\mathrm{e}^{-x^2}}{x} - \sqrt{\pi} \, \mathrm{erf}\,(x) \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2} y(x) = (x^2 + 3)\, y(x)$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2} y(x)$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2} y(x) + (-x^2 - 3)\, y(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$
    $x^m \cdot y(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$
  - Shift index using $k- >k - m$
    $x^m \cdot y(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$
  - Convert $\frac{d^2}{dx^2} y(x)$ to series expansion
    $\frac{d^2}{dx^2} y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$
  - Shift index using $k- >k + 2$
    $\frac{d^2}{dx^2} y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions
  $2a_2 - 3a_0 + (6a_3 - 3a_1)\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) - 3a_k - a_{k-2})\, x^k \right) = 0$

- The coefficients of each power of x must be 0
  $[2a_2 - 3a_0 = 0, 6a_3 - 3a_1 = 0]$
- Solve for the dependent coefficient(s)
  $\left\{a_2 = \frac{3a_0}{2}, a_3 = \frac{a_1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation
  $(k^2+3k+2)\, a_{k+2}-3a_k-a_{k-2}=0$
- Shift index using $k->k+2$
  $\left((k+2)^2+3k+8\right) a_{k+4}-3a_{k+2}-a_k=0$
- Recursion relation that defines the series solution to the ODE
  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^k, a_{k+4}=\frac{3a_{k+2}+a_k}{k^2+7k+12}, a_2=\frac{3a_0}{2}, a_3=\frac{a_1}{2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 30

```
dsolve(diff(diff(y(x),x),x) = (x^2+3)*y(x),y(x),singsol=all)
```

$$y = x\left(c_2 \operatorname{erf}(x)\sqrt{\pi}+c_1\right) e^{\frac{x^2}{2}}+e^{-\frac{x^2}{2}} c_2$$

Mathematica DSolve solution

Solving time : 0.062 (sec)
Leaf size : 46

```
DSolve[{D[y[x],{x,2}]==(x^2+3)*y[x],{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to e^{-\frac{x^2}{2}}\left(-\sqrt{\pi}c_2 e^{x^2} x\operatorname{erf}(x)+c_1 e^{x^2} x-c_2\right)$$

2.1.816 Problem 839

Internal problem ID [9988]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 839
Date solved : `Monday, January 27, 2025 at 06:16:28 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + 2xy' + (x^2+1)\, y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.066 (sec)

Writing the ode as

$$y'' + 2xy' + (x^2+1)\, y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2x \\ C &= x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1554: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - -\infty\\ &= \infty\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{2x}{1}\, dx}\\ &= z_1 e^{-\frac{x^2}{2}}\\ &= z_1 \left(\mathrm{e}^{-\frac{x^2}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{\mathrm{e}^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-x^2}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-\frac{x^2}{2}}\right) + c_2\left(\mathrm{e}^{-\frac{x^2}{2}}(x)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + 2x\left(\frac{d}{dx}y(x)\right) + (x^2+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  - Shift index using $k->k-m$

    $x^m \cdot y(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k->k+2$

    $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 + a_0 + (6a_3 + 3a_1)\,x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)(k+1) + a_k(2k+1) + a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 + a_0 = 0, 6a_3 + 3a_1 = 0]$

- Solve for the dependent coefficient(s)

  $\left\{a_2 = -\frac{a_0}{2}, a_3 = -\frac{a_1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\,a_{k+2} + 2a_k k + a_k + a_{k-2} = 0$

- Shift index using $k->k+2$

  $\left((k+2)^2+3k+8\right)a_{k+4}+2a_{k+2}(k+2)+a_{k+2}+a_k=0$

- Recursion relation that defines the series solution to the ODE

  $\left[y(x)=\sum_{k=0}^{\infty}a_k x^k, a_{k+4}=-\frac{2ka_{k+2}+a_k+5a_{k+2}}{k^2+7k+12}, a_2=-\frac{a_0}{2}, a_3=-\frac{a_1}{2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 16

```
dsolve(diff(diff(y(x),x),x)+2*diff(y(x),x)*x+(x^2+1)*y(x) = 0,y(x),singsol=all)
```

$$y=\mathrm{e}^{-\frac{x^2}{2}}(c_2x+c_1)$$

Mathematica DSolve solution

Solving time : 0.024 (sec)
Leaf size : 22

```
DSolve[{D[y[x],{x,2}]+2*x*D[y[x],x]+(x^2+1)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to e^{-\frac{x^2}{2}}(c_2x+c_1)$$

2.1.817 Problem 840

Internal problem ID [9989]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 840
Date solved : `Monday, January 27, 2025 at 06:16:29 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.143 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1556: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2\left(\frac{\cos(x)}{\sqrt{x}}(\tan(x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right) + x\left(\frac{d}{dx}y(x)\right) + \left(x^2 - \frac{1}{4}\right)y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{(4x^2-1)y(x)}{4x^2} - \frac{\frac{d}{dx}y(x)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x) + \frac{\frac{d}{dx}y(x)}{x} + \frac{(4x^2-1)y(x)}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$

- Multiply by denominators
  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) + 4x\left(\frac{d}{dx}y(x)\right) + (4x^2 - 1)y(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$
- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-2})\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$
- Shift index using $k- >k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$
- Solution for $r = -\frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$
- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.039 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x+(x^2-1/4)*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{\sin(x) c_1 + \cos(x) c_2}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.031 (sec)
Leaf size : 39

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-1/4)*y[x] == 0,{}},y[x],x,IncludeSingularSolution
```

$$y(x) \to \frac{e^{-ix}(2c_1 - ic_2 e^{2ix})}{2\sqrt{x}}$$

2.1.818 Problem 841

Internal problem ID [9990]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 841
Date solved : `Monday, January 27, 2025 at 06:16:30 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' + \left(-8x^2+4x\right)y' + \left(4x^2-4x-1\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.089 (sec)

Writing the ode as

$$4x^2y'' + \left(-8x^2+4x\right)y' + \left(4x^2-4x-1\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -8x^2+4x \\ C &= 4x^2-4x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1558: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-8x^2+4x}{4x^2}\, dx} \\ &= z_1 e^{x - \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{e^x}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^x}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-8x^2+4x}{4x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2x-\ln(x)}}{(y_1)^2}\, dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^x}{\sqrt{x}}\right) + c_2 \left(\frac{e^x}{\sqrt{x}}(x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$4x^2\left(\frac{d^2}{dx^2}y(x)\right) + (-8x^2+4x)\left(\frac{d}{dx}y(x)\right) + (4x^2-4x-1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(4x^2-4x-1)y(x)}{4x^2} + \frac{(2x-1)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) - \frac{(2x-1)\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(4x^2-4x-1)y(x)}{4x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{2x-1}{x}, P_3(x) = \frac{4x^2-4x-1}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{1}{4}$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) - 4x(2x-1)\left(\frac{d}{dx}y(x)\right) + (4x^2-4x-1)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ∘ Shift index using $k->k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ∘ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- ∘ Shift index using $k->k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- ∘ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\, x^r + (a_1(3+2r)\,(1+2r) - 4a_0(1+2r))\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3+2r)\,(1+2r) - 4a_0(1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = \frac{4a_0}{3+2r}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + (-8k - 8r + 4)\, a_{k-1} + 4a_{k-2} = 0$

- Shift index using $k->k+2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + (-8k - 12 - 8r)\, a_{k+1} + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{4(2ka_{k+1} + 2ra_{k+1} - a_k + 3a_{k+1})}{4k^2 + 8kr + 4r^2 + 16k + 16r + 15}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = \frac{4(2ka_{k+1} - a_k + 2a_{k+1})}{4k^2 + 12k + 8}$

- Solution for $r = -\frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = \frac{4(2ka_{k+1} - a_k + 2a_{k+1})}{4k^2 + 12k + 8}, a_1 = 2a_0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = \frac{4(2ka_{k+1} - a_k + 4a_{k+1})}{4k^2 + 20k + 24}$

- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{4(2ka_{k+1} - a_k + 4a_{k+1})}{4k^2 + 20k + 24}, a_1 = a_0\right]$

- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = \frac{4(2ka_{k+1} - a_k + 2a_{k+1})}{4k^2 + 12k + 8}, a_1 = 2a_0, b_{k+2} = \frac{4(2kb_{k+1} - b_k + 4}{4k^2 + 20k + 2}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.022 (sec)
Leaf size : 15

```
dsolve(4*x^2*diff(diff(y(x),x),x)+(-8*x^2+4*x)*diff(y(x),x)+(4*x^2-4*x-1)*y(x) = 0,y(x),
```

$$y = \frac{e^x(c_2x + c_1)}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.042 (sec)
Leaf size : 21

```
DSolve[{4*x^2*D[y[x],{x,2}]+(-8*x^2+4*x)*D[y[x],x]+(4*x^2-4*x-1)*y[x] == 0,{}},y[x],x,IncludeS
```

$$y(x) \to \frac{e^x(c_2x + c_1)}{\sqrt{x}}$$

2.1.819 Problem 843

Internal problem ID [9991]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 843
Date solved : `Monday, January 27, 2025 at 06:16:30 PM`
CAS classification : `[[_2nd_order, _quadrature]]`

Solve

$$y'' = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.031 (sec)

Writing the ode as

$$y'' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1560: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= 1 \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\
&= 1 \int \frac{1}{1} \, dx \\
&= 1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2(1(x))
\end{aligned}
$$

Will add steps showing solving for IC soon.

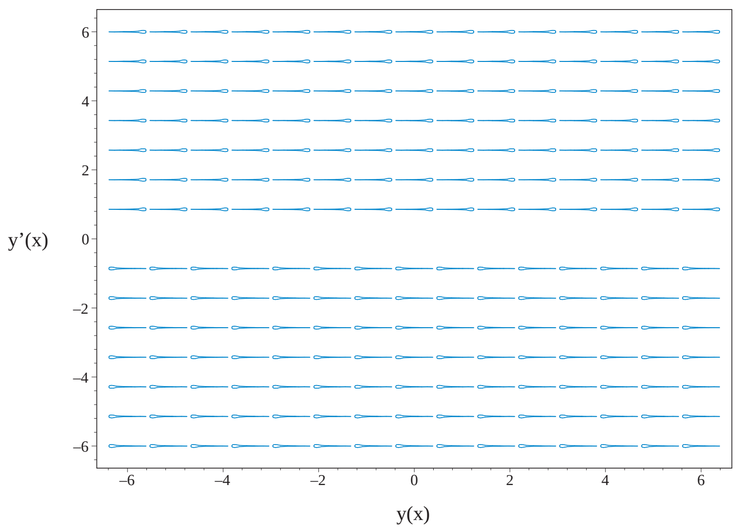

Figure 2.4: Slope field plot
$y'' = 0$

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2} y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2} y(x)$
- Characteristic polynomial of ODE
  $r^2 = 0$
- Use quadratic formula to solve for r
  $r = \frac{0 \pm (\sqrt{0})}{2}$
- Roots of the characteristic polynomial
  $r = 0$
- 1st solution of the ODE
  $y_1(x) = 1$
- Repeated root, multiply $y_1(x)$ by x to ensure linear independence
  $y_2(x) = x$
- General solution of the ODE
  $y(x) = C1 y_1(x) + C2 y_2(x)$
- Substitute in solutions
  $y(x) = C2x + C1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

Maple dsolve solution

Solving time : 0.000 (sec)
Leaf size : 9

```
dsolve(diff(diff(y(x),x),x) = 0,y(x),singsol=all)
```

$$y = c_1 x + c_2$$

Mathematica DSolve solution

Solving time : 0.002 (sec)
Leaf size : 12

```
DSolve[{D[y[x],{x,2}]==((4*(1/2)^2-1)/(4*x^2))*y[x],{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_2 x + c_1$$

2.1.820 Problem 844

Internal problem ID [9992]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 844
Date solved : `Monday, January 27, 2025 at 06:16:31 PM`
CAS classification :
`[[_Emden, _Fowler], [_2nd_order, _linear, '_with_symmetry_[0,F(x)]']]`

Solve

$$y'' = \frac{2y}{x^2}$$

Solved as second order ode using Kovacic algorithm

Time used: 0.097 (sec)

Writing the ode as

$$y'' - \frac{2y}{x^2} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -\frac{2}{x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1562: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= -1 - (-1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x} + (-)(0) \\
&= -\frac{1}{x} \\
&= -\frac{1}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{1}{x} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= \frac{1}{x} \int \frac{1}{\frac{1}{x^2}} \, dx \\ &= \frac{1}{x}\left(\frac{x^3}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x}\left(\frac{x^3}{3}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) = \frac{2y(x)}{x^2}$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x) - \frac{2y(x)}{x^2} = 0$

- Multiply by denominators of the ODE

$x^2\left(\frac{d^2}{dx^2}y(x)\right) - 2y(x) = 0$

- Make a change of variables

$t = \ln(x)$

- □ Substitute the change of variables back into the ODE
  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $\frac{d}{dx}y(x) = \left(\frac{d}{dt}y(t)\right)\left(\frac{d}{dx}t(x)\right)$
  - Compute derivative

    $\frac{d}{dx}y(x) = \frac{\frac{d}{dt}y(t)}{x}$
  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $\frac{d^2}{dx^2}y(x) = \left(\frac{d^2}{dt^2}y(t)\right)\left(\frac{d}{dx}t(x)\right)^2 + \left(\frac{d^2}{dx^2}t(x)\right)\left(\frac{d}{dt}y(t)\right)$
  - Compute derivative

    $\frac{d^2}{dx^2}y(x) = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - 2y(t) = 0$

- Simplify

$\frac{d^2}{dt^2}y(t) - \frac{d}{dt}y(t) - 2y(t) = 0$

- Characteristic polynomial of ODE

$r^2 - r - 2 = 0$

- Factor the characteristic polynomial

$(r+1)(r-2) = 0$

- Roots of the characteristic polynomial

$r = (-1, 2)$

- 1st solution of the ODE

$y_1(t) = e^{-t}$

- 2nd solution of the ODE

$y_2(t) = e^{2t}$

- General solution of the ODE

$y(t) = C1 y_1(t) + C2 y_2(t)$

- Substitute in solutions

$y(t) = C1\, e^{-t} + C2\, e^{2t}$

- Change variables back using $t = \ln(x)$

$y(x) = \frac{C1}{x} + C2\, x^2$

- Simplify

$y(x) = \frac{C1}{x} + C2\, x^2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 15

```
dsolve(diff(diff(y(x),x),x) = 2/x^2*y(x),y(x),singsol=all)
```

$$y = \frac{c_2 x^3 + c_1}{x}$$

Mathematica DSolve solution

Solving time : 0.011 (sec)
Leaf size : 18

```
DSolve[{D[y[x],{x,2}]==((4*(3/2)^2-1)/(4*x^2))*y[x],{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2 x^3 + c_1}{x}$$

2.1.821 Problem 845

Internal problem ID [9993]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 845
Date solved : `Monday, January 27, 2025 at 06:16:31 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$y'' = \frac{6y}{x^2}$$

Solved as second order ode using Kovacic algorithm

Time used: 0.104 (sec)

Writing the ode as

$$y'' - \frac{6y}{x^2} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -\frac{6}{x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{6}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 6 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{6}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1564: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{6}{x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{6}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	3	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -2$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= -2 - (-2) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{2}{x} + (-)(0) \\
&= -\frac{2}{x} \\
&= -\frac{2}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{2}{x}\right)(0) + \left(\left(\frac{2}{x^2}\right) + \left(-\frac{2}{x}\right)^2 - \left(\frac{6}{x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega \, dx} \\ &= e^{\int -\frac{2}{x} dx} \\ &= \frac{1}{x^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{1}{x^2} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= \frac{1}{x^2} \int \frac{1}{\frac{1}{x^4}} \, dx \\ &= \frac{1}{x^2}\left(\frac{x^5}{5}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{1}{x^2}\right) + c_2\left(\frac{1}{x^2}\left(\frac{x^5}{5}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) = \frac{6y(x)}{x^2}$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) - \frac{6y(x)}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) - 6y(x) = 0$

- Make a change of variables

  $t = \ln(x)$

- □ Substitute the change of variables back into the ODE
  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $\frac{d}{dx}y(x) = \left(\frac{d}{dt}y(t)\right)\left(\frac{d}{dx}t(x)\right)$

  - Compute derivative

    $\frac{d}{dx}y(x) = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $\frac{d^2}{dx^2}y(x) = \left(\frac{d^2}{dt^2}y(t)\right)\left(\frac{d}{dx}t(x)\right)^2 + \left(\frac{d^2}{dx^2}t(x)\right)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $\frac{d^2}{dx^2}y(x) = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - 6y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - \frac{d}{dt}y(t) - 6y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - r - 6 = 0$

- Factor the characteristic polynomial

  $(r+2)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (-2, 3)$

- 1st solution of the ODE

  $y_1(t) = e^{-2t}$

- 2nd solution of the ODE

  $y_2(t) = e^{3t}$

- General solution of the ODE

  $y(t) = C1 y_1(t) + C2 y_2(t)$

- Substitute in solutions

  $y(t) = C1\, e^{-2t} + C2\, e^{3t}$

- Change variables back using $t = \ln(x)$

  $y(x) = \frac{C1}{x^2} + C2\, x^3$

- Simplify

  $y(x) = \frac{C1}{x^2} + C2\, x^3$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 15

```
dsolve(diff(diff(y(x),x),x) = 6/x^2*y(x),y(x),singsol=all)
```

$$y = \frac{c_1 x^5 + c_2}{x^2}$$

Mathematica DSolve solution

Solving time : 0.012 (sec)
Leaf size : 18

```
DSolve[{D[y[x],{x,2}]==((4*(5/2)^2-1)/(4*x^2))*y[x],{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2 x^5 + c_1}{x^2}$$

2.2 section 2. Solution found using all possible Kovacic cases

2.2.1 Problem 1

Internal problem ID [9994]
Book : Collection of Kovacic problems
Section : section 2. Solution found using all possible Kovacic cases
Problem number : 1
Date solved : `Monday, January 27, 2025 at 06:16:32 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' = \left(-\frac{3}{16x^2} - \frac{2}{9(x-1)^2} + \frac{3}{16x(x-1)}\right) y$$

Solved as second order ode using Kovacic algorithm

Time used: 1.028 (sec)

Writing the ode as

$$y'' + \frac{(32x^2 - 27x + 27)\, y}{144x^2 (x-1)^2} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= \frac{32x^2 - 27x + 27}{144x^2 (x-1)^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-32x^2 + 27x - 27}{144 (x^2 - x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -32x^2 + 27x - 27 \\ t &= 144 (x^2 - x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-32x^2 + 27x - 27}{144\,(x^2 - x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1566: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 144(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Unable to find solution using case two.

Attempting to find a solution using $n = 4$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{16\,(x-1)} - \frac{2}{9\,(x-1)^2} - \frac{3}{16x} - \frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. This shows that $b = -\frac{3}{16}$. Hence

$$E_c = \left\{6 + \frac{12k}{n}\sqrt{1+4b} \middle| k = 0, \pm 1, \pm 2, \ldots, \pm \frac{n}{2}\right\} \cap \mathbb{Z}$$

Where n for case 3 is $4,6$ or 12. For the current case $n=4$. Hence the above becomes

$$E_c = \{3,6,9\}$$

For the pole at $x=1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. This shows that $b=-\frac{2}{9}$. Hence

$$E_c = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \ldots, \pm \frac{n}{2}\right\} \cap \mathbb{Z}$$

Where n for case 3 is $4,6$ or 12. For the current case $n=4$. Hence the above becomes

$$E_c = \{4,5,6,7,8\}$$

Let

$$E_\infty = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \ldots, \pm \frac{n}{2}\right\} \cap \mathbb{Z} \tag{B1}$$

Where b is the coefficient of $\frac{1}{x^2}$ in the Laurent series for r at ∞ given by

$$r \approx -\frac{2}{9x^2} - \frac{37}{144x^3} - \frac{23}{48x^4} - \frac{101}{144x^5} - \frac{133}{144x^6} - \frac{55}{48x^7} + \cdots$$

The above shows that

$$b = -\frac{2}{9}$$

The value of n in eq. (B1) for case 3 is $4,6$ or 2.For the current case $n=4$. eq. (B1) simplifies to the following, after removing any duplicate and non integer entries in the set.

$$E_\infty = \{4,5,6,7,8\}$$

The following table summarizes the results found so far for poles and for the order of r at ∞ for case 3 of Kovacic algorithm using $n=4$.

pole c location	pole order	set $\{E_c\}$
0	2	$\{3,6,9\}$
1	2	$\{4,5,6,7,8\}$

Order of r at ∞	set $\{E_\infty\}$
2	$\{4,5,6,7,8\}$

Now that E_c sets for all poles are found and E_∞ set is found, the next step is to determine a non negative integer d using the following

$$d = \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)$$

Where in the above e_c is a distinct element from each corresponding E_c. This means all possible tuples $\{e_{c_1}, e_{c_2}, \ldots, e_{c_n}\}$ are tried in the sum above, where e_{c_i} is one element of each E_c found earlier. Using the following family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 3,\ e_2 = 4,\ e_\infty = 7$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{4}{12}(7-(3+(4))) \\ &= 0 \end{aligned}$$

The following rational function is

$$\begin{aligned}\theta &= \frac{n}{12}\sum_{c\in\Gamma}\frac{e_c}{x-c}\\ &= \frac{4}{12}\left(\frac{3}{(x-(0))}+\frac{4}{(x-(1))}\right)\\ &= \frac{1}{x}+\frac{4}{3x-3}\end{aligned}$$

And

$$\begin{aligned}S &= \prod_{c\in\Gamma}(x-c)\\ &= x(x-1)\end{aligned}$$

The polynomial $p(x)$ is now determined. Since the degree of the polynomial is $d = 0$, then let

$$p(x) = 1$$

The following set of equations are set up in order to determine the coefficients a_i (if any) of the above polynomial

$$\begin{aligned}P_n &= -p(x)\\ &= -1\\ P_{i-1} &= -Sp_i' + ((n-i)S' - S\theta)P_i - (n-1)(i+1)S^2rP_{i+1} \qquad i = n, n-1, \ldots, 0 \quad \text{(1A)}\end{aligned}$$

The coefficients a_i are solved for from

$$P_{-1} = 0 \tag{2A}$$

By using method of undetermined coefficients. Carrying the above computation in eq. (1A) gives the following sequence of polynomials P_i (noting that $n = 4$ and $r = \frac{-32x^2+27x-27}{144(x^2-x)^2}$).

$$\begin{aligned}P_4 &= -p\\ &= -1\\ P_3 &= \frac{7x}{3}-1\\ P_2 &= -4x^2+\frac{41}{12}x-\frac{3}{4}\\ P_1 &= \frac{40}{9}x^3-\frac{409}{72}x^2+\frac{5}{2}x-\frac{3}{8}\\ P_0 &= -\frac{64}{27}x^4+\frac{871}{216}x^3-\frac{257}{96}x^2+\frac{13}{16}x-\frac{3}{32}\\ P_{-1} &= 0\end{aligned}$$

Because $P_{-1} = 0$ then $z = e^{\int \omega}$ is a solution. ω is found by finding a solution to the equation generated by the following sum

$$\begin{aligned}\sum_{i=0}^{n} S^i\frac{P_i}{(n-i)!}\omega^i &= 0\\ \sum_{i=0}^{4} S^i\frac{P_i}{(4-i)!}\omega^i &= 0\end{aligned}$$

Where the P_i are the polynomials found earlier. Computing the above sum gives

$$-\frac{8x^4}{81}+\frac{871x^3}{5184}-\frac{257x^2}{2304}+\frac{13x}{384}-\frac{1}{256}+\frac{x(x-1)\left(320x^3-409x^2+180x-27\right)\omega}{432} - \frac{x^2(x-1)^2\left(48x^2-41x+9\right)\omega^2}{24}+x^3(x-1)^3\left(\frac{7x}{3}-1\right)\omega^3-x^4(x-1)^4\omega^4 = 0$$

The solution ω of eq. 3A is found as

$$\omega = \frac{1}{12x(x-1)}\left(7x-3+\sqrt{x^2+\left((x-1)^2x^3\right)^{1/3}-x} + \sqrt{-\frac{2\left(\left(-x^2+x+\frac{\left((x-1)^2x^3\right)^{1/3}}{2}\right)\sqrt{x^2+\left((x-1)^2x^3\right)^{1/3}-x}+x^2(x-1)\right)}{\sqrt{x^2+\left((x-1)^2x^3\right)^{1/3}-x}}}\right) \tag{4A}$$

This ω is used to find a solution to $z''=rz$.

$$z_1(x) = e^{\int \omega\, dx} \tag{5A}$$

Unable to integrate $\int \omega dx$. Leaving the integral unevaluated. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B=0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= e^{\int \omega\, dx} \end{aligned}$$

Where ω given above. The second solution y_2 to the original ode is found using reduction of order

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx \\ &= e^{\int \omega\, dx} \int \frac{e^{\int -\frac{B}{A}\, dx}}{\left(e^{\int \omega\, dx}\right)^2}\, dx \end{aligned}$$

Since $B=0$ then the above reduces to

$$y_2 = e^{\int \omega\, dx} \int \left(e^{\int \omega\, dx}\right)^{-2}\, dx$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(e^{\int \omega\, dx}\right) + c_2\left(e^{\int \omega\, dx} \int \left(e^{\int \omega\, dx}\right)^{-2}\, dx\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) = \left(-\frac{3}{16x^2} - \frac{2}{9(x-1)^2} + \frac{3}{16x(x-1)}\right) y(x)$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(32x^2-27x+27)y(x)}{144x^2(x-1)^2}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(32x^2-27x+27)y(x)}{144x^2(x-1)^2} = 0$

□ Check to see if x_0 is a regular singular point

- ∘ Define functions

  $\left[P_2(x) = 0, P_3(x) = \frac{32x^2-27x+27}{144x^2(x-1)^2}\right]$

- ∘ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 0$

- ∘ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{3}{16}$

- ∘ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $144x^2(x-1)^2\left(\frac{d^2}{dx^2}y(x)\right) + (32x^2 - 27x + 27)\, y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ∘ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ∘ Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ∘ Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$

- ∘ Shift index using $k- >k + 2 - m$

  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $9a_0(-1+4r)(-3+4r)\, x^r + (9a_1(3+4r)(1+4r) - 9a_0(32r^2-32r+3))\, x^{1+r} + \left(\sum_{k=2}^{\infty} (9a_k(4k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $9(-1+4r)(-3+4r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{4}, \frac{3}{4}\right\}$

- Each term must be 0

  $9a_1(3+4r)(1+4r)-9a_0(32r^2-32r+3)=0$
- Solve for the dependent coefficient(s)

  $a_1=\frac{a_0(32r^2-32r+3)}{16r^2+16r+3}$
- Each term in the series must be 0, giving the recursion relation

  $144(a_k+a_{k-2}-2a_{k-1})k^2+144(2(a_k+a_{k-2}-2a_{k-1})r-a_k-5a_{k-2}+6a_{k-1})k+144(a_k+a_{k-2}-$
- Shift index using $k->k+2$

  $144(a_{k+2}+a_k-2a_{k+1})(k+2)^2+144(2(a_{k+2}+a_k-2a_{k+1})r-a_{k+2}-5a_k+6a_{k+1})(k+2)+144$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{144k^2a_k-288k^2a_{k+1}+288kra_k-576kra_{k+1}+144r^2a_k-288r^2a_{k+1}-144ka_k-288ka_{k+1}-144ra_k-288ra_{k+1}+32a_k-27a_{k+1}}{9(16k^2+32kr+16r^2+48k+48r+35)}$
- Recursion relation for $r=\frac{1}{4}$

  $a_{k+2}=-\frac{144k^2a_k-288k^2a_{k+1}-72ka_k-432ka_{k+1}+5a_k-117a_{k+1}}{9(16k^2+56k+48)}$
- Solution for $r=\frac{1}{4}$

  $\left[y(x)=\sum_{k=0}^{\infty}a_kx^{k+\frac{1}{4}},a_{k+2}=-\frac{144k^2a_k-288k^2a_{k+1}-72ka_k-432ka_{k+1}+5a_k-117a_{k+1}}{9(16k^2+56k+48)},a_1=-\frac{3a_0}{8}\right]$
- Recursion relation for $r=\frac{3}{4}$

  $a_{k+2}=-\frac{144k^2a_k-288k^2a_{k+1}+72ka_k-720ka_{k+1}+5a_k-405a_{k+1}}{9(16k^2+72k+80)}$
- Solution for $r=\frac{3}{4}$

  $\left[y(x)=\sum_{k=0}^{\infty}a_kx^{k+\frac{3}{4}},a_{k+2}=-\frac{144k^2a_k-288k^2a_{k+1}+72ka_k-720ka_{k+1}+5a_k-405a_{k+1}}{9(16k^2+72k+80)},a_1=-\frac{a_0}{8}\right]$
- Combine solutions and rename parameters

  $\left[y(x)=\left(\sum_{k=0}^{\infty}a_kx^{k+\frac{1}{4}}\right)+\left(\sum_{k=0}^{\infty}b_kx^{k+\frac{3}{4}}\right),a_{k+2}=-\frac{144k^2a_k-288k^2a_{k+1}-72ka_k-432ka_{k+1}+5a_k-117a_{k+1}}{9(16k^2+56k+48)},a_1=-\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Tetrahedral Galois group A4_SL2.
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.010 (sec)
Leaf size : 30

```
dsolve(diff(diff(y(x),x),x) = (-3/16/x^2-2/9/(x-1)^2+3/16/(x-1)/x)*y(x),y(x),singsol=all
```

$$y=\sqrt{x-1}\,x^{1/4}\left(c_1\,\mathrm{LegendreP}\left(-\frac{1}{6},\frac{1}{3},\sqrt{x}\right)+c_2\,\mathrm{LegendreQ}\left(-\frac{1}{6},\frac{1}{3},\sqrt{x}\right)\right)$$

Mathematica DSolve solution

Solving time : `0.237` (sec)
Leaf size : 550

```
DSolve[{D[y[x],{x,2}]== ( -3/(16*x^2)- 2/(9*(x-1)^2) + 3/(16*x*(x-1))) *y[x],{}},y[x],x,Incl
```

$$
\begin{aligned}
y(x) \to{} & c_1 \exp\left(\int_1^x \operatorname{Root}\left[2048K[1]^4 - 3484K[1]^3 + 2313K[1]^2 - 702K[1]\right.\right. \\
& + \left(20736K[1]^8 - 82944K[1]^7 + 124416K[1]^6 - 82944K[1]^5 + 20736K[1]^4\right) \#1^4 \\
& + \left(-48384K[1]^7 + 165888K[1]^6 - 207360K[1]^5 + 110592K[1]^4 - 20736K[1]^3\right) \#1^3 \\
& + \left(41472K[1]^6 - 118368K[1]^5 + 120096K[1]^4 - 50976K[1]^3 + 7776K[1]^2\right) \#1^2 \\
& + \left(-15360K[1]^5 + 34992K[1]^4 - 28272K[1]^3 + 9936K[1]^2 - 1296K[1]\right) \#1 \\
& \left.\left. + 81\&, 1\right] dK[1]\right) + c_2 \exp\left(\int_1^x \operatorname{Root}\left[2048K[1]^4 - 3484K[1]^3 + 2313K[1]^2 - 702K[1]\right.\right. \\
& + \left(20736K[1]^8 - 82944K[1]^7 + 124416K[1]^6 - 82944K[1]^5 + 20736K[1]^4\right) \#1^4 \\
& + \left(-48384K[1]^7 + 165888K[1]^6 - 207360K[1]^5 + 110592K[1]^4 - 20736K[1]^3\right) \#1^3 \\
& + \left(41472K[1]^6 - 118368K[1]^5 + 120096K[1]^4 - 50976K[1]^3 + 7776K[1]^2\right) \#1^2 \\
& + \left(-15360K[1]^5 + 34992K[1]^4 - 28272K[1]^3 + 9936K[1]^2 - 1296K[1]\right) \#1 \\
& \left.\left. + 81\&, 1\right] dK[1]\right) \int_1^x \exp\left(-2 \int_1^{K[2]} \operatorname{Root}\left[2048K[1]^4 - 3484K[1]^3 + 2313K[1]^2\right.\right. \\
& -702K[1] + \left(20736K[1]^8 - 82944K[1]^7 + 124416K[1]^6 - 82944K[1]^5 + 20736K[1]^4\right) \#1^4 + \left(-48384K[1]^7 \right. \\
& + \left(41472K[1]^6 - 118368K[1]^5 + 120096K[1]^4 - 50976K[1]^3 + 7776K[1]^2\right) \#1^2 + \left(-15360K[1]^5 + 34992K[\right. \\
& \left.\left. + 81\&, 1\right] dK[1]\right) dK[2]
\end{aligned}
$$

2.2.2 Problem 2

Internal problem ID [9995]
Book : Collection of Kovacic problems
Section : section 2. Solution found using all possible Kovacic cases
Problem number : 2
Date solved : `Monday, January 27, 2025 at 06:16:33 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$y'' = \frac{20y}{x^2}$$

Solved as second order ode using Kovacic algorithm

Time used: 0.102 (sec)

Writing the ode as

$$y'' - \frac{20y}{x^2} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -\frac{20}{x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{20}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 20 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{20}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1568: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{20}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 20$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 5 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -4 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{20}{x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 20$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 5 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -4
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{20}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	5	-4

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	5	-4

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -4$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -4 - (-4) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{4}{x} + (-)(0) \\
&= -\frac{4}{x} \\
&= -\frac{4}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(-\frac{4}{x}\right)(0) + \left(\left(\frac{4}{x^2}\right) + \left(-\frac{4}{x}\right)^2 - \left(\frac{20}{x^2}\right)\right) &= 0\\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= pe^{\int \omega\, dx}\\ &= \mathrm{e}^{\int -\frac{4}{x} dx}\\ &= \frac{1}{x^4}\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1\\ &= \frac{1}{x^4}\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}y_2 &= y_1 \int \frac{1}{y_1^2}\, dx\\ &= \frac{1}{x^4} \int \frac{1}{\frac{1}{x^8}}\, dx\\ &= \frac{1}{x^4}\left(\frac{x^9}{9}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2\\ &= c_1\left(\frac{1}{x^4}\right) + c_2\left(\frac{1}{x^4}\left(\frac{x^9}{9}\right)\right)\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) = \frac{20y(x)}{x^2}$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) - \frac{20y(x)}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) - 20y(x) = 0$

- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $\frac{d}{dx}y(x) = \left(\frac{d}{dt}y(t)\right)\left(\frac{d}{dx}t(x)\right)$

  - Compute derivative

    $\frac{d}{dx}y(x) = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $\frac{d^2}{dx^2}y(x) = \left(\frac{d^2}{dt^2}y(t)\right)\left(\frac{d}{dx}t(x)\right)^2 + \left(\frac{d^2}{dx^2}t(x)\right)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $\frac{d^2}{dx^2}y(x) = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - 20y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - \frac{d}{dt}y(t) - 20y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - r - 20 = 0$

- Factor the characteristic polynomial

  $(r+4)(r-5) = 0$

- Roots of the characteristic polynomial

  $r = (-4, 5)$

- 1st solution of the ODE

  $y_1(t) = e^{-4t}$

- 2nd solution of the ODE

  $y_2(t) = e^{5t}$

- General solution of the ODE

  $y(t) = C1 y_1(t) + C2 y_2(t)$

- Substitute in solutions

  $y(t) = C1\, e^{-4t} + C2\, e^{5t}$

- Change variables back using $t = \ln(x)$

  $y(x) = \frac{C1}{x^4} + C2\, x^5$

- Simplify

  $y(x) = \frac{C1}{x^4} + C2\, x^5$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 15

```
dsolve(diff(diff(y(x),x),x) = 20/x^2*y(x),y(x),singsol=all)
```

$$y = \frac{c_1 x^9 + c_2}{x^4}$$

Mathematica DSolve solution

Solving time : 0.012 (sec)
Leaf size : 18

```
DSolve[{D[y[x],{x,2}]==((4*(9/2)^2-1)/(4*x^2))*y[x],{}},y[x],x,IncludeSingularSolutions->Tru
```

$$y(x) \to \frac{c_2 x^9 + c_1}{x^4}$$

2.2.3 Problem 3

Internal problem ID [9996]
Book : Collection of Kovacic problems
Section : section 2. Solution found using all possible Kovacic cases
Problem number : 3
Date solved : `Monday, January 27, 2025 at 06:16:33 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$y'' = \frac{12y}{x^2}$$

Solved as second order ode using Kovacic algorithm

Time used: 0.108 (sec)

Writing the ode as

$$y'' - \frac{12y}{x^2} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -\frac{12}{x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{12}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 12 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{12}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1570: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{12}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 12$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 4 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -3 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{12}{x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 12$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 4\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -3\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{12}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	4	-3

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	4	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -3$ then

$$\begin{aligned}d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right)\\ &= -3 - (-3)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty\\ &= -\frac{3}{x} + (-)(0)\\ &= -\frac{3}{x}\\ &= -\frac{3}{x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{x}\right)(0) + \left(\left(\frac{3}{x^2}\right) + \left(-\frac{3}{x}\right)^2 - \left(\frac{12}{x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= e^{\int -\frac{3}{x} dx} \\ &= \frac{1}{x^3} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{1}{x^3} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \frac{1}{x^3} \int \frac{1}{\frac{1}{x^6}}\, dx \\ &= \frac{1}{x^3}\left(\frac{x^7}{7}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{1}{x^3}\right) + c_2\left(\frac{1}{x^3}\left(\frac{x^7}{7}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) = \frac{12y(x)}{x^2}$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) - \frac{12y(x)}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) - 12y(x) = 0$

- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $\frac{d}{dx}y(x) = \left(\frac{d}{dt}y(t)\right)\left(\frac{d}{dx}t(x)\right)$

  - Compute derivative

    $\frac{d}{dx}y(x) = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $\frac{d^2}{dx^2}y(x) = \left(\frac{d^2}{dt^2}y(t)\right)\left(\frac{d}{dx}t(x)\right)^2 + \left(\frac{d^2}{dx^2}t(x)\right)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $\frac{d^2}{dx^2}y(x) = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - 12y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - \frac{d}{dt}y(t) - 12y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - r - 12 = 0$

- Factor the characteristic polynomial

  $(r+3)(r-4) = 0$

- Roots of the characteristic polynomial

  $r = (-3, 4)$

- 1st solution of the ODE

  $y_1(t) = e^{-3t}$

- 2nd solution of the ODE

  $y_2(t) = e^{4t}$

- General solution of the ODE

  $y(t) = C1 y_1(t) + C2 y_2(t)$

- Substitute in solutions

  $y(t) = C1\, e^{-3t} + C2\, e^{4t}$

- Change variables back using $t = \ln(x)$

  $y(x) = \frac{C1}{x^3} + C2\, x^4$

- Simplify

  $y(x) = \frac{C1}{x^3} + C2\, x^4$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 15

```
dsolve(diff(diff(y(x),x),x) = 12/x^2*y(x),y(x),singsol=all)
```

$$y = \frac{c_2 x^7 + c_1}{x^3}$$

Mathematica DSolve solution

Solving time : 0.011 (sec)
Leaf size : 18

```
DSolve[{D[y[x],{x,2}]==((4*(7/2)^2-1)/(4*x^2))*y[x],{}},y[x],x,IncludeSingularSolutions->Tru
```

$$y(x) \to \frac{c_2 x^7 + c_1}{x^3}$$

2.2.4 Problem 4

Internal problem ID [9997]
Book : Collection of Kovacic problems
Section : section 2. Solution found using all possible Kovacic cases
Problem number : 4
Date solved : `Monday, January 27, 2025 at 06:16:34 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$y'' - \frac{y}{4x^2} = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.200 (sec)

Writing the ode as

$$y'' - \frac{y}{4x^2} = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -\frac{1}{4x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1572: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} + \frac{\sqrt{2}}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} - \frac{\sqrt{2}}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{1}{4x^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2}+\frac{\sqrt{2}}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}-\frac{\sqrt{2}}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}+\frac{\sqrt{2}}{2}$	$\frac{1}{2}-\frac{\sqrt{2}}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}+\frac{\sqrt{2}}{2}$	$\frac{1}{2}-\frac{\sqrt{2}}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}-\frac{\sqrt{2}}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{2}-\frac{\sqrt{2}}{2}-\left(\frac{1}{2}-\frac{\sqrt{2}}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^{-}}{x-c_1}\right)+(-)[\sqrt{r}]_\infty \\
&= \frac{\frac{1}{2}-\frac{\sqrt{2}}{2}}{x}+(-)\,(0) \\
&= \frac{\frac{1}{2}-\frac{\sqrt{2}}{2}}{x} \\
&= -\frac{\sqrt{2}-1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}
$$

Let

$$
p(x)=1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0)+2\left(\frac{\frac{1}{2}-\frac{\sqrt{2}}{2}}{x}\right)(0)+\left(\left(-\frac{\frac{1}{2}-\frac{\sqrt{2}}{2}}{x^2}\right)+\left(\frac{\frac{1}{2}-\frac{\sqrt{2}}{2}}{x}\right)^2-\left(\frac{1}{4x^2}\right)\right)=0
$$

$$
0=0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{\frac{1}{2}-\frac{\sqrt{2}}{2}}{x} dx} \\
&= x^{\frac{1}{2}-\frac{\sqrt{2}}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}
$$

Since $B=0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= x^{\frac{1}{2}-\frac{\sqrt{2}}{2}}
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^{\frac{1}{2}-\frac{\sqrt{2}}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Since $B=0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\
&= x^{\frac{1}{2}-\frac{\sqrt{2}}{2}} \int \frac{1}{x^{1-\sqrt{2}}} \, dx \\
&= x^{\frac{1}{2}-\frac{\sqrt{2}}{2}} \left(\frac{x\sqrt{2}\, x^{\sqrt{2}-1}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^{\frac{1}{2}-\frac{\sqrt{2}}{2}}\right) + c_2\left(x^{\frac{1}{2}-\frac{\sqrt{2}}{2}}\left(\frac{x\sqrt{2}\, x^{\sqrt{2}-1}}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) - \frac{y(x)}{4x^2} = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Multiply by denominators of the ODE

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) - y(x) = 0$

- Make a change of variables

  $t = \ln(x)$

- ☐ Substitute the change of variables back into the ODE

  - ○ Calculate the 1st derivative of y with respect to x , using the chain rule

    $\frac{d}{dx}y(x) = \left(\frac{d}{dt}y(t)\right)\left(\frac{d}{dx}t(x)\right)$

  - ○ Compute derivative

    $\frac{d}{dx}y(x) = \frac{\frac{d}{dt}y(t)}{x}$

  - ○ Calculate the 2nd derivative of y with respect to x , using the chain rule

    $\frac{d^2}{dx^2}y(x) = \left(\frac{d^2}{dt^2}y(t)\right)\left(\frac{d}{dx}t(x)\right)^2 + \left(\frac{d^2}{dx^2}t(x)\right)\left(\frac{d}{dt}y(t)\right)$

  - ○ Compute derivative

    $\frac{d^2}{dx^2}y(x) = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $4x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - y(t) = 0$

- Simplify

  $4\frac{d^2}{dt^2}y(t) - 4\frac{d}{dt}y(t) - y(t) = 0$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = \frac{d}{dt}y(t) + \frac{y(t)}{4}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d^2}{dt^2}y(t) - \frac{d}{dt}y(t) - \frac{y(t)}{4} = 0$

- Characteristic polynomial of ODE

  $r^2 - r - \frac{1}{4} = 0$

- Use quadratic formula to solve for r

  $r = \frac{1\pm\left(\sqrt{2}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(\frac{1}{2} - \frac{\sqrt{2}}{2}, \frac{1}{2} + \frac{\sqrt{2}}{2}\right)$

- 1st solution of the ODE

  $y_1(t) = e^{\left(\frac{1}{2}-\frac{\sqrt{2}}{2}\right)t}$

- 2nd solution of the ODE

  $y_2(t) = e^{\left(\frac{1}{2}+\frac{\sqrt{2}}{2}\right)t}$

- General solution of the ODE

  $y(t) = C1y_1(t) + C2y_2(t)$

- Substitute in solutions

  $y(t) = C1\,e^{\left(\frac{1}{2}-\frac{\sqrt{2}}{2}\right)t} + C2\,e^{\left(\frac{1}{2}+\frac{\sqrt{2}}{2}\right)t}$

- Change variables back using $t = \ln(x)$

  $y(x) = C1\,e^{\left(\frac{1}{2}-\frac{\sqrt{2}}{2}\right)\ln(x)} + C2\,e^{\left(\frac{1}{2}+\frac{\sqrt{2}}{2}\right)\ln(x)}$

- Simplify
  $$y(x) = \sqrt{x}\left(x^{-\frac{\sqrt{2}}{2}} C1 + x^{\frac{\sqrt{2}}{2}} C2\right)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 27

```
dsolve(diff(diff(y(x),x),x)-1/4/x^2*y(x) = 0,y(x),singsol=all)
```

$$y = \sqrt{x}\left(x^{\frac{\sqrt{2}}{2}} c_1 + x^{-\frac{\sqrt{2}}{2}} c_2\right)$$

Mathematica DSolve solution

Solving time : 0.021 (sec)
Leaf size : 32

```
DSolve[{D[y[x],{x,2}]-1/(4*x^2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to x^{\frac{1}{2}-\frac{1}{\sqrt{2}}}\left(c_2 x^{\sqrt{2}} + c_1\right)$$

2.2.5 Problem 5

Internal problem ID [9998]
Book : Collection of Kovacic problems
Section : section 2. Solution found using all possible Kovacic cases
Problem number : 5
Date solved : `Monday, January 27, 2025 at 06:16:35 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' - (2x+2)\,y' + (2+x)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.131 (sec)

Writing the ode as

$$xy'' + (-2x-2)\,y' + (2+x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -2x-2 \\ C &= 2+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1574: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the gcd$(s,t)=1$. This gives $b=2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b} = 2\\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b} = -1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{2}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right)\\
&= -1 - (-1)\\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty\\
&= -\frac{1}{x} + (-)(0)\\
&= -\frac{1}{x}\\
&= -\frac{1}{x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) &= 0\\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{1}{x} dx} \\
&= \frac{1}{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-2x-2}{x} \, dx} \\
&= z_1 e^{x+\ln(x)} \\
&= z_1 (x \, \mathrm{e}^x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-2}{x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{2x+2\ln(x)}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{x \, \mathrm{e}^{2x+2\ln(x)} \mathrm{e}^{-2x}}{3} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2 \left(\mathrm{e}^x \left(\frac{x \, \mathrm{e}^{2x+2\ln(x)} \mathrm{e}^{-2x}}{3} \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dx^2} y(x)\right) x - (2x+2) \left(\frac{d}{dx} y(x)\right) + (x+2) \, y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2} y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2} y(x) = -\frac{(x+2)y(x)}{x} + \frac{2(x+1)\left(\frac{d}{dx} y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x) - \frac{2(x+1)\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(x+2)y(x)}{x} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{2(x+1)}{x}, P_3(x) = \frac{x+2}{x}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = -2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$
  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$
- Multiply by denominators
  $\left(\frac{d^2}{dx^2}y(x)\right)x + (-2-2x)\left(\frac{d}{dx}y(x)\right) + (x+2)\,y(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$
    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k-m$
    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..1$
    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k- >k+1-m$
    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion
    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,x^{k+r-1}$
  - Shift index using $k- >k+1$
    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-3+r)\,x^{-1+r} + (a_1(1+r)(-2+r) - 2a_0(-1+r))\,x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)(k-2+r) - 2\right.$
- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-3+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 3\}$
- Each term must be 0
  $a_1(1+r)(-2+r) - 2a_0(-1+r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)(k-2+r) - 2a_k k - 2a_k r + 2a_k + a_{k-1} = 0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)(k+r-1)-2a_{k+1}(k+1)-2ra_{k+1}+2a_{k+1}+a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{2ka_{k+1}+2ra_{k+1}-a_k}{(k+2+r)(k+r-1)}$
- Recursion relation for $r=0$

  $a_{k+2}=\frac{2ka_{k+1}-a_k}{(k+2)(k-1)}$
- Series not valid for $r=0$, division by 0 in the recursion relation at $k=1$

  $a_{k+2}=\frac{2ka_{k+1}-a_k}{(k+2)(k-1)}$
- Recursion relation for $r=3$

  $a_{k+2}=\frac{2ka_{k+1}-a_k+6a_{k+1}}{(k+5)(k+2)}$
- Solution for $r=3$

  $\left[y(x)=\sum_{k=0}^{\infty}a_kx^{k+3},a_{k+2}=\frac{2ka_{k+1}-a_k+6a_{k+1}}{(k+5)(k+2)},4a_1-4a_0=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 14

```
dsolve(x*diff(diff(y(x),x),x)-(2+2*x)*diff(y(x),x)+(x+2)*y(x) = 0,y(x),singsol=all)
```

$$y=\mathrm{e}^x(c_2x^3+c_1)$$

Mathematica DSolve solution

Solving time : 0.046 (sec)
Leaf size : 25

```
DSolve[{x*D[y[x],{x,2}]-(2*x+2)*D[y[x],x]+(2+x)*y[x] ==0,{}},y[x],x,IncludeSingularSolutions
```

$$y(x)\to\frac{1}{3}e^{x+1}(c_2x^3+3c_1)$$

2.2.6 Problem 6

Internal problem ID [9999]
Book : Collection of Kovacic problems
Section : section 2. Solution found using all possible Kovacic cases
Problem number : 6
Date solved : `Monday, January 27, 2025 at 06:16:35 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$y'' + \frac{y}{x^2} = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.221 (sec)

Writing the ode as

$$y'' + \frac{y}{x^2} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= \frac{1}{x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1576: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -1$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{i\sqrt{3}}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{x^2}$$

Since the gcd$(s,t) = 1$. This gives $b = -1$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{i\sqrt{3}}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{i\sqrt{3}}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{1}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2} + \frac{i\sqrt{3}}{2}$	$\frac{1}{2} - \frac{i\sqrt{3}}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2} + \frac{i\sqrt{3}}{2}$	$\frac{1}{2} - \frac{i\sqrt{3}}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2} - \frac{i\sqrt{3}}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{2} - \frac{i\sqrt{3}}{2} - \left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x} + (-)\,(0) \\
&= \frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x} \\
&= \frac{1 - i\sqrt{3}}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{\frac{1}{2}-\frac{i\sqrt{3}}{2}}{x}\right)(0)+\left(\left(-\frac{\frac{1}{2}-\frac{i\sqrt{3}}{2}}{x^2}\right)+\left(\frac{\frac{1}{2}-\frac{i\sqrt{3}}{2}}{x}\right)^2-\left(-\frac{1}{x^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\,dx}\\
&= \mathrm{e}^{\int \frac{\frac{1}{2}-\frac{i\sqrt{3}}{2}}{x}dx}\\
&= x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B=0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1\\
&= x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}
\end{aligned}$$

Which simplifies to

$$y_1 = x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B=0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx\\
&= x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \int \frac{1}{x^{1-i\sqrt{3}}}\,dx\\
&= x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}\left(-\frac{ix\sqrt{3}\,x^{i\sqrt{3}-1}}{3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2\\
&= c_1\left(x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}\right) + c_2\left(x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}\left(-\frac{ix\sqrt{3}\,x^{i\sqrt{3}-1}}{3}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + \frac{y(x)}{x^2} = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Multiply by denominators of the ODE

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) + y(x) = 0$

- Make a change of variables

  $t = \ln(x)$

- ☐ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $\frac{d}{dx}y(x) = \left(\frac{d}{dt}y(t)\right)\left(\frac{d}{dx}t(x)\right)$

  - Compute derivative

    $\frac{d}{dx}y(x) = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $\frac{d^2}{dx^2}y(x) = \left(\frac{d^2}{dt^2}y(t)\right)\left(\frac{d}{dx}t(x)\right)^2 + \left(\frac{d^2}{dx^2}t(x)\right)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $\frac{d^2}{dx^2}y(x) = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - \frac{d}{dt}y(t) + y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{1\pm(\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, \frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right)$

- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right)$

- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^{\frac{t}{2}}\sin\left(\frac{\sqrt{3}\,t}{2}\right)$

- General solution of the ODE

  $y(t) = C1 y_1(t) + C2 y_2(t)$

- Substitute in solutions

  $y(t) = C1\,\mathrm{e}^{\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right) + C2\,\mathrm{e}^{\frac{t}{2}}\sin\left(\frac{\sqrt{3}\,t}{2}\right)$

- Change variables back using $t = \ln(x)$

  $y(x) = C1\sqrt{x}\cos\left(\frac{\sqrt{3}\ln(x)}{2}\right) + C2\sqrt{x}\sin\left(\frac{\sqrt{3}\ln(x)}{2}\right)$

- Simplify

  $y(x) = \sqrt{x}\left(C1\cos\left(\frac{\sqrt{3}\ln(x)}{2}\right) + C2\sin\left(\frac{\sqrt{3}\ln(x)}{2}\right)\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 29

```
dsolve(diff(diff(y(x),x),x)+1/x^2*y(x) = 0,y(x),singsol=all)
```

$$y = \sqrt{x}\left(c_1 \sin\left(\frac{\sqrt{3}\,\ln(x)}{2}\right) + c_2 \cos\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)\right)$$

Mathematica DSolve solution

Solving time : 0.028 (sec)
Leaf size : 42

```
DSolve[{D[y[x],{x,2}]+1/x^2*y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \sqrt{x}\left(c_1 \cos\left(\frac{1}{2}\sqrt{3}\log(x)\right) + c_2 \sin\left(\frac{1}{2}\sqrt{3}\log(x)\right)\right)$$

2.2.7 Problem 7

Internal problem ID [10000]
Book : Collection of Kovacic problems
Section : section 2. Solution found using all possible Kovacic cases
Problem number : 7
Date solved : `Monday, January 27, 2025 at 06:16:36 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(-x^2+1\right) y''+y'+y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.642 (sec)

Writing the ode as

$$\left(-x^2+1\right) y''+y'+y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{4x^2+4x-3}{4\left(x^2-1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2+4x-3 \\ t &= 4\left(x^2-1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{4x^2+4x-3}{4\left(x^2-1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1578: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{7}{16\,(x-1)} - \frac{3}{16\,(x+1)^2} + \frac{5}{16\,(x-1)^2} - \frac{7}{16\,(x+1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{4x^2 + 4x - 3}{4\left(x^2 - 1\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 1$. Hence

$$\begin{aligned} E_\infty &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{2\} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
1	2	$\{-1, 2, 5\}$
-1	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = -1,\ e_2 = 1,\ e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(2 - (-1 + (1))) \\ &= 1 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x - c} \\ &= \frac{1}{2}\left(\frac{-1}{(x - (1))} + \frac{1}{(x - (-1))}\right) \\ &= -\frac{1}{2(x - 1)} + \frac{1}{2x + 2} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 1$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 1$, then letting

$$p = x + a_0 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$\frac{4a_0 + 6}{(x - 1)^2 (x + 1)} = 0$$

And solving for p gives

$$p = x - \frac{3}{2}$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{x-\frac{3}{2}} - \frac{1}{2\left(x-1\right)} + \frac{1}{2x+2}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{x-\frac{3}{2}} - \frac{1}{2\left(x-1\right)} + \frac{1}{2x+2}\right)w + \frac{-8x^3+4x^2+10x-7}{4\left(x^2-1\right)^2\left(2x-3\right)} = 0$$

Solving for ω gives

$$\omega = \frac{2\sqrt{5}\,\sqrt{\left(x-1\right)\left(x+1\right)}\,x - 2\sqrt{5}\,\sqrt{\left(x-1\right)\left(x+1\right)} + 2x^2 - 2x + 1}{2\left(2x-3\right)\left(x-1\right)\left(x+1\right)}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega\,dx} \\ &= e^{\int \frac{2\sqrt{5}\,\sqrt{(x-1)(x+1)}\,x-2\sqrt{5}\,\sqrt{(x-1)(x+1)}+2x^2-2x+1}{2(2x-3)(x-1)(x+1)}dx} \\ &= \frac{\sqrt{2x-3}\left(x+1\right)^{1/4}\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}}5^{1/4}}{\left(x-1\right)^{1/4}\sqrt{\frac{5\sqrt{x^2-1}+(-2+3x)\sqrt{5}}{\sqrt{x^2-1}\,\sqrt{-\frac{(2x-3)^2}{x^2-1}}}}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{-x^2+1}\,dx} \\ &= z_1 e^{-\frac{\operatorname{arctanh}(x)}{2}} \\ &= z_1\left(\frac{1}{\sqrt{\frac{x+1}{\sqrt{-x^2+1}}}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}}\sqrt{2x-3}\left(5x+5\right)^{1/4}}{\sqrt{\frac{x+1}{\sqrt{-x^2+1}}}\sqrt{\frac{i\left(3\sqrt{5}\,x+5\sqrt{x^2-1}-2\sqrt{5}\right)}{2x-3}}\left(x-1\right)^{1/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{1}{-x^2+1}\,dx}}{\left(y_1\right)^2}\,dx \\ &= y_1 \int \frac{e^{-\operatorname{arctanh}(x)}}{\left(y_1\right)^2}\,dx \\ &= y_1\left(\int \frac{i\left(x+\sqrt{x^2-1}\right)^{-\sqrt{5}}\left(3\sqrt{5}\,x+5\sqrt{x^2-1}-2\sqrt{5}\right)\sqrt{x-1}}{\left(2x-3\right)^2\sqrt{5x+5}}dx\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}}\sqrt{2x-3}\,(5x+5)^{1/4}}{\sqrt{\frac{x+1}{\sqrt{-x^2+1}}}\sqrt{\frac{i\left(3\sqrt{5}\,x+5\sqrt{x^2-1}-2\sqrt{5}\right)}{2x-3}}\,(x-1)^{1/4}}\right) \\
&+ c_2\left(\frac{\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}}\sqrt{2x-3}\,(5x+5)^{1/4}}{\sqrt{\frac{x+1}{\sqrt{-x^2+1}}}\sqrt{\frac{i\left(3\sqrt{5}\,x+5\sqrt{x^2-1}-2\sqrt{5}\right)}{2x-3}}\,(x-1)^{1/4}}\left(\int \frac{i\left(x+\sqrt{x^2-1}\right)^{-\sqrt{5}}\left(3\sqrt{5}\,x+5\sqrt{x^2-1}-2\sqrt{5}\right)\sqrt{\vphantom{x}}}{(2x-3)^2\sqrt{5x+5}}\right.\right.
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(-x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)+\frac{d}{dx}y(x)+y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{y(x)}{x^2-1}+\frac{\frac{d}{dx}y(x)}{x^2-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x)-\frac{\frac{d}{dx}y(x)}{x^2-1}-\frac{y(x)}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=-\frac{1}{x^2-1}, P_3(x)=-\frac{1}{x^2-1}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=\frac{1}{2}$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $(x^2-1)\left(\frac{d^2}{dx^2}y(x)\right)-\frac{d}{dx}y(x)-y(x)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)-\frac{d}{du}y(u)-y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $\frac{d}{du}y(u)$ to series expansion

    $\frac{d}{du}y(u)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1}$

- Shift index using $k->k+1$

  $\frac{d}{du}y(u)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,u^{k+r}$

- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-a_0r(2r-1)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-a_{k+1}(k+1+r)\,(2k+1+2r)+a_k(k^2+2kr+r^2-k-r-1)\right)u^k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(2r-1)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{0,\frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $-2(k+1+r)\left(k+r+\frac{1}{2}\right)a_{k+1}+\left(k^2+(2r-1)\,k+r^2-r-1\right)a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{(k^2+2kr+r^2-k-r-1)a_k}{(k+1+r)(2k+1+2r)}$

- Recursion relation for $r=0$

  $a_{k+1}=\frac{(k^2-k-1)a_k}{(k+1)(2k+1)}$

- Solution for $r=0$

  $\left[y(u)=\sum_{k=0}^{\infty}a_ku^k,a_{k+1}=\frac{(k^2-k-1)a_k}{(k+1)(2k+1)}\right]$

- Revert the change of variables $u=x+1$

  $\left[y(x)=\sum_{k=0}^{\infty}a_k(x+1)^k\,,a_{k+1}=\frac{(k^2-k-1)a_k}{(k+1)(2k+1)}\right]$

- Recursion relation for $r=\frac{1}{2}$

  $a_{k+1}=\frac{(k^2-\frac{5}{4})a_k}{(k+\frac{3}{2})(2k+2)}$

- Solution for $r=\frac{1}{2}$

  $\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k+\frac{1}{2}},a_{k+1}=\frac{(k^2-\frac{5}{4})a_k}{(k+\frac{3}{2})(2k+2)}\right]$

- Revert the change of variables $u=x+1$

  $\left[y(x)=\sum_{k=0}^{\infty}a_k(x+1)^{k+\frac{1}{2}}\,,a_{k+1}=\frac{(k^2-\frac{5}{4})a_k}{(k+\frac{3}{2})(2k+2)}\right]$

- Combine solutions and rename parameters

  $\left[y(x)=\left(\sum_{k=0}^{\infty}a_k(x+1)^k\right)+\left(\sum_{k=0}^{\infty}b_k(x+1)^{k+\frac{1}{2}}\right),a_{k+1}=\frac{(k^2-k-1)a_k}{(k+1)(2k+1)},b_{k+1}=\frac{(k^2-\frac{5}{4})b_k}{(k+\frac{3}{2})(2k+2)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.030 (sec)
Leaf size : 58

```
dsolve((-x^2+1)*diff(diff(y(x),x),x)+diff(y(x),x)+y(x) = 0,y(x),singsol=all)
```

$$y = c_1 \operatorname{hypergeom}\left(\left[\frac{\sqrt{5}}{2}-\frac{1}{2}, -\frac{1}{2}-\frac{\sqrt{5}}{2}\right], \left[\frac{1}{2}\right], \frac{1}{2}+\frac{x}{2}\right) + c_2\sqrt{2+2x}\operatorname{hypergeom}\left(\left[\frac{\sqrt{5}}{2}, -\frac{\sqrt{5}}{2}\right], \left[\frac{3}{2}\right], \frac{1}{2}+\frac{x}{2}\right)$$

Mathematica DSolve solution

Solving time : 5.189 (sec)
Leaf size : 215

```
DSolve[{(1-x^2)*D[y[x],{x,2}]+D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \left(\sqrt{x-1}-\sqrt{x+1}\right)^{\frac{1}{2}\left(-1-\sqrt{5}\right)}\left(\sqrt{x-1}+\sqrt{x+1}\right)^{\frac{1}{2}\left(\sqrt{5}-1\right)}\left(5\sqrt{x-1}-\sqrt{5}\sqrt{x+1}\right)\left(c_2\int_1^x -\frac{2e^{-\operatorname{arctanh}(K[2])}\left(\sqrt{K[2]-1}-\sqrt{K[2]+1}\right)^{\sqrt{5}}\left(\sqrt{K[2]-1}+\sqrt{K[2]+1}\right)^{-\sqrt{5}}}{\left(\sqrt{5}\sqrt{K[2]+1}-5\sqrt{K[2]-1}\right)^2}dK[2] + c_1\right)\exp\left(\frac{1}{2}\left(\operatorname{arctanh}(x)-\int_1^x \frac{1}{1-K[1]^2}dK[1]\right)\right)$$

2.2.8 Problem 8

Internal problem ID [10001]
Book : Collection of Kovacic problems
Section : section 2. Solution found using all possible Kovacic cases
Problem number : 8
Date solved : `Monday, January 27, 2025 at 06:16:37 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2-x\right) y''-x y'+y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.154 (sec)

Writing the ode as

$$\left(x^2-x\right) y''-x y'+y=0 \tag{1}$$

$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=x^2-x\\ B&=-x\\ C&=1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r=\frac{-x+4}{4x\left(x-1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=-x+4\\ t&=4x(x-1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{-x+4}{4x\left(x-1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1580: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x(x-1)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{x-1} + \frac{1}{x} + \frac{3}{4(x-1)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x+4}{4x\left(x-1\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x+4}{4x\left(x-1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{x} - \frac{1}{2\left(x-1\right)} + (-)\left(0\right) \\
&= \frac{1}{x} - \frac{1}{2\left(x-1\right)} \\
&= \frac{x-2}{2x\left(x-1\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x} - \frac{1}{2(x-1)}\right)(0) + \left(\left(-\frac{1}{x^2} + \frac{1}{2(x-1)^2}\right) + \left(\frac{1}{x} - \frac{1}{2(x-1)}\right)^2 - \left(\frac{-x+4}{4x(x-1)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{1}{x} - \frac{1}{2(x-1)}\right) dx} \\ &= \frac{x}{\sqrt{x-1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{x^2-x}\, dx} \\ &= z_1 e^{\frac{\ln(x-1)}{2}} \\ &= z_1\left(\sqrt{x-1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x^2-x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\ln(x-1)}}{(y_1)^2}\, dx \\ &= y_1\left(\frac{1}{x} + \ln(x)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(x) + c_2\left(x\left(\frac{1}{x} + \ln(x)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 14

```
dsolve((x^2-x)*diff(diff(y(x),x),x)-diff(y(x),x)*x+y(x) = 0,y(x),singsol=all)
```

$$y = \ln(x) c_2 x + c_1 x + c_2$$

Mathematica DSolve solution

Solving time : 0.391 (sec)
Leaf size : 75

```
DSolve[{(x^2-x)*D[y[x],{x,2}]-x*D[y[x],x]+y[x]==0,{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \sqrt{1-x} \exp\left(\int_1^x \left(\frac{1}{K[1]} + \frac{1}{2-2K[1]}\right) dK[1]\right) \left(c_2 \int_1^x \exp\left(-2\int_1^{K[2]} \left(\frac{1}{K[1]} + \frac{1}{2-2K[1]}\right) dK[1]\right) dK[2] + c_1\right)$$

2.2.9 Problem 9

Internal problem ID [10002]
Book : Collection of Kovacic problems
Section : section 2. Solution found using all possible Kovacic cases
Problem number : 9
Date solved : `Monday, January 27, 2025 at 06:16:37 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(-x^2+2\right) y''-x\left(4x^2+3\right) y'+\left(-2x^2+2\right) y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.529 (sec)

Writing the ode as

$$\left(-x^4+2x^2\right) y''+\left(-4x^3-3x\right) y'+\left(-2x^2+2\right) y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^4+2x^2 \\ B &= -4x^3-3x \\ C &= -2x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{14x^2+5}{4\left(x^3-2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 14x^2+5 \\ t &= 4\left(x^3-2x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{14x^2+5}{4\left(x^3-2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1581: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 - 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \sqrt{2}$ of order 2. There is a pole at $x = -\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{33}{64 \left(x - \sqrt{2}\right)^2} + \frac{33}{64 \left(x + \sqrt{2}\right)^2} - \frac{43\sqrt{2}}{128 \left(x - \sqrt{2}\right)} + \frac{43\sqrt{2}}{128 \left(x + \sqrt{2}\right)} + \frac{5}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4} \end{aligned}$$

For the pole at $x=\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x-\sqrt{2}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{33}{64}$. Hence

$$
\begin{aligned}
{[\sqrt{r}]_c} &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{11}{8} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b} = -\frac{3}{8}
\end{aligned}
$$

For the pole at $x=-\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x+\sqrt{2}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{33}{64}$. Hence

$$
\begin{aligned}
{[\sqrt{r}]_c} &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{11}{8} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b} = -\frac{3}{8}
\end{aligned}
$$

Since the order of r at ∞ is $4>2$ then

$$
\begin{aligned}
{[\sqrt{r}]_\infty} &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{14x^{2}+5}{4\left(x^{3}-2x\right)^{2}}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$\sqrt{2}$	2	0	$\frac{11}{8}$	$-\frac{3}{8}$
$-\sqrt{2}$	2	0	$\frac{11}{8}$	$-\frac{3}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}+\alpha_{c_3}^{-}\right) \\
&= 1-(-1) \\
&= 2
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{4x} - \frac{3}{8\left(x-\sqrt{2}\right)} - \frac{3}{8\left(x+\sqrt{2}\right)} + (-)\left(0\right) \\
&= -\frac{1}{4x} - \frac{3}{8\left(x-\sqrt{2}\right)} - \frac{3}{8\left(x+\sqrt{2}\right)} \\
&= \frac{-2x^2+1}{2x^3-4x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(2) + 2\left(-\frac{1}{4x} - \frac{3}{8\left(x-\sqrt{2}\right)} - \frac{3}{8\left(x+\sqrt{2}\right)}\right)(2x+a_1) + \left(\left(\frac{1}{4x^2} + \frac{3}{8\left(x-\sqrt{2}\right)^2} + \frac{3}{8\left(x+\sqrt{2}\right)^2}\right) + \left(-\right.\right.
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 1, a_1 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 + 1
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2+1\right) e^{\int \left(-\frac{1}{4x} - \frac{3}{8\left(x-\sqrt{2}\right)} - \frac{3}{8\left(x+\sqrt{2}\right)}\right) dx} \\
&= \left(x^2+1\right) e^{-\frac{3\ln\left(x+\sqrt{2}\right)}{8} - \frac{\ln(x)}{4} - \frac{3\ln\left(x-\sqrt{2}\right)}{8}} \\
&= \frac{x^2+1}{\left(x+\sqrt{2}\right)^{3/8} x^{1/4} \left(x-\sqrt{2}\right)^{3/8}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x^3-3x}{-x^4+2x^2} \, dx} \\
&= z_1 e^{\frac{3\ln(x)}{4} - \frac{11\ln\left(x^2-2\right)}{8}} \\
&= z_1 \left(\frac{x^{3/4}}{\left(x^2-2\right)^{11/8}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^{5/2} + \sqrt{x}}{\left(x^2-2\right)^{11/8} \left(x+\sqrt{2}\right)^{3/8} \left(x-\sqrt{2}\right)^{3/8}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^3-3x}{-x^4+2x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{3\ln(x)}{2}-\frac{11\ln(x^2-2)}{4}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{e^{\frac{3\ln(x)}{2}-\frac{11\ln(x^2-2)}{4}} (x^2-2)^{11/4} \left(x+\sqrt{2}\right)^{3/4} \left(x-\sqrt{2}\right)^{3/4}}{\left(x^{5/2}+\sqrt{x}\right)^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{5/2}+\sqrt{x}}{(x^2-2)^{11/8} \left(x+\sqrt{2}\right)^{3/8} \left(x-\sqrt{2}\right)^{3/8}}\right) + c_2 \left(\frac{x^{5/2}+\sqrt{x}}{(x^2-2)^{11/8} \left(x+\sqrt{2}\right)^{3/8} \left(x-\sqrt{2}\right)^{3/8}} \left(\int \frac{e^{\frac{3\ln(x)}{2}-\frac{11\ln(x^2}{4}}}{}\right.\right.
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2(-x^2+2)\left(\frac{d^2}{dx^2}y(x)\right) - x(4x^2+3)\left(\frac{d}{dx}y(x)\right) + (-2x^2+2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{2(x^2-1)y(x)}{x^2(x^2-2)} - \frac{(4x^2+3)\left(\frac{d}{dx}y(x)\right)}{x(x^2-2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{(4x^2+3)\left(\frac{d}{dx}y(x)\right)}{x(x^2-2)} + \frac{2(x^2-1)y(x)}{x^2(x^2-2)} = 0$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{4x^2+3}{x(x^2-2)}, P_3(x) = \frac{2(x^2-1)}{x^2(x^2-2)}\right]$
  - $x \cdot P_2(x)$ is analytic at $x=0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -\frac{3}{2}$
  - $x^2 \cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 1$
  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

$$x^2(x^2-2)\left(\frac{d^2}{dx^2}y(x)\right)+x(4x^2+3)\left(\frac{d}{dx}y(x)\right)+(2x^2-2)\,y(x)=0$$

- Assume series solution for $y(x)$

$$y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$$

- □ Rewrite ODE with series expansions
  - Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

    $$x^m\cdot y(x)=\sum_{k=0}^{\infty}a_k x^{k+r+m}$$

  - Shift index using $k->k-m$

    $$x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$$

  - Convert $x^m\cdot\left(\frac{d}{dx}y(x)\right)$ to series expansion for $m=1..3$

    $$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$$

  - Shift index using $k->k+1-m$

    $$x^m\cdot\left(\frac{d}{dx}y(x)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

  - Convert $x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m=2..4$

    $$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

  - Shift index using $k->k+2-m$

    $$x^m\cdot\left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_0(-1+2r)\,(-2+r)\,x^r-a_1(1+2r)\,(-1+r)\,x^{1+r}+\left(\sum_{k=2}^{\infty}(-a_k(2k+2r-1)\,(k+r-2)+a_l\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-(-1+2r)\,(-2+r)=0$$

- Values of r that satisfy the indicial equation

  $$r\in\left\{2,\frac{1}{2}\right\}$$

- Each term must be 0

  $$-a_1(1+2r)\,(-1+r)=0$$

- Solve for the dependent coefficient(s)

  $$a_1=0$$

- Each term in the series must be 0, giving the recursion relation

  $$-2\left(k+r-\frac{1}{2}\right)(k+r-2)\,a_k+a_{k-2}(k+r)\,(k+r-1)=0$$

- Shift index using $k->k+2$

  $$-2\left(k+\frac{3}{2}+r\right)(k+r)\,a_{k+2}+a_k(k+r+2)\,(k+r+1)=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2}=\frac{a_k(k+r+2)(k+r+1)}{(2k+3+2r)(k+r)}$$

- Recursion relation for $r=2$

  $$a_{k+2}=\frac{a_k(k+4)(k+3)}{(2k+7)(k+2)}$$

- Solution for $r=2$

  $$\left[y(x)=\sum_{k=0}^{\infty}a_k x^{k+2},a_{k+2}=\frac{a_k(k+4)(k+3)}{(2k+7)(k+2)},a_1=0\right]$$

- Recursion relation for $r=\frac{1}{2}$

  $$a_{k+2}=\frac{a_k\left(k+\frac{5}{2}\right)\left(k+\frac{3}{2}\right)}{(2k+4)\left(k+\frac{1}{2}\right)}$$

- Solution for $r = \frac{1}{2}$

  $$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{a_k \left(k+\frac{5}{2}\right)\left(k+\frac{3}{2}\right)}{(2k+4)\left(k+\frac{1}{2}\right)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = \frac{a_k(4+k)(k+3)}{(2k+7)(k+2)}, a_1 = 0, b_{k+2} = \frac{b_k \left(k+\frac{5}{2}\right)\left(k+\frac{3}{2}\right)}{(2k+4)\left(k+\frac{1}{2}\right)}, b_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returnin
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.089 (sec)
Leaf size : 47

```
dsolve(x^2*(-x^2+2)*diff(diff(y(x),x),x)-x*(4*x^2+3)*diff(y(x),x)+(-2*x^2+2)*y(x) = 0,y(
```

$$y = c_1 x^2 \,\text{hypergeom}\left(\left[\frac{3}{2}, 2\right], \left[\frac{7}{4}\right], \frac{x^2}{2}\right) + \frac{c_2 \sqrt{x}\,(x^2+1)}{(x^2-2)\,(-2x^2+4)^{3/4}}$$

Mathematica DSolve solution

Solving time : 0.606 (sec)
Leaf size : 133

```
DSolve[{x^2*(2-x^2)*D[y[x],{x,2}] - x*(3+4*x^2)*D[y[x],x] + (2-2*x^2)*y[x] == 0,{}},y[x],x,I
```

$$y(x) \to (x^2+1)\exp\left(\int_1^x -\frac{1-2K[1]^2}{4K[1]-2K[1]^3}dK[1] - \frac{1}{2}\int_1^x \frac{4K[2]^2+3}{K[2](K[2]^2-2)}dK[2]\right)\left(c_2\int_1^x \frac{\exp\left(-2\int_1^{K[3]} -\frac{1-2K[1]^2}{4K[1]-2K[1]^3}dK[1]\right)}{(K[3]^2+1)^2}dK[3] + c_1\right)$$

2.3 section 3. Problems from Kovacic related papers

2.3.1 Problem Kovacic 1985 paper. page 13. section 3.2, example 1

Internal problem ID [10003]
Book : Collection of Kovacic problems
Section : section 3. Problems from Kovacic related papers
Problem number : Kovacic 1985 paper. page 13. section 3.2, example 1
Date solved : `Monday, January 27, 2025 at 06:16:38 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' = \frac{(4x^6 - 8x^5 + 12x^4 + 4x^3 + 7x^2 - 20x + 4)\, y}{4x^4}$$

Solved as second order ode using Kovacic algorithm

Time used: 0.533 (sec)

Writing the ode as

$$y'' + \left(-x^2 + 2x - 3 - \frac{1}{x} - \frac{7}{4x^2} + \frac{5}{x^3} - \frac{1}{x^4}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -x^2 + 2x - 3 - \frac{1}{x} - \frac{7}{4x^2} + \frac{5}{x^3} - \frac{1}{x^4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^6 - 8x^5 + 12x^4 + 4x^3 + 7x^2 - 20x + 4}{4x^4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^6 - 8x^5 + 12x^4 + 4x^3 + 7x^2 - 20x + 4 \\ t &= 4x^4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^6 - 8x^5 + 12x^4 + 4x^3 + 7x^2 - 20x + 4}{4x^4} \right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1583: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 6 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^4$. There is a pole at $x = 0$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned} \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) \end{aligned}$$

The partial fraction decomposition of r is

$$r = 3 + x^2 - 2x + \frac{1}{x} + \frac{7}{4x^2} - \frac{5}{x^3} + \frac{1}{x^4}$$

There is pole in r at $x=0$ of order 4, hence $v=2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c=0$ gives

$$[\sqrt{r}]_c \approx \frac{1}{x^2} - \frac{5}{2x} - \frac{9}{4} - \frac{41x}{8} - \frac{443x^2}{32} - \frac{3017x^3}{64} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v=2$ the above becomes

$$[\sqrt{r}]_c = \frac{1}{x^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^2}$ is

$$a = 1$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c=0$. This term becomes $\frac{1}{x^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be -5. Therefore

$$\begin{aligned} b &= (-5) - (0) \\ &= -5 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{1}{x^2} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &&= \frac{1}{2}\left(\frac{-5}{1} + 2\right) &&= -\frac{3}{2} \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &&= \frac{1}{2}\left(-\frac{-5}{1} + 2\right) &&= \frac{7}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x - 1 + \frac{1}{x} + \frac{3}{2x^2} + \frac{15}{8x^3} - \frac{17}{8x^4} - \frac{37}{8x^5} - \frac{85}{16x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v=1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= -1 + x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2 - 2x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^6 - 8x^5 + 12x^4 + 4x^3 + 7x^2 - 20x + 4}{4x^4} \\ &= Q + \frac{R}{4x^4} \\ &= \left(x^2 - 2x + 3\right) + \left(\frac{4x^3 + 7x^2 - 20x + 4}{4x^4}\right) \\ &= x^2 - 2x + 3 + \frac{4x^3 + 7x^2 - 20x + 4}{4x^4} \end{aligned}$$

We see that the coefficient of the term x^3 in the quotient is 3. Now b can be found.

$$\begin{aligned} b &= (3) - (1) \\ &= 2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= -1 + x \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{2}{1} - 1\right) &= \frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{2}{1} - 1\right) &= -\frac{3}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^6 - 8x^5 + 12x^4 + 4x^3 + 7x^2 - 20x + 4}{4x^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	4	$\frac{1}{x^2}$	$-\frac{3}{2}$	$\frac{7}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$-1 + x$	$\frac{1}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+}\right) \\ &= \frac{1}{2} - \left(-\frac{3}{2}\right) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{x^2} - \frac{3}{2x} + (-1+x) \\ &= \frac{1}{x^2} - \frac{3}{2x} - 1 + x \\ &= \frac{1}{x^2} - \frac{3}{2x} - 1 + x\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{1}{x^2} - \frac{3}{2x} - 1 + x\right)(2x + a_1) + \left(\left(-\frac{2}{x^3} + \frac{3}{2x^2} + 1\right) + \left(\frac{1}{x^2} - \frac{3}{2x} - 1 + x\right)^2 - \left(\frac{4x^6 - 8x^5 + 12\right.\right.$$

$$\frac{-2x^3 a_1 + (-4a_0 + 2a_1 - 4)\, x^2 -}{x^2}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= (x^2 - 1)\, e^{\int \left(\frac{1}{x^2} - \frac{3}{2x} - 1 + x\right) dx} \\ &= (x^2 - 1)\, e^{\frac{x^2}{2} - x - \frac{1}{x} - \frac{3\ln(x)}{2}} \\ &= \frac{(x^2 - 1)\, e^{\frac{x^3 - 2x^2 - 2}{2x}}}{x^{3/2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1 \\ &= \frac{(x^2 - 1)\, e^{\frac{x^3 - 2x^2 - 2}{2x}}}{x^{3/2}}\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2-1)\,\mathrm{e}^{\frac{x^3-2x^2-2}{2x}}}{x^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{\mathrm{e}^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \frac{(x^2-1)\,\mathrm{e}^{\frac{x^3-2x^2-2}{2x}}}{x^{3/2}} \int \frac{1}{\frac{(x^2-1)^2\mathrm{e}^{\frac{x^3-2x^2-2}{x}}}{x^3}}\,dx \\
&= \frac{(x^2-1)\,\mathrm{e}^{\frac{x^3-2x^2-2}{2x}}}{x^{3/2}} \left(\int \frac{x^3\mathrm{e}^{-\frac{x^3-2x^2-2}{x}}}{(x^2-1)^2}dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{(x^2-1)\,\mathrm{e}^{\frac{x^3-2x^2-2}{2x}}}{x^{3/2}}\right) + c_2\left(\frac{(x^2-1)\,\mathrm{e}^{\frac{x^3-2x^2-2}{2x}}}{x^{3/2}}\left(\int \frac{x^3\mathrm{e}^{-\frac{x^3-2x^2-2}{x}}}{(x^2-1)^2}dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
```

```
        -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @
   No special function solution was found.
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.732 (sec)
Leaf size : 66

```
dsolve(diff(diff(y(x),x),x) = 1/4*(4*x^6-8*x^5+12*x^4+4*x^3+7*x^2-20*x+4)/x^4*y(x),y(x
```

$$y = \frac{e^{\frac{x^3-2x^2-2}{2x}}\left(x^2-1\right)\left(c_2\left(\int \frac{x^3 e^{\frac{-x^3+2x^2+2}{x}}}{\left(x-1\right)^2\left(x+1\right)^2}dx\right)+c_1\right)}{x^{3/2}}$$

Mathematica DSolve solution

Solving time : 0.596 (sec)
Leaf size : 79

```
DSolve[{D[y[x],{x,2}]== (4*x^6-8*x^5+12*x^4+4*x^3+7*x^2-20*x+4)/(4*x^4)*y[x],{}},y[x],x,Incl
```

$$y(x) \to \frac{e^{\frac{x^2}{2}-x-\frac{1}{x}}\left(x^2-1\right)\left(c_2\int_1^x \frac{e^{-K[1]^2+2K[1]+\frac{2}{K[1]}}K[1]^3}{\left(K[1]^2-1\right)^2}dK[1]+c_1\right)}{x^{3/2}}$$

2.3.2 Problem Kovacic 1985 paper. page 14. section 3.2, example 2

Internal problem ID [10004]
Book : Collection of Kovacic problems
Section : section 3. Problems from Kovacic related papers
Problem number : Kovacic 1985 paper. page 14. section 3.2, example 2
Date solved : Monday, January 27, 2025 at 06:16:39 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$y'' = \left(\frac{6}{x^2} - 1\right) y$$

Solved as second order ode using Kovacic algorithm

Time used: 0.293 (sec)

Writing the ode as

$$y'' + \left(-\frac{6}{x^2} + 1\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -\frac{6}{x^2} + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 6}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2 + 6 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2+6}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1584: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{x^2} - 1$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx i - \frac{3i}{x^2} - \frac{9i}{2x^4} - \frac{27i}{2x^6} - \frac{405i}{8x^8} - \frac{1701i}{8x^{10}} - \frac{15309i}{16x^{12}} - \frac{72171i}{16x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= i \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-x^2 + 6}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-1) + \left(\frac{6}{x^2}\right) \\ &= \frac{6}{x^2} - 1 \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= i \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{0}{i} - 0\right) = 0 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{0}{i} - 0\right) = 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2+6}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	i	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= 0 - (-2) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{2}{x} + (-)(i) \\ &= -\frac{2}{x} - i \\ &= -\frac{2}{x} - i \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(-\frac{2}{x} - i\right)(2x + a_1) + \left(\left(\frac{2}{x^2}\right) + \left(-\frac{2}{x} - i\right)^2 - \left(\frac{-x^2+6}{x^2}\right) \right) = 0$$

$$\frac{2ixa_1 + 4ia_0 - 6x - 4a_1}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -3, a_1 = -3i\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 3ix - 3$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 3ix - 3\right) e^{\int \left(-\frac{2}{x} - i\right) dx} \\
&= \left(x^2 - 3ix - 3\right) e^{-2\ln(x) - ix} \\
&= \frac{\left(x^2 - 3ix - 3\right) e^{-ix}}{x^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= \frac{\left(x^2 - 3ix - 3\right) e^{-ix}}{x^2}
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(x^2 - 3ix - 3\right) e^{-ix}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\
&= \frac{\left(x^2 - 3ix - 3\right) e^{-ix}}{x^2} \int \frac{1}{\frac{\left(x^2 - 3ix - 3\right)^2 e^{-2ix}}{x^4}} \, dx \\
&= \frac{\left(x^2 - 3ix - 3\right) e^{-ix}}{x^2} \left(\frac{\left(ix^2 - 3x - 3i\right) e^{2ix}}{-2x^2 + 6ix + 6}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\left(x^2 - 3ix - 3\right) e^{-ix}}{x^2}\right) + c_2 \left(\frac{\left(x^2 - 3ix - 3\right) e^{-ix}}{x^2} \left(\frac{\left(ix^2 - 3x - 3i\right) e^{2ix}}{-2x^2 + 6ix + 6}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) = \left(\frac{6}{x^2} - 1\right) y(x)$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(x^2-6)y(x)}{x^2}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(x^2-6)y(x)}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = 0, P_3(x) = \frac{x^2-6}{x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 0$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = -6$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) + (x^2 - 6)\, y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ○ Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ○ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k (k + r)(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2 + r)(-3 + r)\, x^r + a_1(3 + r)(-2 + r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k + r + 2)(k + r - 3) + a_{k-2})\, x^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2 + r)(-3 + r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 3\}$

- Each term must be 0

  $a_1(3 + r)(-2 + r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation
  $a_k(k+r+2)(k+r-3)+a_{k-2}=0$
- Shift index using $k->k+2$
  $a_{k+2}(k+4+r)(k+r-1)+a_k=0$
- Recursion relation that defines series solution to ODE
  $a_{k+2}=-\frac{a_k}{(k+4+r)(k+r-1)}$
- Recursion relation for $r=-2$
  $a_{k+2}=-\frac{a_k}{(k+2)(k-3)}$
- Solution for $r=-2$
  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2}=-\frac{a_k}{(k+2)(k-3)}, a_1=0\right]$
- Recursion relation for $r=3$
  $a_{k+2}=-\frac{a_k}{(k+7)(k+2)}$
- Solution for $r=3$
  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2}=-\frac{a_k}{(k+7)(k+2)}, a_1=0\right]$
- Combine solutions and rename parameters
  $\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^{k-2}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), a_{k+2}=-\frac{a_k}{(k+2)(k-3)}, a_1=0, b_{k+2}=-\frac{b_k}{(k+7)(k+2)}, b_1=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.023` (sec)
Leaf size : 41

```
dsolve(diff(diff(y(x),x),x) = (6/x^2-1)*y(x),y(x),singsol=all)
```

$$y=\frac{(c_1x^2+3c_2x-3c_1)\cos(x)+\sin(x)(c_2x^2-3c_1x-3c_2)}{x^2}$$

Mathematica DSolve solution

Solving time : `0.02` (sec)
Leaf size : 21

```
DSolve[{D[y[x],{x,2}]== ( (4*(5/2)^2-1)/(4*x^2)-1)*y[x],{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x)\to x(c_1j_2(x)-c_2y_2(x))$$

2.3.3 Problem Kovacic 1985 paper. page 15. Weber equation

Internal problem ID [10005]
Book : Collection of Kovacic problems
Section : section 3. Problems from Kovacic related papers
Problem number : Kovacic 1985 paper. page 15. Weber equation
Date solved : `Monday, January 27, 2025 at 06:16:40 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' = \left(\frac{x^2}{4} - \frac{11}{2}\right) y$$

Solved as second order ode using Kovacic algorithm

Time used: 0.246 (sec)

Writing the ode as

$$y'' + \left(-\frac{x^2}{4} + \frac{11}{2}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -\frac{x^2}{4} + \frac{11}{2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 22}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 22 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{11}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1586: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{11}{2x} - \frac{121}{4x^3} - \frac{1331}{4x^5} - \frac{73205}{16x^7} - \frac{1127357}{16x^9} - \frac{37202781}{32x^{11}} - \frac{643076643}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 22}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} - \frac{11}{2}\right) + (0) \\ &= \frac{x^2}{4} - \frac{11}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{11}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{11}{2}\right) - (0) \\ &= -\frac{11}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{11}{2}}{\frac{1}{2}} - 1\right) &&= -6 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{11}{2}}{\frac{1}{2}} - 1\right) &&= 5 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{11}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-6	5

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 5$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 5 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 5$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(20x^3 + 12x^2 a_4 + 6x a_3 + 2a_2\right) + 2\left(-\frac{x}{2}\right)\left(5x^4 + 4x^3 a_4 + 3x^2 a_3 + 2x a_2 + a_1\right) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \right.\right.$$

$$a_4 x^4 + 2(10 + a_3)\, x^3 + 3(a_2 + 4a_4)\, x^2 + 2(2a_1 + 3a_3)\, x + 5a_0 +$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0, a_1 = 15, a_2 = 0, a_3 = -10, a_4 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^5 - 10x^3 + 15x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \left(x^5 - 10x^3 + 15x\right) e^{\int -\frac{x}{2} dx} \\ &= \left(x^5 - 10x^3 + 15x\right) e^{-\frac{x^2}{4}} \\ &= x\left(x^4 - 10x^2 + 15\right) e^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= x(x^4 - 10x^2 + 15)\, \mathrm{e}^{-\frac{x^2}{4}} \end{aligned}$$

Which simplifies to

$$y_1 = x(x^4 - 10x^2 + 15)\, \mathrm{e}^{-\frac{x^2}{4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= x(x^4 - 10x^2 + 15)\, \mathrm{e}^{-\frac{x^2}{4}} \int \frac{1}{x^2 (x^4 - 10x^2 + 15)^2\, \mathrm{e}^{-\frac{x^2}{2}}}\, dx \\ &= x(x^4 - 10x^2 + 15)\, \mathrm{e}^{-\frac{x^2}{4}} \left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{x^2 (x^4 - 10x^2 + 15)^2} dx \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(x(x^4 - 10x^2 + 15)\, \mathrm{e}^{-\frac{x^2}{4}} \right) + c_2 \left(x(x^4 - 10x^2 + 15)\, \mathrm{e}^{-\frac{x^2}{4}} \left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{x^2 (x^4 - 10x^2 + 15)^2} dx \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2} y(x) = \left(\frac{x^2}{4} - \frac{11}{2} \right) y(x)$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2} y(x)$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2} y(x) + \left(-\frac{x^2}{4} + \frac{11}{2} \right) y(x) = 0$

- Multiply by denominators

  $4 \frac{d^2}{dx^2} y(x) + (-x^2 + 22)\, y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- Shift index using $k->k-m$

  $x^m \cdot y(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

  $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- Shift index using $k->k+2$

  $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $8a_2 + 22a_0 + (24a_3 + 22a_1)\, x + \left(\sum_{k=2}^{\infty} (4a_{k+2}(k+2)\,(k+1) + 22a_k - a_{k-2})\, x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[8a_2 + 22a_0 = 0, 24a_3 + 22a_1 = 0]$

- Solve for the dependent coefficient(s)

  $\left\{a_2 = -\frac{11a_0}{4}, a_3 = -\frac{11a_1}{12}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $4(k^2 + 3k + 2)\, a_{k+2} + 22a_k - a_{k-2} = 0$

- Shift index using $k->k+2$

  $4\left((k+2)^2 + 3k + 8\right) a_{k+4} + 22a_{k+2} - a_k = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{-22a_{k+2}+a_k}{4(k^2+7k+12)}, a_2 = -\frac{11a_0}{4}, a_3 = -\frac{11a_1}{12}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solut
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special fu
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.022 (sec)
Leaf size : 39

```
dsolve(diff(diff(y(x),x),x) = (1/4*x^2-11/2)*y(x),y(x),singsol=all)
```

$$y = \frac{e^{-\frac{x^2}{4}}\left(15\,\mathrm{hypergeom}\left(\left[-\frac{5}{2}\right], \left[\frac{1}{2}\right], \frac{x^2}{2}\right)c_2 + x c_1 (x^4 - 10x^2 + 15)\right)}{15}$$

Mathematica DSolve solution

Solving time : 0.017 (sec)
Leaf size : 22

```
DSolve[{D[y[x],{x,2}]== (1/4*x^2-1/2-5)*y[x],{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_2 \,\mathrm{ParabolicCylinderD}(-6, ix) + c_1 \,\mathrm{ParabolicCylinderD}(5, x)$$

2.3.4 Problem Kovacic 1985 paper. page 19. section 4.2. Example 1

Internal problem ID [10006]
Book : Collection of Kovacic problems
Section : section 3. Problems from Kovacic related papers
Problem number : Kovacic 1985 paper. page 19. section 4.2. Example 1
Date solved : `Monday, January 27, 2025 at 06:16:41 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' = \left(\frac{1}{x} - \frac{3}{16x^2}\right) y$$

Solved as second order ode using Kovacic algorithm

Time used: 0.155 (sec)

Writing the ode as

$$y'' + \frac{(-16x+3)\, y}{16x^2} = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= \frac{-16x+3}{16x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{16x-3}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 16x - 3 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{16x-3}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1588: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2-1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{x} - \frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{1,2,3\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1,2,3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 1,\ e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(1-(1)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{1}{(x-(0))}\right) \\ &= \frac{1}{2x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d=0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d=0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2x} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2x} + \frac{1-16x}{16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{1+4\sqrt{x}}{4x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= e^{\int \omega \,dx} \\
&= \mathrm{e}^{\int \frac{1+4\sqrt{x}}{4x} dx} \\
&= x^{1/4} \mathrm{e}^{2\sqrt{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= x^{1/4} \mathrm{e}^{2\sqrt{x}}
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^{1/4} \mathrm{e}^{2\sqrt{x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= x^{1/4} \mathrm{e}^{2\sqrt{x}} \int \frac{1}{\sqrt{x}\,\mathrm{e}^{4\sqrt{x}}}\,dx \\
&= x^{1/4} \mathrm{e}^{2\sqrt{x}} \left(-\frac{\mathrm{e}^{-4\sqrt{x}}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{1/4} \mathrm{e}^{2\sqrt{x}}\right) + c_2 \left(x^{1/4} \mathrm{e}^{2\sqrt{x}} \left(-\frac{\mathrm{e}^{-4\sqrt{x}}}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2}y(x) = \left(\frac{1}{x} - \frac{3}{16x^2}\right) y(x)$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = \frac{(16x-3)y(x)}{16x^2}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x) - \frac{(16x-3)y(x)}{16x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - ◦ Define functions

    $\left[P_2(x) = 0, P_3(x) = -\frac{16x-3}{16x^2}\right]$
  - ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 0$
  - ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{3}{16}$
  - ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $16x^2\left(\frac{d^2}{dx^2}y(x)\right) + (-16x+3)\, y(x) = 0$
- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - ◦ Shift index using $k- >k-m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - ◦ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r}$

    Rewrite ODE with series expansions

    $a_0(-1+4r)(-3+4r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(4k+4r-1)(4k+4r-3) - 16a_{k-1}\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+4r)(-3+4r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{4}, \frac{3}{4}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $16\left(k - \frac{3}{4} + r\right)\left(k + r - \frac{1}{4}\right) a_k - 16a_{k-1} = 0$
- Shift index using $k- >k+1$

  $16\left(k + \frac{1}{4} + r\right)\left(k + \frac{3}{4} + r\right) a_{k+1} - 16a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{16a_k}{(4k+1+4r)(4k+3+4r)}$
- Recursion relation for $r = \frac{1}{4}$

  $a_{k+1} = \frac{16a_k}{(4k+2)(4k+4)}$
- Solution for $r = \frac{1}{4}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+1} = \frac{16a_k}{(4k+2)(4k+4)}\right]$
- Recursion relation for $r = \frac{3}{4}$

  $a_{k+1} = \frac{16a_k}{(4k+4)(4k+6)}$

- Solution for $r = \frac{3}{4}$

  $$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{4}}, a_{k+1} = \frac{16a_k}{(4k+4)(4k+6)}\right]$$

- Combine solutions and rename parameters

  $$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{4}}\right), a_{k+1} = \frac{16a_k}{(4k+2)(4k+4)}, b_{k+1} = \frac{16b_k}{(4k+4)(4k+6)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 25

```
dsolve(diff(diff(y(x),x),x) = (1/x-3/16/x^2)*y(x),y(x),singsol=all)
```

$$y = x^{1/4}\left(c_1 \sinh\left(2\sqrt{x}\right) + c_2 \cosh\left(2\sqrt{x}\right)\right)$$

Mathematica DSolve solution

Solving time : 0.055 (sec)
Leaf size : 41

```
DSolve[{D[y[x],{x,2}]== (1/x-3/(16*x^2))*y[x],{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2} e^{-2\sqrt{x}} \sqrt[4]{x} \left(2c_1 e^{4\sqrt{x}} - c_2\right)$$

2.3.5 Problem Kovacic 1985 paper. page 23. section 5.2. Example 1

Internal problem ID [10007]
Book : Collection of Kovacic problems
Section : section 3. Problems from Kovacic related papers
Problem number : Kovacic 1985 paper. page 23. section 5.2. Example 1
Date solved : Monday, January 27, 2025 at 06:16:41 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$y'' = \left(-\frac{3}{16x^2} - \frac{2}{9\left(x-1\right)^2} + \frac{3}{16x\left(x-1\right)}\right) y$$

Solved as second order ode using Kovacic algorithm

Time used: 1.026 (sec)

Writing the ode as

$$y'' + \frac{(32x^2 - 27x + 27)\, y}{144x^2\left(x-1\right)^2} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= \frac{32x^2 - 27x + 27}{144x^2\left(x-1\right)^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-32x^2 + 27x - 27}{144\left(x^2 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -32x^2 + 27x - 27 \\ t &= 144\left(x^2 - x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-32x^2 + 27x - 27}{144 (x^2 - x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1590: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 144(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Unable to find solution using case two.

Attempting to find a solution using $n = 4$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x} - \frac{2}{9 (x-1)^2} + \frac{3}{16 (x-1)} - \frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. This shows that $b = -\frac{3}{16}$. Hence

$$E_c = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \ldots, \pm \frac{n}{2}\right\} \cap \mathbb{Z}$$

Where n for case 3 is $4, 6$ or 12. For the current case $n = 4$. Hence the above becomes

$$E_c = \{3, 6, 9\}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. This shows that $b = -\frac{2}{9}$. Hence

$$E_c = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \dots, \pm \frac{n}{2}\right\} \cap \mathbb{Z}$$

Where n for case 3 is $4, 6$ or 12. For the current case $n = 4$. Hence the above becomes

$$E_c = \{4, 5, 6, 7, 8\}$$

Let

$$E_\infty = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \dots, \pm \frac{n}{2}\right\} \cap \mathbb{Z} \tag{B1}$$

Where b is the coefficient of $\frac{1}{x^2}$ in the Laurent series for r at ∞ given by

$$r \approx -\frac{2}{9x^2} - \frac{37}{144x^3} - \frac{23}{48x^4} - \frac{101}{144x^5} - \frac{133}{144x^6} - \frac{55}{48x^7} + \dots$$

The above shows that

$$b = -\frac{2}{9}$$

The value of n in eq. (B1) for case 3 is $4, 6$ or 2.For the current case $n = 4$. eq. (B1) simplifies to the following, after removing any duplicate and non integer entries in the set.

$$E_\infty = \{4, 5, 6, 7, 8\}$$

The following table summarizes the results found so far for poles and for the order of r at ∞ for case 3 of Kovacic algorithm using $n = 4$.

pole c location	pole order	set $\{E_c\}$
0	2	$\{3, 6, 9\}$
1	2	$\{4, 5, 6, 7, 8\}$

Order of r at ∞	set $\{E_\infty\}$
2	$\{4, 5, 6, 7, 8\}$

Now that E_c sets for all poles are found and E_∞ set is found, the next step is to determine a non negative integer d using the following

$$d = \frac{n}{12}\left(e_\infty - \sum_{c \in \Gamma} e_c\right)$$

Where in the above e_c is a distinct element from each corresponding E_c. This means all possible tuples $\{e_{c_1}, e_{c_2}, \dots, e_{c_n}\}$ are tried in the sum above, where e_{c_i} is one element of each E_c found earlier. Using the following family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 3,\ e_2 = 4,\ e_\infty = 7$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{n}{12}\left(e_\infty - \sum_{c \in \Gamma} e_c\right) \\ &= \frac{4}{12}(7 - (3 + (4))) \\ &= 0 \end{aligned}$$

The following rational function is

$$\begin{aligned}\theta &= \frac{n}{12}\sum_{c\in\Gamma}\frac{e_c}{x-c}\\ &= \frac{4}{12}\left(\frac{3}{(x-(0))}+\frac{4}{(x-(1))}\right)\\ &= \frac{1}{x}+\frac{4}{3x-3}\end{aligned}$$

And

$$\begin{aligned}S &= \prod_{c\in\Gamma}(x-c)\\ &= x(x-1)\end{aligned}$$

The polynomial $p(x)$ is now determined. Since the degree of the polynomial is $d=0$, then let

$$p(x)=1$$

The following set of equations are set up in order to determine the coefficients a_i (if any) of the above polynomial

$$\begin{aligned}P_n &= -p(x)\\ &= -1\\ P_{i-1} &= -Sp_i'+((n-i)S'-S\theta)P_i-(n-1)(i+1)S^2rP_{i+1} \qquad i=n,n-1,\dots,0 \quad \text{(1A)}\end{aligned}$$

The coefficients a_i are solved for from

$$P_{-1}=0 \tag{2A}$$

By using method of undetermined coefficients. Carrying the above computation in eq. (1A) gives the following sequence of polynomials P_i (noting that $n=4$ and $r=\frac{-32x^2+27x-27}{144(x^2-x)^2}$).

$$\begin{aligned}P_4 &= -p\\ &= -1\\ P_3 &= \frac{7x}{3}-1\\ P_2 &= -4x^2+\frac{41}{12}x-\frac{3}{4}\\ P_1 &= \frac{40}{9}x^3-\frac{409}{72}x^2+\frac{5}{2}x-\frac{3}{8}\\ P_0 &= -\frac{64}{27}x^4+\frac{871}{216}x^3-\frac{257}{96}x^2+\frac{13}{16}x-\frac{3}{32}\\ P_{-1} &= 0\end{aligned}$$

Because $P_{-1}=0$ then $z=e^{\int\omega}$ is a solution. ω is found by finding a solution to the equation generated by the following sum

$$\begin{aligned}\sum_{i=0}^{n}S^i\frac{P_i}{(n-i)!}\omega^i &= 0\\ \sum_{i=0}^{4}S^i\frac{P_i}{(4-i)!}\omega^i &= 0\end{aligned}$$

Where the P_i are the polynomials found earlier. Computing the above sum gives

$$-\frac{8x^4}{81}+\frac{871x^3}{5184}-\frac{257x^2}{2304}+\frac{13x}{384}-\frac{1}{256}+\frac{x(x-1)\left(320x^3-409x^2+180x-27\right)\omega}{432} \\ -\frac{x^2(x-1)^2\left(48x^2-41x+9\right)\omega^2}{24}+x^3(x-1)^3\left(\frac{7x}{3}-1\right)\omega^3-x^4(x-1)^4\omega^4=0$$

The solution ω of eq. 3A is found as

$$\omega = \frac{1}{12x(x-1)}\left(7x-3+\sqrt{x^2+\left((x-1)^2 x^3\right)^{1/3}-x} + \sqrt{-\frac{2\left(\left(-x^2+x+\frac{\left((x-1)^2x^3\right)^{1/3}}{2}\right)\sqrt{x^2+\left((x-1)^2 x^3\right)^{1/3}-x}+x^2(x-1)\right)}{\sqrt{x^2+\left((x-1)^2 x^3\right)^{1/3}-x}}}\right) \tag{4A}$$

This ω is used to find a solution to $z''=rz$.

$$z_1(x) = e^{\int \omega\,dx} \tag{5A}$$

Unable to integrate $\int \omega dx$. Leaving the integral unevaluated. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B=0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= e^{\int \omega\,dx} \end{aligned}$$

Where ω given above. The second solution y_2 to the original ode is found using reduction of order

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx \\ &= e^{\int \omega\,dx} \int \frac{e^{\int -\frac{B}{A}\,dx}}{\left(e^{\int \omega\,dx}\right)^2}\,dx \end{aligned}$$

Since $B=0$ then the above reduces to

$$y_2 = e^{\int \omega\,dx} \int \left(e^{\int \omega\,dx}\right)^{-2}\,dx$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(e^{\int \omega\,dx}\right) + c_2\left(e^{\int \omega\,dx}\int \left(e^{\int \omega\,dx}\right)^{-2}\,dx\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2}y(x) = \left(-\frac{3}{16x^2} - \frac{2}{9(x-1)^2} + \frac{3}{16x(x-1)}\right) y(x)$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
$\frac{d^2}{dx^2}y(x) = -\frac{(32x^2-27x+27)y(x)}{144x^2(x-1)^2}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
$\frac{d^2}{dx^2}y(x) + \frac{(32x^2-27x+27)y(x)}{144x^2(x-1)^2} = 0$

□ Check to see if x_0 is a regular singular point
  - Define functions
  $\left[P_2(x) = 0, P_3(x) = \frac{32x^2-27x+27}{144x^2(x-1)^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
  $\left.(x \cdot P_2(x))\right|_{x=0} = 0$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{3}{16}$
  - $x = 0$is a regular singular point
  Check to see if x_0 is a regular singular point
  $x_0 = 0$
- Multiply by denominators
$144x^2(x-1)^2\left(\frac{d^2}{dx^2}y(x)\right) + (32x^2 - 27x + 27)\,y(x) = 0$
- Assume series solution for $y(x)$
$y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$
  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k-m$
  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$
  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,x^{k+r-2+m}$
  - Shift index using $k- >k+2-m$
  $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,x^{k+r}$
  Rewrite ODE with series expansions
  $9a_0(-1+4r)(-3+4r)\,x^r + (9a_1(3+4r)(1+4r) - 9a_0(32r^2 - 32r + 3))\,x^{1+r} + \left(\sum_{k=2}^{\infty} (9a_k(4k\right.$
- $a_0$cannot be 0 by assumption, giving the indicial equation
$9(-1+4r)(-3+4r) = 0$
- Values of r that satisfy the indicial equation
$r \in \left\{\frac{1}{4}, \frac{3}{4}\right\}$

- Each term must be 0

  $$9a_1(3+4r)(1+4r) - 9a_0(32r^2-32r+3) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = \frac{a_0(32r^2-32r+3)}{16r^2+16r+3}$$

- Each term in the series must be 0, giving the recursion relation

  $$144(a_k + a_{k-2} - 2a_{k-1})k^2 + 144(2(a_k + a_{k-2} - 2a_{k-1})r - a_k - 5a_{k-2} + 6a_{k-1})k + 144(a_k + a_{k-2} -$$

- Shift index using $k->k+2$

  $$144(a_{k+2} + a_k - 2a_{k+1})(k+2)^2 + 144(2(a_{k+2} + a_k - 2a_{k+1})r - a_{k+2} - 5a_k + 6a_{k+1})(k+2) + 144$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{144k^2a_k - 288k^2a_{k+1} + 288kra_k - 576kra_{k+1} + 144r^2a_k - 288r^2a_{k+1} - 144ka_k - 288ka_{k+1} - 144ra_k - 288ra_{k+1} + 32a_k - 27a_{k+}}{9(16k^2+32kr+16r^2+48k+48r+35)}$$

- Recursion relation for $r = \frac{1}{4}$

  $$a_{k+2} = -\frac{144k^2a_k - 288k^2a_{k+1} - 72ka_k - 432ka_{k+1} + 5a_k - 117a_{k+1}}{9(16k^2+56k+48)}$$

- Solution for $r = \frac{1}{4}$

  $$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+2} = -\frac{144k^2a_k - 288k^2a_{k+1} - 72ka_k - 432ka_{k+1} + 5a_k - 117a_{k+1}}{9(16k^2+56k+48)}, a_1 = -\frac{3a_0}{8}\right]$$

- Recursion relation for $r = \frac{3}{4}$

  $$a_{k+2} = -\frac{144k^2a_k - 288k^2a_{k+1} + 72ka_k - 720ka_{k+1} + 5a_k - 405a_{k+1}}{9(16k^2+72k+80)}$$

- Solution for $r = \frac{3}{4}$

  $$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{4}}, a_{k+2} = -\frac{144k^2a_k - 288k^2a_{k+1} + 72ka_k - 720ka_{k+1} + 5a_k - 405a_{k+1}}{9(16k^2+72k+80)}, a_1 = -\frac{a_0}{8}\right]$$

- Combine solutions and rename parameters

  $$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{4}}\right), a_{k+2} = -\frac{144k^2a_k - 288k^2a_{k+1} - 72ka_k - 432ka_{k+1} + 5a_k - 117a_{k+1}}{9(16k^2+56k+48)}, a_1 = -\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Tetrahedral Galois group A4_SL2.
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 30

```
dsolve(diff(diff(y(x),x),x) = (-3/16/x^2-2/9/(x-1)^2+3/16/(x-1)/x)*y(x),y(x),singsol=all
```

$$y = \sqrt{x-1}\, x^{1/4}\left(c_1 \operatorname{LegendreP}\left(-\frac{1}{6}, \frac{1}{3}, \sqrt{x}\right) + c_2 \operatorname{LegendreQ}\left(-\frac{1}{6}, \frac{1}{3}, \sqrt{x}\right)\right)$$

Mathematica DSolve solution

Solving time : `0.241` (sec)
Leaf size : 550

```
DSolve[{D[y[x],{x,2}]== ( -3/(16*x^2) - 2/(9*(x-1)^2) + 3/(16*x*(x-1))  )*y[x],{}},y[x],x,In
```

$$
\begin{aligned}
y(x) \to{}& c_1 \exp\left(\int_1^x \text{Root}\big[2048K[1]^4 - 3484K[1]^3 + 2313K[1]^2 - 702K[1]\right. \\
&+ \left(20736K[1]^8 - 82944K[1]^7 + 124416K[1]^6 - 82944K[1]^5 + 20736K[1]^4\right)\#1^4 \\
&+ \left(-48384K[1]^7 + 165888K[1]^6 - 207360K[1]^5 + 110592K[1]^4 - 20736K[1]^3\right)\#1^3 \\
&+ \left(41472K[1]^6 - 118368K[1]^5 + 120096K[1]^4 - 50976K[1]^3 + 7776K[1]^2\right)\#1^2 \\
&+ \left(-15360K[1]^5 + 34992K[1]^4 - 28272K[1]^3 + 9936K[1]^2 - 1296K[1]\right)\#1 \\
&\left. + 81\&, 1\big]\, dK[1]\right) + c_2 \exp\left(\int_1^x \text{Root}\big[2048K[1]^4 - 3484K[1]^3 + 2313K[1]^2 - 702K[1]\right. \\
&+ \left(20736K[1]^8 - 82944K[1]^7 + 124416K[1]^6 - 82944K[1]^5 + 20736K[1]^4\right)\#1^4 \\
&+ \left(-48384K[1]^7 + 165888K[1]^6 - 207360K[1]^5 + 110592K[1]^4 - 20736K[1]^3\right)\#1^3 \\
&+ \left(41472K[1]^6 - 118368K[1]^5 + 120096K[1]^4 - 50976K[1]^3 + 7776K[1]^2\right)\#1^2 \\
&+ \left(-15360K[1]^5 + 34992K[1]^4 - 28272K[1]^3 + 9936K[1]^2 - 1296K[1]\right)\#1 \\
&\left. + 81\&, 1\big]\, dK[1]\right) \int_1^x \exp\left(-2\int_1^{K[2]} \text{Root}\big[2048K[1]^4 - 3484K[1]^3 + 2313K[1]^2\right. \\
&-702K[1] + \left(20736K[1]^8 - 82944K[1]^7 + 124416K[1]^6 - 82944K[1]^5 + 20736K[1]^4\right)\#1^4 + \left(-48384K[1]^7 \cdots\right. \\
&+ \left(41472K[1]^6 - 118368K[1]^5 + 120096K[1]^4 - 50976K[1]^3 + 7776K[1]^2\right)\#1^2 + \left(-15360K[1]^5 + 34992K[\cdots\right. \\
&\left. + 81\&, 1\big]\, dK[1]\right) dK[2]
\end{aligned}
$$

2.3.6 Problem Kovacic 1985 paper. page 25. section 5.2. Example 2

Internal problem ID [10008]
Book : Collection of Kovacic problems
Section : section 3. Problems from Kovacic related papers
Problem number : Kovacic 1985 paper. page 25. section 5.2. Example 2
Date solved : `Monday, January 27, 2025 at 06:16:43 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' = -\frac{(5x^2+27)\,y}{36\,(x^2-1)^2}$$

Solved as second order ode using Kovacic algorithm

Time used: 89.892 (sec)

Writing the ode as

$$y'' + \frac{(5x^2+27)\,y}{36\,(x^2-1)^2} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= \frac{5x^2+27}{36\,(x^2-1)^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5x^2-27}{36\,(x^2-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -5x^2 - 27 \\ t &= 36\left(x^2-1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-5x^2 - 27}{36\,(x^2-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1592: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Unable to find solution using case two.

Attempting to find a solution using $n = 4$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{9\,(x+1)^2} - \frac{11}{72\,(x+1)} - \frac{2}{9\,(x-1)^2} + \frac{11}{72\,(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. This shows that $b = -\frac{2}{9}$. Hence

$$E_c = \left\{6 + \frac{12k}{n}\sqrt{1+4b} \middle| k = 0, \pm 1, \pm 2, \ldots, \pm \frac{n}{2}\right\} \cap \mathbb{Z}$$

Where n for case 3 is 4, 6 or 12. For the current case $n = 4$. Hence the above becomes

$$E_c = \{4, 5, 6, 7, 8\}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. This shows that $b = -\frac{2}{9}$. Hence

$$E_c = \left\{6 + \frac{12k}{n}\sqrt{1+4b} \middle| k = 0, \pm 1, \pm 2, \ldots, \pm \frac{n}{2}\right\} \cap \mathbb{Z}$$

Where n for case 3 is 4, 6 or 12. For the current case $n = 4$. Hence the above becomes

$$E_c = \{4, 5, 6, 7, 8\}$$

Let

$$E_\infty = \left\{6 + \frac{12k}{n}\sqrt{1+4b} \middle| k = 0, \pm 1, \pm 2, \ldots, \pm \frac{n}{2}\right\} \cap \mathbb{Z} \tag{B1}$$

Where b is the coefficient of $\frac{1}{x^2}$ in the Laurent series for r at ∞ given by

$$r \approx -\frac{5}{36x^2} - \frac{37}{36x^4} - \frac{23}{12x^6} - \frac{101}{36x^8} - \frac{133}{36x^{10}} - \frac{55}{12x^{12}} + \cdots$$

The above shows that

$$b = -\frac{5}{36}$$

The value of n in eq. (B1) for case 3 is 4, 6 or 2.For the current case $n = 4$. eq. (B1) simplifies to the following, after removing any duplicate and non integer entries in the set.

$$E_\infty = \{2, 4, 6, 8, 10\}$$

The following table summarizes the results found so far for poles and for the order of r at ∞ for case 3 of Kovacic algorithm using $n = 4$.

pole c location	pole order	set $\{E_c\}$
1	2	$\{4, 5, 6, 7, 8\}$
-1	2	$\{4, 5, 6, 7, 8\}$

Order of r at ∞	set $\{E_\infty\}$
2	$\{2, 4, 6, 8, 10\}$

Now that E_c sets for all poles are found and E_∞ set is found, the next step is to determine a non negative integer d using the following

$$d = \frac{n}{12}\left(e_\infty - \sum_{c \in \Gamma} e_c\right)$$

Where in the above e_c is a distinct element from each corresponding E_c. This means all possible tuples $\{e_{c_1}, e_{c_2}, \ldots, e_{c_n}\}$ are tried in the sum above, where e_{c_i} is one element of each E_c found earlier. Using the following family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 4,\ e_2 = 4,\ e_\infty = 8$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned}
d &= \frac{n}{12}\left(e_\infty - \sum_{c \in \Gamma} e_c\right) \\
&= \frac{4}{12}(8 - (4 + (4))) \\
&= 0
\end{aligned}$$

The following rational function is

$$\begin{aligned}\theta &= \frac{n}{12}\sum_{c\in\Gamma}\frac{e_c}{x-c}\\ &= \frac{4}{12}\left(\frac{4}{(x-(1))}+\frac{4}{(x-(-1))}\right)\\ &= \frac{8x}{3x^2-3}\end{aligned}$$

And

$$\begin{aligned}S &= \prod_{c\in\Gamma}(x-c)\\ &= (x-1)(x+1)\end{aligned}$$

The polynomial $p(x)$ is now determined. Since the degree of the polynomial is $d=0$, then let

$$p(x)=1$$

The following set of equations are set up in order to determine the coefficients a_i (if any) of the above polynomial

$$\begin{aligned}P_n &= -p(x)\\ &= -1\\ P_{i-1} &= -Sp_i' + ((n-i)S' - S\theta)P_i - (n-1)(i+1)S^2rP_{i+1} \qquad i=n,n-1,\ldots,0 \quad \text{(1A)}\end{aligned}$$

The coefficients a_i are solved for from

$$P_{-1}=0 \qquad \text{(2A)}$$

By using method of undetermined coefficients. Carrying the above computation in eq. (1A) gives the following sequence of polynomials P_i (noting that $n=4$ and $r=\frac{-5x^2-27}{36(x^2-1)^2}$).

$$\begin{aligned}P_4 &= -p\\ &= -1\\ P_3 &= \frac{8x}{3}\\ P_2 &= -5x^2-\frac{1}{3}\\ P_1 &= \frac{50}{9}x^3+\frac{14}{9}x\\ P_0 &= -\frac{125}{54}x^4-\frac{67}{27}x^2+\frac{1}{18}\\ P_{-1} &= 0\end{aligned}$$

Because $P_{-1}=0$ then $z=e^{\int\omega}$ is a solution. ω is found by finding a solution to the equation generated by the following sum

$$\begin{aligned}\sum_{i=0}^{n}S^i\frac{P_i}{(n-i)!}\omega^i &= 0\\ \sum_{i=0}^{4}S^i\frac{P_i}{(4-i)!}\omega^i &= 0\end{aligned}$$

Where the P_i are the polynomials found earlier. Computing the above sum gives

$$\frac{(x-1)^2(x+1)^2\left(-5x^2-\frac{1}{3}\right)\omega^2}{2}+\frac{8(x-1)^3(x+1)^3x\,\omega^3}{3} - (x-1)^4(x+1)^4\omega^4-\frac{125x^4}{1296}-\frac{67x^2}{648}+\frac{1}{432}+\frac{25\omega\,x^5}{27}-\frac{2\omega\,x^3}{3}-\frac{7\omega x}{27}=0$$

The solution ω of eq. 3A is found as

$$\omega = \frac{1}{6x^2-6}\left(4x+\sqrt{x^2-1+(x^2-1)^{2/3}} + \sqrt{-\frac{2\left(\left(-x^2+\frac{(x^2-1)^{2/3}}{2}+1\right)\sqrt{x^2-1+(x^2-1)^{2/3}}+x^3-x\right)}{\sqrt{x^2-1+(x^2-1)^{2/3}}}}\right) \tag{4A}$$

This ω is used to find a solution to $z''=rz$.

$$z_1(x) = e^{\int \omega\,dx} \tag{5A}$$

Unable to integrate $\int \omega dx$. Leaving the integral unevaluated. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B=0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= e^{\int \omega\,dx} \end{aligned}$$

Where ω given above. The second solution y_2 to the original ode is found using reduction of order

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx \\ &= e^{\int \omega\,dx} \int \frac{e^{\int -\frac{B}{A}\,dx}}{\left(e^{\int \omega\,dx}\right)^2}\,dx \end{aligned}$$

Since $B=0$ then the above reduces to

$$y_2 = e^{\int \omega\,dx} \int \left(e^{\int \omega\,dx}\right)^{-2}\,dx$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(e^{\int \omega\,dx}\right) + c_2\left(e^{\int \omega\,dx} \int \left(e^{\int \omega\,dx}\right)^{-2}\,dx\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) = -\frac{(5x^2+27)y(x)}{36(x^2-1)^2}$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x) + \frac{(5x^2+27)y(x)}{36(x^2-1)^2} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

  $\left[P_2(x) = 0, P_3(x) = \frac{5x^2+27}{36(x^2-1)^2}\right]$

  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1} = 0$

  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1} = \frac{2}{9}$

  - $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

$36(x^2-1)^2\left(\frac{d^2}{dx^2}y(x)\right) + (5x^2+27)\,y(x) = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

$(36u^4 - 144u^3 + 144u^2)\left(\frac{d^2}{du^2}y(u)\right) + (5u^2 - 10u + 32)\,y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..2$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

  - Shift index using $k->k-m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 2..4$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $16a_0(-1+3r)(-2+3r)\,u^r + (16a_1(2+3r)(1+3r) - 2a_0(72r^2-72r+5))\,u^{1+r} + \left(\sum_{k=2}^{\infty} (16a_k(\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$16(-1+3r)(-2+3r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{\frac{1}{3}, \frac{2}{3}\right\}$

- Each term must be 0
  $16a_1(2+3r)(1+3r)-2a_0(72r^2-72r+5)=0$
- Solve for the dependent coefficient(s)
  $a_1=\frac{a_0(72r^2-72r+5)}{8(9r^2+9r+2)}$
- Each term in the series must be 0, giving the recursion relation
  $36(4a_k+a_{k-2}-4a_{k-1})k^2+36(2(4a_k+a_{k-2}-4a_{k-1})r-4a_k-5a_{k-2}+12a_{k-1})k+36(4a_k+a_{k-}$
- Shift index using $k->k+2$
  $36(4a_{k+2}+a_k-4a_{k+1})(k+2)^2+36(2(4a_{k+2}+a_k-4a_{k+1})r-4a_{k+2}-5a_k+12a_{k+1})(k+2)+3$
- Recursion relation that defines series solution to ODE
  $a_{k+2}=-\frac{36k^2a_k-144k^2a_{k+1}+72kra_k-288kra_{k+1}+36r^2a_k-144r^2a_{k+1}-36ka_k-144ka_{k+1}-36ra_k-144ra_{k+1}+5a_k-10a_{k+1}}{16(9k^2+18kr+9r^2+27k+27r+20)}$
- Recursion relation for $r=\frac{1}{3}$
  $a_{k+2}=-\frac{36k^2a_k-144k^2a_{k+1}-12ka_k-240ka_{k+1}-3a_k-74a_{k+1}}{16(9k^2+33k+30)}$
- Solution for $r=\frac{1}{3}$
  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+\frac{1}{3}}, a_{k+2}=-\frac{36k^2a_k-144k^2a_{k+1}-12ka_k-240ka_{k+1}-3a_k-74a_{k+1}}{16(9k^2+33k+30)}, a_1=-\frac{11a_0}{48}\right]$
- Revert the change of variables $u=x+1$
  $\left[y(x)=\sum_{k=0}^{\infty}a_k(x+1)^{k+\frac{1}{3}}, a_{k+2}=-\frac{36k^2a_k-144k^2a_{k+1}-12ka_k-240ka_{k+1}-3a_k-74a_{k+1}}{16(9k^2+33k+30)}, a_1=-\frac{11a_0}{48}\right]$
- Recursion relation for $r=\frac{2}{3}$
  $a_{k+2}=-\frac{36k^2a_k-144k^2a_{k+1}+12ka_k-336ka_{k+1}-3a_k-170a_{k+1}}{16(9k^2+39k+42)}$
- Solution for $r=\frac{2}{3}$
  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+\frac{2}{3}}, a_{k+2}=-\frac{36k^2a_k-144k^2a_{k+1}+12ka_k-336ka_{k+1}-3a_k-170a_{k+1}}{16(9k^2+39k+42)}, a_1=-\frac{11a_0}{96}\right]$
- Revert the change of variables $u=x+1$
  $\left[y(x)=\sum_{k=0}^{\infty}a_k(x+1)^{k+\frac{2}{3}}, a_{k+2}=-\frac{36k^2a_k-144k^2a_{k+1}+12ka_k-336ka_{k+1}-3a_k-170a_{k+1}}{16(9k^2+39k+42)}, a_1=-\frac{11a_0}{96}\right]$
- Combine solutions and rename parameters
  $\left[y(x)=\left(\sum_{k=0}^{\infty}a_k(x+1)^{k+\frac{1}{3}}\right)+\left(\sum_{k=0}^{\infty}b_k(x+1)^{k+\frac{2}{3}}\right), a_{k+2}=-\frac{36k^2a_k-144k^2a_{k+1}-12ka_k-240ka_{k+1}-3a_k-74a}{16(9k^2+33k+30)}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Tetrahedral Galois group A4_SL2.
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.009 (sec)
Leaf size : 25

```
dsolve(diff(diff(y(x),x),x) = -1/36*(5*x^2+27)/(x^2-1)^2*y(x),y(x),singsol=all)
```

$$y = \sqrt{x^2-1}\left(\text{LegendreP}\left(-\frac{1}{6}, \frac{1}{3}, x\right) c_1 + \text{LegendreQ}\left(-\frac{1}{6}, \frac{1}{3}, x\right) c_2\right)$$

Mathematica DSolve solution

Solving time : 0.038 (sec)
Leaf size : 38

```
DSolve[{D[y[x],{x,2}]== -(5*x^2+27)/(36*(x^2-1)^2)*y[x],{}},y[x],x,IncludeSingularSolutions-
```

$$y(x) \to \sqrt{x^2-1}\left(c_1 P_{-\frac{1}{6}}^{\frac{1}{3}}(x) + c_2 Q_{-\frac{1}{6}}^{\frac{1}{3}}(x)\right)$$

2.3.7 Problem Kovacic 2005 paper. Example 2

Internal problem ID [10009]
Book : Collection of Kovacic problems
Section : section 3. Problems from Kovacic related papers
Problem number : Kovacic 2005 paper. Example 2
Date solved : `Monday, January 27, 2025 at 06:18:13 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$y'' = -\frac{y}{4x^2}$$

Solved as second order ode using Kovacic algorithm

Time used: 0.128 (sec)

Writing the ode as

$$y'' + \frac{y}{4x^2} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= \frac{1}{4x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1594: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=-\frac{1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{2}-\left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^{-}}{x-c_1}\right)+(-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x}+(-)\,(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}
$$

Let

$$
p(x)=1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega \, dx} \\ &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A} dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \sqrt{x} \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} dx \\ &= \sqrt{x} \int \frac{1}{x} dx \\ &= \sqrt{x}(\ln(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\sqrt{x}\right) + c_2\left(\sqrt{x}(\ln(x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) = -\frac{y(x)}{4x^2}$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) + \frac{y(x)}{4x^2} = 0$

- Multiply by denominators of the ODE

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right) + y(x) = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

    $\frac{d}{dx}y(x) = \left(\frac{d}{dt}y(t)\right)\left(\frac{d}{dx}t(x)\right)$

  ◦ Compute derivative

    $\frac{d}{dx}y(x) = \frac{\frac{d}{dt}y(t)}{x}$

  ◦ Calculate the 2nd derivative of y with respect to x , using the chain rule

    $\frac{d^2}{dx^2}y(x) = \left(\frac{d^2}{dt^2}y(t)\right)\left(\frac{d}{dx}t(x)\right)^2 + \left(\frac{d^2}{dx^2}t(x)\right)\left(\frac{d}{dt}y(t)\right)$

  ◦ Compute derivative

    $\frac{d^2}{dx^2}y(x) = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $4x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + y(t) = 0$

- Simplify

  $4\frac{d^2}{dt^2}y(t) - 4\frac{d}{dt}y(t) + y(t) = 0$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = \frac{d}{dt}y(t) - \frac{y(t)}{4}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d^2}{dt^2}y(t) - \frac{d}{dt}y(t) + \frac{y(t)}{4} = 0$

- Characteristic polynomial of ODE

  $r^2 - r + \frac{1}{4} = 0$

- Factor the characteristic polynomial

  $\frac{(2r-1)^2}{4} = 0$

- Root of the characteristic polynomial

  $r = \frac{1}{2}$

- 1st solution of the ODE

  $y_1(t) = e^{\frac{t}{2}}$

- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t\,e^{\frac{t}{2}}$

- General solution of the ODE

  $y(t) = C1 y_1(t) + C2 y_2(t)$

- Substitute in solutions

  $y(t) = C1\,e^{\frac{t}{2}} + C2 t\,e^{\frac{t}{2}}$

- Change variables back using $t = \ln(x)$

  $y(x) = \sqrt{x}\,C1 + C2\ln(x)\sqrt{x}$

- Simplify
  $y(x) = (\ln(x)\, C2 + C1)\sqrt{x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 14

```
dsolve(diff(diff(y(x),x),x) = -1/4/x^2*y(x),y(x),singsol=all)
```

$$y = (c_2 \ln(x) + c_1)\sqrt{x}$$

Mathematica DSolve solution

Solving time : 0.019 (sec)
Leaf size : 24

```
DSolve[{D[y[x],{x,2}]== -1/(4*x^2)*y[x],{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}\sqrt{x}(c_2 \log(x) + 2c_1)$$

2.3.8 Problem David Saunders 1981 paper. Example 1

Internal problem ID [10010]
Book : Collection of Kovacic problems
Section : section 3. Problems from Kovacic related papers
Problem number : David Saunders 1981 paper. Example 1
Date solved : `Monday, January 27, 2025 at 06:18:14 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' = (x^2 + 3)\, y$$

Solved as second order ode using Kovacic algorithm

Time used: 0.207 (sec)

Writing the ode as

$$y'' + (-x^2 - 3)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -x^2 - 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 3}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 3 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = (x^2 + 3)\, z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1596: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x + \frac{3}{2x} - \frac{9}{8x^3} + \frac{27}{16x^5} - \frac{405}{128x^7} + \frac{1701}{256x^9} - \frac{15309}{1024x^{11}} + \frac{72171}{2048x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= x
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = x^2
$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2+3}{1} \\
&= Q + \frac{R}{1} \\
&= \left(x^2+3\right) + (0) \\
&= x^2 + 3
\end{aligned}
$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is 3. Now b can be found.

$$
\begin{aligned}
b &= (3) - (0) \\
&= 3
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= x \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{3}{1} - 1\right) &&= 1 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{3}{1} - 1\right) &&= -2
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = x^2 + 3
$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	x	1	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 1$, and since there are no poles, then

$$
\begin{aligned}
d &= \alpha_\infty^+ \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + (x) \\ &= x \\ &= x\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2(x)(1) + \left((1) + (x)^2 - \left(x^2 + 3\right)\right) &= 0 \\ -2a_0 &= 0\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= (x)\, e^{\int x dx} \\ &= (x)\, e^{\frac{x^2}{2}} \\ &= x\, e^{\frac{x^2}{2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1 \\ &= x\, e^{\frac{x^2}{2}}\end{aligned}$$

Which simplifies to

$$y_1 = x\, e^{\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= x\,\mathrm{e}^{\frac{x^2}{2}} \int \frac{1}{x^2 \mathrm{e}^{x^2}}\, dx \\
&= x\,\mathrm{e}^{\frac{x^2}{2}} \left(-\frac{\mathrm{e}^{-x^2}}{x} - \sqrt{\pi}\,\mathrm{erf}(x)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x\,\mathrm{e}^{\frac{x^2}{2}}\right) + c_2 \left(x\,\mathrm{e}^{\frac{x^2}{2}} \left(-\frac{\mathrm{e}^{-x^2}}{x} - \sqrt{\pi}\,\mathrm{erf}(x)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2}y(x) = (x^2+3)\,y(x)$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dx^2}y(x)$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
$\frac{d^2}{dx^2}y(x) + (-x^2-3)\,y(x) = 0$
- Assume series solution for $y(x)$
$y(x) = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions
  ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$
  $x^m \cdot y(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$
  ◦ Shift index using $k- >k-m$
  $x^m \cdot y(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$
  ◦ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion
  $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\,x^{k-2}$
  ◦ Shift index using $k- >k+2$
  $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\,x^k$

Rewrite ODE with series expansions
$2a_2 - 3a_0 + (6a_3 - 3a_1)\,x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)(k+1) - 3a_k - a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0
$[2a_2 - 3a_0 = 0, 6a_3 - 3a_1 = 0]$
- Solve for the dependent coefficient(s)
$\left\{a_2 = \frac{3a_0}{2}, a_3 = \frac{a_1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation
  $(k^2 + 3k + 2) a_{k+2} - 3a_k - a_{k-2} = 0$
- Shift index using $k- > k+2$
  $\left((k+2)^2 + 3k + 8\right) a_{k+4} - 3a_{k+2} - a_k = 0$
- Recursion relation that defines the series solution to the ODE
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{3a_{k+2}+a_k}{k^2+7k+12}, a_2 = \frac{3a_0}{2}, a_3 = \frac{a_1}{2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 30

```
dsolve(diff(diff(y(x),x),x) = (x^2+3)*y(x),y(x),singsol=all)
```

$$y = x\left(c_2 \operatorname{erf}(x) \sqrt{\pi} + c_1\right) e^{\frac{x^2}{2}} + e^{-\frac{x^2}{2}} c_2$$

Mathematica DSolve solution

Solving time : `0.062` (sec)
Leaf size : 46

```
DSolve[{D[y[x],{x,2}]== (x^2+3)*y[x],{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-\frac{x^2}{2}}\left(-\sqrt{\pi} c_2 e^{x^2} x \operatorname{erf}(x) + c_1 e^{x^2} x - c_2\right)$$

2.3.9 Problem David Saunders 1981 paper. Example 3

Internal problem ID [10011]
Book : Collection of Kovacic problems
Section : section 3. Problems from Kovacic related papers
Problem number : David Saunders 1981 paper. Example 3
Date solved : Monday, January 27, 2025 at 06:18:14 PM
CAS classification : [[_2nd_order, _exact, _linear, _homogeneous]]

Solve

$$x^2y'' = 2y$$

Solved as second order ode using Kovacic algorithm

Time used: 0.100 (sec)

Writing the ode as

$$x^2y'' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 0 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1598: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the gcd$(s,t)=1$. This gives $b=2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=2 \\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=-1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{2}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=-1$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -1-(-1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^-}{x-c_1}\right)+(-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x}+(-)\,(0) \\
&= -\frac{1}{x} \\
&= -\frac{1}{x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}
$$

Let

$$
p(x)=1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0)+2\left(-\frac{1}{x}\right)(0)+\left(\left(\frac{1}{x^2}\right)+\left(-\frac{1}{x}\right)^2-\left(\frac{2}{x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{1}{x} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= \frac{1}{x} \int \frac{1}{\frac{1}{x^2}} \, dx \\ &= \frac{1}{x} \left(\frac{x^3}{3} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{1}{x} \right) + c_2 \left(\frac{1}{x} \left(\frac{x^3}{3} \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) = 2y(x)$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{2y(x)}{x^2}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea

  $\frac{d^2}{dx^2}y(x) - \frac{2y(x)}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) - 2y(x) = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $\frac{d}{dx}y(x) = \left(\frac{d}{dt}y(t)\right)\left(\frac{d}{dx}t(x)\right)$

  - Compute derivative

    $\frac{d}{dx}y(x) = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $\frac{d^2}{dx^2}y(x) = \left(\frac{d^2}{dt^2}y(t)\right)\left(\frac{d}{dx}t(x)\right)^2 + \left(\frac{d^2}{dx^2}t(x)\right)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $\frac{d^2}{dx^2}y(x) = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - 2y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - \frac{d}{dt}y(t) - 2y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - r - 2 = 0$

- Factor the characteristic polynomial

  $(r+1)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 2)$

- 1st solution of the ODE

  $y_1(t) = e^{-t}$

- 2nd solution of the ODE

  $y_2(t) = e^{2t}$

- General solution of the ODE

  $y(t) = C1 y_1(t) + C2 y_2(t)$

- Substitute in solutions

  $y(t) = C1\, e^{-t} + C2\, e^{2t}$

- Change variables back using $t = \ln(x)$

  $y(x) = \frac{C1}{x} + C2\, x^2$

- Simplify

  $y(x) = \frac{C1}{x} + C2\, x^2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 15

```
dsolve(x^2*diff(diff(y(x),x),x) = 2*y(x),y(x),singsol=all)
```

$$y = \frac{c_2 x^3 + c_1}{x}$$

Mathematica DSolve solution

Solving time : 0.012 (sec)
Leaf size : 18

```
DSolve[{x^2*D[y[x],{x,2}]== 2*y[x],{}},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2 x^3 + c_1}{x}$$

2.3.10 Problem Carolyn J. Smith 1984 paper. Appendix B examples and tests. Example 1

Internal problem ID [10012]
Book : Collection of Kovacic problems
Section : section 3. Problems from Kovacic related papers
Problem number : Carolyn J. Smith 1984 paper. Appendix B examples and tests. Example 1
Date solved : `Monday, January 27, 2025 at 06:18:15 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y''+4xy'+\left(4x^2+2\right)y=0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.053 (sec)

Writing the ode as

$$y''+4xy'+\left(4x^2+2\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=1\\ B&=4x\\ C&=4x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s&=0\\ t&=1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.1600: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x}{1}\, dx} \\
&= z_1 e^{-x^2} \\
&= z_1 \left(\mathrm{e}^{-x^2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2x^2}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(e^{-x^2}\right) + c_2\left(e^{-x^2}(x)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + 4x\left(\frac{d}{dx}y(x)\right) + (4x^2+2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

- ☐ Rewrite ODE with series expansions
  - ○ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  - ○ Shift index using $k->k-m$

    $x^m \cdot y(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  - ○ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$

  - ○ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

    $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - ○ Shift index using $k->k+2$

    $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 + 2a_0 + (6a_3 + 6a_1)\,x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 2a_k(2k+1) + 4a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 + 2a_0 = 0, 6a_3 + 6a_1 = 0]$

- Solve for the dependent coefficient(s)
  $\{a_2 = -a_0, a_3 = -a_1\}$
- Each term in the series must be 0, giving the recursion relation
  $(k^2 + 3k + 2) a_{k+2} + 4a_k k + 2a_k + 4a_{k-2} = 0$
- Shift index using $k- >k + 2$
  $\left((k+2)^2 + 3k + 8\right) a_{k+4} + 4a_{k+2}(k+2) + 2a_{k+2} + 4a_k = 0$
- Recursion relation that defines the series solution to the ODE
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{2(2ka_{k+2}+2a_k+5a_{k+2})}{k^2+7k+12}, a_2 = -a_0, a_3 = -a_1\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 16

```
dsolve(diff(diff(y(x),x),x)+4*diff(y(x),x)*x+(4*x^2+2)*y(x) = 0,y(x),singsol=all)
```

$$y = \mathrm{e}^{-x^2}(c_2 x + c_1)$$

Mathematica DSolve solution

Solving time : 0.025 (sec)
Leaf size : 20

```
DSolve[{D[y[x],{x,2}]+4*x*D[y[x],x]+(4*x^2+2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to e^{-x^2}(c_2 x + c_1)$$

2.3.11 Problem Carolyn J. Smith 1984 paper. Appendix B examples and tests. Example 2

Internal problem ID [10013]
Book : Collection of Kovacic problems
Section : section 3. Problems from Kovacic related papers
Problem number : Carolyn J. Smith 1984 paper. Appendix B examples and tests. Example 2
Date solved : `Monday, January 27, 2025 at 06:18:15 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - 2xy' + \left(x^2+2\right)y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.125 (sec)

Writing the ode as

$$x^2y'' - 2xy' + \left(x^2+2\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1602: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\, dx} \\ &= z_1 e^{\ln(x)} \\ &= z_1(x) \end{aligned}$$

Which simplifies to

$$y_1 = x\cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(x\cos(x)) + c_2(x\cos(x)(\tan(x))) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve
$x^2\left(\frac{d^2}{dx^2}y(x)\right) - 2x\left(\frac{d}{dx}y(x)\right) + (x^2+2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{(x^2+2)y(x)}{x^2} + \frac{2\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linea
  $\frac{d^2}{dx^2}y(x) - \frac{2\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(x^2+2)y(x)}{x^2} = 0$
- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{2}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = -2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $x^2\left(\frac{d^2}{dx^2}y(x)\right) - 2x\left(\frac{d}{dx}y(x)\right) + (x^2+2)\,y(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- Shift index using $k- > k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x \cdot \left(\frac{d}{dx} y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r) x^{k+r}$
- Convert $x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1) x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)(-2+r) x^r + a_1 r(-1+r) x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-1)(k+r-2) + a_{k-2}) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$
- Each term must be 0

  $a_1 r(-1+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)(k+r-2) + a_{k-2} = 0$
- Shift index using $k- > k + 2$

  $a_{k+2}(k+1+r)(k+r) + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+1+r)(k+r)}$
- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$
- Solution for $r = 1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$
- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$
- Solution for $r = 2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+2)(k+3)}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 15

```
dsolve(x^2*diff(diff(y(x),x),x)-2*diff(y(x),x)*x+(x^2+2)*y(x) = 0,y(x),singsol=all)
```

$$y = x(\sin(x) c_1 + \cos(x) c_2)$$

Mathematica DSolve solution

Solving time : 0.035 (sec)
Leaf size : 33

```
DSolve[{x^2*D[y[x],{x,2}]-2*x*D[y[x],x]+(x^2+2)*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to c_1 e^{-ix} x - \frac{1}{2} i c_2 e^{ix} x$$

2.3.12 Problem Carolyn J. Smith 1984 paper. Appendix B examples and tests. Example 3

Internal problem ID [10014]
Book : Collection of Kovacic problems
Section : section 3. Problems from Kovacic related papers
Problem number : Carolyn J. Smith 1984 paper. Appendix B examples and tests. Example 3
Date solved : `Monday, January 27, 2025 at 06:18:16 PM`
CAS classification : `[[_2nd_order, _exact, _linear, _homogeneous]]`

Solve

$$(x-2)^2 y'' - (x-2) y' - 3y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.126 (sec)

Writing the ode as

$$(x-2)^2 y'' + (-x+2) y' - 3y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= (x-2)^2 \\ B &= -x+2 \\ C &= -3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15}{4(x-2)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15 \\ t &= 4(x-2)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15}{4(x-2)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.1604: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-2)^2$. There is a pole at $x = 2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4(x-2)^2}$$

For the pole at $x = 2$ let b be the coefficient of $\frac{1}{(x-2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15}{4(x-2)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15}{4(x-2)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
2	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{3}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= -\frac{3}{2} - \left(-\frac{3}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{2(x-2)} + (-)(0) \\
&= -\frac{3}{2(x-2)} \\
&= -\frac{3}{2(x-2)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{2(x-2)}\right)(0) + \left(\left(\frac{3}{2(x-2)^2}\right) + \left(-\frac{3}{2(x-2)}\right)^2 - \left(\frac{15}{4(x-2)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= \mathrm{e}^{\int -\frac{3}{2(x-2)} dx} \\ &= \frac{1}{(x-2)^{3/2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x+2}{(x-2)^2}\, dx} \\ &= z_1 e^{\frac{\ln(x-2)}{2}} \\ &= z_1\left(\sqrt{x-2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x-2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x+2}{(x-2)^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\ln(x-2)}}{(y_1)^2}\, dx \\ &= y_1\left(\frac{(x-2)^4}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{1}{x-2}\right) + c_2\left(\frac{1}{x-2}\left(\frac{(x-2)^4}{4}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Maple step by step solution

Let's solve

$(x-2)^2 \left(\frac{d^2}{dx^2}y(x)\right) - (x-2)\left(\frac{d}{dx}y(x)\right) - 3y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{3y(x)}{(x-2)^2} + \frac{\frac{d}{dx}y(x)}{x-2}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) - \frac{\frac{d}{dx}y(x)}{x-2} - \frac{3y(x)}{(x-2)^2} = 0$

□ Check to see if $x_0 = 2$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{1}{x-2}, P_3(x) = -\frac{3}{(x-2)^2}\right]$

  ◦ $(x-2) \cdot P_2(x)$ is analytic at $x = 2$

    $\left((x-2) \cdot P_2(x)\right)\Big|_{x=2} = -1$

  ◦ $(x-2)^2 \cdot P_3(x)$ is analytic at $x = 2$

    $\left((x-2)^2 \cdot P_3(x)\right)\Big|_{x=2} = -3$

  ◦ $x = 2$is a regular singular point

  Check to see if $x_0 = 2$ is a regular singular point

  $x_0 = 2$

- Multiply by denominators

  $(x-2)^2 \left(\frac{d^2}{dx^2}y(x)\right) + (-x+2)\left(\frac{d}{dx}y(x)\right) - 3y(x) = 0$

- Change variables using $x = u + 2$ so that the regular singular point is at $u = 0$

  $u^2\left(\frac{d^2}{du^2}y(u)\right) - u\left(\frac{d}{du}y(u)\right) - 3y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite DE with series expansions

  ◦ Convert $u \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion

    $u \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r}$

  ◦ Convert $u^2 \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

    $u^2 \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r}$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} a_k(k+r+1)\,(k+r-3)\, u^{k+r} = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+1)\,(k-3) = 0$

- Recursion relation that defines series solution to ODE

  $a_k = 0$

- Recursion relation for $r = 0$

  $a_k = 0$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_k = 0\right]$$

- Revert the change of variables $u = x - 2$

  $$\left[y(x) = \sum_{k=0}^{\infty} a_k (x-2)^k, a_k = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 19

```
dsolve((x-2)^2*diff(diff(y(x),x),x)-(x-2)*diff(y(x),x)-3*y(x) = 0,y(x),singsol=all)
```

$$y = \frac{(x-2)^4 c_1 + c_2}{x-2}$$

Mathematica DSolve solution

Solving time : `0.036` (sec)
Leaf size : 22

```
DSolve[{(x-2)^2*D[y[x],{x,2}]-(x-2)*D[y[x],x]-3*y[x]==0,{}},y[x],x,IncludeSingularSolutions->T
```

$$y(x) \to c_1(x-2)^3 + \frac{c_2}{x-2}$$